2.3 Spanning Trees

Let G be a graph. Recall from Section 1.4 that a subgraph H of G is called a spanning subgraph of G if the vertex set of H is the same as the vertex set of G.

> A **spanning tree** of a graph G is a spanning subgraph of G that is a tree.

Our first result of this section shows that the graphs which have spanning trees are easily described.

Theorem 2.12 *A graph G is connected if and only if it has a spanning tree.*

Proof Suppose that G is connected with n vertices and q edges. Then, by Corollary 2.10, we have $q \geq n-1$. If $q = n-1$ then, by (iii) $\Rightarrow$(i) of Theorem 2.11, G is a tree and so we can take $T = G$ as a spanning tree of G.

If $q > n-1$ then, by Theorem 2.4 (or by (i) $\Rightarrow$ (iii) of Theorem 2.11), G is not a tree and so G must contain a cycle. Let e_1 be an edge of such a cycle. Then the subgraph $G - e_1$ is connected (since e_1 is not a bridge), has n vertices, and has $q-1$ edges. If $q - 1 = \ -1$ then, repeating the above argument gives $T = G - e_1$ as a spanning tree of G.

If $q - 1 > n - 1$ then $G - e_1$ is not a tree so, as before, there is a cycle in $G - e_1$. Removing an edge e_2 from such a cycle gives a subgraph $G - \{e_1, e_2\} = (G - e_1) - e_2$

which is connected, has n vertices and $q-2$ edges. We keep on repeating this process, deleting $q-n+1$ edges altogether, to eventually produce a subgraph T which is connected, has n vertices and $q-(q-n+1)=n-1$ edges. Thus by Theorem 2.11, T is a tree and since it has the same vertex set as G it is a spanning tree of G.

Conversely, if G has a spanning subtree T, then given any two vertices u and v of G then u and v are also vertices of the connected subgraph T. Thus u and v are connected by a path in T and so by a path in G. This shows that G is connected. □

Figures 2.12 and 2.13 illustrate the Theorem.

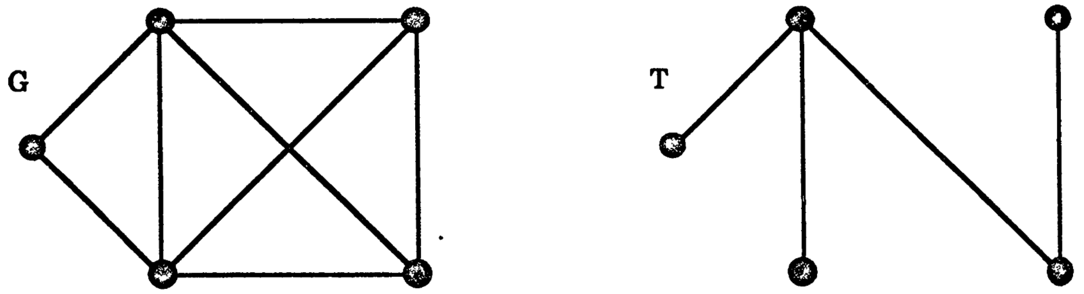

Figure 2.12: A connected graph and a spanning tree.

