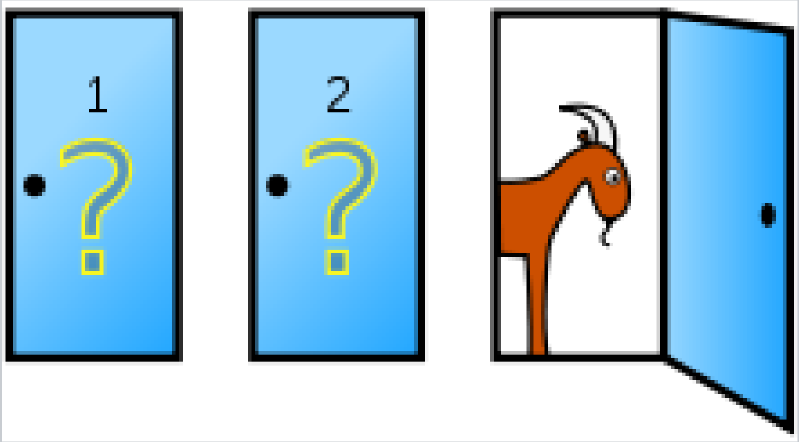
# 2

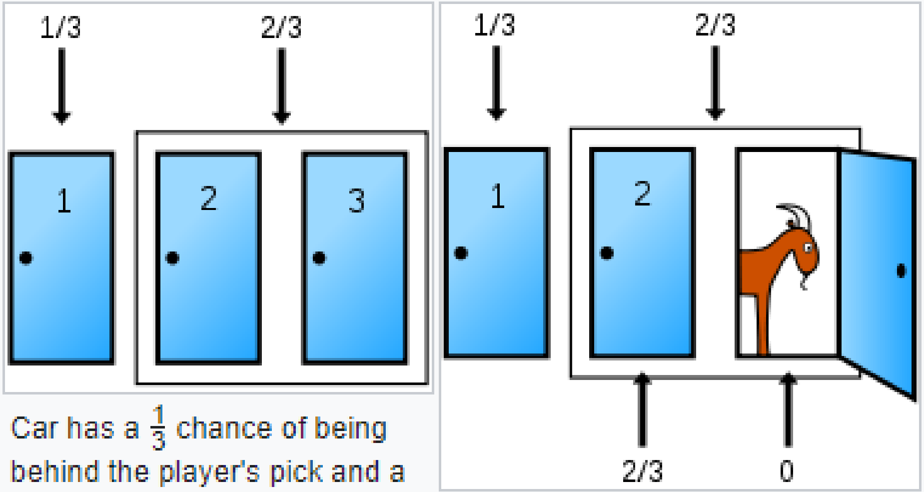
# Probability

# Introduction

- The Monty Hall problem




In search of a new car, the player picks a door, say *1*. The game host then opens one of the other doors, say *3*, to reveal a goat and offers to let the player switch from door *1* to door 2.




Car has a $\frac{1}{3}$ chance of being behind the player's pick and a $\frac{2}{3}$ chance of being behind one of the other two doors.

The host opens a door, the odds for the two sets don't change but the odds move to 0 for the open door and $\frac{2}{3}$ for the closed door.

```
# Monty Hall problem
monty<-function(ndoors=3){
  # Randomly Choose a door to put a car
  randoor<-sample(1:ndoors,1)
  # Choose a car at random
  choice<-sample(1:ndoors,1)
  # Always change the door
  # If at first stage you luckly choose the right one
  # ==> get nothing after changing the door
  # And you will win otherwise
  ifelse(choice==randoor,"goat","car")
}
# 1000 trials
m<-replicate(1000,monty(3))
table(m)/1000
```

```
> table(m)/1000
m
 car goat
0.67 0.33
```

Some results of 1000 trials using R

# Contents

# Outcomes, events, and the sample space

- An *experiment* that can result in different outcomes, even though it is repeated in the same manner every time, is called a *random experiment*.
- The set of all possible outcomes of a random experiment is called the *sample space* of the experiment. The sample space is denoted as S.
- An *event* is a subset of the sample space of a random experiment.

| Notation | |
|---|---|
| | S = sample space<br>Ø = empty event<br>P(E) = probability of event E |

# Probability as relative frequency

Used to quantify likelihood or *chance*

```
> table(sample(0:1,10000,rep=T))

   0    1
4979 5021
```

```
> table(sample(0:1,100000,rep=T))

    0     1
50115 49885
```

```
> table(sample(0:1,1000000,rep=T))

     0      1
499365 500635
```

As number of trials $\rightarrow \infty$,
the result will be 50% - 50%

# Probability

- Used to represent risk or *uncertainty* in engineering applications
- Can be interpreted as our ***degree of belief***
- **Equally Likely Outcomes.** Whenever a sample space consists of N possible outcomes that are *equally likely*, the probability of each outcome is 1/N.

