Chapter 3: Discrete Random Variables and Probability Distributions

LEARNING OBJECTIVES

1. Discrete random variables
2. Probability mass function and cumulative distribution function
3. Mean and Variance
4. Discrete Uniform Distribution
5. Binomial Distribution
6. Geometric and Negative Binomial Distribution
7. Hypergeometric Distribution
6. Poisson Distribution

Definition

A **discrete** random variable is a random variable with a finite or countably infinite range.

Example

1. Roll a die twice: Let X be the number of times 4 comes up then X could be 0, 1, or 2 times.

2. Toss a coin 5 times: Let X be the number of heads then $X = 0, 1, 2, 3, 4$, or 5.

3. X = The number of stocks in the Dow Jones Industrial Average that have share price increases on a given day then X is a discrete random variable because whose share price increases can be counted.

Determining a Discrete Random Variable

Let X be a discrete random variable with possible outcomes $x_1, x_2, \ldots, x_n$.

1. Find the probability of each possible outcome.

2. Check that each probability is between 0 and 1 and that the sum is 1.

3. Summarizing results in following table:

$\boldsymbol{X}$	x_1	x_2	....	x_n
$\boldsymbol{P(x)}$	p_1	p_2	....	p_n

Example

Let the random variable X denote the number of heads in three tosses of a fair coin. Determine the probability distribution of X.

The sample space:

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

The events: $[X = 0] = \{TTT\}$ $\quad [X = 1] = \{HTT, THT, TTH\}$

$[X = 2] = \{HHT, HTH, THH\}$ $\quad [X = 3] = \{HHH\}$

X	0	1	2	3
P(x)	1/8	3/8	3/8	1/8

Definition

For a discrete random variable X with possible values x_1, x_2, …, x_n , a **probability mass function** is a function such that

(1) $f(x_i) \geq 0$

(2) $\sum_{i=1}^{n} f(x_i) = 1$

(3) $f(x_i) = P(X = x_i)$

In above example, we have

$f(0) = 1/8$, $f(1) = 3/8$, $f(2) = 3/8$ and $f(3) = 1/8$.

Cumulative distribution function

Definition

The **cumulative distribution function** of a discrete random variable X, denoted as $F(x)$, is

$$F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i)$$

For a discrete random variable X, F(x) satisfies the following properties.

(1) $0 \le F(x) \le 1$

(2) If $x \le y$, then $F(x) \le F(y)$

Cumulative distribution function

Example

Suppose that a day's production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample.

What is the cumulative distribution function of X?

The first we find the probability mass function of X.

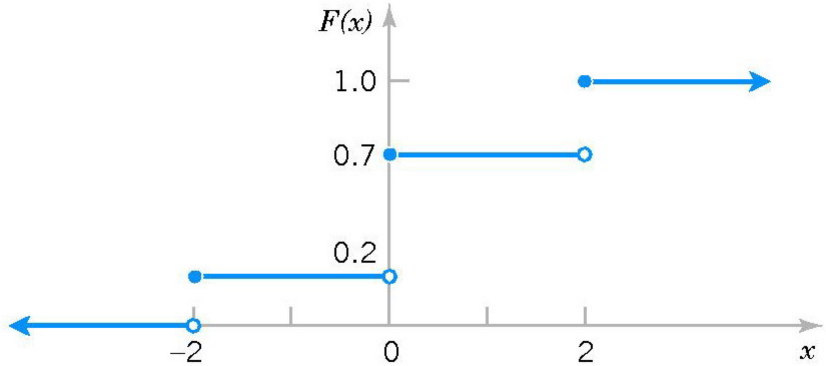
X	0	1	2
$f(x)$	0.886	0.111	0.003

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.886 & \text{if } 0 \le x < 1 \\ 0.997 & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$$

Example

Determine the probability mass function of X from the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.2 & -2 \le x < 0 \\ 0.7 & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



$f(-2) = 0.2 - 0 = 0.2$

$f(0) = 0.7 - 0.2 = 0.5$

$f(2) = 1.0 - 0.7 = 0.3$

Definition

The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$ is

$$\mu = E(X) = \sum_i x_i f(x_i)$$

The **variance** of X, denoted as σ^2 or $V(X)$ is

$$\sigma^2 = V(X) = E(X - \mu)^2$$
$$= \sum_i x_i^2 f(x_i) - \mu^2$$

The **standard deviation** of X is σ.

