MTH 655, Finite Element Methods: Homework 1

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Problem 1

Consider the following two point boundary value problem (BVP). Let $\Omega = (0,1)$. Given $f \in C^0(\Omega)$, find $u \in C^2(\Omega) \cup C^0(\bar{\Omega})$ such that

$$(D2): \begin{cases} -\frac{d^2u}{dx^2} + u = f; \; 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

Obtain the weak or variational formulation for the above problem using test functions $v \in H_0^1(\Omega)$.

Solution

We begin by multiplying each side of $(D2)$ by the test function v and then integrating over the domain, Ω.

$$-\int_\Omega \frac{d^2u}{dx^2} v \; dx + \int_\Omega uv \; dx = \int_\Omega fv \; dx$$

Using integration by parts on the first integral gives

$$\int_\Omega \frac{d^2u}{dx^2} v \; dx = \frac{du}{dx} v|_{v(0)}^{v(1)} - \int_\Omega \frac{du}{dx}\frac{dv}{dx} \; dx$$

Applying the boundary conditions and resubstituting then gives the final variational formulation of $(D2)$

$$\int_\Omega \frac{du}{dx}\frac{dv}{dx} \; dx + \int_\Omega uv \; dx = \int_\Omega fv \; dx; \; v \in H_0^1$$

Problem 2

Consider the space V_k^2 of continuous piecewise quadratic functions on the interval $[0,1]$.

(a) Show that this space has (nodal) basis functions that are defined at the nodes x_i as well as the midpoints $\frac{x_i + x_{i+1}}{2}$ of a partition of $[0,1]$.

(b) For the two point BVP considered in class, i.e., Given $f \in C^0(\Omega)$, find $u \in C^2(\Omega) \cup C^0(\bar{\Omega})$ such that

$$(D) : \begin{cases} -\frac{d^2u}{dx^2} = f;\ 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

compute the local stiffness matrix on any interior element and discuss how these element contributions are used to assemble the stiffness matrix.

Solution

(a) We begin by defining the basis functions

$$V_k^2 = \{v_I = \sum_{i=1}^{k} \alpha_i \phi_i(x_j);\ \phi_i(x_i) = 1;\ \phi_{i-\frac{1}{2}}(x_{i-\frac{1}{2}}) = 1;\ \phi_i(x) = \frac{(x - x_{i-1})(x - x_{i-\frac{1}{2}})}{x_i - x_{i-1})(x_i - x_{i-\frac{1}{2}})},\ x \in [x_{i-1}, x_i],$$
$$\phi_i(x) = \frac{(x - x_{i+\frac{1}{2}})(x - x_{i+1})}{x_i - x_{i+\frac{1}{2}})(x_i - x_{i+1})},\ x \in [x_i, x_{i+1}],\ 0 \text{ otherwise};$$
$$\phi_{i-\frac{1}{2}} = \frac{(x - x_{i-1})(x - x_i)}{x_{i-\frac{1}{2}} - x_{i-1})(x_{i-\frac{1}{2}} - x_i)},\ x \in [x_{i-1}, x_i]\ \ 0 \text{ otherwise}\}$$

To show that this is a nodal basis we first show linear independence by noting that

$$\sum_{i=1}^{k} c_i \phi_i(x_i) = 0$$

implies that $c_i = 0$ since $\phi_i(x_i) = 1$.