1/2

```
> tosscoin(1)
  toss1
1     H
2     T
```

*equally likely*

```
rolldie(1)
X1
 1
 2
 3
 4
 5
 6
```

1/6

# Probability - Ex

A young family plans to have two children. What is the probability of two girls?

Solution 1 (wrong). There are 3 possible families with 2 children: two girls, two boys, and one of each gender. Therefore, the probability of two girls is 1/3.

S = {GG, BG, BB} (NOT equally likely)

➔ P(GG) = 1/3 (!)

Solution 2 (right).

S = {GG, BG, GB, BB} (equally likely)

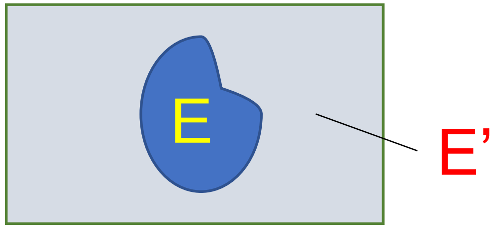
➔ P(GG) = ¼

# Probability - Ex

- For a discrete sample space, the probability of an event E, denoted as P(E), equals the sum of the probabilities of the outcomes in E.

Given

| Outcome | a | b | c | d |
|---|---|---|---|---|
| Probability | 0.1 | 0.3 | 0.5 | 0.1 |




Consider the events A = {a, b}, B= {b, c, d}, C = {d}

a/ Find P(A), P(B), P(C)

b/ Find P(A'), P(B'), P(C')

c/ Find $P(A \cap B)$, $P(A \cup B)$, $P(A \cap C)$

Notation

E' : complement of event E

# Axioms of Probability

If S is the sample space and E is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leqslant P(E) \leqslant 1$

(3) For two events $E_1$ and $E_2$ with $E_1 \cap E_2 = \varnothing$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

# Probability of a Union

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Special case:

If A and B are *mutually exclusive* events,

$$P(A \cup B) = P(A) + P(B)$$

**Ex1.** Roll a six-sided die twice.

The sample space is given by

S = {(1, 1), (1, 2), (1, 3), …, (4, 6), (5, 6), (6, 6)}.

Let $A = \{ (X_1, X_2) \mid X_1 = X_2 \}$ and

$B = \{ (X_1, X_2) \mid X_1 + X_2 > 7 \}$.

➔ P(A) = 1/6, P(B) = 15/36, and $P(A \cap B)$ = 3/36.

What is the probability $P(A \cup B)$ ?

```
> A
   x1 x2
1   1  1
8   2  2
15  3  3
22  4  4
29  5  5
36  6  6

> intersect(A,B)
   x1 x2
22  4  4
29  5  5
36  6  6

> Prob(union(A,B))
[1] 0.5
```

```
> B
   x1 x2
12  6  2
17  5  3
18  6  3
22  4  4
23  5  4
24  6  4
27  3  5
28  4  5
29  5  5
30  6  5
32  2  6
33  3  6
34  4  6
35  5  6
36  6  6
```

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

**Ex2.** Suppose that after 10 years of service, 40% of computers have problems with motherboards (MB), 30% have problems with hard drives (HD), and 15% have problems with both MB and HD. What is the probability that a 10-year old computer still has fully functioning MB and HD?

