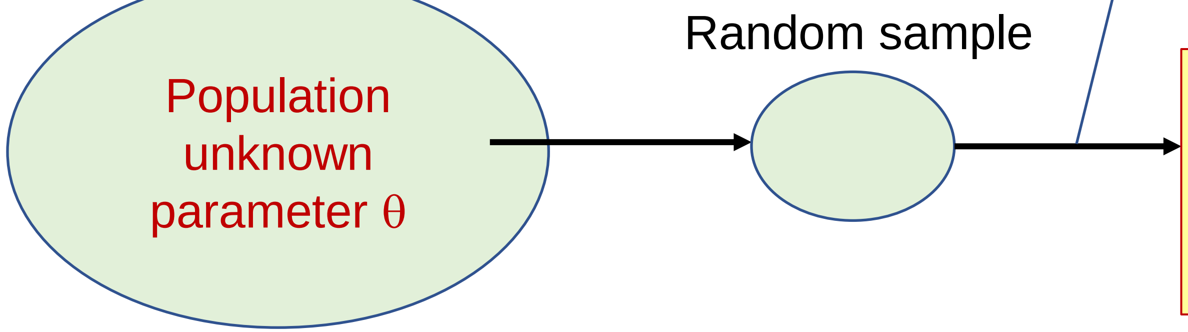
8

Statistical Intervals for a Single Sample

Learning Objectives

- After careful study of this chapter, you should be able to do the following:
  - 1. Construct *confidence intervals* on the mean of a normal distribution, using normal distribution / t distribution method
  - 2. Construct confidence intervals on the variance and standard deviation of a normal distribution
  - 3. Construct confidence intervals on a population proportion
  - 4. Use a general method for constructing an approximate confidence interval on a parameter

Introduction



How to compute l, u from sample data?

$$l \leqslant \theta \leqslant u,$$

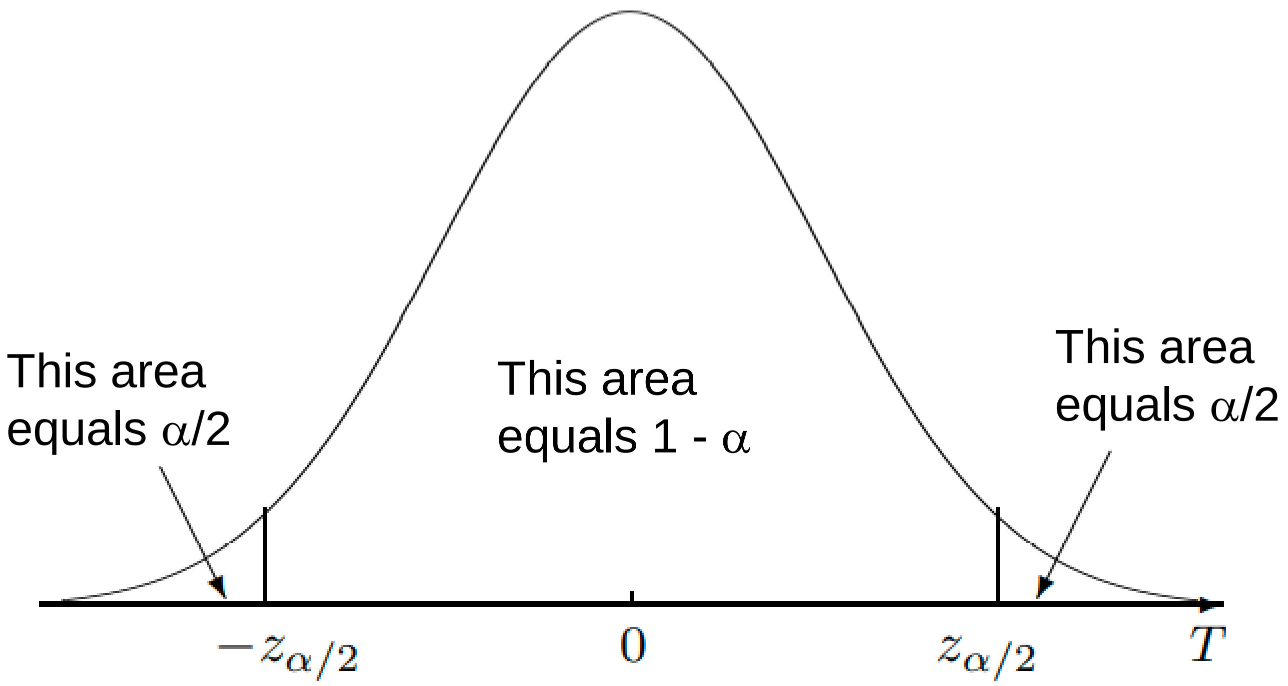
where $P(l \leqslant \theta \leqslant u) = 1 - \alpha$

