An improved equation for the radio refractive index of air

Gordon D. Thayer

*US Department of Commerce, National Oceanic and Atmospheric Administration,
Environmental Research Laboratories,
Office of the Program Manager for Weather Modification, Boulder, Colorado 80302*

New constants are derived for the equation relating the radio refractive index, n, at frequencies below 20 GHz to the pressure, temperature, and humidity of the air. The new equation is

$$(n - 1) \times 10^6 = N = 77.6(P_a/T)Z_a^{-1} + 64.8(e/T)Z_w^{-1} + 3.776 \times 10^5(e/T^2)Z_w^{-1}$$

where P_a is the partial pressure of dry air (mb), T is the absolute temperature (K), e is the partial pressure of water vapor (mb), and Z_a^{-1} and Z_w^{-1} are the inverse compressibility factors for dry air and water vapor, respectively. Improved values of the constants are obtained by considering the relationship between the radio and optical portions of the electromagnetic spectrum. With the use of the full three-term equation including non-ideal gas law effects, the accuracy of the new radio refractive index equation ranges from 0.018%, or about 0.05N, for dry air to 0.048%, or 0.21N, for extremely moist air.

INTRODUCTION

Recent work on the use of multiple frequency radio-optical distance measuring techniques [*Bender, 1967; Thayer, 1967*] has led to a requirement for increased accuracy in the equation relating the atmospheric radio refractive index to pressure, temperature, and humidity. *Bender* [1967] states: "—it seems safe to estimate that sensitivities of one part in 10^7 or better over one-way paths of 30 km or more will be achievable soon under good atmospheric conditions." Multiple frequency techniques of this type are capable of eliminating most atmospheric refraction effects, but the precision is still limited by the accuracies with which the constants in the equation for refractive index are known [*Bender, 1967; Thayer, 1967*]. In addition, in laboratory work *Owens* [1967] points out that reduction of observed wavelengths to *in vacuo* conditions requires that the refractive index be known at least as well as the precision to which the present wavelength standard is defined, 1×10^{-8}. The equation for radio refractive index in present use [*Smith and Weintraub, 1953*] was developed to have an accuracy of roughly 0.5% or 2×10^{-6} ($2N$); this obviously requires some improvement.

In this paper improved values of the constants in the radio refractive index equation for frequencies below 20 GHz are derived, utilizing both the latest laboratory measurements and the interrelationships between the radio and optical refractivities. The accuracy of the improved equation is found to range from 0.05N (5×10^{-8}) for dry air to about 0.2N for extremely moist air. For precise geodetic and laboratory work, use of the full three-term equation with non-ideal gas corrections is recommended. The simplified two-term equation presently in wide use [*Smith and Weintraub, 1953*] actually is found to have an accuracy of 0.5N or better, and its continued use for less precise applications is recommended.

BACKGROUND

The general equation for the radio refractive index, n, of air (neglecting dispersion) is

$$(n - 1) \times 10^6 = N = K_1(P_a/T)Z_a^{-1}$$
$$+ K_2(e/T)Z_w^{-1} + K_3(e/T^2)Z_w^{-1} \quad (1)$$

where N is the refractivity, P_a is the partial pressure of dry air, e is the partial pressure of water vapor, T is the absolute temperature, and Z_a^{-1} and Z_w^{-1} are the inverse compressibility factors (corrections for non-ideal gas behavior) for dry air and water vapor,

respectively (for example, see *Boudouris* [1963] and *Owens* [1967]).

The constants, K_i, in (1) are evaluated by actual measurement of the refractive index. The most recent values of the inverse compressibility factors are those given by *Owens* [1967]:

$$Z_a^{-1} = 1 + P_a[57.90 \times 10^{-8}(1 + 0.52/T)$$
$$- 9.4611 \times 10^{-4}t/T^2] \quad (2)$$

and

$$Z_w^{-1} = 1 + 1650(e/T^3)(1 - 0.01317t + 1.75$$
$$\times 10^{-4}t^2 + 1.44 \times 10^{-6}t^3) \quad (3)$$

where T is the Kelvin temperature, and t the Celsius.

The first two terms in (1) are the result of induced molecular polarization of air and water vapor molecules, respectively; the third term represents the effects of the permanent dipole moment of the water vapor molecule. Theoretically, the dielectric constants of atmospheric constituents caused by induced molecular polarization should be the same at optical and radio frequencies [*Barrell*, 1951; *Smith and Weintraub*, 1953; *Boudouris*, 1963]. Boudouris compares such values for nine different nonpolar gases, finding agreement ranging from 0.07% to 2.0% (for helium). This is not true for the polar term; optical frequencies are well above the natural resonances of the water molecule dipole moments, and there is no measurable effect from this term, i.e., $K_3 = 0$. However, radio frequencies less than 100 GHz are well below the important molecular resonances of water vapor, and the value of K_3 is nearly independent of the impressed field frequency (there is a minor resonance at about 22.2 GHz, but its dispersion is negligible).

