J. SAASTAMOINEN
National Research Council of Canada

CONTRIBUTIONS TO THE THEORY OF ATMOSPHERIC REFRACTION (1)

Abstract

Since the barometer measures the weight of the overlying atmosphere, it follows by the law of Gladstone and Dale that the height integral $\int (n-1)\, dr$ of the atmospheric refractivity for light, taken from ground level up to the top of the atmosphere, is directly proportional to ground pressure. The refractivity integral, therefore, can be determined without detailed knowledge of the height distribution of the refractive index, which not only simplifies the derivation of refraction formulas in which atmospheric models have been used hitherto, but also improves their accuracy. For zenith distances not exceeding about 75 degrees, the correction for astronomical refraction will be given by the standard formula

$$\Delta z_0'' = 16''.271 \tan z \left[1 + 0.0000394 \tan^2 z \left(\frac{p - 0.156e}{T}\right)\right]\left(\frac{p - 0.156e}{T}\right) -$$

$$- 0''.0749 \left(\tan^3 z + \tan z\right)\left(\frac{p}{1000}\right)$$

where z is the apparent zenith distance, p is the total pressure and e is the partial pressure of water vapour, both in millibars, and T is the absolute temperature in degrees Kelvin Part II of the paper contains further applications of the theory to refraction problems in satellite geodesy, including the photogrammetric refraction and the atmospheric corrections in the ranging of artificial satellites.

Part I. Astronomical Refraction

Derivation of General Formula for Astronomical Refraction

In Figure 1, the law of refraction applied at point **P** gives

$$(n + dn) \sin z = n \sin (z + dz) = n (\sin z + \cos z\, dz)$$

from which immediately follows the differential equation $d(\Delta z) = dz = (\tan z / n)\, dn$ and the corresponding integral equation

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$$(0 \leq z_1 \leq 90°) \; ; \qquad \Delta z = \int_1^{n_1} \frac{\tan z}{z} \, dn \tag{1}$$

which is the basic mathematical expression of the correction for astronomical refraction.

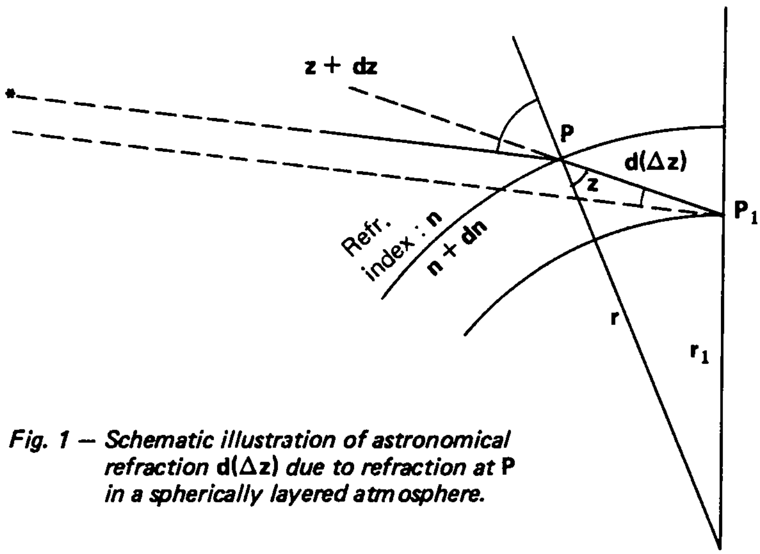

Fig. 1 — Schematic illustration of astronomical refraction d(Δz) *due to refraction at* P *in a spherically layered atmosphere.*

Since z is not, in general, constant along the light path but depends upon the refractive index according to the law of refraction

$$n\, r \sin z = n_1\, r_1 \sin z_1 = const. \tag{2}$$

it will be necessary to find a suitable expression for $\tan z$ that makes (1) integrable. Setting $n_1\, r_1 / (nr) = y$ for brevity, we have from (2)

$$\sin^2 z = y^2 \tan^2 z_1 / (1 + \tan^2 z_1)$$

$$\cos^2 z = (1 + \tan^2 z_1 - y^2 \tan^2 z_1) / (1 + \tan^2 z_1)$$

and

$$\tan z = y \tan z_1 \left[1 + \tan^2 z_1 \, (1 - y^2) \right]^{-\frac{1}{2}} =$$

$$= y \tan z_1 - \frac{1}{2} y (1 - y^2) \tan^3 z_1 + \frac{3}{8} y (1 - y^2)^2 \tan^5 z_1 -$$

$$- \frac{5}{16} y (1 - y^2)^3 \tan^7 z_1 + \frac{35}{128} y (1 - y^2)^4 \tan^9 z_1 - \frac{63}{256} y (1 - y^2)^5 \tan^{11} z_1 + \ldots$$

Neglecting the subsequent terms in the binomial expansion, the first five may be written identically

$$\tan z = \tan z_1 + \left(y - \frac{r_1}{r}\right)(\tan^3 z_1 + \tan z_1) - \left(\frac{r - r_1}{r}\right)(\tan^3 z_1 + \tan z_1) +$$

$$+ \left(1 + \frac{1}{2}y\right)(1-y)^2 \tan^3 z_1 + \frac{3}{8} y (1+y)^2 (1-y)^2 \tan^5 z_1 -$$

$$- \frac{5}{16} y (1+y)^3 (1-y)^3 \tan^7 z_1 + \frac{35}{128} y (1+y)^4 (1-y)^4 \tan^9 z_1$$

into which we substitute the approximation

$$y - \frac{r_1}{r} = \left(\frac{r_1}{r}\right)\left(\frac{n_1 - n}{n}\right) = n_1 - n$$

$$\frac{r - r_1}{r} = \frac{1}{r_1}(r - r_1) - \frac{1}{r_1^2}(r - r_1)^2$$

$$\left(1 + \frac{1}{2}y\right)(1-y)^2 = \frac{3}{2}(1-y)^2 = \frac{3}{2r_1^2}(r - r_1)^2$$

$$\frac{3}{8} y (1+y)^2 (1-y)^2 = \frac{3}{2}(1-y)^2 = \frac{3}{2}\left[\left(1 - \frac{r_1}{r}\right) - (n_1 - n)\right]^2 =$$

$$= \frac{3}{2r_1^2}(r - r_1)^2 - \frac{3}{r_1}(n_1 - n)(r - r_1)$$

$$\frac{5}{16} y (1+y)^3 (1-y)^3 = \frac{5}{2}(1-y)^3 = \frac{5}{2r_1^3}(r - r_1)^3$$

$$\frac{35}{128} y (1+y)^4 (1-y)^4 = \frac{35}{8}(1-y)^4 = \frac{35}{8r_1^4}(r - r_1)^4$$

and obtain

$$\tan z = \tan z_1 + (\tan^3 z_1 + \tan z_1)(n_1 - n) - A_1 (r - r_1) + A_2 (r - r_1)^2 -$$

$$- A_2'(n_1 - n)(r - r_1) - A_3 (r - r_1)^3 + A_4 (r - r_1)^4 \qquad (3)$$

where the coefficients are :

4

$$
\begin{aligned}
A_1 &= (\tan^3 z_1 + \tan z_1)/r_1 \\
A_2 &= (3\tan^5 z_1 + 5\tan^3 z_1 + \dots)/(2\,r_1^2) \\
A_2' &= 3\tan^5 z_1/r_1 \\
A_3 &= 5\tan^7 z_1/(2\,r_1^3) \\
A_4 &= 35\tan^9 z_1/(8\,r_1^4)
\end{aligned}
\tag{4}
$$

