

## CONTRIBUTIONS TO THE THEORY OF ATMOSPHERIC REFRACTION <sup>(1)</sup>

### Abstract

Since the barometer measures the weight of the overlying atmosphere, it follows by the law of Gladstone and Dale that the height integral  $\int (n - 1) dr$  of the atmospheric refractivity for light, taken from ground level up to the top of the atmosphere, is directly proportional to ground pressure. The refractivity integral, therefore, can be determined without detailed knowledge of the height distribution of the refractive index, which not only simplifies the derivation of refraction formulas in which atmospheric models have been used hitherto, but also improves their accuracy. For zenith distances not exceeding about 75 degrees, the correction for astronomical refraction will be given by the standard formula

$$\Delta z_0'' = 16''.271 \tan z \left[ 1 + 0.0000394 \tan^2 z \left( \frac{p - 0.156e}{T} \right) \right] \left( \frac{p - 0.156e}{T} \right) -$$

$$- 0''.0749 (\tan^3 z + \tan z) \left( \frac{p}{1000} \right)$$

where  $z$  is the apparent zenith distance,  $p$  is the total pressure and  $e$  is the partial pressure of water vapour, both in millibars, and  $T$  is the absolute temperature in degrees Kelvin. Part II of the paper contains further applications of the theory to refraction problems in satellite geodesy, including the photogrammetric refraction and the atmospheric corrections in the ranging of artificial satellites.

### Part I. Astronomical Refraction

#### Derivation of General Formula for Astronomical Refraction

In Figure 1, the law of refraction applied at point  $P$  gives

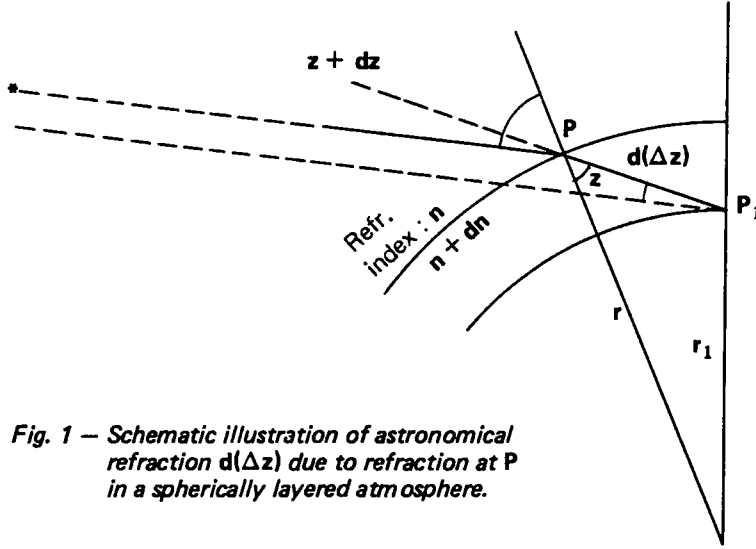
$$(n + dn) \sin z = n \sin (z + dz) = n (\sin z + \cos z dz)$$

from which immediately follows the differential equation  $d(\Delta z) = dz = (\tan z/n) dn$  and the corresponding integral equation

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$$(0 \leq z_1 \leq 90^\circ) ; \quad \Delta z = \int_1^{n_1} \frac{\tan z}{z} dn \quad (1)$$

which is the basic mathematical expression of the correction for astronomical refraction.



*Fig. 1 – Schematic illustration of astronomical refraction  $d(\Delta z)$  due to refraction at P in a spherically layered atmosphere.*

Since  $z$  is not, in general, constant along the light path but depends upon the refractive index according to the law of refraction

$$nr \sin z = n_1 r_1 \sin z_1 = \text{const.} \quad (2)$$

it will be necessary to find a suitable expression for  $\tan z$  that makes (1) integrable. Setting  $n_1 r_1 / (nr) = y$  for brevity, we have from (2)

$$\sin^2 z = y^2 \tan^2 z_1 / (1 + \tan^2 z_1)$$

$$\cos^2 z = (1 + \tan^2 z_1 - y^2 \tan^2 z_1) / (1 + \tan^2 z_1)$$

and

$$\begin{aligned} \tan z &= y \tan z_1 [1 + \tan^2 z_1 (1 - y^2)]^{-\frac{1}{2}} = \\ &= y \tan z_1 - \frac{1}{2} y (1 - y^2) \tan^3 z_1 + \frac{3}{8} y (1 - y^2)^2 \tan^5 z_1 - \\ &- \frac{5}{16} y (1 - y^2)^3 \tan^7 z_1 + \frac{35}{128} y (1 - y^2)^4 \tan^9 z_1 - \frac{63}{256} y (1 - y^2)^5 \tan^{11} z_1 + \dots \end{aligned}$$

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Neglecting the subsequent terms in the binomial expansion, the first five may be written identically

$$\begin{aligned} \tan z &= \tan z_1 + \left(y - \frac{r_1}{r}\right) (\tan^3 z_1 + \tan z_1) - \left(\frac{r - r_1}{r}\right) (\tan^3 z_1 + \tan z_1) + \\ &+ \left(1 + \frac{1}{2} y\right) (1 - y)^2 \tan^3 z_1 + \frac{3}{8} y (1 + y)^2 (1 - y)^2 \tan^5 z_1 - \\ &- \frac{5}{16} y (1 + y)^3 (1 - y)^3 \tan^7 z_1 + \frac{35}{128} y (1 + y)^4 (1 - y)^4 \tan^9 z_1 \end{aligned}$$

into which we substitute the approximation

$$y - \frac{r_1}{r} = \left(\frac{r_1}{r}\right) \left(\frac{n_1 - n}{n}\right) = n_1 - n$$

$$\frac{r - r_1}{r} = \frac{1}{r_1} (r - r_1) - \frac{1}{r_1^2} (r - r_1)^2$$

$$\left(1 + \frac{1}{2} y\right) (1 - y)^2 = \frac{3}{2} (1 - y)^2 = \frac{3}{2 r_1^2} (r - r_1)^2$$

$$\begin{aligned} \frac{3}{8} y (1 + y)^2 (1 - y)^2 &= \frac{3}{2} (1 - y)^2 = \frac{3}{2} \left[ \left(1 - \frac{r_1}{r}\right) - (n_1 - n) \right]^2 = \\ &= \frac{3}{2 r_1^2} (r - r_1)^2 - \frac{3}{r_1} (n_1 - n) (r - r_1) \end{aligned}$$

$$\frac{5}{16} y (1 + y)^3 (1 - y)^3 = \frac{5}{2} (1 - y)^3 = \frac{5}{2 r_1^3} (r - r_1)^3$$

$$\frac{35}{128} y (1 + y)^4 (1 - y)^4 = \frac{35}{8} (1 - y)^4 = \frac{35}{8 r_1^4} (r - r_1)^4$$

and obtain

$$\begin{aligned} \tan z &= \tan z_1 + (\tan^3 z_1 + \tan z_1) (n_1 - n) - A_1 (r - r_1) + A_2 (r - r_1)^2 - \\ &- A_2' (n_1 - n) (r - r_1) - A_3 (r - r_1)^3 + A_4 (r - r_1)^4 \end{aligned} \quad (3)$$

where the coefficients are :

$$\begin{aligned}
A_1 &= (\tan^3 z_1 + \tan z_1) / r_1 \\
A_2 &= (3 \tan^5 z_1 + 5 \tan^3 z_1 + \dots) / (2 r_1^2) \\
A_2' &= 3 \tan^5 z_1 / r_1 \\
A_3 &= 5 \tan^7 z_1 / (2 r_1^3) \\
A_4 &= 35 \tan^9 z_1 / (8 r_1^4)
\end{aligned} \tag{4}$$

By the substitution of (3), integral (1) breaks down into seven terms, of which the first two can be solved at once :

$$\begin{aligned}
\tan z_1 \int_1^{n_1} \frac{dn}{n} &= \tan z_1 \log n_1 = \tan z_1 \log [1 + (n_1 - 1)] = \\
&= \tan z_1 (n_1 - 1) - \frac{1}{2} \tan z_1 (n_1 - 1)^2 + \dots \\
(\tan^3 z_1 + \tan z_1) \int_1^{n_1} (n_1 - n) dn &= \frac{1}{2} (\tan^3 z_1 + \tan z_1) (n_1 - 1)^2
\end{aligned}$$

Since  $n$  is nearly unity ( $1 \leq n < 1.0004$ ), all the terms of higher than second order will be omitted in the first integral, as well as  $n$  in the denominator of the subsequent ones. Equation (1) then becomes

$$\begin{aligned}
\Delta z &= \tan z_1 (n_1 - 1) + \frac{1}{2} \tan^3 z_1 (n_1 - 1)^2 - A_1 \int_1^{n_1} (r - r_1) dn + \\
&+ A_2 \int_1^{n_1} (r - r_1)^2 dn - A_2' \int_1^{n_1} (n_1 - n) (r - r_1) dn - \\
&- A_3 \int_1^{n_1} (r - r_1)^3 dn + A_4 \int_1^{n_1} (r - r_1)^4 dn
\end{aligned} \tag{5}$$

The five remaining atmospheric integrals can be determined, as follows.