DEVELOPMENT

Very accurate measurements have been made of the refractive index of dry air at both radio and optical frequencies; however, the refractive index of water vapor is less accurately known for either spectrum, partly because of the difficulties inherent in measuring and controlling atmospheric humidity. The accepted value of the radio refractivity of dry air of standard composition (i.e., containing 0.03% CO_2) at 1013.25 mb and 273.15 K is [*Smith and Weintraub*, 1953; *Bean and Dutton*, 1966]

$$(n_a - 1) \times 10^6 = N_a = 288.0_4 \pm 0.05 \quad (4)$$

A corresponding value for radio frequencies, N_a', can be calculated by extrapolation from optical refractive index values. The latest value of the optical refractivity for 1013.25 mb, 273.15 K, and infinite wavelength is obtained by substituting a wave number of zero in the appropriate dispersion formula [*Edlén*, 1966], resulting in $N = 287.63 \pm 0.05$. This is converted into refractive index and multiplied by the square root of the magnetic permeability at radio frequencies, yielding a value of refractive index appropriate for the radio spectrum. The value of magnetic permeability used here, $\mu - 1 = 0.37_9 \times 10^{-8}$, was obtained from the latest values for the magnetic susceptibility for the constituents of dry air [*Weast*, 1969]. The resulting radio refractivity value is

$$N_a' = 287.8_2 \pm 0.05 \quad (5)$$

The difference between (4) and (5) is $0.22N$, which has a ratio of 3.1 to the 0.07-N standard deviation of the difference between the two values of N_a. When this ratio is compared with tabulated values of the Student-t distribution for large degrees of freedom (since each of the two values N_a and N_a' is the mean of a number of independent measurements), the difference of $0.22N$ is shown to be significant at better than the one-percent level of confidence. Since the difference between the two values of N_a in (4) and (5) is probably real, the value from (4) should be used alone to determine K_1 in (1); thus,

$$K_1 = (288.04 \pm 0.05)(273.15)/(1013.25)(1.000588)$$
$$= 77.60_{36} \pm 0.01_4 \quad (6)$$

On the other hand, the difference of $0.22N$ between (4) and (5) is only about 0.08% of either value, and similar agreement might be expected between K_2 extrapolated from the optical spectrum and K_2 measured at radio wavelengths. The most accurate value of the optical refractivity of water vapor available, obtained through a redetermination of the data of *Barrell and Sears* [1940] using the accurate dispersion formula of *Erickson* [1962], for zero wave number, 33.433 mb vapor pressure, and 29.479°C, is $7.17_{95} \pm 0.009$ [*Thayer*, 1967]. This becomes 7.16_{85} after correction for the magnetic permeability of water vapor at radio wavelengths. The value of K_2' is then

$$K_2' = (7.1685 \pm 0.009)(302.63)/(33.433)(1.001447)$$
$$= 64.79 \pm 0.08 \quad (7)$$

Comparison of the value of K_2' with a value of K_2 derived from radio data is difficult because the

value of the polar term in the radio refractivity of water vapor averages some twenty times larger than the induced polarization term. Thus, past investigators have measured the radio refractivity of water vapor and then determined the values of K_2 and K_3 by statistical techniques, such as linear least squares regression. This can be accomplished by reducing observed data to the form

$$(n_w - 1) \times 10^6 (T/e_i) = K_2 + K_3(1/T) \qquad (8)$$

where n_w is the refractive index of water vapor, and e_i is the pressure of water vapor corrected to correspond to that of an ideal gas (i.e., so that T/e_i is inversely proportional to the actual density of the water vapor). A good example of this work, representing perhaps the definitive measurements of the radio refractive index of water vapor, is that of *Boudouris* [1963] and his coworkers. Boudouris used data taken at the University of Pisa [*Battaglia et al.*, 1957], and at the Laboratoire de Physique de l'Atmosphere. Their results, in the form of (8), are shown in Figure 1. By performing a least squares fit of these data on the corresponding values of $1/T$, one obtains a regression line where the intercept is equal to K_2 and the slope is equal to K_3. The values obtained by Boudouris and their standard errors of estimate are

$$K_2 = 72.0 \pm 10.5$$

and

$$K_3 = (3.754 \pm 0.030) \times 10^5 \qquad (9)$$

The relative standard errors in (9) are 14.5% for K_2 and 0.8% for K_3; however, the accuracy of the resulting equation for N_w is much better than 0.8% because the two standard errors in (9) are correlated. The standard error of estimate for the regression line in Figure 1 is $\pm$ 2.2 K mb^{-1} (this is closely equal to the prediction error as ordinarily defined); the standard error of estimate for N_w is thus $\pm$ 2.2 e/T, or 0.16% of N_w at 0°C and 0.19% at 50°C. These errors would result in a maximum error in N_w of about $0.3N$ under extreme conditions of temperature and humidity.

