

Exact Cover

Denis Festa

October 19, 2023



Introduction

The following presentation provides a self-contained report on the implementation of the exact cover problem provided by prof. Marina Zanella (University of Brescia, Italy). The language of choice is Python for its simplicity and readability. The code is available at <https://www.kaggle.com/code/denisfesta/exact-cover-problem>. The following points discuss:

- the choice of the data structures for the algorithm in its basic form;
- the exploration of the solutions to different problems;
- the comparison between loading portions of the file and loading the whole file;
- the comparison between the basic algorithm and the *plus* algorithm;
- the application of the algorithm to the sudoku problem.

To pursue the goal of implementing the algorithm, the matrices A and B need to be stored in an appropriate data structure. Since the matrices contain only 0s and 1s, the first idea might be to store boolean values instead of integers, however, using `True` and `False` gives no advantage. One popular library that provides boolean arrays and matrices is `numpy`, which I compare with the less popular `bitarray` library and show the results in figures (4) and (5). The elements of a `numpy` array or matrix occupy less memory than the elements of a `bitarray` array or matrix, however, the `numpy` variable storing the array occupies more memory than the `bitarray` variable storing the array. Given the absence of matrix operations, I don't see any particular advantage in using `numpy` over `bitarray`.

Built-in array of integers

```
1 a = [1]*1000
2 print("Size of a, Size of a[0]")
3 sys.getsizeof(a), sys.getsizeof(a[0])
```

[13] ✓ 0.0s

... Size of a, Size of a[0]

... (8056, 28)

Built-in array of booleans

```
1 b = [True]*1000
2 print("Size of b, Size of b[0]")
3 sys.getsizeof(b), sys.getsizeof(b[0])
```

[14] ✓ 0.0s

... Size of b, Size of b[0]

... (8056, 28)

Numpy array of bits

```
1 p = np.ones(10_000, dtype=bool)
2 print("Size of p, Size of p[0]")
3 sys.getsizeof(p), sys.getsizeof(p[0])
```

[15] ✓ 0.0s

... Size of p, Size of p[0]

... (10112, 25)

Bitarray

```
1 c = bitarray('1'*1000)
2 print("Size of c, Size of c[0]")
3 sys.getsizeof(c), sys.getsizeof(c[0])
```

[16] ✓ 0.0s

... Size of c, Size of c[0]

... (216, 28)

Built-in matrix of integers

```
[17] ✓ 0.0s
1 N = 30
2 M = 1000

1 a = [[1]*M for _ in range(N)]
2 print("Size of a, Size of a[0], Size of a[0][0]")
3 sys.getsizeof(a), sys.getsizeof(a[0]), sys.getsizeof(a[0][0])

[18] ✓ 0.0s
... Size of a, Size of a[0], Size of a[0][0]
... (312, 8056, 28)
```

Built-in matrix of booleans

```
[18] ✓ 0.0s
1 b = [[True]*M for _ in range(N)]
2 print("Size of b, Size of b[0], Size of b[0][0]")
3 sys.getsizeof(b), sys.getsizeof(b[0]), sys.getsizeof(b[0][0])

... Size of b, Size of b[0], Size of b[0][0]
... (312, 8056, 28)
```

Numpy matrix of bits

```
[20] ✓ 0.0s
1 p = np.ones((N, M), dtype=bool)
2 print("Size of p, Size of p[0], Size of p[0][0]")
3 sys.getsizeof(p), sys.getsizeof(p[0]), sys.getsizeof(p[0][0])

... Size of p, Size of p[0], Size of p[0][0]
... (30128, 112, 25)
```

Bitarray matrix

```
[21] ✓ 0.0s
1 c = [bitarray('1'*M) for _ in range(N)]
2 print("Size of c, Size of c[0], Size of c[0][0]")
3 sys.getsizeof(c), sys.getsizeof(c[0]), sys.getsizeof(c[0][0])

... Size of c, Size of c[0], Size of c[0][0]
... (312, 216, 28)
```

I tried to profile the code with `memory-profiler` to find whether the memory occupation advantages we expect to gain from choosing one data structure over the other are actually realized, but I couldn't see any significant difference in the memory usage while executing the code, I guess it's because the memory required to store the matrices is negligible compared to the memory required to store the set of solutions, the set of explored nodes and the stack of the recursive calls.