l: lower-confidence limits (bounds)
u: upper-confidence limits (bounds)
$1 - \alpha$: confidence coefficient

Confidence Interval On The Mean Of A Normal Distribution, σ^2 Known

- Problem. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with unknown mean μ and known variance σ^2.
- A confidence interval estimate for μ is an interval of the form $l \leqslant \mu \leqslant u$
- If $P(L \leqslant \mu \leqslant U) = 1 - \alpha$, $(0 \leqslant \alpha \leqslant 1)$, then
  - $[l, u]$ is called *confidence interval*
  - $1 - \alpha$ is called the *confidence coefficient*

100(1 - α)% CI of A standard normal distribution



For a Standard Normal distribution

$P(-z_{\alpha/2} \leqslant Z \leqslant z_{\alpha/2}) = 1 - \alpha$

Population
unknown mean
μ

Random sample

How to compute l, u
from sample data?

$$L \leqslant \mu \leqslant U,$$

where $P(L \leqslant \mu \leqslant U) = 1 - \alpha$

CLT ➔
$\bar{X}$ is normally
distributed with mean
μ and variance σ^2/n

$$P\left\{-z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right\} = 1 - \alpha$$

$$\Downarrow$$

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Confidence Interval on the Mean, Variance Known

If $\bar{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1 - \alpha)\%$ CI on μ is given by

$$\boxed{\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the standard normal distribution.

Confidence Interval on the Mean, Variance Known - Example

Metallic Material Transition ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at 60ºC are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, and 64.3. Assume that impact energy is normally distributed with $\sigma = 1\text{J}$. We want to find a 95% CI for μ, the mean impact energy. The required quantities are $z_{\alpha/2} = z_{0.025} = 1.96$, $n = 10$, $\sigma = 1$, and $\bar{x} = 64.46$. The resulting 95% CI is found as follows:

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$64.46 - 1.96\frac{1}{\sqrt{10}} \le \mu \le 64.46 + 1.96\frac{1}{\sqrt{10}}$$