By the substitution of (3), integral (1) breaks down into seven terms, of which the first two can be solved at once :

$$
\tan z_1 \int_1^{n_1} \frac{dn}{n} = \tan z_1 \log n_1 = \tan z_1 \log\left[1 + (n_1 - 1)\right] =
$$

$$
= \tan z_1 (n_1 - 1) - \frac{1}{2}\tan z_1 (n_1 - 1)^2 + \dots
$$

$$
(\tan^3 z_1 + \tan z_1)\int_1^{n_1} (n_1 - n)\,dn = \frac{1}{2}(\tan^3 z_1 + \tan z_1)(n_1 - 1)^2
$$

Since n is nearly unity $(1 \leq n < 1.0004)$, all the terms of higher than second order will be omitted in the first integral, as well as n in the denominator of the subsequent ones. Equation (1) then becomes

$$
\begin{aligned}
\Delta z = {} & \tan z_1 (n_1 - 1) + \frac{1}{2}\tan^3 z_1 (n_1 - 1)^2 - A_1 \int_1^{n_1} (r - r_1)\,dn + \\
& + A_2 \int_1^{n_1} (r - r_1)^2\,dn - A_2' \int_1^{n_1} (n_1 - n)(r - r_1)\,dn - \\
& - A_3 \int_1^{n_1} (r - r_1)^3\,dn + A_4 \int_1^{n_1} (r - r_1)^4\,dn
\end{aligned}
\tag{5}
$$

The five remaining atmospheric integrals can be determined, as follows.

$$\text{Integral} \int_{1}^{n_1} (r - r_1)\, dn.$$

In physical meteorology, the atmosphere may be thought of as a mixture of two ideal gases, dry air and water vapour. If we denote the total pressure, the partial pressure of water vapour and the absolute temperature by p, e and T respectively, the densities of the dry–air and water–vapour components are, as stated by the perfect gas law.

$$\rho_d = \frac{p - e}{RT} \quad \text{and} \quad \rho_w = \frac{e}{R_w T}$$

where R and R_w stand for the appropriate gas constants. The density of the mixture is, of course, equal to $\rho_d + \rho_w$, or

$$\rho = \frac{p}{RT} - \left(1 - \frac{R}{R_w}\right) \frac{e}{RT}$$

The atmosphere being in hydrostatic equilibrium, pressure p measured at any height level is equal to the total weight of the air contained in a vertical column of unit cross section, reaching from the point of observation $(r = r_1)$ up to the top of the atmosphere $(r = r')$. Consequently,

$$\int_{r_1}^{r'} \rho\, dr = \frac{1}{R} \int_{r_1}^{r'} \left(\frac{p}{T}\right) dr - \frac{1}{R}\left(1 - \frac{R}{R_w}\right) \int_{r_1}^{r'} \left(\frac{e}{T}\right) dr = \frac{p_1}{g} \tag{6}$$

where g is the local value of gravity at the centroid of the atmospheric column.

The refractivity of moist air for electromagnetic radiation may be written

$$n - 1 = \frac{(n_0 - 1)\, T_0}{p_0} \left(\frac{p}{T}\right) - c_w\, (e/T) + c_{w'}\, (e/T^2) \tag{7}$$

where n_0 is the refractive index of dry air at pressure p_0 and temperature T_0, and c_w and $c_{w'}$ are constants. The corresponding height integral

$$\int_{r_1}^{r'} (n - 1)\, dr = \frac{(n_0 - 1)\, T_0}{p_0} \int_{r_1}^{r'} \left(\frac{p}{T}\right) dr - c_w \int_{r_1}^{r'} \left(\frac{e}{T}\right) dr + c_{w'} \int_{r_1}^{r'} \left(\frac{e}{T^2}\right) dr$$

can be readily determined with the aid of equation (6). This gives

$$\int_{r_1}^{r'} (n-1)\,dr = \frac{(n_0-1)\,RT_0}{p_0\,g}\,p_1 + \left[\frac{(n_0-1)\,T_0}{p_0}\left(1-\frac{R}{R_w}\right) - c_w\right]\int_{r_1}^{r'}\left(\frac{e}{T}\right)dr +$$

$$+ c_{w'}\int_{r_1}^{r'}\left(\frac{e}{T^2}\right)dr \qquad (8)$$

Equation (8) expresses the value of the refractivity integral in terms of ground pressure p_1, with minor corrections included due to the presence of water vapour in the atmosphere.

As far as the astronomical refraction is concerned, the contribution of humidity to the refractivity integral is negligible, and the last two terms in equation (8) can be omitted. Setting

$$u = r - r_1 \qquad v = n - 1$$
$$du = dr \qquad dv = dn$$

and integrating by parts :

$$\int (r-r_1)\,dn = \int u\,dv = uv - \int v\,du = (r-r_1)(n-1) - \int (n-1)\,dr\,,$$

we then obtain from (8) and (7) the important relationships

$$\int_1^{n_1} (r-r_1)\,dn = \int_{r_1}^{r'} (n-1)\,dr = \frac{(n_0-1)\,RT_0}{p_0\,g}\,p_1 = \frac{R}{g}(n_1-1)\,T_1 \qquad (9)$$

$$\textbf{Integral} \int_1^{n_1} (r-r_1)^2\,dn\,.$$

This integral requires some consideration of the vertical distribution of pressure and temperature in the atmosphere. We shall determine its value in two parts, the stratospheric component and the tropospheric component. The state of the atmosphere at the bounding surface, the tropopause, shall be denoted by superscripts p^0, T^0, etc..., and it is assumed to be known.

Throughout the stratosphere, the temperature may be taken as constant, and equal to temperature T^0 at the tropopause. Integration of the hydrostatic equation for fluids, $dp = g\,\rho\,dr$, on the condition $\rho = p/(RT^0)$ gives the pressure as

$$p = p^0\, e^{m\,(r-r^0)} \qquad (10)$$

where e is the base of natural logarithms, and $m = -g/(RT^0)$ is constant. Similarly,

$$n - 1 = (n^0 - 1)\, e^{m(r - r^0)} \tag{11}$$

and differentiating (11)

$$dn = m\,(n^0 - 1)\, e^{m(r - r^0)}\, dr = m\,(n - 1)\, dr \tag{12}$$

Since identically

$$r - r_1 = (r^0 - r_1) + (r - r^0)$$

$$(r - r_1)^2 = (r^0 - r_1)^2 + 2\,(r^0 - r_1)\,(r - r^0) + (r - r^0)^2$$

we have first, using (9)

$$\int_1^{n^0} (r - r_1)^2\, dn = (r^0 - r_1)^2\,(n^0 - 1) + \frac{2R}{g}\,(r^0 - r_1)\,(n^0 - 1)\,T^0 + \int_1^{n^0} (r - r^0)^2\, dn$$