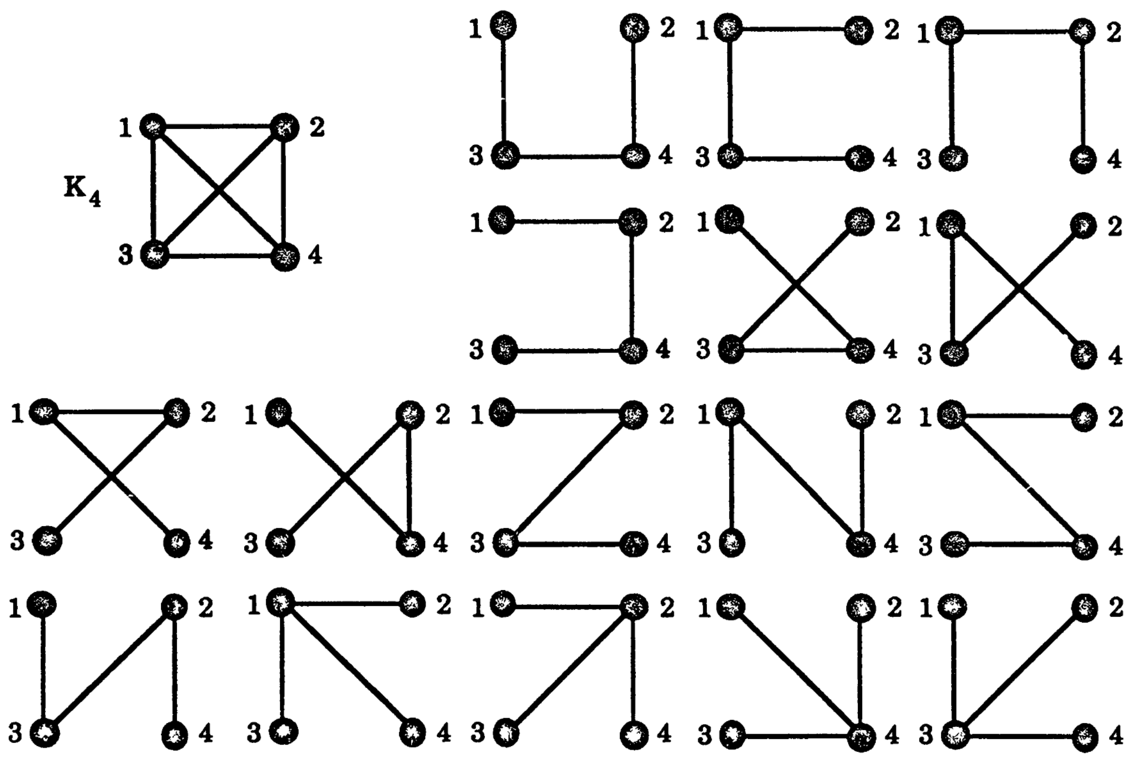

Figure 2.13: K_4 and its 16 different spanning trees.

Note that in the 16 different spanning trees of K_4 shown in Figure 2.13 there are only two non-isomorphic ones — the first 12 shown are isomorphic to each other, while the last four are also isomorphic to each other. K_6, the complete graph on 6 vertices, has 1296 *different* spanning trees, but just 6 *non-isomorphic* ones. Another way of saying this is, given 6 vertices, then there are 1296 different ways of joining these vertices to form a tree if we label the vertices $1, 2, 3, 4, 5, 6$, but if we drop these labels then there are only 6 different ways.

The subject of counting how many spanning trees and non-isomorphic spanning trees there are for a given graph was probably initiated by the English mathematician Arthur Cayley, who used trees to try to count the number of saturated hydrocarbons C_nH_{2n+2} containing a given number of carbon atoms. Cayley was the first person to use the term "tree" (in 1857) and in 1889 [12] he proved the following result which tells us that given n vertices, labelled $1, \ldots, n$, then there are n^{n-2} different ways of joining them to form a tree.

Theorem 2.13 (Cayley, 1889) *The complete graph K_n has n^{n-2} different spanning trees.*

Proof We omit the proof but, for those interested, see pages 32–35 of Bondy and Murty [7] or pages 50–52 of Wilson [65]. □

Exercises for Section 2.3

2.3.1 Give a list of all spanning trees, including isomorphic ones, of the connected graphs of Figure 2.14. How many non-isomorphic spanning trees are there in each case?

