Spotting Graph Theory Problems in Spot It

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Abstract

The card game "Spot It" supports a unique mechanic: every card of the deck has eight different symbols, and shares exactly one symbol of some kind with every other card. The obvious game play ("spot the match') works for children as young as two; the intricacies of deck construction astound these children of forty-two. In hypothetically constructing our own deck, we run across interesting problems in graph theory, number theory, and abstract algebra. Some are solved, some remain unsolved. Until now. Not really. We just prove hard things are hard.

Notably, starting with $g = p^k, p \in \mathbb{P}, k \in \mathbb{N}$ and $n = g^2 + g + 1$, we prove the following four constructions are equivalent:

1. (The Children's Game) A deck of n "Spot It" Cards with $s = g + 1$ symbol slots, where each of $m = n$ symbols occurs exactly $g + 1$ times,
2. (Graph Theory) An edge partition of the complete graph K_n into complete subgraphs K_g,
3. (Abstract Algebra) A finite (Galois) field of order g, and
4. (Number Theory) A Perfect Difference Set[**1**] on n elements.

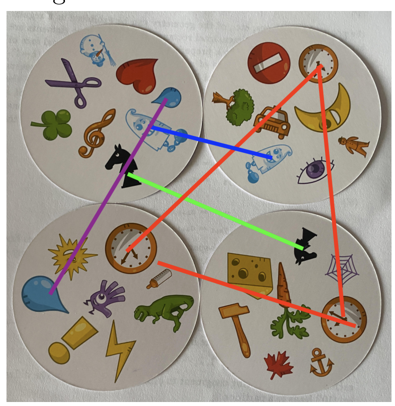
We then examine other configurations of s and g, as well as comment on other reasonable constructions of a Spot It Card deck.

1 The game and the problem

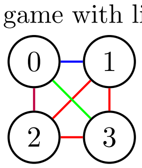
Introduced to us by Ari Steinberg, "Spot It" is a children's game of 55 cards as shown in Figure 1a, featuring eight colorful symbols on each. Though gameplay comes with a few variants in its tiny rulebook, the primary mechanic when presented two cards is simply *spot single the common symbol first.* The game is simple enough for a two- or three-year old to grasp (and win!), but this poses the question: *just how did they construct such a deck?*

*Note: There have been some other investigations here[**2**], but for the purposes of enjoyment, everything in this paper was researched without reference to prior (Spot It) work.*

(a) Four cards in the game



(b) Four cards in the game with links



(c) Four cards graph

Naturally, there are trivial constructions: every symbol occurs only twice, or once, or the count of symbols is so varied that a deck can be constructed almost greedily. However, the Spot It game has uniformly 8 symbol "slots" on each card, which each appear 8 times [1] across the different cards.

[1] As we will see later, there should be 57 cards for this to be true; it's likely two cards were removed

This is what we examine in this paper:

> **The Core Question:** For what choices of g and s can we construct a "Spot It" deck where each of the s symbols on each card appears exactly g times throughout the deck?

1.1 Reframing as a graph

Noticing that every card has a relationship to every other card (notably, the identity of the single symbol shared between them) as in Fig. 1b, we take our first step by reconstructing this problem as an undirected graph as in Fig. 1c.

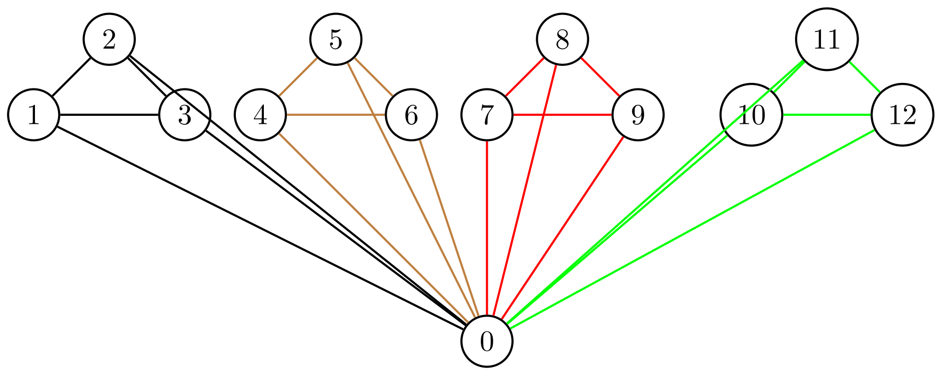


Figure 2: $n = s(g-1)+1$. Here, $s = 4, g = 3$

> **The Graph Representation:** A deck of Spot It Cards each with s symbol slots, where each symbol appears g times c can be represented by a graph G:
>
> 1. With n nodes, where $n = s(g-1)+1$
> 2. With m unique edge colors, where m denotes the number of symbols,
> 3. Where all edges of color m_i form a complete subgraph on g nodes, s, and $m = \binom{n}{2}/\binom{g}{2}$.

