Brilliant: Differential Equations II

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Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 1: Basics

1.1 Chapter 1: Nonlinear Equations

The two types of problems in this course are:

- Nonlinear equations (several equations on one independent variable)
- Partial differential equations (single equation with several independent variables)

Linear equations have solutions like y_1, y_2 that can be combined using any $c \in \mathbb{R}$ like $y_1 + cy_2$.

Example: Bacteria in a dish with a lot of food, no deaths

- $b'(t) = r_b b(t), r_b > 0. r_b$ would be the rate of growth.
- This is linear. Reason 1: $\frac{d}{dt}(y_1 + cy_2) = y_1' + cy_2' = r_b(y_1 + c_y 2)$ since $y' = r_b y(t)$, and same for y2.
- Also, this works because the solution is $b(t) = b(0)e^{r_b t}$, so $b_1(t) + cb_2(t) = b_1(0)e^{r_b t} + cb_2(0)e^{r_b t} = (b_1(0) + cb_2(0))e^{r_b t}$

Example: **Logistic** equation: Bacteria in a dish with a lot of food, limited by carrying capacity M.

- $b'(t) = r_b b(t)[M - b(t)]$.
- This is nonlinear. Reason: $\frac{d}{dt}(y_1' + cy_2') = y_1' + cy_2' = r_b[y_1 + cy_2][M - y_1 - cy_2] = My_1 + Mcy_2 - y_1^2 - 2cy_1y_2 - cy_1^2y_2^2$
- $\neq My_1 - y_1^2 + Mcy_2 - c^2y_2^2$ because of the extra $-2cy_1y_2$ term.

Sidebar: Note that this equation $b' = r_b b[M - b]$ is *separable*, so it can be solved.

- $\frac{db}{dt} = rb[M - b]$
- $\frac{db}{b(M-b)} = rdt$
- $\frac{1}{M}(\frac{1}{b} + \frac{1}{M-b})db = rdt$ after partial fractions work
- $(\ln(b) - \ln(M - b)) = Mrt + C \Rightarrow \ln(\frac{b}{M-b}) = Mrt + C$
- $\frac{b}{M-b} = e^{Mrt}e^{C}$
- Initial conditions $b = b(0), t = 0 \Rightarrow \frac{b}{M-b} = \frac{b(0)}{M-b(0)}e^{Mrt}$
- $b(1 + \frac{b(0)}{M-b(0)}e^{Mrt}) = M\frac{b(0)}{M-b(0)}e^{Mrt}$
- $b(M - b(0) + b(0)e^{Mrt}) = Mb(0)e^{Mrt}$
- $b = \frac{Mb(0)e^{Mrt}}{M+b(0)[e^{Mrt}-1]}$

This logistic solution will taper off to M at some point. Note that $\lim_{t\to\infty} b(t) = M$ since the non-exponential terms stop mattering. Also $b(t) = M$ sticks as a constant solution or **equilibrium** immediately. *These equilibria tell us what matters - the long-term behavior of solutions!*

Another **Example**: Lotka-Volterra equation pairs: Bacteria (b) and bacteria-killing phages (p), with kill rate k.

- The "product" $kb(t)p(t)$ measures the interactions and kills resulting from this.
- $b'(t) = r_b b(t) - kp(t)b(t)$, or the normal growht rate minus kill rate
- $p'(t) = kp(t)b(t)$ since its population grows as it kills bacteria.
- Equilibria include $b = 0, p = 0$ and $b = 0, p > 0$, since these are *constant* solutions, or places where $b'(t) = 0, p'(t) = 0$.

Direction fields, with vector pointing towards $\langle b'(t), p'(t)\rangle$ (TODO - I think) let us follow the arrows to determine the curve over time. In this case, the bacteria will always go extinct.

However, if we add a new death rate term $-d_p p(t)$ so $p'(t) = -d_p p(t) + kp(t)b(t)$:

- We get an equilibrium at $b = \frac{d_p}{k}, p = \frac{r_b}{k}$. (Since $0 = b'(t) = rb - kpb, (\Rightarrow pk = r), 0 = p'(t) - dp + kpb, (\Rightarrow bk = d)$)
- But otherwise the solutions swirl around this point. This is called a **cycle**. TODO What is a **limit cycle**?

Note that there are systems where the "solution particle" neither reaches an equilibrium or cycles around one point. The **Lorenz system** famously has this owl-eye shaped double attractor (an example of **strange sets**) where initially close particles diverge unpredictably if the constants ρ, σ, b are chosen right:

- $x'(t) = \sigma(y - x)$
- $y'(t) = x(\rho - z) - y$
- $z'(t) = xy - bz$
- TODO

1.2 Chapter 1.2: PDEs

Many methods of attack for PDEs

- Separation of variables
- Power series (Note: did we actually touch on this?)
- Fourier Transform

Example: Standing wave, where one end of a rope is fixed.

- Vertical displacement from a line of rope: $u(x, t)$ depends on where (x) and when (tt).
- Rope's **wave equation** is $u_{tt} = v^2 u_{xx}$, where v is the "constant wave speed", and the others are the space, time partials.
- Note that $u = \cos(vt)\sin(x)$ and $u = \sin(vt)cos(x)$ both work.
- If you guess the solution has split variables like $u = X(x)Y(y)T(t)$, then, upon substitution and division by $X(x)Y(y)T(t)$, $\frac{\delta^2 u}{\delta t^2} = v^2[\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}]$ yields $\frac{T''(t)}{T(t)} = v^2[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}]$
- This method may or may not work. But if it does, it means that since x, y, and t are independent variables, each individual piece must be constant.
- So, for example, if we know $\frac{X''(x)}{X(x)} = -4\pi^2$, we can get to $X(x) = \sin(2\pi x)$
- The wave equation is similar in 3D: $u_{tt} = v^2[u_{xx} + u_{yy} + u_{zz}]$, or using the Laplacian, $u_{tt} = v^2\nabla^2 u$. Here, u measures not displacement but expansion/compression of air at (x, y, z), time t.

Using Fourier transforms helps turn difficult PDEs into an easier problem like an ODE. *Fourier transforms work best* when

- The domain is all of $\mathbb{R}^n$
- The function u vanishes at infinity.

The Fourier transform changes the domain of x to that of ω. It comes with the (highly simplified) rule (see Vector Calculus course): $F[\frac{\delta f}{\delta x}] = i\omega F[f]$. **Example**: Drunkard's walk.

- One dimensional: moves left or right in a random way. Starts at $x = 0, t = 0$.
- $u(x,t)$ is probability of being at point x at time t. Naturally, $\int_{x=-\infty}^{x=\infty} u(x,t)dx = 1$.
- Also, it obeys the 1-dD diffusion equation $\frac{\delta u}{\delta t} = \frac{\delta^2 u}{\delta x^2}$
- The Fourier transform doesn't affect t at all.
- So by taking Fourier transform of both sides of diffusion equation we get
  - $F(u_t) = \frac{\delta}{\delta t}F(u)$ since F doesn't care about t.
  - $\frac{\delta^2 u}{\delta x^2} = i\omega F(\frac{\delta u}{\delta x}) = -\omega^2 F(u)$
  - So $\frac{\delta}{\delta t}F(u) = -\omega^2 F(u)$
  - This is solvable as $F(u) = ce^{-\omega^2 t}$. Take it on faith that $c = \frac{1}{2\pi}$ for now. TODO
  - Known fact: $F[Ae^{-\frac{ax^2}{2}}] = \sqrt{\frac{1}{2\pi a}}Ae^{\frac{-\omega^2}{2a}}$
  - This means $t = \frac{1}{2a}$ and $a = \frac{1}{2t}$
  - $F(u) = \frac{1}{2\pi}e^{-\omega^2 t}, F[Ae^{-\frac{ax^2}{2}}] = \sqrt{\frac{1}{2\pi a}}Ae^{\frac{-\omega^2}{2a}}$ so $u = Ae^{\frac{-ax^2}{2}}$
  - Solving, you get $A = \sqrt{\frac{1}{4\pi t}}, a = \frac{1}{2t}$, so $u(x,t) = \sqrt{\frac{1}{4\pi t}}e^{-\frac{x^2}{4t}}$

2 Chapter 2: Nonlinear Equations

2.1 2.1: Lotka-Volterra I

Major ideas:

- **phase plane**: TODO
- **nullcline**: TODO
- **direction field**: TODO
- **equilibria**: TODO

Example: Bacteria vs. phages (again)

- Bacteria unrestrained grow in proportion to their population, so $\frac{db}{dt} = r_b b(t)$ (solved: $b(t) = b(0)e^{r_b t}$)
- Phages unfed decrease in proportion to current size, so $\frac{dp}{dt} = -d_p p(t)$ (solved: $p(t) = p(0)e^{-d_p t}$)
- Bacteria die with likelihood of meeting a phage, and phages increase with likelihood of meeting a bacterium. So the set of equations, for constant k, becomes:
  - $b'(t) = r_b b(t) - kb(t)p(t)$
  - $p'(t) = -d_p p(t) + kb(t)p(t)$
  - *The product of p and b makes our equations nonlinear* (WHY?)
  - I guess, very generally, $b_1 p_1 = k, b_2 p_2 = k$, but $(b_1 + b_2)(p_1 + p_2) = b_1 p_1 + b_2 p_2 + b_1 p_2 + b_2 p_1 = 2k + b_1 p_2 + b_2 p_1 \neq 2k$, so the last two "mixed" terms mean you can't just add solutions (b_1, p_1) and (b_2, p_2).

General thoughts on this solution:

- So a solution $(b(t), p(t))$, traces out a curve on the bp-phase plane (b is x-axis, p is y-axis) as time (unrepresented in the plane) continues.
- If we add a unit tangent vector at every point (B, P) aligned with $(b'(t), p'(t)) = (r_b B - kBP, -d_p P + kBP)$, we can follow the arrows to see the solution over time.
- The above is called a **direction field**
- This is sometimes hard to sketch analytically, so we can look to the **nullclines**: places where one of the components of the direction field is zero.
- In this case, $r_b B - kBP = (r_b - kP)B = 0$ when $P = 0$ or $P = \frac{r_b}{k}$, and $-d_p P + kBP = (kB - d_p)P = 0$ when $P = 0$ or $B = \frac{d_p}{k}$.
- The **upshot of nullclines** (since we don't care about $P, B \leq 0$): The lines $B = \frac{d_p}{k}, P = \frac{r_b}{k}$ *divide the plane into pieces where the components of this (continuous) function pair can't change sign.*
- For instance, $B > \frac{d_p}{k}, P < \frac{r_b}{k}$ means $r_b b - kbp > 0, -d_p p + kdp > 0$, so both populations are growing here. This helps to sketch the curve.
- The curve looks like a counterclockwise whirlpool around the $(B, P) = (\frac{d_p}{k}, \frac{r_b}{k})$. (bacteria grow with low but growing phages; bacteria decrease as phages overwhelm; both decrease as phages starve; bacteria start coming back)
- The center point is a (constant **equilibrium**) solution, and other solutions swirl around it but don't get attracted or repelled.

There are a few types of equilibria:

- This one is a **center** around which solutions circle.
- A **stable equilibrum** would see small upsets come back to an unchanging state.
- An **unstable equilibrum** would see small upsets create wildly divergent paths.

2.2 2.2: Lotka-Volterra II

In the Bacteria-Phage system, we can't yet prove everything rotates around the **center**. Let's do that.

Developing a **conserved quantity** will help to do that. **Example**: Block on a horizontal spring with mass m, spring constant k_s:

- $x(t)$: Displacement from rest position.
- $v(t) = \frac{dx}{dt}$: Horizontal velocity
- $\frac{dv}{dt} = -\frac{k_s}{m}x(t)$ by Hooke's law, I think.
- Suppose there's some Energy function $E(x, v)$. By chain rule $\frac{d}{dt}E(x(t), v(t)) = \frac{dE}{dx}\frac{dx}{dt} + \frac{dE}{dv}\frac{dv}{dt}$
- $= \frac{dE}{dx}v - \frac{k_s}{m}\frac{dE}{dv}x$. If we set E as conserved, as in $E'(t) = 0$, then $\frac{dE}{dx}v = \frac{k_s}{m}\frac{dE}{dv}x$
- We can eyeball and see that $E = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ solves this equation, or we can assume $E(x, v) = F(x) + G(v) \Rightarrow 0 = E'(t) = F'(x)v - \frac{k_s}{m}G'(v)x = 0$ from the above equations and guess from there.
- This means in the xv phase space, that there's a fixed E such that the particle follows the ellipse $E = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ in phase space around the solution point (0,0).

Extended Example: Continuing on finding a conserved quantity for Bacteria / Phage:

- We need to find $U(b(t), p(t))$ such that $U'(t) = 0$, or by chain rule $\frac{\delta U}{\delta b}\frac{\delta b}{\delta t} + \frac{\delta U}{\delta p}\frac{\delta p}{\delta t} = 0$
- Subbing in, $\frac{\delta U}{\delta b}[r_b b - kbp] + \frac{\delta U}{\delta p}[-d_p p + kbp] = 0$
- A hint suggests finding U such that $\frac{\delta U}{\delta b} = -\frac{d_p}{b} + k, \frac{\delta U}{\delta p} = -\frac{r_b}{p} + k$ to make terms cancel.
- Integrating these gives us U as both $-d_p \ln(b) + kb + Q(p)$ and $-r_b \ln(p) + kp + R(b)$ so $U = -d_p \ln(b) - r_b \ln(p) + kb + kp$. This weird curve consistutes a level set in pb-space upon which a solution sits.
- The spring example has an elliptic paraboloid solution. There's an absolute minimum ($E = 0$ at $(0, 0)$) but level sets become closed loops away from it.