Now from (12)

$$\int (r - r^0)^2\, dn = m\,(n^0 - 1) \int (r - r^0)^2\, e^{m(r - r^0)}\, dr =$$

$$= m\,(n^0 - 1)\, \frac{e^{m(r - r^0)}}{m^3}\,[m^2\,(r - r^0)^2 - 2m\,(r - r^0) + 2] + C =$$

$$= (n - 1)\left[(r - r^0)^2 - \frac{2}{m}\,(r - r^0) + \frac{2}{m^2}\right] + C$$

where C is the constant of integration. This gives

$$\int_1^{n^0} (r - r^0)^2\, dn = \frac{2\,(n^0 - 1)}{m^2} = \frac{2R^2}{g^2}\,(n^0 - 1)\,T^{0\,2} \tag{13}$$

and the total stratospheric component is consequently

$$\int_1^{n^0} (r - r_1)^2\, dn = (r^0 - r_1)^2\,(n^0 - 1) + \frac{2R}{g}\,(r^0 - r_1)\,(n^0 - 1)\,T^0 + \frac{2R^2}{g^2}\,(n^0 - 1)\,T^{0\,2} \tag{14}$$

Through most of the troposphere, the temperature decreases with height at a fairly uniform rate which varies slightly with latitude and season, although in the polar regions there exists a permanent inversion in the lower troposphere where the actual temperatures increase with height. Integration of the hydrostatic equation on the conditions $\rho = \mathbf{p/(RT)}$ and

$$\mathbf{T = T_1 + \beta (r - r_1)} \tag{15}$$

where the vertical gradient of temperature, $\beta = \mathbf{dT/dr}$, is assumed constant gives the pressure as

$$\mathbf{p = p_1 \left(\frac{T}{T_1}\right)^{-g/(R\beta)}} \tag{16}$$

and the pressure–temperature ratio as $\mathbf{p/T = (p_1/T_1)\ (T/T_1)^{m'}}$, where $\mathbf{m' = -\,g/(R\beta) - 1}$ is constant. The refractivity is now given by

$$\mathbf{n - 1 = (n_1 - 1)\left(\frac{T}{T_1}\right)^{m'}} \tag{17}$$

and its differential by

$$\mathbf{dn = \frac{m'(n_1 - 1)}{T_1}\left(\frac{T}{T_1}\right)^{m'-1} dT = m'\left(\frac{n-1}{T}\right) dT} \tag{18}$$

Since from (15)

$$\mathbf{r - r_1 = \frac{T - T_1}{\beta} = \frac{T_1}{\beta}\left(\frac{T}{T_1} - 1\right)} \quad \text{and} \quad \mathbf{(r - r_1)^2 = \frac{T_1^2}{\beta^2}\left(\frac{T}{T_1} - 1\right)^2}$$

we have

$$\mathbf{\int (r - r_1)^2\, dn = \frac{m'(n_1 - 1)\, T_1}{\beta^2}\int \left(\frac{T}{T_1} - 1\right)^2 \left(\frac{T}{T_1}\right)^{m'-1} dT =}$$

$$\mathbf{= \frac{m'(n_1 - 1)\, T_1^2}{\beta^2}\left[\left(\frac{1}{m'+2}\right)\left(\frac{T}{T_1}\right)^{m'+2} - \left(\frac{2}{m'+1}\right)\left(\frac{T}{T_1}\right)^{m'+1} + \frac{1}{m'}\left(\frac{T}{T_1}\right)^{m'}\right] + C =}$$

$$\mathbf{= \frac{(n - 1)\, T_1^2}{\beta^2}\left[\left(\frac{m'}{m'+2}\right)\left(\frac{T}{T_1}\right)^2 - \left(\frac{2m'}{m'+1}\right)\left(\frac{T}{T_1}\right) + 1\right] + C =}$$

$$= (r - r_1)^2 (n-1) + \frac{(n-1)\,T_1^2}{\beta^2}\left[\left(\frac{2}{m'+1}\right)\left(\frac{T}{T_1}\right) - \left(\frac{2}{m'+2}\right)\left(\frac{T}{T_1}\right)^2\right] + C$$

where C is the constant of integration, and the preceding term, transformed step by step, is

$$\frac{(n-1)\,T_1^2}{\beta^2}\left[\left(\frac{2}{m'+1}\right)\left(\frac{T}{T_1}\right) - \left(\frac{2}{m'+2}\right)\left(\frac{T}{T_1}\right)^2\right] = \frac{2\,(n-1)\,T_1\,T}{\beta^2\,(m'+1)}\left[1 - \left(\frac{m'+1}{m'+2}\right)\left(\frac{T}{T_1}\right)\right] =$$

$$= \frac{2\,(n-1)\,T_1\,T}{\beta^2\,(m'+1)}\left[\left(\frac{1}{m'+2}\right)\left(\frac{T}{T_1}\right) - \left(\frac{T}{T_1} - 1\right)\right] = \frac{2\,(n-1)\,T}{\beta\,(m'+1)}\left[\frac{T}{\beta\,(m'+2)} - (r - r_1)\right] =$$

$$= \frac{2\,(n-1)\,RT}{g}\left[(r - r_1) - \frac{T}{\beta\,(m'+2)}\right] = \frac{2R}{g}\,(r - r_1)\,(n-1)\,T + \frac{2R^2}{g^2\,(1 - R\beta/g)}\,(n-1)\,T^2$$

The tropospheric component is accordingly

$$\int_{n^0}^{n_1} (r - r_1)^2\,dn = -(r^0 - r_1)^2\,(n^0 - 1) - \frac{2R}{g}\,(r^0 - r_1)\,(n^0 - 1)\,T^0 + \\ + \frac{2R^2}{g^2\,(1 - R\beta/g)}\left[(n_1 - 1)\,T_1^2 - (n^0 - 1)\,T^{02}\right] \qquad (19)$$

which equation holds for any constant value of $\beta \neq g/R$, including $\beta = 0$.

The sum of component integrals (14) and (19) gives the total value of the integral

$$\int_1^{n_1} (r - r_1)^2\,dn = \frac{2\,R^2}{g^2}\left[\frac{(n_1 - 1)\,T_1^2 - (n^0 - 1)\,T^{02}}{1 - R\beta/g} + (n^0 - 1)\,T^{02}\right] \qquad (20)$$

under normal atmospheric conditions where the vertical distribution of temperature throughout the troposphere is substantially a linear function of height.

Integral $\int_1^{n_1} (n_1 - n)\,(r - r_1)\,dn$.

Integration by parts using the substitutions $u = r - r_1$ and $v = [\,n_1 - 1 - (n-1)\,]^2$ gives first, in view of (9),

$$\int_1^{n_1} (n_1 - n)(r - r_1)\,dn = \frac{R}{g}(n_1 - 1)^2 T_1 - \frac{1}{2}\int_{r_1}^{r'} (n-1)^2\,dr \qquad (21)$$

the latter integral being more conveniently determined.

In the stratosphere, equation (11) gives

$$\int (n-1)^2\,dr = (n^0 - 1)^2 \int e^{2m(r-r^0)}\,dr =$$

$$= (n^0 - 1)^2 \frac{e^{2m(r-r^0)}}{2m} + C = -\frac{R}{2g}(n-1)^2 T^0 + C$$

and

$$\int_{r^0}^{r'} (n-1)^2\,dr = \frac{R}{2g}(n^0 - 1)^2 T^0 \qquad (22)$$

whereas in the troposphere, applying (17)

$$\int (n-1)^2\,dr = \frac{(n_1 - 1)^2}{\beta}\int \left(\frac{T}{T_1}\right)^{2m'} dT =$$

$$= \frac{(n_1 - 1)^2 T_1}{\beta(2m'+1)}\left(\frac{T}{T_1}\right)^{2m'+1} + C = -\left(\frac{R}{2g + R\beta}\right)(n-1)^2 T + C$$

and

$$\int_{r_1}^{r^0} (n-1)^2\,dr = \left(\frac{R}{2g + R\beta}\right)[(n_1 - 1)^2 T_1 - (n^0 - 1)^2 T^0] \qquad (23)$$

The sum of (22) and (23) substituted into equation (21) finally gives the total value of the integral

$$\int_1^{n_1} (n_1 - n)(r - r_1)\,dn = \frac{R}{g}(n_1 - 1)^2 T_1 - \frac{R}{2(2g + R\beta)}[(n_1 - 1)^2 T_1 +$$

$$+ \frac{1}{2}(R\beta/g)(n^0 - 1)^2 T^0] \qquad (24)$$

again assuming that the vertical gradient of temperature is constant in the troposphere.

