CHAPTER 2

SETS, FUNCTIONS, SEQUENCES, SUMS

Why study this chapter?

Many important discrete structures are built using sets

{ } graphs

relations

Functions

$f(x)$ used to represent complexity of algorithms

needed for cryptography, machine learning

Sequences

Summations $\sum a_n$

Learning Objectives

- Manipulate set operations
- Represent sets by bit strings
- Determine whether a rule is a function
- Determine whether a function is 1-1, onto, bijective
- Find composite functions
- Find next terms, general term of a sequence
- Express a sum as sigma notation

{ } sets

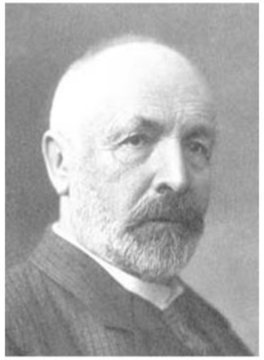
A collection of objects, called elements/members

{a, b, {c} } is a set with 3 members a, b, and {c}

$\mathbb{N}$ denotes the set of natural numbers 0, 1, 2, 3, …

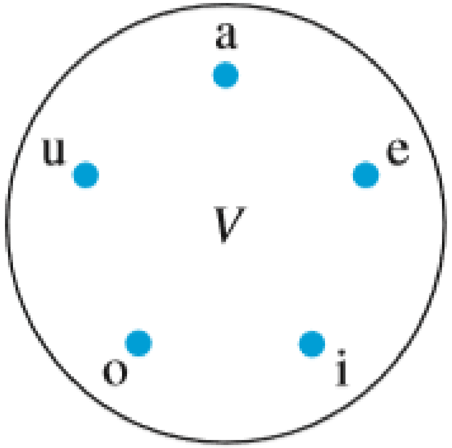
$\mathbb{Z}$ is the of integers …, -2, -1, 0, 1, 2, …

$\mathbb{R}$ is the set of real numbers



Georg Cantor
(1845–1918)

Venn diagrams



Venn diagram for the set of vowels $V = \{a, e, i, o, u\}$

$\in$ membership relation

$a \in \{a, b, \{c\}\}$

$c \notin \{a, b, \{c\}\}$

$a \in A$ means

a is an element/member of the set A

a is in A

a belongs to the set A

= equality

Two sets are called equally if and only if they have the same elements

$$\{1, 2, 3\} = \{2, 1, 3\} = \{1, 2, 3, 3\}$$

$$\{1, 2, 3\} \neq \{2, 3, 4\}$$

$$\{a, b\} \neq \{a, \{b\}\}$$

$\varnothing$ empty set

The set with no elements

$\{x \mid x > 3 \text{ and } x < 2\} = \{\ \} = \varnothing$

$\varnothing = \{\varnothing\}$

$\varnothing \in \{3\}$

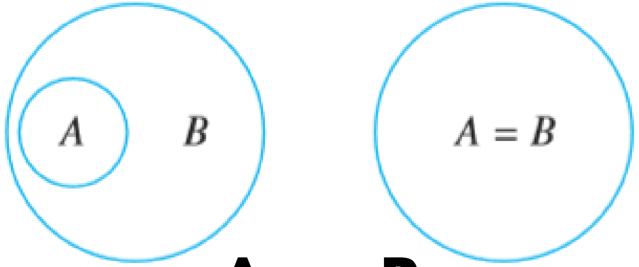
$\{a, \varnothing\} = \{a\}$

$\subseteq$ subset

$A \subseteq B$

$\Leftrightarrow \forall x (x \in A \Rightarrow x \in B)$

A is called a subset of B, e.g., $\{a, b\} \subseteq \{a, b, c\}$



$A \subseteq B$

$\subset$ proper subset

Write $A \subset B$ if

$A \subseteq B$, and $A \neq B$

A is called a proper subset of B, e.g., $\{a, b\} \subset \{a, b, c\}$

$\{a, b, c\} \subseteq \{a, c, b\}$

$\{a, b, c\} \subset \{a, c, b\}$

properties

$$\emptyset \subseteq S$$

Proof.

$\forall x(x \in \emptyset \Rightarrow x \in S)$

$\mathbf{F} \Rightarrow \mathbf{F}$

$\mathbf{F} \Rightarrow \mathbf{T}$

$\emptyset \subseteq \emptyset$

$\emptyset \subseteq \{a\}$

$\emptyset \in \{a\}$

$\emptyset \in \{a, \emptyset\}$

$$S \subseteq S$$

Proof.