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$$\text{Integral} \int_1^{n_1} (r - r_1) \, dn.$$

In physical meteorology, the atmosphere may be thought of as a mixture of two ideal gases, dry air and water vapour. If we denote the total pressure, the partial pressure of water vapour and the absolute temperature by  $p$ ,  $e$  and  $T$  respectively, the densities of the dry-air and water-vapour components are, as stated by the perfect gas law.

$$\rho_d = \frac{p - e}{RT} \quad \text{and} \quad \rho_w = \frac{e}{R_w T}$$

where  $R$  and  $R_w$  stand for the appropriate gas constants. The density of the mixture is, of course, equal to  $\rho_d + \rho_w$ , or

$$\rho = \frac{p}{RT} - \left(1 - \frac{R}{R_w}\right) \frac{e}{RT}$$

The atmosphere being in hydrostatic equilibrium, pressure  $p$  measured at any height level is equal to the total weight of the air contained in a vertical column of unit cross section, reaching from the point of observation ( $r = r_1$ ) up to the top of the atmosphere ( $r = r'$ ). Consequently,

$$\int_{r_1}^{r'} \rho \, dr = \frac{1}{R} \int_{r_1}^{r'} \left( \frac{p}{T} \right) dr - \frac{1}{R} \left( 1 - \frac{R}{R_w} \right) \int_{r_1}^{r'} \left( \frac{e}{T} \right) dr = \frac{p_1}{g} \quad (6)$$

where  $g$  is the local value of gravity at the centroid of the atmospheric column.

The refractivity of moist air for electromagnetic radiation may be written

$$n - 1 = \frac{(n_0 - 1) T_0}{p_0} \left( \frac{p}{T} \right) - c_w (e/T) + c_{w'} (e/T^2) \quad (7)$$

where  $n_0$  is the refractive index of dry air at pressure  $p_0$  and temperature  $T_0$ , and  $c_w$  and  $c_{w'}$  are constants. The corresponding height integral

$$\int_{r_1}^{r'} (n - 1) \, dr = \frac{(n_0 - 1) T_0}{p_0} \int_{r_1}^{r'} \left( \frac{p}{T} \right) dr - c_w \int_{r_1}^{r'} \left( \frac{e}{T} \right) dr + c_{w'} \int_{r_1}^{r'} \left( \frac{e}{T^2} \right) dr$$

can be readily determined with the aid of equation (6). This gives

$$\int_{r_1}^{r'} (n-1) dr = \frac{(n_0-1) RT_0}{p_0 g} p_1 + \left[ \frac{(n_0-1) T_0}{p_0} \left( 1 - \frac{R}{R_w} \right) - c_w \right] \int_{r_1}^{r'} \left( \frac{e}{T} \right) dr +$$

$$+ c_w \int_{r_1}^{r'} \left( \frac{e}{T^2} \right) dr \quad (8)$$

Equation (8) expresses the value of the refractivity integral in terms of ground pressure  $p_1$ , with minor corrections included due to the presence of water vapour in the atmosphere.

As far as the astronomical refraction is concerned, the contribution of humidity to the refractivity integral is negligible, and the last two terms in equation (8) can be omitted. Setting

$$u = r - r_1 \quad v = n - 1$$

$$du = dr \quad dv = dn$$

and integrating by parts :

$$\int (r - r_1) dn = \int u dv = uv - \int v du = (r - r_1) (n - 1) - \int (n - 1) dr ,$$

we then obtain from (8) and (7) the important relationships

$$\int_1^{n_1} (r - r_1) dn = \int_{r_1}^{r'} (n - 1) dr = \frac{(n_0-1) RT_0}{p_0 g} p_1 = \frac{R}{g} (n_1 - 1) T_1 \quad (9)$$

$$\text{Integral} \int_1^{n_1} (r - r_1)^2 dn .$$

This integral requires some consideration of the vertical distribution of pressure and temperature in the atmosphere. We shall determine its value in two parts, the stratospheric component and the tropospheric component. The state of the atmosphere at the bounding surface, the tropopause, shall be denoted by superscripts  $p^0$ ,  $T^0$ , etc..., and it is assumed to be known.

Throughout the stratosphere, the temperature may be taken as constant, and equal to temperature  $T^0$  at the tropopause. Integration of the hydrostatic equation for fluids,  $dp = g \rho dr$ , on the condition  $\rho = p/(RT^0)$  gives the pressure as

$$p = p^0 e^{m(r-r^0)} \quad (10)$$

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where  $e$  is the base of natural logarithms, and  $m = -g/(RT^0)$  is constant. Similarly,

$$n - 1 = (n^0 - 1) e^{m(r - r^0)} \quad (11)$$

and differentiating (11)

$$dn = m (n^0 - 1) e^{m(r - r^0)} dr = m (n - 1) dr \quad (12)$$

Since identically

$$r - r_1 = (r^0 - r_1) + (r - r^0)$$

$$(r - r_1)^2 = (r^0 - r_1)^2 + 2(r^0 - r_1)(r - r^0) + (r - r^0)^2$$

we have first, using (9)

$$\int_1^{n^0} (r - r_1)^2 dn = (r^0 - r_1)^2 (n^0 - 1) + \frac{2R}{g} (r^0 - r_1) (n^0 - 1) T^0 + \int_1^{n^0} (r - r^0)^2 dn$$

Now from (12)

$$\begin{aligned} \int (r - r^0)^2 dn &= m (n^0 - 1) \int (r - r^0)^2 e^{m(r - r^0)} dr = \\ &= m (n^0 - 1) \frac{e^{m(r - r^0)}}{m^3} [m^2 (r - r^0)^2 - 2m (r - r^0) + 2] + C = \\ &= (n - 1) \left[ (r - r^0)^2 - \frac{2}{m} (r - r^0) + \frac{2}{m^2} \right] + C \end{aligned}$$

where  $C$  is the constant of integration. This gives

$$\int_1^{n^0} (r - r^0)^2 dn = \frac{2(n^0 - 1)}{m^2} = \frac{2R^2}{g^2} (n^0 - 1) T^{02} \quad (13)$$

and the total stratospheric component is consequently

$$\begin{aligned} \int_1^{n^0} (r - r_1)^2 dn &= (r^0 - r_1)^2 (n^0 - 1) + \frac{2R}{g} (r^0 - r_1) (n^0 - 1) T^0 + \\ &+ \frac{2R^2}{g^2} (n^0 - 1) T^{02} \end{aligned} \quad (14)$$

Through most of the troposphere, the temperature decreases with height at a fairly uniform rate which varies slightly with latitude and season, although in the polar regions there exists a permanent inversion in the lower troposphere where the actual temperatures increase with height. Integration of the hydrostatic equation on the conditions  $\rho = p/(RT)$  and

$$T = T_1 + \beta(r - r_1) \quad (15)$$

where the vertical gradient of temperature,  $\beta = dT/dr$ , is assumed constant gives the pressure as

$$p = p_1 \left( \frac{T}{T_1} \right)^{-g/(R\beta)} \quad (16)$$

and the pressure-temperature ratio as  $p/T = (p_1/T_1) (T/T_1)^{m'}$ , where  $m' = -g/(R\beta) - 1$  is constant. The refractivity is now given by

$$n - 1 = (n_1 - 1) \left( \frac{T}{T_1} \right)^{m'} \quad (17)$$

and its differential by

$$dn = \frac{m'(n_1 - 1)}{T_1} \left( \frac{T}{T_1} \right)^{m'-1} dT = m' \left( \frac{n-1}{T} \right) dT \quad (18)$$

Since from (15)

$$r - r_1 = \frac{T - T_1}{\beta} = \frac{T_1}{\beta} \left( \frac{T}{T_1} - 1 \right) \quad \text{and} \quad (r - r_1)^2 = \frac{T_1^2}{\beta^2} \left( \frac{T}{T_1} - 1 \right)^2$$

we have

$$\begin{aligned} \int (r - r_1)^2 dn &= \frac{m'(n_1 - 1) T_1}{\beta^2} \int \left( \frac{T}{T_1} - 1 \right)^2 \left( \frac{T}{T_1} \right)^{m'-1} dT = \\ &= \frac{m'(n_1 - 1) T_1^2}{\beta^2} \left[ \left( \frac{1}{m'+2} \right) \left( \frac{T}{T_1} \right)^{m'+2} - \left( \frac{2}{m'+1} \right) \left( \frac{T}{T_1} \right)^{m'+1} + \frac{1}{m'} \left( \frac{T}{T_1} \right)^{m'} \right] + C = \\ &= \frac{(n_1 - 1) T_1^2}{\beta^2} \left[ \left( \frac{m'}{m'+2} \right) \left( \frac{T}{T_1} \right)^2 - \left( \frac{2m'}{m'+1} \right) \left( \frac{T}{T_1} \right) + 1 \right] + C = \end{aligned}$$