Integrals $\int_1^{n_1} (r-r_1)^3 \, dn$ and $\int_1^{n_1} (r-r_1)^4 \, dn$.

For the stratospheric component of the first integral we have from (12)

$$\int (r-r^0)^3 \, dn = m\,(n^0-1) \int (r-r^0)^3 \, e^{m(r-r^0)} \, dr =$$

$$= m\,(n^0-1)\left[\frac{1}{m}(r-r^0)^3 \, e^{m(r-r^0)} - \frac{3}{m}\int (r-r^0) \quad e^{m(r-r^0)} \, dr\right] =$$

$$= (r-r^0)^3 \, (n-1) - \frac{3}{m}\int (r-r^0)^2 \, dn$$

and further, in view of (13)

$$\int_1^{n^0} (r-r^0)^3 \, dn = -\frac{3}{m}\int_1^{n^0} (r-r^0)^2 \, dn = \frac{6R^3}{g^3}\,(n^0-1)\,T^{03} \qquad (25)$$

Using the identity $(r-r_1)^3 = (r^0-r_1)^3 + 3(r^0-r_1)^2(r-r^0) + 3(r^0-r_1)(r-r^0)^2 + (r-r^0)^3$ and applying integrals (9), (13), and (25), the stratospheric component is obtained as

$$\int_1^{n^0} (r-r_1)^3 \, dn = (r^0-r_1)^3 \, (n^0-1) + \frac{3R}{g}(r^0-r_1)^2 \, (n^0-1)\,T^0 + \frac{6R^2}{g^2}(r^0-r_1)\,(n^0-1)\,T^{02} + \frac{6R^3}{g^3}(n^0-1)\,T^{03} \qquad (26)$$

For the tropospheric component of the same integral we have from (15) and (18)

$$\int (r-r_1)^3 \, dn = \frac{m'(n_1-1)\,T_1^{\,2}}{\beta^3}\int \left(\frac{T}{T_1}-1\right)^3 \left(\frac{T}{T_1}\right)^{m'-1} dT =$$

$$= \frac{m'(n_1-1)\,T_1^{\,3}}{\beta^3}\left[\left(\frac{1}{m'+3}\right)\left(\frac{T}{T_1}\right)^{m'+3} - \left(\frac{3}{m'+2}\right)\left(\frac{T}{T_1}\right)^{m'+2} + \left(\frac{3}{m'+1}\right)\left(\frac{T}{T_1}\right)^{m'+1} - \frac{1}{m'}\left(\frac{T}{T_1}\right)^{m'}\right] + C = \frac{(n-1)\,T_1^{\,3}}{\beta^3}\left[\left(\frac{m'}{m'+3}\right)\left(\frac{T}{T_1}\right)^3 - \left(\frac{3m'}{m'+2}\right)\left(\frac{T}{T_1}\right)^2 + \left(\frac{3m'}{m'+1}\right)\left(\frac{T}{T_1}\right) - 1\right] + C =$$

$$= (r-r_1)^3\,(n-1) - \frac{3\,(n-1)\,T_1^2\,T}{\beta^3\,(m'+1)}\left[\left(\frac{m'+1}{m'+3}\right)\left(\frac{T}{T_1}\right)^2 - \left(\frac{2m'+2}{m'+2}\right)\left(\frac{T}{T_1}\right) + 1\right] + C =$$

$$= (r-r_1)^3\,(n-1) + \frac{3R}{g}\,(r-r_1)^2\,(n-1)\,T + \frac{6\,(n-1)\,T_1\,T^2}{\beta^3\,(m'+1)\,(m'+2)}\left[\left(\frac{m'+2}{m'+3}\right)\left(\frac{T}{T_1}\right) - 1\right] + C$$

$$= (r-r_1)^3\,(n-1) + \frac{3R}{g}\,(r-r_1)^2\,(n-1)\,T + \frac{6R^2}{g^2\,(1-R\,\beta/g)}\,(r-r_1)\,(n-1)\,T^2 +$$

$$+ \frac{6R^3}{g^3\,(1-R\,\beta/g)\,(1-2R\,\beta/g)}\,(n-1)\,T^3 + C$$

The tropospheric component is accordingly

$$\int_{n^0}^{n_1} (r-r_1)^3\,dn = -(r^0-r_1)^3\,(n^0-1) - \frac{3R}{g}\,(r^0-r_1)^2\,(n^0-1)\,T^0 -$$
$$- \frac{6R^2}{g^2\,(1-R\,\beta/g)}\,(r^0-r_1)\,(n^0-1)\,T^{02} + \tag{27}$$
$$+ \frac{6R^3}{g^3\,(1-R\,\beta/g)\,(1-2R\,\beta/g)}\left[(n_1-1)\,T_1^3 - (n^0-1)\,T^{03}\right]$$

which added to stratospheric component (26) gives the total integral

$$\int_1^{n_1} (r-r_1)^3\,dn = \frac{6R^3}{g^3}\left[\frac{(n_1-1)\,T_1^3 - (n^0-1)\,T^{03}}{(1-R\,\beta/g)\,(1-2R\,\beta/g)} + (n^0-1)\,T^{03}\right] +$$
$$+ \frac{6R^2}{g^2}\left[1 - \frac{1}{1-R\,\beta/g}\right](r^0-r_1)\,(n^0-1)\,T^{02} \tag{28}$$

Similarly,

$$\int_1^{n_1} (r-r_1)^4\,dn = \frac{24R^4}{g^4}\left[\frac{(n_1-1)\,T_1^4 - (n^0-1)\,T^{04}}{(1-R\,\beta/g)\,(1-2R\beta/g)\,(1-3R\,\beta/g)} + (n^0-1)\,T^{04}\right] +$$
$$+ \frac{24R^3}{g^3}\left[1 - \frac{1}{(1-R\,\beta/g)\,(1-2R\,\beta/g)}\right](r^0-r_1)\,(n^0-1)\,T^{03} +$$
$$+ \frac{12R^2}{g^2}\left(1 - \frac{1}{1-R\,\beta/g}\right)(r^0-r_1)^2\,(n^0-1)\,T^{02} \tag{29}$$

is obtained for the explicit value of the last integral considered in the expression for astronomical refraction (5).

Due to insolational heating of the ground during the day and its radiational cooling during the night, temperature gradients within the first few kilometres of the troposphere next to the ground frequently differ significantly from the approximately constant value of β above that level. Consequently integrals (20), (24), (28) and (29) should be modified by subdividing their respective tropospheric components. But since only a small contribution to these integrals comes from the lower levels, it will be quite sufficient merely to extend the constant temperature gradient of the free troposphere down to the ground level, neglecting the small error thus involved. This requires that the actual values of $\mathbf{T_1}$ and $\mathbf{n_1} - \mathbf{1}$ be replaced by

$$T_1' = T^0 - \beta (r^0 - r_1) \tag{15'}$$

$$n_1' - 1 = (n^0 - 1)(T_1'/T^0)^{m'} \tag{17'}$$

and the prevailing temperature gradient of the lower troposphere can be disregarded.

