The Flags Problem

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5/29/24

Abstract

This is a problem I thought about while driving around my town, yelling at pedestrians. Turning them into an abstraction proves useful here but it may not hold up in traffic court.

1 The Problem

Pedestrians are crossing a city street, either going left or right. Orange handheld flags sit in racks on either side of the street to them to provide visibility to cars (Fig. 1). If there is a flag in their side's rack, a pedestrian will take it across the street with them (Fig. 2) and place it in the opposite rack (Fig. 3). If their side's rack is empty, they cross the street anyway and don't touch the flags (Fig. 4).

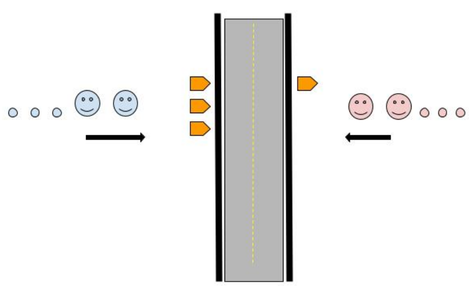

Figure 1: Before Crossing

We can assume:

- A very large number of pedestrians cross over the course of the day.
- Pedestrians arrive and cross immediately, one at a time.

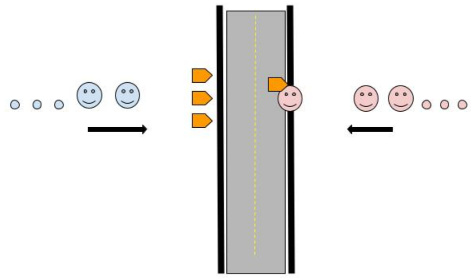

Figure 2: Carry the flag

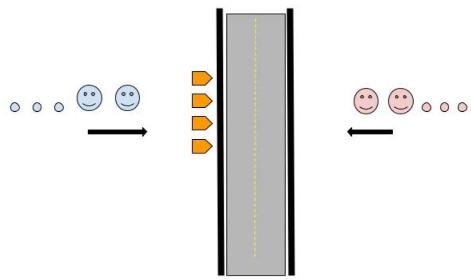

Figure 3: After Crossing

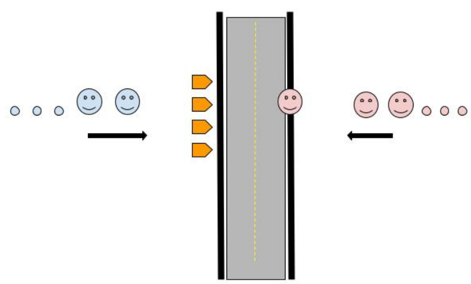

Figure 4: Flagless Crossing

- Pedestrians cross once, either right to left or left to right, and vanish for the rest of the day.
- Each crossing is an independent event. The probability that the next pedestrian crosses left to right is some p, which doesn't change over the day. Right-to-left is

then $1-p$.

Questions:

1. At the end of the day, suppose we see three flags on the left and one on the right. What's the most likely value of p?[1]

2. Suppose we see k flags on the left and $n-k$ flags on the right. What's the most likely value of p?

2 Solution

2.1 Markov Chain Model

Rather than think of a line of pedestrians waiting to cross from either side, we should instead model this as a series of crossings from the left and right, with probabilities p and $1-p$, respectively. For example, the sequence $LRLLR$ would have probability $p(1-p)p^2(1-p)$.

To describe the flags, we also have states $A_0 ... A_n$, corresponding to 0 flags on the right side of the street, 1 flag on the right, up to n flags on the right. We denote the probability of being in state A_k as a_k.

$$\begin{bmatrix} 1-p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & p & p \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

(a) Transition matrix from state $\vec{a}$

Then, starting at time $t=0$ from state distribution vector $\vec{a}$, we can use a Markov model approach to describe the probability of being in each state at time $t=1$. The matrix M in Fig. 5a, when left-multiplied with state vector $\vec{a}$ at $t=0$, produces $\vec{a}$ at $t=1$. M_{ij}, meaning the element in row i and column j, gives the probability of transitioning from state A_j to A_i.[2]. The first column, then, says that we have a $1-p$ probability of transitioning from state A_0 to A_0 (a flagless left crossing) and a p probability of transitioning from state A_0 to A_1 (a crossing taking one of the $n-0=n$ flags on the left with us).

[1] Given that we have no priors, we could call this the *maximum likelihood estimate*. Or, the value of p that makes this most probable.

[2] Note that I zero-index matrices, a CS-based habit I've not been able to break

2.2 Finding Equilibrium

A Markov matrix will always have a largest eigenvalue of $\lambda = 1$. Though there are other proofs of this, since repeated application of the Markov chain will neither grow nor shrink a probability vector without bound, this must be so.

With an eigenvalue $\lambda = 1$, this means we must have an eigenvector $\vec{a_\lambda}$ such that $M \cdot \vec{a_\lambda} = \lambda \cdot \vec{a_\lambda}$, or $M \cdot \vec{a_\lambda} = \vec{a_\lambda}$.

We assert without proof that M is aperiodic (we're not going to get into a cycle), so by Markov chain theory, repeated application of M will yield the equilibrium behavior no matter the initial state distribution $\vec{a}$, as long as the sum of the vector's values are 1. Therefore, we need to find the equilibrium solution as shown in Fig. 6.

$$\begin{bmatrix} 1-p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & p & p \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Figure 6: Equilibrium assertion

2.3 Finding Equilibrium

Solve for eigenvectors

$$pa_3 + pa_4 = a_5 \Rightarrow a_4 = \frac{p}{1-p}a_3 \tag{1}$$

$$pa_2 + (1-p)a_4 = a_3 \Rightarrow pa_2 + pa_3 = a_3 \Rightarrow a_3 = \frac{p}{1-p}a_2 \tag{2}$$

$$pa_1 + (1-p)a_3 = a_2 \Rightarrow pa_1 + pa_2 = a_2 \Rightarrow a_2 = \frac{p}{1-p}a_1 \tag{3}$$

$$pa_0 + (1-p)a_2 = a_1 \Rightarrow pa_0 + pa_1 = a_1 \Rightarrow a_1 = \frac{p}{1-p}a_0 \tag{4}$$

$$(1-p)a_0 + (1-p)a_1 = a_0 \Rightarrow (1-p)a_0 + pa_0 = a_0 \tag{5}$$

$$\tag{6}$$

We quickly see a pattern emerge where for $0 < i \leq n$, each $a_i = \frac{p}{1-p}a_{i-1}$. This last line means a_0 is unconstrained, but of course the sum of the a_i components are 1. Therefore each of these can normalized by dividing out $a_0(1 + (\frac{p}{1-p})^1 + (\frac{p}{1-p})^2 + ... + (\frac{p}{1-p})^{n-1})$. Setting $r = \frac{p}{1-p}$ for convenience, each term becomes $r^k \frac{1-r}{1-r^n}$.

2.4 Solving for p at Equilibrium