Florida Atlantic University **Instructor: Hongwei Long**

STA4032 Midterm Exam

July 11, 2012

No books, notes or other aids, except calculators and the provided formula sheet, are allowed.

1. A batch contains 36 bacteria cells. Assume that 12 of the cells are not capable of cellular replication and 24 of the cells are able to replicate. Six cells are selected at random, without replacement, to be checked for replication.

(5) (a) What is the probability that all six cells of the selected cells are able to replicate?

(5) (b) What is the probability that at least one of the selected cells is able to replicate?

(5) (c) What is the probability that at least one of the selected cells is not capable of replication?

(5) (d) What is the mean and variance of the number of cells in the sample that can replicate?

Solution. (a) Let X be the number of cells in the sample that are able to replicate. Then, X has a hypergeometric distribution with $K = 24, N = 36, n = 6$. The desired probability is

$$P(X = 6) == \frac{\binom{24}{6} \cdot \binom{12}{0}}{\binom{36}{6}} = 0.069.$$

(b)

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{24}{0} \cdot \binom{12}{6}}{\binom{36}{6}} = 0.9995.$$

(c) Note that the event that there are at least one of the selected cells is not capable of replication is equivalent to the event that at most 5 of the seleced cells in the sample are able to replicate, i.e. $\{X \leq 5\}$. So, we have

$$P(X \leq 5) = 1 - P(X = 6) = 1 - 0.069 = 0.931.$$

(d) $E(X) = n\frac{K}{N} = 6 \cdot \frac{24}{36} = 4$ and

$$\sigma^2 = V(X) = 6 \cdot \left(\frac{24}{36}\right)\left(1 - \frac{24}{36}\right)\left(\frac{30}{35}\right) = 1.1428.$$

2. A factory produces its products with three machines. Machine I, II, and III produces 50%, 30%, and 20% of the products, but 4%, 2%, and 4% of their products are defective, respectively.

(10) (a) What is the probability that a randomly selected product is defective?

(5) (b) If a randomly selected product was found to be defective, what is the probability that this product was produced by machine I?

(5) (c) If a randomly selected product was found to be good, what is the probability that this product was produced by machine I?

Solution. (a) Let I, II, and III denote the events that the selected product is produced by machine I, II, and III, respectively. Let D be the event that the selected product is defective. Then, $P(I) = 0.5, P(II) = 0.3, P(III) = 0.2, P(D|I) = 0.04, P(D|II) = 0.02, P(D|III) = 0.04$. So, by the total probability rule, we have

$$\begin{aligned} P(D) &= P(D|I)P(I) + P(D|II)P(II) + P(D|III)P(III) \\ &= (0.04)(0.50) + (0.02)(0.30) + (0.04)(0.20) = 0.034. \end{aligned}$$

(b) By Bayes' theorem, we find

$$P(I|D) = \frac{P(D|I)P(I)}{P(D)} = \frac{(0.04)(0.50)}{0.034} = 0.5882.$$

(c) By Bayes' theorem, we find

$$P(I|D') = \frac{P(D'|I)P(I)}{P(D')} = \frac{(1-0.04)(0.50)}{1-0.034} = 0.4969.$$

3. Let the probability mass function of X be given by

$$f(x) = \frac{2x-1}{16}, \quad x = 1, 2, 3, 4.$$

(10) (a) Find the cumulative distribution function $F(x)$ of X.
(10) (b) Find the mean, variance, and standard deviation of X.
Solution. (a) It is easy to find that $F(1) = f(1) = 1/16$, $F(2) = F(1) + f(2) = 1/16 + 3/16 = 1/4$, $F(3) = F(2) + f(3) = 1/4 + 5/16 = 9/16$, and $F(4) = F(3) + f(4) = 9/16 + 7/16 = 1$. So, the cumulative distribution function is given by

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{16}, & 1 \le x < 2 \\ \frac{1}{4}, & 2 \le x < 3 \\ \frac{9}{16}, & 3 \le x < 4 \\ 1, & x \ge 4. \end{cases}$$

(b)

$$\mu = E(X) = \sum_{x=1}^{4} x f(x) = 1 \cdot (1/16) + 2 \cdot (3/16) + 3 \cdot (5/16) + 4 \cdot (7/16) = 3.125,$$
$$E(X^2) = 1^2 \cdot (1/16) + 2^2 \cdot (3/16) + 3^2 \cdot (5/16) + 4^2 \cdot (7/16) = 10.625,$$
$$\sigma^2 = E(X^2) - \mu^2 = 10.625 - 3.125^2 = 0.859375.$$
$$\sigma = \sqrt{\sigma^2} = \sqrt{0.859375} = 0.927.$$

4. The probability that your call to a service line is answered in less than 30 seconds is 0.60. Assume that your calls are independent.
(5) (a) If you call 10 times, what is the probability that at least 6 calls are answered in less than 30 seconds?
(5) (b) If you call 10 times, what is the mean number of calls that are answered in less than 30 seconds?
(5) (c) What is the probability that you must call four times to obtain the first answer in less than 30 seconds?
(5) (d) What is the mean number of calls until you are answered in less than 30 seconds?
Solution. (a) Let X be the number of calls in the 10 calls that are answered in less than 30 seconds. Then, X has a binomial distribution with $n = 10$ and $p = 0.60$.
The desired probability is

$$P(X \ge 6) = 1 - P(X \le 5) = 1 - \sum_{x=0}^{5} \binom{10}{x} (0.6)^x (0.4)^{10-x} = 1 - 0.3669 = 0.6331.$$

(b) $E(X) = np = 10 \cdot (0.6) = 6$.
(c) Let Y be the number of calls needed to obtain an answer in less than 30 seconds.

Then Y has a geometric distribution with parameter $p = 0.6$. So the desired probability is $P(Y = 4) = (1 - 0.6)^3(0.6) = 0.0384$.
(d) $E(Y) = 1/p = 1/(0.6) = 1.667$.

5. The number of errors in a textbook follows a Poisson distribution with a mean of 0.1 error per page.
(8) (a) What is the probability that there are three or less errors in 10 pages?
(7) (b) What is the probability of at least one error in 10 pages?
(5) (c) What is the expected number of errors in 100 pages?
Solution. (a) Let X be the number of errors in 10 pages. Then X has a Poisson distribution with $\lambda = (0.1)(10) = 1$. So, the desired probability is

$$\begin{aligned} P(X \leq 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= e^{-1} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!} + \frac{e^{-1}(1)^3}{3!} = 0.9810. \end{aligned}$$

(b) $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1} = 1 - 0.3679 = 0.6321$.
(c) The expected number of errors in 100 pages is equal to $(0.1)(100) = 10$.