Example

The number of messages sent per hour over a computer network has the following distribution:

X	10	11	12	13	14	15
$f(x)$	0.08	0.15	0.30	0.20	0.20	0.07

Determine the mean and standard deviation of the number of messages sent per hour.

$$E(X) = 10(0.08) + 11(0.15) + \cdots + 15(0.07) = 12.5$$

$$V(X) = 10^2(0.08) + 11^2(0.15) + \cdots + 15^2(0.07) - 12.5^2 = 1.85$$

$$\sigma = \sqrt{V(X)} = \sqrt{1.85} = 1.36$$

D.R. variables
PMF & CDF
Mean & Variance
Uniform Dis.
Binomial Dis.
Poisson Dis.
Summary

Mean of a Function of a Discrete Random Variable

If X is a discrete random variable with probability mass function $f(x)$ then for any function $h(x)$

$$E[h(x)] = \sum_i h(x_i) f(x_i)$$

Corollary

$$E(aX + b) = aE(X) + b$$

$$V(aX + b) = a^2 V(X)$$

Definition

A random variable X has a **discrete uniform distribution** if each of the n values in its range, say, $x_1, x_2, \ldots, x_n$ has equal probability. Then,

$$f(x_i) = 1/n$$

Mean and Variance

Suppose X is a discrete uniform random variable on the consecutive integers $a, a+1, \ldots, b$ for $a \leq b$. The mean and variance of X

$$\mu = E(X) = (a + b)/2 \qquad \sigma^2 = \frac{(b-a+1)^2 - 1}{12}$$

Definition

A random experiment consists of n Bernoulli trials such that

(1) The trials are independent

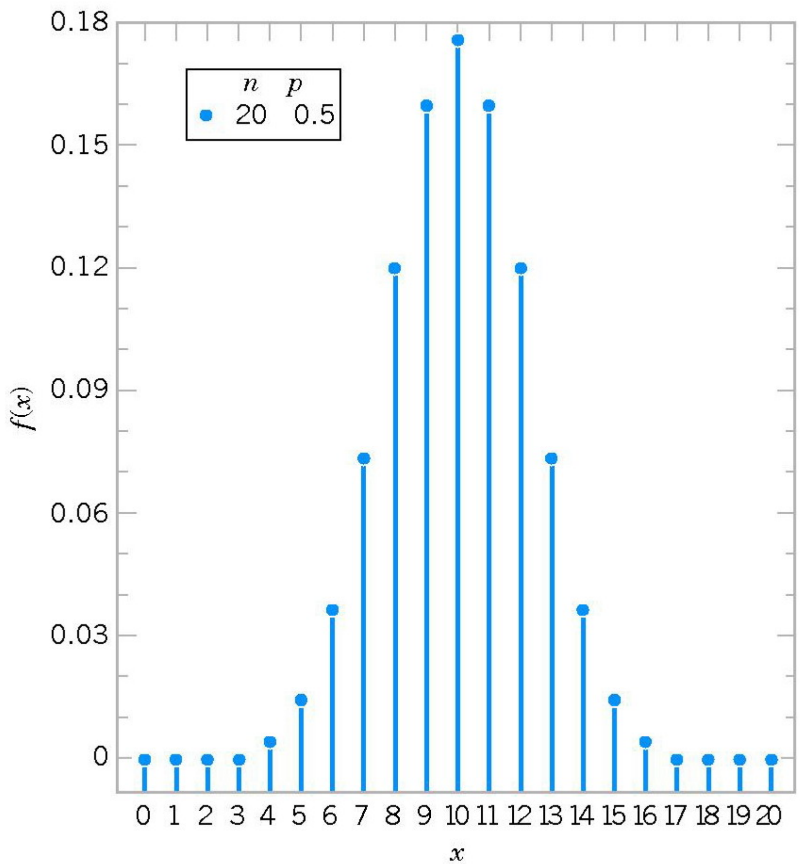
(2) Each trial results in only two possible outcomes, labeled as "success" and "failure"