Exploring the solutions

Different choices of the cardinality of the domain (columns of the matrix A) and the number of sets (rows of the matrix A, rows and columns of the matrix B) lead to different values of:

- the number of solutions found;
- the number of explored nodes to find the solutions;
- the time required to find the solutions.

In the following slides I show the values of these quantities for different automatically generated instances of the problem, for time constraints I kept the number of sets in a range between 20 and ~ 400 and the cardinality of the domain in a range between 5 and 14.

It's intuitive that the number of solutions found increases with the number of sets and decrease with the cardinality of the domain. In this case the intuition is confirmed by the solutions found (1).



Figure: The number of solutions found increases with the number of sets and decreases with the cardinality of the domain.

Explored nodes

It might seem intuitive that, similarly to the previous case, the number of explored nodes increases with the number of sets and decreases with the cardinality of the domain, however, the same problems that were solved in the previous case display a different and less intuitive behaviour in this case (2). The interpretation of this behaviour is that the algorithm has to work harder, that is to explore more nodes, to find the solutions for more complex problems, that is those problems with a higher number of rows and a higher number of columns. Notice that the graphs in figure (2) follow the same pattern as the graphs in figure (4), which shows the time required to find all the solutions for the same problems. Why, fixing the number of rows, a small number of columns implies a higher number of explored nodes?

This behaviour is probably due to the fact that the rows of the matrix A are generated by sampling from a uniform distribution the values $\{0, 1\}$ (the slide *Generating Exact Cover problems* explains how the exact cover problems used here were generated), when the number of columns is small each row is more likely to be compatible with other rows, so it has to be explored more deeply, whereas when the number of columns is high, given that each row contains around half 1s and half 0s, it is more likely that a row is incompatible with other rows, so less rows have to be explored. Later in the presentation, in the section *More or less 1s*, an alternative is presented, which makes use of different distributions to generate the rows of the matrix A .



Figure: The number of explored nodes increases with the number of sets and increases when the cardinality of the domain is too small or too large. The only interpretation found is that the number of explored nodes increases with the complexity of the problem.

Explorable nodes

The implemented algorithm chooses to prune many subtrees that will not lead to a plausible solution. Professor Marina Zanella showed that if the algorithm was blind and didn't prune any subtree, then the number of nodes to be explored would be:

$$\sum_{i=1}^N \sum_{j=1}^i \binom{i}{j} = 2^N - 1.$$

In figure (3) the enormous difference between the number of explored nodes and the number of explorable nodes is shown. The number of explorable nodes is not always exactly $2^N - 1$ because the actual computation to find the explorable nodes takes into account only rows of A different from an all-zero row (which would be, by definition, compatible with any other row).

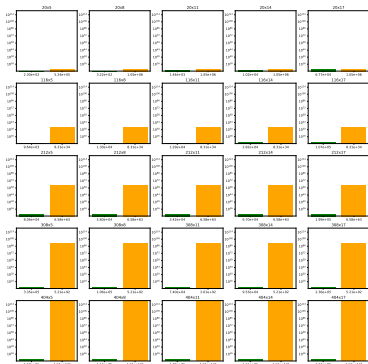
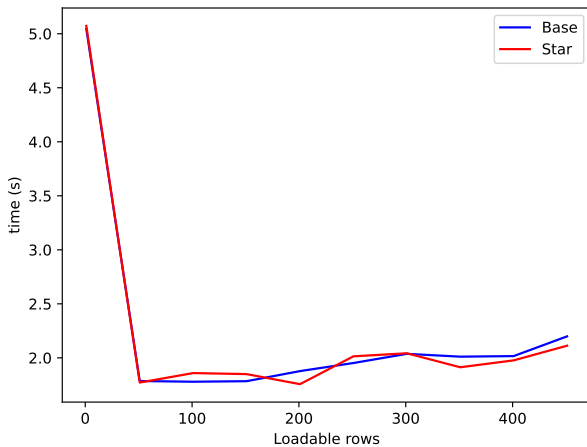
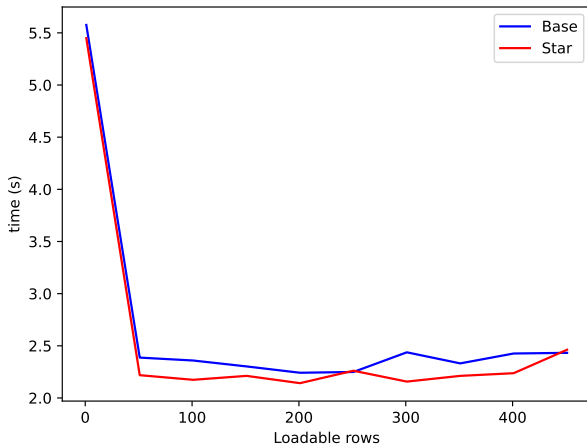