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$$= (r - r_1)^2 (n - 1) + \frac{(n - 1) T_1^2}{\beta^2} \left[ \left( \frac{2}{m' + 1} \right) \left( \frac{T}{T_1} \right) - \left( \frac{2}{m' + 2} \right) \left( \frac{T}{T_1} \right)^2 \right] + C$$

where  $C$  is the constant of integration, and the preceding term, transformed step by step, is

$$\begin{aligned} \frac{(n - 1) T_1^2}{\beta^2} \left[ \left( \frac{2}{m' + 1} \right) \left( \frac{T}{T_1} \right) - \left( \frac{2}{m' + 2} \right) \left( \frac{T}{T_1} \right)^2 \right] &= \frac{2 (n - 1) T_1 T}{\beta^2 (m' + 1)} \left[ 1 - \left( \frac{m' + 1}{m' + 2} \right) \left( \frac{T}{T_1} \right) \right] = \\ &= \frac{2 (n - 1) T_1 T}{\beta^2 (m' + 1)} \left[ \left( \frac{1}{m' + 2} \right) \left( \frac{T}{T_1} \right) - \left( \frac{T}{T_1} - 1 \right) \right] = \frac{2 (n - 1) T}{\beta (m' + 1)} \left[ \frac{T}{\beta (m' + 2)} - (r - r_1) \right] = \\ &= \frac{2 (n - 1) R T}{g} \left[ (r - r_1) - \frac{T}{\beta (m' + 2)} \right] = \frac{2 R}{g} (r - r_1) (n - 1) T + \frac{2 R^2}{g^2 (1 - R \beta / g)} (n - 1) T^2 \end{aligned}$$

The tropospheric component is accordingly

$$\begin{aligned} \int_{n^0}^{n_1} (r - r_1)^2 dn &= - (r^0 - r_1)^2 (n^0 - 1) - \frac{2 R}{g} (r^0 - r_1) (n^0 - 1) T^0 + \\ &+ \frac{2 R^2}{g^2 (1 - R \beta / g)} \left[ (n_1 - 1) T_1^2 - (n^0 - 1) T^{02} \right] \end{aligned} \quad (19)$$

which equation holds for any constant value of  $\beta \neq g/R$ , including  $\beta = 0$ .

The sum of component integrals (14) and (19) gives the total value of the integral

$$\int_1^{n_1} (r - r_1)^2 dn = \frac{2 R^2}{g^2} \left[ \frac{(n_1 - 1) T_1^2 - (n^0 - 1) T^{02}}{1 - R \beta / g} + (n^0 - 1) T^{02} \right] \quad (20)$$

under normal atmospheric conditions where the vertical distribution of temperature throughout the troposphere is substantially a linear function of height.

$$\text{Integral} \int_1^{n_1} (n_1 - n) (r - r_1) dn.$$

Integration by parts using the substitutions  $u = r - r_1$  and  $v = [n_1 - 1 - (n - 1)]^2$  gives first, in view of (9),

$$\int_1^{n_1} (n_1 - n) (r - r_1) \, dn = \frac{R}{g} (n_1 - 1)^2 T_1 - \frac{1}{2} \int_{r_1}^{r'} (n - 1)^2 \, dr \quad (21)$$

the latter integral being more conveniently determined.

In the stratosphere, equation (11) gives

$$\begin{aligned} \int (n - 1)^2 \, dr &= (n^0 - 1)^2 \int e^{2m(r-r^0)} \, dr = \\ &= (n^0 - 1)^2 \frac{e^{2m(r-r^0)}}{2m} + C = -\frac{R}{2g} (n - 1)^2 T^0 + C \end{aligned}$$

and

$$\int_{r^0}^{r'} (n - 1)^2 \, dr = \frac{R}{2g} (n^0 - 1)^2 T^0 \quad (22)$$

whereas in the troposphere, applying (17)

$$\begin{aligned} \int (n - 1)^2 \, dr &= \frac{(n_1 - 1)^2}{\beta} \int \left( \frac{T}{T_1} \right)^{2m'} \, dT = \\ &= \frac{(n_1 - 1)^2 T_1}{\beta (2m' + 1)} \left( \frac{T}{T_1} \right)^{2m' + 1} + C = -\left( \frac{R}{2g + R\beta} \right) (n - 1)^2 T + C \end{aligned}$$

and

$$\int_{r_1}^{r^0} (n - 1)^2 \, dr = \left( \frac{R}{2g + R\beta} \right) [ (n_1 - 1)^2 T_1 - (n^0 - 1)^2 T^0 ] \quad (23)$$

The sum of (22) and (23) substituted into equation (21) finally gives the total value of the integral

$$\begin{aligned} \int_1^{n_1} (n_1 - n) (r - r_1) \, dn &= \frac{R}{g} (n_1 - 1)^2 T_1 - \frac{R}{2(2g + R\beta)} [ (n_1 - 1)^2 T_1 + \\ &+ \frac{1}{2} (R\beta/g) (n^0 - 1)^2 T^0 ] \end{aligned} \quad (24)$$

again assuming that the vertical gradient of temperature is constant in the troposphere.

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$$\text{Integrals } \int_1^{n_1} (r-r_1)^3 \, dn \text{ and } \int_1^{n_1} (r-r_1)^4 \, dn.$$

For the stratospheric component of the first integral we have from (12)

$$\begin{aligned} \int (r-r^0)^3 \, dn &= m (n^0 - 1) \int (r-r^0)^3 e^{m(r-r^0)} \, dr = \\ &= m (n^0 - 1) \left[ \frac{1}{m} (r-r^0)^3 e^{m(r-r^0)} - \frac{3}{m} \int (r-r^0) e^{m(r-r^0)} \, dr \right] = \\ &= (r-r^0)^3 (n-1) - \frac{3}{m} \int (r-r^0)^2 \, dn \end{aligned}$$

and further, in view of (13)

$$\int_1^{n^0} (r-r^0)^3 \, dn = -\frac{3}{m} \int_1^{n^0} (r-r^0)^2 \, dn = \frac{6R^3}{g^3} (n^0 - 1) T^{03} \quad (25)$$

Using the identity  $(r-r_1)^3 = (r^0-r_1)^3 + 3(r^0-r_1)^2(r-r^0) + 3(r^0-r_1)(r-r^0)^2 + (r-r^0)^3$  and applying integrals (9), (13), and (25), the stratospheric component is obtained as

$$\begin{aligned} \int_1^{n^0} (r-r_1)^3 \, dn &= (r^0-r_1)^3 (n^0-1) + \frac{3R}{g} (r^0-r_1)^2 (n^0-1) T^0 + \\ &+ \frac{6R^2}{g^2} (r^0-r_1) (n^0-1) T^{02} + \frac{6R^3}{g^3} (n^0-1) T^{03} \end{aligned} \quad (26)$$

For the tropospheric component of the same integral we have from (15) and (18)

$$\begin{aligned} \int (r-r_1)^3 \, dn &= \frac{m'(n_1-1) T_1^2}{\beta^3} \int \left( \frac{T}{T_1} - 1 \right)^3 \left( \frac{T}{T_1} \right)^{m'-1} dT = \\ &= \frac{m'(n_1-1) T_1^3}{\beta^3} \left[ \left( \frac{1}{m'+3} \right) \left( \frac{T}{T_1} \right)^{m'+3} - \left( \frac{3}{m'+2} \right) \left( \frac{T}{T_1} \right)^{m'+2} + \left( \frac{3}{m'+1} \right) \left( \frac{T}{T_1} \right)^{m'+1} - \right. \\ &\left. - \frac{1}{m'} \left( \frac{T}{T_1} \right)^{m'} \right] + C = \frac{(n-1) T_1^3}{\beta^3} \left[ \left( \frac{m'}{m'+3} \right) \left( \frac{T}{T_1} \right)^3 - \left( \frac{3m'}{m'+2} \right) \left( \frac{T}{T_1} \right)^2 + \left( \frac{3m'}{m'+1} \right) \left( \frac{T}{T_1} \right) - 1 \right] + C = \end{aligned}$$

$$\begin{aligned}
&= (r-r_1)^3 (n-1) - \frac{3(n-1) T_1^2 T}{\beta^3 (m'+1)} \left[ \left( \frac{m'+1}{m'+3} \right) \left( \frac{T}{T_1} \right)^2 - \left( \frac{2m'+2}{m'+2} \right) \left( \frac{T}{T_1} \right) + 1 \right] + C = \\
&= (r-r_1)^3 (n-1) + \frac{3R}{g} (r-r_1)^2 (n-1) T + \frac{6(n-1) T_1 T^2}{\beta^3 (m'+1) (m'+2)} \left[ \left( \frac{m'+2}{m'+3} \right) \left( \frac{T}{T_1} \right) - 1 \right] + C \\
&= (r-r_1)^3 (n-1) + \frac{3R}{g} (r-r_1)^2 (n-1) T + \frac{6R^2}{g^2 (1-R\beta/g)} (r-r_1) (n-1) T^2 + \\
&\quad + \frac{6R^3}{g^3 (1-R\beta/g) (1-2R\beta/g)} (n-1) T^3 + C
\end{aligned}$$