The determination of K_3 can be improved if the value of K_2' derived earlier, (7), is adopted for the radio spectrum; although the precise value adopted is not crucial, this seems to be the most appropriate number available. This value falls well within the error limits for K_2 in (9); in fact, it is within the estimated accuracy limits for K_2 for any published
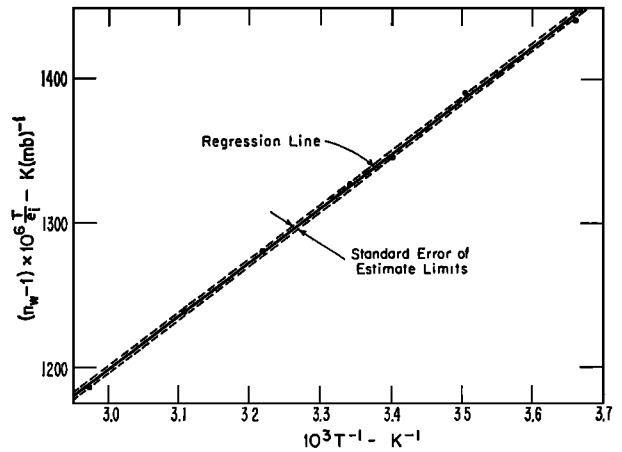

Fig. 1. Determination of the equation for the radio refractive index of water vapor. The regression line has coefficients $a = 72.0 \pm 10.5$ and $b = (3.754 \pm 0.030) \times 10^5$, and equation $(n_w - 1) \times 10^6\ T/e = 72.0 + 3.75 \times 10^5/T \pm 2.2$.

results [*Bean and Dutton*, 1966]. In addition, it is in good agreement with the values used by some earlier investigators [*Strickland*, 1942; *Essen and Froome*, 1951]. Using this value for K_2, K_3 can be calculated directly from the observed data as

$$K_3 = (N_w/e_i)T^2 - K_2T \qquad (10)$$

Boudouris [1963] gives measurement accuracies of 0.2 mm Hg for his vapor pressure values and 5 $\times$ 10^{-8}, or $0.05N$, for his values of the refractive index of water vapor (this required a precision of only about 10^{-8} in measuring $n_w - 1$). Using these values, an estimate of the variance caused by measurement uncertainties can be calculated for each value of K_3 obtained with (10). A weighted average value of K_3 can then be calculated, weighting each value by the inverse of its assigned uncertainty variance. Since the measurements at the higher vapor pressures were inherently more accurate and there were more individual measurements performed for them, the weighting process selects strongly toward these values. The weighted mean value of K_3 obtained from Boudouris' data in this manner is

$$K_3 = (3.77_6 \pm 0.004) \times 10^5 \qquad (11)$$

where the stated uncertainty is the standard error of the weighted mean, as calculated from the measurement accuracies given by *Boudouris* [1963]. The standard error of the weighted mean, as calculated from the standard deviation of each of the mean values at the seven different temperatures from the

overall mean, is only about $\pm 0.002 \times 10^5$, implying that Boudouris' stated accuracies were sufficiently conservative. The value of the Debye constant resulting from this determination of K_3 is $1.853_3 \pm 0.001$ Debye; this is also within the error limits of most published results [Boudouris, 1963].

Combining the results from (6), (7), and (11), the recommended complete equation for the radio refractivity of air is

$$N = 77.6(P_a/T)Z_a^{-1} + 64.8(e/T)Z_w^{-1}$$
$$+ 3.776 \times 10^5(e/T^2)Z_w^{-1} \qquad (12)$$