$$63.84 \le \mu \le 65.08$$

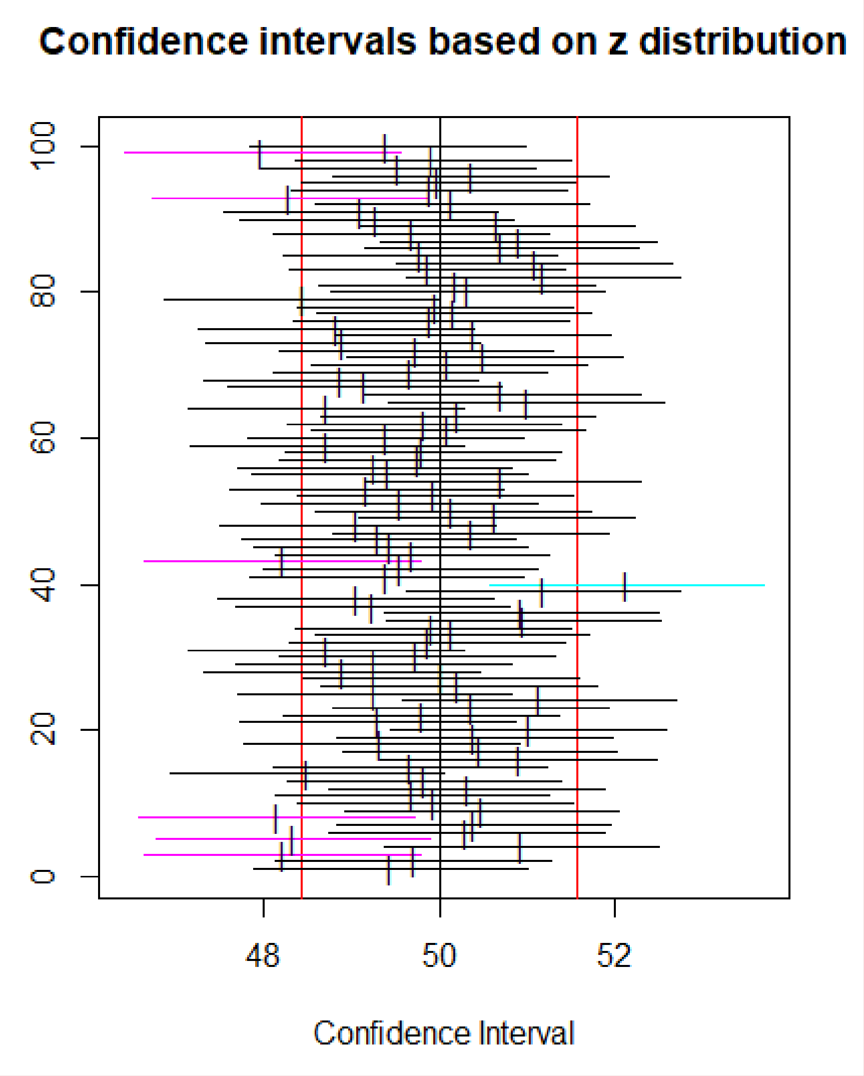
Practical Interpretation: Based on the sample data, a range of highly plausible values for mean impact energy for A238 steel at 60°C is $63.84\text{J} \leqslant \mu \leqslant 65.08\text{J}$.

Interpreting a Confidence Interval

- If an infinite number of random samples are collected and a 100(1)% confidence interval for is computed from each sample, 100(1 - α)% of these intervals will contain the true value of μ.
- We don’t know if the statement is true for this specific sample, but the method used to obtain the interval [l, u] yields correct statements 100(1 - α)% of the time.

Simulated confidence intervals

100 samples of size 25 were generated from a norm(mean = 50, sd = 4) distribution, and each sample was used to find a 95% confidence interval for the population mean. The 100 confidence intervals are represented above by horizontal lines, and the respective sample means are denoted by vertical slashes. Confidence intervals that "cover" the true mean $\mu = 50$ are plotted in black; those that fail to cover are plotted in a lighter color. In the plot we see that 7 of the simulated intervals out of the 100 failed to cover $\mu = 50$, which is a success rate of 93%. If the number of generated samples were to increase from 100 to 1000 to 10000, . . . , then we would expect our success rate to approach the exact value of 95%.



Choice of sample size

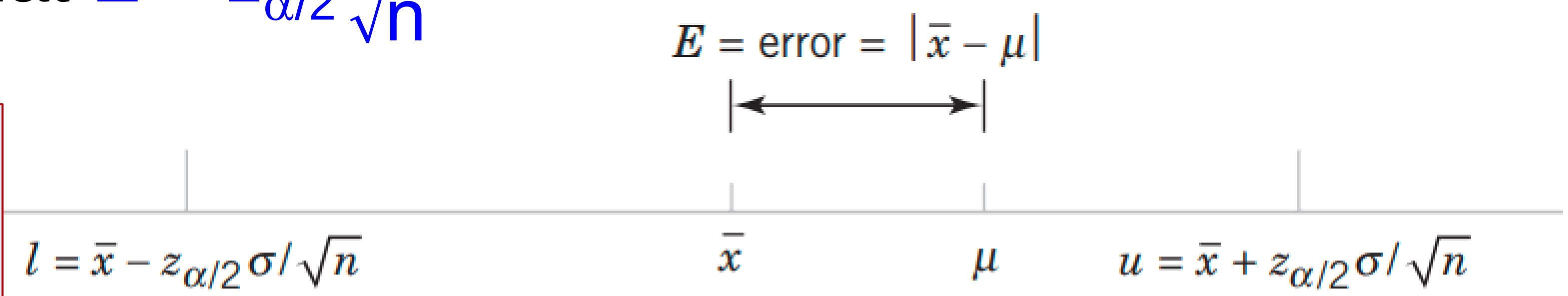
- From the 100(1 - α)% CI

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- We have E = error = $|\bar{x} - \mu| \leqslant z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