The tropospheric component is accordingly

$$\begin{aligned}
\int_{n^0}^{n_1} (r-r_1)^3 dn &= -(r^0-r_1)^3 (n^0-1) - \frac{3R}{g} (r^0-r_1)^2 (n^0-1) T^0 - \\
&\quad - \frac{6R^2}{g^2 (1-R\beta/g)} (r^0-r_1) (n^0-1) T^{02} + \\
&\quad + \frac{6R^3}{g^3 (1-R\beta/g) (1-2R\beta/g)} [(n_1-1) T_1^3 - (n^0-1) T^{03}]
\end{aligned} \quad (27)$$

which added to stratospheric component (26) gives the total integral

$$\begin{aligned}
\int_1^{n_1} (r-r_1)^3 dn &= \frac{6R^3}{g^3} \left[ \frac{(n_1-1) T_1^3 - (n^0-1) T^{03}}{(1-R\beta/g) (1-2R\beta/g)} + (n^0-1) T^{03} \right] + \\
&\quad + \frac{6R^2}{g^2} \left[ 1 - \frac{1}{1-R\beta/g} \right] (r^0-r_1) (n^0-1) T^{02}
\end{aligned} \quad (28)$$

Similarly,

$$\begin{aligned}
\int_1^{n_1} (r-r_1)^4 dn &= \frac{24R^4}{g^4} \left[ \frac{(n_1-1) T_1^4 - (n^0-1) T^{04}}{(1-R\beta/g) (1-2R\beta/g) (1-3R\beta/g)} + (n^0-1) T^{04} \right] + \\
&\quad + \frac{24R^3}{g^3} \left[ 1 - \frac{1}{(1-R\beta/g) (1-2R\beta/g)} \right] (r^0-r_1) (n^0-1) T^{03} + \\
&\quad + \frac{12R^2}{g^2} \left( 1 - \frac{1}{1-R\beta/g} \right) (r^0-r_1)^2 (n^0-1) T^{02}
\end{aligned} \quad (29)$$

## THEORY OF ATMOSPHERIC REFRACTION

is obtained for the explicit value of the last integral considered in the expression for astronomical refraction (5).

Due to insolation heating of the ground during the day and its radiational cooling during the night, temperature gradients within the first few kilometres of the troposphere next to the ground frequently differ significantly from the approximately constant value of  $\beta$  above that level. Consequently integrals (20), (24), (28) and (29) should be modified by subdividing their respective tropospheric components. But since only a small contribution to these integrals comes from the lower levels, it will be quite sufficient merely to extend the constant temperature gradient of the free troposphere down to the ground level, neglecting the small error thus involved. This requires that the actual values of  $T_1$  and  $n_1 - 1$  be replaced by

$$T_1' = T^0 - \beta(r^0 - r_1) \quad (15')$$

$$n_1' - 1 = (n^0 - 1) (T_1' / T^0)^m \quad (17')$$

and the prevailing temperature gradient of the lower troposphere can be disregarded.

We may now combine the results from the preceding discussion, and write on the basis of equation (5) the following expression for the correction for astronomical refraction (in seconds of arc) :

$$\begin{aligned} \Delta z'' = \rho'' \tan z_1 \left[ 1 + \frac{1}{2} \tan^2 z_1 (n_1 - 1) \right] (n_1 - 1) - \\ - \frac{\rho'' R}{r_1 g} (\tan^3 z_1 + \tan z_1) (n_1 - 1) T_1 + \delta_1'' - \delta_2'' - \delta_3'' + \delta_4'' \end{aligned} \quad (30)$$

where

$$\begin{aligned} \delta_1'' &= \frac{\rho'' R^2}{r_1^2 g^2} (3 \tan^5 z_1 + 5 \tan^3 z_1) \left[ \frac{(n_1' - 1) T_1'^2 - (n^0 - 1) T^{02}}{1 - R \beta / g} + (n^0 - 1) T^{02} \right] \\ \delta_2'' &= \frac{3 \rho'' R}{r_1 g} \tan^5 z_1 \left[ (n_1' - 1)^2 T_1' - \frac{(n_1' - 1)^2 T_1' + \frac{1}{2} (R \beta / g) (n^0 - 1)^2 T^0}{2 (2 + R \beta / g)} \right] \\ \delta_3'' &= \frac{15 \rho'' R^3}{r_1^3 g^3} \tan^7 z_1 \left[ \frac{(n_1' - 1) T_1'^3 - (n^0 - 1) T^{03}}{(1 - R \beta / g) (1 - 2 R \beta / g)} + (n^0 - 1) T^{03} \right] + \\ &+ \frac{15 \rho'' R^2}{r_1^3 g^2} \tan^7 z_1 \left( 1 - \frac{1}{1 - R \beta / g} \right) (r^0 - r_1) (n^0 - 1) T^{02} \end{aligned} \quad (31)$$

$$\begin{aligned}
\delta_4'' = & \frac{105 \rho'' R^4}{r_1^4 g^4} \tan^9 z_1 \left[ \frac{(n_1' - 1) T_1'^4 - (n^0 - 1) T^{04}}{(1 - R\beta/g)(1 - 2R\beta/g)(1 - 3R\beta/g)} + (n^0 - 1) T^{04} \right] + \\
& + \frac{105 \rho'' R^3}{r_1^4 g^3} \tan^9 z_1 \left[ 1 - \frac{1}{(1 - R\beta/g)(1 - 2R\beta/g)} \right] (r^0 - r_1) (n^0 - 1) T^{03} + \\
& + \frac{105 \rho'' R^2}{2 r_1^4 g^2} \tan^9 z_1 \left( 1 - \frac{1}{1 - R\beta/g} \right) (r^0 - r_1)^2 (n^0 - 1) T^{02}
\end{aligned}$$

represent minor terms dependent on the vertical structure of the atmosphere. Up to zenith distance  $z_1 = 80^\circ$ , equation (30) will give the value of integral (1) accurately enough for all practical purposes, as can best be demonstrated by test computations on atmospheric models based upon the formulas previously derived (see Tables Ia - c and IIa - c).

Table 1a.

Atmospheric Model No. 1  
(Tropical Zone)

$r$ , km 6360+	$p$ , mb	$T$ , °K	$(n-1)10^6$	$-\left(\frac{dn}{dr}\right) 10^6, \text{ km}^{-1}$	$r$ , km 6360+	$p$ , mb	$T$ , °K	$(n-1) 10^6$	$-\left(\frac{dn}{dr}\right) 10^6, \text{ km}^{-1}$
0	1010.00	299.85	265.72	24.8210	21.3	45.19	198.00	18.01	3.0983
0.8	921.55	295.00	246.43	23.3982	22.2	38.71	198.00	15.42	2.6538
1.6	839.57	290.15	228.26	22.0353	23.1	33.15	198.00	13.21	2.2730
2.4	763.68	285.30	211.16	20.7308	24	28.40	198.00	11.31	1.9469
3.2	693.53	280.45	195.08	19.4832	26	20.13	198.00	8.02	1.3800
4	628.76	275.60	179.97	18.2908	28	14.27	198.00	5.68	0.9782
4.8	569.05	270.75	165.80	17.1522	30	10.11	198.00	4.03	0.6933
5.6	514.08	265.90	152.52	16.0658	32	7.17	198.00	2.86	0.4914
6.4	463.55	261.05	140.08	15.0300	34	5.08	198.00	2.02	0.3483
7.7	390.19	253.17	121.58	13.4513	36	3.60	198.00	1.43	0.2469
9	326.65	245.29	105.05	11.9962	38	2.55	198.00	1.02	0.1750
10.3	271.88	237.41	90.34	10.6585	40	1.81	198.00	0.72	0.1241
11.6	224.89	229.52	77.29	9.4323	44	0.91	198.00	0.36	0.0623
12.9	184.80	221.64	65.77	8.3116	48	0.46	198.00	0.18	0.0313
14.2	150.77	213.76	55.64	7.2906	52	0.23	198.00	0.09	0.0157
15.5	122.08	205.88	46.78	6.3636	56	0.12	198.00	0.05	0.0079
16.8	98.03	198.00	39.06	5.5250	60	0.06	198.00	0.02	0.0040
16.8	98.03	198.00	39.06	6.7209	64	0.03	198.00	0.01	0.0020
17.7	83.97	198.00	33.45	5.7566	68	0.01	198.00	0.006	0.0010
18.6	71.92	198.00	28.65	4.9307	72	0.007	198.00	0.003	0.0005
19.5	61.60	198.00	24.54	4.2233					
20.4	52.76	198.00	21.02	3.6173					

$$r_1 = 6360 \text{ km}$$

$$r^0 = 6376.8 \text{ km}$$

$$p_1 = 1010 \text{ mb}$$

$$T_1 = 299.85^\circ \text{K}$$

$$n_1 = 1.000265717 (\lambda = 0.574 \mu)$$

$$\beta = -6.0625^\circ \text{K km}^{-1}$$

$$R = 2.8704 \times 10^6 \text{ erg g}^{-1} \text{ }^\circ \text{K}^{-1}$$

$$g = 97.8 \times 10^1 \text{ cm sec}^{-2}$$

Table Ib.