We may now combine the results from the preceding discussion, and write on the basis of equation (5) the following expression for the correction for astronomical refraction (in seconds of arc) :

$$\Delta z'' = \rho'' \tan z_1 \left[1 + \frac{1}{2} \tan^2 z_1 \, (n_1 - 1)\right] (n_1 - 1) -$$

$$- \frac{\rho'' R}{r_1 g} (\tan^3 z_1 + \tan z_1)(n_1 - 1) T_1 + \delta_1'' - \delta_2'' - \delta_3'' + \delta_4'' \tag{30}$$

where

$$\delta_1'' = \frac{\rho'' R^2}{r_1^2 g^2} (3 \tan^5 z_1 + 5 \tan^3 z_1) \left[\frac{(n_1' - 1) T_1'^2 - (n^0 - 1) T^{02}}{1 - R\beta/g} + (n^0 - 1) T^{02}\right]$$

$$\delta_2'' = \frac{3 \rho'' R}{r_1 g} \tan^5 z_1 \left[(n_1' - 1)^2 T_1' - \frac{(n_1' - 1)^2 T_1' + \frac{1}{2} (R\beta/g)(n^0 - 1)^2 T^0}{2(2 + R\beta/g)}\right]$$

$$\delta_3'' = \frac{15 \rho'' R^3}{r_1^3 g^3} \tan^7 z_1 \left[\frac{(n_1' - 1) T_1'^3 - (n^0 - 1) T^{03}}{(1 - R\beta/g)(1 - 2R\beta/g)} + (n^0 - 1) T^{03}\right] +$$

$$+ \frac{15 \rho'' R^2}{r_1^3 g^2} \tan^7 z_1 \left(1 - \frac{1}{1 - R\beta/g}\right)(r^0 - r_1)(n^0 - 1) T^{02} \tag{31}$$

$$\delta_4'' = \frac{105\,\rho''\,R^4}{r_1^4\,g^4}\,\tan^9 z_1 \left[\frac{(n_1'-1)\,T_1'^4-(n^0-1)\,T^{04}}{(1-R\beta/g)\,(1-2R\beta/g)\,(1-3R\beta/g)}+(n^0-1)\,T^{04}\right]+$$

$$+\frac{105\,\rho''\,R^3}{r_1^4\,g^3}\,\tan^9 z_1 \left[1-\frac{1}{(1-R\beta/g)\,(1-2R\beta/g)}\right](r^0-r_1)\,(n^0-1)\,T^{03}+$$

$$+\frac{105\,\rho''\,R^2}{2\,r_1^4\,g^2}\,\tan^9 z_1 \left(1-\frac{1}{1-R\beta/g}\right)(r^0-r_1)^2\,(n^0-1)\,T^{02}$$

represent minor terms dependent on the vertical structure of the atmosphere. Up to zenith distance $z_1 = 80^\circ$, equation (30) will give the value of integral (1) accurately enough for all practical purposes, as can best be demonstrated by test computations on atmospheric models based upon the formulas previously derived (see Tables Ia – c and IIa – c).

Table Ia.

Atmospheric Model No. 1

(Tropical Zone)

r, km 6360+	p, mb	T, °K	$(n-1)10^6$	$-(\frac{dn}{dr})\,10^6$, km^{-1}
0	1010.00	299.85	265.72	24.8210
0.8	921.55	295.00	246.43	23.3982
1.6	839.57	290.15	228.26	22.0353
2.4	763.68	285.30	211.16	20.7308
3.2	693.53	280.45	195.08	19.4832
4	628.76	275.60	179.97	18.2908
4.8	569.05	270.75	165.80	17.1522
5.6	514.08	265.90	152.52	16.0658
6.4	463.55	261.05	140.08	15.0300
7.7	390.19	253.17	121.58	13.4513
9	326.65	245.29	105.05	11.9962
10.3	271.88	237.41	90.34	10.6585
11.6	224.89	229.52	77.29	9.4323
12.9	184.80	221.64	65.77	8.3116
14.2	150.77	213.76	55.64	7.2906
15.5	122.08	205.88	46.78	6.3636
16.8	98.03	198.00	39.06	5.5250
16.8	98.03	198.00	39.06	6.7209
17.7	83.97	198.00	33.45	5.7566
18.6	71.92	198.00	28.65	4.9307
19.5	61.60	198.00	24.54	4.2233
20.4	52.76	198.00	21.02	3.6173

r, km 6360+	p, mb	T, °K	$(n-1)\,10^6$	$-(\frac{dn}{dr})\,10^6$, km^{-1}
21.3	45.19	198.00	18.01	3.0983
22.2	38.71	198.00	15.42	2.6538
23.1	33.15	198.00	13.21	2.2730
24	28.40	198.00	11.31	1.9469
26	20.13	198.00	8,02	1.3800
28	14.27	198.00	5.68	0.9782
30	10.11	198.00	4.03	0.6933
32	7.17	198.00	2.86	0.4914
34	5.08	198.00	2.02	0.3483
36	3.60	198.00	1.43	0.2469
38	2.55	198.00	1.02	0.1750
40	1.81	198.00	0.72	0.1241
44	0.91	198.00	0.36	0.0623
48	0.46	198.00	0.18	0.0313
52	0.23	198.00	0.09	0.0157
56	0.12	198.00	0.05	0.0079
60	0.06	198.00	0.02	0.0040
64	0.03	198.00	0.01	0.0020
68	0.01	198.00	0.006	0.0010
72	0.007	198.00	0.003	0.0005

$r_1 = 6360$ km

$r^0 = 6376.8$ km

$p_1 = 1010$ mb

$T_1 = 299.85$ °K

$n_1 = 1.000265717$ ($\lambda = 0.574\,\mu$)

$\beta = -6.0625$ °K km^{-1}

$R = 2.8704 \times 10^6$ erg g^{-1} °K^{-1}

$g = 97.8 \times 10^1$ cm sec^{-2}

Table Ib.

Atmospheric Model No. 2

(Temperate Zone)