where the constants in (1) and their standard errors are

$$K_1 = 77.60 \pm 0.014, \text{ or } 0.018\%$$

$$K_2 = 64.8 \pm 0.08, \text{ or } 0.12\%$$

and

$$K_3 = (3.776 \pm 0.004) \times 10^5, \text{ or } 0.11\% \qquad (13)$$

Although the terms in (12) involving water vapor have a standard error on the order of 0.1%, in practice the errors in the total value of N will never approach this figure. Error curves for the temperature range $-65°$ to $+57°C$ are shown in Figure 2 for various relative humidities. The largest errors slightly exceed $0.2N$, but they are calculated for conditions representing the highest dewpoint and relative hu-
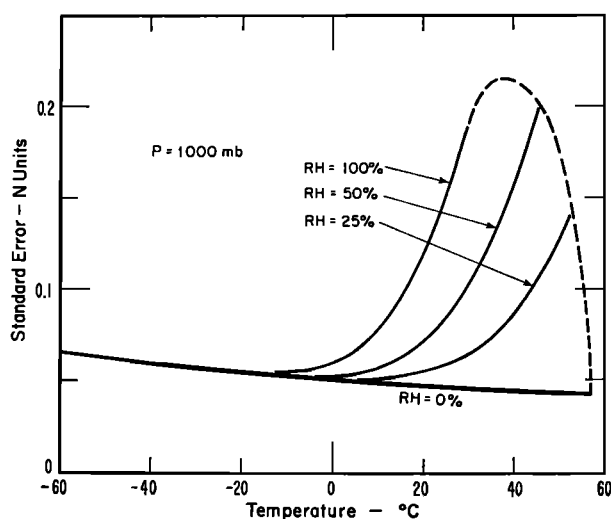

Fig. 2. Error limits for the improved equation for the radio refractive index of air. The dashed curve indicates the approximate limit of ambient temperature-dewpoint combinations based on world records of temperature and humidity [Dodd, 1969].

midity combinations occurring in some of the hottest and most humid locations in the world [Dodd, 1969]. Data for temperate climates, e.g., Miami, Florida, and Mobile, Alabama, indicate that dewpoints in these regions will exceed 25°C only about 1% of the time during the warmest months. The errors from Figure 2 for these conditions would not exceed $0.15N$, or about 0.04%. The percentage errors in N represented by the curves in Figure 2 range from 0.018% for completely dry air (or air below $-20°C$) to a maximum of about 0.05% for the extreme limit curve.

DISCUSSION OF RESULTS

The improved equation for the radio refractive index of air presented in this paper has an accuracy about one order of magnitude better than the 0.5% assumed by Smith and Weintraub [1953] in the development of their equation, which has been in standard use for many years. Their equation

$$N = 77.6(P/T) + 3.73 \times 10^5 e/T^2 \qquad (14)$$

has the constant K_2 absorbed in K_1 and the value of K_3 adjusted slightly to cover the difference between K_1 and K_2 (i.e., $-5.6 e/T$). In addition, (14) was designed to be used with $0°C = 273$ K, and the inverse compressibility factors, Z^{-1}, were neglected. Neglecting the inverse compressibility factors leads to errors of about $0.04N$ at pressures near 500 mb in a region where the accuracy of (12) is about $0.02N$; neglecting Z_w^{-1} can lead to errors of as much as $0.1N$ at high humidities. A comparison of values from (14) with those from (12) indicates that the actual overall error in using (14) at sea level pressures will range from about $0.14N$ for very dry air to a maximum of about $0.5N$ for very moist air; the corresponding relative errors are 0.05 to 0.1% in N. Thus, the continued use of the Smith-Weintraub equation is recommended for general use, and (12) is recommended for work requiring the highest precision.

In the derivation of (12) no allowance was made for variations in the composition of dry air; in particular, atmospheric CO_2 may vary by an order of magnitude from its standard value of 0.03% [Owens, 1967]; this would result in an error of about $0.5N$ at sea level pressures. The accuracies given for (12) are contingent on the value of CO_2 remaining below 0.06%; if doubt exists that this is the case, the CO_2 content should be monitored and the refractivity of CO_2 calculated separately as

$$N(CO_2) = 133 P_c/T \qquad (15)$$

where P_c is the partial pressure of CO_2 in mb [*Boudouris*, 1963]. The derivation of (12) also implicitly assumes the validity of the Lorenz-Lorentz equation; *Böttcher* [1952] believes that this assumption may introduce errors as large as $0.03N$ into the refractive index at ordinary pressures and temperatures. However, errors of this size are somewhat smaller than the standard error of (12) and are probably not significant.

In conclusion, an improved equation for the radio refractive index of air has been derived, which has an accuracy of $0.2N$ or 0.05% of N, or better, throughout the range of pressures, temperatures, and humidities likely to be encountered anywhere in the earth's atmosphere. This equation is recommended for use in precise applications such as geodesy or laboratory work. Continued use of the familiar Smith-Weintraub equation is recommended for less precise applications.

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