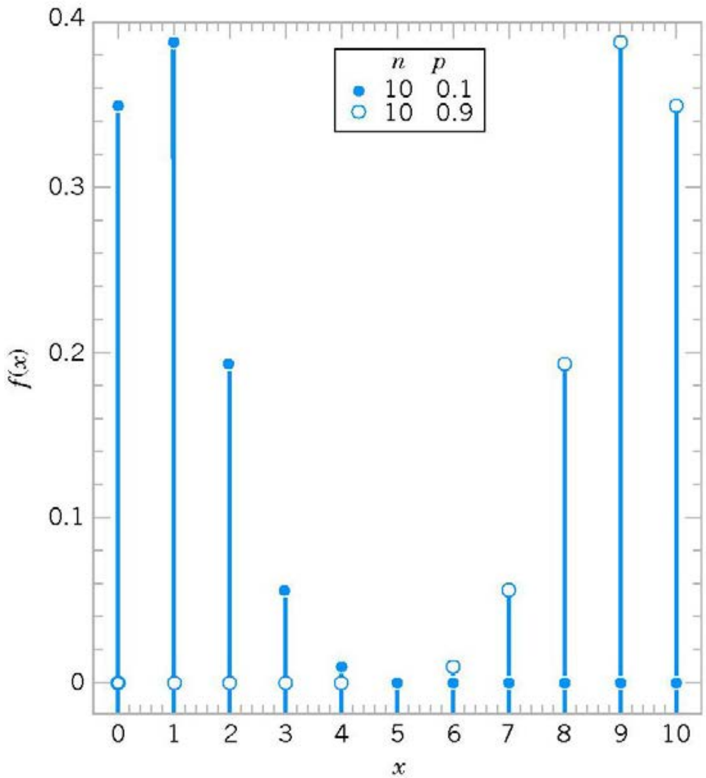
(3) The probability of a success in each trial, denoted as p, remains constant

The random variable X that equals the number of trials that result in a success has a **binomial random variable** with parameters $0<p<1$ and n=1,2,…The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n.$$

Binomial Distribution





Binomial distributions for selected values of n and p.

D.R. variables
PMF & CDF
Mean & Variance
Uniform Dis.
Binomial Dis.
Poisson Dis.
Summary

Mean and Variance

$$\mu = E(X) = np \qquad \sigma^2 = V(X) = np(1-p)$$

Example

Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant.

(a) Find the probability that in the next 18 samples, exactly 2 contain the pollutant.

(b) Determine the probability that at least four samples contain the pollutant.

(c) Determine the probability that $3 \leq X < 7$.

Solution. Let X = the number of samples that contain the pollutant in the next 18 samples analyzed. Then X is a binomial random variable with p = 0.1 and n = 18.