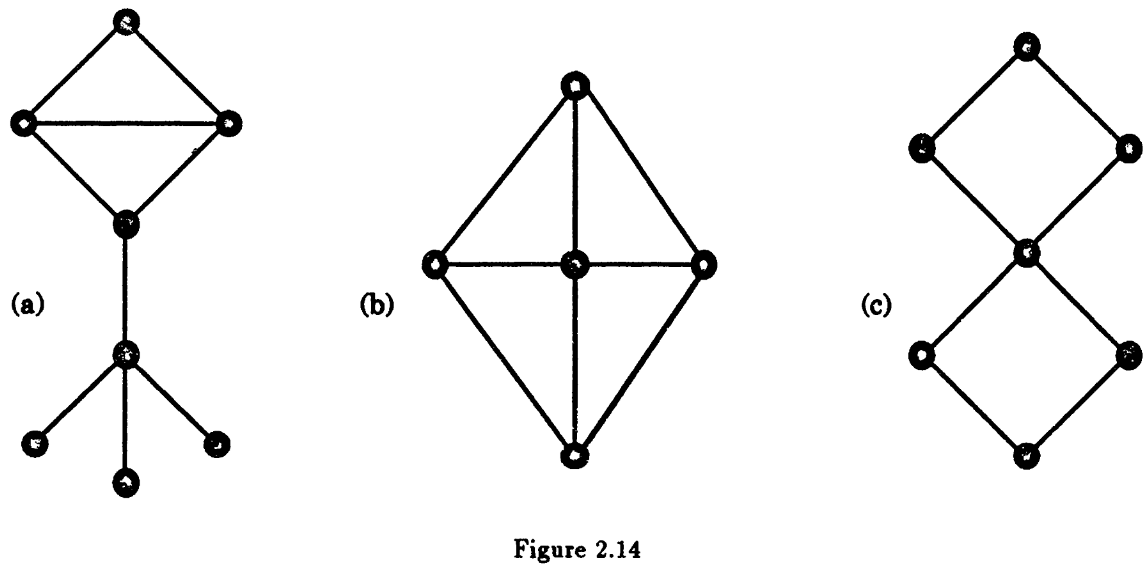

Figure 2.14

2.3.2 Let T be a tree with at least k edges, $k \geq 2$. How many connected components are there in the subgraph of T obtained by deleting k edges of T?

2.3.3 Let G be a connected graph which is not a tree and let C be a cycle in G. Prove that the complement of any spanning tree of G contains at least one edge of C.

2.3.4 Let e be an edge of the connected graph G.

(a) Prove that e is a bridge if and only if it is in every spanning tree of G.

(b) Prove that e is a loop if and only if it is in no spanning tree of G.

2.3.5 Let G be a graph with exactly one spanning tree. Prove that G is a tree.

2.3.6 An edge e (not a loop) of a graph G is said to be **contracted** if it is deleted and then its end vertices are fused (identified). The resulting graph is denoted by $G * e$. Figure 2.15 illustrates this.

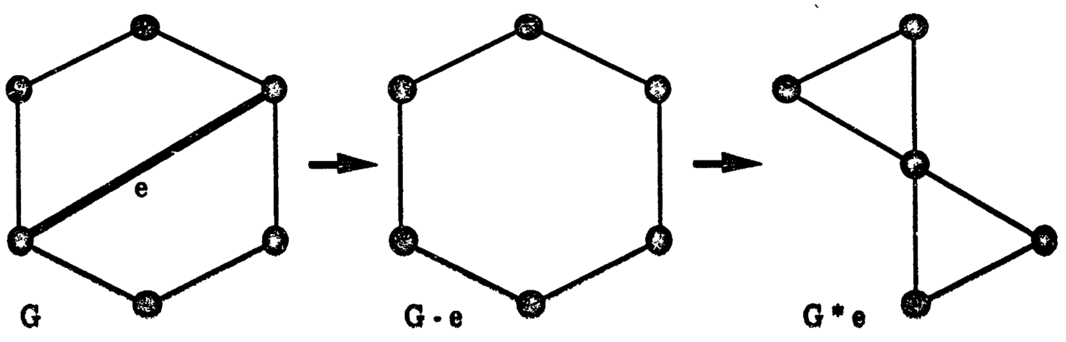

Figure 2.15

(a) Prove that if T is a spanning tree of G which contains e then $T * e$ is a spanning tree of $G * e$.

(b) Prove that if U is a spanning tree of $G * e$ then there is a unique spanning tree T of G which contains e and is such that $U = T * e$.

(c) Let $\tau(G)$ denote the number of different (not necessarily non-isomorphic) spanning trees of the connected graph G. Prove, using (a) and (b), that if e is an edge of G which is not a loop then

$$\tau(G) = \tau(G - e) + \tau(G * e).$$

2.3.7 Part (c) of Exercise 2.3.6 provides a way of calculating $\tau(G)$ for any connected graph G. Following the presentation given in Bondy and Murty [7], we illustrate it with an example in Figure 2.16 where the number of spanning trees for each graph is denoted pictorially by the graph itself. The idea is to break down the initial graph by a series of edge contractions to produce a collection of trees or "trees with loops". The number of graphs in this collection is then $\tau(G)$ for the initial graph G. Part (c) of Exercise 2.3.6 is used in each step of the breakdown. The contracted edges are shown thicker. The final "expression" consists of 11 graphs which are either trees or "trees with loops" and so we can conclude that $\tau(G) = 11$.