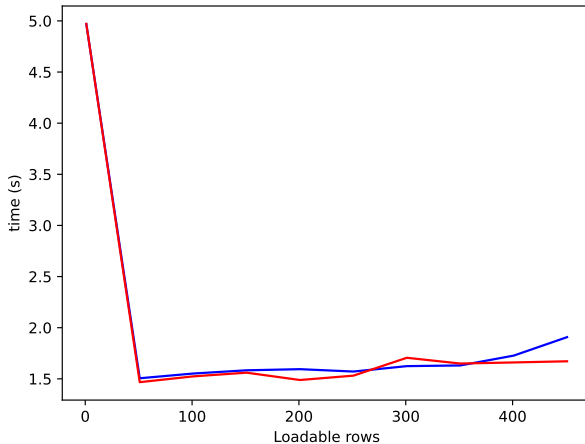
Figure: The number of nodes that would be explored by a blind algorithm is much greater than the number of nodes that are actually explored by the implemented algorithm.

Loading portions

The problem to solve might require a huge matrix A , to prevent the computer from running out of memory one of the possible solutions is to load portions of the matrix and solve the problem for each portion, growing incrementally the set of solutions. What is expected is that loading the whole matrix and solving at once is faster than loading portions of the matrix. The results shown in figure (16), (17) and (18) show something different. What can be noticed is that the greatest advantage that comes from decreasing the dimension of the chunk of loaded rows is gained when the dimension is small, then there is no clear advantage, sometimes loading a greater number of rows at once is faster and sometimes it's slower. To make the order in which the problems were solved as influential as possible on the results, the problems were first solved with an increasing number of loadable rows (figure (16)), then with a decreasing number of loadable rows (figure (17)) and finally with a random number of loadable rows (figure (18)).







Base and *plus* algorithms

The results show that the *plus* algorithm, most of the times, is slightly faster than the *base* algorithm, but it remains unclear why sometimes this is not the case. The same unpredictable behaviour is shown in figure (4).

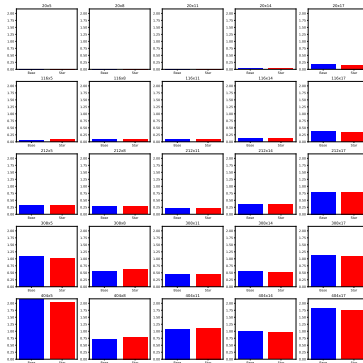


Figure: This figure shows the time required to find all the solutions for the same problems whose results were shown before, in particular it follows the same pattern in figure (2). The *plus* algorithm is slightly faster than the *base* algorithm, but it remains unclear why sometimes this is not the case.

A series of manual tests on different dimensions of the chunk of loadable rows on a larger problem (1000 rows and 15 columns) brought to the following results:

- when loading 1 row at a time the algorithm takes 24.5 seconds;
- when loading 2 rows it takes 15.3 seconds;
- when loading 3 rows it takes 12.3 seconds;
- when loading 5 rows it takes 9.7 seconds;
- when loading 10 rows it takes 7.9 seconds;
- when loading 20 rows it takes 6.9 seconds;
- when loading 50 rows it takes 6.5 seconds;
- when loading 100 rows it takes 6.5 seconds;
- when loading 400 rows it takes 6.6 seconds;
- when loading 900 rows it takes 7.1 seconds.
- when loading 1000 rows it takes 7.6 seconds