Note: Singletons (groups where $g = 1$) would be rendered as self-edges. These are uninteresting and are generally ignored in this paper

Proof:

1. As in Fig. 2, node n_0's adjacencies are exactly s monocolor cliques of size $g-1$ (excluding n_0 itself). In a complete graph, these adjacencies comprise the total node set, so $n = (g-1)s+1$ when adding n_0 back in. Using any other node is equivalent.

2. As in Fig. 1b, while card 1 and card 2 having the relationship “clock”, node 1 and node 2 instead share an edge with the color red. This is the same relationship between nodes 2 and 3, and nodes 1 and 3. “Drop”, “knight”, and “ghost” would be colors purple, green, and blue, respectively. This works because every edge has exactly one color (corresponding 1:1 with a symbol) in this formulation, and every card pair has exactly one symbol shared.

3. All cards with a given symbol (say, “clock”) must correspond to nodes be linked with the color red to all other nodes whose card has a clock; this is a complete subgraph. A complete graph K_n has $\binom{n}{2}$ edges. A monocolor clique of size g is a complete graph as well, with $\binom{g}{2}$ edges. K_n's edges are exactly these equal-sized cliques, so there are therefore $m = \frac{\binom{n}{2}}{\binom{g}{2}}$ of them, corresponding to colors.

And since every edge in our complete graph K_n is in exactly one monocolor clique of size K_g, this becomes a crisp graph theory problem.

The Core Question in Graph Terms: Given s and g as before, can we construct an edge partition (colloquially here, “coloring”)[2] of K_n into a set of complete subgraphs of size g (denoted K_g)?

Though exhaustive research wasn't done, this graph problem does not appear to have a clear analytical solution out there.[3]

Since m and n are determined from s and g, we'll start by looking at possibly candidate configurations of s and g.

2 The Candidate Theorem: $g|s(s-1), g \leq s$

Suppose that that every symbol s has exactly g cards containing it [4]. Then

1. $g|s(s-1)$.
2. If $s > 1$ and $g > 1$ then $g \leq s$
3. Corollary to (2): $m = (\frac{s}{g})n$ and therefore $m \geq n$
4. All candidate configurations of g, s are $g \leq s$, $g|s(s-1)$.

Proof:

[3]Or people who care about publishing it within the reach of lazy hobbyists, anyway!

[4]for example, all $s = 7$ symbols in Fig. **??** correspond to cliques of size $g = 3$

1.

$$\binom{g}{2}\Big|\binom{n}{2} \Rightarrow \frac{n(n-1)}{g(g-1)} \in \mathbb{N} \Rightarrow g(g-1)|n(n-1) \tag{1}$$

$$n = (g-1)s+1 \Rightarrow g(g-1)\big|(sg-s+1)(sg-s) = (sg-s+1)s(g-1) \tag{2}$$

$$\Rightarrow g|s^2g - s^2 + s \Rightarrow g|(1-s)s \Rightarrow g|s(s-1) \tag{3}$$

2. Any node n_i is adjacent to s monocolor cliques of size g. These cliques $C_1...C_s$, containing non-n_i nodes if $g > 1$, comprise all nodes, and any other cliques can contain no more than one of each K_i. This means that clique of size g greater than s cannot be formed, since the only place to find nodes are these $C_1...C_s$. The other trivial case, $s = 1$, means there is only one color in the whole graph.
This means we need not consider configurations like $g = 6, s = 3$ even though $6|3(3-1)$.

3. Another corollary here is that $\boxed{m \geq n}$, since:

$$n = (sg - s + 1) \tag{4}$$

$$m = \frac{\binom{(sg-s+1)(sg-s)}{2}}{\binom{g}{2}} = \frac{(sg-s+1)(sg-s)}{g(g-1)} = \frac{(sg-s+1)s}{g} \tag{5}$$

$$\Rightarrow m = (\frac{s}{g})n \tag{6}$$

$$s \geq g \Rightarrow m \geq n \tag{7}$$

4. This is just a combination of (1) and (2). But for example, a tiling of triangles ($g = 3$) means that either $s \equiv 0 \mod 3$ (see Fig. **??**) or $s \equiv 1 \mod 3$ (see Fig, **??**).

