Spotting Graph Theory Problems in Spot It

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3/29/23

Abstract

Main plan:

- Explain the game
- Modeling with a graph. Tiling C_n with C_g's.
- Set up the problem - s and g determine everything, what combos work?
- The Candidate Theorem: What combos CAN work?
- Showing combos with 3. TODO: Build completr here.
- Complete graph of setup g = s - 1. Introduce: round robin squares. Show s = 5, g=4
- Complete graphs of setup g = s. Show s = 3, g = 3.
- Rotator types. Maybe this will prove $s = kg$ and $s = kg + 1$?
- Proving g=4,s=5 cannot work. (Need to prove that the graph has to have four independent sets)
- Notes on nonuniform g sizes (removal, inception, the actual game of Spot It)

1 The game and the problem

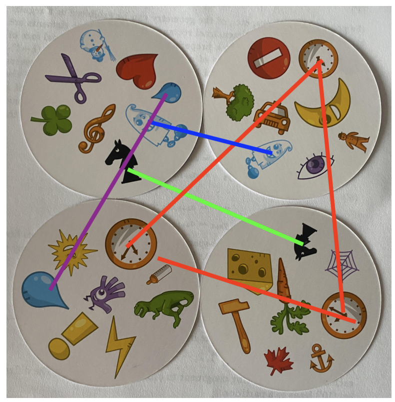
The children's game spot it has a simple mechanic at the cost of card construction: each card of 8 symbols shares exactly one with every other card.

constructing a deck like this, where each symbol occurs exactly g times along n cards, can be reformulated as constructing a partition of of complete graph K_n by graphs K_g

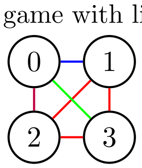
TODO: Fig: 4-node graph

Note: singletons would be self-edges

(a) Four cards in the game



(b) Four cards in the game with links



(c) Four cards graph

2 The candidates

We look to satisfy all constraints with s *slots,* g *occurrences per color. If* s *is uniform across cards and all symbols occur in groups of* g*, these two variables determine the number of cards (nodes)* n *and the number of symbols (colors)* m*. They also restrict the possible combinations to those where* $g|s$ *or* $g|s-1$

Fig: example s=6, g=3

Fig: example s=7, g=3

3 Constructing $g = s - 1$ over a field

We find that if g is a prime power, we can explicitly construct a graph of any size that satisfies our game.

Fig: 5x5 with green

Fig: 4x4 field table

Todo proof:

- Assume $g_y g_{x_1} + c = g_y^* g^{x_1} + c^*$, and $g_y g_{x_2} + c = g_y^* g^{x_2} + c^*$ for $g_{x_1} \neq g_{x_2}$
- Subtract the two to get $g_y(g_{x_1} - g_{x_2}) = g_y^*(g_{x_1} - g_{x_2})$
- The field $\mathcal{F}$ requires the nonzero $(g_{x_1} - g_{x_2}) \in \mathcal{F}$ to have an inverse.
- Multiplying both sides by that inverse, we have $g_y = g_y^*$

This is most obvious for primes, but can be constructed from, say, $GF(4)$:

+	0	1	B	D
0	0	1	B	D
1	1	0	D	B
B	B	D	0	1
D	D	B	1	0

(a) Addition table GF(4)

·	0	1	B	D
0	0	0	0	0
1	0	1	B	D
B	0	B	D	1
D	0	D	1	B

(b) Multiplication table GF(4)

4 Constructing $g = s$ from $g = s - 1$

With the previous construction in hand, we can easily construct a $g = s$ graph with the same restrictions on g.

TODO Fig: augment 5x5 to with 5+1

5 Alternative: Constructing $g = s$ with perfect difference sets

Though it will be shown equivalent to the last construction, we can use another concept to build these graphs: perfect difference sets. Notably, these are proven to exist for $g = p^k$ by Singer. TODO Cite

TODO Fig: n=31 finite difference

6 Interlude: Graph Equivalence up to relabeling

proving all $g = s$ graph colorings are the same up to relabeling. This means that the graphs constructed in sections X and Y are the same.