We then show that $\{\phi_i\}$ spans V_k^2. This is true if for $v \in V_k^2 \Rightarrow v = v_I$. Since

$$v_I = \sum_{i=1}^{k} v(x_i)\phi_i$$

we must show $v - v_I = 0$ identically. This is determined by noting that v and v_I are both quadratic on each $[x_{i-1}, x_i]$ and zero at the endpoints of the interval. Thus, v_I spans V_k^2 and is a nodal basis function.
(b) We begin by defining the local stiffness matrix as

$$A^l = \begin{bmatrix} \int_{I_l} (\frac{dl_1}{dx})^2\ dx & \int_{I_l} \frac{dl_1}{dx}\frac{dl_2}{dx}\ dx & \int_{I_l} \frac{dl_1}{dx}\frac{dl_3}{dx}\ dx \\ \int_{I_l} \frac{dl_2}{dx}\frac{dl_1}{dx}\ dx & \int_{I_l} (\frac{dl_2}{dx})^2\ dx & \int_{I_l} \frac{dl_2}{dx}\frac{dl_3}{dx}\ dx \\ \int_{I_l} \frac{dl_3}{dx}\frac{dl_1}{dx}\ dx & \int_{I_l} \frac{dl_3}{dx}\frac{dl_2}{dx}\ dx & \int_{I_l} (\frac{dl_3}{dx})^2\ dx \end{bmatrix}$$

where we note $A_{ij} = A_{ji}$. We then shift each segment of the nodal basis functions l_1, l_2, l_3 to the interval $[0, h]$. This gives

$$l_1 = \frac{x^2 - (\frac{3}{2})hx + \frac{h^2}{2}}{\frac{h^2}{2}}$$
$$l_2 = \frac{xh - x^2}{\frac{h^2}{4}}$$
$$l_3 = \frac{x^2 - (\frac{h}{2})x}{\frac{h^2}{2}}$$

Differentiating each of these basis functions with respect to x gives

$$\frac{l_1}{dx} = \frac{4x}{h^2} - \frac{3}{h}$$
$$\frac{l_2}{dx} = \frac{4}{h} - \frac{8x}{h^2}$$
$$\frac{l_3}{dx} = \frac{4x}{h^2} - \frac{1}{h}$$

Then the integrals in the local stiffness matrix are as follows

$$\int_{I_l} (\frac{dl_1}{dx})^2 \, dx = \int_0^h \frac{4x^2 - 6xh + \frac{9}{4}h^2}{\frac{h^4}{4}} \, dx = \frac{7}{3h}$$
$$\int_{I_l} (\frac{dl_2}{dx})^2 \, dx = \int_0^h \frac{4x^2 - 4xh + h^2}{\frac{h^4}{16}} \, dx = \frac{16}{3h}$$
$$\int_{I_l} (\frac{dl_3}{dx})^2 \, dx = \int_0^h \frac{4x^2 - 2xh + \frac{1}{4}h^2}{\frac{h^4}{4}} \, dx = \frac{7}{3h}$$
$$\int_{I_l} \frac{dl_1}{dx}\frac{dl_2}{dx} \, dx = \int_0^h \frac{-4x^2 + 5xh - \frac{3}{2}h^2}{\frac{h^4}{8}} \, dx = \frac{-8}{3h}$$
$$\int_{I_l} \frac{dl_1}{dx}\frac{dl_3}{dx} \, dx = \int_0^h \frac{4x^2 - 4xh - \frac{3}{4}h^2}{\frac{h^4}{4}} \, dx = \frac{1}{3h}$$
$$\int_{I_l} \frac{dl_2}{dx}\frac{dl_3}{dx} \, dx = \int_0^h \frac{-4x^2 + 3xh - \frac{1}{2}h^2}{\frac{h^4}{8}} \, dx = \frac{-8}{3h}$$

Thus, the local stiffness matrix is

$$A^l = \frac{1}{3h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

The global stiffness matrix would be assembled as follows to form a pentagonal matrix.

$$A = \begin{bmatrix} A^1_{22} & A^1_{23} & 0 & 0 & 0 & 0 & ... \\ A^1_{32} & A^1_{33} + A^2_{11} & A^2_{12} & A^2_{13} & 0 & 0 & ... \\ 0 & A^2_{21} & A^2_{22} & A^2_{23} & 0 & 0 & ... \\ 0 & A^3_{31} & A^2_{32} & A^2_{33} + A^3_{11} & A^3_{12} & A^3_{13} & ... \\ ... & ... & ... & ... & ... & ... & ... \end{bmatrix}$$
$$= \begin{bmatrix} 16 & -8 & 0 & 0 & 0 & 0 & ... \\ -8 & 14 & -8 & 1 & 0 & 0 & ... \\ 0 & -8 & 16 & -8 & 0 & 0 & ... \\ 0 & 1 & -8 & 14 & -8 & 1 & ... \\ ... & ... & ... & ... & ... & ... & ... \end{bmatrix}$$