$\forall x(x \in S \Rightarrow x \in S)$

$\mathbf{F} \Rightarrow \mathbf{F}$

$\mathbf{T} \Rightarrow \mathbf{T}$

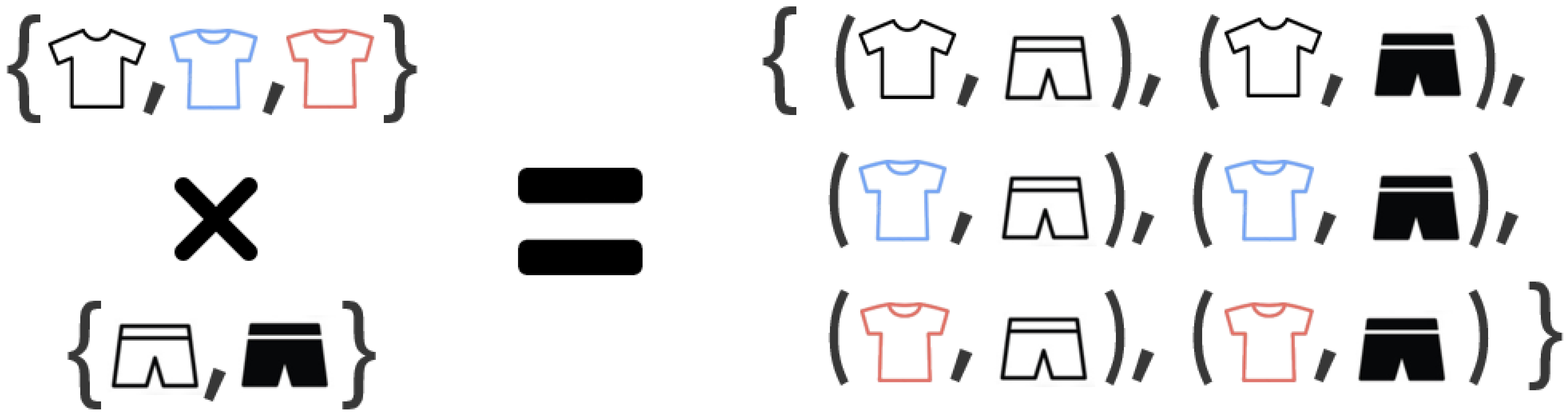
Set equality

$$A = B \Leftrightarrow \begin{cases} A \subseteq B \\ B \subseteq A \end{cases}$$