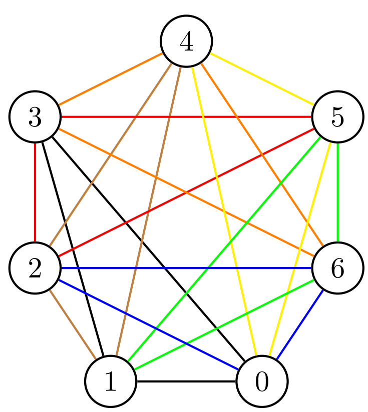


Figure 3: s=3, g=3, n=7, m=7. Cliques of form (n_i, n_{i+1}, n_{i+3}). for all i

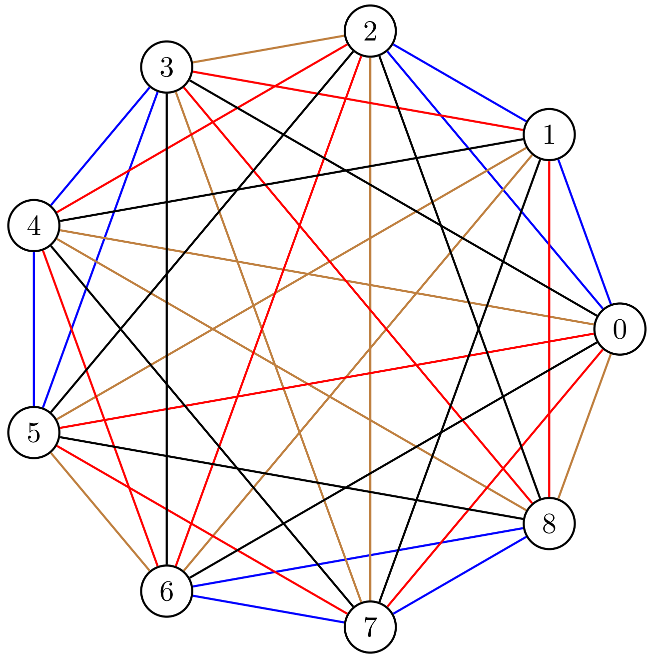


Figure 4: s=4, g=3, n=9, m=12.
Cliques: $(n_i, n_{i+3}, n_{i+6}), (n_i, n_{i+1}, n_{i+2}), (n_i, n_{i+4}, n_{i+8}), (n_i, n_{i+5}, n_{i+7}), 3|i$

3 Constructing $g = s - 1$ over a field

Though we presented a few legitimate examples of complete graphs tiled by uniformly sized complete subgraphs in Fig. **??** and Fig. **??**, these are not easy to find by hand once s becomes much larger. The whole problem of graph partitioning admits many algorithms, most approximations[**3**], though usually over an arbitrary graph instead of a relatively simple complete graph, and many referring to separating actual *nodes* into partitions, rather than edges.

We can, however, systematically find an edge partition if g is a prime power.

> **g=s-1 construction**: If g is a prime power p^k, we can explicitly construct a graph that satisfies our game with $g = s - 1$.

As the combination of s, g determine the shape of the graph entirely, $g = s-1$ implies:

- The graph has $n = s(g-1) + 1 = (g+1)(g-1) + 1 = g^2$ nodes.
- Those nodes can be grouped into g groups of size g.

- There are $m = (\frac{s}{g})n = (\frac{s}{g})g^2 = sg = (g+1)g$ colors in the graph.

We can start with the easiest way to see this: $g = p, p \in \mathbb{P}$.

Construct the multiplication table for the finite field on $p = 3$, as in Fig. 5.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	1	0

(a) Addition table GF(3)

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

(b) Multiplication table GF(3)

Figure 5: Field tables for GF(3)

To construct the $m = g(g-1)$ colors (symbol cliques) in the graph:

- Construct a finite field $\mathcal{GF}(\})$, which is of size g.
- Divide the g^2 nodes into g groups $(G_0, G_1, G_2 ... G_{g-1})$, where $[0...g-1] \in \mathcal{F}$, of g nodes each $((G_{0,0}, G_{0,1} ... G_{0,g-1}) ... G_{g-1,0}, G_{g-1,1} ... G_{g-1,g-1})$
- For all $c \in [0, g-1]$:
  - For all $y \in [0, g-1]$:
    - Set $C_{c,y}$ to the empty set.
    - For all $x \in [0, g-1]$, add $G_{x, G_x \cdot G_y + c}$ to clique $C_{c,y}$, where $+$ and $\cdot$ refer to addition and multiplication rules for $\mathcal{F}$.
- The g cliques formed from the node groups $(G_0, G_1, G_2 ... G_{g-1})$, plus the g^2 cliques like C_c, y form the $(g+1)g$ cliques or "colors".