Use this method to find $\tau(G)$ for the graph G of Figure 2.15.

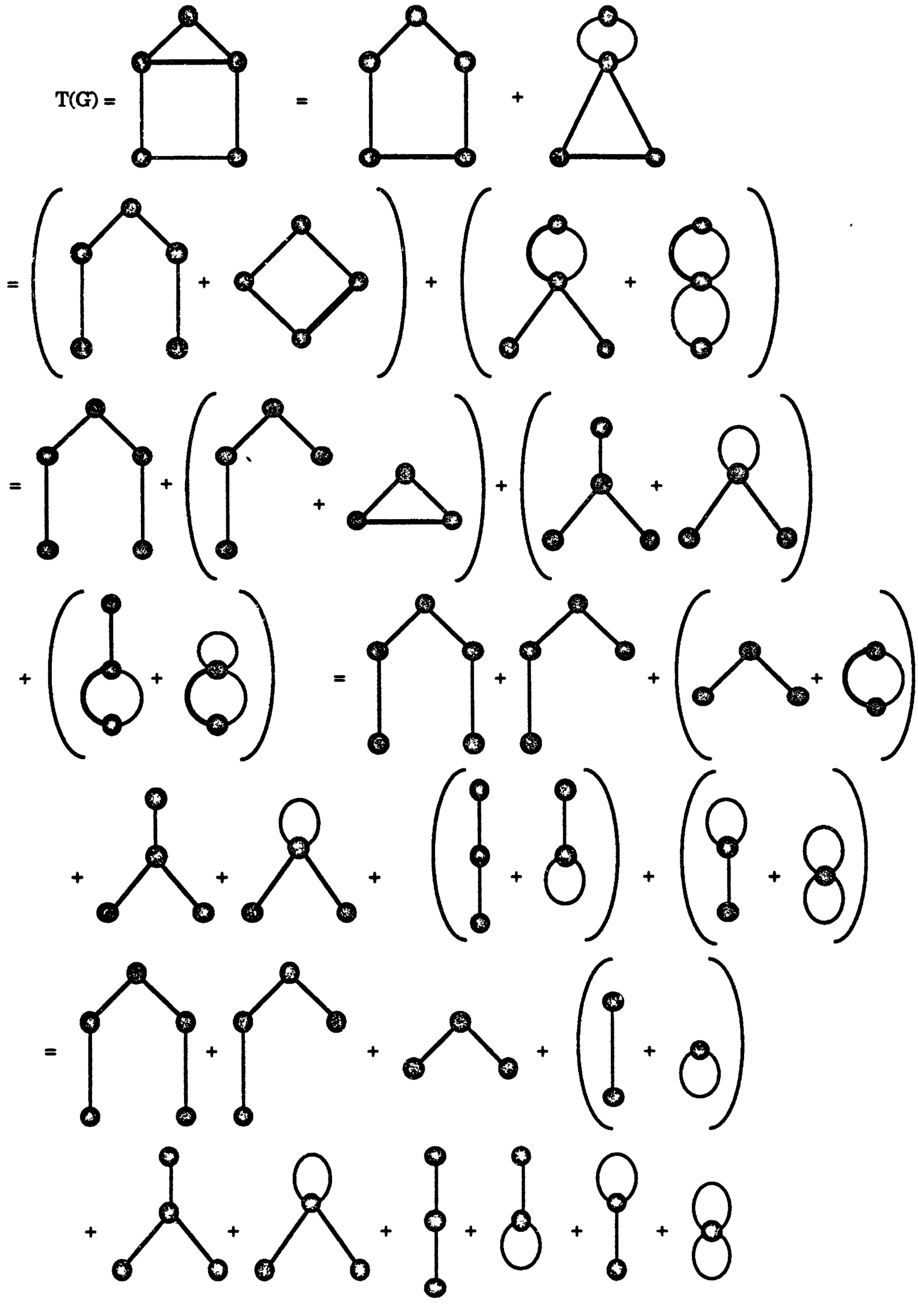

Figure 2.16: Calculation of $\tau(G)$ using contractions.