- For the Lotka example, there is a critical point ($\nabla U = \vec{0}$) when $\nabla U(b,p) = (\frac{\delta U}{\delta b}, \frac{\delta U}{\delta p}) = (k - \frac{d_p}{b}, k - \frac{r_b}{p})$, which is $(0,0)$ at our known center $(\frac{d_p}{k}, \frac{r_b}{k})$
- Showing that we always increase gong away from the point $(\frac{d_p}{k}, \frac{r_b}{k})$ should guarantee us closed level sets.
- One method: Assume we're picking a unit vector $\vec{v} = \langle \hat{v_b}, \hat{v_p} \rangle$ so that our line from our center is $\vec{v} = \langle \frac{d_p}{k} + tv_b, \frac{r_b}{k} + tv_b \rangle$. $U = F(b) + G(p)$ in this case, so sub the b part into F to get $F(\frac{d_p}{k} + t\hat{v_b}) = d_p[1 - \ln(\frac{d_p}{k} + t\hat{v_b})] + kt\vec{v_b}$. Taking derivative of that w.r.t t shows it is always positive. Same goes for the G(p) portion of U.
- Another (DF) method: Note that $\nabla U = (k - \frac{d_p}{b}, k - \frac{r_b}{p})$'s grad (second derivative) is always positive. So derivative always has positive curvature (maybe using that term wrong), and we'll always increase around this point.
- Also, we know that the particle travels around the level set (loop) and doesn't reverse course, because then, $b'(t) = p'(t) = 0$, and we only have that at the center point (nullcline intersection).

2.3 2.3: Linearization

Extended Example: Suppose there's a limit to bacterial growth, so we cap our population at M_b.

- If $b(t) << M_b$, things should be similar. If $b(t)$ is nearly M_b, then growth should approach 0. So, this implies $\frac{db}{dt} = r_b b(t) \rightarrow \frac{db}{dt} = r_b b(t)(1 - \frac{b(t)}{M_b})$. Note: This isn't the only possibility but we'll use it.
- This updates our Lotka-Volterra model to something more complicated:
  - $b'(t) = r_b b(t)(1 - \frac{b(t)}{M_b}) - kb(t)p(t)$
  - $p'(t) = -d_p p(t) + kb(t)p(t)$
- Other than $b = 0, p = 0$, the meaningful nullclines are solved by setting $b'(t) = 0$ (yielding $r_b(1 - \frac{b}{M_b}) - kp = 0$) and $p'(t) = 0$ (yielding $b = \frac{d_p}{k}$)
- Note: We'll clean up through some MAGIC non-dimensionalization (how to derive?) to simplify:
  - $x(t) = \frac{1}{M_b} b(\frac{t}{r_b}), y(t) = \frac{k}{r_b}(\frac{t}{r_b}), \alpha = \frac{d_p}{r_b}, \beta = \frac{kM_b}{r_b}$
  - Gives us new equations: $\frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \frac{dy}{dt} = -\alpha y(t) + \beta x(t)y(t)$
  - And new nullclines: $x + y = 1, x = \frac{a}{b}$

- So there's an equilibrium point in the positive xy quadrant if: $y = 1 - x = 1 - \frac{\alpha}{\beta}$ and $y > 0$ implies $1 - \frac{\alpha}{\beta} > 0 \Rightarrow \frac{\alpha}{\beta} < 1$
- Looking at the direction field, it appears solutions swirl around and are attracted *into* the center point $(\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta})$, making it a **stable equilibrum**

This is similar to the block-spring example, if a damping term $-\frac{\gamma}{m}v$ is added.

- $\frac{dx}{dt} = v, \frac{dv}{dt} = -\frac{k_s}{m}x - \frac{\gamma}{m}v$
- This can be thought of in matrix terms: $\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{\gamma}{m} \end{pmatrix}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ Call the matrix A.
- From Diff Eq I, the solution is $exp(tA)$ (matrix exponential), making x(t) a linear combination of $e^{\lambda t}$ or possibly $te^{\lambda t}$ terms, with the eigenvalues as λs.
- The eigenvalues in this case, using the quadratic formula, could be:
  - Two real, distinct, negative roots. So, these $e^{\lambda t}$ terms decay, and x(t) levels off.
  - Two distinct complex roots with real part $-\frac{\gamma}{2m} < 0$. This ends up being some sines and cosines multiplied by $e^{-\frac{\gamma t}{2m}}$, which decays too.
  - Finally, if we have a repeated negative real eigenvalue, we have solution $x(t) = Ae^{-\frac{\gamma t}{2m}} + Bte^{-\frac{\gamma t}{2m}}$, also decaying.
  - So any disturbance in the spring will oscillate and come to rest at $x(t) = v(t) = 0$ quickly.

  So with linear systems $\vec{x}'(t) = A\vec{x}(t)$, the eigenvalues determine what happens around the equilibrium point. However, the **bacteria-phage model is non-linear**. Here is **how we linearize** for nearby solutions in a nonlinear system:

  - Set small disturbance $\delta x(t) << 1, \delta y(t) <<$ so $x(t) = \frac{\alpha}{\beta} + \delta x(t), y(t) = 1 - \frac{\alpha}{\beta} + \delta y(t)$
  - Since they're small, all powers like $\delta x(t)^2$ and $\delta x(t)\delta y(t)$ are considered basically zero.
  - So substitute $x(t) \rightarrow \frac{\alpha}{\beta} + \delta x(t), y(t) \rightarrow = 1 - \frac{\alpha}{\beta} + \delta y(t)$ into our $\frac{dx}{dt}$ and $\frac{dy}{dt}$ equations.
  - This gives us the A solving $\frac{d}{dt}\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = A\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$, which is $A = \begin{pmatrix} -\frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \\ \beta - \alpha & 0 \end{pmatrix}$ after working through the substitution.

  - Finding the eigenvalues here yields the same situation as the block-spring example: decays in all situations.

It turns out through the **Hartman-Grobman Theorem** that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$, for some continuously differential vector field F, if we linearize near equilibrium x_0, then what falls out of this A approach works if the eigenbalues *aren't all purely imaginary.*

It turns out the uncapped bacteria system from before looks like $\frac{d}{dt}\begin{pmatrix}\delta x(t) \\ \delta y(t)\end{pmatrix} = \begin{pmatrix}0 & -1 \\ \alpha & 0\end{pmatrix}\begin{pmatrix}\delta x(t) \\ \delta y(t)\end{pmatrix}$, with characteristic equation $\lambda^2 + \alpha = 0, \alpha > 0$. This means both values are imaginary, and we had to use the conserved quantity appraoach!

2.4 2.4: Hartman-Grobman Theorem

Extended Example: Consider a phage that dies off quicky:

- $\frac{db}{dt} = r_b b(t) - k_b b(t)p(t), \frac{dp}{dt} = -r_p p(t) = 0 \cdot b(t)p(t)$, where k_p is the zero (phages don't increase), and k_b is still the kill factor for the bacteria.
- In this base, $b(t) = p(t) = 0$ is the only equilibrium.
- Non-dimensionalize as $x(t) = b(\frac{t}{r_b}), y(t) = \frac{k_b}{r_b}p(\frac{t}{r_b}), \alpha = \frac{r_p}{r_b}$
- This makes the equations $x'(t) = x(t) - x(t)y(t), y'(t) = -\alpha y(t)$, and the nullclines therefore $x(t) = 0, y(t) = 1, y(t) = 0$
- Looking at this six-section dierection field, we see that solutions exactly on the y-axis are attracted to equilibrium $(0, 0)$, and other are repelled.
- This makes sense since if the bacteria is 0, the phage die and approach $(0, 0)$, otherwise the bacteria multiply and win (so it's a *saddle point*)
- The way to tell: linearize the equations. $\frac{d}{dt}\begin{pmatrix}x(t) \\ y(t)\end{pmatrix} \approx \begin{pmatrix}1 & 0 \\ 0 & -\alpha\end{pmatrix}\begin{pmatrix}x(t) \\ y(t)\end{pmatrix}$ since, if $x(t), y(t) << 1, x(y)t(t) = 0$.
- Then the eigenvalues are $\lambda = 1, -\alpha$ so the solution is Ae^t, $Be^{-\alpha t}$ for $x(t), y(t)$ (TODO respectively?) **Hartman-Grobman ensures this is the general solution**.

However, let's solve directly and see if we come to the same result.

- $y'(t) = -\alpha y(t), y(0) = y_0 \Rightarrow y(t) = y_0 e^{-\alpha t}$
- With this in hand, $\frac{dx}{dt} = x(t) - = x(y)y(t) = x(t)[1 - y_0 e^{-\alpha t}]], x(0) = x_0$ separates out to
  - $\frac{dx}{x} = [1 - y_0 e^{-\alpha t}]dt$

- $\ln(x) = [t + \frac{y_0}{\alpha}e^{-\alpha t}] + C$
- $x = e^C e^t \exp(\frac{y_0}{\alpha}e^{-\alpha t})$
- $x(0) = x_0 \Rightarrow e^C = x_0 e^{-\frac{y_0}{\alpha}}$
- $\Rightarrow x(t) = x_0 e^t \exp(\frac{y_0}{\alpha}(e^{-\alpha t} - 1))$

But how do we deform the phase plane so this looks linear? We need some mapping $\vec{h}(x, y) = \langle u(x, y), v(x, y)\rangle$ that is continuous and invertible (so we don't "damage" the phase plane). This is called a **homeomorphism**.

- So near the equilibrium $(0, 0)$, the equations $y'(t) = -\alpha y(t), y(0) = y_0 \Rightarrow y(t) = y_0 e^{-\alpha t}$ linearized for $\delta x, \delta y$ must be similar to those for $u(x(t), y(t), v(x(t), y(t))$
- This means we need $\frac{du}{dt} = u$, $\frac{dv}{dt} = -\alpha v$
- After doing the substitution, we see that $v = v_0 e^{-\alpha t}$ exactly mimics $y(t) = y_0 e^{-\alpha t}$ for the phage solution. So we take $v = y$.
- Therefore, we know that since $u = u_0 e^t$ and $x(ty) = x_0 \exp(t + \frac{y_0}{\alpha}(e^{-\alpha t} - 1))$, that we need $u(x(t), y(t) = u(x_0, y_0)e^t$
- And this is satisfied if we guess $u(x, y) = xe^{-y}\alpha$ and work it out.
- This function $\vec{h}(x, y) = (u, v) = \langle xe^{-\frac{y}{\alpha}}, y\rangle$ is invertible by $(x, y) = \langle ue^{\frac{v}{\alpha}}, v\rangle$, which is continuous.

2.5 2.5: Application - Lasers

Lasers create excited atoms, which then emit photos while transitioning to an unexcited state. This system has a close analogue with the previous phages (like photons) and bacteria (like atoms) model.