For an example, compare identical graphs Fig. **??** (generated from the algorithm) and Fig. **??**. The mapping between node (i, j) in Fig. **??** and node k in Fig. **??** is $(i, j) \rightarrow k = i + 3j$, with inverse $k \rightarrow (k \mod 3, \lfloor k/3 \rfloor)$.

1. Cliques of form $G_i : G_0, G_1, G_2$: these are the triangles in black. These correspond to nodes $n \mod i$ in Fig. **??**; for example $G_0 = \{0, 3, 6\}$.
2. Cliques of form $C_{j,0} : C_{0,0}, C_{1,0}, C_{2,0}$: these are the triangles in blue. Fig. **??** sees these connect between nodes with the same $n \mod i$ in Fig. **??**; for example $C_{1,0} = \{1, 4, 7\}$.
3. Cliques of form $C_{j,1} : C_{0,1}, C_{1,1}, C_{2,1}$: these are the triangles in brown. Fig. **??** sees

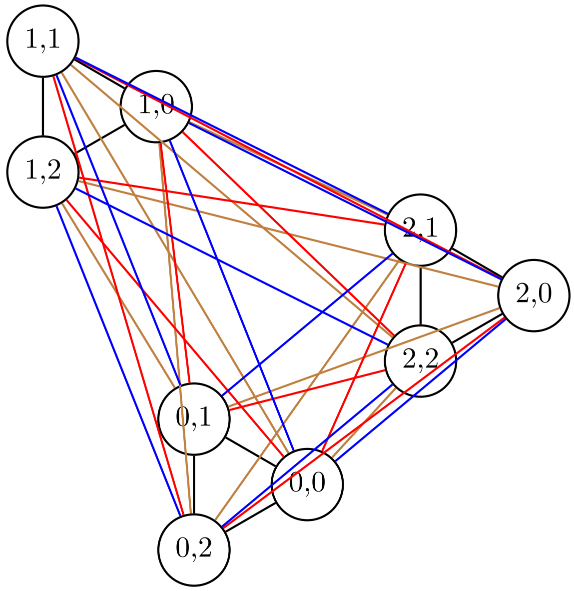


Figure 6: whole K_9: $s = 4, g = 3, n = 9, m = 12$

these connect between nodes that "advance one" with each hop between black cliques in **??**; for example $C_{0,1} = \{0, 4, 8\}$.

4. Cliques of form $C_{j,2} : C_{0,1}, C_{1,1}, C_{2,1}$: these are the triangles in red. Fig. **??** sees these connect between nodes that "go back one" with each hop between black cliques in **??**; for example $C_{0,1} = \{0, 5, 7\}$.

Every node is connected to every other node once in Fig. **??**, since for nodes x, y and z, w:

- $x = z$ and and they're in the same "black" clique G_x.
- $y = w$ and they're in a "blue" clique $C_{0,y}$
- There is some "increment" a where starting from node x, y, $z - x$ "hops" away lands you at node w, like the brown ($a = 1$) or red ($a = 2 \equiv -1 \mod 3$) cliques.

The last statement is deliberately informal. For prime numbers, it's clear that every increment in $[1, p-1]$ traverses a different, non-overlapping path $a, 2a, 3a...(a-1)p$.

The leap comes in realizing that these are not additive increments, but journeys through a field's multiplication table; the brown groups correspond to walking through the second row of Fig. **??** and adding each to either $c = 0, 1, or2$; the reds correspond to the third. The blues (0 "increment") are actually a walk through the top row!

This is most obvious for primes, but can be constructed from, say, $GF(4)$:

+	0	1	B	D
0	0	1	B	D
1	1	0	D	B
B	B	D	0	1
D	D	B	1	0

(a) Addition table GF(4)

·	0	1	B	D
0	0	0	0	0
1	0	1	B	D
B	0	B	D	1
D	0	D	1	B

(b) Multiplication table GF(4)

Unlike a prime-order finite field, the addition table is not cyclic, so in a sense, our indices for $G_{0,i}$ say are not $i \in [0, 3]$ but $i \in \{0, 1, B, D\}$! This is why the phrase "where $[0...g-1] \in \mathcal{F}$" is important in the algorithm.