➔ Choose n such that $E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

➔ $$\boxed{n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2}$$



Choice of sample size - Example

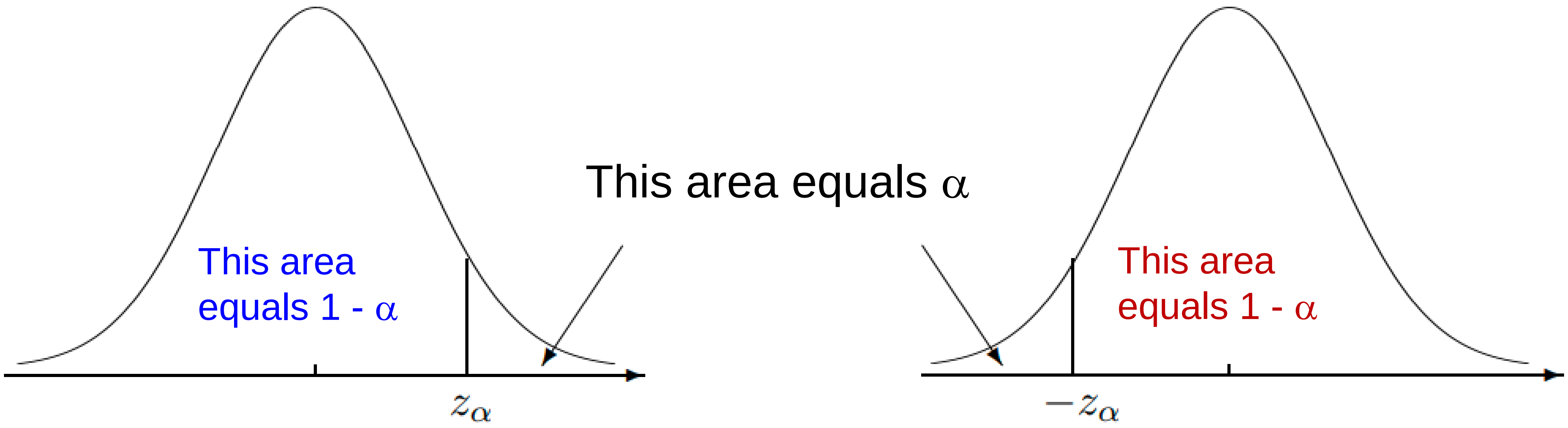
Suppose that we wanted to determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut at 60°C has a length of at most 1.0J.

Since the bound on error in estimation E is one-half of the length of the CI, that is, $E \leqslant ½$, to determine n we use $n = \lceil (\frac{z_{\alpha/2}\sigma}{E})^2 \rceil$ with E = ½, $z_{\alpha/2} = 1.96$.

$$\left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left[\frac{(1.96)1}{0.5}\right]^2 = 15.37$$

The required sample size is 16.

One-Sided Confidence Bounds



100(1-α)% lower-confidence bound for μ is

$$\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \infty$$

100(1-α)% upper-confidence bound for μ is

$$-\infty \leqslant \mu \leqslant \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Confidence Interval on the Mean Unknown σ^2 ← Large sample size

- What if σ is unknown? We instead use the interval

$$\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}$$

where s (used to estimate σ) is the sample standard deviation and n is large enough

- What if n is small?

