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Chapter 1

Resonator response functions

1.1 Using this document

- This document contains expressions for many relevant parameters such as loaded, internal and external quality factors Q, the different decay rates κ, the reflection coefficients $\Gamma = S_{11}$ and transmission parameters S_{12} for many different kinds of lumped element or transmission line based electrical/electromagnetic resonators.
- Most of the parameters have individual expressions for each type of resonator. They are always just defined for the subsection, in which they are used. For example, an expression which describes Q_{ext} for a particular resonator is only valid for this particular resonator, although the external quality factor is just denoted as Q_{ext} for all types of resonators considered here.
- The calculations, formulas and expression are encoded in three different colors. Black are in most cases less important but maybe helpful steps in the calculations. Green are important relations, which are exact within the used model. Blue are important expressions, which are approximations and not generally valid. Sometimes they follow from a strict Taylor expansion and are good approximations in a certain parameter regime, sometimes they follow from the neglection of small terms, sometimes they follow from a combination of both approximation approaches.

1.2 Response functions of parallel RLC circuits

1.2.1 Simple parallel RLC circuit

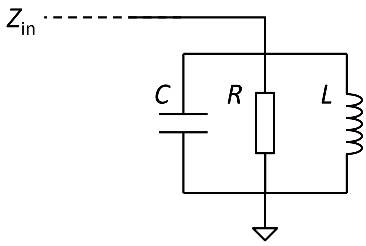

Figure 1.1: Schematic of a parallel RLC circuit.

First, we consider a circuit in which the components R, L and C are connected in parallel to a source, cf. Fig. 1.1. Then, the input impedance, i.e., the impedance between the connection points of the source to the circuit (here: between the input point and ground), is given by

$$\frac{1}{Z_{\text{in}}} = \frac{1}{R} + \frac{1}{i\omega L} + i\omega C, \tag{1.1}$$

which, with the resonance frequency $\omega_0^2 = 1/LC$, can be expressed as

$$Z_{\text{in}} = \frac{R}{1 + iRC\omega_0 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}. \tag{1.2}$$

The last expression can be approximated around resonance $\omega \approx \omega_0$ by means of a Taylor expansion as

$$Z_{\text{in}} \approx \frac{R}{1 + 2iRC(\omega - \omega_0)}. \tag{1.3}$$

The (internal) quality factor, defined with the energy stored in the resonator W and the dissipated power P_{loss} as

$$Q = \omega_0 \frac{W}{P_{\text{loss}}} \tag{1.4}$$

of the parallel RLC circuit is given by

$$Q = \omega_0 RC = \frac{R}{\omega_0 L}. \tag{1.5}$$

because $W = \frac{1}{2}CV^2$ and $P_{\text{loss}} = V^2/2R$.
With this and with $\Delta\omega = \omega - \omega_0$, the input impedance can be written as

$$
\begin{aligned}
Z_{\text{in}} &= \frac{R}{1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} && (1.6) \\
&\approx \frac{R}{1 + 2iQ\frac{\Delta\omega}{\omega_0}} && (1.7) \\
&= \frac{R}{1 + 2i\frac{\Delta\omega}{\kappa}} && (1.8)
\end{aligned}
$$

where in the last step we have used the general relation for the decay rate $\kappa = \omega_0/Q$, which implies

$$
\kappa = \frac{1}{RC} = \frac{\omega_0^2 L}{R} \qquad (1.9)
$$

for the parallel RLC circuit.

1.2.2 Parallel RLC circuit directly coupled on one side to a transmission line

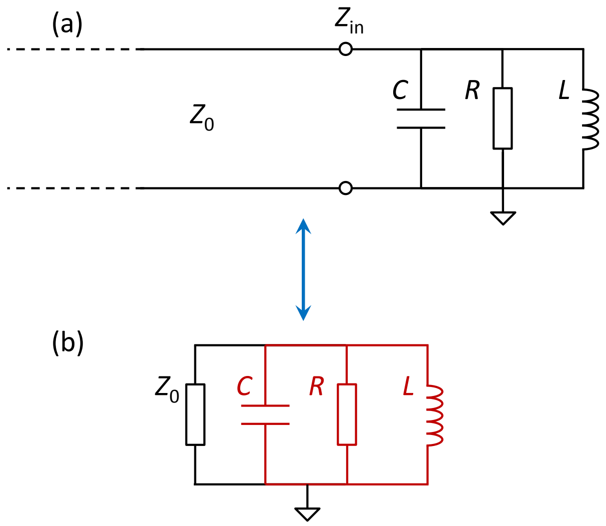

Figure 1.2: (a) Schematic of a parallel RLC circuit coupled by one port to a transmission line with real-valued characteristic impedance Z_0. (b) Equivalent circuit of (a), where the transmission line has been replaced by a lumped element resistor Z_0. In red the internal parts of the circuit are drawn, while the external part is drawn in black.

As next step, we consider a parallel RLC circuit directly connected to a transmission line with characteristic impedance Z_0 (we assume Z_0 to be real-valued here), by which it can be driven and characterized, cf. Fig. 1.2 (a). When a voltage wave is sent to the circuit, a part of the wave is reflected and a part is going into the circuit. First, the quality factor of the resonator is changed due to the additional loss channel, the transmission line with impedance Z_0, which can just be viewed as an additional resistor in parallel. cf. Fig. 1.2 (b). The new total resistance is given by

$$R_{\mathrm{tot}} = \frac{RZ_0}{R + Z_0} \tag{1.10}$$

and the new quality factor by

$$Q_{\mathrm{L}} = \omega_0 R_{\mathrm{tot}} C = \frac{R_{\mathrm{tot}}}{\omega_0 L}. \tag{1.11}$$

This loaded quality factor Q_{L} can be separated into an internal part $Q_{\mathrm{int}} = \omega_0 RC$ due to R and an external part $Q_{\mathrm{ext}} = \omega_0 Z_0 C$ due to Z_0. To see this, we consider the loaded loss factor

$$\frac{1}{Q_{\mathrm{L}}} = \frac{R + Z_0}{\omega_0 R Z_0 C} = \frac{1}{\omega_0 RC} + \frac{1}{\omega_0 Z_0 C} = \frac{1}{Q_{\mathrm{int}}} + \frac{1}{Q_{\mathrm{ext}}}. \tag{1.12}$$

Note, that Q-factors add reciprocally.
For the total decay rate κ we hence get

$$\kappa_{\text{tot}} = \frac{1}{RC} + \frac{1}{Z_0 C} = \kappa_{\text{int}} + \kappa_{\text{ext}}. \tag{1.13}$$

and the relations

$$\frac{\kappa_{\text{int}}}{\kappa_{\text{ext}}} = \frac{Z_0}{R}, \qquad \frac{\kappa_{\text{int}}}{\kappa_{\text{tot}}} = \frac{Z_0}{R+Z_0}, \qquad \frac{\kappa_{\text{ext}}}{\kappa_{\text{tot}}} = \frac{R}{R+Z_0}. \tag{1.14}$$

For transmission line connected circuits, however, there are some more interesting response functions such as reflection and transmission parameters. The reflection coefficient at the input of the circuit is given by

$$\Gamma = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \frac{\kappa_{\text{ext}} - \kappa_{\text{int}} - i\omega_0(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})}{\kappa_{\text{ext}} + \kappa_{\text{int}} + i\omega_0(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})} \tag{1.15}$$

$$\approx \frac{\kappa_{\text{ext}} - \kappa_{\text{int}} - 2i\Delta\omega}{\kappa_{\text{ext}} + \kappa_{\text{int}} + 2i\Delta\omega}. \tag{1.16}$$

where we have used

$$Z_{\text{in}} = \frac{R}{1 + i\frac{\omega_0}{\kappa_{\text{int}}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \tag{1.17}$$

$$\approx \frac{R}{1 + 2i\frac{\Delta\omega}{\kappa_{\text{int}}}}, \tag{1.18}$$

cf. the RLC circuit without transmission line (previous subsection). Of course, there exists no expression for a transmission parameter here, as transmission is not possible for a one-port circuit.
For a derivation of the reflection coefficient formula, see the later section on transmission lines. The reflection parameter Γ is the same as the scattering parameter S_{11}. We will use both, to describe the reflection throughout this document.

1.2.3 Parallel RLC circuit directly embedded in a transmission line

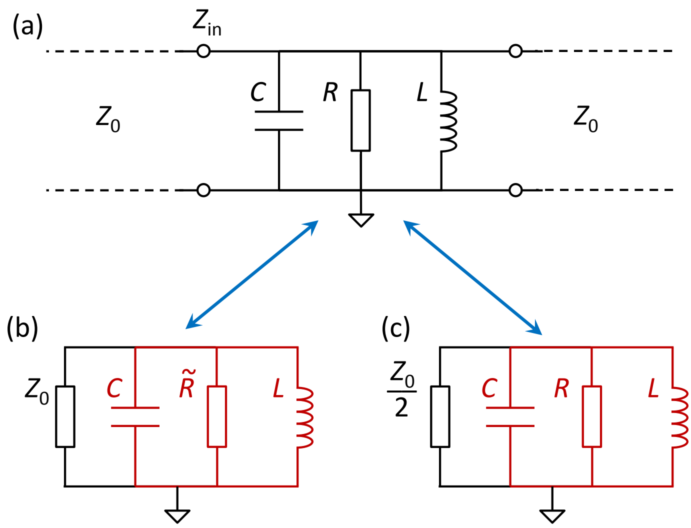

Figure 1.3: (a) Schematic of a parallel RLC circuit embedded into a transmission line with real-valued characteristic impedance Z_0. (b) Equivalent circuit of (a), where one of the transmission lines has been replaced by a lumped element resistor Z_0 while the second transmission line has been formally absorbed into the internal resistance $\tilde{R}$. (c) Equivalent circuit of (a), where both transmission lines have been replaced by a single lumped element resistor Z_0. In (b) and (c) the effective internal parts of the circuit are drawn in red, while the external parts are drawn in black.

The parallel RLC circuit can not only be connected to a transmission line on one side, but on both sides, i.e., it is embedded into a transmission line, cf. Fig. 1.3 (a). The two parallel equivalent external resistors with impedance Z_0, corresponding to the two transmission lines, can be formally combined into a single one with $\tilde{Z}_0 = Z_0/2$, cf. Fig. 1.3 (c). Then, R_{tot} as well as the Qs and κs can be directly calculated with the expressions from the previous section, just replacing Z_0 by $\tilde{Z}_0$. We get for the total resistance

$$R_{\mathrm{tot}} = \frac{R\tilde{Z}_0}{R + \tilde{Z}_0} = \frac{RZ_0}{2R + Z_0}, \tag{1.19}$$

for the quality factors

$$Q_{\mathrm{L}} = \omega_0 R_{\mathrm{tot}} C, \qquad Q_{\mathrm{int}} = \omega_0 RC, \qquad Q_{\mathrm{ext}} = \frac{\omega_0 Z_0 C}{2} \tag{1.20}$$

and for the κs

$$\kappa_{\mathrm{tot}} = \kappa_{\mathrm{int}} + \kappa_{\mathrm{ext}} = \frac{1}{RC} + \frac{2}{Z_0 C} \tag{1.21}$$

as well as

$$\frac{\kappa_{\text{int}}}{\kappa_{\text{ext}}} = \frac{Z_0}{2R}, \qquad \frac{\kappa_{\text{int}}}{\kappa_{\text{tot}}} = \frac{Z_0}{2R + Z_0}, \qquad \frac{\kappa_{\text{ext}}}{\kappa_{\text{tot}}} = \frac{2R}{2R + Z_0}. \tag{1.22}$$

For the input impedance and reflection parameter, we reconsider our calculations. The input impedance is almost identical to the previous consideration, but with a modified resistance value. Due to the second transmission line, the new resistor is $\tilde{R} = RZ_0/(R + Z_0)$, i.e., the total resistor of the one-port coupled version, cf. also Fig. 1.3 (b). The input impedance hence reads

$$Z_{\text{in}} = \frac{\tilde{R}}{1 + i\tilde{Q}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \tag{1.23}$$

$$\approx \frac{\tilde{R}}{1 + 2i\tilde{Q}\frac{\Delta\omega}{\omega_0}} \tag{1.24}$$

$$= \frac{\tilde{R}}{1 + 2i\frac{\Delta\omega}{\tilde{\kappa}}} \tag{1.25}$$

with $\tilde{Q} = \omega_0 \tilde{R} C$ and $\tilde{\kappa} = 1/\tilde{R}C$. If the losses into the second transmission line are absorbed into the internal losses, the reflection coefficient reads exactly the same as for the one-port coupled resonator but with $Q_{\text{int}} = \tilde{Q}$ and $\kappa_{\text{int}} = \tilde{\kappa}$.
If, however, all losses into both transmission lines are considered as external losses, we get with $\tilde{\kappa} = \kappa_{\text{int}} + \kappa_{\text{ext}}/2$

$$\Gamma = \frac{\frac{\kappa_{\text{ext}}}{2} - \tilde{\kappa} - i\omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}{\frac{\kappa_{\text{ext}}}{2} + \tilde{\kappa} + i\omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} = \frac{-\kappa_{\text{int}} - i\omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}{\kappa_{\text{ext}} + \kappa_{\text{int}} + i\omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \tag{1.26}$$

$$\approx \frac{-\kappa_{\text{int}} - 2i\Delta\omega}{\kappa_{\text{ext}} + \kappa_{\text{int}} + 2i\Delta\omega}. \tag{1.27}$$

The results of this section can easily be generalized for transmission lines with different impedances Z_0 and Z_1 on the two sides of the circuit.

1.2.4 Parallel RLC circuit capacitively coupled on one side to a transmission line

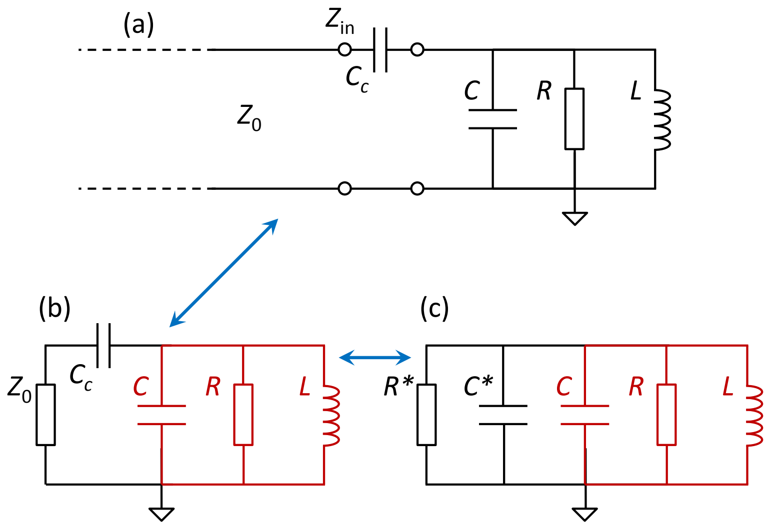

Figure 1.4: (a) Schematic of a parallel RLC circuit capacitively coupled by one port to a transmission line with real-valued characteristic impedance Z_0. (b) Equivalent circuit of (a), where the transmission line has been replaced by a lumped element resistor Z_0. (c) Parallel circuit equivalent of (b), where the series combination of coupling capacitor C_c and transmission line impedance Z_0 has been transformed into parallel equivalents C^* and R^*. In (b) and (c) internal parts of the circuit are drawn in red, while the external part is drawn in black.

As next step, we couple the resonator to a feedline with Z_0 by a series coupling capacitor C_c, cf. Fig. 1.4 (a). From the viewpoint of the resonator, cf. Fig. 1.4 (b), the coupling capacitor and the feedline impedance are in series and have the total (external) impedance

$$Z_e = Z_0 + \frac{1}{i\omega C_c} \tag{1.28}$$

or just rewritten and separated into real and imaginary part

$$\frac{1}{Z_e} = \frac{\omega^2 C_c^2 Z_0}{1 + \omega^2 C_c^2 Z_0^2} + i\omega \frac{C_c}{1 + \omega^2 C_c^2 Z_0^2} \tag{1.29}$$

Now, we can transform this series combination into a parallel combination of an equivalent resistor R^* and an equivalent capacitor C^*, cf. Fig. 1.4 (c). If there was a parallel combination of a resistor and a capacitor, its external impedance would be

$$\frac{1}{Z_e} = \frac{1}{R^*} + i\omega C^*. \tag{1.30}$$

Thus, if we choose

$$R^* = \frac{1+\omega^2 C_c^2 Z_0^2}{\omega^2 C_c^2 Z_0}, \qquad C^* = \frac{C_c}{1+\omega^2 C_c^2 Z_0^2} \tag{1.31}$$

for the parallel equivalent combination, we get the same external impedance as for the series combination of C_c and Z_0. For our further considerations, we assume $\omega^2 C_c^2 Z_0^2 \ll 1$ (reasonable for most relevant cases) and $\omega \approx \omega_0$, the new resonance frequency of the circuit. We get

$$R^* \approx \frac{1}{\omega_0^2 C_c^2 Z_0}, \qquad C^* \approx C_c \tag{1.32}$$

From this point, we can just repeat our analysis from above to get the corresponding external, internal and total Qs and κs. We get

$$Q_\mathrm{L} = \omega_0 R_\mathrm{tot} C_\mathrm{tot}. \tag{1.33}$$

with $R_\mathrm{tot} = RR^*/(R+R^*)$, $C_\mathrm{tot} = C + C_c$ and $\omega_0 = 1/\sqrt{L(C+C_c)}$.
Within our approximation, we can separate the total Q_L into internal and external contributions

$$Q_\mathrm{int} \approx \omega_0 R(C+C_c), \qquad Q_\mathrm{ext} \approx \omega_0 R^*(C+C_c). \tag{1.34}$$

For the decay rates we get

$$\kappa_\mathrm{tot} = \frac{1}{R_\mathrm{tot}C_\mathrm{tot}} \approx \frac{1}{R(C+C_c)} + \frac{1}{R^*(C+C_c)} = \kappa_\mathrm{int} + \kappa_\mathrm{ext} \tag{1.35}$$

Hence, the equations are almost identical to the directly coupled circuit, except for the fact that the coupling capacitor has transformed the characteristic impedance Z_0 to a large equivalent resistor R^* and that we have to add the coupling capacitor to the intrinsic capacitance.
As it will be useful in the next step, we give the external quality factor and the external decay rate also as

$$Q_\mathrm{ext} \approx \frac{C+C_c}{\omega_0 C_c^2 Z_0}, \qquad \kappa_\mathrm{ext} \approx \frac{Z_0 C_c^2}{L(C+C_c)^2}. \tag{1.36}$$

Remember, that these approximations are valid for $\omega \approx \omega_0$ and $\omega_0^2 C_c^2 Z_0^2 \ll 1$.
The accurate expressions imply that the equivalent impedances R^* and C^* as well as the derived expressions for Q_L and even ω_0 are frequency dependent[1].
For the input impedance and the reflection parameter, we cannot just adopt the results from above, because the coupling capacitor considerably modifies the input impedance. With the expression of the bare resonator input impedance, it is given by

[1]Whatever it means that the resonance frequency is frequency dependent? Seems not to make too much sense...

$$Z_{\text{in}} = \frac{R}{1 + iR\left(\omega C - \frac{1}{\omega L}\right)} + \frac{1}{i\omega C_c}. \tag{1.37}$$

The calculation of the new resonance frequency unfortunately is not straightforward when a resistor is included. Hence, we use a trick introduced in Pozar. First, we calculate the resonance frequency of the lossless case and then replace the resonance frequency in the expression for the input impedance by

$$\omega_0 \rightarrow \omega_0 \left(1 + \frac{i}{2Q_{\text{int}}}\right) = \omega_0 + i\frac{\kappa_{\text{int}}}{2}. \tag{1.38}$$

Below, we will describe a somewhat different method to get to the same result. The input impedance of the capacitively coupled LC circuit is

$$\begin{aligned}
Z_{\text{in}} &= \left(\frac{1}{i\omega L} + i\omega C\right)^{-1} + \frac{1}{i\omega C_c} && (1.39)\\
&= \frac{i\omega L}{1 - \omega^2 LC} + \frac{1}{i\omega C_c} && (1.40)\\
&= i\frac{\omega^2 L(C + C_c) - 1}{\omega C_c(1 - \omega^2 LC)}. && (1.41)
\end{aligned}$$

The input impedance vanishes for the resonance frequency

$$\omega_0 = \frac{1}{\sqrt{L(C + C_c)}}. \tag{1.42}$$

Now, we can rewrite the input impedance as

$$Z_{\text{in}} = i\frac{\frac{\omega^2}{\omega_0^2} - 1}{\omega C_c(1 - \omega^2 LC)} \tag{1.43}$$

and make a Taylor approximation around $\omega \approx \omega_0$

$$\begin{aligned}
Z_{\text{in}} &\approx i\frac{2L(C + C_c)}{C_c(1 - \omega_0^2 LC)}(\omega - \omega_0) && (1.44)\\
&= 2iL\frac{(C + C_c)^2}{C_c^2}\Delta\omega. && (1.45)
\end{aligned}$$

The approximated input impedance looks like the input impedance of a series LC circuit with the inductance $L(C + C_c)^2/C_c^2$, i.e., the coupling capacitor acts as an impedance inverter. When we now include the losses according to Pozar's complex frequency trick, we get

$$Z_{\text{in}} \approx \frac{L(C + C_c)^2}{C_c^2}\left(\kappa_{\text{int}} + 2i\Delta\omega\right) \tag{1.46}$$

with which we can also express the reflection parameter as

$$\Gamma = \frac{\frac{L(C+C_c)^2}{C_c^2}(\kappa_{\text{int}} + 2i\Delta\omega) - Z_0}{\frac{L(C+C_c)^2}{C_c^2}(\kappa_{\text{int}} + 2i\Delta\omega) + Z_0} \tag{1.47}$$

$$= \frac{\kappa_{\text{int}} + 2i\Delta\omega - \frac{Z_0 C_c^2}{L(C+C_c)^2}}{\kappa_{\text{int}} + 2i\Delta\omega + \frac{Z_0 C_c^2}{L(C+C_c)^2}}. \tag{1.48}$$

The last term in nominator and denominator can be rewritten as (see above)

$$\frac{Z_0 C_c^2}{L(C+C_c)^2} = \kappa_{\text{ext}}. \tag{1.49}$$

So we get for the reflection

$$\Gamma = \frac{\kappa_{\text{int}} - \kappa_{\text{ext}} + 2i\Delta\omega}{\kappa_{\text{int}} + \kappa_{\text{ext}} + 2i\Delta\omega}. \tag{1.50}$$

We note here, that we get exactly the same result for the approximated input impedance and the reflection parameter of the full capacitively coupled RLC circuit by the following procedure:

- Separate the full input impedance including R into real and imaginary part.
- Determine the resonance condition from the condition of a vanishing imaginary part. It is given by

$$1 + R^2\left(\omega_0 C - \frac{1}{\omega_0 L}\right)^2 = R^2\omega_0 C_c\left(\frac{1}{\omega_0 L} - \omega_0 C\right). \tag{1.51}$$

- Taylor expand the imaginary part of the input impedance around the frequency corresponding to this resonance condition point (it is not necessary to have an expression for ω_0 to do so).
- Insert the approximate resonance frequency $\omega_0 = 1/\sqrt{L(C+C_c)}$ into the exact real and the approximated imaginary part of the input impedance.