More or less 1s

For a better comprehension of how the number of explored nodes depends on the structure of the problem, one possibility is to generate random exact cover problems varying the sparsity of the matrix A . The intuition is that the more 1s there are in each row of the matrix A , the more difficult it is for the problem to have different rows that are compatible, hence, the more difficult it is for the algorithm to find the solutions and the more nodes it has to explore. The more 0s there are in each row of the matrix A , the more likely it is for the problem to find solutions, each partition will contain more rows since each row contains fewer 1s, hence, the algorithm will have less difficulties in finding the solutions and will explore less nodes. The figures (5), (6) and (7) show the results of the experiments and, the larger the number of columns, the more our intuition is confirmed.



Figure: The number of explored nodes increases with the number of rows and decreases with the number of 1s in each row, this is because the more 1s there are, the less likely it is for the problem to have compatible rows, hence, the less sets are to be explored, the more 0s there are, the more likely it is for the problem to have compatible rows, hence, the more sets are to be explored.



Figure: Once the problem has enough columns, the number of solutions found increases with the number of 0s in each row, the reason is the same as before (figure (5)).

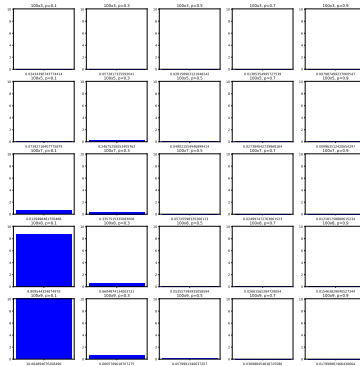


Figure: Again, the time required to find the solutions follows the same pattern as the number of explored nodes, shown in figure (5).

In the following slides the mechanism wrapping the EC algorithm is presented. The idea is to load a portion of the matrix A , execute the EC algorithm and store the solutions found in a set, then load the next portion of the matrix A , execute the EC algorithm and grow the set of solutions. The important thing to keep in mind is that when the algorithm is executing on a portion of the matrix A , say from row i to row j , the rows from 0 to $i - 1$ are not immediately available, they will be read from within the EC algorithm, again in portions. This explains the need for two variables: **offset** and **reading offset** (the same colors will be used in the schema and the animation to follow.)

- **offset** is the index of the first row of the portion of the matrix A that has been passed to the EC algorithm from outside
- **reading offset** is the index of the first row of the portion of the matrix A that is being read by the EC algorithm, reading offset will always start from 0 and will be incremented until the rows are read from 0 to the last row of the portion of the matrix A that has been passed to the EC algorithm from outside.

```

function INCREMENTALEXACTCOVER(filename)
  rows  $\leftarrow$  GETROWS(filename)
  columns  $\leftarrow$  GETCOLUMNS(filename)
  B  $\leftarrow$  ZEROS(rows, columns)
  COV  $\leftarrow$  SET( $\emptyset$ )
  offset  $\leftarrow$  0
  while true do
    A  $\leftarrow$  GETPORTIONOFA(filename, offset, loadableRows)
    if A is empty then
      break
    end if
    EC(A, B, COV, offset, filename, loadableRows)
  end while
  return COV
end function

```

function EC(*A*, *B*, *COV*, *offset*, *filename*, *loadableRows*)

$N \leftarrow \text{ROWS}(A)$

$M \leftarrow \text{COLUMNS}(A)$

for $i \leftarrow 0$ to $N - 1$ **do**

$row \leftarrow A[i]$

if $\text{SUM}(row) = 0$ **then**

for $t \leftarrow 0$ to $N - 1$ **do**

$B[t][i + offset] \leftarrow 0$

$B[i + offset][t] \leftarrow 0$

end for

continue

end if

if $\text{SUM}(row) = M$ **then**

for $t \leftarrow 0$ to $N - 1$ **do**

$B[t][i + offset] \leftarrow 0$

$B[i + offset][t] \leftarrow 0$

end for

$COV \leftarrow COV \cup \{(offset + i,)\}$

continue

end if

$readingOffset \leftarrow 0$

WHILECYCLE(*A*, *B*, *COV*, *offset*, *filename*, *loadableRows*, *i*, *readingOffset*)