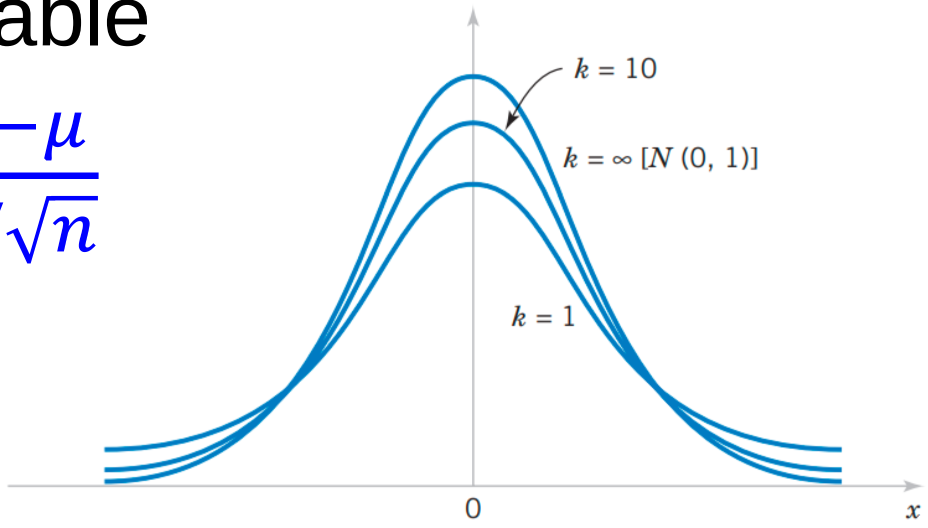
t distribution

- Let X_1, X_2, … , X_n be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

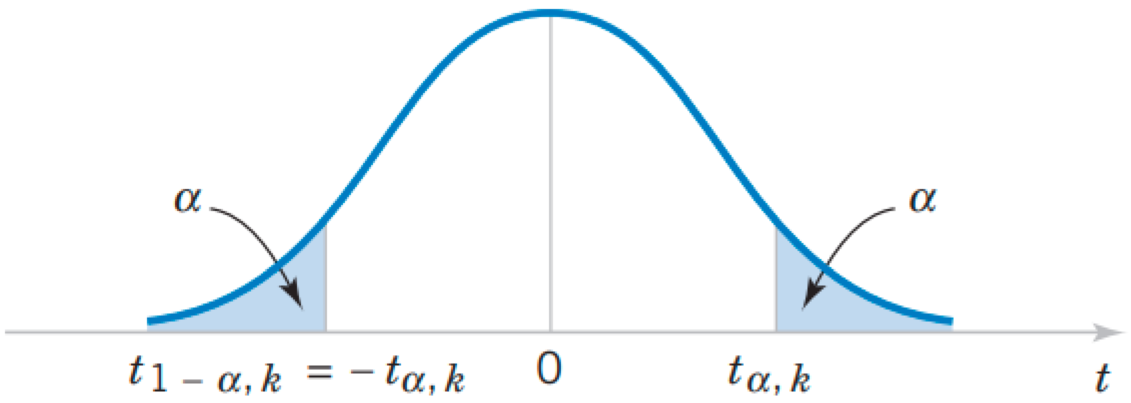
has a t distribution

with n - 1 degrees of freedom.



Probability density functions of several t distributions

t Confidence Interval on μ



Percentage points of the t distribution

$$P\left(-t_{\alpha/2,n-1} \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{\alpha/2,n-1}\right) = 1-\alpha$$

$$\boxed{\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} \leqslant \mu \leqslant \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}}$$

Remark. One-sided confidence bounds on the mean are found by replacing $t_{\alpha/2,n-1}$ with $t_{\alpha,n-1}$.

t Confidence Interval on μ - Example

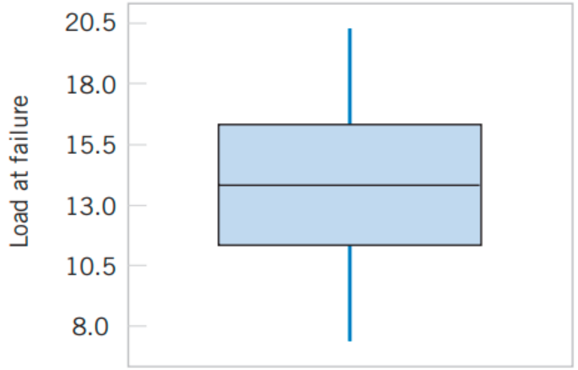
An article in the journal Materials Engineering (1989, Vol. II, No. 4, pp. 275–281) describes the results of tensile adhesion tests on 22 U-700 alloy specimens. The load at specimen failure is as follows (in megapascals):