It is, however, a long calculation without new insights, so we just mention the possibility and that we get the same result. But it demonstrates, that Pozar's complex frequency trick corresponds to using the losses at the resonance frequency for all frequencies, although in general also the losses are frequency dependent. For a high Q resonator, however, only a narrow frequency interval is relevant and assuming constant losses in this small interval is a good approximation.

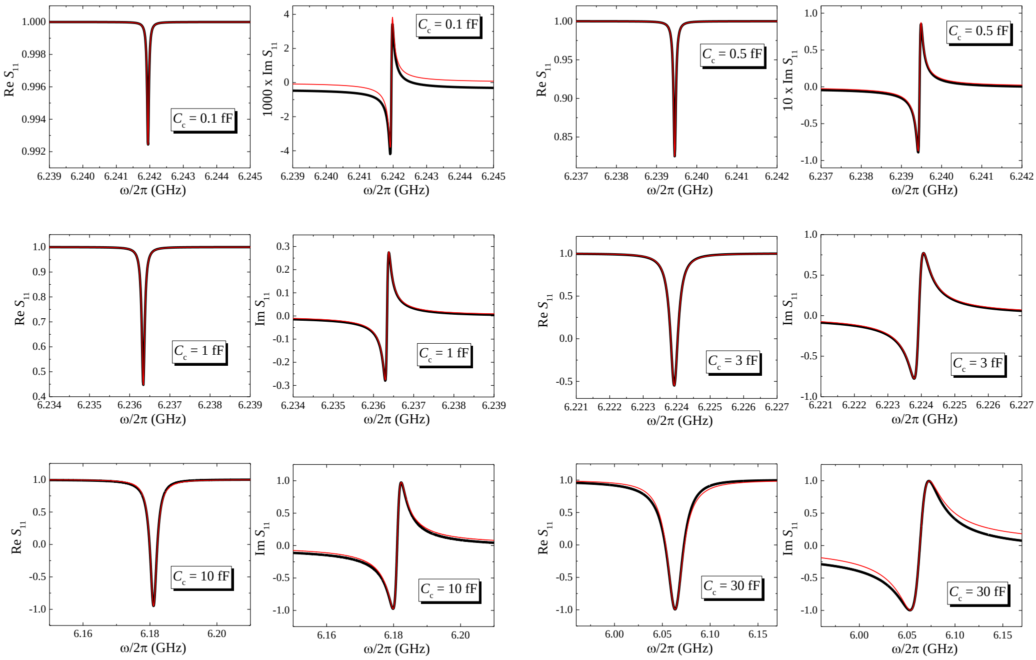

Figure 1.5: Real and imaginary parts of the reflection parameter in direct comparison between QUCS results in black and the approximate expression given by Eq. (1.50) in red for six different coupling capacitors C_c. As circuit parameters, we chose $R = 5\,M\Omega$, $C = 500\,fF$, $L = 1300\,pH$ and $Z_0 = 50\,\Omega$.

The reflection parameter - QUCS vs approximation

To get an idea of how good the derived expression for the reflection parameter is compared to an unapproximated result, we now plot the result of the approximate formula in comparison with the result of a QUCS simulation, which probably is very close to being exact. We do this for different coupling capacitors and plot real and imaginary parts separately.

Figure 1.5 shows the results for six different coupling capacitors C_c, going from strongly undercoupled ($\kappa_{\text{ext}} \ll \kappa_{\text{int}}$) to strongly overcoupled ($\kappa_{\text{ext}} \gg \kappa_{\text{int}}$).

We can see that the approximate expression seems to be very good for the intermediate coupling capacitors, but shows some deviations for very small or very large C_c. For very small C_c (upper left in the figure), there seems to be an offset in the imaginary part, which we have lost during the approximation. For very large C_c, however, both real and imaginary part deviate from the QUCS result with increasing detuning from the resonance frequency and the deviation is not an offset.

The origin of the deviations is a kind of background in the reflection, which is caused by the coupling element. During the approximations done in deriving Γ we have lost this background, but we can compensate for it by multiplying the reflection coefficient of the whole circuit with the reflection coefficient of the coupling element

$$\Gamma_{\mathrm{tot}} = \Gamma \cdot \Gamma_c \tag{1.52}$$

with

$$\Gamma_c = \frac{\frac{1}{i\omega C_c} - Z_0}{\frac{1}{i\omega C_c} + Z_0}. \tag{1.53}$$

Figure 1.6 shows the QUCS results together with the corrected reflection according to this formula.[2] The curves now fit on each other much better than without the correction.

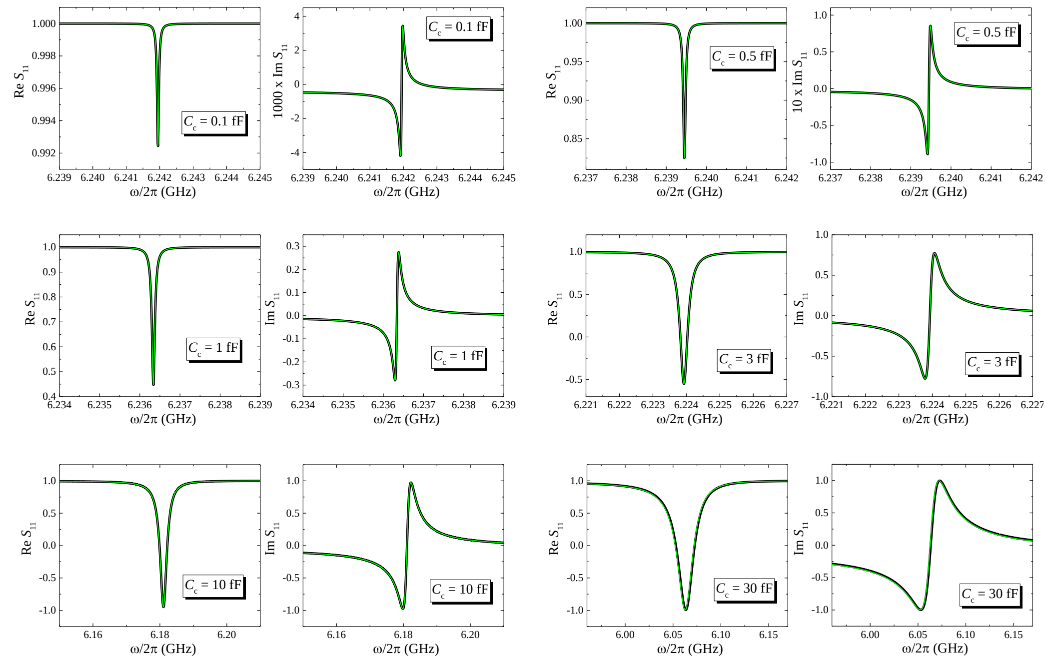

Figure 1.6: Real and imaginary parts of the reflection parameter in direct comparison between QUCS results in black and the corrected approximate expression $\Gamma_{\mathrm{tot}} = \Gamma \cdot \Gamma_c$ in green for six different coupling capacitors C_c. As circuit parameters, we chose $R = 5\,M\Omega$, $C = 500\,fF$, $L = 1300\,pH$ and $Z_0 = 50\,\Omega$.

[2] Unfortunately, I cannot derive or justify this compensation factor strictly. Another possibility is to use the factor e^{2Z_0/Z_c} instead of Γ_c, which in the shown parameter range gives almost the same results.

1.2.5 Parallel RLC circuit capacitively coupled on both sides to a transmission line

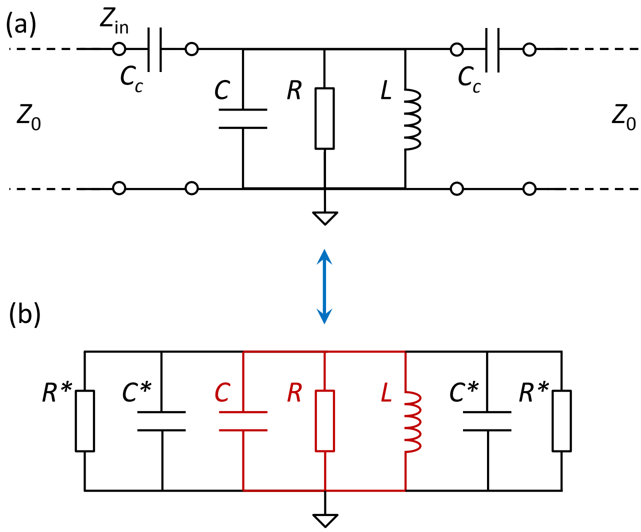

Figure 1.7: (a) Schematic of a parallel RLC circuit capacitively coupled by two ports to a transmission line with real-valued characteristic impedance Z_0. (b) Norton equivalent of (a), where the transmission lines and coupling capacitors have been replaced by equivalent lumped element resistors R^* and capacitors C^*. In (b) internal parts of the circuit are shown in red, while external parts are drawn in black.

Often, the resonator is not only coupled on one side to a feedline but on both sides. Figure 1.7 (a) shows a parallel RLC circuit capacitively coupled by a coupling capacitor C_c to a feedline with characteristic impedance Z_0 on both sides. For the equivalent resistors and capacitors, cf. Fig. 1.7 (b), we can directly take the expressions from the previous section. For the same approximations as in the previous section, we get the resonance frequency

$$\omega_0 = \frac{1}{\sqrt{L(C + 2C_c)}}, \tag{1.54}$$

and with $C_{\text{tot}} = C + 2C_c$ and $R_{\text{tot}} = RR^*/(2R + R^*)$ we get the quality factors

$$Q_{\text{L}} = \omega_0 R_{\text{tot}} C_{\text{tot}}, \qquad Q_{\text{int}} = \omega_0 R(C + 2C_c), \qquad Q_{\text{ext}} = \frac{\omega_0 R^*(C + 2C_c)}{2}. \tag{1.55}$$

Finally, we calculate again the decay rates to be given by

$$\kappa_{\text{tot}} = \frac{1}{R_{\text{tot}} C_{\text{tot}}} \approx \frac{1}{R(C + 2C_c)} + \frac{2}{R^*(C + 2C_c)} = \kappa_{\text{int}} + \kappa_{\text{ext}}. \tag{1.56}$$

For the ratio of the decay rates we get

$$\frac{\kappa_{\text{int}}}{\kappa_{\text{ext}}} = \frac{R^*}{2R}, \qquad \frac{\kappa_{\text{int}}}{\kappa_{\text{tot}}} = \frac{R^*}{2R + R^*}, \qquad \frac{\kappa_{\text{ext}}}{\kappa_{\text{tot}}} = \frac{2R}{2R + R^*}. \tag{1.57}$$

Again, we also give the useful expressions

$$Q_{\text{ext}} \approx \frac{C + 2C_c}{2\omega_0 C_c^2 Z_0}, \qquad \kappa_{\text{ext}} \approx \frac{2C_c^2 Z_0}{L(C + 2C_c)^2} \tag{1.58}$$

for the external Q and κ.
Next, we calculate the input impedance. First, we define

$$\tilde{C} = C + C_c, \qquad \tilde{R} = \frac{RR^*}{R + R^*}, \qquad \tilde{\kappa} = \frac{1}{\tilde{R}(\tilde{C} + C_c)} = \kappa_{\text{int}} + \frac{\kappa_{\text{ext}}}{2}. \tag{1.59}$$

With these, we can – analogously to the directly embedded RLC circuit – include the coupling capacitor and transmission line of the second port into the internal resistor and capacitor and repeat exactly the analysis of the one-port capacitively coupled resonator. For the approximate input impedance we hence get

$$Z_{\text{in}} \approx \frac{L(\tilde{C} + C_c)^2}{C_c^2} (\tilde{\kappa} + 2i\Delta\omega). \tag{1.60}$$

The reflection parameter now is given by

$$\begin{aligned}
\Gamma &\approx \frac{\tilde{\kappa} + 2i\Delta\omega - \frac{Z_0 C_c^2}{L(\tilde{C}+C_c)^2}}{\tilde{\kappa} + 2i\Delta\omega + \frac{Z_0 C_c^2}{L(\tilde{C}+C_c)^2}} && (1.61)\\
&= \frac{\kappa_{\text{int}} + \frac{\kappa_{\text{ext}}}{2} + 2i\Delta\omega - \frac{Z_0 C_c^2}{L(C+2C_c)^2}}{\kappa_{\text{int}} + \frac{\kappa_{\text{ext}}}{2} + 2i\Delta\omega + \frac{Z_0 C_c^2}{L(C+2C_c)^2}} && (1.62)\\
&= \frac{\kappa_{\text{int}} + \frac{\kappa_{\text{ext}}}{2} + 2i\Delta\omega - \frac{\kappa_{\text{ext}}}{2}}{\kappa_{\text{int}} + \frac{\kappa_{\text{ext}}}{2} + 2i\Delta\omega + \frac{\kappa_{\text{ext}}}{2}} && (1.63)\\
&= \frac{\kappa_{\text{int}} + 2i\Delta\omega}{\kappa_{\text{int}} + \kappa_{\text{ext}} + 2i\Delta\omega}. && (1.64)
\end{aligned}$$

The results can easily be generalized to different coupling capacitors and different feedline impedances on the both sides.

1.2.6 Parallel RLC circuit inductively coupled on one side to a transmission line

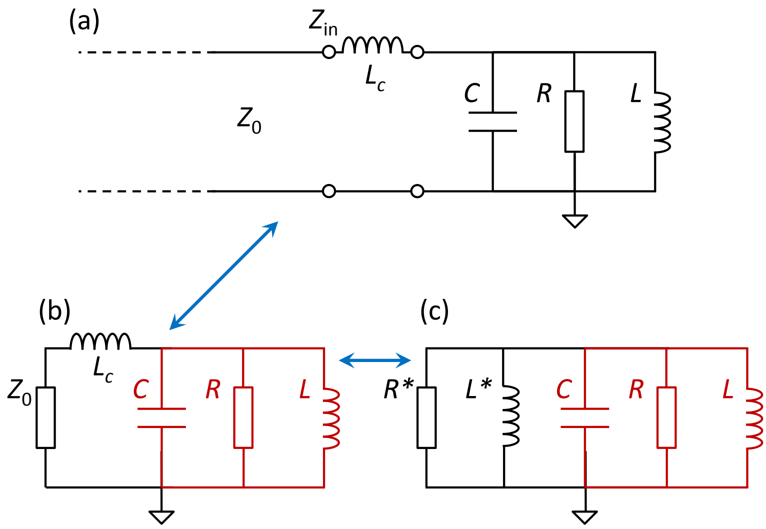

Figure 1.8: (a) Schematic of a parallel RLC circuit inductively coupled at one port to a transmission line with real-valued characteristic impedance Z_0. (b) Equivalent circuit of (a), where the transmission line has been replaced by a lumped element resistor Z_0. (c) Parallel circuit equivalent (sometimes called Norton equivalent) of (b), where the series combination of coupling inductor L_c and transmission line impedance Z_0 has been transformed into parallel equivalents L^* and R^*. In (b) and (c) internal parts of the circuit are drawn in red, while the external part is drawn in black.

Another possibility to couple a parallel RLC circuit reactively to a transmission line is via a series inductor, cf. Fig. 1.8 (a). The analysis of such a device is very similar to the analysis of the capacitively coupled version. The input impedance of the coupling inductor and the feedline from the resonator viewpoint is given by

$$Z_e = Z_0 + i\omega L_c \tag{1.65}$$

which can be rewritten into

$$\frac{1}{Z_e} = \frac{Z_0}{Z_0^2 + \omega^2 L_c^2} + \frac{1}{i\omega} \frac{\omega^2 L_c}{Z_0^2 + \omega^2 L_c^2} = \frac{1}{R^*} + \frac{1}{i\omega L^*}. \tag{1.66}$$

with

$$R^* = \frac{Z_0^2 + \omega^2 L_c^2}{Z_0}, \qquad L^* = \frac{Z_0^2 + \omega^2 L_c^2}{\omega^2 L_c}, \tag{1.67}$$

cf. also Fig. 1.8 (b). For $\omega L_c \gg Z_0$ and $\omega \approx \omega_0$, these relations can be approximated as

$$R^* \approx \frac{\omega_0^2 L_c^2}{Z_0}, \qquad L^* \approx L_c. \tag{1.68}$$

The resonance frequency is hence given by

$$\omega_0 = \frac{1}{\sqrt{L_{\mathrm{tot}} C}}, \tag{1.69}$$

and the loaded quality factor by

$$Q_{\mathrm{L}} = \omega_0 R_{\mathrm{tot}} C = \frac{R_{\mathrm{tot}}}{\omega_0 L_{\mathrm{tot}}} \tag{1.70}$$

with $R_{\mathrm{tot}} = RR^*/(R+R^*)$ and $L_{\mathrm{tot}} = LL_c/(L+L_c)$.
Note, that in contrast to the capacitive coupling where the resonance frequency is shifted downwards by the coupling capacitor, the resonance frequency for the inductive coupling is shifted upwards, because the total equivalent inductance is smaller than the bare resonator L.
The quality factor can be separated into

$$Q_{\mathrm{int}} = \frac{R}{\omega_0 L_{\mathrm{tot}}}, \qquad Q_{\mathrm{ext}} = \frac{R^*}{\omega_0 L_{\mathrm{tot}}} = \frac{\omega_0 L_c (L+L_c)}{Z_0 L}. \tag{1.71}$$

The decay rates are given by

$$\kappa_{\mathrm{tot}} = \frac{1}{R_{\mathrm{tot}} C} = \frac{\omega_0^2 L_{\mathrm{tot}}}{R_{\mathrm{tot}}}, \qquad \kappa_{\mathrm{int}} = \frac{1}{RC} = \frac{\omega_0^2 L_{\mathrm{tot}}}{R}, \qquad \kappa_{\mathrm{ext}} = \frac{1}{R^* C} = \frac{Z_0 L}{L_c(L+L_c)}. \tag{1.72}$$

To calculate the input impedance we follow the same procedure as for the capacitively coupled case, i.e., we start with the input impedance of the corresponding LC resonator (no R) and later replace the resonance frequency by its complex counterpart (see Pozar or above)

$$\begin{aligned} Z_{\mathrm{in}} &= \left(\frac{1}{i\omega L} + i\omega C\right)^{-1} + i\omega L_c && (1.73)\\ &= \frac{i\omega L}{1-\omega^2 LC} + i\omega L_c && (1.74)\\ &= i\omega(L+L_c)\frac{1-\omega^2 \frac{LL_c}{L+L_c} C}{1-\omega^2 LC}. && (1.75) \end{aligned}$$

Again, we find the resonance frequency from the vanishing nominator at

$$\omega_0 = \sqrt{\frac{L+L_c}{LL_c C}} = \frac{1}{\sqrt{L_{\mathrm{tot}} C}}. \tag{1.76}$$