# Probability of a Union

For three events A, B, C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

**Ex.** If A, B, and C are *mutually exclusive* events with $P(A) = 0.2$, $P(B) = 0.3$, $P(C) = 0.4$

Determine the probabilities

a/ $P(A \cap B)$

b/ $P(A \cap B \cap C)$

c/ $P(A \cup B \cup C)$

d/ $P[(A \cup B) \cap C]$

e/ $P(A' \cap B' \cap C')$

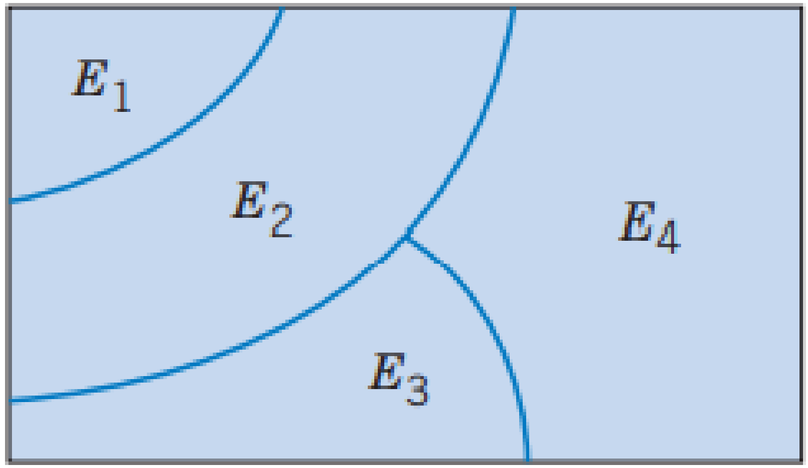
A collection of events $E_1, E_2, \ldots, E_k$ is said to be *mutually exclusive* if for all pairs,

$$E_i \cap E_j = \varnothing$$

# Mutually Exclusive Events

For a collection of *mutually exclusive* events

$$P(E_1 \cup E_1 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k)$$



# Conditional probability

- The conditional probability of B given A,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \text{ for } P(A) > 0$$

**Ex.** Toss a coin twice.
The sample space is given by S = {HH, HT, TH, TT}.
Let A = {a head occurs} and
B = {a head and tail occur}.
➔ $P(A) = 3/4$, $P(B) = 2/4$, and $P(A \cap B) = 2/4$.
What are the probabilities $P(A \mid B)$ and $P(B \mid A)$?

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{2/4}{3/4} = 2/3$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{2/4}{2/4} = 1$$

# Multiplication Rule

- For events A, B,

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

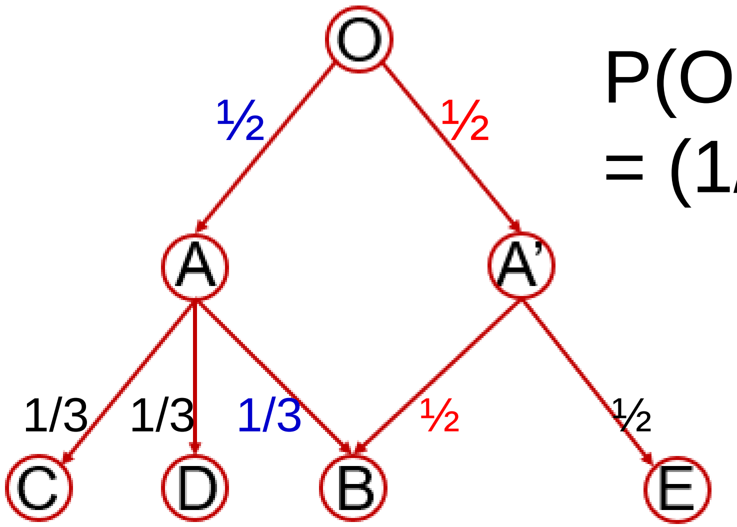
- For events $A_1$, $A_2$, $A_3$,

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_2 \cap A_3 \mid A_1)P(A_1) \\ &= P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_2 \cap A_1) \end{aligned}$$

**Ex.** Consider an urn with 10 balls inside, **7** of which are **red** and **3** of which are **green**. Select 3 balls successively from the urn. Let $A_1$ = "1st ball is red", $A_2$ = "2nd ball is red", and $A_3$ = "3rd ball is red".