- $n(t)$: number of photons in the laser; r_g: rate of photons gained (created by excited atoms transitioning to unexcited state); r_l: rate of photons lost (emitted)
- $\Rightarrow \frac{dn}{dt} = r_g - r_l$ by definition.
- We can assume we're losing a constant k (kill?) portion of photons per unit time, so $\frac{dn}{dt} = r_g - kn(t)$
- $e(t)$: number of excited atoms (that will maybe create photons). Atoms are excited by external energy pump.
- Excited atoms radiate when meeting a photon (which survives the meeting)
- So we can use the same setup from the bacteria: with I the constant of meeting (intersect?), $r_g = Ie(t)n(t) \Rightarrow n'(t) = Ie(t)n(t) - kn(t)$

Mini example: Assume no photons leave (cap the end of the laser)

- $k = 0$ in this scenario.
- So every meeting creates one more photon ($n \to n+1$) while enervating one excited atom ($e \to e-1$). This implies, equivalently:
  - $e + n$ is a conserved quantity,
  - $e(t) + n(t) = e(0) + n(0)$,
  - $[e(t) + n(t)]' = 0$
  - Then, if$k = 0, n'(t) = Ie(t)n(t) - kn(t)$, and coupled with $e'(t) + n'(t) = 0$ above, we have $e'(t) = -Ie(t)n(t)$

Extended example: Atoms spontaneously lose energy. This is actually what happens

- From quantum physics, we have a rate s of atoms just (s)pontaneously losing energty.
- We also have an energy (p)ump that energizes atoms with quantity p.
- Then, our change in (e)xcited atoms is $e'(t) = p - s - Ie(n)(t)$
- So our **final laser equations** are $e'(t) = p - s - Ie(n)(t), n'(t) = Ie(t)n(t) - kn(t)$
- If we want to find the smallest p guaranteeing $n \geq 1$ (there's at least one photo) at equilibrium ($e'(t) = n'(t) = 0$):
  - $n'(t) = 0 \Rightarrow Ien = kn \Rightarrow n(Ie - k) = 0$. If $n \neq 0, \Rightarrow e = \frac{k}{I}$
  - $e'(t) = 0 \Rightarrow Ien = p - se$
  - Together, $p - se = Ien = kn \Rightarrow kn + se = p \Rightarrow kn + s\frac{k}{I} = p$
  - $n \geq 1 \Rightarrow p \leq k + \frac{ks}{I}$
  - **Another tactic**: We could also assume we *start out at equilibrium*, so n_0, e_0 are constant solutions.
  - Solving $n' = 0 = Ie_0n_0 - kn_0, e' = 0 = Ie_0n_0 - se_0 + p$, we find equilibria $n_0 = \frac{p}{k} - \frac{s}{I}, e_0 = \frac{k}{I}$
  - Then, $n_0 \geq 1 \Rightarrow \frac{p}{k} - \frac{s}{I} \geq 1 \Rightarrow p \geq k + \frac{ks}{I}$

Non-dimensionalization time:

- Scale against $e_0(= \frac{k}{I}), n_0(= \frac{p}{k} - \frac{s}{I})$ like this: $x(t) = \frac{n(\alpha t)}{n_0}, y(t) = \frac{e(\alpha t)}{e_0}$

- NOTE: What does this do? This makes (1,1) the equilibrium, as $x(t) = \frac{n_0}{n_0} = 1, y(t) = \frac{e_0}{e_0} = 1$!
- What α lets us tke $n' = Ien - kn, e' = -Ien - se + p$ and write
  - $\frac{dx}{dt} = x(t)y(t) - x(t)$
  - $\frac{dy}{dt} = \frac{1}{k}(\frac{pI}{k} - s)[1 - x(t)y(t)] + \frac{s}{k}[1 - y(t)]$
  - $x' = \frac{\alpha n'(\alpha t)}{n_0} = xy - x = \frac{Ie(\alpha t)n(\alpha t)}{kn_0} - \frac{n(\alpha t)}{n_0}$
  - $\frac{\alpha Ien - \alpha kn(\alpha t)}{n_0} = \frac{Ie(\alpha t)n(\alpha t)}{kn_0} - \frac{n(\alpha t)}{n_0}$
  - $\alpha Ie - \alpha k = \frac{Ie(\alpha t)}{k} - 1 \Rightarrow \alpha(Ie - k) = \frac{Ie - k}{k} \Rightarrow \alpha = \frac{1}{k}$
  - This solves the x equation, and I suppose it can be validated in the y equation (tediously).
  - If we chunk up our (somehow positive?) constants as $c = \frac{1}{k}(\frac{pI}{k} - s), d = \frac{s}{k}$, we end up with $y' = c[1 - xy] + d[1 - y]$
  - We only care about $x, y > 0$, so $x' = 0 = xy - x = x(y - 1)$ implies $y = 1$ is a nullcline
  - $y' = 0 = c[1 - xy] + d[1 - y] = c - cxy + d - dy \Rightarrow c + d = y(d + cx) \Rightarrow y = \frac{c+d}{d+cx}$, a scaled and shifted hyperbola.

  Look at the solutions:

  - It appears we have a counterclockwise swirl around $(1, 1)$, and nearby solutions tend toward this equilibrium.
  - Hartman-Grobman: rewrite our linearized solution in neighborhood of $(1, 1)$ as $x(t) = 1 + \delta x(t), y(t) = 1 + \delta y(t)$
  - Using $x' = xy - x, y' = c[1 - xy] + d[1 - y]$ and $\frac{d}{dt}\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = A \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$, we can solve and write $A = \begin{pmatrix} 0 & 1 \\ -c & -c - d \end{pmatrix}$
  - Eigenvalues: $\lambda = \frac{1}{2}(-c - d \pm \sqrt{(c + d)^2 - 4c})$
    * If square root term is zero, we have repeated eigenvalue, so $\delta x(t), \delta y(t)$ are combos of $e^{-\frac{c+d}{2}}, te^{-\frac{c+d}{2}}$, which decays
    * If square root term is greater than zero, we have two distinct real, negative eigenvalues (since c, d are positive), so this decays.

* If square root term is less than zero, we have distinct complex eigenvalues, but combos of $e^{-\frac{c+d}{2}}\cos(\frac{1}{2}\sqrt{-(c+d)^2+4c}), e^{-\frac{c+d}{2}}\sin(\frac{1}{2}\sqrt{-(c+d)^2+4c})$ decay too
* Note : I suppose Hartman-Grobman can't work in purely imaginary eigenvalue scenario, because these kinds of functions don't converge or diverge without a term outside the sin or cos
* And in any case, since these lambdas aren't strictly imaginary, Hartman-Grobman works.

2.6 2.6: Liapunov Equations

We had some intuition that "nearby" solutions would fall into an equilibrium, but what does "nearby" mean? **Liapunov Equations** help us here. What is the "basin of attraction"?

- Suppose we turn the pump off ($p = 0$), and set spontaneous enervation equal to photon leak $s = k$.
- (TODO?) Somehow we can rescale to $\frac{dx}{dt} = Ie(t)n(t) - kn(t)$, $\frac{dy}{dt} = -Ie(t)n(t) - kn(t)$ which (TODO??) gives us $\frac{dx}{dt} = xy - x$, $\frac{dy}{dt} = xy - y$
- This means equilibria ($x' = y' = 0$) exist at $(0,0),(1,1)$
- If we're turning the pump off, we're looking at equilibrium (0,0). Linearizing, we get $x' = -\delta x, y' = -\delta y$, so a matrix of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
- With repeated non-imaginary (H-G applies!) eigenvalues $-1,-1$, we can see that both e^{-t}, te^{-t} decay, and we get sucked into the origin.

But how do we prove this? Let's find a conserved quantity $U'(x(t),y(t)) = 0$

- $U'(x(t),y(t)) = \frac{\delta U}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta U}{\delta y}\frac{\delta y}{\delta t} = \frac{\delta U}{\delta x}x(y-1) + \frac{\delta U}{\delta y}y(x-1)$
- Setting $\frac{\delta U}{\delta x}x = -x+1, \frac{\delta U}{\delta y}y = y-1$ makes this zero
- Solving those two by separating variables and combining, we get $U = -x+y+\ln(|\frac{x}{y}|)$
- So if we're stabiling $f = (x-y)$ (why?), we see $(x-y)' = x'-y' = (xy-x)-(xy-y) = x-y = f \Rightarrow f = e^{-t}$
- With $x(0) = x_0, y(0) = y_0 \Rightarrow f(0) = x_0 - y_0, f = x(t) - y(t) = (x_0 - y_0)e^{-t}$
- How to express $y(t)$ while eliminating $x(t)$, knowing $x(y) - y(t) = (x_0 - y_0)e^{-t}$ and $U(x,y) = y - x + \ln(|\frac{x}{y}|)$ is conserved? **The trick:** $U(x_0,y_0) = U(x,y)$ since *it doesn't change*!

- $y_0 - x_0 + \ln(|\frac{x}{y}|) = y - x + \ln(|\frac{x}{y}|) = -(x_0 - y_0)e^{-t} + \ln(|\frac{x}{y}|)$
- $(1 - e^{-t})(y_0 - x_0) = \ln(\frac{x/y}{x_0/y_0})$
- Defining for convenience, $f = \exp((1 - e^{-t})(y_0 - x_0))$, then $f\frac{y}{y_0} = \frac{x}{x_0}$
- Sub in to $x - y = (x_0 - y_0)e^{-t} : y[\frac{x_0}{y_0}f - 1] = (x_0 - y_0)e^{-t}$
- Solve for $y : y = \frac{y_0(x_0 - y_0)e^{-t}}{x_0 f(t) - y_0}$
- Combine with above to get $x = \frac{x_0(x_0 - y_0)e^{-t}f(t)}{x_0 f(t) - y_0}$

- So with equilibria $(0, 0), (1, 1)$, the direction field computer plot shows us attracted to $(0, 0)$ (no laser action) pretty much anywhere left and down from $(1, 1)$ in the x, y phase plane.
- Apparently the linearized solutions near $0, 0$ are $x_{lin} = x_0 e^{-t}, y_{lin} = y_0 e^{-t}$ (WHY?)
- Looking above, if $(x_0 - y_0) \approx 0$, then $f(t) \approx 1$, and $x, y \to x_{lin}, y_{lin}$

On to **Liapunov** functions, which will tell us perhaps the size of the "basin of convergence", unlike Hartman-Grobman, which just says there is a neighborhood.

A **Liapunov** function $U(x, y)$ is

- Continuously differentiable
- With a unique minimum (x_0, y_0), usually aligned to be U's only zero.
- $U'(x(t), y(t)) \leq 0$. Everything "flows downhill";
- Tailor made for the problem, hard to find.

Back to the rescaled laser example

- $x'(t) = x(t)y(t) - x(t)$
- $y'(t) = c[1 - x(t)y(t)] + d[1 - y(t)], c, d > 0$
  - **Analogy: The damped-block spring system** $\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{\gamma}{m} \end{pmatrix}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$
  - When $\gamma = 0$, we know $E(x, v) = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ is conserved when looking at E'
  - $\gamma = 0 \Rightarrow x' = v, v' = -\frac{k_s}{m}x$
  - $\frac{dE}{dt} = (\frac{1}{2}k_s x^2 + \frac{1}{2}mv^2)' = 0$ since $\frac{1}{2}(k_s xx' + mvv') = \frac{1}{2}(kxv + mv\frac{-k}{m}x) = 0$
  - But if $\gamma \neq 0$, $\frac{d}{dt}E(x(t), v(t)) = \frac{d}{dt}[\frac{1}{2}k_s x^2 + \frac{1}{2}mv^2] = kxx' + mvv'$

- $= kx(v) + mv(\frac{-k_s}{m}x - \frac{\gamma}{m}v) = -\gamma v(t)^2 = \frac{dE}{dt}$
- Total spring energy is then decreasing in the fluid.
- Brilliant has Cool visualization of spiraling down into the "bowl" of x, y with E as the z dimension, equilibrium $(0, 0, 0)$
- We need to choose a γ-fied $E-$like function that decreases for pairs $\delta x(t), \delta y(t)$. We can choose, like E, some $u(\delta x, \delta y) = \frac{1}{2}C_1[\delta x]^2 + C_2[\delta y]^2$.
- Choosing $C_1 = c, C_2 = 1$ gives us $\frac{d}{dt}u(\delta x(t), \delta y(t)) = \frac{d}{dt}(\frac{c}{2}[\delta x(t)]^2 + \frac{1}{2}[\delta y(t)]^2)$
- $= c(dx)(dx)' + dy(dy)' = c(dx)(dy) + dy(-c(dx) - (c+d)(dy)) = -(c+d)[\delta y(t)]^2$
- So $u = \frac{d}{dt}(\frac{c}{2}[\delta x(t)]^2 + \frac{1}{2}[\delta y(t)]^2)$ is an energy function that could work for the laser.

Finally, we want to construct a function that

- Doesn't increase (derivative negative) on any pairs $x, y > 0$ (pulls down)
- Is near equal to $u = \frac{c}{2}(x-1)^2 + \frac{1}{2}(y-1)^2$ near $(1, 1)$. (the energy function for block-spring above)
- With $x' - = xy - x, y' = c - cxy + d - dy$, plus the identity near $z \approx 1$ of $\ln(z) \approx (z-1) - \frac{1}{2}(z-1)^2 ...$
- You can find $U(x, y) = c(x-1) + (y-1) - c\ln(x) - \ln(y)$ that satisfies all of these
- It therefore shows that pumped laser solutions tend to equilibrium $(1, 1)$ in the long term.

TODO: So this is enough to establish a convergence to an equilibrium?

- Find an equilibrium (x_0, y_0)
- Find an energy function u that decreases for all pairs $(\delta x(t)\delta y(t))$ near the minimum.
- Find a Liapunov function U function that decreases EVERYWHERE along $x(t), y(t)$ (in our domain, like $x, y > 0$)
- Ensure that $U = u$ in the neighborhood of the equilibrium.
- Then Liapunov's theorem somehow makes this work (TODO)?

2.7 2.7: Dog chasing a duck (Limit Cycles)

This is a pair of nonlinear equations to determine if a dog in the pond's interior catches a duck who skates along the border.