Fig: 5 nodes each with four neighbors

7 Constructing $g = s - 1$ from the previous

Removing one clique and all edges should suffice.

Todo: are these all the same, looking at complement.color graph $g - 1$ cliques left behind?

8 Considering wider $g|s$ and $g|s - 1$: inception.

Figure: s=4, g=3 has 9 nodes. S=g=9 has 9-cliques which could be broken down

9 Considering wider $g|s$: the schoolgirl problem

Link: `https://en.m.wikipedia.org/wiki/Kirkman\%27s_schoolgirl_problem`

10 Nonuniform g: deletion and partial inception

11 Another option question

Can it be true that $g|s(s - 1)$ but not true that $g|s$ or $g|s - 1$?—

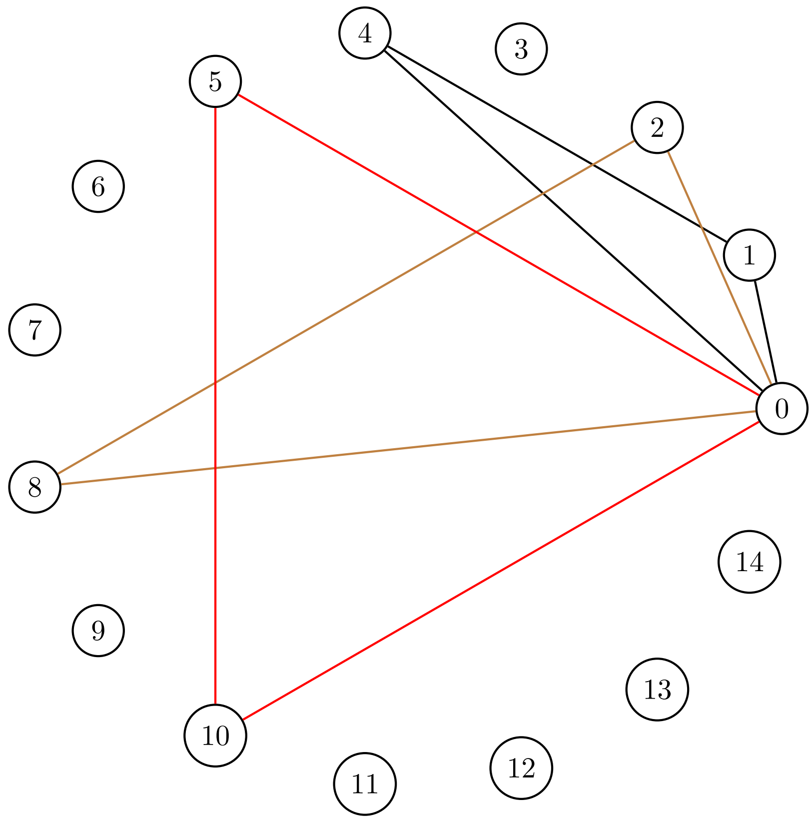


Figure 3: s=7, g=3, n=15, m=35, node 0 adjacencies. Rule: (i, i+5, i+10) x 3, (i, i+1, i+4) and (i, i+2, i+8)

12 THIS IS THE END

13 The Game

14 The Graph Theorem

A deck of n Spot It cards with m symbols over s slots can be represented by a graph G on n nodes of degree s and edges of m unique colors, and for any color m_i, the edges of that color (and all adjacent nodes) form a complete subgraph of G_i of G.

Note: Self-loop objection.

15 The Core Question

For what choices of g and s can graphs be constructed that satisfy our constraints?

TODO

16 The Candidate Theorem

Suppose further that that every symbol s has exactly g cards containing it [1]. Then

1. Total nodes $n = (g-1)s + 1$,
2. Total colors $m = \frac{\binom{n}{2}}{\binom{g}{2}}$.
3. $g|s(s-1)$.
4. If $s > 1$ and $g > 1$ then $g \leq s$
5. All candidate configurations of g, s are $g \leq s$, $g|s(s-1)$.