Problem 3

Consider a uniform grid over the interval $[0, 1]$ with step size h and an associated finite element space V_k^1 of continuous piecewise linear functions that satisfy zero boundary conditions.

(a) Find the linear interpolant $I_h v$ for $v(x) = x^2$.

(b) Compute analytically the norm of the interpolation error $v - I_h v$ in the L^2 and energy norms.

Solution

(a) A linear interpolant using piecewise Lagrange polynomials may be represented as follows

$$l_1(x) = -\frac{x-h}{h}$$
$$l_2(x) = \frac{x}{h}$$

which gives for $I_i v$

$$I_i v = (x_i)^2 \frac{x}{h} - (x_{i-1})^2 (\frac{x-h}{h})$$

Thus, $I_h v = \cup_{i=1}^k I_i v$.
(b) The L^2 error was calculated with the aid of Mathematica as

$$\begin{aligned} \|v - I_h v\|_{L^2} &= (\sum_{i=1}^{k} [\int_0^h ((x + (i-1)h)^2 - i^2 hx + (i-1)^2 h(x-h))^2 \; dx])^{\frac{1}{2}} \\ &= (\sum_{i=1}^{k} \frac{h^5}{30})^{\frac{1}{2}} \\ &= (\frac{kh^5}{30} = \frac{h^2}{\sqrt{30}} \end{aligned}$$

Similarly, the Energy norm error was also calculated using Mathematica as

$$\begin{aligned}
\|v - I_h v\|_E &= (\sum_{i=1}^{k}[\int_0^h (\frac{d}{dx}[v - I_i v])^2])^{\frac{1}{2}} \\
&= (\sum_{i=1}^{k}[\int_0^h h^2 - 4hx + 4x^2 \ dx])^{\frac{1}{2}} \\
&= (\sum_{i=1}^{k} \frac{h^3}{3})^{\frac{1}{2}} = (\frac{kh^3}{3})^{\frac{1}{2}} = \frac{h}{\sqrt{3}}
\end{aligned}$$

These error estimates correspond with the interpolation theory that states that the L_2 and Energy errors are second-order and first-order in h, respectively.

Problem 4

Consider the two point BVP that we studied in class, i.e.,

For $\Omega = (0, 1)$, given $f \in C^0(\Omega)$, find $u \in C^2(\Omega) \cup C^0(\bar{\Omega})$ such that

$$(D) : \begin{cases} -\frac{d^2u}{dx^2} = f; \ 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

Compute f so that the exact solution for the BVP is $u(x) = sin(\pi x)$. Write a MATLAB code that uses a uniform mesh and linear finite elements and plot the discrete solution for $h = 2^{-j}, j = 1, ..., 3$. Also, plot the exact solution for comparison.

Solution

f may be found as follows

$$f = -\frac{d^2u}{dx^2} = -\frac{d^2}{dx^2}[\sin(\pi x)] = \pi^2 \sin(\pi x)$$

Solving (D) via MATLAB using finite elements produced the following solutions for the three mesh sizes, h.

The grid convergence in Figure 2 shows second-order convergence in the max norm.