When we Taylor expand the input impedance around $\omega = \omega_0$, we get

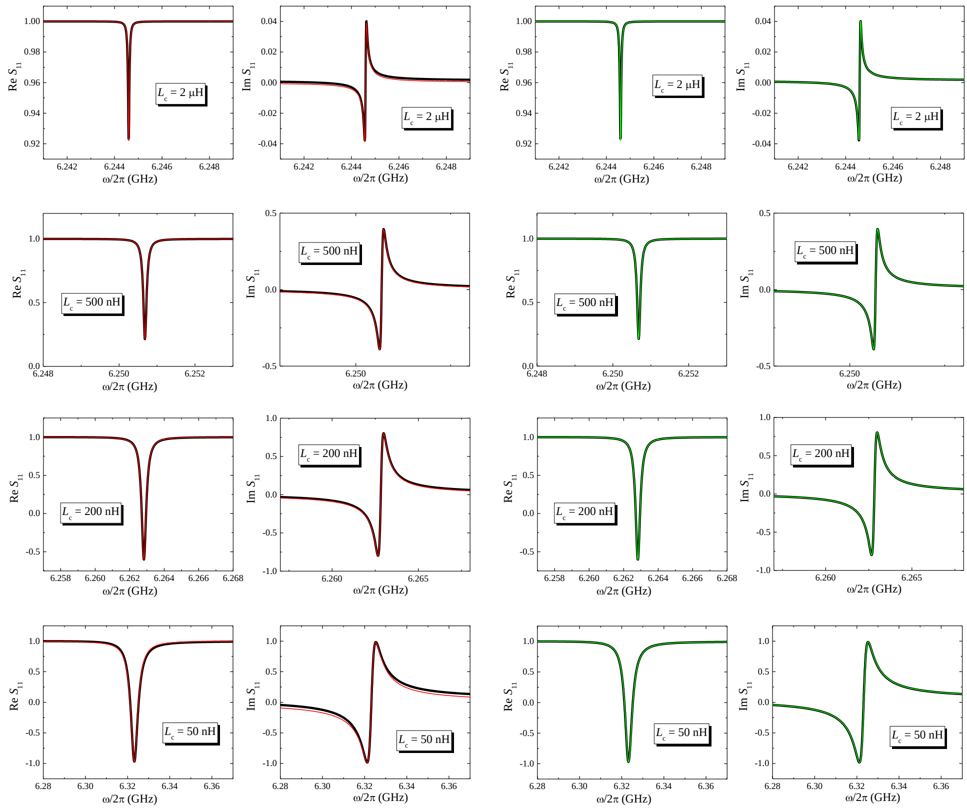

Figure 1.9: Real and imaginary parts of the reflection parameter in direct comparison between QUCS results in black and the approximate expression given by Eq. (1.50) in red for four different coupling inductors L_c (left side). Right side shows the comparison between the QUCS results and $\Gamma_{\text{tot}} = \Gamma \cdot \Gamma_c$. As circuit parameters, we chose $R = 5\,M\Omega$, $C = 500\,fF$, $L = 1300\,pH$ and $Z_0 = 50\,\Omega$.

$$Z_{\text{in}} \approx -2i\frac{L+L_c}{1-\omega_0^2 LC}(\omega-\omega_0) \tag{1.77}$$

$$= 2i\frac{L_c(L+L_c)}{L}\Delta\omega. \tag{1.78}$$

After replacing the real resonance frequency by its complex counterpart, we get

$$Z_{\text{in}} \approx \frac{L_c(L+L_c)}{L}\left(\kappa_{\text{int}} + 2i\Delta\omega\right). \tag{1.79}$$

For the reflection, we receive

$$\Gamma = \frac{\frac{L_c(L+L_c)}{L}(\kappa_{\text{int}} + 2i\Delta\omega) - Z_0}{\frac{L_c(L+L_c)}{L}(\kappa_{\text{int}} + 2i\Delta\omega) + Z_0} \tag{1.80}$$

$$= \frac{\kappa_{\text{int}} + 2i\Delta\omega - \frac{Z_0 L}{L_c(L+L_c)}}{\kappa_{\text{int}} + 2i\Delta\omega + \frac{Z_0 L}{L_c(L+L_c)}} \tag{1.81}$$

$$= \frac{\kappa_{\text{int}} - \kappa_{\text{ext}} + 2i\Delta\omega}{\kappa_{\text{int}} + \kappa_{\text{ext}} + 2i\Delta\omega}. \tag{1.82}$$

The reflection parameter - QUCS vs approximation

As for the capacitively coupled case, Fig. 1.9 shows in comparison the result for the reflection according to Eq. (1.82) and the result of a QUCS simulation for different coupling inductors. The agreement between the two again is quite good in the range of parameters shown. For very small or large coupling inductors, however, there is the same kind of deviation between the curves as for the capacitively coupled case. These deviations can also be compensated for in good approximation by the reflection factor of the coupling inductor as shown in the right half of the figure.

1.2.7 Parallel RLC circuit inductively coupled on both sides to a transmission line

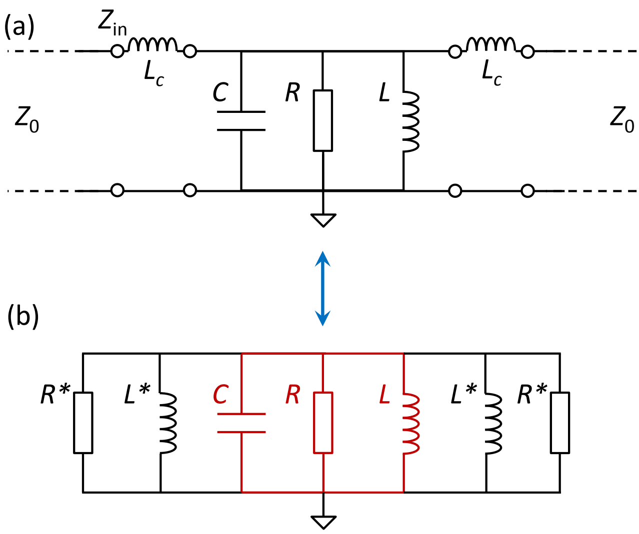

Figure 1.10: (a) Schematic of a parallel RLC circuit inductively coupled by two ports to a transmission line with real-valued characteristic impedance Z_0. (b) Norton equivalent of (a), where the transmission lines and coupling inductors have been replaced by equivalent lumped element resistors R^* and inductors L^*. In (b) internal parts of the circuit are shown in red, while external parts are drawn in black.

Now, the parallel RLC circuit inductively coupled by series inductances L_c on both sides to a transmission line with a characteristic impedance Z_0. Analogous to the cases of directly or capacitively coupled two-port circuits, we can easily generalize our analysis from the one-port considerations. On each side, the combination of inductor L_c and effective resistor Z_0 are transformed into their parallel circuit equivalents by

$$R^* = \frac{Z_0^2 + \omega^2 L_c^2}{Z_0}, \qquad L^* = \frac{Z_0^2 + \omega^2 L_c^2}{\omega^2 L_c}. \tag{1.83}$$

These can be approximated for $\omega L_c \gg Z_0$ and $\omega \approx \omega_0$ as

$$R^* \approx \frac{\omega_0^2 L_c^2}{Z_0}, \qquad L^* \approx L_c. \tag{1.84}$$

With the total resistance $R_{\text{tot}} = RR^*/(2R+R^*)$ and the total inductance $L_{\text{tot}} = LL_c/(2L+L_c)$ we get for the loaded quality factor

$$Q_{\text{L}} = \frac{R_{\text{tot}}}{\omega_0 L_{\text{tot}}} \tag{1.85}$$

and for the internal and external quality factors

$$Q_{\mathrm{int}} = \frac{R}{\omega_0 L_{\mathrm{tot}}}, \qquad Q_{\mathrm{ext}} = \frac{R^*}{2\omega_0 L_{\mathrm{tot}}} = \frac{\omega_0 L_c (2L + L_c)}{2 Z_0 L}. \tag{1.86}$$

For the decay rates, we get

$$\kappa_{\mathrm{tot}} = \frac{\omega_0^2 L_{\mathrm{tot}}}{R_{\mathrm{tot}}}, \qquad \kappa_{\mathrm{int}} = \frac{\omega_0^2 L_{\mathrm{tot}}}{R}, \qquad \kappa_{\mathrm{ext}} = \frac{2\omega_0^2 L_{\mathrm{tot}}}{R^*} = \frac{2 Z_0 L}{L_c (2L + L_c)}. \tag{1.87}$$

For the calculation of the input impedance, we first define as above

$$\tilde{L} = \frac{L L_c}{L + L_c}, \qquad \tilde{R} = \frac{R R^*}{R + R^*}, \qquad \tilde{\kappa} = \kappa_{\mathrm{int}} + \frac{\kappa_{\mathrm{ext}}}{2}. \tag{1.88}$$

By this, we absorb the right transmission line including the one coupling inductor into the internal circuit elements. Then, we can just take the input impedance from the one-port coupled case and replace the former variables by the new ones

$$Z_{\mathrm{in}} \approx \frac{L_c(\tilde{L} + L_c)}{\tilde{L}} (\tilde{\kappa} + 2i\Delta\omega) \tag{1.89}$$

$$= \frac{L_c(2L + L_c)}{L} (\tilde{\kappa} + 2i\Delta\omega). \tag{1.90}$$

For the reflection, we get

$$\Gamma = \frac{\frac{L_c(2L+L_c)}{L}(\tilde{\kappa} + 2i\Delta\omega) - Z_0}{\frac{L_c(2L+L_c)}{L}(\tilde{\kappa} + 2i\Delta\omega) + Z_0} \tag{1.91}$$

$$= \frac{\tilde{\kappa} + 2i\Delta\omega - \frac{Z_0 L}{L_c(2L+L_c)}}{\tilde{\kappa} + 2i\Delta\omega + \frac{Z_0 L}{L_c(2L+L_c)}} \tag{1.92}$$

$$= \frac{\tilde{\kappa} + 2i\Delta\omega - \frac{\kappa_{\mathrm{ext}}}{2}}{\tilde{\kappa} + 2i\Delta\omega + \frac{\kappa_{\mathrm{ext}}}{2}} \tag{1.93}$$

$$= \frac{\kappa_{\mathrm{int}} + 2i\Delta\omega}{\kappa_{\mathrm{int}} + \kappa_{\mathrm{ext}} + 2i\Delta\omega}. \tag{1.94}$$

1.2.8 Parallel RLC circuit capacitively side-coupled to a transmission line

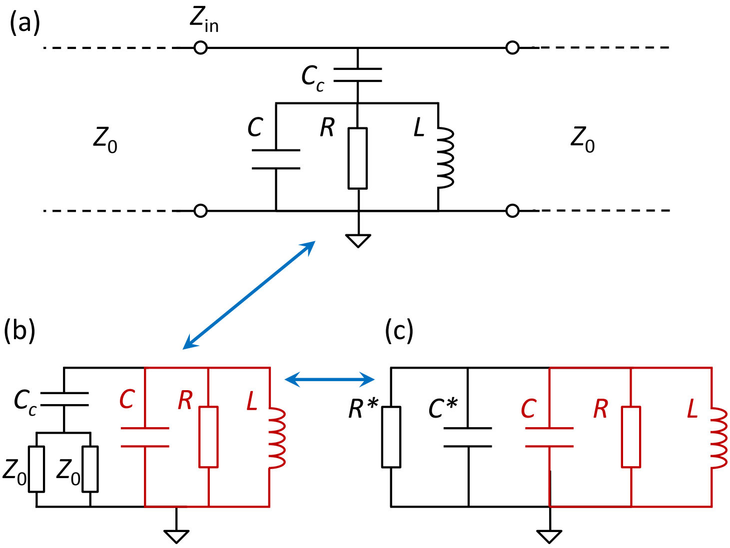

Figure 1.11: (a) Schematic of a parallel RLC circuit capacitively side-coupled to a transmission line with real-valued characteristic impedance Z_0. (b) lumped element equivalent of (a), where the transmission lines have been replaced by lumped element resistors Z_0. (c) Norton equivalent of (b), where the transmission lines and the coupling capacitor have been replaced by an equivalent lumped element resistor R^* and capacitor C^*. In (b) and (c) internal parts of the circuit are shown in red, while external parts are drawn in black.

The last type of parallel RLC resonator we discuss here is the capacitively side-coupled one, cf. Fig. 1.11 (a). It looks similar to the directly coupled RLC circuit coupled at two ports to a transmission line. However, it is separated from the signal conductor by a coupling capacitor C_c. So first, let us again see, how the coupling capacitor and the transmission lines look from the circuit viewpoint. Their combined impedance, cf. Fig. 1.11 (b), is given by

$$Z_e = \left(\frac{1}{Z_0} + \frac{1}{Z_0}\right)^{-1} + \frac{1}{i\omega C_c} = \frac{Z_0}{2} + \frac{1}{i\omega C_c}. \tag{1.95}$$

Hence, we can apply for the first part of the analysis our results from the capacitively one-port coupled resonator above but have to replace Z_0 by $Z_0/2$. The equivalent elements are given by

$$R^* = \frac{4 + \omega^2 C_c^2 Z_0^2}{2\omega^2 C_c^2 Z_0}, \qquad C^* = \frac{4C_c}{4 + \omega^2 C_c^2 Z_0^2}. \tag{1.96}$$

or, for $\omega \approx \omega_0$ and $\omega C_c \ll Z_0$, by

$$R^* \approx \frac{2}{\omega_0^2 C_c^2 Z_0}, \qquad\qquad C^* \approx C_c. \tag{1.97}$$

With $R_{\rm tot} = RR^*/(R+R^*)$, $C_{\rm tot} = C+C_c$ and $\omega_0 = 1/\sqrt{L(C+C_c)}$. We hence can separate the total quality factor

$$Q_{\rm L} = \omega_0 R_{\rm tot} C_{\rm tot} \tag{1.98}$$

into internal and external contributions again

$$Q_{\rm int} = \omega_0 R(C+C_c), \qquad\qquad Q_{\rm ext} = \omega_0 R^*(C+C_c) = \frac{2(C+C_c)}{\omega_0 C_c^2 Z_0}. \tag{1.99}$$

The decay rates become

$$\kappa_{\rm int} = \frac{1}{R(C+C_c)}, \qquad \kappa_{\rm ext} = \frac{1}{R^*(C+C_c)} = \frac{\omega_0^2 C_c^2 Z_0}{2(C+C_c)}. \tag{1.100}$$

So far so good. Next step is the input impedance, which is different than the previous ones, because the second transmission line is in parallel to the whole resonator including the coupling capacitor. So what we do now is take the approximate expression for the input impedance of the one-port capacitively coupled RLC circuit and put a Z_0 resistor in parallel, which leads to

$$\frac{1}{Z_{\rm in}} = \frac{C_c^2}{L(C+C_c)^2(\kappa_{\rm int} + 2i\Delta\omega)} + \frac{1}{Z_0} \tag{1.101}$$

or

$$\begin{aligned} Z_{\rm in} &= \frac{Z_0 L(C+C_c)^2(\kappa_{\rm int} + 2i\Delta\omega)}{Z_0 C_c^2 + L(C+C_c)^2(\kappa_{\rm int} + 2i\Delta\omega)} && (1.102) \\ &= \frac{Z_0(\kappa_{\rm int} + 2i\Delta\omega)}{\frac{Z_0 C_c^2}{L(C+C_c)^2} + \kappa_{\rm int} + 2i\Delta\omega} && (1.103) \\ &= Z_0 \frac{\kappa_{\rm int} + 2i\Delta\omega}{2\kappa_{\rm ext} + \kappa_{\rm int} + 2i\Delta\omega}. && (1.104) \end{aligned}$$

For the reflection we find

$$\begin{aligned} \Gamma &= \frac{Z_0(\kappa_{\rm int} + 2i\Delta\omega) - Z_0(2\kappa_{\rm ext} + \kappa_{\rm int} + 2i\Delta\omega)}{Z_0(\kappa_{\rm int} + 2i\Delta\omega) + Z_0(2\kappa_{\rm ext} + \kappa_{\rm int} + 2i\Delta\omega)} && (1.105) \\ &= \frac{-\kappa_{\rm ext}}{\kappa_{\rm int} + \kappa_{\rm ext} + 2i\Delta\omega}. && (1.106) \end{aligned}$$

As it is rather simple for this case, we calculate also the transmission parameters S_{11} and S_{12} for the capacitively side-coupled parallel RLC circuit. To do so, we have to use the $ABCD$-matrix formalism. The whole resonator including the coupling capacitor is a single shunt impedance between center conductor and ground conductor and hence has the $ABCD$ matrix (cf. e.g. Pozar)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/Z_{\text{in}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{C_c^2}{L(C+C_c)^2(\kappa_{\text{int}}+2i\Delta\omega)} & 1 \end{pmatrix}. \tag{1.107}$$

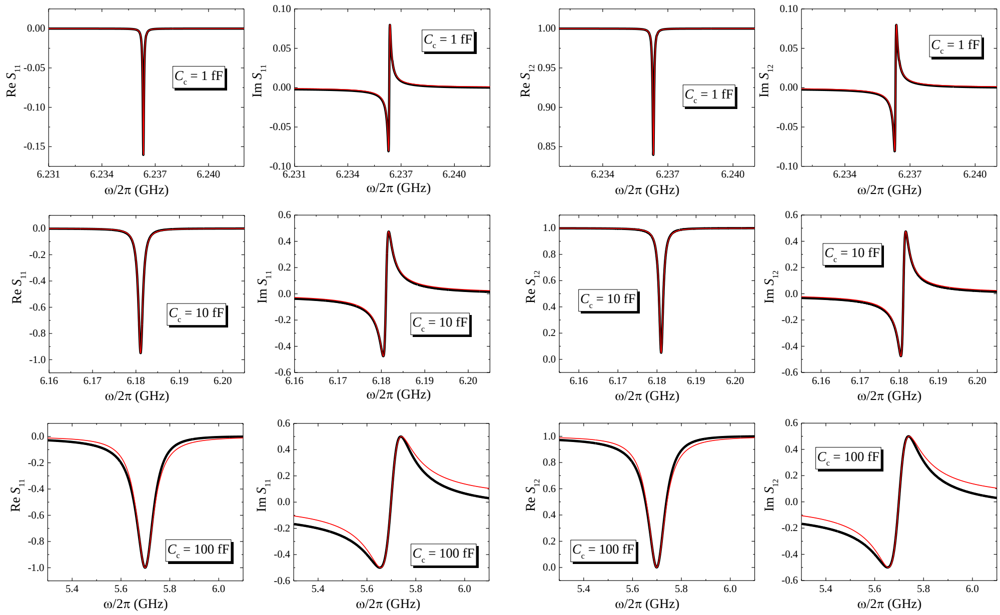

Figure 1.12: Real and imaginary parts of reflection (left side) and transmission (right side) parameter in direct comparison between QUCS results in black and the approximate expression given by Eqs. (1.110), (1.114) in red for three different coupling capacitors C_c. As circuit parameters, we chose $R = 5\,M\Omega$, $C = 500\,fF$, $L = 1300\,pH$ and $Z_0 = 50\,\Omega$.

What is really important here is the fact, that we take the input impedance without the second feedline. The shunting by the second feedline happens automatically by the next step, the calculation of S_{11} via

$$S_{11} = \frac{A + B/Z_0 - CZ_0 - D}{A + B/Z_0 + CZ_0 + D} \tag{1.108}$$

$$= \frac{-Z_0/Z_{\text{in}}}{2 + Z_0/Z_{\text{in}}}. \tag{1.109}$$

Inserting our input impedance expression gives

$$S_{11} = \frac{-\kappa_{\text{ext}}}{\kappa_{\text{int}} + \kappa_{\text{ext}} + 2i\Delta\omega} = \Gamma, \tag{1.110}$$

i.e., the identical expression as for Γ, which is not surprising because $\Gamma = S_{11}$ is general. For the transmission parameter S_{12} we get (first formula again from Pozar)

$$\begin{aligned} S_{12} &= \frac{2(AD - BC)}{A + B/Z_0 + CZ_0 + D} && (1.111) \\ &= \frac{2}{2 + Z_0/Z_{\text{in}}} && (1.112) \\ &= 1 + S_{11} && (1.113) \\ &= \frac{\kappa_{\text{int}} + 2i\Delta\omega}{\kappa_{\text{int}} + \kappa_{\text{ext}} + 2i\Delta\omega}. && (1.114) \end{aligned}$$

Reflection and transmission parameter - QUCS vs approximation

In Fig. 1.12 the reflection parameter and the transmission parameter are plotted in a direct comparison between QUCS results and the approximate formulas derived above for three different coupling capacitors. Note, that the reflection parameter on the left side has exactly the same shape as the transmission parameter on the right side, but the real part of the transmission is shifted by 1 as derived above $S_{12} = 1 + S_{11}$. Except for large coupling capacitors, where again a deviation between the two curves is found for large detunings, there is an excellent agreement between the exact QUCS result and the approximate expression.