(a) $$P(x=2)=\binom{18}{2}0.1^2(1-0.1)^{18-2}=0.284$$

(b) $$P(x\geq 4)=\sum_{x\geq 4}\binom{18}{x}0.1^x(1-0.1)^{18-x}=0.098$$

(c) $$P(3\leq x<7)=\sum_{x=3}^{6}\binom{18}{x}0.1^x(1-0.1)^{18-x}=0.265$$

Definition

A random experiment consists of n Bernoulli trials such that

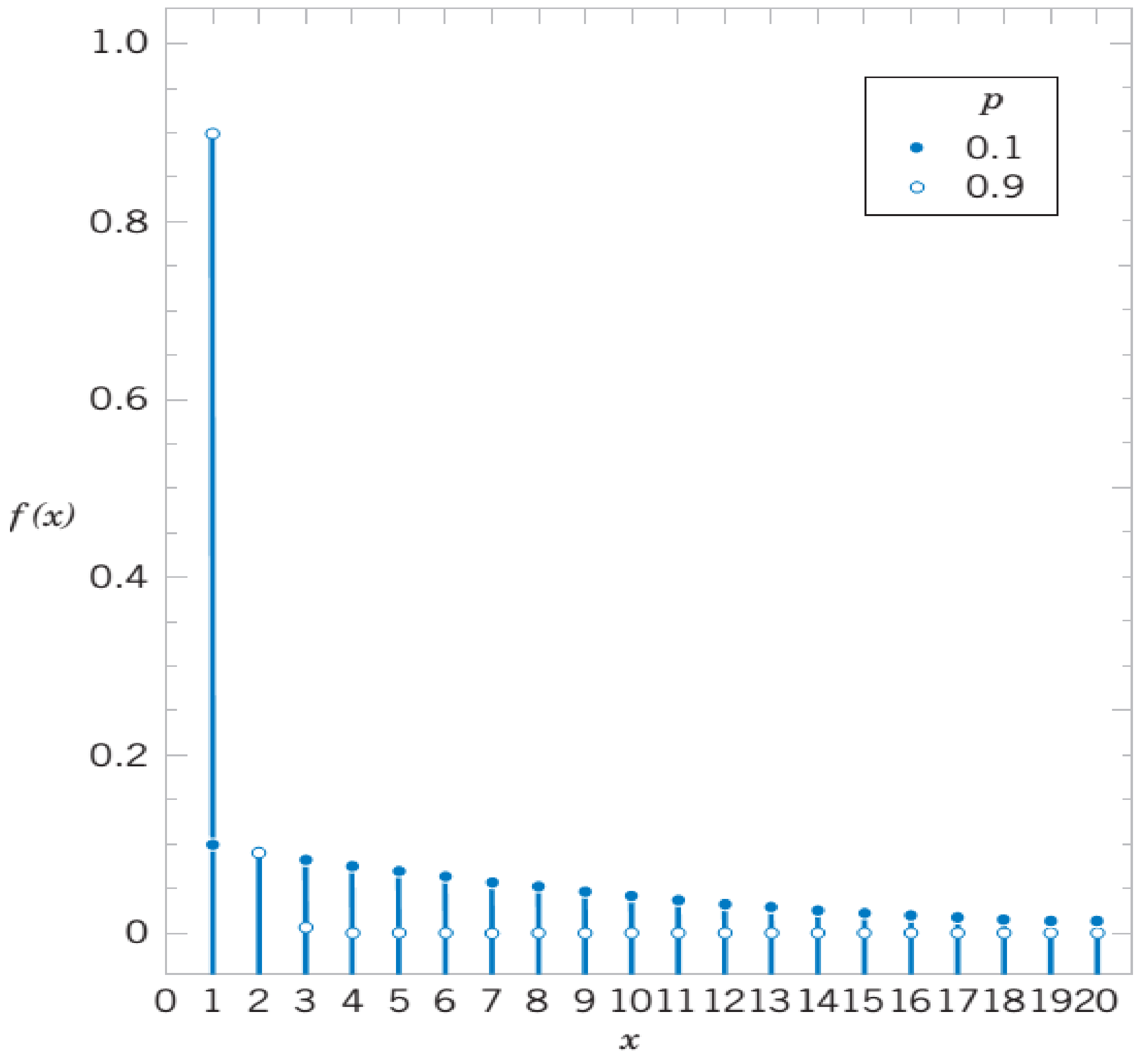
(1) The trials are independent

(2) Each trial results in only two possible outcomes, labeled as "success" and "failure"

(3) The probability of a success in each trial, denoted as p, remains constant

Let the random variable X denote the number of trials until the first success. Then X is a **geometric random variable** with

$$f(x) = (1-p)^{x-1}p \qquad x = 1, 2, \ldots$$


1.0
0.8
0.6
0.4
0.2
0
f(x)
p
0.1
0.9
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
x

Mean and Variance

If X is a geometric random variable with parameter p,

$$\mu = E(X) = 1/p \qquad \text{and} \qquad \sigma^2 = V(X) = (1-p)/p^2$$

EXAMPLE 3-20 Digital Channel

The probability that a bit transmitted through a digital transmission channel is received in error is 0.1. Assume the transmissions are independent events, and let the random variable. ***X*** **denote the number of bits transmitted** ***until*** **the first error.**