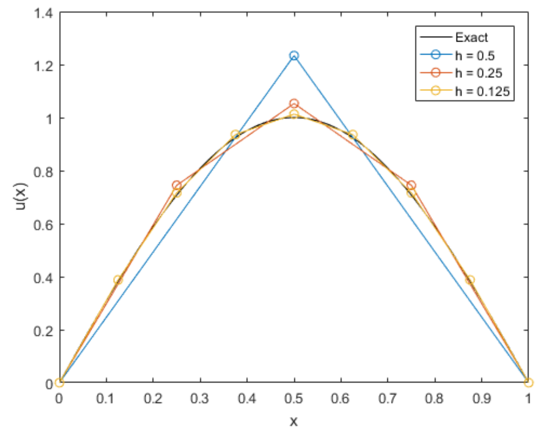


Figure 1: The FEM solution at various mesh sizes compared to the exact solution.

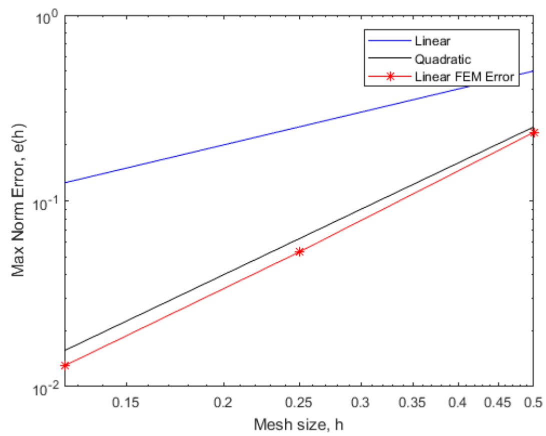
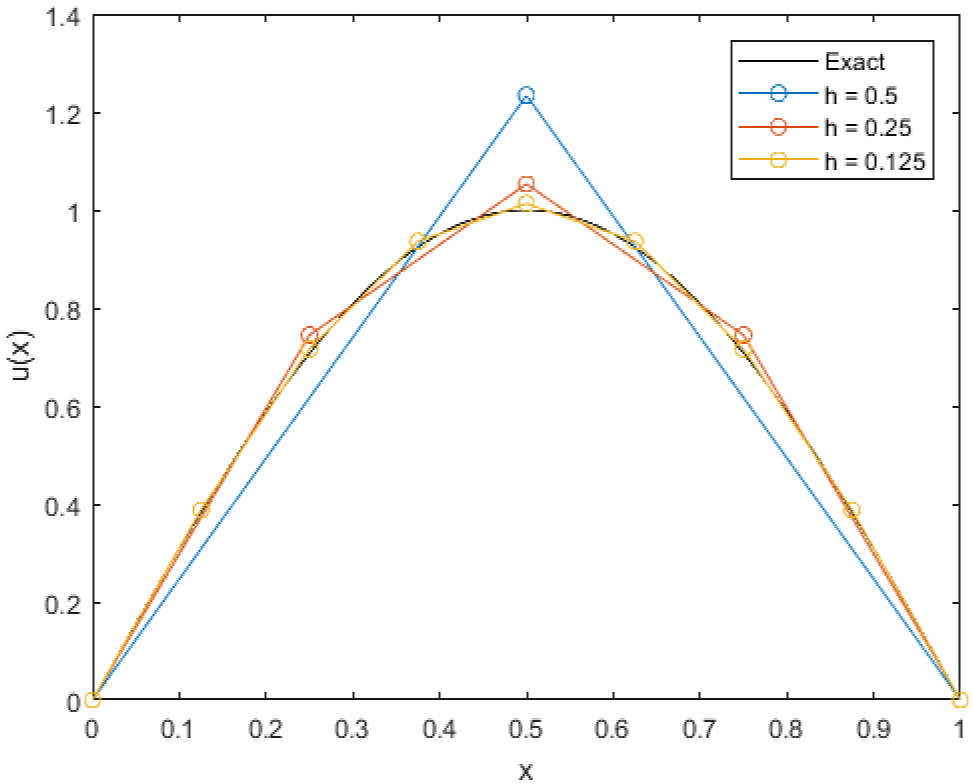
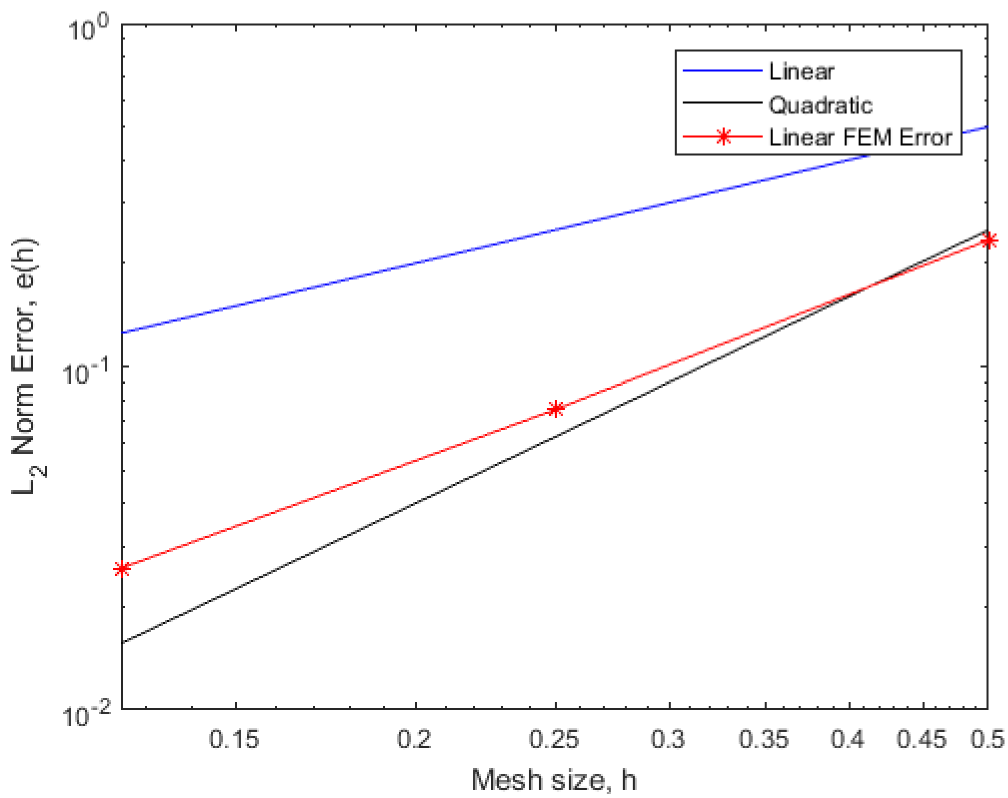