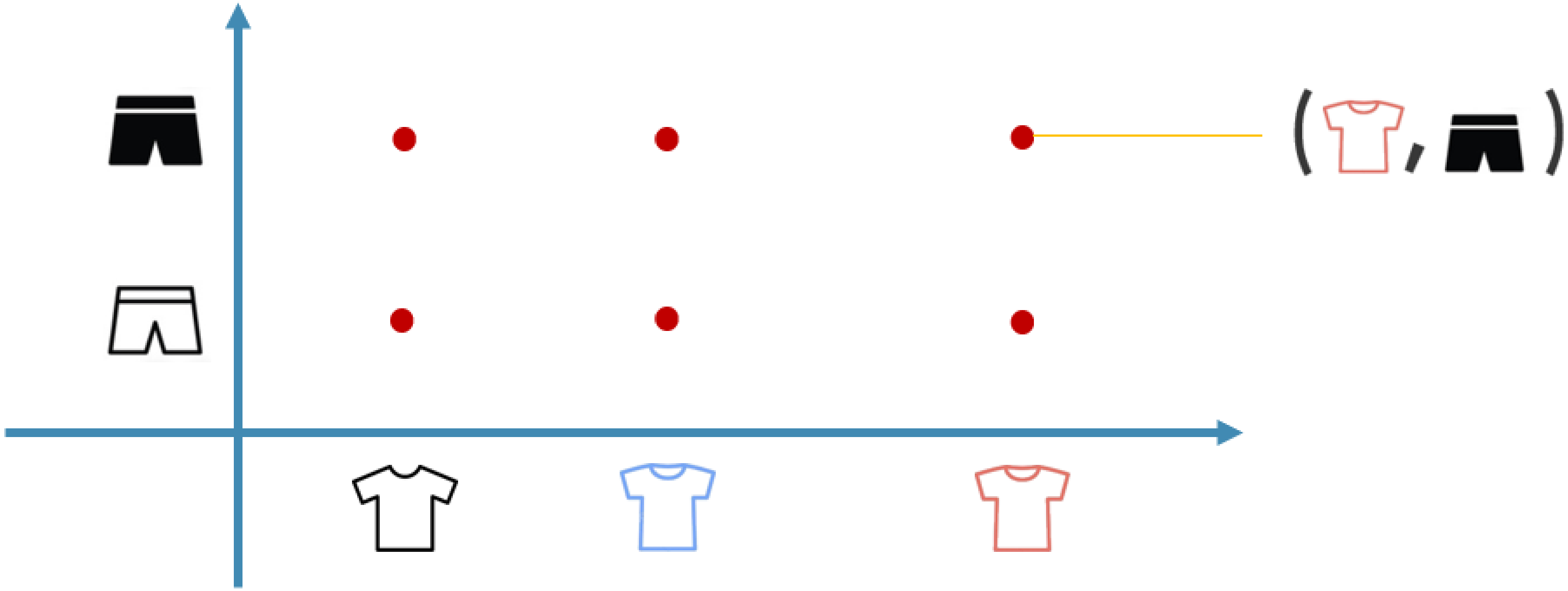
power set

set S	subsets	power set P(S)	|P(S)|
∅	∅	{∅}	1
{1}	∅, {1}	{∅, {1}}	2
{1, 2}	∅, {1}, {2}, {1, 2}	{∅, {1}, {2}, {1, 2}}	4
{1, 2, 3}	∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}	{∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}	8 2^n

× Cartesian product



⚠ (a, b) ≠ (b, a)



Cartesian product

Given two sets $A = \{a, b, c\}$, and $B = \{1, 2, 3, 4\}$

How many elements does $A \times B$ have?

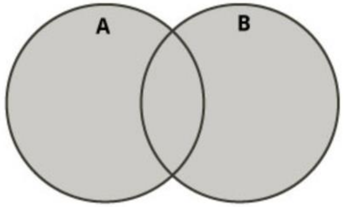
$(3, a) \in A \times B$

$\{b, 2\} \in A \times B$

$\{(a, 1), (c, 3)\} \subseteq A \times B$

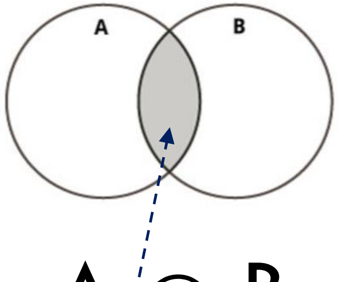
$A \times B = B \times A$

Set operations



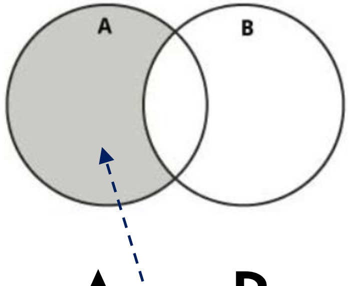
$A \cup B$

union



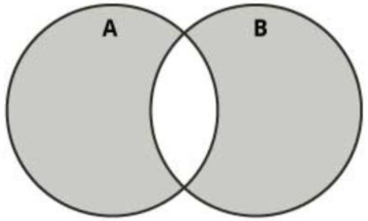
$A \cap B$

intersection



$A - B$

difference



$A \oplus B$

symmetric difference

Universal set and complement

Universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{\quad 2, \qquad 5, 6, \quad 8\}$