r, km 6380+	p, mb	T, °K	$(n-1)10^6$	$-(\frac{dn}{dr})10^6$, km^{-1}	r, km 6380+	p, mb	T, °K	$(n-1)10^6$	$-(\frac{dn}{dr})10^6$, km^{-1}
0	1015.00	285.08	280.87	27.2824	18.9	64.81	218.00	23.45	3.6727
0.5	955.68	281.86	267.48	26.2791	20.6	49.66	218.00	17.97	2.8142
1	899.20	278.63	254.58	25.3018	22.3	38.05	218.00	13.77	2.1564
1.5	845.46	275.40	242.17	24.3501	24	29.16	218.00	10.55	1.6523
2	794.36	272.18	230.23	23.4236	26	21.32	218.00	7.71	1.2080
2.5	745.79	268.96	218.74	22.5219	28	15.58	218.00	5.64	0.8832
3	699.65	265.73	207.70	21.6447	30	11.39	218.00	4.12	0.6457
3.5	655.86	262.50	197.10	20.7916	32	8.33	218.00	3.01	0.4720
4	614.32	259.28	186.91	19.9622	34	6.09	218.00	2.20	0.3451
4.8	552.31	254.12	171.45	18.6834	36	4.45	218.00	1.61	0.2523
5.6	495.48	248.96	157.00	17.4629	38	3.25	218.00	1.18	0.1844
6.4	443.48	243.80	143.50	16.2990	40	2.38	218.00	0.86	0.1348
7.2	396.01	238.64	130.91	15.1902	44	1.27	218.00	0.46	0.0721
8	352.74	233.48	119.18	14.1351	48	0.68	218.00	0.25	0.0385
8.8	313.38	228.32	108.28	13.1322	52	0.36	218.00	0.13	0.0206
9.6	277.67	223.16	98.15	12.1799	56	0.19	218.00	0.07	0.0110
10.4	245.33	218.00	88.78	11.2767	60	0.10	218.00	0.04	0.0059
10.4	245.33	218.00	88.78	13.9033	64	0.06	218.00	0.02	0.0031
12.1	187.98	218.00	68.02	10.6535	68	0.03	218.00	0.01	0.0017
13.8	144.04	218.00	52.12	8.1633	72	0.02	218.00	0.006	0.0009
15.5	110.37	218.00	39.94	6.2551					
17.2	84.57	218.00	30.60	4.7930					

$r_1 = 6380$ km

$r^0 = 6390.4$ km

$p_1 = 1015$ mb

$T_1 = 285.08$ °K

$n_1 = 1.000280868$ ($\lambda = 0.574\,\mu$)

$\beta = -6.45$ °K km^{-1}

$R = 2.8704 \times 10^6$ erg g^{-1} °K^{-1}

$g = 98 \quad \times 10^1$ cm sec^{-2}

Table Ic.

Atmospheric Model No. 3

(Arctic Zone)

r, km 6400 +	p, mb	T, °K	$(n-1)\,10^6$	$-(\frac{dn}{dr})\,10^6$, km^{-1}
0	1020.00	252.50	318.67	56.9646
0.2	992.85	254.68	307.53	54.5009
0.4	966.64	256.87	296.86	52.1633
0.6	941.34	259.06	286.65	49.9446
0.8	916.90	261.24	276.88	47.8378
1	893.30	263.42	267.51	45.8362
1.2	870.48	265.61	258.53	43.9339
1.4	848.44	267.80	249.93	42.1252
1.6	827.12	269.98	241.68	40.4047
1.6	827.12	269.98	241.68	24.7840
2.5	737.04	264.11	220.15	23.0779
3.4	655.07	258.24	200.11	21.4549
4.3	580.64	252.36	181.50	19.9125
5.2	513.21	246.49	164.25	18.4486
6.1	452.26	240.62	148.27	17.0609
7	397.31	234.74	133.52	15.7471
7.9	347.89	228.87	119.91	14.5050
8.8	303.56	223.00	107.39	13.3324
8.8	303.56	223.00	107.39	16.4745
10.7	226.81	223.00	80.23	12.3089
12.6	169.46	223.00	59.95	9.1967
14.5	126.61	223.00	44.79	6.8713
16.4	94.60	223.00	33.46	5.1339

r, km 6400 +	p, mb	T, °K	$(n-1)\,10^6$	$-(\frac{dn}{dr})\,10^6$, km^{-1}
18.3	70.68	223.00	25.00	3.8358
20.2	52.81	223.00	18.68	2.8659
22.1	39.46	223.00	13.96	2.1413
24	29.48	223,00	10.43	1.5999
26	21.69	223.00	7.67	1.1771
28	15.96	223.00	5.65	0.8661
30	11.74	223.00	4.15	0.6373
32	8.64	223.00	3.06	0.4689
34	6.36	223.00	2.25	0.3450
36	4.68	223.00	1.65	0.2538
38	3.44	223.00	1.22	0.1868
40	2.53	223.00	0.90	0.1374
44	1.37	223.00	0.48	0.0744
48	0.74	223.00	0.26	0.0403
52	0.40	223.00	0.14	0.0218
56	0.22	223.00	0.08	0.0118
60	0.12	223.00	0.04	0.0064
64	0.06	223.00	0.02	0.0035
68	0.03	223.00	0.01	0.0019
72	0.02	223.00	0.007	0.0010

$r_1 = 6400$ km

$r^i = 6401.6$ km

$r^0 = 6408.8$ km

$p_1 = 1020$ mb

$T_1 = 252.5$ °K

$n_1 = 1.000318670$ ($\lambda = 0.574\,\mu$)

$\beta^i = +10.925$ °K km^{-1}

$\beta = -6.525$ °K km^{-1}

$R = 2.8704 \times 10^6$ erg g^{-1} °K^{-1}

$g = 98.2 \times 10^1$ cm sec^{-2}

Table IIa.
Atmospheric Model No. 1
(Tropical Zone)

Astronomical Refraction, $\Delta z'' = -\rho'' \int_{r_1}^{r'} \left(\frac{dn/dr}{n}\right) \tan z \, dr$, *for* $z = 60^\circ, 70^\circ$ & 80°

r, km 6360+	$-\rho''\left(\frac{dn/dr}{n}\right)$	tan z $z_1 = 60^\circ$	tan z $z_1 = 70^\circ$	tan z $z_1 = 80^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1 = 60^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1 = 70^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1 = 80^\circ$
0	5.11834	1.73205	2.74748	5.67128	8.8652	14.0626	29.0276
0.8	4.82503	1.73131	2.74498	5.65136	8.3536	13.2446	27.2679
1.6	4.54407	1.73057	2.74246	5.63143	7.8638	12.4619	25.5896
2.4	4.27514	1.72982	2.73993	5.61152	7.3962	11.7136	23.9900
3.2	4.01791	1.72906	2.73737	5.59163	6.9472	10.9986	22.4667
4	3.77208	1.72830	2.73481	5.57178	6.5193	10.3159	21.0172
4.8	3.53731	1.72753	2.73222	5.55197	6.1106	9.6647	19.6390
5.6	3.31330	1.72676	2.72963	5.53221	5.7213	9.0441	18.3299
6.4	3.09973	1.72598	2.72702	5.51252	5.3501	8.4530	17.0873
7.7	2.77419	1.72471	2.72276	5.48066	4.7847	7.5534	15.2044
9	2.47413	1.72342	2.71847	5.44902	4.2640	6.7258	13.4816
10.3	2.19828	1.72213	2.71416	5.41762	3.7867	5.9665	11.9094
11.6	1.94541	1.72082	2.70983	5.38648	3.3477	5.2717	10.4789
12.9	1.71428	1.71951	2.70549	5.35562	2.9477	4.6380	9.1810
14.2	1.50371	1.71819	2.70113	5.32506	2.5837	4.0617	8,0073
15.5	1.31252	1.71687	2.69677	5.29482	2.2534	3.5396	6.9496
16.8	1.13958	1.71554	2.69240	5.26491	1.9550	3.0682	5.9996
16.8	1.38823	1.71554	2.69240	5.26491	2.3781	3.7323	7.2984
17.7	1.18736	1.71463	2.68940	5.24464	2.0369	3.1933	6.2271
18.6	1.01700	1.71371	2.68639	5.22427	1.7428	2.7320	5.3131
19.5	0.87109	1.71278	2.68337	5.20414	1.4920	2.3374	4.5333
20.4	0.74611	1.71186	2.68035	5.18415	1.2772	1.9998	3.8679