Here are the resulting tables for $s = 5, g = 4$. For example, in Fig. **??**, consider the third row as saying "clique $C_{0,B}$ contains nodes $G_{0,0}, G_{1,B}, G_{B,D}, G_{D,1}$".

c	y	0	1	B	D
0	0	0	0	0	0
0	1	0	1	B	D
0	B	0	B	D	1
0	D	0	D	1	B

Figure 8: Groups $G_{0,i}$

c	y	0	1	B	D
1	0	1	1	1	1
1	1	1	0	D	B
1	B	1	D	B	0
1	D	0	B	0	D

Figure 9: Groups $G_{1,i}$

c	y	0	1	B	D
B	0	B	D	0	1
B	1	B	0	1	D
B	B	B	1	D	0
B	D	B	B	B	B

Figure 10: Groups $G_{B,i}$

(Of course, once the multiplication is defined, feel free to substitute 2 for B and 3 for D.)

c	y	0	1	B	D
D	0	D	D	D	D
D	1	D	B	1	0
D	B	D	1	0	B
D	D	D	0	B	1

Figure 11: Groups $G_{D,i}$

Including groups like $G_B = \{G_{0,B}, G_{1,B}, G_{2,B}, G_{3,B}\}$, we see that there are no repeated edges, and all edges are accounted for.

In general, to prove that we have no edges overlapping, we need to assert that for every pair of elements, say, (B, D), and every pair of columsn, say 2 and 4, that B and D appear in the same row in positions 2 and 4 exactly once. There must be a row with columns with the difference $D - B$ once in the multiplciation talbe; then, it's a matter of finding the unique c that causes the exact match. As to duplicates, consider two columns x_1 and x_2 that have a repeated value pair in some table, once on row g_y, once on g_y^*:

- Assume $g_y g_{x_1} + c = g_y^* g_{x_1} + c^*$, and $g_y g_{x_2} + c = g_y^* g_{x_2} + c^*$ for $g_{x_1} \neq g_{x_2}$
- Subtract the two to get $g_y(g_{x_1} - g_{x_2}) = g_y^*(g_{x_1} - g_{x_2})$
- The field $\mathcal{F}$ requires the nonzero $(g_{x_1} - g_{x_2}) \in \mathcal{F}$ to have an inverse.
- Multiplying both sides by that inverse, we have $g_y = g_y^*$

4 Constructing $g = s$ from $g = s - 1$

With the previous construction ($g = s - 1$) in hand, we can easily construct a $g = s$ graph with the same restrictions on g.

- Create node n_y for $y \in 0, [g - 1]$
- Add n_y to all cliques $C_{c,y}$
- Create node n_g. Add this to every G_i.
- Create clique of $n_y, y \in [0, g]$