Then $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_2 \cap A_1)$ = (5/8)(6/9)(7/10)

**Ex.** Someone wants to walk from O to B. What is the probability that he will reach B after two random steps?

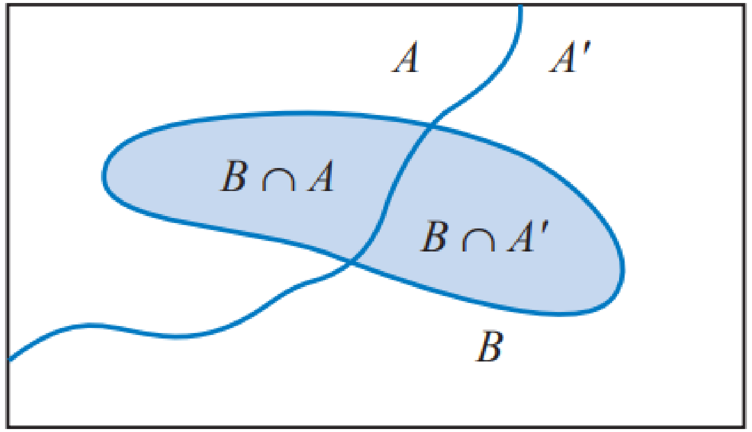



$$P(O \rightarrow B) = P(O \rightarrow A \rightarrow B) + P(O \rightarrow A' \rightarrow B)$$
$$= (1/2)(1/3) + (1/2)(1/2) = 1/6 + 1/4 = 5/12$$

# Total Probability Rule (two events)

For any events $A$ and $B$,

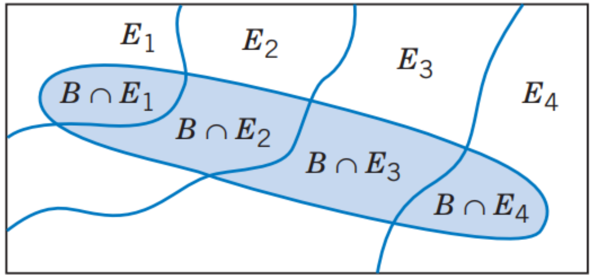
$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A')$$



# Total Probability Rule (multiple events)

Assume $E_1, E_2, \ldots, E_k$ are $k$ mutually exclusive and exhaustive sets. Then

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k)$$
$$= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k)$$



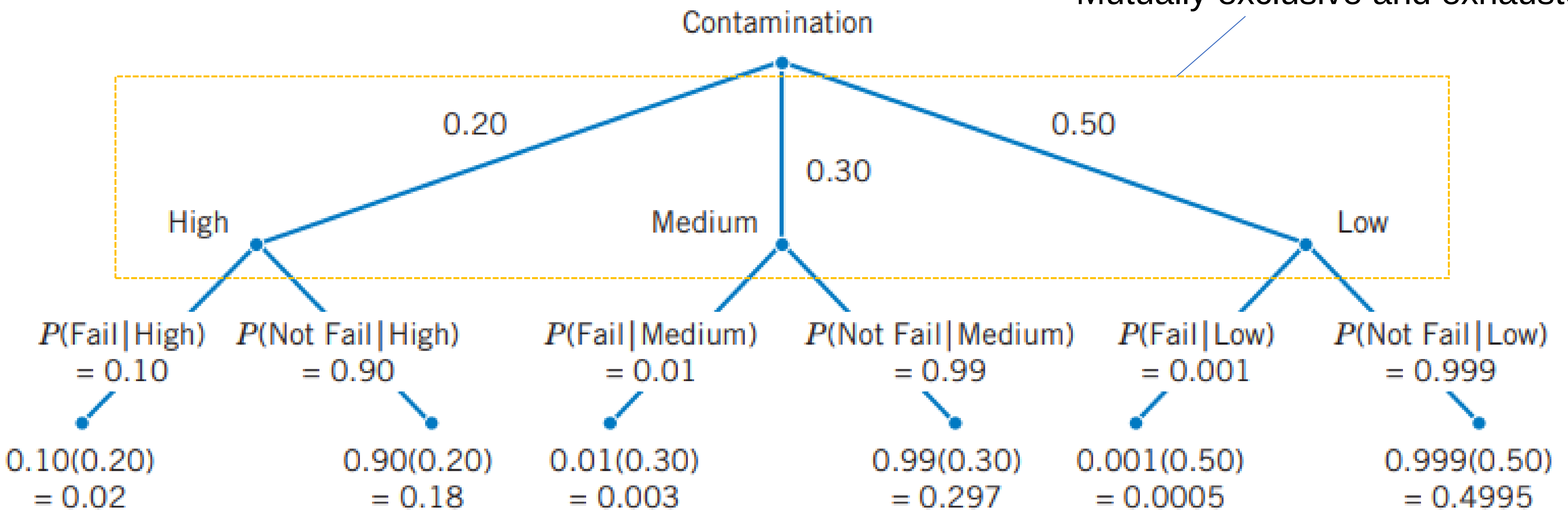

**Ex**. (Semiconductor Failures) Assume the following probabilities for product failure subject to levels of contamination in manufacturing:

| Probability of Failure | Level of Contamination |
|---|---|
| 0.10 | High |
| 0.01 | Medium |
| 0.001 | Low |

In a particular production run, 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination.
What is the probability that a product using one of these chips fails?