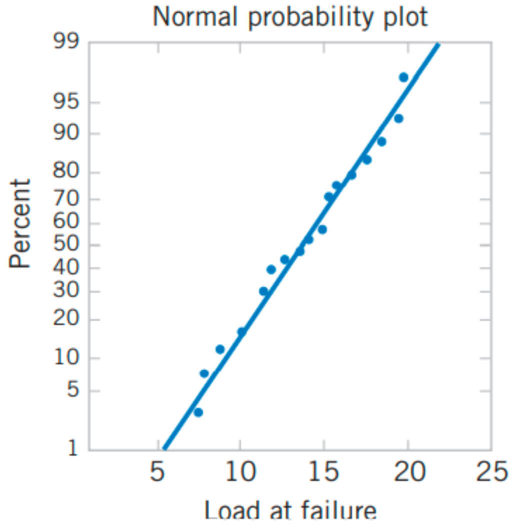
19.8 10.1 14.9 7.5 15.4 15.4

15.4 18.5 7.9 12.7 11.9

11.4 11.4 14.1 17.6 16.7

15.8 19.5 8.8 13.6 11.9 11.4





The sample mean is $\bar{x} = 13.71$, and the sample standard deviation is $s = 3.55$.

We want to find a 95% CI on μ. Since n = 22, we have n - 1 = 21 degrees of freedom for t, so $t_{0.025,21} = 2.080$.

$$\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} \leqslant \mu \leqslant \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} \rightarrow 12.14 \leqslant \mu \leqslant 15.28$$

t Confidence Interval on μ - Example

$$12.14 \leqslant \mu \leqslant 15.28$$

- Practical Interpretation: The CI is fairly wide because there is *a lot of variability* in the tensile adhesion test measurements. A *larger sample size* would have led to a shorter interval.

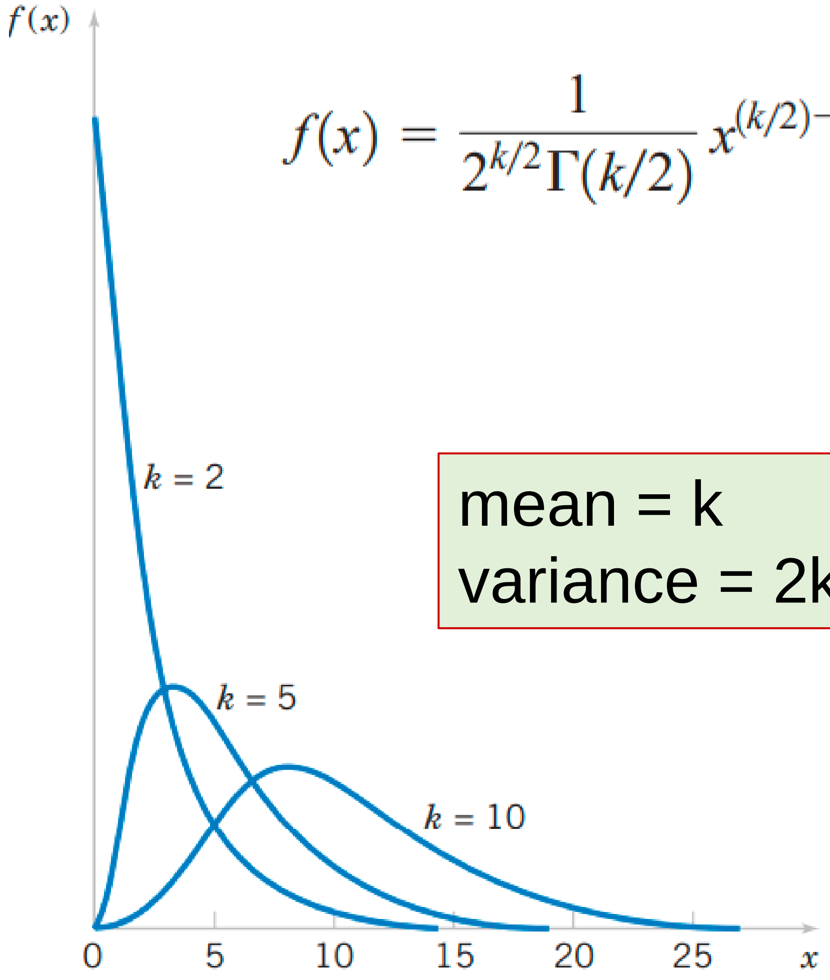
Confidence Interval for σ^2 of a Normal Distribution