- Variables:
  - r_p: Radius of pond.
  - $\vec{r_H}$: Distance of duck to center (always the radius of the pond)
  - $\vec{l}$: Displacement of dog from duck, which is of some length R at any point.
  - θ: Duck's position in the lake (think polar coordinates)
  - ϕ: Angle between $\vec{r_H}$ and $\vec{l}$.
  - Duck always swims at speed $r_p\theta'(t)$, and dog swims at $k > 0$ times this, or $kr_p\theta'(t)$.
- Therefore $\vec{r_H} = \langle r_p\cos(\theta), r_p\sin(\theta)\rangle$. It's just the polar coordinates.
- Doing some geometry gets you $\vec{l} = R\langle\cos(\theta+\phi, \sin(\theta+\phi)\rangle$
- We can establish $\vec{T} = \vec{r_H} - \vec{l}$ and dog's speed squared $\|T'(t)\|^2 = (\vec{r_H}' - \vec{l}') \cdot (\vec{r_H}' - \vec{l}') = \|\vec{r_H}'\|^2 + \|\vec{l}'\|^2 - 2\vec{r_H}'\vec{l}'$
- Naturally, this $\|T'\|^2$ is also equal to the constant $(kr_p\theta')^2$. Our diff equations will fall out of these.
- $\vec{r_H}^2 = r_p^2[\theta'(t)]^2$ since duck's speed is constant. $\vec{l} = (R')^2 + R^2[\theta' + \phi']^2$ after working it out.
- Finally, after using identities $\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi), \sin(\theta+\phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)$, we can work out $-2\vec{r_H}'\vec{l}' = -2r_p\theta'[R\cos(\phi)(\theta'+\phi') + \sin(\phi)R']$
- After rescaling R to ρ such that $\frac{R}{r_p} = \rho$ and diving our speed equation by constant $r_p\theta'$, we end up with speed equation $k^2 = [\rho(1 + \frac{d\phi}{d\theta} - \cos(\phi)]^2 + (\frac{d\rho}{d\theta} - \sin(\phi))^2$
- We propose that there are some solutions here for the **pursuit equations**. We'll ignore the generalized form an focus on one set
  - $\rho(1 + \frac{d\phi}{d\theta}) - \cos(\phi) = 0, \frac{d\rho}{d\theta} - \sin(\phi) = -k$ do work in the above. (Doesn't prove others don't work)
  - This leaves our equations as $\frac{d\phi}{d\theta} = \frac{\cos(\phi)}{\rho} - 1, \frac{d\rho}{d\theta} = -k + \sin(\phi)$
  - However, there *aren't simple equilibria here.* In no world with $k \neq 0$ does the dog sit still (or the duck).

- Supposing $k < 1$ and R, ϕ are fixed (dog never gets closer and just loops), this means he's going in a circle, since the two legs of a triangle $(\vec{l}, \vec{r_p})$ and the interior angle (ϕ) are fixed, so this fixes length of the third leg, which is a radius
- You can also use dog's position vectors $x(t) = r_p \cos(\theta) - R\cos(\theta + \phi), y(t) = r_p \sin(\theta) - R\sin(\theta + \phi)$ and trig identities to prove $x(t)^2 + y(t)^2 = r_p^2 + R^2 - 2r_p R\cos(\phi)$
- If $k < 1$, then solving $\frac{d\rho}{d\theta} = 0 = -k + \sin(\phi) \Rightarrow \sin(\phi) = k \Rightarrow \phi = \sin^{-1}(k)$ and $\rho = \cos(\phi) = \cos(\sin^{-1}(k)) = \sqrt{1-k^2}$
    * Quick proof of $\cos(\sin^{-1}(k)) = \sqrt{1-k^2}$:
    * $\cos^2(\sin^{-1}(k)) + \sin^2(\sin^{-1}(k)) = 1 \Rightarrow \cos(\sin^{-1}(k)) = 1 - \sin^2(\sin^{-1}(k))$
    * $= 1 - k^2 \Rightarrow \cos(\sin^{-1}(k)) = \sqrt{1-k^2}$
- When $k < 1$, the direction field seems to have attractive equilibriua but **GOTCHA**: there are ϕ angles that differ by 2π units, so they're the same. The direction field is a cylinder with circumference 2π, and there are other solutions tracking toward $(\sin^{-1}(k), \sqrt{1-k^2})$
- Linearizing, assume we are near our equilibirum point and $\phi = \sin^{-1} k + \delta\phi, \rho = \sqrt{1-k^2} + \delta\rho$.
- We can also remember that $f(x + \delta x) \approx f(x) + f'(x)\delta x$ from calculus.
- $\frac{d}{d\theta}[\delta\rho] = \frac{d}{d\theta}[\rho - \sqrt{1-k^2}] = \frac{d\rho}{d\theta} - \frac{d}{d\theta}\sqrt{1-k^2} = -k + \sin(\phi)$
- $= -k + \sin(\sin^{-1}(k) + \delta\phi)$ and by the calculus rule $\frac{d}{d\theta}[\delta\rho] = -k + \sin(\sin^{-1}(k)) + \cos(\sin^{-1}(k))\delta\phi = \sqrt{1-k^2}\delta\phi$
- And for $\frac{d}{d\theta}[\delta\phi] = \frac{d}{d\theta}\phi - \frac{d}{d\theta}(\sin^{-1}(k)) = \frac{\cos(\phi)}{\rho} - 1$
- Using multivariable hint $f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y$,
- $f = \frac{\cos(\sin^{-1}(k)+\delta\phi)}{\sqrt{1-k^2}+\delta\rho} - 1 \approx \frac{\sqrt{1-k^2}}{\sqrt{1-k^2}} - 1 + \frac{-\sin(\sin^{-1}(k)}{\sqrt{1-k^2}}\delta\phi - \frac{\cos(\sin^{-1}(k))}{1-k^2}\delta\rho$
- $= -\frac{k\delta\phi+\delta\rho}{\sqrt{1-k^2}}$
- So $\frac{d}{d\theta}\begin{pmatrix}\delta\phi \\ \delta\rho\end{pmatrix}\begin{pmatrix}-\frac{k}{\sqrt{1-k^2}} & -\frac{1}{\sqrt{1-k^2}} \\ \sqrt{1-k^2} & 0\end{pmatrix}\begin{pmatrix}\delta\phi \\ \delta\rho\end{pmatrix}$, and the eigenvalues aren't purely imeginary, and the real part is negative, so all decay. Therefore, the equilibrium at $(\sin^{-1}(k), \sqrt{1-k^2})$ attracts nearby solutions.

  - There aren't solutions (for $K < 1?$), but numerically solved, the dog catches at $k > 1$, and for $k \leq 1$, swims out to a path approaching a circle. This is a **limit cycle**, an isolated trajectory that closes on itself.

2.8 Poincare-Bendixson Theorem

Limit cycles in the real world: a chemical reaction in perpetual osciallation!

Key concept - **trapping region**: a region in phase plane on some region D, with differential solutions touching every point, where the direction field sees every boundary arrow point IN. This means:

- The solution has to stay in D.
- Any solution that self-intersects forms a cycle in the phase plane.
- The three conceivable ways a solution can "snake" around forever (the **Poincare-Bendixson theorem** says it):
  - Approaches a closed loop in D.
  - Approaches a fixed point in D (possibly a special case of the last bullet)
  - Cycle: Snake eats its own tail
- A non-cycling solution is the only other possibility - just a point equilibirum.

Example: Chemical oscillatory reaction.

- x is concentration of I^-, y is concentration of ClO_2^- ions in some reaction.
- a is positive, and clearly $x, y \geq 0$ in the physical world.
- Otherwise meaningless equations: $\frac{dx}{dt} = 5a - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(\frac{4y}{1+x^2})$
- Solve for equilibria by setting $\frac{dx}{dt} = \frac{dy}{dt} = 0$
  - Denote $Q = \frac{y}{1+x^2}$
  - First equation implies $x(1 + 4Q) = 5a$
  - Second euqation, plus knowing $x \neq 0$, $\Rightarrow x(1 - Q) = 0 \Rightarrow Q = 1$
  - $Q = 1 \Rightarrow 5x = 5a \Rightarrow x = a$
  - $\Rightarrow 1 = \frac{y}{1+x^2} \Rightarrow y = 1 + a^2$
  - Only solution pair is $(a, 1 + a^2)$

Linearizing the solution around $(a, 1 + a^2)$

- $x = a + \delta x, y = 1 + a^2 + \delta y \Rightarrow \frac{dx}{dt} = \frac{d[\delta x]}{dt}, \frac{dy}{dt} = \frac{d[\delta y]}{dt}$
- Call $f = \frac{d[\delta x]}{dt} = 5a - x - \frac{4xy}{1+x^2}$,
- Approximate $f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y$
- $f(x, y)(a, 1 + a^2) = 5a - x - \frac{4xy}{1+x^2}(a, 1 + a^2) = 5a - a - 4(a\frac{1+a^2}{1+a^2}) = 0$
- $\frac{\delta f}{\delta x}\delta x(a, 1+a^2) = (-1-\frac{(1+x^2)(4y-2x4xy)}{(1+x^2)^2})\delta x(a, 1+a^2)) = (-1-4-\frac{8a^2}{1+a^2})\delta x = \frac{-5+3a^2}{1+a^2}\delta x$
- $\frac{\delta f}{\delta y}\delta y(a, 1 + a^2) = \frac{-4x}{1+x^2}\delta y(a, 1 + a^2)) = \frac{-4a}{1+a^2}\delta y$
- Call $g = \frac{d[\delta y]}{dt} = x - \frac{xy}{1+x^2}$
- Approximate $g(x + \delta x, y + \delta y) \approx g(x, y) + \frac{\delta g}{\delta x}\delta x + \frac{\delta g}{\delta y}\delta y$
- $g(x, y)(a, 1 + a^2) = x - \frac{xy}{1+x^2}(a, 1 + a^2) = a - a\frac{1+a^2}{1+a^2} = 0$
- $\frac{\delta g}{\delta x}\delta x(a, 1 + a^2) = (1 - \frac{(1+x^2)y-xy2x}{(1+x^2)^2})\delta x(= 1 - \frac{(1+a^2)^2-2a^2(1+a^2)}{(1+a^2)^2})\delta x = 2a^2\delta x$
- $\frac{\delta g}{\delta y}\delta y(a, 1 + a^2) = \frac{-x}{1+x^2}\delta y = \frac{-a}{1+a^2}\delta y$
- $\Rightarrow \frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \frac{1}{1+a^2}\begin{pmatrix}3a^2-5 & -4a\\ 2a^2 & -a\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$
- Let's arbitrarily choose $a = 2 \Rightarrow (a, 1 + a^2) = (2, 5)$. The coefficient matrix ends up being $\frac{1}{5}\begin{pmatrix}7 & -8\\ 8 & -2\end{pmatrix}$, which has eigenvalues with a positive real $\pm$ some i component. So, Hartman-Grobman applies and we don't decay into our point but push away.

We want to **build the trapping region**.

- Remember, $\frac{dx}{dt} = 10 - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(1 - \frac{y}{1+x^2})$ subbing in 2 for a)
- On the left, if $x = 0$ we see $\frac{dx}{dt} = 10, \frac{dy}{dt} = 0$. So we're pointing right (into the first quadrant region)
- On the bottom, if $y = 0$, we're pointing at $\langle 10 - x, x\rangle$ (into the region).
- On the right, for some $x = b$, $10 - b - \frac{4b}{1+b^2}y$ will make sure we point left.
- On the top, for some $y = c$, $x(1 - \frac{c}{1+x^2} < 0$ makes sure we point down.
- Assume, since we're encircling $(2, 5)$, that $b \geq 3, c \geq 6$ for comfort.
- To satisfy all of these, note $x(1 - \frac{c}{1+x^2}) < 0 \Rightarrow 1 - \frac{c}{1+x^2} < 0 \Rightarrow c > 1 + x^2, 0 < x < b \Rightarrow c > 1 + b^2 \Rightarrow \sqrt{c-1} > b$

- And for $0 < y < c$, note that $10 - x - \frac{4xy}{1+x^2} < 10 - b < 0$.
- Pick $b = 11$, say, implying $11 < \sqrt{c-1}$, so then $123 < c$. So $(b, c) = (11, 124)$ ensures oscillation around $(2, 5)$ without leaving that region.

Tricky: How to reduce this region? No real way except simulation or some tricks. If we PRESUME a cycle, we can prove the cycle extens to the left of $x = 3$ or $x_{min} < 3$

- **META trick**: Don't worry if you have unsolvable integrals - maybe you can cancel them out. **Run with what you have.**
- Trick: Assume $x(t+T) = x(t), y(t+T) = y(t)$ for some $T > 0$, or that there's a PERIOD T.
- $\int_0^T \frac{dx}{dt} dt = x(T) - x(0) = 0, \int_0^T \frac{dy}{dt} dt = y(T) - y(0) = 0$ by fundamental theorem.
- Our equations again: $\frac{dx}{dt} = 10 - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(1 - \frac{y}{1+x^2})$
- So $0 = \int_0^T [10 - \int x(t) - 4 \int \frac{x(t)y(t)}{1+x(t)^2}]dt$ by the first equation
- $0 = \int x(t) - \int \frac{x(t)y(t)}{1+x(t)^2}]dt$ by the second.
- Subtract four times the second from the first to get $0 = 10T - 5\int_0^T x(t) \Rightarrow 2T = \int_0^T x(t)dt \geq int_0^T x_{min} dt = Tx_{min}$
- So $2 \geq x_{min}$

2.9 Chaos and the Lorenz Equation

What enabled mathematical **chaos** (unpredictability in nonlinear differential equations) was really computers and seeing simulated solutions.

The (simplified) **Lorenz system** are these equations

- $\frac{dx}{dt} = \sigma(y - x)$
- $\frac{dy}{dt} = x(\rho - z) - y$
- $\frac{dx}{dt} = xy - bz$
- All with $\sigma, \rho, b > 0$

Solving the equations, we see equilibria for these are:

- $(0, 0)$ always
- The two solutions $(\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1)$ when $\rho > 1$.