# Example (cont.)

Mutually exclusive and exhausted




$P(\text{Fail}) = 0.02 + 0.003 + 0.0005 = 0.0235$

# Bayes' Theorem

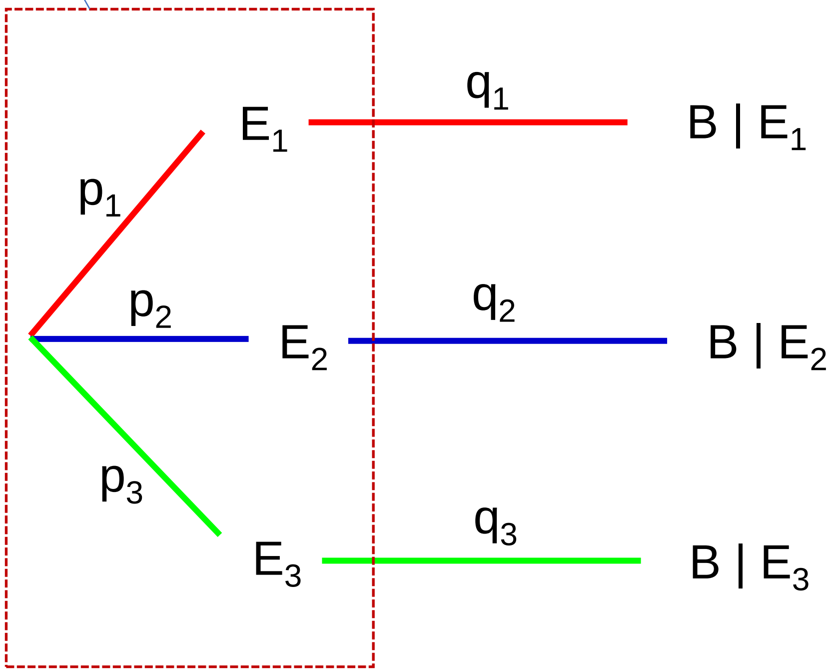
If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event with $P(B) > 0$,

$$P(E_i \mid B) = \frac{P(B \mid Ei)P(Ei)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \cdots + P(B \mid Ek)P(Ek)}$$

Usually in applications we are given (or know) a *priori* probabilities $P(E_i)$. We collect some evidence, which we represent by the event B.
Bayes' rule can be used to *update* $P(E_i)$ to $P(E_i \mid B)$

# Bayes' Theorem

mutually exclusive and exhaustive




$$P(B) = p_1q_1 + p_2q_2 + p_3q_3$$

$$P(E_1 \mid B) = \frac{p_1q_1}{p_1q_1 + p_2q_2 + p_3q_3}$$

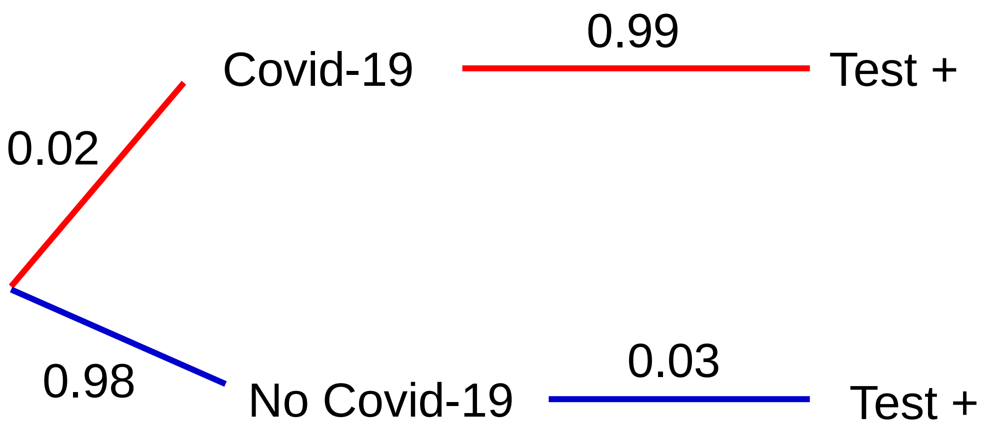
$$P(E_2 \mid B) = \frac{p_2q_2}{p_1q_1 + p_2q_2 + p_3q_3}$$