$$P(X = 5) = P(OOOOE) = 0.9^4 0.1 = 0.066$$

E(X)-1/0.1=10 and $\sigma = [(1 - 0.1)/0.1^2]^{1/2} = 9.49$

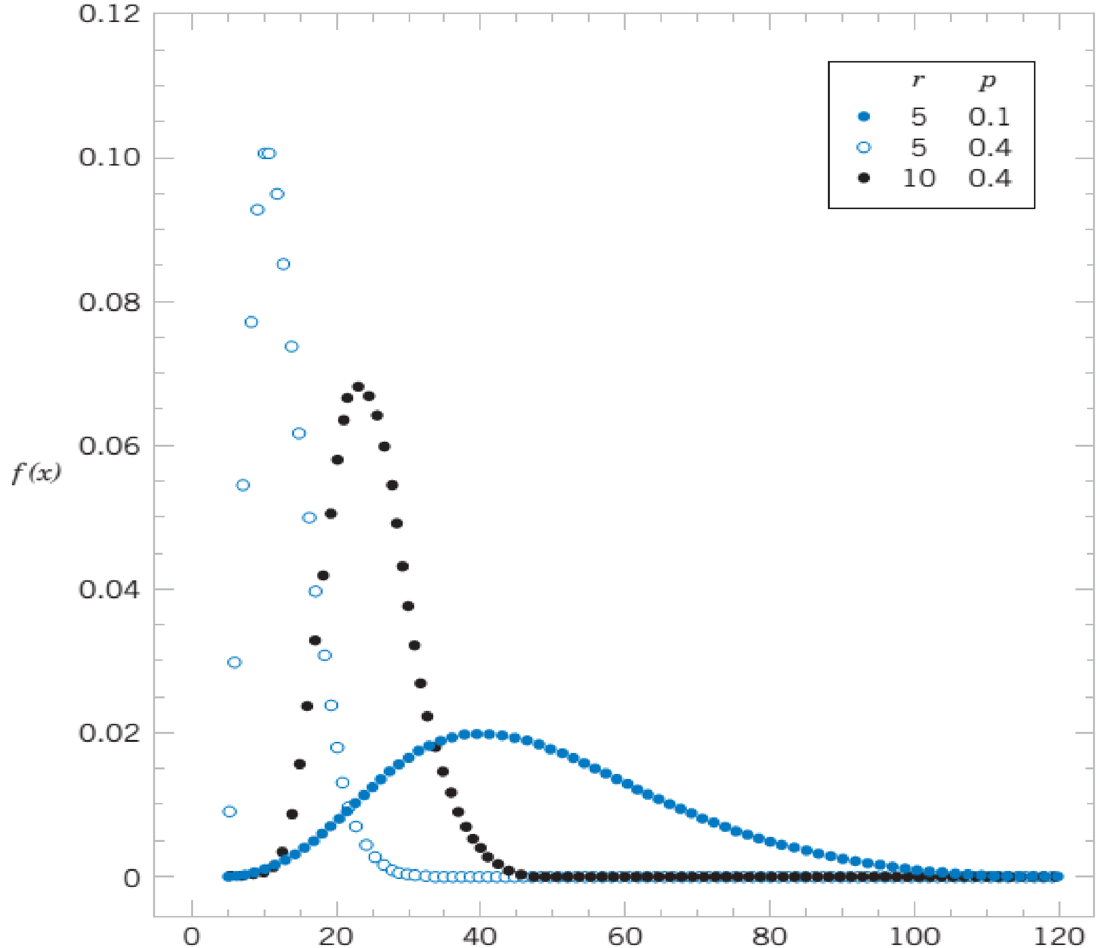
Definition

In a series of Bernoulli trials (independent trials with constant probability ***p*** of a success), let the random variable ***X*** denote the number of trials until ***r*** successes occur. Then ***X*** is a **negative binomial random variable** with parameters **0 < p < 1** and ***r*** **= 1, 2, 3, …**

The pmf:

$$f(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r \qquad x = r, r+1, r+2, \ldots$$