1.3 Response functions of series RLC circuits

1.3.1 Simple series RLC circuit

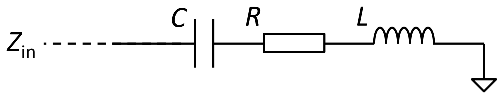

Figure 1.13: Schematic of a series RLC circuit.

Next, we consider a resonant RLC circuit, where R, L, and C are in series, cf. Fig. 1.13. The input impedance is given by

$$
\begin{aligned}
Z_{\text{in}} &= R + i\omega L + \frac{1}{i\omega C} && (1.115)\\
&= R + iL\omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) && (1.116)
\end{aligned}
$$

with the resonance frequency $\omega_0^2 = 1/LC$. Around resonance $\omega \approx \omega_0$ it can be approximated by

$$Z_{\text{in}} \approx R + 2iL\Delta\omega, \tag{1.117}$$

where again $\Delta\omega = \omega - \omega_0$. For the quality factor, we get

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC} \tag{1.118}$$

due to $W = \frac{1}{2}LI^2$ and $P_{\text{loss}} = \frac{1}{2}RI^2$. Now, the input impedance reads

$$
\begin{aligned}
Z_{\text{in}} &= R + iQR\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) && (1.119)\\
&\approx R + 2iQR\frac{\Delta\omega}{\omega_0} && (1.120)\\
&= R + 2iR\frac{\Delta\omega}{\kappa}. && (1.121)
\end{aligned}
$$

with

$$\kappa = \frac{R}{L} = \omega_0^2 RC. \tag{1.122}$$

1.3.2 Series RLC circuit directly coupled on one side to a transmission line

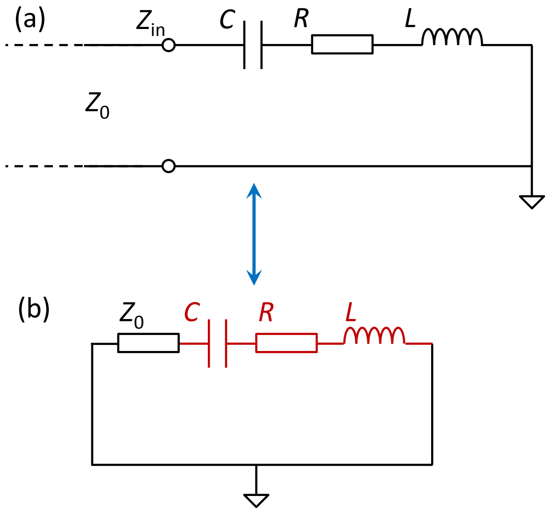

Figure 1.14: (a) Schematic of a series RLC circuit coupled by one port to a transmission line with real-valued characteristic impedance Z_0. (b) Equivalent circuit of (a), where the transmission line has been replaced by a lumped element resistor Z_0. In red the internal parts of the circuit are drawn, while the external part is drawn in black.

Now, we consider a series RLC circuit connected to a transmission line with characteristic impedance Z_0 on one side. The transmission line impedance can be represented as a lumped element resitor, with which the total resistance of the circuit is given by

$$R_{\text{tot}} = R + Z_0. \tag{1.123}$$

The total quality factor is

$$Q_{\text{L}} = \frac{\omega_0 L}{R_{\text{tot}}} = \frac{1}{\omega_0 R_{\text{tot}} C} \tag{1.124}$$

which can be separated into

$$\frac{1}{Q_{\text{L}}} = \frac{R + Z_0}{\omega_0 L} = \frac{R}{\omega_0 L} + \frac{Z_0}{\omega_0 L} = \frac{1}{Q_{\text{int}}} + \frac{1}{Q_{\text{ext}}}. \tag{1.125}$$

The decay rate is given by

$$\kappa_{\text{tot}} = \kappa_{\text{int}} + \kappa_{\text{ext}} = \frac{R}{L} + \frac{Z_0}{L} \tag{1.126}$$

and we can find the relations

$$\frac{\kappa_{\text{int}}}{\kappa_{\text{ext}}} = \frac{R}{Z_0}, \qquad \frac{\kappa_{\text{int}}}{\kappa_{\text{tot}}} = \frac{R}{R+Z_0}, \qquad \frac{\kappa_{\text{ext}}}{\kappa_{\text{tot}}} = \frac{Z_0}{R+Z_0}, \tag{1.127}$$

i.e., exactly the reciprocal expressions from the corresponding parallel RLC circuit. Finally, we calculate the reflection parameter as

$$\Gamma = \frac{\kappa_{\text{int}} - \kappa_{\text{ext}} + i\omega_0(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})}{\kappa_{\text{int}} + \kappa_{\text{ext}} + i\omega_0(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})} \tag{1.128}$$

$$\approx \frac{\kappa_{\text{int}} - \kappa_{\text{ext}} + 2i\Delta\omega}{\kappa_{\text{int}} + \kappa_{\text{ext}} + 2i\Delta\omega}. \tag{1.129}$$

1.3.3 Series RLC circuit directly embedded in a transmission line

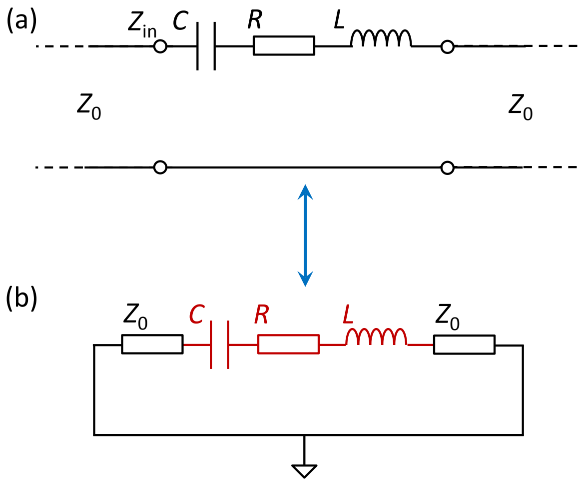

Figure 1.15: (a) Schematic of a series RLC circuit embedded into a transmission line with real-valued characteristic impedance Z_0. (b) Equivalent circuit of (a), where both transmission lines have been replaced by lumped element resistors Z_0. In (b) internal parts of the circuit are drawn in red, while external parts are drawn in black.

Again, we also consider the situation where the circuit is coupled to a transmission line on both sides, cf. Fig. 1.15 (a). When we replace in the schematic each transmission line by a lumped element resistor Z_0, cf. Fig 1.15 (b), we get for the total resistance

$$R_{\mathrm{tot}} = R + 2Z_0. \tag{1.130}$$

For the quality factors we get

$$Q_{\mathrm{L}} = \frac{\omega_0 L}{R + 2Z_0}, \qquad Q_{\mathrm{int}} = \frac{\omega_0 L}{R}, \qquad Q_{\mathrm{ext}} = \frac{\omega_0 L}{2Z_0}, \tag{1.131}$$

and the decay rates become

$$\kappa_{\mathrm{tot}} = \frac{R + 2Z_0}{L}, \qquad \kappa_{\mathrm{int}} = \frac{R}{L}, \qquad \kappa_{\mathrm{ext}} = \frac{2Z_0}{L}. \tag{1.132}$$

With $\tilde{R} = R + Z_0$ and $\tilde{\kappa} = \kappa_{\mathrm{int}} + \kappa_{\mathrm{ext}}/2$, we get for the input impedance

$$\begin{aligned} Z_{\mathrm{in}} &= \tilde{R} + i\frac{\tilde{R}}{\tilde{\kappa}}\omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) && (1.133)\\ &\approx \tilde{R}\left(1 + 2i\frac{\Delta\omega}{\tilde{\kappa}}\right), && (1.134) \end{aligned}$$

which finally leads to the reflection

$$\begin{aligned}
\Gamma &\approx \frac{\tilde{R}\left(1+2i\frac{\Delta\omega}{\tilde{\kappa}}\right)-Z_0}{\tilde{R}\left(1+2i\frac{\Delta\omega}{\tilde{\kappa}}\right)+Z_0} && (1.135)\\
&= \frac{\tilde{\kappa}+2i\Delta\omega-\frac{Z_0\tilde{\kappa}}{\tilde{R}}}{\tilde{\kappa}+2i\Delta\omega+\frac{Z_0\tilde{\kappa}}{\tilde{R}}} && (1.136)\\
&= \frac{\kappa_{\text{int}}+2i\Delta\omega}{\kappa_{\text{int}}+\kappa_{\text{ext}}+2i\Delta\omega}. && (1.137)
\end{aligned}$$

1.3.4 Series RLC circuit capacitively coupled on one side to a transmission line

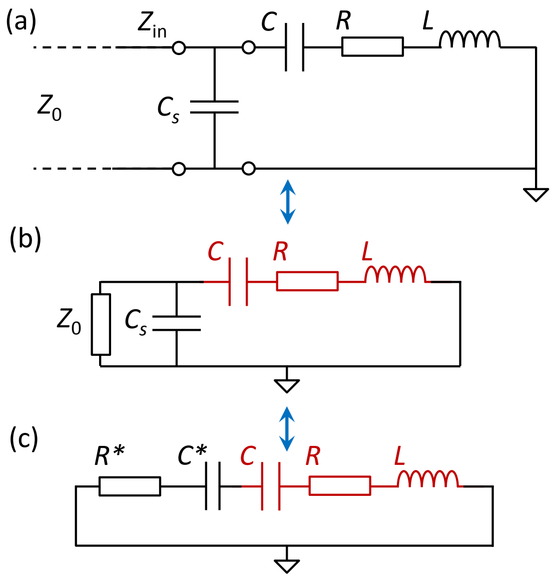

Figure 1.16: (a) Schematic of a series RLC circuit capacitively coupled on one port to a transmission line with real-valued characteristic impedance Z_0. (b) Equivalent circuit of (a), where the transmission line has been replaced by a lumped element resistor Z_0. (c) Thevenin equivalent of (b), where the parallel combination of coupling shunt capacitor C_s and transmission line impedance Z_0 has been transformed into series equivalents C^* and R^*. In (b) and (c) internal parts of the circuit are drawn in red, while the external part is drawn in black.

Similar to the parallel RLC circuit, we can couple the series RLC circuit via an capacitor to the feedline instead of directly. However, the capacitor now has to be a shunt capacitor with typically $C_s \gg C$ between the signal line and ground, cf. Fig. 1.16 (a), and not a series capacitor in the signal line as above for the parallel RLC circuit.[3] The shunt capacitor and the feedline impedance have the total impedance

$$Z_e = \left(\frac{1}{Z_0} + i\omega C_s\right)^{-1} = \frac{Z_0}{1 + i\omega C_s Z_0} \tag{1.138}$$

seen from the resonator, cf. Fig. 1.16 (b). This can be separated into real and imaginary part

[3]If we chose a series capacitor here instead, we would just add capacitance and change the circuit but would not speak of coupling the circuit to a feedline. However, there is not a perfectly clear difference for me between the two cases...

$$Z_e = \frac{Z_0}{1+\omega^2 C_s^2 Z_0^2} + \frac{1}{i\omega}\frac{\omega^2 C_s Z_0^2}{1+\omega^2 C_s^2 Z_0^2} = R^* + \frac{1}{i\omega C^*}. \tag{1.139}$$

with the equivalent series lumped elements

$$R^* = \frac{Z_0}{1+\omega^2 C_s^2 Z_0^2}, \qquad C^* = \frac{1+\omega^2 C_s^2 Z_0^2}{\omega^2 C_s Z_0^2}, \tag{1.140}$$

cf. also Fig. 1.19 (c). As for this type of coupling usually $\omega C_s Z_0 \gg 1$, we can approximate the equivalent circuit elements around the resonance frequency $\omega \approx \omega_0$ by

$$R^* \approx \frac{1}{\omega_0^2 C_s^2 Z_0}, \qquad C^* \approx C_s, \tag{1.141}$$

which is analogous to the corresponding parallel circuit considerations. The total resistance is hence

$$R_{\text{tot}} = R + R^*, \tag{1.142}$$

the total capacitance is

$$C_{\text{tot}} = \frac{CC_s}{C+C_s} \tag{1.143}$$

and the resonance frequency is

$$\omega_0 = \frac{1}{\sqrt{LC_{\text{tot}}}} = \sqrt{\frac{C+C_s}{LCC_s}}. \tag{1.144}$$

Note, that here – similar to the inductively coupled parallel RLC circuit – the resonance frequency is shifted upwards by the coupling capacitor, not downwards. In this approximation, we can again separate the total quality factor

$$Q_{\text{L}} = \frac{1}{\omega_0 R_{\text{tot}} C_{\text{tot}}} \tag{1.145}$$

into internal and external contributions via

$$Q_{\text{int}} = \frac{C+C_s}{\omega_0 RCC_s}, \qquad Q_{\text{ext}} = \frac{C+C_s}{\omega_0 R^* CC_s} = \frac{\omega_0 Z_0 C_s (C+C_s)}{C}. \tag{1.146}$$

For the decay rates, we get

$$\kappa_{\text{int}} = \frac{\omega_0^2 RCC_s}{C+C_s}, \qquad \kappa_{\text{ext}} = \frac{\omega_0^2 R^* CC_s}{C+C_s} = \frac{C}{Z_0 C_s (C+C_s)}. \tag{1.147}$$

Now, we derive the input impedance by Pozar's trick and start with the input impedance of the lossless capacitively coupled series RLC circuit. It is given by the relation

$$\frac{1}{Z_{\text{in}}} = \left(i\omega L + \frac{1}{i\omega C}\right)^{-1} + i\omega C_s \tag{1.148}$$

$$= i\frac{\omega(C + C_s) - \omega^3 LCC_s}{1 - \omega^2 LC}. \tag{1.149}$$

and can be written as

$$Z_{\text{in}} = i\frac{1 - \omega^2 LC}{\omega^3 LCC_s - \omega(C + C_s)} \tag{1.150}$$

$$= i\frac{1 - \omega^2 LC}{\omega(C + C_s)(\omega^2 L\frac{CC_s}{C+C_s} - 1)}. \tag{1.151}$$

As we are dealing with a parallel resonance, i.e., with a diverging imaginary part of the input impedance, the resonance frequency is given by a vanishing denominator, i.e., by

$$\omega_0 = \sqrt{\frac{C + C_s}{LCC_s}} \tag{1.152}$$

as already found above. With this, we rewrite the input impedance as

$$\frac{1}{Z_{\text{in}}} = i\frac{\omega(C + C_s)}{1 - \omega^2 LC}\left(1 - \frac{\omega^2}{\omega_0^2}\right) \tag{1.153}$$

and approximate it by means of a Taylor expansion around $\omega = \omega_0$ as

$$\frac{1}{Z_{\text{in}}} \approx 2i\frac{C_s(C + C_s)}{C}\Delta\omega. \tag{1.154}$$

Hence, the input impedance is

$$Z_{\text{in}} \approx \frac{1}{2i\frac{C_s(C+C_s)}{C}\Delta\omega}. \tag{1.155}$$

When we insert the complex resonance frequency now, we get

$$Z_{\text{in}} \approx \frac{1}{\frac{C_s(C+C_s)}{C}(\kappa_{\text{int}} + 2i\Delta\omega)} \tag{1.156}$$

$$= \frac{C}{C_s(C + C_s)}\frac{1}{\kappa_{\text{int}} + 2i\Delta\omega}. \tag{1.157}$$

Hence, we get for the reflection parameter

$$\Gamma = \frac{1 - Z_0\frac{C_s(C+C_s)}{C}(\kappa_{\text{int}} + 2i\Delta\omega)}{1 + Z_0\frac{C_s(C+C_s)}{C}(\kappa_{\text{int}} + 2i\Delta\omega)} \tag{1.158}$$

$$= \frac{\kappa_{\text{ext}} - \kappa_{\text{int}} - 2i\Delta\omega}{\kappa_{\text{ext}} + \kappa_{\text{int}} + 2i\Delta\omega}. \tag{1.159}$$

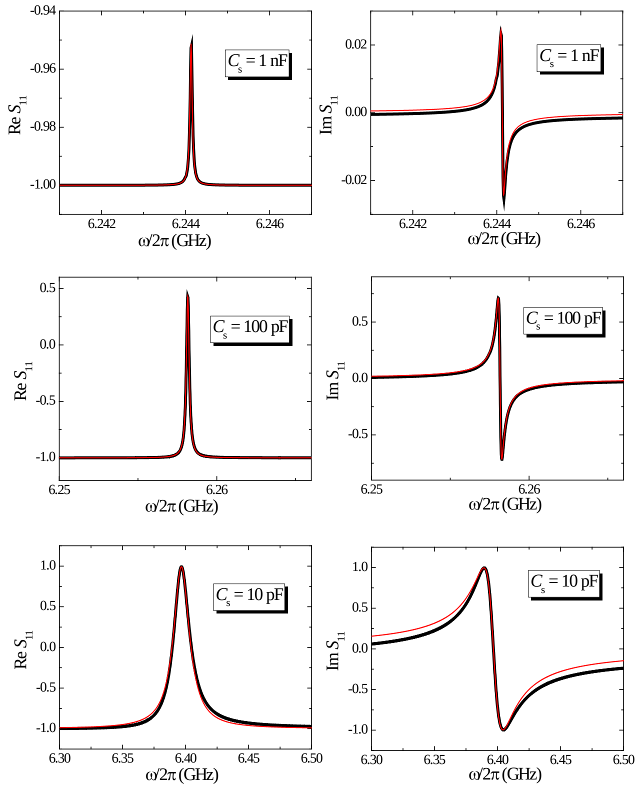

Figure 1.17: Real and imaginary parts of the reflection parameter in direct comparison between QUCS results in black and the approximate expression given by Eq. (1.159) in red for three different shunt capacitors C_s. As circuit parameters, we chose $R = 520\,\mu\Omega$, $C = 500\,fF$, $L = 1300\,pH$ and $Z_0 = 50\,\Omega$.

The reflection parameter - QUCS vs approximation

In Fig. 1.17, the reflection parameter derived above is plotted in comparison with the QUCS result for three different shunt capacitors C_s. The agreement is very good for the intermediate shunt capacitor and deviations can be seen for very small and very large coupling. For small C_s, there is an offset in the imaginary part and for large C_s there occurs an asymmetry in the exact curves, which is not captured by the approximate expressions. This is very similar to the observations for the series coupled parallel RLCs, cf. above.

1.3.5 Series RLC circuit capacitively coupled on both sides to a transmission line

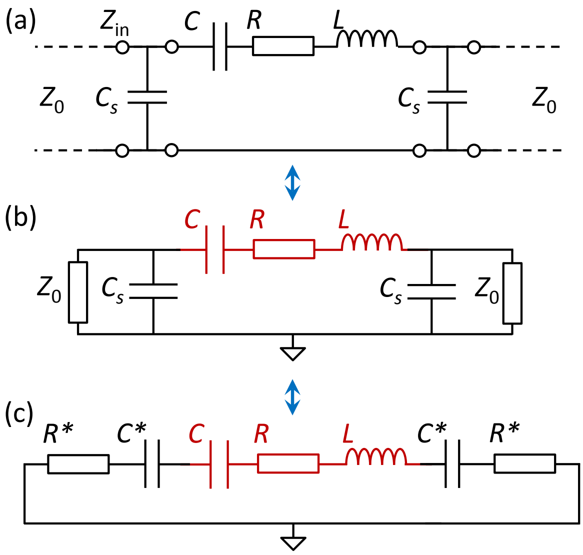

Figure 1.18: (a) Schematic of a series RLC circuit capacitively coupled on two ports to a transmission line with real-valued characteristic impedance Z_0. (b) Thevenin equivalent of (a), where the transmission lines and coupling shunt capacitors C_s have been transformed into their series equivalents C^* and R^*. In (b) and (c) internal parts of the circuit are drawn in red, while the external part is drawn in black.