Atmospheric Model No. 2  
(Temperature Zone)

$r, \text{ km}$ 6380 +	$p, \text{ mb}$	$T, ^\circ\text{K}$	$(n-1)10^6$	$-\left(\frac{dn}{dr}\right) 10^6, \text{ km}^{-1}$	$r, \text{ km}$ 6380 +	$p, \text{ mb}$	$T, ^\circ\text{K}$	$(n-1)10^6$	$-\left(\frac{dn}{dr}\right) 10^6, \text{ km}^{-1}$
0	1015.00	286.08	280.87	27.2824	18.9	64.81	218.00	23.45	3.6727
0.5	955.68	281.86	267.48	26.2791	20.6	49.66	218.00	17.97	2.8142
1	899.20	278.63	254.58	25.3018	22.3	38.05	218.00	13.77	2.1564
1.5	845.46	275.40	242.17	24.3501	24	29.16	218.00	10.55	1.6523
2	794.36	272.18	230.23	23.4236	26	21.32	218.00	7.71	1.2080
2.5	745.79	268.96	218.74	22.5219	28	15.58	218.00	5.64	0.8832
3	699.65	266.73	207.70	21.6447	30	11.39	218.00	4.12	0.6457
3.5	655.86	262.50	197.10	20.7916	32	8.33	218.00	3.01	0.4720
4	614.32	259.28	186.91	19.9622	34	6.09	218.00	2.20	0.3451
4.8	552.31	254.12	171.45	18.6834	36	4.45	218.00	1.61	0.2523
5.6	495.48	248.96	157.00	17.4629	38	3.25	218.00	1.18	0.1844
6.4	443.48	243.80	143.50	16.2990	40	2.38	218.00	0.86	0.1348
7.2	396.01	238.64	130.91	15.1902	44	1.27	218.00	0.46	0.0721
8	352.74	233.48	119.18	14.1361	48	0.68	218.00	0.25	0.0385
8.8	313.36	228.32	108.28	13.1322	52	0.36	218.00	0.13	0.0206
9.6	277.67	223.16	98.15	12.1799	56	0.19	218.00	0.07	0.0110
10.4	245.33	218.00	88.78	11.2767	60	0.10	218.00	0.04	0.0059
10.4	245.33	218.00	88.78	13.9033	64	0.06	218.00	0.02	0.0031
12.1	187.98	218.00	68.02	10.6535	68	0.03	218.00	0.01	0.0017
13.8	144.04	218.00	52.12	8.1633	72	0.02	218.00	0.006	0.0009
15.5	110.37	218.00	39.94	6.2551					
17.2	84.57	218.00	30.60	4.7930					

$$r_1 = 6380 \text{ km}$$

$$r^0 = 6390.4 \text{ km}$$

$$p_1 = 1015 \text{ mb}$$

$$T_1 = 286.08 ^\circ\text{K}$$

$$n_1 = 1.000280868 \text{ } (\lambda = 0.574 \mu)$$

$$\beta = -6.45 ^\circ\text{K km}^{-1}$$

$$R = 2.8704 \times 10^6 \text{ erg g}^{-1} ^\circ\text{K}^{-1}$$

$$g = 98 \times 10^1 \text{ cm sec}^{-2}$$



Table 1c.

## Atmospheric Model No. 3

(Arctic Zone)

$r$ , km 6400 +	$p$ , mb	$T$ , °K	$(n-1) 10^6$	$-\left(\frac{dn}{dr}\right) 10^6, \text{km}^{-1}$	$r$ , km 6400 +	$p$ , mb	$T$ , °K	$(n-1) 10^6$	$-\left(\frac{dn}{dr}\right) 10^6, \text{km}^{-1}$
0	1020.00	252.50	318.67	56.9646	18.3	70.68	223.00	25.00	3.8358
0.2	992.85	254.68	307.53	54.5009	20.2	52.81	223.00	18.68	2.8659
0.4	966.64	256.87	296.86	52.1633	22.1	39.46	223.00	13.96	2.1413
0.6	941.34	259.06	286.65	49.9446	24	29.48	223.00	10.43	1.5999
0.8	916.90	261.24	276.88	47.8378	26	21.69	223.00	7.67	1.1771
1	893.30	263.42	267.51	45.8362	28	15.96	223.00	5.65	0.8661
1.2	870.48	265.61	258.53	43.9339	30	11.74	223.00	4.15	0.6373
1.4	848.44	267.80	249.93	42.1252	32	8.64	223.00	3.06	0.4689
1.6	827.12	269.98	241.68	40.4047	34	6.36	223.00	2.25	0.3450
1.6	827.12	269.98	241.68	24.7840	36	4.68	223.00	1.65	0.2538
2.5	737.04	264.11	220.15	23.0779	38	3.44	223.00	1.22	0.1868
3.4	655.07	258.24	200.11	21.4549	40	2.53	223.00	0.90	0.1374
4.3	580.64	262.36	181.50	19.9126	44	1.37	223.00	0.48	0.0744
5.2	513.21	246.49	164.25	18.4486	48	0.74	223.00	0.26	0.0403
6.1	452.26	240.62	148.27	17.0609	52	0.40	223.00	0.14	0.0218
7	397.31	234.74	133.52	15.7471	56	0.22	223.00	0.08	0.0118
7.9	347.89	228.87	119.91	14.5050	60	0.12	223.00	0.04	0.0064
8.8	303.56	223.00	107.39	13.3324	64	0.06	223.00	0.02	0.0035
8.8	303.56	223.00	107.39	16.4745	68	0.03	223.00	0.01	0.0019
10.7	226.81	223.00	80.23	12.3089	72	0.02	223.00	0.007	0.0010
12.6	169.46	223.00	59.95	9.1967					
14.5	126.61	223.00	44.79	6.8713					
16.4	94.60	223.00	33.46	5.1339					

$$r_1 = 6400 \text{ km}$$

$$r_1^i = 6401.6 \text{ km}$$

$$r_1^0 = 6408.8 \text{ km}$$

$$p_1 = 1020 \text{ mb}$$

$$T_1 = 252.5^\circ \text{K}$$

$$n_1 = 1.000318670 \quad (\lambda = 0.574 \mu)$$

$$\beta^i = +10.925^\circ \text{K km}^{-1}$$

$$\beta = -6.526^\circ \text{K km}^{-1}$$

$$R = 2.8704 \times 10^6 \text{ erg g}^{-1} \text{ }^\circ \text{K}^{-1}$$

$$g = 98.2 \times 10^1 \text{ cm sec}^{-2}$$

Table IIa.  
Atmospheric Model No. 1  
(Tropical Zone)

Astronomical Refraction,  $\Delta r'' = -\rho'' \int_0^z \left( \frac{dn}{dr} \right) \tan z \, dz$ , for  $z = 60^\circ, 70^\circ, 80^\circ$