Negative Binomial Distribution



Mean and Variance

If X is a geometric random variable with parameter p,

$$\mu = E(X) = 1/p \qquad \text{and} \qquad \sigma^2 = V(X) = (1-p)/p^2$$

EXAMPLE 3-20 Digital Channel

The probability that a bit transmitted through a digital transmission channel is received in error is 0.1. Assume the transmissions are independent events, and let the random variable. ***X* denote the number of bits transmitted *until* the first error.**

$$P(X = 5) = P(OOOOE) = 0.9^4 0.1 = 0.066$$

E(X)=1/0.1=10 and $\sigma = [(1-0.1)/0.1^2]^{1/2} = 9.49$

Mean and Variance

If X is a negative binomial random variable with parameters p and r,

$$\mu = E(X) = r/p \qquad \text{and} \qquad \sigma^2 = V(X) = r(1-p)/p^2$$

EXAMPLE 3-25 Web Servers

A Web site contains three identical computer servers. Only one is used to operate the site, and the other two are spares that can be activated in case the primary system fails. The probability of a failure in the primary computer (or any activated spare system) from a request for service is 0.0005. Assuming that each request represents an independent trial, what is the mean number of requests until failure of all three servers?

$$E(X) = 3/0.0005 = 6000 \text{ requests}$$

EXAMPLE 3-25 Web Servers

What is the probability that all three servers fail within five requests? The probability is $P(X \leq 5)$ and because X denotes the number of requests to the third failure $P(X \leq 2) = 0$. Therefore,

$$\begin{aligned} P(X \leq 5) &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= 0.0005^3 + \binom{3}{2} 0.0005^3(0.9995) \\ &\quad + \binom{4}{2} 0.0005^3\,(0.9995)^2 \\ &= 1.25 \times 10^{-10} + 3.75 \times 10^{-10} + 7.49 \times 10^{-10} \\ &= 1.249 \times 10^{-9} \end{aligned}$$