r, km 6360+	$-\rho''\left(\frac{dn/dr}{n}\right)$	tan z $z_1 = 60^\circ$	tan z $z_1 = 70^\circ$	tan z $z_1 = 80^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1 = 60^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1 = 70^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1 = 80^\circ$
21.3	0.63906	1.71093	2.67732	5.16432	1.0934	1.7110	3.3003
22.2	0.54738	1.71000	2.67430	5.14464	0.9360	1.4638	2.8161
23.1	0.46884	1.70907	2.67128	5.12514	0.8013	1.2524	2.4029
24	0.40157	1.70814	2.66826	5.10581	0.6859	1.0715	2.0504
26	0.28464	1.70607	2.66157	5.06349	0.4856	0.7576	1.4413
28	0.20176	1.70400	2.65490	5.02207	0.3438	0.5357	1.0133
30	0.14301	1.70193	2.64827	4.98155	0.2434	0.3787	0.7124
32	0.10137	1.69986	2.64168	4.94192	0.1723	0.2678	0.5010
34	0.07185	1.69780	2.63513	4.90317	0.1220	0.1893	0.3523
36	0.05093	1.69575	2.62861	4.86527	0.0864	0.1339	0.2478
38	0.03610	1.69370	2.62214	4.82821	0.0611	0.0947	0.1743
40	0.02559	1.69166	2.61572	4.79196	0.0433	0.0669	0.1226
44	0.01286	1.68759	2.60299	4.72178	0.0217	0.0335	0.0607
48	0.00646	1.68355	2.59044	4.65454	0.0109	0.0167	0.0301
52	0.00325	1.67953	2.57806	4.59005	0.0055	0.0084	0.0149
56	0.00163	1.67554	2.56585	4.52812	0.0027	0.0042	0.0074
60	0.00082	1.67158	2.55380	4.46859	0.0014	0.0021	0.0037
64	0.00041	1.66764	2.54191	4.41131	0.0007	0.0010	0.0018
68	0.00021	1.66373	2.53018	4.35815	0.0003	0.0005	0.0009
72	0.00010	1.65984	2.51860	4.30297	0.0002	0.0003	0.0004

Integrals * ;

0 – 6.4 km :	44.8045	70.9438	145.0468
6.4 – 16.8 km :	35.8720	56.5197	112.6666
16.8 – 24 km :	9.8000	15.3615	29.7592
24 – 40 km :	3.7214	5.7954	10.9443
40 – 72 km :	0.2506	0.3853	0.6957
Astronomical Refraction :	94".45	149".00	299".11

Formula (30) ;

1st term :	94.968	150.735	312.160
2nd term :	– 0.525	– 1.781	– 14.264
δ_1 :	0.007	0.052	1.668
$-\delta_2$:	– 0.001	– 0.007	– 0.257
$-\delta_3$:	– 0.000	– 0.002	– 0.271
δ_4 :	0.000	0.000	0.061
	94".45	149".00	299".10

* Integration formula (Newton–Cotes) :

$$\int_a^{a+8\Delta} f(r)\,dr = \frac{8\Delta}{2.835}\Big\{0.0989\,[\,f(a)+f(a+8\Delta)\,]+0.5888\,[\,f(a+\Delta)+f(a+7\Delta)\,]-0.0928\,[\,f(a+2\Delta)+f(a+6\Delta)\,]+1.0496\,[\,f(a+3\Delta)+f(a+5\Delta)\,]-0.454\,f(a+4\Delta)\Big\}$$

Table IIb.

Atmospheric Model No. 2

(Temperate Zone)

Astronomical Refraction, $\Delta z'' = -\rho'' \int_{r_1}^{r'} \left(\frac{dn/dr}{n}\right) \tan z \, dr$, *for* $z = 60°, 70° \ \& \ 80°$

r, km 6380+	$-\rho''\left(\frac{dn/dr}{n}\right)$	tan z			$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$		
		$z_1 = 60°$	$z_1 = 70°$	$z_1 = 80°$	$z_1 = 60°$	$z_1 = 70°$	$z_1 = 80°$
0	5.62582	1.73205	2.74748	5.67128	9.7442	15.4568	31.9056
0.8	5.41900	1.73160	2.74595	5.65910	9.3836	14.8803	30.6667
1	5.21754	1.73116	2.74442	5.64690	9.0323	14.3191	29.4629
1.5	5.02136	1.73069	2.74288	5.63470	8.6904	13.7729	28.2938
2	4.83035	1.73023	2.74132	5.62248	8.3576	13.2415	27.1585
2.5	4.64446	1.72977	2.73976	5.61026	8.0339	12.7247	26.0566
3	4.46361	1.72931	2.73820	5.59804	7.7190	12.2223	24.9875
3.5	4.28773	1.72884	2.73662	5.58582	7.4128	11.7339	23.9504
4	4.11672	1.72837	2.73504	5.57360	7.1152	11.2594	22.9449
4.8	3.85307	1.72761	2.73250	5.55407	6.6566	10.5286	21.4002
5.6	3.60141	1.72685	2.72994	5.53456	6.2191	9.8316	19.9322
6.4	3.36143	1.72608	2.72736	5.51510	5.8021	9.1676	18.5386
7.2	3.13280	1.72531	2.72477	5.49568	5.4051	8.5362	17.2169
8	2.91523	1.72453	2.72217	5.47631	5.0274	7.9358	15.9647
8.8	2.70841	1.72375	2.71956	5.45701	4.6686	7.3657	14.7798
9.6	2.51203	1.72296	2.71693	5.43778	4.3281	6.8250	13.6599
10.4	2.32578	1.72217	2.71430	5.41862	4.0054	6.3129	12.6026
10.4	2.86751	1.72217	2.71430	5.41862	4.9383	7.7833	15.5380
12.1	2.19729	1.72050	2.70874	5.37871	3.7804	5.9519	11.8186
13.8	1.68371	1.71879	2.70311	5.33890	2.8939	4.5512	8.9891
15.5	1.29016	1.71707	2.69743	5.29938	2.2153	3.4801	6.8371
17.2	0.98860	1.71534	2.69172	5.26028	1.6958	2.6610	5.2003

r, km 6380+	$-\rho''\left(\frac{dn/dr}{n}\right)$	tan z			$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$		
		$z_1 = 60°$	$z_1 = 70°$	$z_1 = 80°$	$z_1 = 60°$	$z_1 = 70°$	$z_1 = 80°$
18.9	0.75753	1.71359	2.68600	5.22171	1.2981	2.0347	3.9556
20.6	0.58046	1.71184	2.68028	5.18372	0.9937	1.5558	3.0089
22.3	0.44478	1.71008	2.67456	5.14636	0.7606	1.1896	2.2890
24	0.34082	1.70832	2.66886	5.10964	0.5822	0.9096	1.7415
26	0.24917	1.70626	2.66218	5.06730	0.4251	0.6633	1.2626
28	0.18216	1.70419	2.65552	5.02588	0.3104	0.4837	0.9155
30	0.13318	1.70212	2.64891	4.98538	0.2267	0.3528	0.6639
32	0.09736	1.70007	2.64233	4.94578	0.1655	0.2573	0.4815
34	0.07118	1.69801	2.63579	4.90708	0.1209	0.1876	0.3493
36	0.05204	1.69596	2.62929	4.86919	0.0883	0.1368	0.2534
38	0.03805	1.69392	2.62284	4.83216	0.0644	0.0998	0.1838
40	0.02781	1.69188	2.61643	4.79594	0.0471	0.0728	0.1334
44	0.01487	1.68783	2.60373	4.72581	0.0251	0.0387	0.0703
48	0.00795	1.68380	2.59121	4.65862	0.0134	0.0206	0.0370
52	0.00425	1.67979	2.57886	4.59416	0.0071	0.0110	0.0195
56	0.00227	1.67581	2.56668	4.53226	0.0038	0.0058	0.0103
60	0.00121	1.67186	2.55466	4.47276	0.0020	0.0031	0.0054
64	0.00065	1.66793	2.54279	4.41550	0.0011	0.0016	0.0029
68	0.00035	1.66403	2.53109	4.36036	0.0006	0.0009	0.0015
72	0.00019	1.66015	2.51953	4.30720	0.0003	0.0005	0.0008