χ^2 Distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2, and let S^2 be the sample variance. Then the random variable

$$X^2 = \frac{(n-1)S^2}{\sigma^2}$$

has a chi-square (χ^2) distribution with n - 1 degrees of freedom.

χ^2 distributions

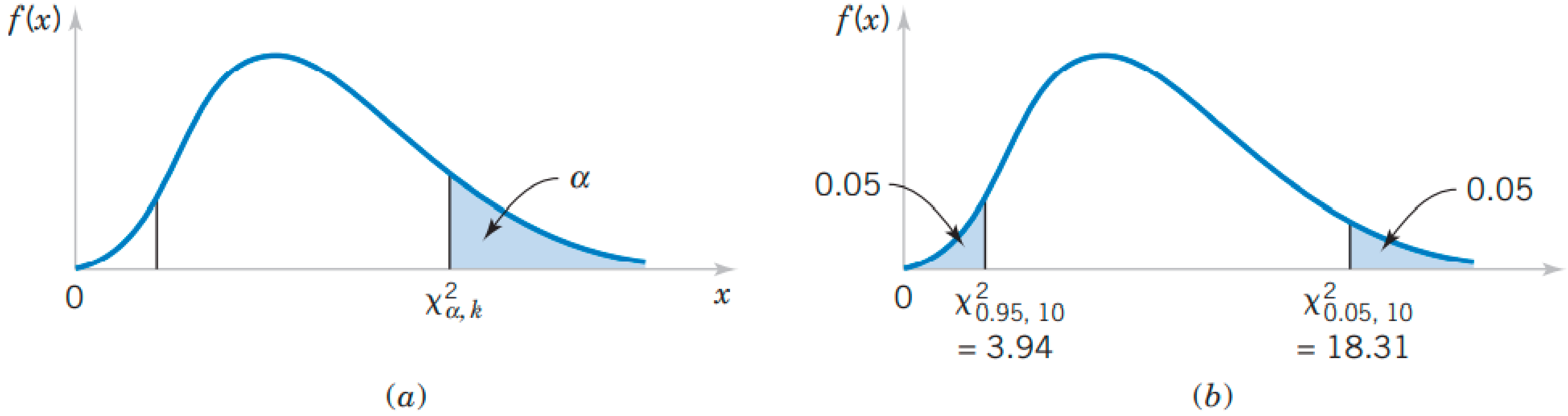


$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2} \qquad x > 0$$

mean = k
variance = 2k

Pdf of several χ^2 distributions.
➔ skewed to the right

Percentage point of the χ^2 distribution.



(a) The percentage point $\chi^2_{\alpha,k}$.
(b) The upper percentage point $\chi^2_{0.05,10} = 18.31$ and the lower percentage point $\chi^2_{0.95,10} = 3.94$

Confidence Interval on the Variance σ^2

$$X^2 = \frac{(n-1)S^2}{\sigma^2}$$

$$P\left(\chi^2_{1-\alpha/2,n-1} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{\alpha/2,n-1}\right) = 1-\alpha$$

$$P\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right) = 1-\alpha$$

If s^2 is the sample variance from a random sample of n observations from a normal distribution with unknown variance σ^2, then a 100(1 - α)% CI on σ^2 is

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

Challenge

Find the One-Sided Confidence Bounds on the Variance

CI on the Variance σ^2 - Example

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of s^2 0.0153 (fluid ounce)^2. If the variance of fill volume is too large, an unacceptable proportion of bottles will be under- or overfilled. We will assume that the fill volume is approximately normally distributed. A 95% upper confidence bound is

$$\sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{0.95,19}}$$

$$\sigma^2 \leq \frac{(19)0.0153}{10.117} = 0.0287$$

$$\sigma \leq 0.17$$

Practical Interpretation: At the 95% level of confidence, the data indicate that the process standard deviation could be as large as 0.17 fluid ounce. The process engineer or manager now needs to determine if a standard deviation this large could lead to an operational problem with under-or over filled bottles.