complement of A $\overline{A} = \{1, \quad 3, 4, \qquad ,7 \quad \}$

$A \cup \overline{A} = U$

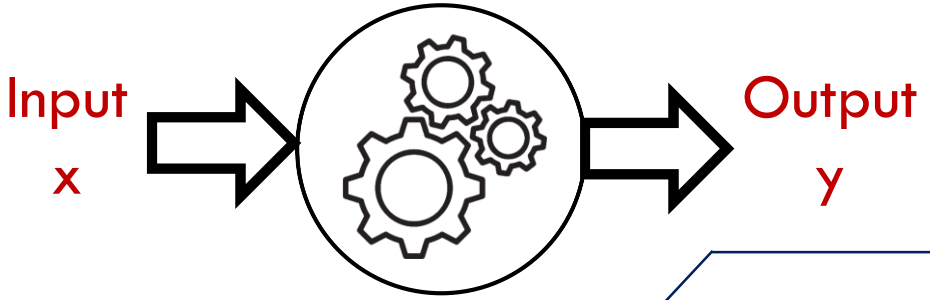
$A \cap \overline{A} = \varnothing$

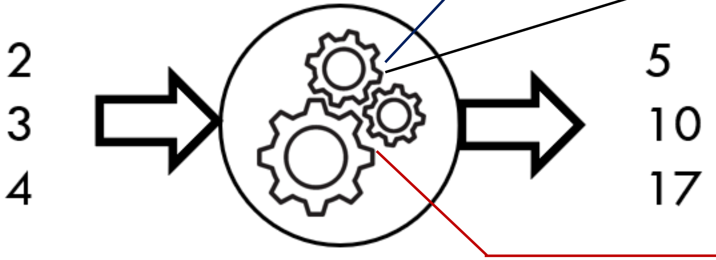
Computer representation of sets

U = {	1,	2,	3,	4,	5,	6,	7,	8 }
u[n]	1	1	1	1	1	1	1	1
A = {		2,	3,		5,		7	}
a[n]	0	1	1	0	1	0	1	0
B = {	1,	2,		4,	5,			8 }
b[n]	1	1	0	1	1	0	0	1

What is the bit string for $\overline{A}$? $A \cup B$? $A \cap B$? $A - B$? $A \oplus B$?

f(x) functions



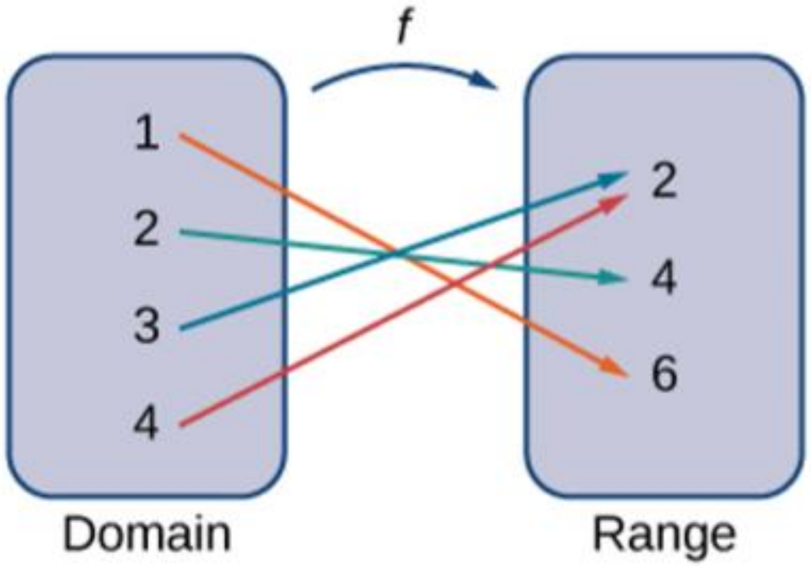


Input	Output
2	5
3	10
4	17

$f = \{(2, 5), (3, 10), (4, 17)\}$

$f(x) = x^2 + 1$
$x \in \{2, 3, 4\}$

Domain and range



f = {(1, 6), (2, 4), (3, 2), (4, 2)}

f: A $\rightarrow$ B

Read: function f

from A to B

A: domain of f

f(A): range of f

function = mapping

= transformation

function or not

$$f:\mathbb{R}\to\mathbb{R},\quad f(x)=\frac{1}{x^2-2}$$

$$f(\sqrt{2})=\frac{1}{0}\ !$$

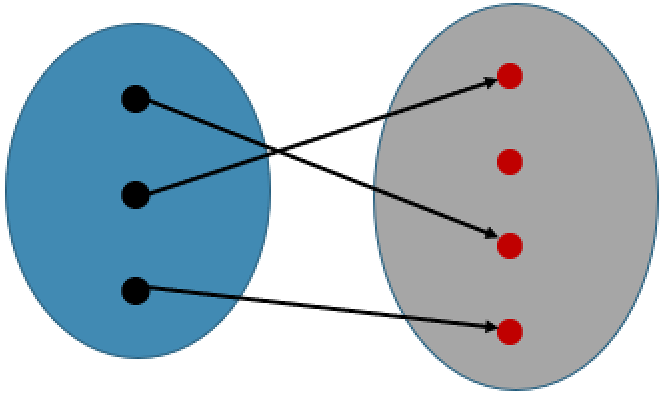
$$f:\mathbb{Z}\to\mathbb{R},\quad f(x)=\frac{1}{x^2-2}$$