Looking at $0 < \rho < 1$ specifically:

- Linearaizing is simple, with $x(t) = \delta x(t), y(t) = \delta y(t), z(t) = \delta z(t)$ and linearized system:
- $\frac{d[\delta x]}{dt} = \sigma(\delta y - \delta x)$
- $\frac{d[\delta y]}{dt} = \rho\delta x - \delta y$
- $\frac{d[\delta z]}{dt} = -b\delta z$
- $\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix} \approx \begin{pmatrix}-\sigma & \sigma & 0\\ \rho & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix}$
- Characteristic equation is $(-b-\lambda)[(1+\lambda)(\sigma+\lambda) - \sigma\rho] = 0$
- Eigenvalues are $-b < 0$ and $\lambda = \frac{1}{2}[-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}]$
- If $\rho < 1$, we have distinct, real, negative eigenvalues, and a locally attractive equilibrium by Hartman-Grobman.

But if $\rho < 1$ globally attactive? Find a Liapunov function.

- Requirement is that the function $U(x(t), y(t), z(t))$ is minimized at the equilibrium, and that as time progresses, U decreases (so we're sucked into the bowl)
- We suppose that $U(x, y, z) = ax^2 + y^2 + z^2$ and using $xy \le \frac{1}{2}x^2 + \frac{1}{2}y^2$:
  - Identity derivation: $0 \le (x-y)^2 \Rightarrow 0 \le x^2 - 2xy + y^2 \Rightarrow xy \le \frac{1}{2}(x^2 + y^2)$
- $\frac{\delta U}{\delta x}x'(t) + \frac{\delta U}{\delta y}y'(t) + \frac{\delta U}{\delta z}z'(t) = 2ax\sigma(y-x) + 2yx(\rho - z) - 2y^2x + 2zxy - 2bz^2$
- $= 2(a\sigma + \rho)xy - 2a\sigma x^2 - 2y^2 - 2bz^2$
- **GOTHCA**: We can't choose $a = -\frac{\rho}{\sigma}$ since then $U = -\frac{\rho}{\sigma}x^2 + y^2 + z^2$ isn't minimized at $(0, 0, 0)$! So a needs to be positive.
- Choosing $a = \frac{1}{\sigma} \Rightarrow a\sigma = 1$, with $\rho < 1 \Rightarrow 2(a\sigma + \rho)xy - 2a\sigma x^2 - 2y^2 - 2bz^2 < 2a\sigma(2xy - x^2 - y^2) - 2bz^2 \le -2bz^2$ by the identity above.
- Then $U = \frac{1}{\sigma}x^2 + y^2 + z^2$ decreases as $t \to \infty$ and is minimized at the globally attractive $(0, 0, 0)$

If $\rho > 1$ things get chaotic. Instead of one equilibrium, we have two new ones at $(\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1)$. Everything **bifurcates**, or qualitatively shifts when inching past $\rho = 1$:

- We have three equilibria.
- The origin turns into a saddle equilibrium.

- Linearizing around $(\alpha, \alpha, \rho-1)$ with α denoting $\sqrt{b(\rho-1)}$ (pretty straightforward), we get characteristic equation for A of $-\lambda^3-(\sigma+b+1)\lambda^2-b(\sigma+\rho)\lambda-2\sigma b(\rho-1)=0$
- Problem is, setting $\rho=1$ drops the $(\rho-1)$ term and we have $-\lambda(\lambda^2+(\sigma+1+b)\lambda+b(\sigma+1))=0$, with solutions $\lambda=0,-b,-\sigma-1$.
- The last two solutions are attractive, but the zero doesn't work for Hartman-Grobman.
- If we set $\lambda=(\rho-1)\Delta r$ when nudging ρ just over 1, we ignore all $\lambda^2, \lambda^3 ...$ as negligible and get $-b(\sigma+\rho)(\rho-1)\Delta r-2\rho b(\rho-1) \approx 0$
- This means $\Delta r \approx -\frac{2\rho}{\rho+\sigma}$, or that this nudged root has to be negative when ρ is near 1.
- More rigorously, we could have proven the roots of the equation are negative for small $\rho-1>0$
- In any case, this means that the near-zero root is negative, so $(\alpha, \alpha, \rho-1)$ attracts locally.
- We can show that this applies the same for $(-\alpha,-\alpha, \rho-1)$

How do equilibria change as we change ρ?

- We saw the What about as we dial past $\rho=1$, our origin equilibrium changes from globally attractive to saddle point.
- In going from a stable equilibrium with negative real-part eigenvalues (attractors) to $(0,0,0)$ as a saddle (mix of negative and positive real parts), we necessarily have a point where the eigenvalues' real parts are zero.
- In other words, $\lambda=ia$ for some real a.
- Subbing ia into our $-\lambda^3-(\sigma+b+1)\lambda^2 b(\sigma+\rho)\lambda-2\sigma b(\rho-1)=0$, we end up getting $[(\sigma+b+1)a^2-2\sigma b(\rho-1)]+i[a^3-(b(\sigma+\rho)a]=0$
- Then we need $a^3-b(\sigma+\rho)a=0 \Rightarrow a=0, a=\pm, \sqrt{b(\sigma+\rho)}$
- If $a=0$. the real part isn't zero. But subbing $a=\pm\sqrt{b(\sigma+\rho)}$ gives us solutions for a set of $\rho=\frac{\sigma(\sigma+b+3)}{\sigma-b-1}$
- So, when moving past this value, our two new equilibria change from locally attractive to saddles too.

Can we create a trapping region?

- The hint: The solutions have to pass through every ellipsoid of form $R^2=\rho x^2+]sigmay^2+\sigma(z-2p)^2$

- What we need to prove: At every point on the boundary, the direction field points "in", or more specificlally, *the angle between inward normal and direction field is acute.*
- This also means that the gradient ∇g of the level set $R^2 = \rho x^2 + \sigma y^2 + \sigma(z - 2r\rho)^2$ is the normal. This is $\langle 2\rho, 2\sigma, 2(z - 2\rho)\rangle$
- So $-\nabla g \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle > 0 \Rightarrow ... \Rightarrow 2\rho\sigma x^2 + 2\sigma y^2 + 2\sigma z^2 - 4\rho\sigma z > 0$
- Use $R^2 - \rho x^2 - \sigma(z - 2p)^2 = \sigma y^2 \Rightarrow \rho\sigma x^2 + 2\sigma y^2 + 2\sigma z^2 - 4\rho\sigma z > 0$
- This simplifies the dot product to $2R^2 - 8\rho^2\sigma + 4\sigma\rho z + 2\rho(\sigma - 1)x^2 > 0$
- Since the x^2 term is always positive, we just need to set R to clear zero when z is its most negative (x = 0, y = 0, z = $2\rho - \frac{R}{\sqrt{\sigma}}$). If we churn a little more we can see that setting $R > 2\sqrt{\sigma}\rho$ will provide a trapping region.

Question: Do the confined solutions fill up the whole (ellipsoid) container?

- Looking at the divergence (volume change of a cube over time) of the solution will tell us.
- For unspecified reasons, $\frac{1}{v(t)}v'(t) = \nabla \cdot \langle \sigma(y - x), x(\rho - z) - y, xy - bz\rangle = -\sigma - 1 - b$
- Solving, $v(t) = v(0)e^{-t(\sigma+1+b)}$
- This means the volume decays to 0, so therefore, our line is confined to a smaller and smaller space (but not just a point, I guess?)

3 Partial Differential Equations

3.1 1D Wave Equation and D'Lambert's Formula

General set up: A rope with a fixed right end (boundary condition and a moving left end), moving up and down.

Start out with special case: no boundary condition (infinite rope, pulse in the middle)

- $u(x, t)$ measures the vertical displacement form the x-axis of the rope at point x, time t
- Physical observation gives us the PDE rule $\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 u}{\delta t^2}$ (or $u_{xx} = u_{tt}$)
- $g(x) = u(x, 0)$ is the initial shape of the rope.
- It's assumed that the rope is not moving initally, so $u_t(x, 0) = 0$

Beginning to solve this:

- $u_{tt} = u_{xx} \Rightarrow u_{tt} - u_{xx} = 0$
- Sort of like $a^2 - b^2 = 0 \Rightarrow (a+b)(a-b) = 0$, we have $0 = (\frac{\delta}{\delta t} \pm \frac{\delta}{\delta x})(u_t \mp u_x) = u_{tt} - u_{xt} + u_{tx} - u_{xx} = u_{tt} - u_{xx}$
- This means the solution is either $u_+ = u_t + u_x$ or $u_- = u_t - u_x$. Note - we don't solve these simultaneously, since that just gives us $u(x,t) = 0$.
- These can be written as, e.g. $0 = u_t + u_x = \langle 1, 1\rangle \cdot \langle u_x, u_t\rangle = \langle 1, 1\rangle \cdot \nabla u$
- TRICK: This is a directional derviative along $\langle 1, 1\rangle$. Introducing a variable like s (accelerant along $\langle 1, 1\rangle$?) below does nothing interesting:
  - $\frac{d}{ds}[u(x+sb, t+sc)] = \frac{\delta u}{\delta x}(x+sb, t+sc)b + \frac{\delta u}{\delta t}(x+sb, t+sc)c = \langle b, c\rangle \cdot \nabla u(x+sb, t+sc)$
  - So if we set $b = c = 1$, we see that $\frac{d}{ds}[u(x+s, t+s)] = \langle 1, 1\rangle \cdot \nabla u(x+s, t+s)$
  - However, since in our world, $u_x + u_t = 0$, then this dot product is zero, and $\frac{d}{ds}u = 0$. This is then *constant in s.*
  - So then $u(x,t) = u(x+s, t+s)$, and shifting x forward by s (seconds?) and t by the same changes nothing. *Interpretation: $u(x,t) = u(x+s, t+s)$ means that s ("shift") seconds later, the point $x+s$ will see the same displacement as x. The wave goes "right" down the line.*
  - From this, we see that $u_+(x,t) = u_+(x-t, 0)$ as well. So, our function at t is what happened t seconds ago at the origin.
- Note: We can't have one solution satisfy both conditions $u_+ = g(x), (u_+)_t = 0$, since then $g'(x) = 0$ which only works if g is a constant.
- Also, $u_{tt} - u_{xx} = 0$ is a linear PDE, in that solutons $u_1(x,t), u_2(x,t)$ see that $\frac{\delta^2}{\delta t^2}[c_1u_1(x,t) + c_2u_2(x,t)] - \frac{\delta^2}{\delta x^2}[c_1u_1(x,t) + c_2u_2(x,t)] = 0$. Multiply by a constant or add solutions together and it's still zero.
- If we set $t = 0$, we get $u(x,t) = c_+g(x+t) + c_-g(x-t) \Rightarrow u(x,0) = c_+g(x+0) + c_-(x-0) = (c_+ + c_-)g(x) = u(x,0)$,so $(c_+ + c_-) = 1$
- Differentiating by t, $u_t(x,t) = c_+g'(x+t) - c_-g'(x-t)$ so $u_t(x,0) = (c_+ - c_-)g'(x) \Rightarrow (c_+ - c_-) = 0$. So $c_+ = c_- = \frac{1}{2}$, and **our solution with initial shape $g(x)$ with** $g'(x) = u_t(x,0) = 0$ is $u = \frac{1}{2}g(x+t) - \frac{1}{2}g(x-t)$
- **This no-initial-velocity wave function translates** into "my displacement at time 3, say, is the average of the initial displacements 3 to my left and 3 to my right" (as those urges meet at "me" 3 seconds from the start). Conceptually, along the fixed initial curve $g(x)$, each point sends out two sensors, one left, one right, and averages

the initial values at those points to find itself at time t. So the top of a hill will start dipping down, becoming two hills pushing out, for example.

With inverted conditions $u(x,0) = 0, u_t(x,0) = f(x)$, we can use the fact that $u(x,t)$ solving the wave equation implies u_t solves it as well!

- $\frac{\delta^2 u}{\delta t^2} - \frac{\delta^2 u}{\delta x^2} = 0 \Rightarrow \frac{\delta}{\delta t}[\frac{\delta^2 u}{\delta t^2} - \frac{\delta^2 u}{\delta x^2}] = 0 \Rightarrow \frac{\delta^2 u_t}{\delta t^2} - \frac{\delta^2 u_t}{\delta x^2} = 0$.
- Therefore, $u_t(x,0) = f(x)$ admits the same solution $u_t(x,t) = \frac{1}{2}[f(x+t) - f(x-t)]$
- Since $u(x,t) - u(x,0) = \int \frac{1}{2}[f(x+t) - f(x-t)]dt$, and $u(x,0) = 0$ by assumption in this setup, $u(x,t) = \int_{s=0}^{s=t}[f(x+s) - f(x-s)]ds$, which is $\frac{1}{2}\int f(s)$ from $x-t$ to $x+t$ $= \frac{1}{2}\int_{s=x-t}^{s=x+t} f(s)ds$

And because the region of the intergral for a point x gets wider as $t \to \infty$, on a flat rope with a pulse in the middle at $x = 0$, we see $u(x,t)$ sitting at 0 until the wave meets it, at which point it rises and then stays at the peak (integral of the whole thing).