end for

end function

▷ Number of rows of *A*

▷ Number of columns of *A*

▷ If the row is all 0s

▷ set the relative row and column of *B* to 0

▷ If the row is all 1s

▷ set the relative row and column of *B* to 0

```

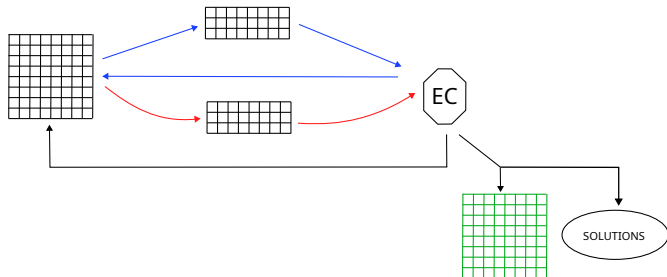
function WHILECYCLE(A, B, COV, offset, filename, loadableRows, i, readingOffset)
  while readingOffset < i + offset do
    oldA ← GETPORTIONOFA(filename, readingOffset, loadableRows)
    numRows ← ROWS(oldA)
    endVal ← MIN(numRows, i + offset − readingOffset) − 1
    for j ← 0 to endVal do
      oldRow ← oldA[j]
      if SUM(oldRow) ∈ {0, M} then
        continue
      end if
      if INTERSECT(oldRow, row) then
        B[j + readingOffset][i + offset] ← 0
      else
        I ← (offset + i, j + readingOffset)
        U ← UNION(oldRow, row)
        if SUM(U) = M then
          COV ← COV ∪ {I}
          B[j + readingOffset][i + offset] ← 0
        else
          B[j + readingOffset][i + offset] ← 1
          intersect ← []
          for k ← 0 to j + readingOffset − 1 do
            if B[k][i + offset] = 1 and B[k][j + readingOffset] = 1 then
              APPEND(intersect, k)
            end if
          end for
          if intersect ≠ ∅ then
            kA ← GETSPECIFICROWSFROMA(filename, intersect)
            EXPLORE(I, U, intersect, COV, kA, B, offset)
          end if
        end if
      end if
    end if
  end for
end while
end function

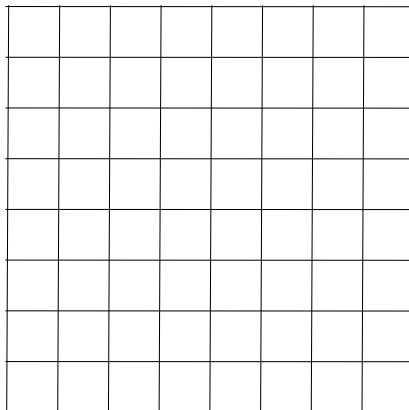
```

```

function EXPLORE( $I, U, intersect, COV, kA, B, offset$ )
  for  $k \in intersect$  do
     $ltmp \leftarrow I \cup \{(k)\}$ 
     $kRow \leftarrow kA[k]$ 
     $Utmp \leftarrow \text{UNION}(U, kRow)$ 
    if  $\text{SUM}(Utmp) == M$  then
       $COV \leftarrow COV \cup \{ltmp\}$ 
    else
       $intersectTmp \leftarrow \{I \mid I \in intersect \text{ and } I < k \text{ and } B[I][k]\}$ 
      if  $intersectTmp \neq \emptyset$  then
        EXPLORE( $ltmp, Utmp, intersectTmp, COV, kA, B, offset$ )
      end if
    end if
  end for
end function