$$f:\mathbb{Z}\to\mathbb{Z},\quad f(x)=\frac{1}{x^2-2}$$

$$f(2)=\frac{1}{2}\notin\mathbb{Z}$$

one-to-one

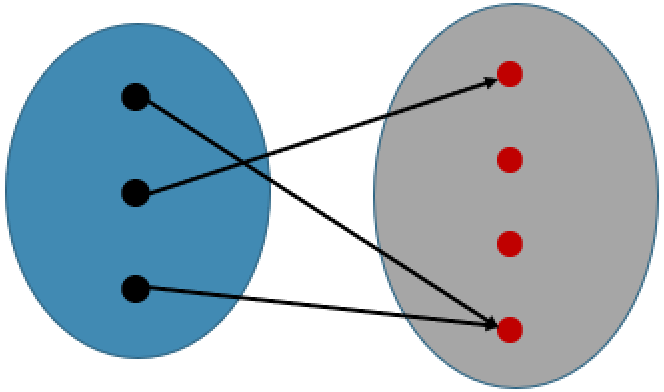
f is called one-to-one ⇔
$x \neq y \Rightarrow f(x) \neq f(y)$ for all x, y



one-to-one

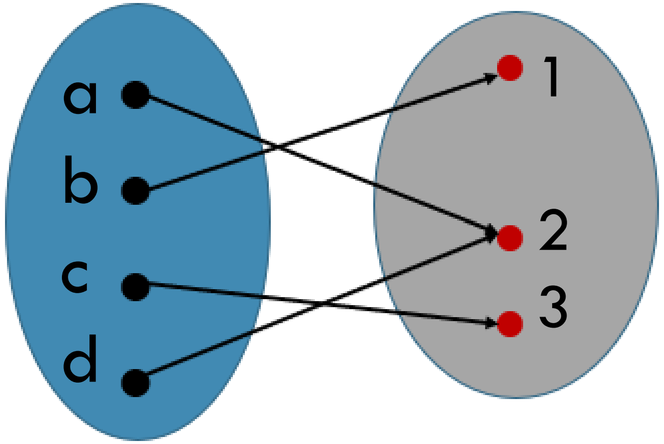
ASCII

$	→	36
@	→	64
A	→	65
B	→	66
C	→	67



Not one-to-one

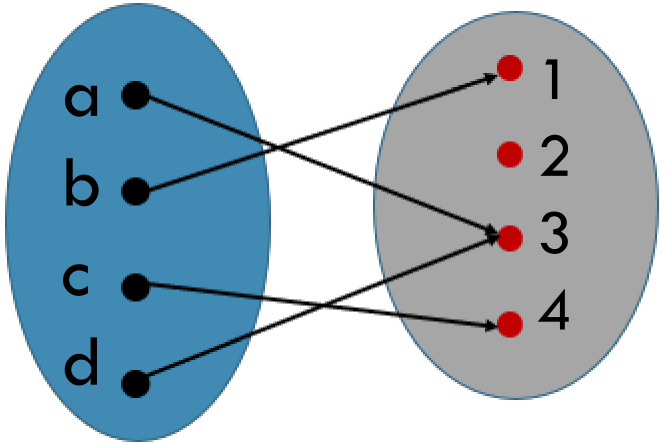
onto



Onto

$f(A) = \{1, 2, 3\} = B$

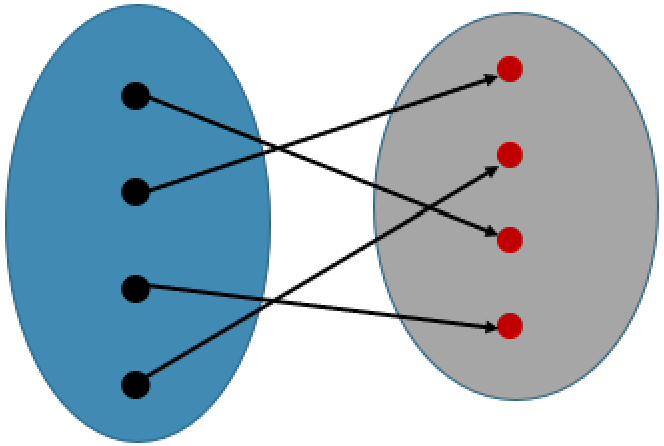
$f: A \to B$ is onto if $f(A) = B$



Not onto

$f(A) = \{1, 3, 4\} \neq B$

Bijection and inverse

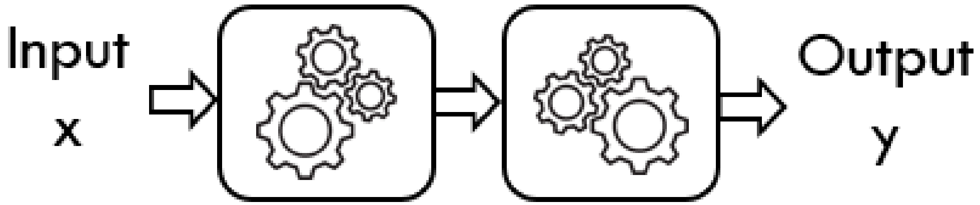


bijection = 1-1 + onto

ASCII

bijection	inverse
$ → 36	$ ←36
@ → 64	@ ← 64
A → 65	A ← 65
B → 66	B ← 66
C → 67	C ← 67

composite function



$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & x+3 \arrow[r, "g"] & (x+3)^2 \end{tikzcd}$$

$f(x) = ?$

$g(x) = ?$

$(g\circ f)(x) = ?$

composite function

Given f = {(a, 2), (b, 1), (c, 2)}

g = {(1, a), (2, b), (3, c)}

Find g∘f and f∘g

a ⇨ 2 ⇨ b   b ⇨ 1 ⇨ a   c ⇨ 2 ⇨ b

g∘f = {(a, b), (b, a), (c, b)}

1 ⇨ a ⇨ 2   2 ⇨ b ⇨ 1   3 ⇨ c ⇨ 2

f∘g = {(1, 2), (2, 1), (3, 2)}

sequences

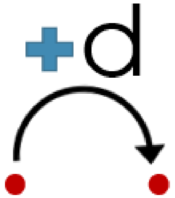
$a_1, a_2, a_3, a_4, \ldots$ sequence $\{a_n\}$