$$P(E_3 \mid B) = \frac{p_3q_3}{p_1q_1 + p_2q_2 + p_3q_3}$$

$$P(E_i \mid B) = \frac{P(B \mid Ei)P(Ei)}{P(B)}$$

# Ex. (Reliability of a Covid-19 Test)

Suppose the probability that a test correctly identifies someone with the Covid-19 as positive is 0.99, and the probability that it correctly identifies someone without the illness as negative is 0.97. The proportion of the Covid-19 in the world is 0.02. A person takes the Covid-19 test, and the result is positive. What is the probability that he/she has the Covid-19, that is, P(Covid-19 | Test +) ?




$$P(\text{Covid-19} \mid \text{Test} +) = \frac{0.02(0.99)}{0.02(0.99) + 0.98(0.03)}$$

$$\approx 0.40$$

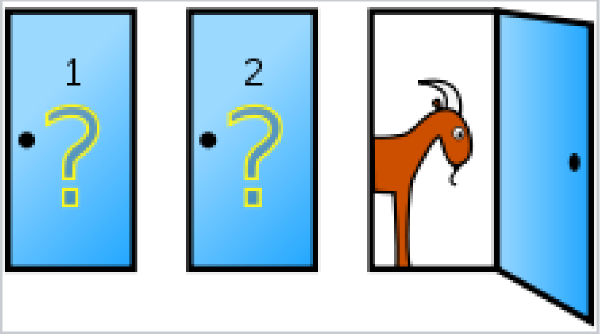
# Ex - Bayesian Network

- Bayesian networks are used on the Web sites of high technology manufacturers to allow customers to quickly diagnose problems with products. An oversimplified example is presented here. A printer manufacturer obtained the following probabilities from a database of test results. Printer failures are associated with three types of problems: hardware, software, and other (such as connectors), with probabilities 0.1, 0.6, and 0.3, respectively. The probability of a printer failure given a hardware problem is 0.9, given a software problem is 0.2, and given any other type of problem is 0.5. If a customer enters the manufacturer's Web site to diagnose a printer failure, what is the most likely cause of the problem?

# Ex. Monty Hall Problem

(wikipedia.org/wiki/Monty_Hall_problem)

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?




In search of a new car, the player picks a door, say *1*. The game host then opens one of the other doors, say *3*, to reveal a goat and offers to let the player switch from door *1* to door *2*.

P(door No. 2 has a car | open door No. 3) = 2/3

# Monty Hall Problem solution

| Prior: P(car behind a door)<br><br>P(Car@...) | Event: you choose door 1, Monty Hall open door 3<br><br>P(Open door 3 \| Car@...) | Posterior: probability of car behind a door after the event<br><br>P(Car@... \| Opened door 3) |
|---|---|---|
| P(Car@door 1) = 1/3 | P(Open door 3 \| Car@door 1) = 1/2 | P(Car@ door 1 \| Opened door 3) = $\frac{(1/3)(1/2)}{(1/3)(1/2) + (1/3)1 + (1/3)0} = 1/3$ |
| P(Car@door 2) = 1/3 | P(Open door 3 \| Car@door 2) = 1 | P(Car@ door 2 \| Opened door 3) = $\frac{(1/3)(1)}{(1/3)(1/2) + (1/3)1 + (1/3)0} = 2/3$ |
| P(Car@door 3) = 1/3 | P(Open door 3 \| Car@door 3) = 0 | P(Car@ door 3 \| Opened door 3) = $\frac{(1/3)(0)}{(1/3)(1/2) + (1/3)1 + (1/3)0} = 0$ |

Given $P(B \mid A) = 0.5$, $P(A) = 0.3$, $P(B \mid A') = 0.6$

a/ Find $P(A \cap B)$

b/ Find $P(B)$, $P(A \cup B)$

c/ Find $P(A \mid B)$, $P(A' \mid B)$

# Independence (two events)

- Two events A, B are *independent* if any one of the following equivalent statements is true:
  - (1) $P(A \mid B) = P(A)$
  - (2) $P(B \mid A) = P(B)$
  - (3) $P(A \cap B) = P(A)P(B)$