Next is the series RLC circuit capacitively shunt coupled to transmission lines on both sides, cf. Fig. 1.18 (a). On each side, we can transform the shunt capacitor C_s and the transmission line impedance Z_0 to their series equivalents C^* and R^*, cf. Fig. 1.18 (b) and (c) and the previous subsection, with

$$R^* = \frac{Z_0}{1+\omega^2 C_s^2 Z_0^2} \approx \frac{1}{\omega_0^2 C_s^2 Z_0}, \qquad C^* = \frac{1+\omega^2 C_s^2 Z_0^2}{\omega^2 C_s Z_0^2} \approx C_s. \tag{1.160}$$

The approximations are valid for $\omega \approx \omega_0$ and $\omega C_s Z_0 \gg 1$. With

$$R_{\text{tot}} = R + 2R^*, \qquad C_{\text{tot}} = \frac{CC_s}{2(C+C_s)}, \qquad \omega_0 = \frac{1}{\sqrt{LC_{\text{tot}}}} = \sqrt{\frac{2C+C_s}{LCC_s}} \tag{1.161}$$

we can write the quality factors as

$$Q_{\text{int}} = \frac{2C+C_s}{\omega_0 RCC_s}, \qquad Q_{\text{ext}} = \frac{2C+C_s}{2\omega_0 R^* CC_s} = \frac{\omega_0 Z_0 C_s (2C+C_s)}{2C}, \tag{1.162}$$

and the decay rates as

$$\kappa_{\text{int}} = \frac{\omega_0^2 R C C_s}{2C + C_s}, \qquad \kappa_{\text{ext}} = \frac{2\omega_0^2 R^* C C_s}{2C + C_s} = \frac{2C}{Z_0 C_s (2C + C_s)}, \tag{1.163}$$

For the input impedance, we can use the trick we always use when going from the one-port coupled version to the two-port coupled one. We first absorb the second port into the effective internal parameters $\tilde{R} = R + R^*$, $\tilde{C} = CC_s/(C + C_s)$ and $\tilde{\kappa} = \kappa_{\text{int}} + \kappa_{\text{ext}}/2$ and replace the original internal parameters by these effective ones in the input impedance of the one-port coupled version

$$\begin{aligned} Z_{\text{in}} &\approx \frac{\tilde{C}}{C_s(\tilde{C} + C_s)} \frac{1}{\tilde{\kappa} + 2i\Delta\omega} && (1.164) \\ &= \frac{C}{C_s(2C + C_s)} \frac{1}{\tilde{\kappa} + 2i\Delta\omega}. && (1.165) \end{aligned}$$

With this, the reflection parameter becomes

$$\begin{aligned} \Gamma &= \frac{1 - Z_0 \frac{C_s(2C+C_s)}{C} (\tilde{\kappa} + 2i\Delta\omega)}{1 + Z_0 \frac{C_s(2C+C_s)}{C} (\tilde{\kappa} + 2i\Delta\omega)} && (1.166) \\ &= \frac{1 - \frac{2}{\kappa_{\text{ext}}} (\tilde{\kappa} + 2i\Delta\omega)}{1 + \frac{2}{\kappa_{\text{ext}}} (\tilde{\kappa} + 2i\Delta\omega)} && (1.167) \\ &= \frac{\frac{\kappa_{\text{ext}}}{2} - \tilde{\kappa} - 2i\Delta\omega}{\frac{\kappa_{\text{ext}}}{2} + \tilde{\kappa} + 2i\Delta\omega} && (1.168) \\ &= \frac{-\kappa_{\text{int}} - 2i\Delta\omega}{\kappa_{\text{int}} + \kappa_{\text{ext}} + 2i\Delta\omega}. && (1.169) \end{aligned}$$

1.3.6 Series RLC circuit inductively coupled on one side to a transmission line

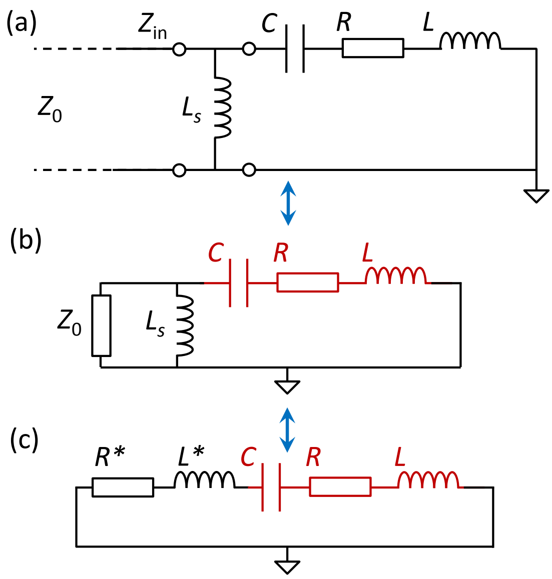

Figure 1.19: (a) Schematic of a series RLC circuit inductively coupled on one port to a transmission line with real-valued characteristic impedance Z_0. (b) Equivalent circuit of (a), where the transmission line has been replaced by a lumped element resistor Z_0. (c) Thevenin equivalent of (b), where the parallel combination of coupling shunt inductor L_s and transmission line impedance Z_0 has been transformed into series equivalents L^* and R^*. In (b) and (c) internal parts of the circuit are drawn in red, while the external part is drawn in black.

As was the case for coupling a parallel RLC circuit to a transmission line, we can also couple the series circuit by means of a shunt inductance instead of a shunt capacitance, cf. Fig. 1.19 (a). Analogously to the cases before, we first transform the parallel combination of coupling shunt inductance L_s and characteristic transmission line impedance Z_0 to their series equivalents, cf. Fig. 1.19 (b), (c). Their impedance from the resonator viewpoint is

$$\frac{1}{Z_e} = \frac{1}{Z_0} + \frac{1}{i\omega L_s} = \frac{Z_0 + i\omega L_s}{i\omega Z_0 L_s}, \tag{1.170}$$

or

$$Z_e = \frac{i\omega Z_0 L_s}{Z_0 + i\omega L_s} = \frac{\omega^2 L_s^2 Z_0}{Z_0^2 + \omega^2 L_s^2} + i\omega \frac{L_s Z_0^2}{Z_0^2 + \omega^2 L_s^2} = R^* + i\omega L^* \tag{1.171}$$

with

$$R^* = \frac{\omega^2 L_s^2 Z_0}{Z_0^2 + \omega^2 L_s^2} \approx \frac{\omega_0^2 L_s^2}{Z_0}, \qquad L^* = \frac{L_s Z_0^2}{Z_0^2 + \omega^2 L_s^2} \approx L_s. \tag{1.172}$$

The approximations of the last expressions have been made assuming $\omega L_s \ll Z_0$ and $\omega \approx \omega_0$. Then, the resonance frequency is given by

$$\omega_0 = \frac{1}{\sqrt{(L + L_s)C}}, \tag{1.173}$$

i.e., it is shifted downwards compared to the bare circuit resonance frequency. The total resistance is $R_{\text{tot}} = R + R^*$, the total inductance is $L_{\text{tot}} = L + L_s$ and the quality factors are

$$Q_{\text{int}} = \frac{\omega_0 (L + L_s)}{R}, \qquad Q_{\text{ext}} = \frac{\omega_0 (L + L_s)}{R^*} = \frac{Z_0 (L + L_s)}{\omega_0 L_s^2}, \tag{1.174}$$

and the decay rates read

$$\kappa_{\text{int}} = \frac{R}{L + L_s}, \qquad \kappa_{\text{ext}} = \frac{R^*}{L + L_s} = \frac{\omega_0^2 L_s^2}{Z_0 (L + L_s)}. \tag{1.175}$$

Now we derive the input impedance and begin with the lossless equivalent (as always done above), which has an input impedance defined by

$$\frac{1}{Z_{\text{in}}} = \left(i\omega L + \frac{1}{i\omega C} \right)^{-1} + \frac{1}{i\omega L_s} \tag{1.176}$$

$$= \frac{i\omega C}{1 - \omega^2 LC} + \frac{1}{i\omega L_s} \tag{1.177}$$

$$= i\frac{\omega^2 (L + L_s)C - 1}{\omega L_s (1 - \omega^2 LC)}. \tag{1.178}$$

The nominator of this expression vanishes[4] for

$$\omega_0 = \frac{1}{\sqrt{(L + L_s)C}}, \tag{1.179}$$

the same expression as derived above. Now, we can approximate the input admittance $1/Z_{\text{in}}$ around this resonance frequency by means of a Taylor expansion as

$$\frac{1}{Z_{\text{in}}} \approx 2i\frac{(L + L_s)^2}{L_s^2} C\Delta\omega, \tag{1.180}$$

which is equivalent to the input impedance

$$Z_{\text{in}} \approx \frac{1}{2i\frac{(L+L_s)^2}{L_s^2}\Delta\omega}, \tag{1.181}$$

[4] Remember here that for parallel resonances, the imaginary part of the impedance diverges on resonance.

Finally, we include the losses by the complex frequency trick and get

$$Z_{\text{in}} \approx \frac{L_s^2}{C(L+L_s)^2}\frac{1}{\kappa_{\text{int}}+2i\Delta\omega}. \tag{1.182}$$

For the reflection parameter, we get

$$\begin{aligned}\Gamma &= \frac{1-\frac{Z_0C(L+L_s)^2}{L_s^2}(\kappa_{\text{int}}+2i\Delta\omega)}{1+\frac{Z_0C(L+L_s)^2}{L_s^2}(\kappa_{\text{int}}+2i\Delta\omega)} && (1.183)\\ &= \frac{\kappa_{\text{ext}}-\kappa_{\text{int}}-2i\Delta\omega}{\kappa_{\text{ext}}+\kappa_{\text{int}}+2i\Delta\omega}. && (1.184)\end{aligned}$$

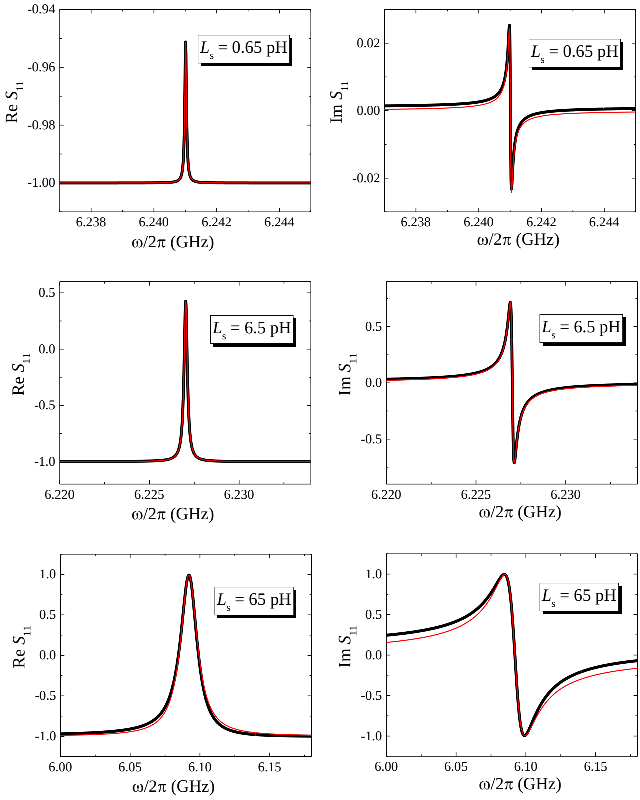

Figure 1.20: Real and imaginary parts of reflection parameter in direct comparison between QUCS results in black and the approximate expression given by Eq. (1.184) in red for three different shunt inductors L_s. As circuit parameters, we chose $R = 520\,\mu\Omega$, $C = 500\,fF$, $L = 1300\,pH$ and $Z_0 = 50\,\Omega$.

The reflection parameter - QUCS vs approximation

In Fig. 1.20, the reflection parameter derived above is plotted in comparison with the QUCS result for three different shunt inductors L_s. The agreement is very good for the intermediate shunt inductor and deviations can be seen for very small and very large coupling. For small L_s, there is an offset in the imaginary part and for large L_s there occurs an asymmetry in the exact curves, which is not captured by the approximate expressions. This is very similar to the observations for the series coupled parallel RLCs, cf. above.

1.3.7 Series RLC circuit inductively coupled on both sides to a transmission line

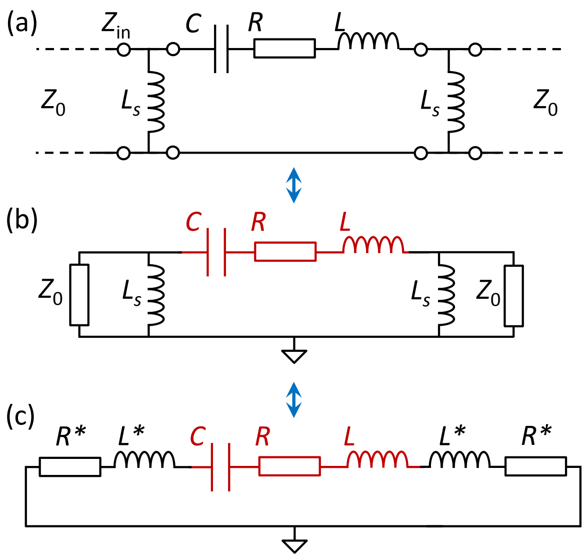

Figure 1.21: (a) Schematic of a series RLC circuit inductively coupled on both ports to a transmission line with real-valued characteristic impedance Z_0. (b) Thevenin equivalent of (a), where the transmission lines and coupling shunt inductors L_s have been transformed into their series equivalents L^* and R^*. In (b) and (c) internal parts of the circuit are drawn in red, while the external part is drawn in black.

Finally, we consider the series RLC circuit coupled by shunt inductances L_s on both sides to feedlines with characteristic impedances Z_0 as shown in Fig. 1.21 (a). The transformation to the series equivalents of the coupling elements is equal to the corresponding single-port case with

$$R^* = \frac{\omega^2 L_s^2 Z_0}{Z_0^2 + \omega^2 L_s^2} \approx \frac{\omega_0^2 L_s^2}{Z_0}, \qquad L^* = \frac{L_s Z_0^2}{Z_0^2 + \omega^2 L_s^2} \approx L_s. \tag{1.185}$$

Hence, we get with the total inductance $L_{\text{tot}} = L + 2L_s$ and the total resistance $R_{\text{tot}} = R + 2R^*$ the resonance frequency

$$\omega_0 = \frac{1}{\sqrt{(L + 2L_s)C}}, \tag{1.186}$$

the quality factors

$$Q_{\text{int}} = \frac{\omega_0 (L + 2L_s)}{R}, \qquad Q_{\text{ext}} = \frac{\omega_0 (L + 2L_s)}{2R^*} = \frac{Z_0 (L + 2L_s)}{2\omega_0 L_s^2}, \tag{1.187}$$

and the decay rates

$$\kappa_{\rm int} = \frac{R}{L+2L_s}, \qquad \kappa_{\rm ext} = \frac{2R^*}{L+2L_s} = \frac{2\omega_0^2 L_s^2}{Z_0(L+2L_s)}. \tag{1.188}$$

When we again define $\tilde{R} = R + R^*$, $\tilde{L} = L + L_s$ and $\tilde{\kappa} = \tilde{R}/(\tilde{L}+L_s) = \kappa_{\rm int} + \kappa_{\rm ext}/2$, we can directly use the input impedance from the one-port coupled case but with the new variables

$$\begin{aligned} Z_{\rm in} &\approx \frac{L_s^2}{C(\tilde{L}+L_s)^2}\frac{1}{\tilde{\kappa}+2i\Delta\omega} && (1.189)\\ &= \frac{L_s^2}{C(L+2L_s)^2}\frac{1}{\tilde{\kappa}+2i\Delta\omega}. && (1.190)\end{aligned}$$

Finally, we calculate the reflection parameter as

$$\begin{aligned} \Gamma &= \frac{1-\frac{Z_0C(L+2L_s)^2}{L_s^2}(\tilde{\kappa}+2i\Delta\omega)}{1+\frac{Z_0C(L+2L_s)^2}{L_s^2}(\tilde{\kappa}+2i\Delta\omega)} && (1.191)\\ &= \frac{1-\frac{2}{\kappa_{\rm ext}}(\tilde{\kappa}+2i\Delta\omega)}{1+\frac{2}{\kappa_{\rm ext}}(\tilde{\kappa}+2i\Delta\omega)} && (1.192)\\ &= \frac{-\kappa_{\rm int}-2i\Delta\omega}{\kappa_{\rm ext}+\kappa_{\rm int}+2i\Delta\omega}. && (1.193)\end{aligned}$$

1.4 Transmission parameter for lumped element RLC circuits

When we are interested in transmission parameters, we usually cannot work with input impedances (except for the side-coupled case) but have to work with the $ABCD$-matrix formalism. We have already used this above, when we derived the transmission parameter for the capacitively side-coupled parallel RLC circuit. Now, we introduce this formalism officially and calculate the transmission for all types of two-port coupled circuits considered above. To do our considerations with lumped elements, we need only two different $ABCD$-matrices, the matrix for impedances in the signal line

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \tag{1.194}$$

and the matrix for a shunt impedance between signal line and ground

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{Z} & 1 \end{pmatrix}. \tag{1.195}$$

Then, the matrices for different elements can just be multiplied to get the total matrix. With the relations between the $ABCD$ matrices and the transmission parameter

$$S_{12} = \frac{2(AD - BC)}{A + B/Z_0 + CZ_0 + D} \tag{1.196}$$

we can then calculate S_{12} for arbitrary impedances embedded into a transmission line with characteristic impedance Z_0.

1.4.1 S_{12} of the directly embedded parallel RLC circuit

First, we treat the simplest version, the directly embedded parallel RLC circuit, cf. Fig.1.3. To illustrate the formalism, we give two different possibilities for calculating the total $ABCD$ matrix here. First, we give the matrix of the total parallel impedance

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\mathrm{RLC}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{R} + \frac{1}{i\omega L} + i\omega C & 1 \end{pmatrix}. \tag{1.197}$$

The second possibility is to take the matrices of the individual elements

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\mathrm{RLC}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{R} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{i\omega L} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ i\omega C & 1 \end{pmatrix}. \tag{1.198}$$

Of course, both methods lead to the same result. Now, we can put the parameters into the equation for S_{12} and get

$$S_{12} = \frac{2}{2 + Z_0\left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C\right)} \tag{1.199}$$

$$\approx \frac{2}{2 + \frac{Z_0}{R}(1 + 2iRC\Delta\omega)} \tag{1.200}$$

$$= \frac{2}{2 + 2\frac{\kappa_{\mathrm{int}}}{\kappa_{\mathrm{ext}}} + 4i\frac{\Delta\omega}{\kappa_{\mathrm{ext}}}} \tag{1.201}$$

$$= \frac{\kappa_{\mathrm{ext}}}{\kappa_{\mathrm{int}} + \kappa_{\mathrm{ext}} + 2i\Delta\omega}. \tag{1.202}$$

In the second step, we have used our usual approximation around the resonance frequency and in the last two steps, we have used the equations for the decay rates derived above for this type of circuit.

1.4.2 S_{12} of the series coupled parallel RLC circuit

The next circuit, for which we calculate the transmission parameter, is the series coupled parallel RLC circuit. On both sides of the circuit, it is coupled to a transmission line by either a coupling capacitor C_c or by a coupling inductor L_c, cf. Figs. 1.7 and 1.10. In the calculation, we do not discriminate between capacitor and inductor in the first part, but just consider the coupling element by its impedance Z_c. First, we have to calculate the $ABCD$-matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z_c \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{Z_p} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & Z_c \\ 0 & 1 \end{pmatrix} \tag{1.203}$$

$$= \begin{pmatrix} 1 + \frac{Z_c}{Z_p} & 2Z_c + \frac{Z_c^2}{Z_p} \\ \frac{1}{Z_p} & 1 + \frac{Z_c}{Z_p} \end{pmatrix} \tag{1.204}$$

with the input impedance of the lossless parallel RLC circuit $1/Z_p = i(\omega C - 1/\omega L)$. We will include the internal losses again at the end by Pozar's complex frequency trick.[5]
Hence, the nominator of the transmission parameter is given by

$$2(AD - BC) = 2. \tag{1.205}$$

For the denominator, we need some more considerations. It is given by

$$A + \frac{B}{Z_0} + CZ_0 + D = 2 + 2\frac{Z_c}{Z_p} + 2\frac{Z_c}{Z_0} + \frac{Z_c^2}{Z_0 Z_p} + \frac{Z_0}{Z_p} \tag{1.206}$$

$$= 2 + 2\frac{Z_c}{Z_0} + \frac{(Z_0 + Z_c)^2}{Z_0 Z_p} \tag{1.207}$$

Now, we use our earlier methods, results and approximations for real and imaginary part of the denominator and do also the discrimination between coupling capacitor and inductor.
For coupling capacitors, we have found in our earlier analysis of this circuit, that the resonance frequency is approximately given by $\omega_0 = 1/\sqrt{L(C + 2C_c)}$. As we are mainly interested in the response around resonance, we approximate the real part as its value at $\omega = \omega_0$ and get

$$\mathrm{Re}[A + \frac{B}{Z_0} + CZ_0 + D] = 2 + \frac{2}{\omega C_c}\left(\omega C - \frac{1}{\omega L}\right) \tag{1.208}$$

$$\approx -2. \tag{1.209}$$

In the last step, we have set $\omega = \omega_0$.
Now we perform a Taylor approximation of the imaginary part of the denominator around $\omega = \omega_0 = 1/\sqrt{L(C + 2C_c)}$ to the linear term

[5] I am not sure if this is allowed here, although the final result looks fine. Should be maybe checked somehow...

$$\begin{aligned}\mathrm{Im}[A + \frac{B}{Z_0} + CZ_0 + D] &= \left(Z_0 - \frac{1}{Z_0\omega^2 C_c^2}\right)\left(\omega C - \frac{1}{\omega L}\right) - \frac{2}{Z_0\omega C_c} && (1.210)\\ &\approx -2\frac{C + 2C_c}{Z_0\omega_0^2 C_c^2}(\omega - \omega_0) && (1.211)\\ &= -4\frac{\Delta\omega}{\kappa_{\mathrm{ext}}}. && (1.212)\end{aligned}$$