$r, \text{ km}$ 5380+	$-\rho'' \left( \frac{dn}{dr} \right) \tan z$			$-\rho'' \left( \frac{dn}{dr} \right) \tan z$			$-\rho'' \left( \frac{dn}{dr} \right) \tan z$		
	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$
0	5.11834	1.73205	2.74748	5.87128	8.8652	14.0525	29.0275	21.3	0.63806
0.8	4.82503	1.73131	2.74488	5.86136	8.85336	13.2448	27.2678	22.2	0.64738
1.6	4.54407	1.73057	2.74246	5.85143	7.8638	12.4619	25.5895	23.1	0.65684
2.4	4.27814	1.72982	2.73983	5.84152	7.3652	11.7136	23.9900	24	0.66187
3.2	4.01791	1.72906	2.73737	5.83163	6.9472	10.9986	22.4657	26	0.26464
4	3.77208	1.72830	2.73481	5.82178	6.5183	10.3180	21.0172	28	0.20176
4.8	3.53731	1.72763	2.73222	5.81197	6.1108	9.6647	19.6380	30	0.14301
5.6	3.31330	1.72678	2.72963	5.83221	5.7213	9.0441	18.3299	32	0.10137
6.4	3.09973	1.72598	2.72702	5.81252	5.3501	8.4530	17.0673	34	0.07185
7.2	2.77419	1.72471	2.72276	5.49068	4.7847	7.8534	15.2044	36	0.05093
8	2.47413	1.72342	2.71847	5.44902	4.2640	6.7258	13.4816	38	0.03610
10.3	2.18628	1.72213	2.71416	5.41762	3.7857	5.9655	11.9084	40	0.02559
11.6	1.94541	1.72082	2.70983	5.38648	3.3477	5.2717	10.4789	44	0.01286
12.9	1.71428	1.71961	2.70549	5.35562	2.9477	4.6380	9.1810	46	0.00646
14.2	1.50371	1.71819	2.70113	5.32506	2.5837	4.0817	8.0073	52	0.00325
15.5	1.31252	1.71687	2.69677	5.29482	2.2534	3.5396	6.9498	56	0.00183
16.8	1.13958	1.71554	2.69240	5.26491	1.9550	3.0882	5.9998	60	0.00082
18.3	0.98823	1.71428	2.68796	5.23491	1.6781	2.7323	5.2084	64	0.00041
17.7	1.18736	1.71463	2.68940	5.24454	2.0366	3.1933	6.2771	68	0.00021
18.6	1.01700	1.71371	2.68636	5.22427	1.7428	2.7320	5.3131	72	0.00010
19.5	0.87109	1.71278	2.68337	5.20414	1.4920	2.3374	4.5333		
20.4	0.74611	1.71186	2.68035	5.18415	1.2772	1.9998	3.8679		

Integrals \* ;

0 — 6.4 km :	44.8045	70.9438	145.0468
6.4 — 16.8 km :	35.8720	56.5197	112.6666
16.8 — 24 km :	9.8000	15.3615	29.7692
24 — 40 km :	3.7214	5.7954	10.9443
40 — 72 km :	0.2605	0.3883	0.6867
Astronomical Refraction :	94'45	148'00	299'11

Formula (30) :

1st term :	94.968	150.735	312.160
2nd term :	-0.525	-1.781	-14.264
$\delta_1$ :	0.007	0.062	1.668
$-\delta_2$ :	-0.001	-0.007	-0.257
$-\delta_3$ :	-0.000	-0.002	-0.271
$\delta_4$ :	0.000	0.000	0.061
	94'45	148'00	298'10

\* Integration formula (Newton-Cotes) :

$$\int_a^b f(x) dx = \frac{b-a}{2.535} \left\{ 0.0689 [f(a) + f(a+8\Delta)] + 0.5888 [f(a+\Delta) + f(a+7\Delta)] - 0.0928 [f(a+2\Delta) + f(a+6\Delta)] + 1.0496 [f(a+3\Delta) + f(a+5\Delta)] - 0.484 f(a+4\Delta) \right\}$$

Table 11b.

Atmospheric Model No. 2  
(Temperate Zone)

Astronomical Refraction,  $\Delta z'' = -\rho'' \int_{r_1}^r \left( \frac{dn}{dr} \right) \tan z \, dr$ , for  $z = 60^\circ, 70^\circ, 80^\circ$

$r, \text{km}$ 6380+	$-\rho'' \left( \frac{dn}{dr} \right) \tan z$			$-\rho'' \left( \frac{dn}{dr} \right) \tan z$			$-\rho'' \left( \frac{dn}{dr} \right) \tan z$		
	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$
0	5.62582	1.72206	2.74748	5.67128	9.7442	15.4568	31.9056	18.9	1.71359
0.5	5.41900	1.73180	2.74695	5.65910	9.3936	14.8903	30.687	20.6	1.71184
1	5.21754	1.73115	2.74442	5.64680	9.0323	14.3181	29.4829	22.3	1.71008
1.5	5.02136	1.73069	2.74288	5.63470	8.6804	13.7729	28.2838	24	1.70832
2	4.83035	1.73023	2.74132	5.62248	8.3576	13.2415	27.1886	26	1.70656
2.5	4.64448	1.72977	2.73978	5.61028	8.0339	12.7247	26.0966	28	1.70481
3	4.46381	1.72931	2.73820	5.59804	7.7190	12.2223	24.9875	30	1.70318
3.5	4.28773	1.72884	2.73662	5.58582	7.4128	11.7339	23.8504	32	1.70156
4	4.11672	1.72837	2.73504	5.57360	7.1152	11.2594	22.6948	34	1.69991
4.5	3.95307	1.72781	2.73350	5.56147	6.8268	10.7985	21.4002	36	1.69826
5	3.80141	1.72685	2.72994	5.54958	6.5458	10.3518	19.9322	38	1.69662
5.5	3.66143	1.72608	2.72736	5.53780	6.2721	9.9186	18.5366	40	1.69498
6	3.53280	1.72531	2.72477	5.52618	6.0081	9.5001	17.2168	42	1.69334
6.5	3.41523	1.72453	2.72217	5.51463	5.7527	9.0958	15.9647	44	1.69170
7	3.30841	1.72376	2.71956	5.50314	5.5066	8.7057	14.7798	46	1.69006
7.5	3.21203	1.72298	2.71693	5.49178	5.2691	8.3290	13.6598	48	1.68842
8	3.12578	1.72221	2.71430	5.48045	5.0398	7.9626	12.6026	50	1.68678
8.5	3.04911	1.72144	2.71167	5.46914	4.8178	7.6063	11.6074	52	1.68514
9	2.98184	1.72067	2.70904	5.45784	4.6025	7.2600	10.6738	54	1.68350
9.5	2.92417	1.71990	2.70641	5.44654	4.3938	6.9237	9.8002	56	1.68186
10	2.87581	1.71913	2.70378	5.43524	4.1909	6.5974	9.0000	58	1.68022
10.5	2.83695	1.71836	2.70115	5.42394	3.9938	6.2811	8.2732	60	1.67858
11	2.80759	1.71759	2.69852	5.41264	3.8025	5.9748	7.6196	62	1.67694
11.5	2.78773	1.71682	2.69589	5.40134	3.6169	5.6785	7.0380	64	1.67530
12	2.76787	1.71605	2.69326	5.39004	3.4362	5.3922	6.5288	66	1.67366
12.5	2.74801	1.71528	2.69063	5.37874	3.2605	5.1159	6.0912	68	1.67202
13	2.72815	1.71451	2.68800	5.36744	3.0898	4.8496	5.6746	70	1.67038
13.5	2.70829	1.71374	2.68537	5.35614	2.9241	4.5933	5.2780	72	1.66874
14	2.68843	1.71297	2.68274	5.34484	2.7634	4.3470	4.8914	74	1.66710