Note you can create s=g=8 this way

c	y	G_0	G_1	G_B	G_D
0	0	0	0	0	0
0	1	0	1	B	D
0	B	0	B	D	1
0	D	0	D	1	B

(a) Groups $G_{0,i}$

c	y	G_0	G_1	G_B	G_D
1	0	1	1	1	1
1	1	1	0	D	B
1	B	1	D	B	0
1	D	0	B	0	D

(b) Groups $G_{1,i}$

c	y	G_0	G_1	G_B	G_D
B	0	B	D	0	1
B	1	B	0	1	D
B	B	B	1	D	0
B	D	B	B	B	B

(c) Groups $G_{B,i}$

c	y	G_0	G_1	G_B	G_D
D	0	D	D	D	D
D	1	D	B	1	0
D	B	D	1	0	B
D	D	D	0	B	1

(d) Groups $G_{D,i}$

Figure 12: s=5, g=4 adjacency tables

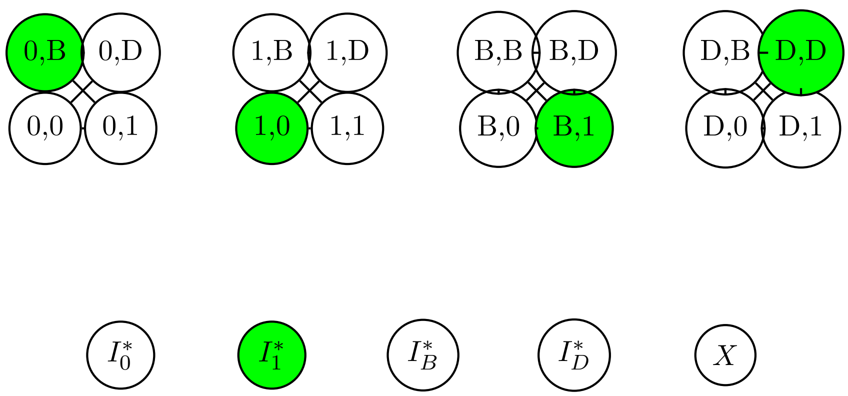


Figure 13: s=5, g=5 from s=5, g=4

5 Alternative: Constructing $g = s$ with perfect difference sets

Though it will be shown equivalent to the last construction, we can use another concept to build these graphs: perfect difference sets. Notably, these are proven to exist for $g = p^k$ by Singer. TODO Cite

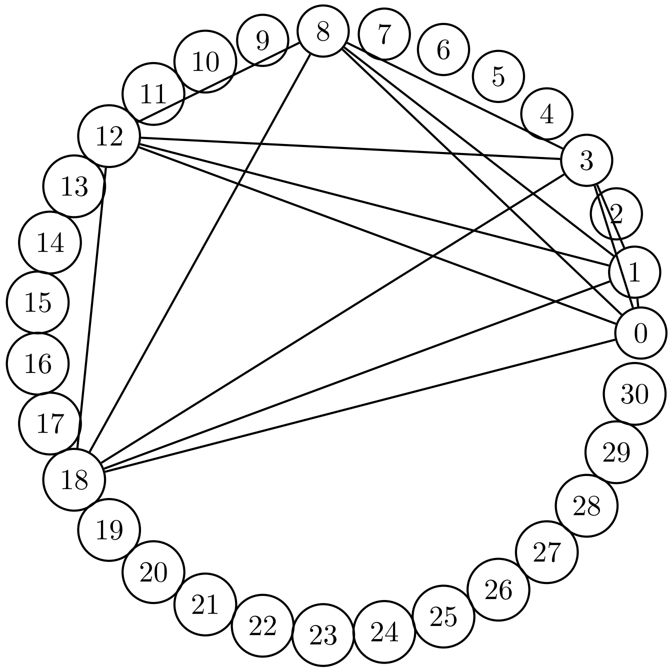


Figure 14: perfect Set difference on g=6, s=6, n=m=31

6 Interlude: Graph Equivalence up to relabeling

proving all $g = s$ graph colorings are the same up to relabeling. This means that the graphs constructed in sections X and Y are the same.

Fig: 5 nodes each with four neighbors

7 Constructing $g = s - 1$ from the previous

Removing one clique and all edges should suffice.

Todo: are these all the same, looking at complement.color graph $g - 1$ cliques left behind?

8 Considering wider $g|s$ and $g|s - 1$: Inception

Rule: $g = 3, s = 4, n = 9, m = 12$

Note that since $81 = 3^4, 3 \in \mathbb{P}$, K_{81} can be partitioned into 9-cliques (complete graphs of size K_9). Because we can recursively construct K_9s from the rule below (or "incept" the graph), we can turn the graph $n = 81, s = 10, g = 9, m = 90$ into one of $n = 81, s = 40, g = 3, m = 1080$, in which every node still has a uniform number of attached colors, and every color has the same number of node members. This graph has not been attempted.

Figure: s=4, g=3 has 9 nodes. S=g=9 has 9-cliques which could be broken down

Note: You can take the s=7, g=3, n=15, m=35 and change the n, n+5, n+10 triangles into unique colors for s=8, m=45, n=15 and g in 2,3 for example.