**Ex1.** Toss a fair coin twice.
➔ S = {HH, HT, TH, TT}.
Consider events A = {HT, HH}, B = {HT, TT}, C = {HH}

- $P(A \mid B) = ½ = P(A)$ ➔ A and B are independent
- $P(C \mid A) = ½ \neq ¼ = P(C)$ ➔ A and C are NOT independent
- $P(C \cap B) = 0 \neq 1/8 = P(C)P(B)$ ➔ B and C are NOT independent

**Ex2.** Roll a fair die. Consider two events
A = “the throw is equal or less than 3”
B = “the throw is even”
Are A and B *independent* events?

No. $P(A \mid B) = 1/3 \neq 1/2 = P(A)$

# Independence - Ex1

A computer program is tested by 3 independent tests. When there is an error, these tests will discover it with probabilities 0.2, 0.3, and 0.5, respectively. Suppose that the program contains an error. What is the probability that it will be found by at least one test?

P(at least one test will discover the error) = 1 – P(no test can discover the error)

= 1 – P(1$^{st}$ test will not discover the error) P(2$^{nd}$ test will not discover the error)P(3$^{rd}$ test will not discover the error) = 1 – (0.8)(0.7)(0.5)

# Independence – Ex2

**Ex.** (Series Circuit) The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?

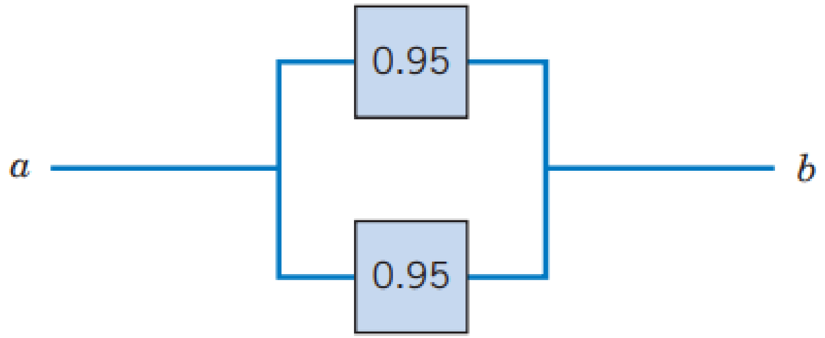

P(the circuit operates) = P(both devices operate)

= P(the left device operates) P(the right device operates) = 0.8(0.9) = 0.72

# Independence – Ex3

(Parallel Circuit) The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?

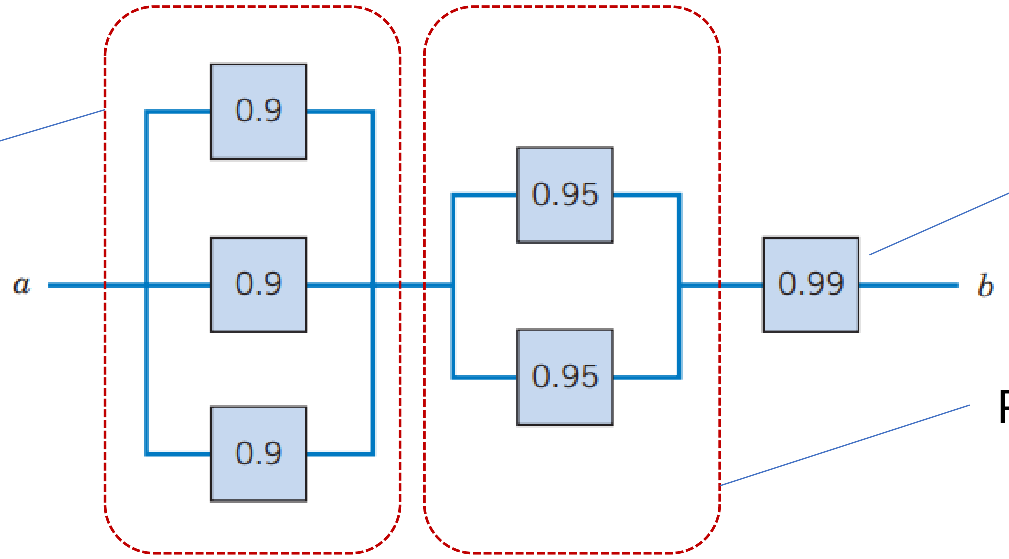
P(the circuit operates) = 1 – P(both devices fail)
= 1 – P(the top device fails)P(the bottom device fails) = 1 – 0.05(0.05) = 0.9975



# Independence – Ex4

(Advanced Circuit) The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?