14.5	2.66857	1.71220	2.68011	5.33354	2.6077	4.1007	4.5148	76	1.66546
15	2.64871	1.71143	2.67748	5.32224	2.4570	3.8644	4.1532	78	1.66382
15.5	2.62885	1.71066	2.67485	5.31094	2.3113	3.6281	3.8066	80	1.66218
16	2.60899	1.70989	2.67222	5.29964	2.1706	3.3918	3.4650	82	1.66054
16.5	2.58913	1.70912	2.66959	5.28834	2.0349	3.1555	3.1284	84	1.65890
17	2.56927	1.70835	2.66696	5.27704	1.9042	2.9192	2.8018	86	1.65726
17.5	2.54941	1.70758	2.66433	5.26574	1.7785	2.6829	2.4852	88	1.65562
18	2.52955	1.70681	2.66170	5.25444	1.6578	2.4466	2.1786	90	1.65398
18.5	2.50969	1.70604	2.65907	5.24314	1.5421	2.2103	1.8820	92	1.65234
19	2.48983	1.70527	2.65644	5.23184	1.4314	1.9740	1.5954	94	1.65070
19.5	2.46997	1.70450	2.65381	5.22054	1.3257	1.7377	1.3188	96	1.64906
20	2.45011	1.70373	2.65118	5.20924	1.2250	1.5014	1.0522	98	1.64742
20.5	2.43025	1.70296	2.64855	5.19794	1.1293	1.2651	0.7956	100	1.64578
21	2.41039	1.70219	2.64592	5.18664	1.0386	1.0288	0.5490	102	1.64414
21.5	2.39053	1.70142	2.64329	5.17534	0.9529	0.7925	0.3024	104	1.64250
22	2.37067	1.70065	2.64066	5.16404	0.8722	0.5562	0.0558	106	1.64086
22.5	2.35081	1.70000	2.63803	5.15274	0.7965	0.3199	0.0000	108	1.63922
23	2.33095	1.69923	2.63540	5.14144	0.7258	0.0836	0.0000	110	1.63758
23.5	2.31109	1.69846	2.63277	5.13014	0.6601	0.0000	0.0000	112	1.63594
24	2.29123	1.69769	2.63014	5.11884	0.6000	0.0000	0.0000	114	1.63430
24.5	2.27137	1.69692	2.62751	5.10754	0.5449	0.0000	0.0000	116	1.63266
25	2.25151	1.69615	2.62488	5.09624	0.4948	0.0000	0.0000	118	1.63102
25.5	2.23165	1.69538	2.62225	5.08494	0.4487	0.0000	0.0000	120	1.62938
26	2.21179	1.69461	2.61962	5.07364	0.4070	0.0000	0.0000	122	1.62774
26.5	2.19193	1.69384	2.61699	5.06234	0.3693	0.0000	0.0000	124	1.62610
27	2.17207	1.69307	2.61436	5.05104	0.3356	0.0000	0.0000	126	1.62446
27.5	2.15221	1.69230	2.61173	5.03974	0.3059	0.0000	0.0000	128	1.62282
28	2.13235	1.69153	2.60910	5.02844	0.2802	0.0000	0.0000	130	1.62118
28.5	2.11249	1.69076	2.60647	5.01714	0.2585	0.0000	0.0000	132	1.61954
29	2.09263	1.69000	2.60384	5.00584	0.2408	0.0000	0.0000	134	1.61790
29.5	2.07277	1.68923	2.60121	4.99454	0.2271	0.0000	0.0000	136	1.61626
30	2.05291	1.68846	2.59858	4.98324	0.2174	0.0000	0.0000	138	1.61462
30.5	2.03305	1.68769	2.59595	4.97194	0.2107	0.0000	0.0000	140	1.61298
31	2.01319	1.68692	2.59332	4.96064	0.2070	0.0000	0.0000	142	1.61134
31.5	1.99333	1.68615	2.59069	4.94934	0.2053	0.0000	0.0000	144	1.60970
32	1.97347	1.68538	2.58806	4.93804	0.2070	0.0000	0.0000	146	1.60806
32.5	1.95361	1.68461	2.58543	4.92674	0.2107	0.0000	0.0000	148	1.60642
33	1.93375	1.68384	2.58280	4.91544	0.2174	0.0000	0.0000	150	1.60478
33.5	1.91389	1.68307	2.58017	4.90414	0.2271	0.0000	0.0000	152	1.60314
34	1.89403	1.68230	2.57754	4.89284	0.2408	0.0000	0.0000	154	1.60150
34.5	1.87417	1.68153	2.57491	4.88154	0.2585	0.0000	0.0000	156	1.60000
35	1.85431	1.68076	2.57228	4.87024	0.2802	0.0000	0.0000	158	1.59846
35.5	1.83445	1.68000	2.56965	4.85894	0.3059	0.0000	0.0000	160	1.59692
36	1.81459	1.67923	2.56702	4.84764	0.3356	0.0000	0.0000	162	1.59538
36.5	1.79473	1.67846	2.56439	4.83634	0.3693	0.0000	0.0000	164	1.59384
37	1.77487	1.67769	2.56176	4.82504	0.4070	0.0000	0.0000	166	1.59230
37.5	1.75501	1.67692	2.55913	4.81374	0.4487	0.0000	0.0000	168	1.59076
38	1.73515	1.67615	2.55650	4.80244	0.4948	0.0000	0.0000	170	1.58922
38.5	1.71529	1.67538	2.55387	4.79114	0.5449	0.0000	0.0000	172	1.58768
39	1.69543	1.67461	2.55124	4.77984	0.5993	0.0000	0.0000	174	1.58614
39.5	1.67557	1.67384	2.54861	4.76854	0.6570	0.0000	0.0000	176	1.58460
40	1.65571	1.67307	2.54598	4.75724	0.7185	0.0000	0.0000	178	1.58306
40.5	1.63585	1.67230	2.54335	4.74594	0.7848	0.0000	0.0000	180	1.58152
41	1.61599	1.67153	2.54072	4.73464	0.8569	0.0000	0.0000	182	1.58000
41.5	1.59613	1.67076	2.53809	4.72334	0.9340	0.0000	0.0000	184	1.57846
42	1.57627	1.67000	2.53546	4.71204	1.0171	0.0000	0.0000	186	1.57692
42.5	1.55641	1.66923	2.53283	4.70074	1.1074	0.0000	0.0000	188	1.57538
43	1.53655	1.66846	2.53020	4.68944	1.2049	0.0000	0.0000	190	1.57384
43.5	1.51669	1.66769	2.52757	4.67814	1.3093	0.0000	0.0000	192	1.57230
44	1.49683	1.66692	2.52494	4.66684	1.4208	0.0000	0.0000	194	1.57076
44.5	1.47697	1.66615	2.52231	4.65554	1.5393	0.0000	0.0000	196	1.56922
45	1.45711	1.66538	2.51968	4.64424	1.6648	0.0000	0.0000	198	1.56768
45.5	1.43725	1.66461	2.51705	4.63294	1.7983	0.0000	0.0000	200	1.56614
46	1.41739	1.66384	2.51442	4.62164	1.9398	0.0000	0.0000	202	1.56460
46.5	1.39753	1.66307	2.51179	4.61034	2.0893	0.0000	0.0000	204	1.56306
47	1.37767	1.66230	2.50916	4.59904	2.2468	0.0000	0.0000	206	1.56152
47.5	1.35781	1.66153	2.50653	4.58774	2.4123	0.0000	0.0000	208	1.55998
48	1.33795	1.66076	2.50390	4.57644	2.5858	0.0000	0.0000	210	1.55844
48.5	1.31809	1.66000	2.50127	4.56514	2.7673	0.0000	0.0000	212	1.55690
49	1.29823	1.65923	2.49864	4.55384	2.9568	0.0000	0.0000	214	1.55536
49.5	1.27837	1.65.							

