THE FOURIER TRANSFORMS OF THE ORIENTED FLUX AND THE FLUX ANTI-SYMMETRY FILTERS

F. BENMANSOUR

1. Definitions and notations

- The normalised oriented flux matrix of an image I is defined as follows [Law08] :

$$\forall x \in \mathbb{R}^n, \forall r > 0,\ \mathcal{F}(x,r) = (I * \partial_{i,j} g_\sigma * \mathbb{1}_r)(x) \tag{1}$$

  where, g_σ is the Gausian function with standard deviation σ, and $\mathbb{1}_r = 1/\mathcal{N}(r)$ if $\|x\| < r$ and 0 otherwise, where $\mathcal{N}(r)$ is the flux normalisation term. Typically, for $n = 3$, $\mathcal{N}(r) = 4\pi r^2$ and for $n = 2$, $\mathcal{N}(r) = 2\pi r$.
- The normalised anti-symmetry flux vector is defined as follows [Law10]:

$$\forall x \in \mathbb{R}^n, \forall r > 0,\ \mathcal{AS}(x,r) = (I * \partial_i g_\sigma * \Delta_r)(x) \tag{2}$$

  where $\Delta_r(x) = 1/\mathcal{N}(r)$ if $\|x\| = r$ and 0 otherwise.

2. Fourier transforms: definitions and useful properties

We are interested to compute the Fourier transforms of the previous filters using ordinary frequencies, that is the Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{R}$ writes

$$\forall u \in \mathbb{R}^n, \hat{f}(u) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i <x,u>} \mathrm{d}^n x \tag{3}$$

Let us recall the following properties:

- $\widehat{f * g} = \hat{f}.\hat{g}$.
- $\widehat{\partial_k f} = 2\pi i u_k \hat{f}$.
- According to theorem 3 in `http://math.arizona.edu/ faris/methodsweb/hankel.pdf`, for a spherically symmetric function f, by setting $\mathbf{k} = 2\pi u$ and hence $s = 2\pi\|u\|$ (and by setting $t = r$ in the integral), we have,

$$\forall u \in \mathbb{R}^n, \hat{f}(u) = 2\pi\|u\|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(2\pi\|u\|t)\ t^{\frac{n}{2}}\ F(t)\ \mathrm{d}t \tag{4}$$

where $F(\|x\|) = f(x)$ and J_d is the Bessel function of first kind of order d.

3. Computing the Fourier transforms of the flux filters

3.1. Fourier transforms of the derivative terms.

- Gaussian

$$\forall u \in \mathbb{R}^n, \widehat{g_\sigma}(u) = \frac{1}{\sqrt{2\pi\sigma^2}^n}\sqrt{2\pi\sigma^2}^n \exp(-2(\pi\sigma\|u\|)^2) = \exp(-2(\pi\sigma\|u\|)^2). \tag{5}$$

- First derivative

$$\forall u \in \mathbb{R}^n, \widehat{\partial_k g_\sigma}(u) = 2\pi i u_k \widehat{g_\sigma}(u) = 2\pi i u_k \exp(-2(\pi\sigma\|u\|)^2). \tag{6}$$

- Second derivative

$$\forall u \in \mathbb{R}^n, \widehat{\partial_{j,k} g_\sigma}(u) = (2\pi i u_j)(2\pi i u_k)\widehat{g_\sigma}(u) = -(2\pi)^2 u_k u_j \exp(-2(\pi\sigma\|u\|)^2). \tag{7}$$

3.2. Fourier transforms of the oriented flux kernel term.

- if the dimension is even, that is $n = 2p$, then, from equation (4)

$$\forall u \in \mathbb{R}^n, \widehat{\mathbb{1}_r}(u) = \frac{2\pi\|u\|^{1-p}}{\mathcal{N}(r)}\int_0^r J_{p-1}(2\pi\|u\|t)\ t^p\ \mathrm{d}t \tag{8}$$

by change of variable $t' = 2\pi\|u\|t$ and thanks to $\frac{d}{dx}[x^n J_n] = x^n J_{n-1}$ (see equation (59) in `http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html`), we obtain:

$$\forall u \in \mathbb{R}^n, \widehat{\mathbb{1}_r}(u) = \frac{r^p}{\mathcal{N}(r)}\frac{J_p(2\pi\|u\|r)}{\|u\|^p} = 2^{p-1}(p-1)!\ r\frac{J_p(2\pi\|u\|r)}{(2\pi\|u\|r)^p} \tag{9}$$

- if the dimension is odd, that is $n = 2p+1$, then, from equation (4)

$$\forall u \in \mathbb{R}^n, \widehat{\mathbb{1}_r}(u) = \frac{2\pi\|u\|^{-p+\frac{1}{2}}}{\mathcal{N}(r)}\int_0^r J_{p-\frac{1}{2}}(2\pi\|u\|t)\ t^{p+\frac{1}{2}}\ \mathrm{d}t. \tag{10}$$

Thanks to $j_p(x) = \sqrt{\frac{\pi}{2x}}J_{p+\frac{1}{2}}(x)$, and by the same previous change of variable, and thanks to $\frac{d}{dx}[x^{p+1}j_p(x)] = j_{p-1}(x)x^{p+1}$, we have:

$$\forall u \in \mathbb{R}^n, \widehat{\mathbb{1}_r}(u) = \frac{4\pi\|u\|^{1-p}}{\mathcal{N}(r)}\frac{1}{(2\pi\|u\|)^{p+2}}[t'^{p+1}j_p(t')]_0^{2\pi\|u\|r} = \frac{(2p)!}{2^p p!}\ r\ \frac{j_p(2\pi\|u\|r)}{(2\pi\|u\|r)^p}. \tag{11}$$

We are mostly interested by the case $n = 3$, that is $p = 1$, we have:

$$\forall u \in \mathbb{R}^n, \widehat{\mathbb{1}_r}(u) = \frac{1}{2\pi\|u\|}j_1(2\pi\|u\|r) = \frac{1}{2\pi\|u\|}\frac{\sin(2\pi\|u\|r) - (2\pi\|u\|r)\cos(2\pi\|u\|r))}{(2\pi\|u\|r)^2} \tag{12}$$

If one normalises equation (8) in [Law08] by dividing it by $4\pi r^2$, then we obtain the exact same expression using equations (12) and (7).