$P(L) = 1 - (0.9)^3$




$P(R) = 0.99$

$P(M) = 1 - (0.95)^2$

➔ $P(L \cap M \cap R)$
$= (1 - 0.9^3)(1 - 0.95^2)(0.99)$
$= 0.9875$

# Ex5 – Birthday Problem

Suppose that there are n = 50 people together in a room. Each person announces the date of his/her birthday in turn. What is the probability of at least one match?

Assume all the years have 365 days for simplicity.

P(at least one match) = 1 – P(no match)

$$= 1 - \frac{364}{365}\frac{363}{365}\cdots\frac{316}{365} = 0.97 = 97\%$$

```
birthday<-function(n){
  p<-1
  for (i in 1:n){
    p<-p*(365-i+1)/365
  }
  return(1-p)
}
> birthday(50)
[1] 0.9703736
```

```
> table(replicate(1000,2 %in% table(sample(1:365,50,replace=T))))

FALSE  TRUE
   35   965
```

# Random Variables

- A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.
- Two types
  - Discrete random variables: random variables with a finite (or countably infinite) range.
  - Continuous random variable: random variables with an interval (either finite or infinite) of real numbers for its range (electrical current, length, pressure, temperature, time, voltage, weight, etc.)

# Random Variables

- A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.

$$X : S \to \mathbb{R}$$

$$X(\omega) \in \mathbb{R}$$

- Based on the range of X, we have two types
  - Discrete random variables: random variables with a finite (or countably infinite) range. X(S) is a finite set $\{x_1, x_2, \ldots, x_k\}$ or countably infinite set $\{x_1, x_2, \ldots\}$.
  - Continuous random variable: random variables with an interval (either finite or infinite) of real numbers for its range (electrical current, length, pressure, temperature, time, voltage, weight, etc.)

# Random variables – Ex1

Flipping a coin twice.

➔ The sample space is S = {HH, HT, TH, TT}.

$$X : S \rightarrow \mathbb{R}$$

X($\omega$) = *number of heads* in each outcome $\omega$

| $\omega \in S$ | HH | HT | TH | TT |
|---|---|---|---|---|
| $X(\omega)$ | 2 | 1 | 1 | 0 |

Discrete random variable

# Discrete Random Variables – Ex2

**Ex.** In a semiconductor manufacturing process, two wafers from a lot are tested. Each wafer is classified as *pass* or *fail*. Assume that the probability that a wafer passes the test is 0.8 and that wafers are independent.

$S = \{pp, fp, pf, ff\}$

$X : S \to \mathbb{R}$

$X(\omega) = x =$ *number of* wafers that pass in $\omega \in S$

| Outcome | | | |
|---|---|---|---|
| Wafer 1 | Wafer 2 | Probability | $x$ |
| Pass | Pass | 0.64 | 2 |
| Fail | Pass | 0.16 | 1 |
| Pass | Fail | 0.16 | 1 |
| Fail | Fail | 0.04 | 0 |

$X(S) = \{0, 1, 2\}$

X is discrete RV

# Discrete Random Variables - Ex2

Flip a fair coin until it shows a head. Let X be the number of flips in the experiment.

Then the sample space is

Outcomes

NOT equally likely!

$$S = \{H, TH, TTH, TTTH, \ldots\}$$

The range of the random variable is $\{1, 2, 3, \ldots\}$

$X(H) = 1$, $X(TH) = 2$, $X(TTH) = 3$, $X(TTTTTH) = 6$, …

Question: $P(X = 6) = ?$

# Summary

- 2.1 Sample Spaces and Events
- 2.2 Interpretations of Probability
- 2.3 Addition Rules
- 2.4 Conditional Probability
- 2.5 Multiplication and Total Probability Rules
- 2.6 Independence
- 2.7 Bayes' Theorem
- 2.8 Random Variables