To make the approximation, we have performed a Taylor approximation and in addition assumed $1 - Z_0^2\omega_0^2 C_c^2 \approx 1$.
With this, the transmission parameter of the capacitively coupled parallel RLC circuit becomes

$$\begin{aligned}S_{12} &= \frac{2}{-2 - 4i\frac{\Delta\omega}{\kappa_{\mathrm{ext}}}} && (1.213)\\ &= \frac{-\kappa_{\mathrm{ext}}}{\kappa_{\mathrm{ext}} + 2i\Delta\omega} && (1.214)\end{aligned}$$

or including the losses

$$S_{12} = \frac{-\kappa_{\mathrm{ext}}}{\kappa_{\mathrm{int}} + \kappa_{\mathrm{ext}} + 2i\Delta\omega}. \qquad (1.215)$$

For the inductive coupling, we just give the results of the calculations. Note however, that the resonance frequency is given by

$$\omega_0^2 = \frac{2L + L_c}{LL_cC}. \qquad (1.216)$$

Doing similar approximations as for the capacitively case, we get

$$\mathrm{Re}[A + \frac{B}{Z_0} + CZ_0 + D] \approx -2. \qquad (1.217)$$

and

$$\begin{aligned}\mathrm{Im}[A + \frac{B}{Z_0} + CZ_0 + D] &\approx -2\frac{L_c(2L + L_c)}{Z_0 L}(\omega - \omega_0) && (1.218)\\ &= -4i\frac{\Delta\omega}{\kappa_{\mathrm{ext}}}. && (1.219)\end{aligned}$$

Hence, we get the same transmission parameter as for the capacitively coupled case

$$S_{12} = \frac{-\kappa_{\mathrm{ext}}}{\kappa_{\mathrm{ext}} + 2i\Delta\omega} \qquad (1.220)$$

or including the losses

$$S_{12} = \frac{-\kappa_{\mathrm{ext}}}{\kappa_{\mathrm{int}} + \kappa_{\mathrm{ext}} + 2i\Delta\omega}. \qquad (1.221)$$

1.4.3 S_{12} of the directly embedded parallel RLC circuit

Analogously to the parallel RLC ciruit, we treat now the series RLC circuit in a transmission line and determine the transmission paramater. The $ABCD$-matrix of the series RLC circuit is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & R + i\omega L + \frac{1}{i\omega C} \\ 0 & 1 \end{pmatrix}. \tag{1.222}$$

With this, we determine the transmission parameter as

$$\begin{aligned} S_{12} &= \frac{2}{2 + \frac{1}{Z_0}\left(R + i\omega L + \frac{1}{i\omega C}\right)} && (1.223) \\ &\approx \frac{2}{2 + \frac{R}{Z_0} + 2i\frac{L}{Z_0}\Delta\omega} && (1.224) \\ &= \frac{2}{2 + 2\frac{\kappa_{\text{int}}}{\kappa_{\text{ext}}} + 4i\frac{\Delta\omega}{\kappa_{\text{ext}}}} && (1.225) \\ &= \frac{\kappa_{\text{ext}}}{\kappa_{\text{int}} + \kappa_{\text{ext}} + 2i\Delta\omega}. && (1.226) \end{aligned}$$

1.4.4 S_{12} of the shunt coupled series RLC circuit

Now, we calculate the transmission parameter for the shunt coupled series RLC circuit, cf. Figs 1.18 and 1.21. The $ABCD$-matrix is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{Z_s} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & Z_i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{Z_s} & 1 \end{pmatrix} \tag{1.227}$$

$$= \begin{pmatrix} 1+\frac{Z_i}{Z_s} & Z_i \\ \frac{2}{Z_s}+\frac{Z_i}{Z_s^2} & 1+\frac{Z_i}{Z_s} \end{pmatrix} \tag{1.228}$$

Here Z_s represents the shunt coupler impedance $Z_s = i\omega L_s$ for a shunt inductance and $Z_s = 1/i\omega C_s$ for a shunt capacitance. The impedance of the series RLC circuit is denotes as $Z_i = R + i\omega L + 1/i\omega C$.
For the nominator of S_{12} we get

$$2(AD - BC) = 2 \tag{1.229}$$

again. The denominator is given by

$$A + \frac{B}{Z_0} + CZ_0 + D = 2 + 2\frac{Z_i}{Z_s} + 2\frac{Z_0}{Z_s} + \frac{Z_i Z_0}{Z_s^2} + \frac{Z_i}{Z_0}. \tag{1.230}$$

Again, we use our earlier methods, results and approximations and do the calculations explicitly for shunt capacitor and shunt inductor separately. From here, we ignore also the internal losses ($R = 0$) as usually and include them at the end by Pozar's complex frequency trick again. The resonance frequency for the shunt capacitor is given by $\omega_0 = \sqrt{(2C + C_s)/LCC_s}$. For the real part of the denominator of S_{12} we get by inserting $\omega = \omega_0$

$$\begin{aligned} \mathrm{Re}[A + \frac{B}{Z_0} + CZ_0 + D] &= 2 - 2\omega C_s\left(\omega L - \frac{1}{\omega C}\right) && (1.231) \\ &\approx 2 - 2C_s\left(\omega_0^2 L - \frac{1}{C}\right) && (1.232) \\ &= -2. && (1.233) \end{aligned}$$

For the imaginary part we get

$$\begin{aligned} \mathrm{Im}[A + \frac{B}{Z_0} + CZ_0 + D] &= 2Z_0\omega C_s + \left(\frac{1}{Z_0} - \omega^2 C_s^2 Z_0\right)\left(\omega L - \frac{1}{\omega C}\right) && (1.234) \\ &\approx 2Z_0\omega C_s - Z_0\omega^2 C_s^2\left(\omega L - \frac{1}{\omega C}\right). && (1.235) \end{aligned}$$

The approximation has been performed assuming $\omega^2 C_s^2 Z_0 \gg 1$ as we also did this in the earlier considerations of the corresponding resonator. A Taylor approximation of the imaginary part around $\omega \approx \omega_0$ gives

$$\mathrm{Im}[A + \frac{B}{Z_0} + CZ_0 + D] \approx -2Z_0 \frac{C_s(2C + C_s)}{C}(\omega - \omega_0) \tag{1.236}$$

$$\approx -4\frac{\Delta\omega}{\kappa_{\mathrm{ext}}}. \tag{1.237}$$

Hence, the transmission of the lossless series RLC ciruit coupled by two shunt capacitors to feedlines is given by

$$S_{12} = \frac{-\kappa_{\mathrm{ext}}}{\kappa_{\mathrm{ext}} + 2i\Delta\omega}, \tag{1.238}$$

i.e., by exactly the same expression as for the parallel circuits in the previous subsection. When the losses are included, we get

$$S_{12} = \frac{-\kappa_{\mathrm{ext}}}{\kappa_{\mathrm{int}} + \kappa_{\mathrm{ext}} + 2i\Delta\omega}. \tag{1.239}$$

At this point, it is not a big surprise that we also get the identical expression for the inductively shunt coupled series RLC circuit. Of course, one would have to use the correct expressions for ω_0 and the κs, but the general expression is identical for all four types of symmetrically coupled resonators.
Later, we will also discuss the transmission through a transmission line resonator which is coupled differently on both sides. For lumped element resonators, this is not straightforward, because one would have to build a combination of series and parallel resonators to couple them differently on both sides.

The transmission parameters - QUCS vs approximation

Figure 1.22 shows the transmission parameters derived above in comparison with the QUCS result. For each of the four coupling types, one example is shown. Note, that the magnitude for all four resonators is a peak and not a dip, although the real part might give this impression.

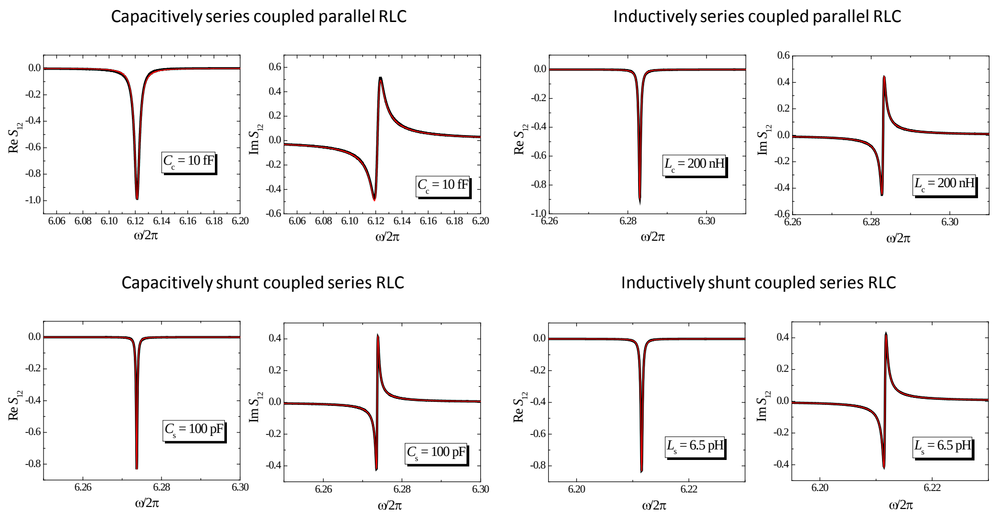

Figure 1.22: Real and imaginary parts of transmission parameters in direct comparison between QUCS results in black and the approximate expressions given above in red. For all four types of two-port coupled resonators one example is shown. As circuit parameters, we chose $R = 5\,M\Omega$, $C = 500\,fF$, $L = 1300\,pH$ and $Z_0 = 50\,\Omega$ for the parallel RLC circuits and $R = 520\,\mu\Omega$, $C = 500\,fF$, $L = 1300\,pH$ and $Z_0 = 50\,\Omega$ for the series RLC circuits.

1.5 Input impedance of terminated transmission lines

1.5.1 The lossless transmission line

Let us consider a lossless transmission line spatially extended to $z < 0$ with characteristic impedance

$$Z_0 = \sqrt{L'/C'} \tag{1.240}$$

and propagation constant

$$\beta = 2\pi f \sqrt{L'C'} = \frac{\omega}{v_{\mathrm{ph}}}, \tag{1.241}$$

where f is the frequency, $\omega = 2\pi f$, $v_{\mathrm{ph}} = 1/\sqrt{L'C'}$ is the phase velocity, and L' and C' are inductance and capacitance per unit length of the line, respectively. The line is terminated at a location $z = 0$ with an arbitrary load impedance Z_{L}, cf. Fig. 1.23.

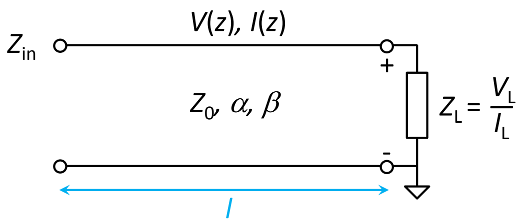

Figure 1.23: Schematic picture of a transmission line with characteristic impedance Z_0, damping constant α, propagation constant β and length l. The line is terminated by an arbitrary load impedance Z_{L}.

An incident voltage wave of the form $V(z) = V_0^+ e^{-i\beta z}$, coming from $z < 0$ (we omit here the time dependence for simplicity, but still talk about "waves"), is partly reflected from and partly transferred into Z_{L}. To see this, we write the resulting wave at z as a superposition of incident and reflected wave

$$V(z) = V_0^+ e^{-i\beta z} + V_0^- e^{i\beta z} \tag{1.242}$$

and

$$I(z) = \frac{V_0^+}{Z_0} e^{-i\beta z} - \frac{V_0^-}{Z_0} e^{i\beta z} \tag{1.243}$$

with the voltage and current amplitudes of the incident and reflected waves V_0^+, $I_0^+ = V_0^+/Z_0$ and V_0^-, $I_0^- = -V_0^-/Z_0$, respectively[6]. For $z = 0$, i.e., at the load, the impedance is given by Z_{L} and thus the ratio of voltage and current amplitudes has to fulfill

[6]The signs follow from the transmission line equations and can be understood as considering the directionality of the current along the line, cf. Pozar

$$Z_\mathrm{L} = \frac{V_\mathrm{L}}{I_\mathrm{L}} = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0. \tag{1.244}$$

Solving for V_0^- gives

$$V_0^- = \frac{Z_\mathrm{L} - Z_0}{Z_\mathrm{L} + Z_0} V_0^+ = \Gamma V_0^+. \tag{1.245}$$

where the last identity defines the (complex) reflection coefficient Γ, which we already have used a lot in the first part of the document.
Now, we can calculate the local or input impedance of the line Z_in at an arbitrary distance $z = -l$ from the load impedance as the local ratio of voltage and current including the phase difference between them according to

$$\begin{aligned} Z_\mathrm{in}(-l) &= \frac{V(-l)}{I(-l)} = Z_0 \frac{(Z_\mathrm{L} + Z_0)e^{i\beta l} + (Z_\mathrm{L} - Z_0)e^{-i\beta l}}{(Z_\mathrm{L} + Z_0)e^{i\beta l} - (Z_\mathrm{L} - Z_0)e^{-i\beta l}} && (1.246) \\ &= Z_0 \frac{Z_\mathrm{L} \cos\beta l + iZ_0 \sin\beta l}{Z_0 \cos\beta l + iZ_\mathrm{L} \sin\beta l} && (1.247) \\ &= Z_0 \frac{Z_\mathrm{L} + iZ_0 \tan\beta l}{Z_0 + iZ_\mathrm{L} \tan\beta l}. && (1.248) \end{aligned}$$

1.5.2 The lossy line

Above, we have considered only lossless lines. If the line is slightly lossy, we can consider the losses by using a complex propagation constant

$$\gamma = \alpha + i\beta \tag{1.249}$$

with the attenuation constant α and the phase constant β. Then, the input impedance can be obtained by an analogous calculation as for the lossless line, cf. Pozar, and it is given by

$$Z_\mathrm{in}(-l) = Z_0 \frac{Z_\mathrm{L} + Z_0 \tanh\gamma l}{Z_0 + Z_\mathrm{L} \tanh\gamma l}. \tag{1.250}$$

For low losses, the characteristic impedance is still given by $Z_0 = \sqrt{L'/C'}$ in good approximation.

1.5.3 Open ended transmission lines

With the results of the previous section, we can derive the input impedance of an open transmission line, i.e., for the particular case of an infinitely large load impedance $Z_\mathrm{L} = \infty$. For the lossless line, we get

$$Z_\mathrm{in} = -iZ_0 \cot \beta l, \tag{1.251}$$

and for the lossy line

$$Z_\mathrm{in} = Z_0 \coth \gamma l = Z_0 \frac{1 + i \tan \beta l \tanh \alpha l}{\tanh \alpha l + i \tan \beta l}. \tag{1.252}$$

From the expression of the input impedance of the lossless line, it can be seen that there are two groups of particular points, where the imaginary part of the input impedance either becomes zero or diverges. These points correspond to the voltage nodes and antinodes of the standing wave on the line, resulting from a total reflection at an open end wih $\Gamma = 1$. As we have seen in the considerations of the lumped element RLC circuits, a vanishing or diverging input impedance corresponds to a series or parallel resonance, respectively.

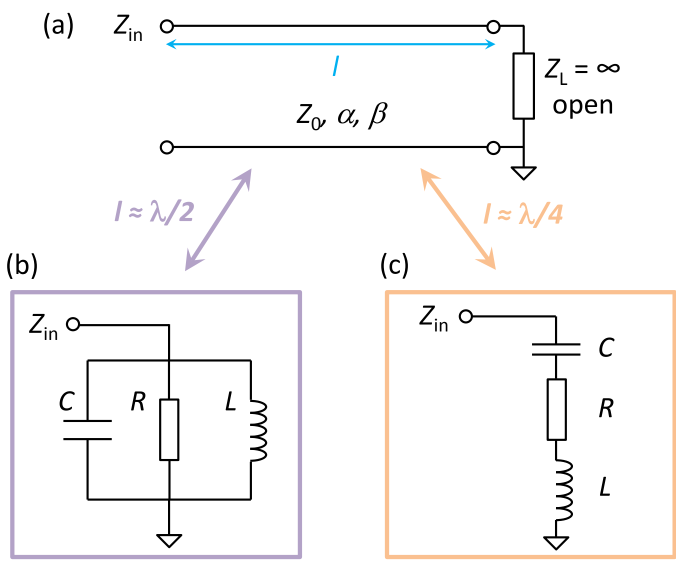

Figure 1.24: (a) Schematic picture of a transmission line with characteristic impedance Z_0, damping constant α, propagation constant β and length l. The line is terminated by an open, i.e., by a load impedance $Z_\mathrm{L} \to \infty$. (b) shows the circuit equivalent of (a) close to frequencies, for which $l \approx \lambda/2$ (in general $l \approx n\lambda/2$). (c) shows the circuit equivalent of (a) close to frequencies, for which $l \approx \lambda/4$ (in general $l \approx (2n-1)\lambda/4$). Note that the equivalent lumped elements R, L and C in (b) and (c) are not identical, but differ considerably between (b) and (c). The circuit equivalents also differ from those in Fig. 1.25 and also between different n. For more details see text.

Figure 1.24 (a)-(c) shows a schematic of the open-ended transmission line and the equivalent circuit representation around the two different resonances. In the following, we will investigate in more detail the two kinds of resonators and find expression for the equivalent lumped elements.

Open ended $\lambda/4$ lines

One kind of resonance occurs, when the length of the open-ended transmission line l is a quarter wavelength of the voltage wave, i.e., $l = \lambda_0/4$ (or odd-numbered multiples of it) at the angular frequency $\omega = \omega_0$. For $l = \lambda_0/4$, we get

$$\cot \beta l = \cot \frac{\pi}{2} = 0, \tag{1.253}$$

i.e. the imaginary part of the input impedance vanishes. In general, this happens for $\beta l = (2n-1)\pi/2$ with n an arbitrary integer, but we restrict all our calculations here to $n = 1$.
Now, we calculate an approximate expression for the input impedance around resonance for a low-loss line, i.e., for $\alpha l \ll 1$, which implies $\tanh \alpha l \approx \alpha l$.

$$\begin{aligned} Z_{\text{in}} &= Z_0 \frac{\tanh \alpha l - i \cot \beta l}{1 - i \tanh \alpha l \cot \beta l} && (1.254) \\ &\approx Z_0 \frac{\alpha l - i \cot \beta l}{1 - i\alpha l \cot \beta l}. && (1.255) \end{aligned}$$

Next, we perform a Taylor approximation of the input impedance around the resonance condition $\cot \beta l = 0$, i.e., around $\omega \approx \omega_0$

$$Z_{\text{in}} \approx Z_0 \alpha l + i Z_0 (1 - \alpha^2 l^2) \frac{\pi}{2} \frac{\Delta\omega}{\omega_0}. \tag{1.256}$$

Due to the assumed smallness of αl, we set as last step $(1 - \alpha^2 l^2) \approx 1$, and get[7]

$$Z_{\text{in}} \approx Z_0 \left(\alpha l + i \frac{\pi}{2} \frac{\Delta\omega}{\omega_0} \right). \tag{1.257}$$

This expression looks formally identical to the approximated input impedance of a series RLC circuit

$$Z_{\text{in}} \approx R + 2iL\Delta\omega. \tag{1.258}$$

This means we can approximate the open-ended transmission line around its quarter wave resonances as a series RLC circuit with the equivalent lumped elements

$$R = Z_0 \alpha l, \qquad L = \frac{Z_0 \pi}{4\omega_0}, \qquad C = \frac{1}{\omega_0^2 L} = \frac{4}{\pi \omega_0 Z_0}. \tag{1.259}$$

[7]This does not look as a very strict approximation, but we note here, that you get exactly the same result, when you first do a first order Taylor approximation for αl and afterwards a second first order Taylor approximation for ω around ω_0.

With this analogy, we can now directly adopt the expressions for the quality factor

$$Q = \frac{\omega_0 L}{R} = \frac{\pi}{4\alpha l} \tag{1.260}$$

and the decay rate

$$\kappa = \frac{R}{L} = \frac{4\omega_0 \alpha l}{\pi} \tag{1.261}$$

from the series RLC circuit.
Note that with the general relations for a transmission line resonator

$$\frac{1}{Z_0\omega_0} = \frac{\lambda_0}{2\pi}C', \qquad\qquad \frac{Z_0}{\omega_0} = \frac{\lambda_0}{2\pi}L' \tag{1.262}$$

we can also write the equivalent reactances as

$$L = \frac{\pi}{4}\frac{4l}{2\pi}L' = \frac{L'l}{2}, \qquad C = \frac{4}{\pi}\frac{4l}{2\pi}C' = \frac{8C'l}{\pi^2} \tag{1.263}$$

where we have used the resonance wavelength $\lambda_0 = 4l$ These relations demonstrate an important thing. The equivalent resonance impedance $Z_r = \sqrt{L/C}$, which normally gives the ratio between voltage and current on resonance in an RLC circuit, is not the same as the characteristic impedance Z_0. In a transmission line resonator, however, the ratio between voltage and current amplitude on resonance is given by Z_0.