So **d'Alembert's formula** is the superposition of the initially flat wave with the initially still wave, which accomodates *all* solutions:

$$\boxed{u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{s=x-t}^{s=x+t} f(s)ds}$$

For the case of no boundary conditions, this solves $u_{tt} = u_{xx}$ for $u(x,0) = g(x), u_t(x,0) = f(t)$. In this instance, the propagation speed is clearly finite.

Note: This complete of a PDE solution is unusual.

3.2 Sources and Boundary conditions

Scenario 1: Here, we fix the infinite rope at the origin, with the wave coming in from the negative x-axis.

Looking at **boundary conditions**, or constraints on spatial edges of a PDE problem:

- A free boundary (a loop that can shift up and down a pole) will cause a reflected wave to travel backwards.
- A fixed boundary (setting $u(0,t) = 0, t \geq 0$) will cause an inverted pulse backwards.

We set up a function $\tilde{u}(x,t) = \{u(x,t), x \leq 0; = -u(-x,t), x \geq 0\}$ using **extension by odd reflection**. So an inverted ghost rope exists to the right of the origin.

Note: This seems to be more about cleverly encoding a boundary behavior (we will invert our wave) with this ghost rope than proving we'll have that behavior with math.

- And if $u_t(x,0) = 0, g(0) = 0, u(x,0) = g(x)$ extended to $x > 0$ as $\tilde{g}(x)$, then d'Alembert's applies: $\tilde{u}(x,t) = \frac{1}{2}[\tilde{g}(x+t) + \tilde{g}(x-t)]$

- So when $x \leq 0, x \leq t \Rightarrow -t \leq x \leq 0$: (meaning, negative x, close enough to the origin to be affected by time t)
  - $\tilde{u}(x,t) = u(x,t)$ here, since there's no inversion on the left side.
  - $(x+t)$ is positive , so $\tilde{g}(x+t) = -g(-(x+t))$ by definition of $\tilde{u}$.
  - $(x-t)$ is negative , so $\tilde{g}(x-t) = g(x-t)$ by definition of $\tilde{u}$.
  - So d'Alembert's reduces to $u(x,t) = \frac{1}{2}[-g(-(x+t)) + g(x-t)]$. This means *I'm the average of the starting position to my left t seconds ago, and the inverted right-of-origin ghost position to my right t seconds ago*

This means that for the part of the rope we care about, $x \leq 0$:

- For $x \leq -t$ (parts of the line unaffected by the reflection so far), $u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)]$
- For $x \geq -t$ (parts affected by the reflection) $u(x,t) = \frac{1}{2}[-g(-(x+t)) + g(x-t)]$
- The intuition, still hard to visualize: if I'm zero at point -10, and wave crests at -11, then
  - First my left sensor will eat the left wave and I'll go up and over.
  - Then much later my right sensor will eat the right shadow wave and I'll do the inverted behavior.
  - These in total mean I'll get a reflection.
  - For the intuition, keep moving my point closer to the origin - nothing changes.

Scenario 2: Here, we let the rope slide up and down at the origin, but bound the total energy:

- One hand on the rope at $x = -L$, very far away:
- Our energy is the sum of kinetic (change in u based on time?) and elastic (change in u based on x?) energies.
- $E = \int_{x=-L}^{x=0} [(\frac{\delta u}{\delta t})^2 + (\frac{\delta u}{\delta x})^2] dx$
- We can't gain or lose energy. This means $\frac{dE}{dt} = 0$. Solving that:
  - $0 = \frac{d}{dt}(\frac{1}{2}\int_{x=-L}^{x=0} [(\frac{\delta u}{\delta t})^2 + (\frac{\delta u}{\delta x})^2] dx) = \frac{1}{2}\int_{x=-L}^{x=0} [\frac{\delta}{\delta t}(\frac{\delta u}{\delta t})^2 + \frac{\delta}{\delta t}(\frac{\delta u}{\delta x})^2] dx)$
  - $= \int_{x=-L}^{x=0} [u_t u_{tt} + u_x \frac{\delta u_t}{\delta x}] dx)$.
  - (Do integration by parts on the second term with $U = u_x, dV = \frac{\delta u_t}{\delta x}$): $0 = \int_{x=-L}^{x=0} [u_t u_{tt}] dx + u_x u_t - \int_{x=-L}^{x=0} [u_t u_{xx}] dx = \int_{x=-L}^{x=0} u_t [u_{tt} - u_{xx}] dx + u_x u_t$

- Since $u_{tt} - u_{xx} = 0$ (REMEMBER YOUR PROBLEM-SPECIFIC IDENTITIES!), $u_x(0,t)u_t(0,t) = 0$
- Saying the displacement can't change with respect to t there gives us the fixed rope case above, so that's uninteresting.
- Therefore, if there's no energy change as the rope vibrates, we know $u_x(0,t) = 0$

Note: Dirichlet conditions are constraints on the value of the function at the boundary (like $u(0,t) = 0$). Neumann constraints are on the derivatives at the boundary.

So redoing d'Alembert with the energy conservation, and therefore the "Neumann" condition $u_x(0,t) = 0$:

- We know if u solves $u_{tt} - u_{xx} = 0$, then u_x does too, since $0 = u_{tt} - u_{xx} \implies 0 = \frac{d}{dx}[u_{tt} - u_{xx}] = [[u_x]_{tt} - [u_x]_{xx}] = 0$.
- We know $u_x(0,t) = 0$ by given constaints, so then we enforce this through odd reflection on u_x as well: $\tilde{u}_x = \{u_x(x,t), x \leq 0; -u_x(-x,t), x \geq 0\}$
- By D'Alembert, this solves the wave equation with $u_x(x,0) = g'(x)$, so $\tilde{u}_x = \frac{1}{2}[\tilde{g}'(x+t) + \tilde{g}'(x-t)]$
- Therefore at $-t \leq x \leq 0 =, \tilde{u}_x(x,t) = \frac{1}{2}[\tilde{g}'(x+t) + \tilde{g}'(x-t)] = \frac{1}{2}[-g'(-x-t) + g'(x-t)]$
- Then integrating, we drop the minus sign in the first term! $u(x,t) = \frac{1}{2}[g(-x-t) + g(x-t)] + C$. Note that $u(x,0) = \frac{1}{2}[g(x) + g'(x))] \Rightarrow C = 0$!

(Note: a nonzero initial velocity profile $u_t(x,0) = f(x)$ can be handled as well. We skip it).

Remember the 1D springs Wave Equation, where springs are initially l apart, have displacement from this measured by $u(x,t)$, have Hooke's coefficient k?

- Force pushing from the left on ball x: $F_L = k[u(x-l,t) - u(x,t)]$
- Force pushing from the right on ball x: $F_R = k[u(x+l,t) - u(x,t)]$
- Additional "source" force $F(x,t)$ means total force $F_{tot} = F_L(x,t) + F_R(x,t) + F(x,t)$
- $F_{tot} = ma = mu_{tt}$
- The Taylor-ish formula $f(x+\delta x) \approx f(x) + f'(x)(\delta x) + f''(x)(\delta x)^2$ means $F_L + F_R \approx kl^2 f''(x) = kl^2 u_{xx}$
- Therefore, $mu_{tt} = kl^2 u_{xx} + F(x,t) \Rightarrow F(x,t) = u_{tt} - \frac{kl^2}{m}u_{xx}$. Set $1 = v = \frac{kl^2}{m}$, $f(x,t) = \frac{F(x,t)}{m}$ to get a simplified all-purpose wave equation. $f(x,t) = u_{tt} - u_{xx}$, with f as the source force-per-unit-mass.

New setup: Source force $f(x,t)$, ignore boundary conditions, and set $u(x,0) = 0, u_t(x,0) = 0$ (still, flat (infinite) rope).

- Part 1: We can relate $f(x,t)$ to a made-up intermediate function $I(x,t)$ which has properties motivated by $u_{tt} - u_{xx} = (\frac{\delta}{\delta t} - \frac{\delta}{\delta x})(\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$
- $I(x,t) = (\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$
- $u_{tt} - u_{xx} = f(x,t) \Rightarrow (\frac{\delta I}{\delta t} - \frac{\delta I}{\delta x}) = f(x,t)$
- We can derive that $u(x,0) = 0, u_t(x,0) = 0$ means that at $x = 0$, $I(x,0) = u_t(x,0) + u_x(x,0) = u_x(x,0)$
- Since $u(x,0) = 0$ and $I(x,0) = u_x(x,0)$, $I(x,0) = 0$.

We can relate $f(x,t)$ and $I(x,t)$:

- Use the dummy variable trick, and look at $f(x-s, t+s)$. We know also that $\frac{\delta I}{\delta t} - \frac{\delta I}{\delta x} = f(x,t)$
- $f(x-s,t+s) = \frac{\delta I}{\delta t}(x-s,t+s) - \frac{\delta I}{\delta x}(x-s,t+s) = \frac{d}{ds}[I(x-s,t+s)]$ by chain rule.
- Integrating both sides: $\int_{s=-t}^{s=0} f(x-s,t+s)ds = I(x,t) - I(x+t,0) = I(x,t)$
- We can rewrite, using $k = -s$, as $I(x,t) = \int_{k=t}^{k=0} f(x+k,t-k)d(-k) = \int_{s=0}^{t} f(x+s,t-s)ds$

Using the same technique, we can relate $I(x,t)$ and $u(x,t)$ since $I(x,t) = (\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$

- Use the dummy variable trick with variable s' and look at $f(x-s',t-s')$. We know also that $\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x} = I(x,t)$
- $I(x-s',t-s') = \frac{\delta u}{\delta t}(x-s,t-s) - \frac{\delta u}{\delta x}(x-s',t-s') = \frac{d}{ds'}[u(x-s',t-s)']$ by chain rule.
- Integrating both sides: $\int_{s'=-t}^{s'=0} f(x-s',t-s')ds = u(x,t) - I(x-t,0) = u(x,t)$
- We can rewrite, using $j = -s'$, as $u(x,t) = \int_{j=t}^{j=0} f(x-j,t-j)d(-j) = \int_{j=0}^{t} I(x-j,t-j)dj$
- So, $u(x,t) = \int_{s'=t}^{s'=0} f(x-s',t-s')ds' = \int_{s'=0}^{t} I(x-s',t-s')ds'$

Combining these, $u(x,t) = \int_{s'=0}^{t} I(x-s',t-s')ds'$, and $I(x,t) = \int_{s=0}^{t} f(x+s,t-s)ds$:

- $I(x-s',t-s') = \int_{s=0}^{t} f(x+s-s',t-s-s')ds$:
- So $u(x,t) = \int_{s'=0}^{t} \int_{s=0}^{t} f(x+s-s',t-s-s')dsds'$

- Change of variables, $\boxed{y = x + s - s', w = s' + s \Rightarrow u(x,t) = \frac{1}{2}\int_{w=0}^{t}\int_{y=x-w}^{y=x+w} f(y, t-w)dydw}$
  - TODO: The $\frac{1}{2}$ term apparently comes from the Jacobian (TODO) $\|\frac{\delta(s',s)}{\delta(w,y)}\|$
- This together means that the *points that can influence* $u(x,t)$ in the $xt-$plane are a triangle with (x,t) as the top, reaching down to $t = 0$, slope 1. So the "wave speed" in this setup is 1.

3.3 2D and 3D (Compression) Waves

(Note: The 2D equation will fall out of the 3D one).

Major setup for 3D compression waves:

- Air molecules compress together from sound, so $u(x,y,z,y)$ measures the density of air at that point.
- Let's assume $g(x \pm t)$ plays the same role as last time: the initial wave state. (Note: The setup implies we're looking at waves that propagate at "one x per t".)
- Sound can come from multiple directions, so the expanded version should look like $u(\vec{x},t) = g(\hat{n} \cdot \vec{x} \pm t)$, with $\hat{n}$ some fixed direction in $\mathbb{R}^3$.
- The equation is $\boxed{u_{tt} = u_{xx} + u_{yy} + u_{zz}}$ since:
  - Setting $\hat{n} = \hat{i}$ or any other basis vector, $g(\hat{i} \cdot \vec{x} \pm) = g(x \pm t)$, which means if the other dims are zero, then $u_{tt} = u_{xx}$ (reduce to 1D case). That checks out (necessary, not sufficient)
  - $\frac{\delta}{\delta t}[g(\hat{n} \cdot \vec{x} \pm t)] = \pm g'(\hat{n} \cdot \vec{x} \pm t)$, same for g'' and $\frac{\delta^2}{\delta t^2}$
  - $\frac{\delta^2}{\delta x^2}[g(\hat{n} \cdot \vec{x} \pm t)] = \hat{n}_x^2 g''(\hat{n} \cdot \vec{x} \pm t)$, same for y, z
  - Since $[\hat{n}_x^2 + hatn_y^2 + hatn_z^2 = 1], u_{tt} = u_{xx} + u_{yy} + u_{zz}$ works out.
  - This can also be written $u_{tt} - \nabla^2 u$

Setup with a forcing function:

- $u_{tt} - \nabla^2 u = f(\vec{x},t), u(\vec{x},0) = 0, u_t(\vec{x},0) = 0$.
- So with a still, blank initial state f is going to be a POP at the origin for a brief time.
- Taking our experience from actual sound, we expect it to decrease away from the origin, and for there to be a finite propagation speed.
- It should also be spherically symmetric.