9 Considering wider $g|s$: Kirkman's schoolgirl problem

NOTE: We have the rotator of size g iff $g|s-1$, since $gk = s-1 \Rightarrow s = gk+1 \Rightarrow m = \frac{gk+1}{s}(sg-s+1)$. This means (I think) that there are $k(sg-s+1)$ cliques, or k rooted at each node, plus $\frac{sg-s+1}{g}$ other rotator cliques, being $s - \frac{s-1}{g}$ of size g that are like the island triangles

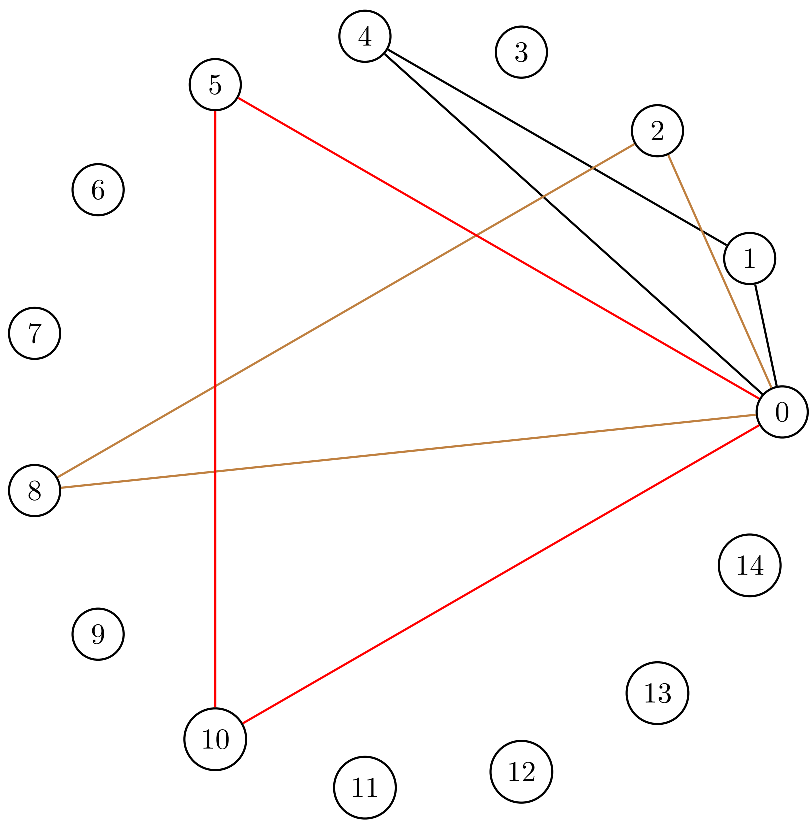


Figure 15: s=7, g=3, n=15, m=35, node 0 adjacencies. Rule: (i, i+5, i+10) x 3, (i, i+1, i+4) and (i, i+2, i+8)