Open ended $\lambda/2$ lines

Another kind of resonance occurs, when the length of the open-ended line corresponds to half a wavelength, i.e., when $l = \lambda_0/2$ (or integer multiples of it) at the resonance frequency $\omega = \omega_0$. Note, that we do not discriminate between the quarter wave and the half wave resonance frequencies in the notation for simplicity, although, for a fixed l, they of course are not the same. In contrast to the quarter wave case, for which the imaginary part of the input impedance vanishes on resonance, the imaginary part of the input impedance for the half wave resonance diverges. This corresponds to the resonance of a parallel RLC circuit. The resonance condition is now given by

$$\tan\beta l = 0 \qquad \text{or} \qquad \beta l = n\pi. \tag{1.264}$$

Again, we only consider the case $n = 1$. When we do a Taylor approximation of the input admittance

$$\frac{1}{Z_{\text{in}}} = \frac{1}{Z_0}\frac{\alpha l + i\tan\beta l}{1 + i\alpha l\tan\beta l} \tag{1.265}$$

around $\omega = \omega_0$ defined by

$$\beta_0 l = \omega_0 \frac{l}{v_{\text{ph}}} = \pi \tag{1.266}$$

we get

$$\frac{1}{Z_{\text{in}}} \approx \frac{\alpha l}{Z_0} + \frac{i}{Z_0}(1-\alpha^2 l^2)\pi\frac{\Delta\omega}{\omega_0} \tag{1.267}$$

$$\approx \frac{1}{Z_0}\left(\alpha l + i\pi\frac{\Delta\omega}{\omega_0}\right). \tag{1.268}$$

In the last step, we have again again assumed $(1-\alpha^2 l^2) \approx 1$.
Hence, we receive the input impedance of the open-ended half wavelength line around resonance as

$$Z_{\text{in}} \approx \frac{Z_0}{\alpha l + i\pi\frac{\Delta\omega}{\omega_0}}. \tag{1.269}$$

which is equivalent to the approximate input impedance of a parallel RLC circuit around resonance

$$Z_{\text{in}} \approx \frac{R}{1+2iRC\Delta\omega}. \tag{1.270}$$

The open-ended half wavelength transmission line resonator can thus be approximated around its resonance by a parallel RLC circuit with the equivalent lumped elements

$$R = \frac{Z_0}{\alpha l}, \qquad C = \frac{\pi}{2Z_0\omega_0}, \qquad L = \frac{1}{\omega_0^2 C} = \frac{2Z_0}{\pi\omega_0}. \tag{1.271}$$

Again, we can express the equivalent reactances also as

$$C = \frac{\pi}{2}\frac{\lambda_0}{2\pi}C' = \frac{C'l}{2}, \qquad L = \frac{2}{\pi}\frac{\lambda_0}{2\pi}L' = \frac{2L'l}{\pi^2}, \tag{1.272}$$

but here now with $\lambda_0 = 2l$ Similar to the quarter wave resonances, we can now derive expressions for the quality factor and the decay rate from the prallel RLC circuit equivalent and find

$$Q = \omega_0 RC = \frac{\pi}{2\alpha l}, \qquad \kappa = \frac{1}{RC} = \frac{2\omega_0\alpha l}{\pi}. \tag{1.273}$$

1.5.4 Short ended transmission lines

The second important transmission line termination besides the open configuration is the short-end, i.e., $Z_{\mathrm{L}} = 0$. Then, the input impedance of the lossless line is given by

$$Z_{\mathrm{in}} = iZ_0 \tan \beta l \tag{1.274}$$

and for the lossy line, we get

$$Z_{\mathrm{in}} = Z_0 \tanh \gamma l = Z_0 \frac{\tanh \alpha l + i \tan \beta l}{1 + i \tan \beta l \tanh \alpha l}. \tag{1.275}$$

Similar to the open-ended line input impedance, there are two types of resonances, but there are interchanged with respect to the open-ended case. There are conditions, for which the imaginary part diverges (parallel resonances) and conditions, for which the imaginary part vanishes (series resonances).

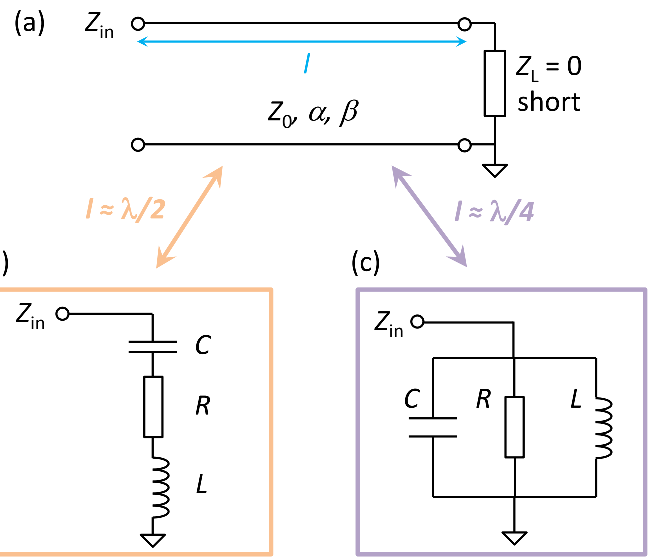

Figure 1.25: (a) Schematic picture of a transmission line with characteristic impedance Z_0, damping constant α, propagation constant β and length l. The line is terminated by a short, i.e., by a load impedance $Z_{\mathrm{L}} = 0$. (b) shows the circuit equivalent of (a) close to frequencies, for which $l \approx \lambda/2$ (in general $l \approx n\lambda/2$). (c) shows the circuit equivalent of (a) close to frequencies, for which $l \approx \lambda/4$ (in general $l \approx (2n-1)\lambda/4$). Note that the equivalent lumped elements R, L and C in (b) and (c) are not identical, but differ considerably between (b) and (c). The circuit equivalents also differ from those in Fig. 1.24 and also between different n. For more details see text.

These conditions and resonances will now be discussed in more detail.

Short ended $\lambda/4$ lines

From Eq. (1.275), we can – analogously to the open-ended lines – calculate the input impedance of the short-ended line for the case that its length corresponds to a quarter wavelength $l = \lambda_0/4$. We again assume low losses, i.e., $\tanh \alpha l \approx \alpha l \ll 1$, and a frequency close to the resonance frequency $\omega \approx \omega_0$.
The quarter wave resonances correspond to the condition

$$\cot \beta l = 0 \qquad \text{or} \qquad \beta l = (2n-1)\frac{\pi}{2}. \tag{1.276}$$

Here, we restrict our considerations again to $n = 1$ again.
The resonance condition implies, that in this case we are dealing with a parallel resonance and that we have to approximate the input admittance $1/Z_{\text{in}}$ around the resonance frequency.
We get for $\tanh \alpha l \ll 1$

$$\begin{aligned} \frac{1}{Z_{\text{in}}} &= \frac{1}{Z_0}\frac{1 + i\tan \beta l \tanh \alpha l}{\tanh \alpha l + i \tan \beta l} &(1.277)\\ &\approx \frac{1}{Z_0}\frac{\alpha l - i\cot \beta l}{1 - i\alpha l \cot \beta l}, &(1.278)\end{aligned}$$

which to first order Taylor approximation becomes

$$\frac{1}{Z_{\text{in}}} \approx \frac{1}{Z_0}\alpha l + \frac{i}{Z_0}(1 - \alpha^2 l^2)\frac{\pi}{2}\frac{\Delta\omega}{\omega_0} \tag{1.279}$$

cf. Eqs. (1.255), (1.256).
With $(1 - \alpha^2 l^2) \approx 1$, we get for the input impedance

$$Z_{\text{in}} \approx \frac{Z_0}{\alpha l + i\frac{\pi}{2}\frac{\Delta\omega}{\omega_0}}, \tag{1.280}$$

i.e., almost the identical result as for the open-ended $\lambda/2$ case. The only difference is the factor of $1/2$ in the denominator, indicating that this resonator has half the length of the corresponding open-ended case. This equivalency is not surprising, because at a half wavelength away from an open end the standing voltage wave looks identical to the quarter wavelength distance at an shorted end. Both have voltage antinodes and currrent nodes at the corresponding input point.
The above input impedance is also equivalent to the approximated input impedance of a lumped element parallel circuit around resonance

$$Z_{\text{in}} \approx \frac{R}{1 + 2iRC\Delta\omega} \tag{1.281}$$

with the correspondencies

$$R = \frac{Z_0}{\alpha l}, \qquad C = \frac{\pi}{4Z_0\omega_0}, \qquad L = \frac{4Z_0}{\pi\omega_0} \tag{1.282}$$

and

$$Q = \omega_0 RC = \frac{\pi}{4\alpha l}, \qquad \kappa = \frac{1}{RC} = \frac{4\omega_0 \alpha l}{\pi}. \tag{1.283}$$

Short ended $\lambda/2$ lines

The second kind of resonances at the input port of a short-ended transmission line correspond to a vanishing imaginary part of the input impedances, i.e., to

$$\tan\beta l = 0 \qquad \text{or} \qquad \beta l = n\pi. \tag{1.284}$$

For $n = 1$ (fundamental half wave mode), we first approximate again $\tanh \alpha l \approx \alpha l$, corresponding to low losses, and get for the input impedance

$$Z_{\text{in}} \approx Z_0 \frac{\alpha l + i\tan\beta l}{1 + i\alpha l \tan\beta l}. \tag{1.285}$$

A Taylor approximation around $\omega \approx \omega_0$ gives (for more details see previous subsections)

$$Z_{\text{in}} \approx Z_0 \alpha l + iZ_0(1 - \alpha^2 l^2)\pi \frac{\Delta\omega}{\omega_0} \tag{1.286}$$

$$\approx Z_0 \alpha l + iZ_0 \pi \frac{\Delta\omega}{\omega_0}. \tag{1.287}$$

In the last step, we have again neglected the term $\alpha^2 l^2$ due to its smallness (see also above). Except for the factor 2 in the imaginary part, this input impedance is identical to the input impedance of the open-ended $\lambda/4$ transmission line. As we can see now, there is a cross-symmetry between the resonances of the open-ended and the resonances of the short-ended line. Half wave resonances of the one termination type look – with respect to the input impedance – like quarter resonances of the other type, except for a factor of 2 corresponding to the different lengths.
Hence, the input impedance of the short-ended $\lambda/2$ line has also the same form as a series RLC circuit around resonance

$$Z_{\text{in}} \approx R + 2iL\Delta\omega \tag{1.288}$$

For the equivalent parameters, we get

$$R = Z_0 \alpha l, \qquad L = \frac{Z_0 \pi}{2\omega_0}, \qquad C = \frac{2}{\pi \omega_0 Z_0} \tag{1.289}$$

and

$$Q = \frac{\omega_0 L}{R} = \frac{\pi}{2\alpha l}, \qquad \kappa = \frac{2\omega_0 \alpha l}{\pi} \tag{1.290}$$

We demonstrate here exemplarily, how we can also use Pozar's complex frequency trick for transmission line resonators. First we Taylor approximate the input impedance of the lossless case $\alpha = 0$, which is given by

$$Z_{\text{in}} \approx iZ_0\pi\frac{\Delta\omega}{\omega_0}. \tag{1.291}$$

Then, in $\Delta\omega = \omega - \omega_0$ (and **only** in $\Delta\omega$) we make the substitution

$$\omega_0 \to \omega_0\left(1 + i\frac{\alpha}{\beta_0}\right) = \omega_0 + i\frac{\kappa}{2} \tag{1.292}$$

and get

$$Z_{\text{in}} \approx Z_0\alpha l + iZ_0\pi\frac{\Delta\omega}{\omega_0}. \tag{1.293}$$

1.6 Response functions of single-port coupled transmission line resonators

1.6.1 Construction of response functions

For many transmission line resonators, we have now all ingredients to construct the response functions. The approach described here works for resonators, which have one port and are terminated at the other end by an open or a short. Another requirement is that the coupler impedance $Z_c \gg Z_0$ or the shunt impedance $Z_s \ll Z_0$, respectively. It is not perfectly suitable for resonators which have a finite termination at the end, e.g. a second port. These kinds of resonators we will discuss in more detail below.

When we have a port-coupled transmission line resonator, we first have to find the lumped element equivalents for the bare transmission line with length l, damping α and characteristic impedance Z_0 by applying the equivalent expressions derived above for the particular type of resonator.

Open-ended C_c coupled resonator

The open-ended C_c coupled resonator with $1 \ll Z_0\omega_0 C_c$ behaves like a half wave resonator with open ends at both sides. Hence, its equivalent lumped elements are given by

$$R = \frac{Z_0}{\alpha l}, \qquad C = \frac{\pi}{2Z_0\omega_{0\mathrm{uc}}}, \qquad L = \frac{2Z_0}{\pi\omega_{0\mathrm{uc}}} \tag{1.294}$$

and the uncoupled resonance frequency by

$$\omega_{0\mathrm{uc}} = \pi\frac{v_{\mathrm{ph}}}{l} = \frac{\pi}{l}\frac{c_0}{\sqrt{\epsilon_{\mathrm{eff}}}}. \tag{1.295}$$

The last relation with the vacuum speed of light c_0 holds for a transmission line with an effective dielectric constant of ϵ_{eff} and no magnetic material included $\mu = 1$.

When the phase velocity of the transmission line is known, all relevant response functions of the transmission line resonator can be calculated in good approximation by means of the lumped element equivalent. We demonstrate the equivalency by calculating the exact reflection parameter of a transmission line resonator on the one hand (with homemade LabView script) and compare it to the QUCS result of the lumped element equivalent on the other hand. First, we choose the transmission line parameters to be $l = 10\,\mathrm{mm}$, $\alpha = 0.001\,\mathrm{m}^{-1}$ and $\epsilon_{\mathrm{eff}} = 5.5$ (approximately the number for a coplanar waveguide on a sapphire wafer). From that, we calculate ω_0, R, L and C and put these numbers to a lumped element QUCS simulation. For both calculations, we use three different coupling capacitors C_c. The result of both is plotted in direct comparison in Fig. 1.26 and the agreement is quite good.

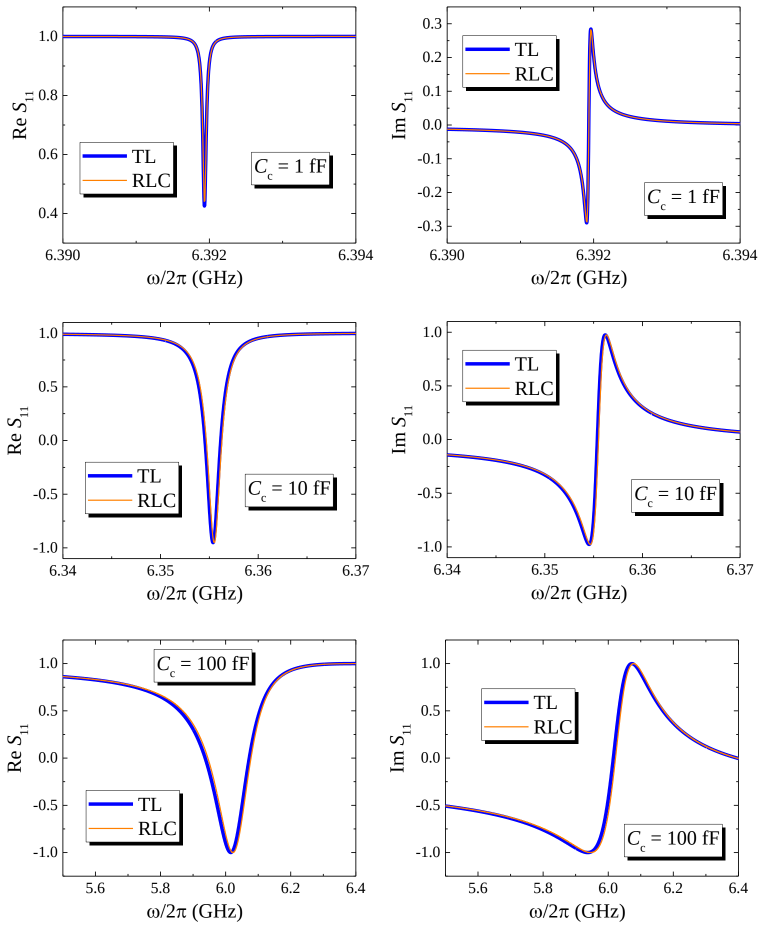

Figure 1.26: Real and imaginary part of the reflection parameter of a capacitively gap coupled transmission line resonator (blue line) in direct comparison with the corresponding RLC equivalent (orange line) for different coupling capacitors C_c. The single-port transmission line resonator is terminated by an open and the line parameters are $l = 10\,mm$, $\alpha = 0.001\,m^{-1}$, $Z_0 = 50\,\Omega$ and $\epsilon_{\text{eff}} = 5.5$ and its reflection was calculated with a homemade LabView program. The lumped element equivalent reflection was calculated by means of QUCS after determining the lumped elements by the relations in the main text.

For completeness and as below we demonstrate a second possibility to derive this result, we also give the expression for the input impedance of the transmission line resonator. Its calculation is based on the corresponding lumped element expression and the equivalent lumped element conversion relations and yields

$$Z_{\text{in}} = \frac{L(C+C_c)^2}{C_c^2}\left(\frac{1}{R(C+C_c)} + 2i\Delta\omega\right) \tag{1.296}$$

$$= \frac{1}{Z_0^2\omega_0^2 C_c^2} \cdot Z_0 \left(\alpha l + i\pi \frac{\omega_{0\mathrm{uc}}}{\omega_0}\frac{\Delta\omega}{\omega_0}\right) \tag{1.297}$$

$$= -\frac{Z_c^2}{Z_0^2} \cdot Z_0 \left(\alpha l + i\pi \frac{\omega_{0\mathrm{uc}}}{\omega_0}\frac{\Delta\omega}{\omega_0}\right) \tag{1.298}$$

This is the same result as Pozar derives by a different approach except for the factor $\omega_{0\mathrm{uc}}/\omega_0$ in the imaginary part.[8]

Short-ended C_s coupled resonator

We can do the same as before for the series coupled resonator now for the capacitively shunt coupled resonator with $Z_0\omega_0 C_s \ll 1$ and a short end at the second side. As this configuration corresponds to the short-ended $\lambda/2$ line, we find the equivalent lumped elements by

$$R = Z_0\alpha l, \qquad C = \frac{2}{\pi Z_0 \omega_{0\mathrm{uc}}}, \qquad L = \frac{Z_0\pi}{2\omega_{0\mathrm{uc}}}. \tag{1.299}$$

The uncoupled resonance frequency is the same as for the open-ended case above, because it only depends on material and length. A comparison between the results of a calculation of the reflection performed for the full transmission line and the series RLC equivalent is shown in Fig. 1.27 and shows the same degree of agreement as the open-ended case before. With the input impedance from the lumped element section, the input impedance of the transmission line resonator is given by

$$Z_{\mathrm{in}} = \frac{C}{C_s(C+C_s)} \cdot \frac{1}{\kappa_{\mathrm{int}} + 2i\Delta\omega} \tag{1.300}$$

$$= \frac{1}{Z_0^2\omega_0^2 C_s^2} \cdot \frac{Z_0}{\alpha l + i\pi \frac{\Delta\omega}{\omega_{0\mathrm{uc}}}} \tag{1.301}$$

$$= -\frac{Z_s^2}{Z_0^2} \cdot \frac{Z_0}{\alpha l + i\pi \frac{\Delta\omega}{\omega_{0\mathrm{uc}}}}. \tag{1.302}$$

We do not derive and compare all possible resonator couplings and termination here, but this approach presumably works very fine for all one-port resonators discussed so far. Cases where more care has to be taken (deviations from open and short terminations) will be briefly discussed below.

[8] I think this result here is more precise, as Pozar assumes $\omega_{0\mathrm{uc}} \approx \omega_0$ hidden in the assumption $l\omega_0 = \pi v_{\mathrm{ph}}$.