Proof:

1. As in Fig. 1, node n_0's adjacencies are exactly s monocolor cliques of size $g-1$ (excluding n_0 itself). In a complete graph, these adjacencies comprise the total node set, so $n = (g-1)s + 1$ when adding n_0 back in. Using any other node is equivalent.
2. A complete graph C_n's has $\binom{n}{2}$ edges. A monocolor clique of size g is a complete graph as well, with $\binom{g}{2}$ edges. C_n's edges are exactly these equal-sized cliques, so there are therefore $m = \frac{\binom{n}{2}}{\binom{g}{2}}$ of them.
3. $$\binom{g}{2} | \binom{n}{2} \Rightarrow \frac{n(n-1)}{g(g-1)} \in \mathbb{N} \Rightarrow g(g-1)|n(n-1) \quad (1)$$
$$n = (g-1)s + 1 \Rightarrow g(g-1)|(sg - s + 1)(sg - s) = (sg - s + 1)s(g-1) \quad (2)$$
$$\Rightarrow g|s^2 g - s^2 + s \Rightarrow g|(1-s)s \Rightarrow g|s(s-1) \quad (3)$$
4. Any node n_i is adjacent to s monocolor cliques of size g. These cliques $C_1 ... C_g$, containing non-n_i nodes if $g > 1$, comprise all nodes, and any other cliques can contain no more than one of each C_i. This means that clique of size g greater than s cannot

be formed, since the only place to find nodes are these $C_1 ... C_g$. The other trivial case, $s = 1$, means there is only one color in the whole graph.

This means we need not consider configurations like $g = 6, s = 3$ even though $6|3(3-1)$.
Another corollary here is that $\boxed{m \geq n}$, since:

$$n = (sg - s + 1) \tag{4}$$

$$m = \frac{\binom{\frac{(sg-s+1)(sg-s)}{2}}{}}{\binom{g}{2}} = \frac{(sg-s+1)(sg-s)}{g(g-1)} = \frac{(sg-s+1)s}{g} \tag{5}$$

$$\Rightarrow m = (\frac{s}{g})n \tag{6}$$

$$s \geq g \Rightarrow m \geq n \tag{7}$$

For example, a tiling of triangles ($g = 3$) means that either $s \equiv 0 \mod 3$ or $s \equiv 1 \mod 3$. Since $n = (g-1)s + 1 = 2s + 1$ or $n \equiv 1 \mod 2$, then $n = 2(3k) + 1 = 6k + 1$ or $n = 2(3k+1) + 1 = 6k + 3$, meaning $n \in \{1, 3\} \mod 6$.

17 Some examples with $g = 3$

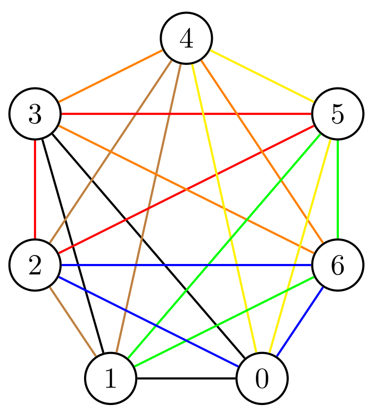


Figure 4: s=3, g=3, n=7, m=7. Rule: (i, i+1, i+3) for all i.

- Rule: $g = 3, s = 3, n = 7, m = 7 : (0, 1, 3)$
- Rule: $g = 3, s = 4, n = 9, m = 12 : (0, 3, 6) \cdot 3, (0, 1, 2) \cdot 3, (0, 4, 8) \cdot 3, (0, 5, 7) \cdot 3$ (this is 3 separate K_3, then circuit inc 0, inc1, inc 2)
- Rule: Another example:$g = 3, s = 6, n = 13, m = 26$. 3-graphs are at $(i, i+2, i+8)$ and $(i, i+1, i+4)$, addition being $\mod 13$.. NOTE: Is this a subset of $s = 6, g = 6$?