Definition

A set of N objects contains

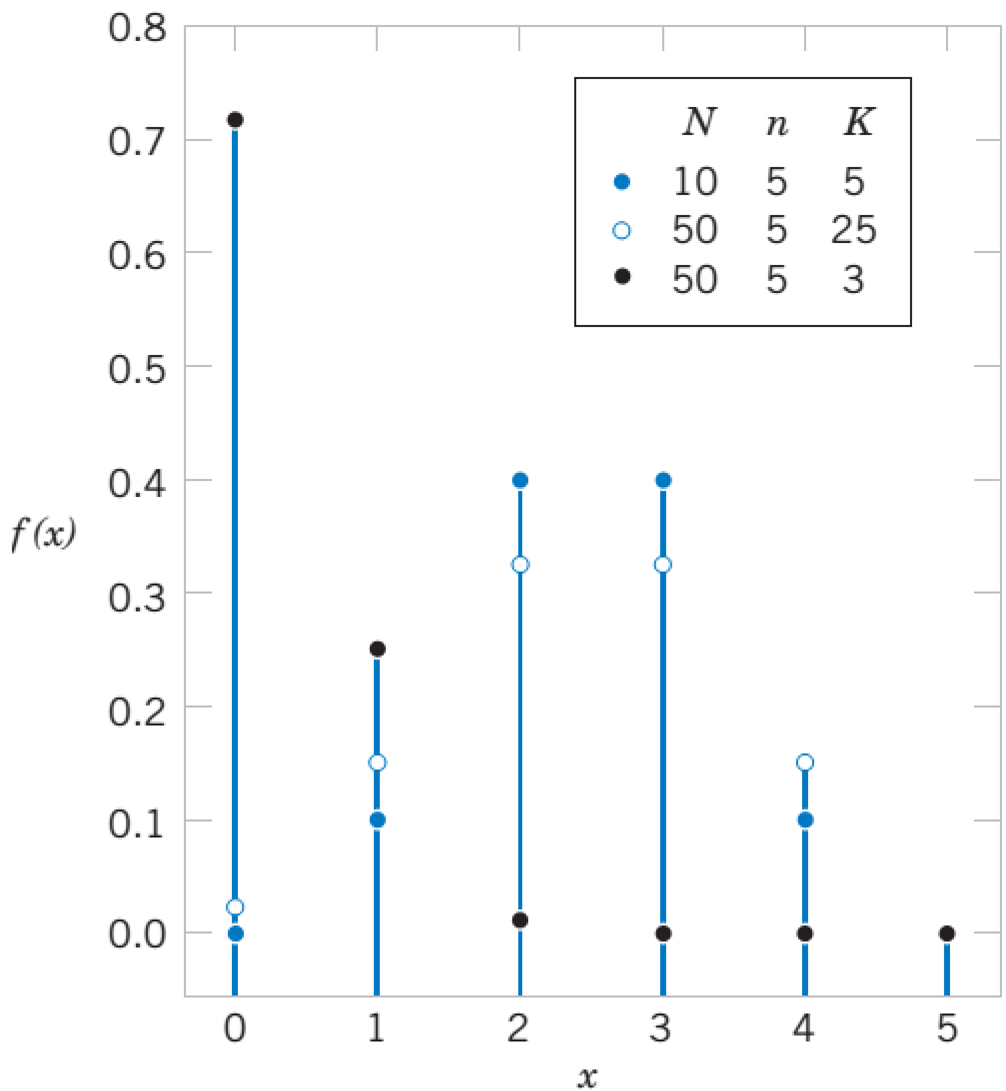
K objects classified as successes

$N - K$ objects classified as failures

A sample of size n objects is selected randomly (without replacement) from the N objects, where $K \le N$ and $n \le N$.

Let the random variable X denote the number of successes in the sample. Then X is a **hypergeometric random variable** and

$$f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \qquad x = \max\{0, n + K - N\} \text{ to } \min\{K, n\}$$


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
f(x)
0
1
2
3
4
5
x
N n K
10 5 5
50 5 25
50 5 3

EXAMPLE 3-27 Parts from Suppliers

A batch of parts contains 100 parts from a local supplier of tubing and 200 parts from a supplier of tubing in the next state. If four parts are selected randomly and without replacement, what is the probability they are all from the local supplier?

Let *X* equal the number of parts in the sample from the local supplier. Then, *X* has a hypergeometric distribution and the requested probability is P(X=4). Consequently,

$$P(X=4) = \frac{\binom{100}{4}\binom{200}{0}}{\binom{300}{4}} = 0.0119$$

Mean and Variance

If X is a hypergeometric random variable with parameters N, K, and n, then

$$\mu = E(X) = np \qquad \text{and} \qquad \sigma^2 = V(X) = np(1-p)\left(\frac{N-n}{N-1}\right) \tag{3-14}$$

where $p = K/N$.

EXAMPLE 3-28

In the previous example, the sample size is four. The random variable X is the number of parts in the sample from the local supplier. Then, $p = 100/300 = 1/3$. Therefore,

$$E(X) = 4(100/300) = 1.33$$

and

$$V(X) = 4(1/3)(2/3)[(300 - 4)/299] = 0.88$$

Definition: Poisson Process and Distribution

Consider an interval *T* of real numbers partitioned into subintervals Δt of small length and assume that Δt as tends to zero,