a[1], a[2], a[3], a[4], … array a[n]

Lucas numbers

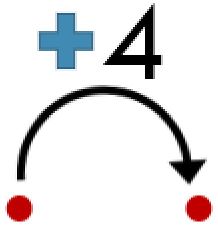
n	1	2	3	4	5	…
a_n	1	3	4	7	11	…

What are the next two terms?



Arithmetic progression

Given the sequence 3, 7, 11, 15, 19, 23, 27, 31, …



Find the next two terms of the sequence

35, 39

What is the general term a_n, the n^{th} term?

$a_n = 4n - 1$

Formulae of sequences

1/ 5, 11, 17, 23, 29, 35, 41, …

$a_n = 6n - 1$

2/ 1, 3, 4, 7, 11, 18, 29, 47, …

$L_n = L_{n-1} + L_{n-2}$ with $L_1 = 1$, $L_2 = 3$ // Lucas sequence

3/ 1, 7, 25, 79, 241, 727, 2185, ...

$a_n = 3^n - 2$

Recursive sequence

Given $a_1 = 3$, $a_2 = 2$, and $a_n = 3a_{n-1} - a_{n-2}$ when $n > 2$

Find a_3, a_4, a_5.

n	**1**	**2**	**3**	**4**	**5**
a_n	3	2	3	7	18

Σ summations

$$\sum_{i=m}^{n} a_i$$

read: the sum from $i = m$ to $i = n$ of a_i

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

```
s := 0
for i :=1 to n do
    s := s + i
return(s)
```

```
return(n*(n+1)/2)
```

Σ summations

Write each of these sums as sigma notation

1/ 1 + 3 + 5 + 7 + 9 + 11 $\sum_{1}^{6}(2i-1)$

2/ 3 + 3 + 3 + 3 + 3 $\sum_{1}^{5}3$

3/ 1 + 4 + 7 + 10 + 13 + 16 + 19 $\sum_{1}^{7}(3i-2)$

summary

- Sets
- Functions
- Sequences
- Summations

THANKS