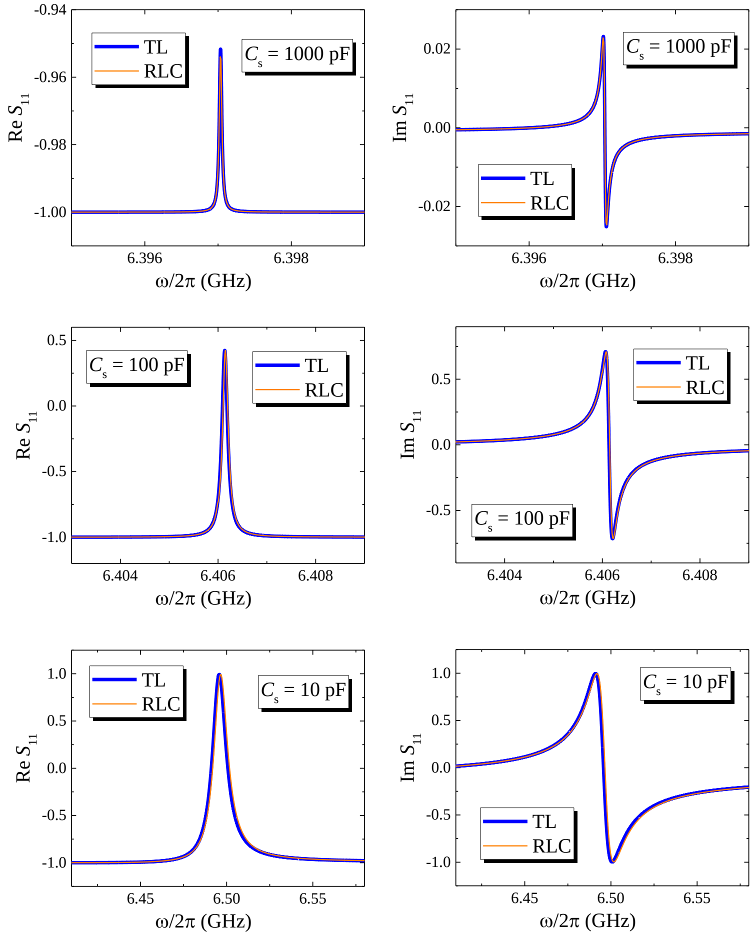

Figure 1.27: Real and imaginary part of the reflection parameter of a capacitively shunt coupled transmission line resonator (blue line) in direct comparison with the corresponding RLC equivalent (orange line) for different shunt capacitors C_s. The single-port transmission line resonator is terminated by a short and the line parameters are $l = 10\,mm$, $\alpha = 0.001\,m^{-1}$, $Z_0 = 50\,\Omega$ and $\epsilon_{\text{eff}} = 5.5$ and its reflection was calculated with a homemade LabView program. The lumped element equivalent reflection was calculated by means of QUCS after determining the lumped elements by the relations in the main text.

Different characteristic impedances

Another very nice application of this approach is, that it is very easy to consider cases where the characteristic impedance of the feedline Z_0 is different from the characteristic impedance of the resonator Z_1. We do this here exemplarily for the inductively shunt coupled $\lambda/2$ resonator. First, we calculate the equivalent lumped elements for this short ended $\lambda/2$ resonator

$$R = Z_1 \alpha l, \qquad C = \frac{2}{\pi Z_1 \omega_{0\mathrm{uc}}}, \qquad L = \frac{Z_1 \pi}{2\omega_{0\mathrm{uc}}}. \tag{1.303}$$

Then, we use these lumped elements and calculate the response functions with the formulas from the corresponding lumped element section. Any Z_0 appearing there, e.g., in the external κ, remains a Z_0 because in the lumped element case any appearance of Z_0 comes from the feedline(s). Figure 1.28 shows the reflection for an inductively coupled transmission line resonator calculated with the LabView software in comparison with the reflection of the corresponding RLC circuit calculated with QUCS. In the calculations, the feedline impedance was first set to two different values $Z_0 = 5\,\Omega$ and $Z_0 = 500\,\Omega$ while the resonator impedance was held constant at $Z_1 = 50\,\Omega$. In the second part, the feedline impedance was held constant at $Z_0 = 50\,\Omega$ and the resonator characteristic impedance was set to $Z_1 = 200\,\Omega$ and $1000\,\Omega$.[9] As third line, you can see the result of the approximate lumped element formula, but with ω_0, κ_{ext} and κ_{int} calculated by hand with the corresponding equivalent expressions. For large or small couplings, ther appear again the well-known deviations between the approximate expression and the exact simulation results.
The input impedance of the resonator considered here is given by

$$Z_{\mathrm{in}} = \frac{L_s^2}{C(L+L_s)^2} \frac{1}{\kappa_{\mathrm{int}} + 2i\Delta\omega} \tag{1.304}$$

$$= \frac{\omega_0^2 L_s^2}{Z_0^2} \frac{Z_0}{\alpha l + i\pi \frac{\omega_{0\mathrm{uc}}}{\omega_0} \frac{\Delta\omega}{\omega_0}}. \tag{1.305}$$

So again, there appears a factor of $\omega_{0\mathrm{uc}}\omega_0$ in the imaginary part of the denominator, which we would not get by Pozar's approximation.

[9]We do not show the "boring" case of both $50\,\Omega$ here.

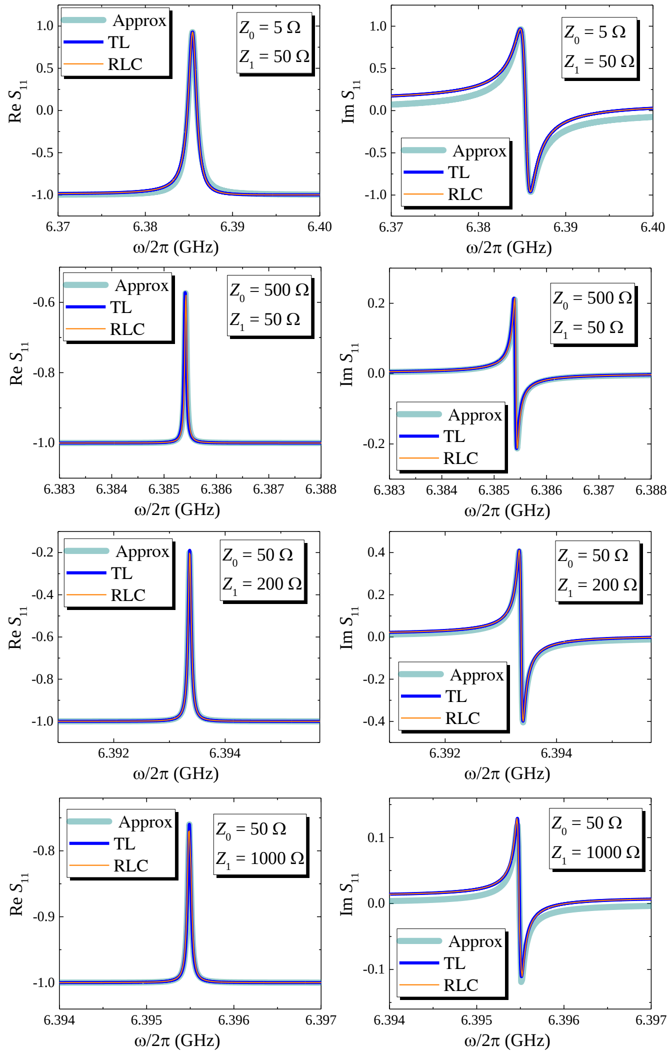

Figure 1.28: Real and imaginary part of the reflection parameter of an inductively shunt coupled transmission line resonator (blue line) in direct comparison with the corresponding RLC equivalent (orange line) for different feedline impedances Z_0 and characteristic resonator impedances Z_1. In addition, the fat bright cyan line shows the result of the approximated expression given in this document after calculating the equivalent lumped elements and the resonance frequency from the transmission line paramaters by hand. The single-port transmission line resonator is terminated by a short and the line parameters are $l = 10\,mm$, $\alpha = 0.001\,m^{-1}$, and $\epsilon_{\mathrm{eff}} = 5.5$. Its reflection was calculated with a homemade LabView program. The lumped element equivalent reflection was calculated by means of QUCS after determining the lumped elements by the relations in the main text. The coupling inductor for all curves is $L_s = 6.5\,pH$.

1.7 The zoo of impedances

Definitions, explanations, examples

There are several different kinds of impedances in electromagnetic and electric circuit theory, which should not be confused with each other, as they in general are different quantities. Sometimes they are related to each other or even equal, but that is not the case in general and so much care has to be taken, when talking about impedance matching issues etc. Here, a definition and explanation of the different relevant impedances in microwave theory:

- **Wave impedance** Z_w:

  The wave impedance of an electromagnetic wave is the ratio of the transverse components of the electric and magnetic fields (see e.g. Pozar)

  $$Z_w = \frac{E_0^-}{H_0^-}. \tag{1.306}$$

  It is hence defined for single points in space. For a TEM plane wave travelling through a homogeneous medium, the wave impedance is everywhere equal to the intrinsic impedance (see next item) of the medium. The wave impedance is a characteristic of the particular type of wave. In a waveguide, TEM, TM, and TE waves have each different wave impedances, which may depend on the type of line or guide, the material and the operating frequency.

- **Intrinsic impedance** η:

  The intrinsic impedance is a characteristic of a medium, which is dependent only on the material properties of the medium (and not on the type of wave). In a conducting material with conductivity σ, permittivity ϵ and permeability μ it is given by

  $$\eta = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}}. \tag{1.307}$$

  Note, that in general conductivity σ, permittivity ϵ and permeability μ can be complex tensors[10]. For a homogeneous nonconducting medium, the intrinsic impedance reduces to

  $$\eta = \sqrt{\frac{\mu}{\epsilon}}. \tag{1.308}$$

  In particular, we get for free space the famous vacuum impedance value

  $$\eta_0 = Z_{w0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega. \tag{1.309}$$

[10] Does then also the intrinsic impedance get a complex tensor? Guess so.

For TEM plane waves always $\eta = Z_w$[11].

- **Characteristic impedance** Z_0:

  The characteristic impedance is defined as the ratio of voltage to current amplitude for a travelling wave on a transmission line

$$Z_0 = \frac{V_0}{I_0} = \sqrt{\frac{R' + i\omega L'}{G' + i\omega C'}} \tag{1.310}$$

  For lossless transmission lines $R' = G' = 0$, this reduces to the well-known

$$Z_0 = \sqrt{\frac{L'}{C'}}. \tag{1.311}$$

  The characteristic impedance is, however, a fundamentally different quantity than wave impedance or intrinsic impedance. It needs always two conductors to get a transmission line and the characteristic impedance depends on the geometry of these conductors, whilst wave and intrinsic impedance do neither need conductors nor depend on their geometry. Another way to say that is that voltage and current (characteristic impedance) are connected to the electric and magnetic fields (wave impedance) via the conductor geometry.

  Example: A (lossless) coaxial cable has the wave impedance/intrinsic impedance

$$Z_w = \frac{E_\rho}{H_\phi} = \sqrt{\frac{\mu}{\epsilon}} = \eta \tag{1.312}$$

  with μ and ϵ of the filling material between the conductors. Without material filling, the wave impedance of the coaxial cable would be just the vacuum wave impedance of $\sim 377\,\Omega$, independent of geometry. The characteristic impedance, however, is given by

$$Z_0 = \frac{V_0}{I_0} = \sqrt{\frac{\mu}{\epsilon}} \frac{\ln(b/a)}{2\pi} = \frac{Z_w}{2\pi} \ln\left(\frac{b}{a}\right) \tag{1.313}$$

  with the radius of the center conductor a and the radius of the outer conductor b, cf. Pozar p. 56. Hence, the characteristic impedance can take any value $0 < Z_0 < \infty$ for constant and fixed Z_w.[12]

[11] And otherwise? Is there a general relation between η and Z_w?

[12] Here arises an interesting question. According to transmission line theory, a partial reflection of a travelling wave occurs at a step of the characteristic impedance in a coaxial cable (or any other transmission line). Although the wave impedance doesn't change. What happens at a point, where the wave impedance changes but not the characteristic impedance? Exactly that happens at a connection point of two coaxial cables with different dielectrics. And nothing seems to be reflected there. Is this ok with full electromagnetism theory? So is it possible to bring an electromagnetic wave from one medium to another without reflections by using transmission lines? Why does the characteristic impedance discontinuity seem to win over the wave impedance discontinuity? Is this effect somehow related to metamaterials, where tiny circuits dominate the wave properties?

Because voltage and current are uniquely defined for TEM waves, the characteristic impedance of a TEM wave is unique. TE and TM waves, however, do not have a uniquely defined voltage and current, so the characteristic impedance for such waves may be defined in different ways. This is important for waveguides which can support all three kind of modes, as e.g. parallel plate waveguides (also coaxial cables or coplanar waveguides at high frequencies).[13]

- **Electrical impedance** Z:

  The electrical impedance is probably the most familiar impedance from circuit theory. It relates voltage and current including their phase difference via $Z = V/I$ in a point-like (lumped) circuit element or a combination of lumped circuit elements. It usually depends on frequency and the type of circuit element. The three most familiar linear circuit elements resistor, cpacitor and inductor have the electrical impedances

$$Z_{\mathrm{R}} = R, \qquad Z_{\mathrm{C}} = \frac{1}{i\omega C}, \qquad Z_{\mathrm{L}} = i\omega L. \tag{1.314}$$

  In networks of circuit elements, their impedances can be combined according to Kirchhoff's laws. For example, the electrical impedance of a series RLC circuit is given by

$$Z = R + i\omega L + \frac{1}{i\omega C} \tag{1.315}$$

  and the corresponding impedance of a parallel RLC circuit is

$$Z = \left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C\right)^{-1}. \tag{1.316}$$

  In any case, the electrical impedance is the proportionality factor between voltage and current in a lumped element circuit at any moment in time.

- **Resonance impedance** Z_r:

  Another quantity of a lumped element circuit, which has the unit of impedance and looks similar to the characteristic impedance of a transmission line is

$$Z_r = \sqrt{\frac{L}{C}}. \tag{1.317}$$

  However, it is not to be confused with the characteristic impedance, because the physical meaning is different. While Z_0 of a transmission line gives the ratio between

[13] The fact that the wave impedance as well as the intrinsic impedance seem to be uniquely defined for the different kinds of waves suggests that there can most probable not be a general relation between wave impedance/intrinsic impedance and characteristic impedance. But maybe there is one for TEM waves?

the amplitudes of current and voltage of a travelling voltage and current wave, the quantity Z_r has only a physical meaning at the resonance frequency of an LC circuit[14] and is not related to something travelling. As there seems not to be a particular term for Z_r in literature, we just call it resonance impedance here.

The resonance impedance expresses the ratio between voltage and current amplitudes (or average values) at the resonance frequency of an RLC resonant circuit, which can be seen as follows. On resonance, the stored energy in a resonant circuit oscillates back and forth between electric and magnetic field, where the average electric energy is given by $E_{\mathrm{el}}^{\mathrm{av}} = \frac{1}{4}CV^2$ and the average magnetic energy is given by $E_{\mathrm{mag}}^{\mathrm{av}} = \frac{1}{4}LI^2$. Hence, the ratio between voltage and current in resonance is given by

$$\frac{V}{I} = \sqrt{\frac{E_{\mathrm{el}}^{\mathrm{av}} L}{E_{\mathrm{mag}}^{\mathrm{av}} C}} \tag{1.318}$$

which on resonance, i.e., for $E_{\mathrm{el}}^{\mathrm{av}} = E_{\mathrm{mag}}^{\mathrm{av}}$, reduces to

$$\frac{V}{I} = \sqrt{\frac{L}{C}} = Z_r. \tag{1.319}$$

Alternatively, one could use the maximum electric and magnetic energies $E_{\mathrm{el}}^{\mathrm{max}} = \frac{1}{2}CV^2$ and $E_{\mathrm{mag}}^{\mathrm{max}} = \frac{1}{2}LI^2$. Note, that the resonance impedance does not consider a phase delay between voltage and current, it relates only amplitudes and only at a single frequency.

In a certain sense there might be a relation to the characteristic impedance of a transmission line, because the transmission line can be seen as a combination of an infinite number of infinitesimally small LC circuits. Nevertheless, they should better not be mixed up, in particular not for the case of transmission line resonators as discussed below in more detail.

- **Input impedance** Z_{in}:

  Another very useful concept is the input impedance. The input impedance of a given microwave circuit made of lumped circuit elements and/or transmission lines is defined as the ratio of voltage and current at a particular point in space, the input point. For pure lumped element circuits, the input impedance is equal to the electrical impedance

$$Z_{\mathrm{in}} = Z \tag{1.320}$$

  and for infinite transmission lines it is equal to the characteristic impedance

$$Z_{\mathrm{in}} = Z_0. \tag{1.321}$$

  But for transmission line circuits, the input impedance is in general very different from the characteristic impedance. When a transmission line is terminated by a load

[14] At least I do not see any other meaning, someone else maybe?

impedance $Z_\mathrm{L} \neq Z_0$ or when an impedance step occurs on the line, a part of a travelling voltage wave on the line is reflected at that point. Then, on the line, the incoming and reflected waves superimpose. The input impedance at a given point on the line, i.e., at a given distance from the reflection point, is then defined as the local ratio of voltage to current V/I (combination of incoming and reflected wave). It is in some sense the equivalent to the electrical impedance of a lumped element circuit, i.e., it gives the ratio including the relative phase between voltage and current at that point.

The input impedance is very useful, because for many purposes it allows to consider the transmission line including the reflection etc just as a single lumped element impedance Z_in. For a transmission line with characteristic impedance Z_0 and propagation constant γ, which is terminated with a load impedance Z_L, the input impedance at a distance $-l$ from the load impedance is given by

$$Z_\mathrm{in} = \frac{V(-l)}{I(-l)} = Z_0 \frac{Z_\mathrm{L} + Z_0 \tanh \gamma l}{Z_0 + Z_\mathrm{L} \tanh \gamma l}. \tag{1.322}$$

For the derivation, see e.g. Pozar or above.

- **Equivalent impedance Z_e:**

  The equivalent impedance is the last kind of impedance discussed here. It is a concept closely related to the input impedance and is mainly used for transmission line resonators, but can in principle also be applied to 3D microwave cavities. At the input port of a coplanar resonator, the input impedance close to the resonance frequency is almost identical to the input impedance of an RLC circuit around its resonance. To illustrate this, we give an example (see also main text on transmission line resonators, where many more examples are discussed). The input impedance of a series RLC circuit around resonance can be approximated by

$$Z_\mathrm{in} \approx R + 2iL\Delta\omega. \tag{1.323}$$

On the other hand, the input impedance of a short-ended transmission line close to its first half wavelength resonance is given by

$$Z_\mathrm{in} \approx Z_0 \left(\alpha l + i\pi \frac{\Delta\omega}{\omega_0} \right) = Z_0 \alpha l + i\pi Z_0 \frac{\Delta\omega}{\omega_0}. \tag{1.324}$$

with α the damping constant of the transmission line. This means, that around resonance the short-ended half wavelength transmission line resonator looks at its input port like a series lumped element resonator made from the equivalent lumped elements

$$R_\mathrm{e} = Z_0 \alpha l, \qquad L_\mathrm{e} = \frac{\pi Z_0}{2\omega_0}, \qquad C_\mathrm{e} = \frac{2}{\pi Z_0 \omega_0}. \tag{1.325}$$

The equivalent capacitance was calculated by $C_e = 1/L_e \omega_0^2$.

With $Z_0 = \sqrt{L'/C'}$, the phase velocity of the transmission line $v_{\text{ph}} = 1/\sqrt{L'C'}$, the resonance condition $l = \lambda_0/2$ and the relation $\omega_0 = v_{\text{ph}}\pi/l$ we can also relate the equivalent circuit elements L_e and C_e to the transmission line parameters via

$$L_{\text{e}} = \frac{L'l}{2}, \qquad C_{\text{e}} = \frac{2C'l}{\pi^2}. \tag{1.326}$$

To model the transmission line resonator as a lumped element circuit around resonance is often useful to get a better physical understanding of some aspects as quality factors or resonance frequency shifts by coupling the resonator to feedlines.

It is important to note, however, that the equivalent lumped elements do not directly relate to physical properties of the transmission line. For example, if one is interested in the ratio of voltage and current on the transmission line in resonance, one cannot use the resonance impedance constructed by L_e and C_e. The ratio between the maximum voltage $2V_0$ and the maximum current $2I_0$ (at their corresponding antinodes) is given by Z_0 and not by

$$Z_{e,r} = \sqrt{\frac{L_e}{C_e}} = \frac{\pi}{2}\sqrt{\frac{L'}{C'}} = \frac{\pi}{2} Z_0 \neq Z_0. \tag{1.327}$$

Also, it is not possible to use L_e and C_e for the calculation of electric fields or voltages on the line, in particular not for the zero point fluctuation voltage or the coupling strength to a transmon qubit.

This non-equivalence becomes also clear, when considering higher modes of the transmission line resonator. Then, for the equivalent reactances and the equivalent impedance, we get with the mode number n

$$L_{e,n} = \frac{L'l}{2}, \qquad C_{e,n} = \frac{2C'l}{n^2\pi^2}, \qquad Z_{e,r,n} = \frac{n\pi}{2} Z_0 \tag{1.328}$$

while the characteristic impedance of the line is mode number independent. Note also, that equivalent lumped elements differ between different resonator types (depending on $\lambda/4$ vs $\lambda/2$ and on the termination, i.e., on open- vs short-ended resonators), even when the transmission line itself is exactly the same.

The example of a coaxial cable resonator

As final example let us consider a half wavelength transmission line resonator, e.g. a piece of lossless coaxial cable with length l (nearly) shorted at both ends and filled with air. Then, the wave impedance as well as the intrinsic impedance are given by

$$\eta = Z_w = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega \tag{1.329}$$

Let us further assume that we have designed the coaxial cable in a way that it has a characteristic impedance of

$$Z_0 = 50\,\Omega, \tag{1.330}$$

i.e. with $b \approx 2.3a$, which sounds reasonable. When we consider the mode $n = 20$, we get an equivalent resonance impedance of

$$Z_{e,r} = 1.57\,\mathrm{k}\Omega. \tag{1.331}$$

So the different impedances have very different values for the same resonator and they depend on different parameters.