(1) the probability of more than one event in a subinterval tends to zero,

(2) the probability of one event in a subinterval tends to $\lambda\Delta t/T$,

(3) the event in each subinterval is independent of other subintervals

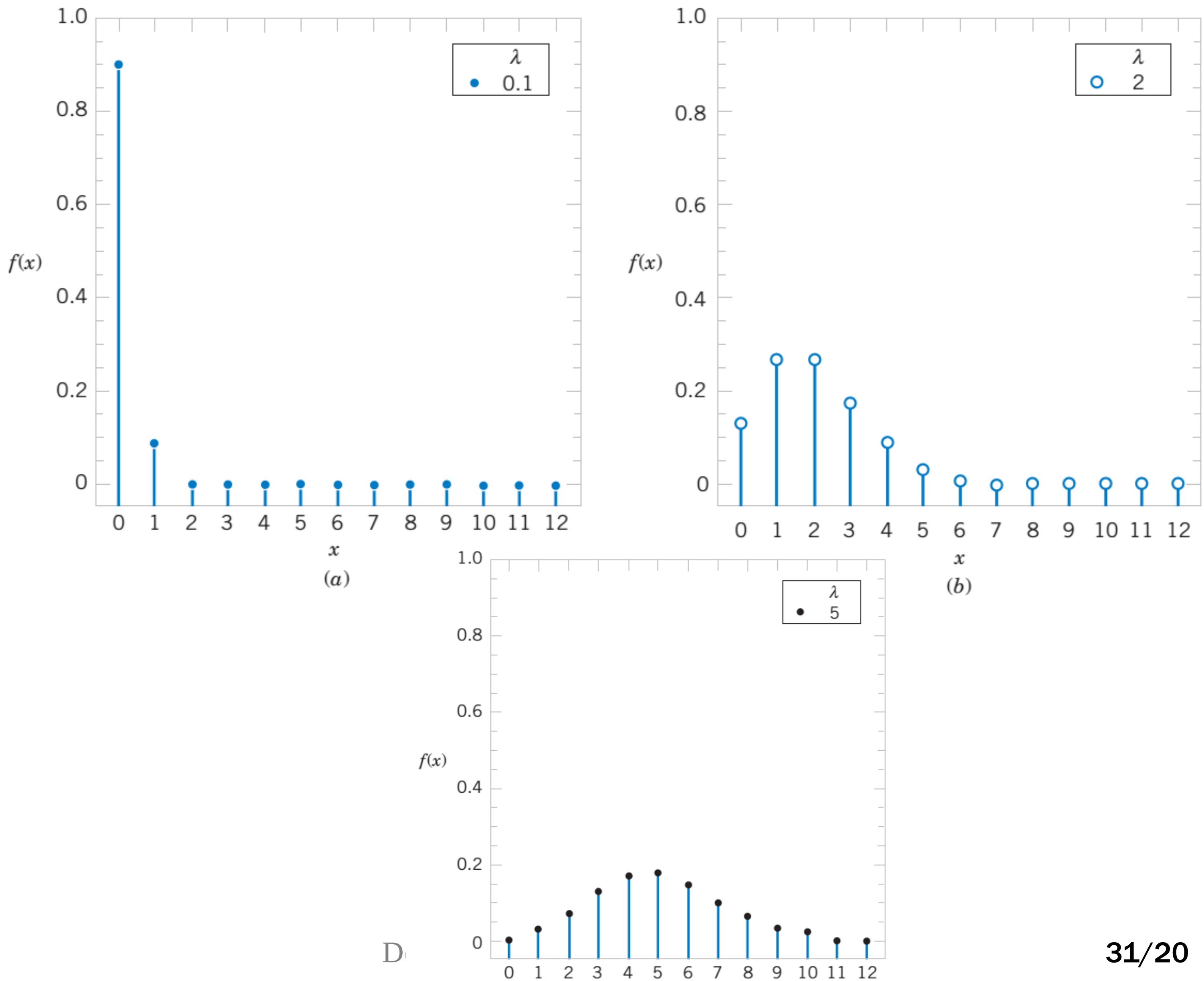
A random experiment with these properties is called a **Poisson process**.

Definition: Poisson Distribution

The random variable X that equals the number of events in a Poisson process is a **Poisson random variable** with parameter $\lambda > 0$, and the probability mass function of X is

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0, 1, 2, \ldots$$

Poisson Distribution



Mean and Variance

If X is a Poisson random variable with parameter λ, then

$$\mu = E(X) = \lambda \qquad \sigma^2 = V(X) = \lambda$$

Example

For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per millimeter. Determine the probability of exactly 2 flaws in 1 millimeter of wire.

Determine the probability of at least 1 flaw in 2 millimeters of wire.

Example

For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per millimeter.
(1) Determine the probability of exactly 2 flaws in 1 millimeter of wire.

(2) Determine the probability of at least 1 flaw in 2 millimeters of wire.

Sol: Let *X*, *Y* denote the variables of number of flaws in 1,2 millimeters of wire. Then, *X*, *Y* has a Poisson distribution with $\lambda=2.3$ and $\lambda=4.6$

$$> Ps := \frac{\exp(1)^{-a} \cdot a^{x}}{x!}:$$

$$> subs(x = 2, a = 2.3, Ps) : evalf(\%)$$

$$0.2651846419$$

$$> 1 - subs(x = 0, a = 4.6, Ps) : evalf(\%)$$

$$0.9899481642$$

We have studied:

1. Discrete random variables
2. Probability mass function and cumulative distribution function
3. Mean and Variance
4. Discrete Uniform Distribution
5. Binomial Distribution
6. Geometric and Negative Binomial Distribution
7. Hypergeometric Distribution
8. Poisson Distribution