Switching to spherical coordinates (u depends on r, θ, ϕ), and using the multivariable chain rule from vecctor calculus, we get

- $\nabla^2 u = \frac{1}{r^2}\frac{\delta}{\delta r}[r^2 \frac{\delta u}{\delta r}] + \frac{1}{r^2 \sin^2(\phi)}\frac{\delta^2 u}{\delta \theta^2} + \frac{1}{r^2 \sin(\phi)}\frac{\delta}{\delta \phi}[\sin(\phi)\frac{\delta u}{\delta \phi}]$
- If we're taking this to be spherically symmetric, then we can zero out ϕ, θ terms: $\nabla^2 u = \frac{1}{r^2}\frac{\delta}{\delta r}[r^2 \frac{\delta u}{\delta r}]$
- Expanding this out, this means that $u_{tt} - \frac{1}{r^2}\frac{\delta}{\delta r}[r^2 \frac{\delta u}{\delta r}] = u_{tt} - \frac{2}{r}u_r - u_{rr} = f(r,t)$
- If we set $U = ru$, and churn through with e.g. $u_r = \frac{\delta}{\delta r}[\frac{U}{r}] = -\frac{U}{r^2} + \frac{U_r}{r}$, etc., we end up with $U_{tt} - U_{rr} = rf(r,t)$. Note that $\frac{U}{r} = u$ means that the solution diminishes u with distance.
- So now we're solving with $U(r,0) = U_t(r,0) = 0$ since $U = ru$.
- Though only $r \geq 0$ matters, we need to keep $U(0,t)$ at zero through odd reflection. Note that if f is even, $rf(r,t)$ is odd.
- The result from last quiz implies: $U(x,t) = \frac{1}{2}\int_{s=0}^{t}\int_{\rho=r-s}^{\rho=r+s} \rho f(\rho, t-s) d\rho ds$, with s subbing for dummy w and ρ being the distance instead of y. Note that our function is really ρf now instead of f.
- Building a "Dirac delta snap" for a symmetric pop at the origin, set $f(r,t) = \frac{1}{\epsilon}e^{-\pi^2 \frac{r^2}{\epsilon^2}}\chi(t)$ for tiny ϵ
- Define $\delta(\rho) = \frac{\exp(-\frac{\pi^2 \rho^2}{\epsilon^2})}{\epsilon^2}$, change $s' = t - s$
- Eventually the math reduces to a delta pop at $r - t + s'$ (in range) and $r + t - s'$ (outside the interval)
- The math reduces to $\int_{s'=0}^{s'=t} \chi(s')\delta(r - t + s')ds'$, or just $\chi(t - r)$ (due to the integral of δ being one exactly at $t - r$.
- Therefore, $\boxed{U(r,t) = \{\frac{\epsilon^3}{(2\pi)^2}\chi(t-r), t - r > 0; 0, t - r \leq 0\}}$
- Looking at this, we confirm that disturbance diminishes with distance, and has a finite propagation speed.
- *TODO: So I guess χ is the actual initial wave function of time at the origin?* The delta was I suppose there to "center" it?

What if we don't have spherical symmetry?

- in the general case, all points $\vec{x}$ influence fixed point $\vec{P}$ through
  - Distance separating points $r = \|\vec{x} - \vec{P}\|$

- Normalized direction $\frac{\vec{x}-\vec{P}}{\|\vec{x}-\vec{P}\|}$

- However, we can *average* u over all points r away: $U(r,t;P) = \frac{1}{4\pi r^2}\iint_{S(\vec{P},r} u(\vec{x},t)d\sigma(\vec{x},$ with S being the r-sphere around P
- Getting the r-partials requires writing each $\vec{x}$ as some $\vec{P}+r\hat{n}$ over all directions, and using the divergence theorem.
- LOTS OF ALGEBGRA IN HERE to get $U_{tt}-U_{rr}-\frac{2}{r}U_r = F(r,t)$ with a dependence on $r,t.\vec{P}$. This F is an even function, and it approaches $u(x,t)$ as r approaches zero.
- Our equation ens up being $u(\vec{P},t) = \frac{1}{4\pi}\iiint_{B(\vec{P},t)} \frac{f(\vec{y},t-\|\vec{y}-\vec{P}\|)}{\|\vec{y}-\vec{P}\|}d\vec{y}$
- Like the other case, we see that we points in space affecting U are a "4D cone" with vertex at $(\vec{P},t)$
- Also, if we consider that f doesn't depend on z, we can flatten this spherical integral to a 2D one by looking at columns of z over the disc of radius t
- This ends up being $u(\vec{P},t) = \frac{1}{2\pi}\iint_{B_2(\vec{P},t)} f(\vec{y})\ln(\sqrt{(\frac{t}{\|\vec{y}-\vec{P}\|})^2-1}+\frac{t}{\|\vec{y}-\vec{P}\|})d\vec{y}$
- This **method of descent** is really just "reducing" our 2D case from a 3D one.
- Also, the **spherical averages** let us reduce a 3D problem to a 1D problem, given the assumptions of the problem.

3.4 2D waves (boundary constrained): Separation of Variables

Main idea: "Guess" that a function like $u(x,t)$ can be factored into $u(x,t) = X(x)T(t)$ and work from there. You can do this recursively as well like $u(x,y,t) = S(x,y)T(t), S(x,y) = X(x)Y(y)$.

Main Setup:

- Rectangular drumhead from $[0,0]$ to $[w,l]$
- Vertical (z) displacement is $u(x,y,t)$, with Dirichlet condition $u(x,y,t) = 0$ enforced on the boundary.
- Known that $u_{tt} = u_{xx} + u_{yy}$.

Solving for u by guessing that there's a split solution $u(x,y,t) = S(x,y)T(t)$.

- $u_{tt} = u_{xx} + u_{yy}$.
- So $S(x,y)T''(t) = \frac{d^2S(x,y)}{dx^2}T(t) + \frac{d^2S(x,y)}{dy^2}T(t)$
- $\Rightarrow \frac{T''(t)}{T(t)} = \frac{\nabla^2 S(x,y)}{S(x,y)}$

- **Big A-ha**: left hand side is a function of t, and right hand of x,y. If they are to be equal, they must both be consta

To solve for T:

- We know $\frac{T''(t)}{T(t)}$ is a constant, so equate it to $-k^2$ for some constant k.
- Supposing the solution $T(t) = e^{rt} \Rightarrow T'' = r^2 e^{rt}$, we know $r^2 = -k^2 \Rightarrow r = \pm ik$.
- The solution $T(t) = Me^{ikt} + Ne^{-ikt}$ is equivalent to $T(t) = A\cos(kt) + B\sin(kt)$ since you can express $\sin, \cos$ as linear combos of $e^{ikt} = \cos(kt) + i\sin(kt)$ and $e^{-ikt} = \cos(kt) - i\sin(kt)$

To solve for X, Y:

- $\frac{\nabla^2 S(x,y)}{S(x,y)} = -k^2$ by its equality with $\frac{T''(t)}{T(t)}$.
- Suppose we can formulate a solution so that $S(x, y) = X(x)Y(y) \Rightarrow -k^2 X(x)Y(y) = Y(y)X''(x) + Y''(y)X(x) \Rightarrow -k^2 = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}$.
- So, for some j, $\frac{X''(x)}{X(x)} = -j^2 \Rightarrow X(x) = C\cos(jx) + D\sin(jx)$ by the T solution above.
- However, $X(x) = 0$ and $X \neq 0$ (not a constant function) forces us to conclude $C = 0 \Rightarrow X(x) = D\sin(jx)$.
- Also, the boundary condition $X(w) = 0$ also means for every $j = \frac{n\pi}{w}$, $n \in \mathbb{N}$, $X(x) = D\sin(\frac{\pi n}{w}x)$
- Following this identical logic for Y over length l, y $q = \frac{m\pi}{l}$,$Y(y) = F\sin(\frac{\pi m}{l}x)$
- Considering that $-k^2 = -j^2 - q^2$, this means $k = \sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}$
- Then, $u = T(t)X(x)Y(y)$
- $\Rightarrow u = (A_{mn}\cos(\sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}) + B_{mn}\sin(\sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}))(\sin(\frac{\pi n}{w}x))(\sin(\frac{\pi m}{l}x))$
- This runs over all $m, n \in \mathbb{N}$

Turning to a **circular membrane** with radius r_0, displacement described by $z = u(r, \theta, t)$:

- Boundary condition is then Dirichlet condition $u(r_0, \theta, t) = 0$
- $u_{tt} = \nabla^2 u \implies u_{tt} = u_r r + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$
- (Note: I suppose this magic (forgotten from vector calculus) is because $u_{tt} = u_{xx} + u_{yy}$, and we're converting between x, y and r, θ)
- $T(t) = A\sin(kt) + B\cos(kt)$ by identical logic to the rectangular drum, where $\frac{T''(t)}{T(t)}$ also was a constant.

- Assuming similarly that $S(r,\theta) = R(r)\Theta(\theta)$, you end up with $R(r)\Theta(\theta)[\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2] = 0$
- Finally, if $\frac{\Theta''}{\Theta} = \kappa$ for $\kappa \in \mathbb{R}$, then $\Theta(\theta) = Ce^{\sqrt{\kappa}\theta} + De^{-\sqrt{\kappa}\theta}$. H
- However, we have an additional condition that we have to be able to rotate the whole scene by $\theta = m2\pi$ radians and have it remain the same, or $\Theta(\theta + 2\pi) = \Theta(\theta)$. This means $\kappa < 0$ since we're in "imaginary exponents yielding sign an cosine" territory.
- This implies $\kappa = -m^2$ for an integer m, and therefore, $\boxed{\Theta(\theta) = C\cos(m\theta) + D\sin(m\theta)}$.
- We know also that $\boxed{T(t) = A\cos(kt) + B\sin(kt)}$
- Setting $\frac{\Theta''}{\Theta} = \kappa = -m^2$, and with $R(r)\Theta(\theta)[\frac{R''}{R}+\frac{R'}{rR}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+k^2] = 0 \Rightarrow \boxed{r^2R'' + rR' + [k^2r^2 - m^2]R}$ (a solution we'll look for later), we have the circular drum solution.

Notes to self:

- Need a clear intuition for a lot of things. What do the variables and their derivatives physically mean?
- Need more symbolic comfort with how integration works.

3.5 Fundamental Solutions

Main idea: The fundamental solution for the problem seems to:

- Solve the differential equation. Here, it is $u_t = \nabla^2 u$ or, specifically, $u_t = u_{xx} + u_{yy}$
- Solve the initial conditions
- More specifically, as $t \to 0_+, u(x,y,t) \to g(x,y)$, or we can "rewind back" to the initial displacement setup.
- Somehow , this is the function from which all heat equations (presumably specified as $g(x,y)$?) are built.

Motivation for heat equation (random 1D walk)

- Drunkard starts at lamppost (position 0) and walks left or right every Δt, each with probability $\frac{1}{2}$
- Probability of being i steps away from lamppost at time $n\Delta t$ is $p(i, n\Delta t)$.
- This depends only on $p(i-1, n\Delta t), p(i+1, n\Delta t)$ as $p(i, (n+1)\Delta t) = \frac{1}{2}p(i-1, n\Delta t) + \frac{1}{2}p(i+1, n\Delta t)$

- We can do the same thing we did the the 1D spring equation and approximate p_x, p_{xx} by means of its relation to small perturbances. Then we can relate this to p_t.
  - Assume we're going to shrink these moves to ΔX instead of 1.
  - We know $p(x,(n+1)\Delta t) = \frac{1}{2}p(x+\Delta x, n\Delta t) + \frac{1}{2}p(x-\Delta x, n\Delta t)$.
  - We know the calculus rule for smooth f, small Δx: $f(x+\Delta x) \approx f(x)+f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2$
  - Applying this, this means $\frac{1}{2}p(x,(n+1)\Delta t) + \frac{1}{2}p(x-\Delta x, n\Delta t) = \frac{1}{2}[p(x,n\Delta t) + p_x(x,n\Delta t)\Delta x + \frac{1}{2}p_{xx}(x,n\Delta t)(\Delta x)^2] + \frac{1}{2}[p(x,n\Delta t) + p_x(x,n\Delta t)\Delta - x + \frac{1}{2}p_{xx}(x,n\Delta t)(\Delta - x)^2] = p(x,n\Delta t) + \frac{1}{2}[\Delta x]^2 p_{xx}(x,n\Delta t)$
  - So for small steps, $p(x,(n+1)\Delta t) \approx p(x,n\Delta t) + \frac{1}{2}[\Delta x]^2 p_{xx}(x,n\Delta t)]$. *So we have this rule for p relating to derivatives along x.*
  - Now let's relate to derivatives along t! For small Δt, the definition of $\frac{\delta p}{\delta t} = \frac{p(x,(n+1)\Delta t) - p(x,n\Delta t)}{\Delta t} \approx \frac{[\Delta x]^2}{2\Delta t}\frac{\delta^2 p}{\delta x^2}(x,n\Delta t)$
  - Special case: If we set $t = n\Delta t$, then in this case, we can say $\frac{\delta p}{\delta t} = D\nabla^2 p, D > 0$ if $D = \frac{[\delta x]^2}{2\Delta t}$, the **diffusion equation**. Note: Since we're pushing $\Delta x, \Delta t \to 0$, is this, uh, $D \approx \frac{[\Delta x]^2}{2\Delta t}$?