Integrals * :			
0 – 4 km :	33.5267	53.1215	108.9898
4 – 10.4 km :	34.9234	55.1647	111.3757
10.4 – 24 km :	27.7120	43.5495	85.7216
24 – 40 km :	3.4041	5.3011	10.0089
40 – 72 km :	0.2989	0.4591	0.8274
Astronomical Refraction:	99".86	167".80	316".92
Formula (30) ;			
1st term :	100.386	159.339	330.039
2nd term :	– 0.525	– 1.781	– 14.260
δ_1 :	0.006	0.050	1.599
$-\delta_2$:	– 0.001	– 0.007	– 0.272
$-\delta_3$:	– 0.000	– 0.002	– 0.258
δ_4 :	0.000	0.000	0.069
	99".87	167".60	316".91

* – See footnote to Table IIa.

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Table IIc.
Atmospheric Model No. 3
(Arctic Zone)

Astronomical Refraction, $\Delta z'' = -\rho'' \int_{r_1}^{r'} \left(\frac{dn/dr}{n}\right) \tan z \, dr$, *for* $z = 60^\circ, 70^\circ$ & 80°

r, km 6400+	$-\rho''\left(\frac{dn/dr}{n}\right)$	tan z $z_1=60^\circ$	tan z $z_1=70^\circ$	tan z $z_1=80^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1=60^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1=70^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1=80^\circ$
0	11.74605	1.73205	2.74748	5.67128	20.3448	32.2720	66.6152
0.2	11.23815	1.73191	2.74701	5.66750	19.4636	30.8713	63.6923
0.4	10.75627	1.73177	2.74652	5.66364	18.6274	29.5423	60.9197
0.6	10.29897	1.73162	2.74603	5.65971	17.8339	28.2813	58.2891
0.8	9.86452	1.73147	2.74552	5.65569	17.0802	27.0833	55.7907
1	9.45187	1.73132	2.74501	5.65162	16.3642	25.9455	53.4183
1.2	9.05968	1.73117	2.74449	5.64748	15.6838	24.8642	51.1643
1.4	8.68678	1.73101	2.74396	5.64327	15.0369	23.8362	49.0219
1.6	8.33206	1.73086	2.74342	5.63902	14.4216	22.8584	46.9846
1.6	5.11083	1.73086	2.74342	5.63902	8.8461	14.0212	28.8200
2.5	4.75912	1.73003	2.74064	5.61712	8.2334	13.0430	26.7325
3.4	4.42450	1.72920	2.73783	5.59521	7.6508	12.1136	24.7560
4.3	4.10651	1.72836	2.73500	5.57330	7.0975	11.2313	22.8968
5.2	3.80468	1.72751	2.73216	5.55140	6.5726	10.3949	21.1213
6.1	3.51864	1.72665	2.72927	5.52963	6.0753	9.6030	19.4559
7	3.24764	1.72579	2.72638	5.50770	5.6047	8.8543	17.8870
7.9	2.99151	1.72492	2.72346	5.48594	5.1601	8.1473	16.4112
8.8	2.74970	1.72404	2.72054	5.46424	4.7406	7.4807	15.0250
8.8	3.39774	1.72404	2.72054	5.46424	5.8578	9.2437	18.5661
10.7	2.53870	1.72220	2.71440	5.41936	4.3721	6.8910	13.7582
12.6	1.89683	1.72032	2.70815	5.37450	3.2632	5.1369	10.1945
14.5	1.41724	1.71840	2.70183	5.32990	2.4354	3.8291	7.5538
16.4	1.05891	1.71647	2.69646	5.29580	1.8176	2.8542	5.5972

r, km 6400+	$-\rho''\left(\frac{dn/dr}{n}\right)$	tan z $z_1=60^\circ$	tan z $z_1=70^\circ$	tan z $z_1=80^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1=60^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1=70^\circ$	$-\rho''\left(\frac{dn/dr}{n}\right)\tan z$ $z_1=80^\circ$
18.3	0.79117	1.71453	2.68907	5.24233	1.3565	2.1275	4.1476
20.2	0.59113	1.71257	2.68268	5.19960	1.0124	1.5858	3.0736
22.1	0.44167	1.71061	2.67630	5.15766	0.7555	1.1820	2.2780
24	0.32999	1.70865	2.66994	5.11653	0.5638	0.8811	1.6884
26	0.24280	1.70659	2.66326	5.07414	0.4144	0.6466	1.2320
28	0.17865	1.70453	2.65662	5.03269	0.3045	0.4746	0.8991
30	0.13145	1.70247	2.65002	4.99215	0.2238	0.3483	0.6562
32	0.09672	1.70042	2.64345	4.95251	0.1645	0.2557	0.4790
34	0.07116	1.69837	2.63693	4.91375	0.1209	0.1876	0.3497
36	0.05236	1.69633	2.63044	4.87586	0.0888	0.1377	0.2553
38	0.03852	1.69429	2.62400	4.83879	0.0653	0.1011	0.1864
40	0.02835	1.69226	2.61760	4.80254	0.0480	0.0742	0.1361
44	0.01535	1.68821	2.60493	4.73236	0.0259	0.0400	0.0726
48	0.00831	1.68419	2.59244	4.66509	0.0140	0.0215	0.0388
52	0.00450	1.68020	2.58011	4.60057	0.0076	0.0116	0.0207
56	0.00243	1.67623	2.56794	4.53861	0.0041	0.0063	0.0111
60	0.00132	1.67228	2.55594	4.47906	0.0022	0.0034	0.0059
64	0.00071	1.66837	2.54410	4.42174	0.0012	0.0018	0.0032
68	0.00039	1.66447	2.53241	4.36654	0.0006	0.0010	0.0017
72	0.00021	1.66060	2.52068	4.31333	0.0003	0.0005	0.0009

Integrals *;			
0 – 1.6 km :	27.4896	43.5899	89.8025
1.6 – 8.8 km :	47.8525	75.7050	153.9958
8.8 – 24 km :	34.3796	54.0825	106.9880
24 – 40 km :	3.3500	5.2165	9.8585
40 – 72 km :	0.3092	0.4769	0.8610
Astronomical Refraction :	113".38	179".07	361".51

Formula (30) ;			
1st term :	113.903	180.810	374.686
2nd term :	– 0.525	– 1.780	– 14.257
δ_1 :	0.008	0.048	1.561
$-\delta_2$:	– 0.001	– 0.009	– 0.273
$-\delta_3$:	– 0.000	– 0.002	– 0.252
δ_4 :	0.000	0.000	0.058
	113".38	179".07	361".52

* See footnote to Table IIa.