3.3. **Fourier transforms of the flux anti-symmetry kernel.** Thanks to equation (4)

$$\forall u \in \mathbb{R}^n, \widehat{\Delta_r}(u) = 2\pi \|u\|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(2\pi\|u\|r)\ \frac{r^{\frac{n}{2}}}{\mathcal{N}(r)} \tag{13}$$

from `http://en.wikipedia.org/wiki/N-sphere#Volume_and_surface_area`, one has $\mathcal{N}(r) = \mathcal{N}_n(r) = \frac{2\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(\frac{n}{2})}$. Again, one can distinguish between the 2 following cases:

- if the dimension is even, that is $n = 2p$, then,

$$\forall u \in \mathbb{R}^n, \widehat{\Delta_r}(u) = (p-1)! 2^{p-1} \frac{J_{p-1}(2\pi\|u\|r)}{(2\pi\|u\|r)^{p-1}} \tag{14}$$

- if the dimension is odd, that is $n = 2p+1$, then,

$$\forall u \in \mathbb{R}^n, \widehat{\Delta_r}(u) = 2^{p-\frac{1}{2}} \Gamma\left(p + \frac{1}{2}\right) \frac{J_{p-\frac{1}{2}}(2\pi\|u\|r)}{(2\pi\|u\|r)^{p-\frac{1}{2}}} \tag{15}$$

In addition, we have the following relation to spherical Bessel functions: $j_p(x) = \sqrt{\frac{\pi}{2x}} J_{p+\frac{1}{2}}(x)$. Finally,

$$\forall u \in \mathbb{R}^n, \widehat{\Delta_r}(u) = \frac{2^p \Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}} \frac{j_{p-1}(2\pi\|u\|r)}{(2\pi\|u\|r)^{p-1}} = \frac{(2p)!}{2^p p!} \frac{j_{p-1}(2\pi\|u\|r)}{(2\pi\|u\|r)^{p-1}} \tag{16}$$

Typically, for $n = 3$, that is $p = 1$, one has (thanks to $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$):

$$\forall u \in \mathbb{R}^n, \widehat{\Delta_r}(u) = j_0(2\pi\|u\|r) = \frac{\sin(2\pi\|u\|r)}{2\pi\|u\|r} \tag{17}$$

Compared to equation (9) in [Law10], we obtain the exact same expression by multiplying equations (17) and (6).

3.4. **Implementation.**

- For function J_n with n integer, use the function `vnl_bessel`.
- For the function j_n with n integer, use the function `boost::math::tr1::sph_bessel`.

3.5. **Summary.** This table summarises the implementation.

Filter	Fourier Transform using ordinary frequencies
• Oriented Flux Filter $\forall x \in \mathbb{R}^n, \mathcal{F}(x,r) = (I * \partial_{i,j} g_\sigma * \mathbb{1}_r)(x)$ with $g_\sigma(x) = \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n} e^{-\frac{\\|x\\|^2}{2\sigma^2}}$ and $\mathbb{1}_r(x) = \begin{cases} \frac{1}{\mathcal{N}(r)} & \text{if } \\|x\\| \leq r \\ 0 & \text{otherwise} \end{cases}$	$\widehat{\partial_{i,j} g_\sigma}(u) = -(2\pi)^2 u_i u_j \exp(-2(\pi\sigma\\|u\\|)^2)$ $\widehat{\mathbb{1}_r}(u) = \begin{cases} 2^{p-1}(p-1)! \quad r \, \dfrac{J_p(2\pi\\|u\\|r)}{(2\pi\\|u\\|r)^p} & \text{if} \quad n = 2p \\ \dfrac{(2p)!}{2^p p!} \quad r \, \dfrac{j_p(2\pi\\|u\\|r)}{(2\pi\\|u\\|r)^p} & \text{if} \quad n = 2p+1 \end{cases}$
• Oriented Flux Filter $\forall x \in \mathbb{R}^n, \mathcal{F}(x,r) = (I * \partial_i g_\sigma * \Delta_r)(x)$ with $g_\sigma(x) = \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n} e^{-\frac{\\|x\\|^2}{2\sigma^2}}$ and $\Delta_r(x) = \begin{cases} \frac{1}{\mathcal{N}(r)} & \text{if } \\|x\\| = r \\ 0 & \text{otherwise} \end{cases}$	$\widehat{\partial_j g_\sigma}(u) = i 2\pi u_j \exp(-2(\pi\sigma\\|u\\|)^2)$ $\widehat{\Delta_r}(u) = \begin{cases} 2^{p-1}(p-1)! \quad \dfrac{J_{p-1}(2\pi\\|u\\|r)}{(2\pi\\|u\\|r)^{p-1}} & \text{if} \quad n = 2p \\ \dfrac{(2p)!}{2^p p!} \quad \dfrac{j_{p-1}(2\pi\\|u\\|r)}{(2\pi\\|u\\|r)^{p-1}} & \text{if} \quad n = 2p+1 \end{cases}$