ANOTHER ONE

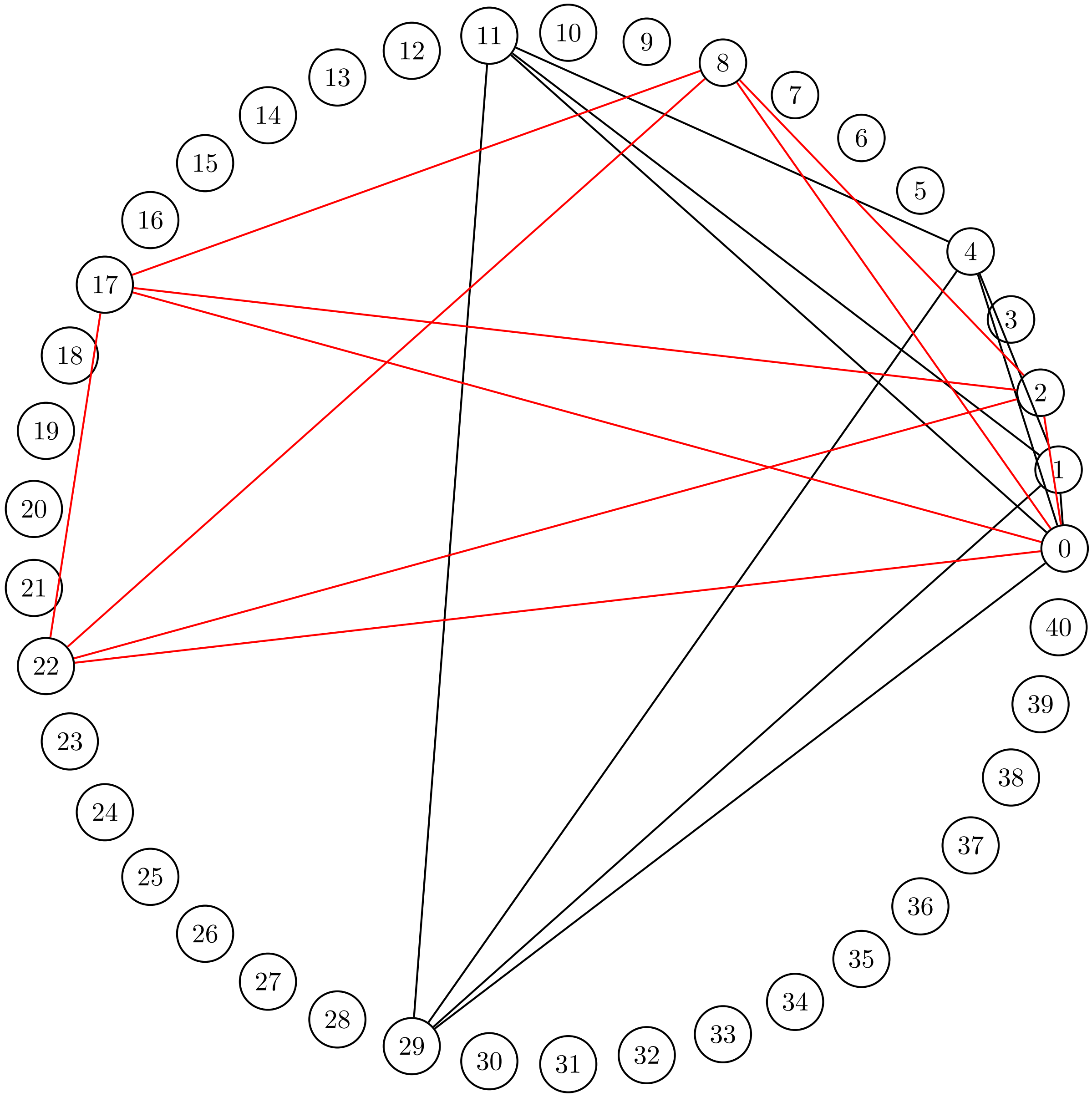


Figure 16: Perfect Difference Set on s=10, g=5, n=41, m=82: (0 1 4 11 29), (0 2 8 17 22)

10 Nonuniform g: deletion and partial inception

11 Another question

Can it be true that $g|s(s-1)$ but not true that $g|s$ or $g|s-1$?—

12 THIS IS THE END

13 Boneyard: Some examples with $g = 3$

- Rule: $g = 3, s = 3, n = 7, m = 7 : (0, 1, 3)$
- Rule: $g = 3, s = 4, n = 9, m = 12 : (0, 3, 6) \cdot 3, (0, 1, 2) \cdot 3, (0, 4, 8) \cdot 3, (0, 5, 7) \cdot 3$ (this is 3 separate K_3, then circuit inc 0, inc1, inc 2)
- Rule: Another example:$g = 3, s = 6, n = 13, m = 26$. 3-graphs are at $(i, i + 2, i + 8)$ and $(i, i + 1, i + 4)$, addition being $\mod 13$.. NOTE: Is this a subset of $s = 6, g = 6$?
- Rule: $g = 3, s = 7, n = 15, m = 35, (i, i + 5, i + 10) \cdot 3, (i, i + 1, i + 4), (i, i + 2, i + 8)$
- Rule: $g = 3, s = 9, n = 19, m = 57 : (0, 1, 6), (0, 2, 10), (0, 3, 7)$
- Rule: $g = 3, s = 10, n = 21, m = 70 : (0, 7, 14) \cdot 3, (0, 2, 10), (0, 1, 5), (0, 3, 9)$
- Rule: $s = g = 4, n = m = 13 : (0, 1, 3, 9)$
- Rule: $(0, 1, 4, 14, 16) on s = g = 5, n = 21$
- Rule: $s = g = 6, n = m = 31 : (0, 1, 3, 8, 12, 18)$
- NONE on $s = 7 = g$
- Rule: $s = g = 8, n = m = 57 - (0, 1, 3, 13, 32, 36, 43, 52)$
- Rule: $s = g = 9, n = m = 73 - (0, 1, 3, 7, 15, 31, 36, 54, 63)$; NOTE - these are all 2^{n-1} for a bit
- Rule: $s = g = 10, n = m = 91 - (0, 1, 3, 9, 27, 49, 56, 61, 77, 81)$
- NONE on $s = g = 11$
- Rule: $g = s = 12, m = n = 133 - (01, 3, 12, 20, 34, 38, 81, 88, 94, 104, 109)$
- NOTE All (g, s=g+1) don't seem to work with the rotators
- TODO Build something for the 2-rotators

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