```

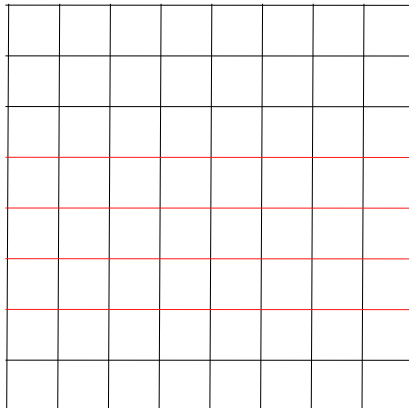




Offset	Reading offset
0	0

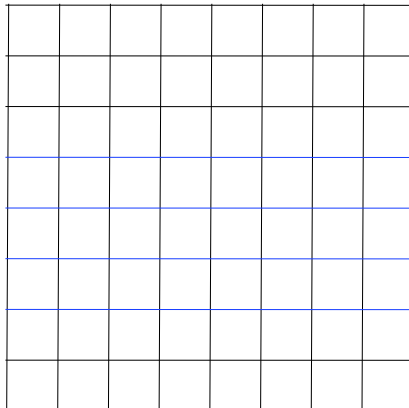
Offset	Reading offset
0	0

Offset	Reading offset
3	0



Offset	Reading offset
3	0

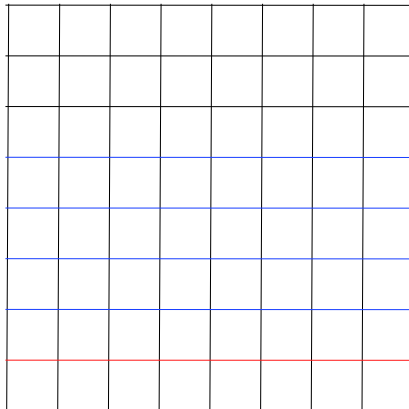
Offset	Reading offset
3	3



Offset	Reading offset
6	0

Offset	Reading offset
6	0

Offset	Reading offset
6	3



Offset	Reading offset
6	6

Any instance of the sudoku problem can be mapped to an instance of the exact cover problem. Independently of the size of the sudoku grid, which we call N (where N is the number of rows, the number of columns, the number of boxes and the number of possible elements to put in a cell), the exact cover problem will have 4 constraints:

- ① every cell contains exactly one of the N possible elements;
- ② every row has exactly one of the N possible elements in each of its N cells;
- ③ every column has exactly one of the N possible elements in each of its N cells;
- ④ every $N \times N$ box has exactly one of the N possible elements in each of its N cells.

¹The following content is taken from
<http://www.ams.org/publicoutreach/feature-column/fcarchive-kanoodle>

Each of these constraints maps to N^2 columns of the matrix A , that is to say, every set of the exact cover problem has $4 \cdot N^2$ elements. The number of sets of the exact cover problem is N^3 , where N^2 is the number of cells and N is the number of possible elements to put in a cell. This is how a 1 is mapped in a row of A , where

$$r, c, n \in \{0, \dots, N-1\}, i \in \{0, \dots, 4 \cdot N^2 - 1\},$$

r is the row,

c is the column,

n is the element,

i is the index of the column of A :

- ① a 1 in the i -th cell from the first N^2 cells of a row of A means that the r -th row and c -th cell of the sudoku grid has been filled with an element, without specifying which one, where $i = r \cdot N + c$;
- ② a 1 in the i -th cell from the second N^2 cells of a row of A means that the r -th row has been filled with the n -th element, where $i = N^2 + r \cdot N + n$;
- ③ a 1 in the i -th cell from the third N^2 cells of a row of A means that the c -th column has been filled with the n -th element, where $i = 2 \cdot N^2 + c \cdot N + n$;
- ④ a 1 in the i -th cell from the fourth N^2 cells of a row of A means that the r -th row has been filled with the n -th element, where $i = 3 \cdot N^2 + \left(\left\lfloor \frac{r}{\sqrt{N}} \right\rfloor \sqrt{N} + \left\lfloor \frac{c}{\sqrt{N}} \right\rfloor \right) \cdot N + n$.

In a sudoku problem there always are some numbers already written in the grid, the presence of the numbers make the problem easier to solve, this reflects on the exact cover problem being easier to solve, the way it becomes easier to solve is that the number of sets to explore decreases. One possibility is to simply remove the sets that correspond to the cells that already have a number written in them but, in order to be able to write an algorithm that maps the solution to the exact cover problem back to the solution to the sudoku, it's easier to keep the sets and make them all 0s rows, we can do this because our algorithm ignores the only 0s rows, so it's like pruning the tree.

```

function SUDOKU_TO_EXACT_COVER(sudoku[], N) ▷ sudoku is an array
of length  $N^2$  containing the numbers written in the sudoku grid
    nConstraints  $\leftarrow 4$ 
    coverMatrix  $\leftarrow \text{ZEROS}(N^3, N^2 \cdot nConstraints)$ 
    FILL(coverMatrix, N)
    rowsToRemove  $\leftarrow \text{SET}(\emptyset)$ 
    PRUNE_ROWS(rowsToRemove, sudoku, N)
    for idx  $\in$  rowsToRemove do
        coverMatrix[idx]  $\leftarrow \text{ZEROS}(N^2 \cdot nConstraints)$ 
    end for
    return coverMatrix
end function