Normal Approximation for a Binomial Proportion

- p: a population proportion
- $\hat{P}$ = X/n: a point estimator of p
- When n is large enough, $X/n \sim \text{Normal}(\text{mean} = p, \text{variance} = p(1-p)/n)$, if p is not too close to either 0 or 1.
- Requirement for approximation: $np,\ n(1 - p) \geqslant 5$.

If n is large, the distribution of

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximately standard normal.

Approximate Confidence Interval on a Binomial Proportion

$$P(-z_{\alpha/2} \le Z \le z_{\alpha/2}) \simeq 1 - \alpha$$

$$P\left(-z_{\alpha/2} \le \frac{\hat{P} - p}{\sqrt{\dfrac{p(1-p)}{n}}} \le z_{\alpha/2}\right) \simeq 1 - \alpha$$

If $\hat{p}$ is the proportion of observations in a random sample of size n that belongs to a class of interest, an approximate 100(1- α)% CI on the proportion p of the population that belongs to this class is

$$\hat{p} - z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

Approximate Confidence Interval on a Binomial Proportion - Example

- In a random sample of 85 automobile engine crankshaft bearings,10 have a surface finish that is rougher than the specifications allow. Therefore, a point estimate of the proportion of bearings in the population that exceeds the roughness specification is $\hat{p} = x/n = 10/85 = 0.12$. A 95% two-sided confidence interval for p is

$$\hat{p} - z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\boxed{0.05 \leq p \leq 0.19}$$

$$0.12 - 1.96\sqrt{\frac{0.12(0.88)}{85}} \leq p \leq 0.12 + 1.96\sqrt{\frac{0.12(0.88)}{85}}$$

Choice of Sample Size

- Error = E := $|p - \hat{P}|$.
- $E \leqslant z_{\alpha/2}\sqrt{p(1-p)/n}$

$$\Rightarrow n \geqslant \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1\text{-}p)$$

Note that $p(1 - p) \leqslant 0.25$

So, n can be chosen using

$$n \geqslant \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25)$$

(In practice, use $\hat{p}$ as an estimate of p in this formula)

Example. In a random sample of 85 automobile engine crankshaft bearings,10 have a surface finish that is rougher than the specifications allow. How large a sample is required if we want to be 95% confident that the error in using $\hat{p}$ to estimate p is less than 0.05?

$$n \geqslant \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1\text{-}p) = \left(\frac{1.96}{0.05}\right)^2 0.12(1 - 0.12) = 163$$

Or $n \geqslant \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25) = 385$

Practical Interpretation: if we have information concerning the value of p, we could use a smaller sample

One-Sided Confidence Bounds

The approximate $100(1 - \alpha)\%$ lower and upper confidence bounds are

$$\hat{p} - z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \qquad \text{and} \qquad p \le \hat{p} + z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

respectively.

Summary

- CONFIDENCE INTERVAL ON THE MEAN OF A NORMAL DISTRIBUTION, VARIANCE KNOWN
- CONFIDENCE INTERVAL ON THE MEAN OF A NORMAL DISTRIBUTION, VARIANCE UNKNOWN
- CONFIDENCE INTERVAL ON THE VARIANCE AND STANDARD DEVIATION OF A NORMAL DISTRIBUTION
- LARGE-SAMPLE CONFIDENCE INTERVAL FOR A POPULATION PROPORTION

$$(3x + 17) \bmod 26 + 46$$

$$(5x + 17) \bmod 26 + 46$$

$$(7x + 17) \bmod 26 + 46$$