Table IIc.

Atmospheric Model No. 3

(Arctic Zone)

$$\text{Astronomical Refraction, } \Delta z'' = -\rho'' \int_1^{r'} \left( \frac{dn}{dn} \right) \tan z \, dz, \text{ for } z = 60^\circ, 70^\circ, \text{ and } 80^\circ$$

$r, \text{ km}$ 9400 +	$\tan z$			$-\rho'' \left( \frac{dn}{dn} \right) \tan z$			$\tan z$			$-\rho'' \left( \frac{dn}{dn} \right) \tan z$		
	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$	$z_1 = 60^\circ$	$z_1 = 70^\circ$	$z_1 = 80^\circ$
0	11.74008	1.73205	2.74748	6.67128	32.2720	66.6152	18.3	0.79117	1.71453	2.69007	5.24233	1.3555
0.2	11.23815	1.73191	2.74701	5.66780	30.8713	63.8923	20.2	0.59113	1.71267	2.68288	5.19960	1.0124
0.4	10.79627	1.73177	2.74652	5.66364	29.5423	60.9197	22.1	0.44167	1.71081	2.67530	5.15766	0.7655
0.6	10.39867	1.73162	2.74603	5.65971	28.2813	58.2801	24	0.32969	1.70895	2.66804	5.11953	0.5811
0.8	9.96482	1.73147	2.74552	5.65589	27.0833	55.7907	26	0.24280	1.70698	2.66038	5.07414	0.4144
1	9.48187	1.73132	2.74501	5.65182	25.9465	53.4183	28	0.17865	1.70483	2.65252	5.02986	0.3045
1.2	9.00668	1.73117	2.74449	5.64748	24.8642	51.1643	30	0.13145	1.70247	2.64500	4.98218	0.2238
1.4	8.60878	1.73101	2.74396	5.64327	23.8362	49.0219	32	0.09872	1.70042	2.63748	4.93251	0.1645
1.6	8.32209	1.73085	2.74342	5.63902	22.8694	46.9948	34	0.07116	1.69837	2.63033	4.91376	0.1208
1.8	8.11083	1.73068	2.74342	5.63902	21.9312	45.0200	36	0.05236	1.69633	2.63044	4.87586	0.0888
2	7.96912	1.73053	2.74304	5.63712	21.0430	43.0870	38	0.03852	1.69429	2.62400	4.83879	0.0653
2.2	7.89450	1.73039	2.73783	5.63521	20.2136	41.1650	40	0.02835	1.69226	2.61760	4.80264	0.0742
2.4	7.80851	1.73026	2.73600	5.63330	19.4373	39.2498	42	0.02135	1.69021	2.61063	4.77236	0.0726
2.6	7.70468	1.72975	2.73215	5.63140	18.7049	37.3413	44	0.01635	1.68819	2.60344	4.74609	0.0388
2.8	7.58184	1.72965	2.72927	5.62953	18.0030	35.4369	46	0.01240	1.68620	2.59611	4.72007	0.0207
3	7.44764	1.72959	2.72638	5.62770	17.3270	33.5360	48	0.00940	1.68423	2.58874	4.69361	0.0111
3.2	7.30161	1.72949	2.72349	5.62586	16.6763	31.6382	50	0.00732	1.68228	2.58139	4.66704	0.0059
3.4	7.14470	1.72940	2.72054	5.62404	16.0497	29.7435	52	0.00571	1.68037	2.57410	4.64067	0.0018
3.6	6.97874	1.72930	2.71764	5.62224	15.4443	27.8507	54	0.00459	1.67847	2.56684	4.61436	0.0011
3.8	6.80374	1.72920	2.71474	5.62044	14.8580	25.9582	56	0.00389	1.67658	2.55960	4.58801	0.0034
4	6.62074	1.72910	2.71184	5.61864	14.2897	24.0660	58	0.00321	1.67470	2.55236	4.56174	0.0017
4.2	6.42974	1.72900	2.70894	5.61684	13.7374	22.1740	60	0.00261	1.67283	2.54510	4.53541	0.0008
4.4	6.23074	1.72890	2.70604	5.61504	13.1999	20.2782	62	0.00211	1.67098	2.53784	4.50906	0.0001
4.6	6.02374	1.72880	2.70314	5.61324	12.6770	18.3795	64	0.00171	1.66913	2.53058	4.48271	0.0000
4.8	5.80874	1.72870	2.70024	5.61144	12.1682	16.4762	66	0.00141	1.66728	2.52332	4.45636	0.0000
5	5.58184	1.72860	2.69734	5.60964	11.6724	14.5680	68	0.00111	1.66543	2.51606	4.43001	0.0000
5.2	5.35474	1.72850	2.69444	5.60784	11.1895	12.6540	70	0.00081	1.66358	2.50880	4.40366	0.0000
5.4	5.11574	1.72840	2.69154	5.60604	10.7195	10.7340	72	0.00051	1.66173	2.50154	4.37731	0.0000
5.6	4.86874	1.72830	2.68864	5.60424	10.2624	8.8070	74	0.00021	1.65988	2.49428	4.35096	0.0000
5.8	4.61374	1.72820	2.68574	5.60244	9.8174	6.8700	76	0.00001	1.65803	2.48702	4.32461	0.0000
6	4.35074	1.72810	2.68284	5.60064	9.3844	4.9230	78	0.00000	1.65618	2.47976	4.29826	0.0000
6.2	4.07974	1.72800	2.67994	5.59884	8.9624	2.9760	80	0.00000	1.65433	2.47250	4.27191	0.0000
6.4	3.80074	1.72790	2.67704	5.59704	8.5514	1.0290	82	0.00000	1.65248	2.46524	4.24556	0.0000
6.6	3.51374	1.72780	2.67414	5.59524	8.1514	-0.9180	84	0.00000	1.65063	2.45798	4.21921	0.0000
6.8	3.21874	1.72770	2.67124	5.59344	7.7624	-2.9710	86	0.00000	1.64878	2.45072	4.19286	0.0000
7	2.91574	1.72760	2.66834	5.59164	7.3844	-5.0240	88	0.00000	1.64693	2.44346	4.16651	0.0000
7.2	2.60474	1.72750	2.66544	5.58984	7.0174	-7.0770	90	0.00000	1.64508	2.43620	4.14016	0.0000
7.4	2.28574	1.72740	2.66254	5.58804	6.6614	-9.1300	92	0.00000	1.64323	2.42894	4.11381	0.0000
7.6	1.95874	1.72730	2.65964	5.58624	6.3164	-11.1830	94	0.00000	1.64138	2.42168	4.08746	0.0000
7.8	1.62374	1.72720	2.65674	5.58444	5.9824	-13.2360	96	0.00000	1.63953	2.41442	4.06111	0.0000
8	1.28074	1.72710	2.65384	5.58264	5.6584	-15.2890	98	0.00000	1.63768	2.40716	4.03476	0.0000
8.2	0.92974	1.72700	2.65094	5.58084	5.3444	-17.3420	100	0.00000	1.63583	2.40000	4.00841	0.0000
8.4	0.57074	1.72690	2.64804	5.57904	5.0404	-19.3950	102	0.00000	1.63398	2.39274	3.98206	0.0000
8.6	0.20474	1.72680	2.64514	5.57724	4.7464	-21.4480	104	0.00000	1.63213	2.38548	3.95571	0.0000
8.8	-0.16874	1.72670	2.64224	5.57544	4.4624	-23.5010	106	0.00000	1.63028	2.37822	3.92936	0.0000
9	-0.53274	1.72660	2.63934	5.57364	4.1884	-25.5540	108	0.00000	1.62843	2.37096	3.90301	0.0000
9.2	-0.88774	1.72650	2.63644	5.57184	3.9244	-27.6070	110	0.00000	1.62658	2.36370	3.87666	0.0000
9.4	-1.23274	1.72640	2.63354	5.57004	3.6704	-29.6600	112	0.00000	1.62473	2.35644	3.85031	0.0000
9.6	-1.56774	1.72630	2.63064	5.56824	3.4264	-31.7130	114	0.00000	1.62288	2.34918	3.82396	0.0000
9.8	-1.89274	1.72620	2.62774	5.56644	3.1924	-33.7660	116	0.00000	1.62103	2.34192	3.79761	0.0000
10	-2.20774	1.72610	2.62484	5.56464	2.9684	-35.8190	118	0.00000	1.61918	2.33466	3.77126	0.0000
10.2	-2.51274	1.72600	2.62194	5.56284	2.7544	-37.8720	120	0.00000	1.61733	2.32740	3.74491	0.0000
10.4	-2.80774	1.72590	2.61904	5.56104	2.5504	-39.9250	122	0.00000	1.61548	2.32014	3.71856	0.0000
10.6	-3.09274	1.72580	2.61614	5.55924	2.3564	-41.9780	124	0.00000	1.61363	2.31288	3.69221	0.0000
10.8	-3.36774	1.72570	2.61324	5.55744	2.1724	-44.0310	126	0.00000	1.61178	2.30562	3.66586	0.0000
11	-3.63274	1.72560	2.61034	5.55564	1.9984	-46.0840	128	0.00000	1.60993	2.29836	3.63951	0.0000
11.2	-3.88774	1.72550	2.60744	5.55384	1.8344	-48.1370	130	0.00000	1.60808	2.29110	3.61316	0.0000
11.4	-4.13274	1.72540	2.60454	5.55204	1.6804	-50.1900	132	0.00000	1.60623	2.28384	3.58681	0.0000
11.6	-4.36774	1.72530	2.60164	5.55024	1.5364	-52.2430	134	0.00000	1.60438	2.27658	3.56046	0.0000
11.8	-4.59274	1.72520	2.59874	5.54844	1.4024	-54.2960	136	0.00000	1.60253	2.26932	3.53411	0.0000
12	-4.80774	1.72510	2.59584	5.54664	1.2784	-56.3490	138	0.00000	1.60068	2.26206	3.50776	0.0000
12.2	-5.01274	1.72500	2.59294	5.54484	1.1644	-58.4020	140	0.00000	1.59883	2.25480	3.48141	0.0000
12.4	-5.20774	1.72490	2.59004	5.54304	1.0604	-60.4550	142	0.00000	1.59698	2.24754	3.45506	0.0000
12.6	-5.39274	1.72480	2.58714	5.54124	0.9664	-62.5080	144	0.00000	1.59513	2.24028	3.42871	0.0000
12.8	-5.56774	1.72470	2.58424	5.53944	0.8824	-64.5610	146	0.00000	1.59328	2.23302	3.40236	0.0000
13	-5.73274	1.72460	2.58134	5.53764	0.8084	-66.6140	148	0.00000	1.59143	2.22576	3.37601	0.0000
13.2	-5.88774	1.72450	2.57844	5.53584	0.7444	-68.6670	150	0.00000	1.58958	2.21850	3.34966	0.0000
13.4	-6.03274	1.72440	2.57554	5.53404	0.6904	-70.7200	152	0.00000	1.58773	2.21124	3.32331	0.0000
13.6	-6.16774	1.72430	2.57264	5.53224	0.6464	-72.7730	154	0.00000	1.58588	2.20398	3.29696	0.0000
13.8	-6.29274	1.72420	2.56974	5.53044	0.6124	-74.8260	156	0.00000	1.58403	2.19672	3.27061	0.0000
14	-6.40774	1.72410	2.56684	5.52864	0.5884	-76.8790	158	0.00000	1.58218	2.18946	3.24426	0.0000
14.2	-6.51274	1.72400	2.56394	5.52684	0.5744	-78.9320	160	0.00000	1.58033	2.18220	3.21791	0.0000
14.4	-6.60774	1.72390	2.56104	5.52504	0.5604	-80.9850	162	0.00000	1.57848	2.17494	3.19156	0.0000
14.6	-6.69274	1.72380	2.55814	5.52324	0.5564	-83.0380	164	0.00000	1.57663	2.16768	3.16521	0.0000
14.8	-6.76774	1.72370	2.55524	5.52144	0.5524	-85.0910	166	0.00000	1.57478	2.16042	3.13886	0.0000
15	-6.83274	1.72360	2.55234	5.51964	0.5484	-87.1440	168	0.00000	1.57293	2.15316	3.11251	0.0000
15.2</												