- Rule: $g = 3, s = 7, n = 15, m = 35, (i, i+5, i+10) \cdot 3, (i, i+1, i+4), (i, i+2, i+8)$
- Rule: $g = 3, s = 9, n = 19, m = 57 : (0, 1, 6), (0, 2, 10), (0, 3, 7)$
- Rule: $g = 3, s = 10, n = 21, m = 70 : (0, 7, 14) \cdot 3, (0, 2, 10), (0, 1, 5), (0, 3, 9)$
- Rule: $s = g = 4, n = m = 13 : (0, 1, 3, 9)$
- Rule: $(0, 1, 4, 14, 16) ons = g = 5, n = 21$
- Rule: $s = g = 6, n = m = 31 : (0, 1, 3, 8, 12, 18)$
- NONE on $s = 7 = g$
- Rule: $s = g = 8, n = m = 57 - (0, 1, 3, 13, 32, 36, 43, 52)$
- Rule: $s = g = 9, n = m = 73 - (0, 1, 3, 7, 15, 31, 36, 54, 63)$; NOTE - these are all 2^{n-1} for a bit
- Rule: $s = g = 10, n = m = 91 - (0, 1, 3, 9, 27, 49, 56, 61, 77, 81)$
- NONE on $s = g = 11$
- Rule: $g = s = 12, m = n = 133 - (01, 3, 12, 20, 34, 38, 81, 88, 94, 104, 109)$
- NOTE All (g, s=g+1) don't seem to work with the rotators
- TODO Build something for the 2-rotators

18 Generating $g = s - 1$, $g \in \mathbb{P}$

G_0	G_1	G_2	G_3
0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3

(a) I_0

G_0	G_1	G_2	G_3
0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

(b) I_1

G_0	G_1	G_2	G_3
0	2	3	1
1	3	0	2
2	0	1	3
3	1	2	0

(c) I_2

G_0	G_1	G_2	G_3
0	3	1	2
1	0	2	3
2	1	3	0
3	2	0	1

(d) I_3

Figure 5: s=5, g=4 adjacency tables

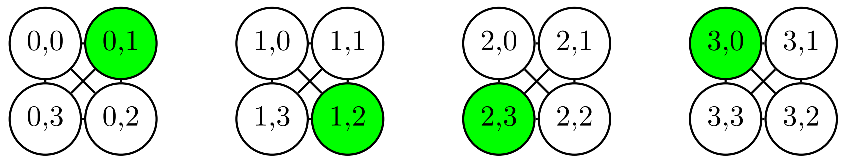


Figure 6: TODO: Busted: (TURN THIS TO g=5, s=6) s=5, g=4, n=16, m=20

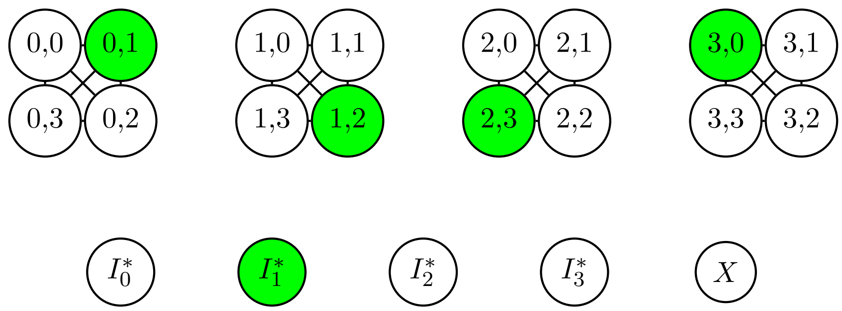


Figure 7: s=5, g=5

19 Generating $g = s$ for $g - 1 \in \mathbb{P}$

TODO: The James construction.

TODO Proof

20 A nonexistence result: g=4, s=5

If we have g=4 and s=5, then there exists a clique of size 4. Every node of that clique is adjacent to four additional colors, and none of those colors can be shared (else double color edge). Thus, of the 20 colors, each color has 3 ”non-adjacent neighbors”.

This forms a graph of n nodes, where each node is a color, and nodes are adjacent if colors share a node in the original graph. Each node in this graph has degree 3.

Brooks's Theorem[**1**] states that if every node has degree Δ or less, than since this is not a complete graph and not an odd cycle, the nodes can be vertex-colored with Δ or fewer colors, or in this case, 3. This means that there must NOT be a complete subgraph K_4, or four mutually non-adjacent colors. NOTE: This does not apply as with prime, we end up with connected components of size K_g.