```

```

function FILL(coverMatrix, N)
  for  $r \leftarrow 0$  to  $N - 1$  do
    for  $c \leftarrow 0$  to  $N - 1$  do
      for  $n \leftarrow 0$  to  $N - 1$  do
         $idx \leftarrow (r \cdot N + c) \cdot N + n$ 
         $coverMatrix[idx][r \cdot N + c] \leftarrow 1$ 
         $coverMatrix[idx][N^2 + r \cdot N + n] \leftarrow 1$ 
         $coverMatrix[idx][2 \cdot N^2 + c \cdot N + n] \leftarrow 1$ 
         $boxRow \leftarrow \lfloor r / \sqrt{N} \rfloor$ 
         $boxCol \leftarrow \lfloor c / \sqrt{N} \rfloor$ 
         $b \leftarrow boxRow \cdot \sqrt{N} + boxCol$ 
         $coverMatrix[idx][3 \cdot N^2 + b \cdot N + n] \leftarrow 1$ 
      end for
    end for
  end for
end function

```



```

function PRUNEROWS(rowsToRemove, sudoku, N)
  for  $r \leftarrow 0$  to  $N - 1$  do
    for  $c \leftarrow 0$  to  $N - 1$  do
       $symbol \leftarrow sudoku[r \cdot N + c] - 1$       ▷ Numbers in a sudoku
      usually start from 1, but in the matrix A they start from 0
      if num is not NULL then
         $startIdx \leftarrow (r \cdot N + c) \cdot N$ 
        for  $i \leftarrow 0$  to  $N - 1$  do
          if  $i \neq num$  then
             $rowsToRemove \leftarrow rowsToRemove \cup \{startIdx + i\}$ 
          end if
        end for
      end if
    end for
  end for
end function

```

```

function EXACTCOVERSOLUTIONTOSUDOKU(ecSolution[], N)
    sudokuSolution  $\leftarrow$  ZEROS(N, N)
    for idx  $\in$  ecSolution do
        r  $\leftarrow \left\lfloor \frac{idx}{(N^2)} \right\rfloor$ 
        c  $\leftarrow \left\lfloor \left( \frac{idx}{N} \right) \bmod N \right\rfloor$ 
        n  $\leftarrow (idx \bmod N) + 1$ 
        sudokuSolution[r][c]  $\leftarrow$  n
    end for
    return sudokuSolution
end function

```

Generating Exact Cover problems

Having a module that generates exact cover problems given the number of sets and the cardinality of the domain of the sets is useful for testing purposes. The idea of my choice to generate the exact cover problem is to ensure that the problem has at least a solution (that is, at least a set of sets that cover all the elements of the domain creating a partition), this is done by setting randomly M rows of the matrix A to the canonical base of \mathbb{R}^M . The precondition for this is that $N \geq M$. The rows of the matrix A that are not set to the canonical base are set randomly, each of their M cells is set to 1 or 0 with a custom probability, the default one is $p = 0.5$.

function GENERATEEXACTCOVER(N, M)

if $N < M$ **then**

return NULL

end if

$A \leftarrow \text{ZEROS}(N, M)$

$\text{idxCanonicalBase}[] \leftarrow \text{RANDOMSAMPLE}(\{0, \dots, N-1\}, M)$ ▷

 Sample from $\{0, \dots, N-1\}$ without replacement for M times

for $i \leftarrow 0$ to $M-1$ **do**

$A[\text{idxCanonicalBase}[i]][i] \leftarrow 1$

end for

for $i \leftarrow 0$ to $N-1$ **do**

if $i \notin \text{idxCanonicalBase}$ **then**

for $j \leftarrow 0$ to $M-1$ **do**

$A[i][j] \leftarrow \text{RANDOMCHOICE}(\{0, 1\})$

end for

end if

end for

return A

end function