Figure 2: The FEM max norm error exhibits second-order behavior.

```
% Main.m
% Peter Ferrero, Oregon State University, MTH655, 1/31/2018
% The main file for the FEM 1D method to solve Problem 4 of Homework 1
 for
% MTH 655.

clear all

n = [2,4,8];
N = length(n);
x = [0:0.001:1]';
ExactSol = Exact(x);

figure(1)
plot(x, ExactSol, 'k')
xlabel('x')
ylabel('u(x)')
hold on

for i = 1:N

    [FemSol, x] = SimpleFEM1D(n(i));
    ExactSol = Exact(x');
    error(i) = norm(ExactSol-FemSol, 2);
    plot(x,FemSol,'-o')

end

a = 0; % left endpoint
b = 1; % right endpoint
h = (b-a)./n; % uniform mesh size

legend('Exact', 'h = 0.5', 'h = 0.25', 'h = 0.125')
hold off

figure(2)
loglog(h,h,'b-',h,h.^2,'k-',h,error,'*-r')
xlabel('Mesh size, h')
ylabel('L_2 Norm Error, e(h)')
legend('Linear', 'Quadratic', 'Linear FEM Error')
```


Exact
h = 0.5
h = 0.25
h = 0.125
u(x)
x


Linear
Quadratic
Linear FEM Error
L2 Norm Error, e(h)
Mesh size, h

Published with MATLAB® R2017a