Each color offers up exactly 3 non-adjacencies, so we have $20 * 3/2 = 30$ non-adjacency edges in the grapjh.

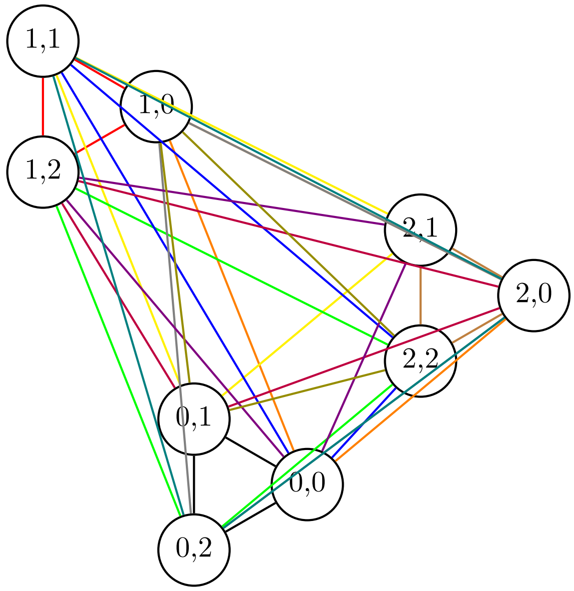


Figure 8: whole C_9: $s = 4, g = 3, n = 9, m = 12$

Our "ring" construction on which the (g, s) configurations $(p, p+1)$ and $(p+1, p+1)$, $p \in \mathbb{P}$ does not work always if $g \not\mathbb{P}$.

Though we have yet to prove nonexistence for all composite g, we can show that $g = 4, s = 5$ cannot work. This is through the proof:

1. The graph defined by $g = 4, s = 5 (n = 16, m = 20)$ must contain four C_4 monocolor cliques S_0, S_1, S_2, S_3 with no pairwise no overlapping nodes.
2. Any coloring of the graph requires choosing four cliques S_0, S_1, S_2, S_3 plus sixteen cliques $S_i = \{s_{0,i}, s_{1,j}, s_{2,k}, s_{3,l}\}$, with $s_{0,i}$ signifying some node in S_0.
3. Such a graph does not exist.

Proof:

1. 1. Consider that the cliques corresponding to each of the $m = 20$ colors must have pairwise overlap of zero or one of the 16 nodes (if they share two nodes, they share an edge, and thus an edge has two colors). Let's create another graph G where each node n_i corresponds to a color $C_i, i \in [0, 19]$, and an edge (n_i, n_j) exists iff $\{n_i, n_j\} \subseteq C_i, C_j$. Suppose G has a maximum of three pairwise nonoverlapping cliques. Then, there can be at most 15 colors, TODO: Did I get this wrong too?
2. TODO

21 Constructions with mixed g

21.1 Trivial

TODO

21.2 Chopping

TODO

21.3 Inception

- what about solutions with mixed sized subgraphs? You can take the s=7, g=3, n=15, m=35 and change the n, n+5, n+10 triangles into unique colors for s=8, m=45, n=15 and g in 2,3 for example.

22 The main question: Are all candidates viable?

Note: Can drop from$s = 4, g = 4, n = 13, m = 13$ to $s = 4, g = 3, n = 9, m = 12$ by dropping last $g-$sized clique and all adjacent edges.

- Perfect difference sets: `https://oeis.org/search?q=0+1+3+9+27+49+56+61+77+81&sort=&language=english&go=Search`, `https://mathworld.wolfram.com/PerfectDifferenceSet.html`.
- Necessary for $n = k^2 + k + 1$. Sufficient is k being a prime power.
- We have the rotator of size g iff $g|s-1$, since $gk = s-1 \Rightarrow s = gk+1 \Rightarrow m = \frac{gk+1}{s}(sg-s+1)$. This means (I think) that there are $k(sg-s+1)$ cliques, or k rooted at each node, plus $\frac{sg-s+1}{g}$ other rotator cliques, being $s - \frac{s-1}{g}$ of size g that are like the island triangles

23 Further questions

References

[1] Wikipedia: `https://en.wikipedia.org/wiki/Brooks\%27_theorem`