We consider that p should be peaked (sharply) around 0, since it's just as likely to go left as right. We hypothesize that we should use a (sharp) bell curve to make that happen. (Note: Is this sort of like the Dirac Delta function?)

- Looking at $D = \frac{[\Delta x]^2}{2\Delta t}$, *we hypothesize that the operative units must be* $\frac{x^2}{t}$
- Therefor rour bell curve looks like $p(x,t) = C(t)\exp\{-\frac{x^2}{\sigma^2 t}\}$
- Since it's a probability measure along the real line, $\int_{x=-\infty}^{x=\infty} p(x,t)dx = 1$. At time t, the particle is at *some* x.
- Using identity $\int_{u=-\infty}^{u=\infty} e^{-au^2}du = \sqrt{\frac{\pi}{a}}, a > 0 \Rightarrow a = \frac{1}{\sigma^2 t} \Rightarrow C(t) = \frac{1}{\sigma\sqrt{\pi t}}$
- Even though our guess was unsubstantiated, we see $p(x,t) = \frac{1}{\sigma\sqrt{\pi t}}\exp\{-\frac{x^2}{\sigma^2 t}\}$ solves $p_t = Dp_{xx}$. To solve for D:
- To solve for D, find $p_t = \frac{\delta}{\delta t}[\frac{1}{\sigma\sqrt{\pi t}}\exp\{-\frac{x^2}{\sigma^2 t}\}]$, same for p_{xx}. Through lots of chain rule churning, we see $p_t = \frac{\sigma^2}{4}p_{xx} \Rightarrow D = \frac{\sigma^2}{4} \Rightarrow \sigma = 2\sqrt{D}$. *Note that p can't be negative, so ignore -2$\sqrt{D}$.*
- This implies our solution is $p(x,t) = [\frac{1}{\sqrt{4\pi Dt}}\exp\{-\frac{1}{4D}\frac{x^2}{t}\}], t > 0$

To say our solution is a **fundamental diffusion solution** means, further, that we can start with any initial conditions $p(x,0) = g(x)$, watch the equation unfold over t, and we'll still have $p_t = Dp_{xx}$.

- A **convolution** $(\Phi \star g)(x,t)$ of (probability function) Φ and initial state g is the function that combines the two at point x, or $(f \star g)(x) \int_{y=-\infty}^{y=\infty} f(x-y)g(y)dy = \int_{y=-\infty}^{y=\infty} f(y)g(x-y)dy = ...$ (TODO)
- In this case, $(\Phi \star g) = p(x,t) = \int_{y=-\infty}^{y=\infty} \frac{1}{\sqrt{4\pi Dt}} \exp\{-\frac{1}{4D}\frac{(x-y)^2}{t}\}g(y)dy, t > 0$
- It has the properties:
  - $\lim_{t\to 0^+} p(x,t) = g(x)$ for every reasonable choice of $p(\vec{x},0) = g(\vec{x})$.
  - $p_t = Dp_{xx}$
- Note we can find a fundamental solution for the heat equation $u_t = \nabla^2 u$ as well
  - For some v, we can hypothesize a distribution function $\Phi(\vec{x},t) = t^{-a}v(\frac{\|x\|^2}{t})$
  - Taking derivates $\frac{\delta}{\delta t}, \frac{\delta^2}{\delta x_i^2}$ lets us use $u_t = ux_0x_0 + u_{x_1x_1} + ...$
  - Calling the argument $z = \frac{\|x\|^2}{t}$, we take derivatives and find $0 = av(z) + (z + 2n)v'(z) + 4zv''(z)$. Note that n is from $\mathbb{R}^n$, since $\nabla(\|\vec{x}\|^2) = 2\vec{x}$ and $\nabla(2\vec{x}) = 2n$.
  - How do we choose a so that the equation can be solved as $[\frac{1}{4}v(z) + v'(z)$?
  - Note that if $[\frac{1}{4}v(z) + v'(z)] = 0$, then its derivative $\frac{d}{dz}[\frac{1}{4}v(z) + v'(z)] = [\frac{1}{4}v'(z) + v''(z)] = 0$
  - Setting $a = \frac{n}{2}$, we get $0 = 2n[\frac{1}{4}v(z) + v'(z)] + \frac{z}{4}\frac{d}{dz}[\frac{1}{4}v'(z) + v''(z)]$
  - We can rearrange to see $[av(z) = 2nv'(z)] + z[v'(z) + 4v''(z)] = 0$.
  - Therefore the solution is $\Phi(\vec{x},t) = t^{-\frac{n}{2}}v(\frac{\|x\|^2}{t})$. To find v, solve $[\frac{1}{4}v(z) + v'(z)] = 0$
  - This is obviously $v(z) = Ce^{-frac14z}$, or $\Phi(\vec{x},t) = Ct^{-\frac{n}{2}}v(\frac{\|x\|^2}{t})$, but we must find C so that $\int \Phi = 1$.
  - The identity $\int_{\mathbb{R}^n} e^{-a\|x\|^2} = (\frac{\pi}{a})^{\frac{n}{2}}$ helps us find $\lim_{t\to 0^+} \frac{C}{t^{-\frac{n}{2}}} \int e^{-\frac{\|x\|^2}{4t}} d\vec{x} = \lim_{t\to 0^+} \frac{C}{t^{-\frac{n}{2}}}(4\pi t)^{\frac{n}{2}} = C(4\pi)^{\frac{n}{2}}$
  - So the fundamental heat solution is $\Phi(\vec{x},t) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{\|x\|^2}{4t}}$
  - As an example, take $g(x,y) = u(x,y,0) = u_0e^{-x^2+y^2}$, like a candle had heated the origin.

– Solving $u(x,y) = (\Phi \star g)(\vec{x},t) = \frac{u_0}{4\pi t}\int_{\mathbb{R}^2} e^{\frac{-\|x-y\|^2}{4t}} e^{-\|y\|^2} d\vec{y}$ basically requires completing the square to get rid of the $2\vec{x}\cdot\vec{y}$ components, then changing variables $\vec{v} = \vec{y} - \frac{1}{1+4t}\vec{x}$, which, over the whole plane, is the same integral.

– You end up with $u(x,y,t) = \frac{u_0}{1+4t} e^{-\frac{x^2+y^2}{1+4t}}$

– Computing partial derivatives u_t, u_{xx}, u_{yy}, checking $u_t = u_{xx} + u_{yy}$ and seeing that $\lim_{t\to 0^+} u(x,y,t) = g(x,y)$ validates it.

– And it turns out, using identity $\int_{\mathbb{R}^2} e^{-a(x^2+y^2)}dxdy = \frac{\pi}{a}$ (the square of the 1d case), that $u_tot = \frac{u_0}{1+4t}\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{1+4t}}dxdy = \frac{u_0}{1+4t}(\pi(1+4t)) = \pi u_0$, so the total energy doesn't depend on time. It is conserved.

3.6 Fun with Functionals

Note that the heat solution ($u_t = nabla^2 u$) and the diffusion equation ($p_t = D\nabla^2 p$) are very similar. The first measures how much heat (density) $u(x,y,z,t)$ exists at a spot u in time; the second measures the probability density of a (particular) particle being in that spot at a time. This means *we can think about one in terms of the other.*

If, for example, we confined our space G to an ellipsoid (or really, any closed region) in $\mathbb{R}^3$, then saying "it's insulated to any heat going in our out" and "particles are confined to not leave or enter" is the same thing. They would have these things in common:

- The solutions *feel* like they should settle down to a constant, uniform value as $t \to \infty$
- The transfer at the "skin" δG is zero so on the boundary, for normal $\hat{n}, D_{\hat{n}}|_{\delta G} = \nabla u \cdot n|_{\delta G} = 0$. Otherwise, the heat transfer, say, would be nonzero. (Note that this is an example of a Neumann condition.)
- This means that $\frac{d}{dt}\iiint_G u d\vec{x} = 0$. For one, no heat entering or leaving means the total has to stay the same. Also consider that $\frac{d}{dt}\iiint_G u d\vec{x} = \iiint_G u_t d\vec{x} = \iiint_G \nabla^2 u d\vec{x}$ (by heat equation definition) $= \iiint_G \nabla\cdot(\nabla u) d\vec{x} = \iint_{\delta G} \nabla u \cdot \hat{n} d\sigma(\vec{x})$ (by divergence them on function ∇u) $= 0$.
- We can define a cost function $C = \frac{1}{2}\iiint_G [(\nabla u)\cdot(\nabla u)] d\vec{x} = \frac{1}{2}\iiint_G [(\nabla u)]^2 d\vec{x}$, since this is zero exactly when $\nabla u \cdot \nabla u = 0$ all over the space.

The idea of **functionals** is that these setups for "energy" ($E[u] = \iiint_G u d\vec{x}$) and "cost" ($C[u] = \frac{1}{2}\iiint_G [\nabla u]^2 d\vec{x}$) take in *functions* to become scalar-producing functions themselves.

We can prove a few things: the cost decreases over time, TODO

Cost decreases over time:

- $\frac{d}{dt}[\frac{1}{2}\iiint_G [\nabla u]^2 dx]$
- $= \frac{1}{2}[\iiint_G \frac{\delta}{\delta t}[\nabla u]^2 dx]$ by... Fubini's maybe ?
- $=\iiint_G [\nabla u \cdot \nabla u_t]dx$ by the chain rule (applied to each dimension, basically)
- Do a multivariable product rule $\nabla \cdot [fV] = \nabla f + V \cdot f \nabla \cdot V \Rightarrow \nabla f \cdot V = \nabla(fV) - f\nabla V$, setting $f = u_t, V = \nabla u$
- This gives $C[u] = \frac{1}{2}\iiint_G [\nabla u]^2 dx = \iiint_G \{\nabla \cdot [u_t \cdot \nabla u] - u_t \nabla \cdot \nabla u\}dx$
- Using divergence on the first term gives $= \iiint_G \{\nabla \cdot [u_t \cdot \nabla u] = \iint_{\delta G}[u_t \cdot \nabla u] = 0$ by the boundary Neumann conditions
- The second term $-\iiint_G (u_t \nabla \cdot \nabla u)dx = -\iiint_G (u_t \nabla^2 u)dx = -\iint_G [\nabla^2 u]^2 dx$ by definotion of heat equation. The integrand must be non-negative, so the whole thing has to decrease or stay constant.

Note that if $\iint_G [\nabla^2 u]^2 dx = 0$ (exact equality), then u must be constant.

- The above equation implies that $\nabla^2 u = 0$ throughout G.
- Use the product rule in reverse again, with $f = u, V = \nabla u : \nabla \cdot (u\nabla u) = \nabla u \cdot \nabla u + u\nabla^2 u$
- We proved the second term is zero. (The first term is equivalent to $\|\nabla u\|^2$, by the way.)
- The divergence theorem say $\iiint_G \nabla \cdot (u\nabla u)dx = \iint_{\delta G} u\nabla u \cdot \hat{n}d\sigma = 0$.
- Therefore the cost $C[u]$ is 0, therefore u is constant.

If it's an inequality, $C[u]$ must decrease forever.

Example: Unit cube $[0,1]\times[0,1]\times[0,1]$, intial condition $u(x,y,z,0) = u_1 \cos(\pi x)\cos(2\pi y)\cos(3\pi z) + u_0$

- Note that the boundary does satisfy $\nabla u \cdot \hat{n}|_{\delta G} = 0$:
- Notice that $\nabla u = \langle K\sin(\pi x), L\sin(2\pi y), M\sin(3\pi z)\rangle$ for some messy constants, and those are all 0 at $x, y, z \in \{0, 1\}$ (the boundaries)
- Also, we can solve this equation $u_t = \nabla u$:
  - Note that $u_x x = -\pi^2(u - u_0), u_y y = -4\pi^2(u - u_0), u_y y = -9\pi^2(u - u_0)$, so $\nabla^2 u = -14\pi^2(u - u_0)$
  - $\frac{du}{dt} = 1 - 4\pi^2(u - u_0)$
  - $\frac{du}{(u-u_0)} = -14\pi^2 dt$

- $\ln(u - u_0) = -14\pi^2 t + C$
- $u = u_0 + De^{-14\pi^2 t}$. Only $D = u_1 \cos(\pi x)\cos(2\pi y)\cos(3\pi z)$ satisfies initial conditions.
- So $u = u_0 + u_1 \cos(\pi x)\cos(2\pi y)\cos(3\pi z)e^{-14\pi^2 t}$

* We see also that energy is conserved, or that $E[u] = \iiint_G u(x, y, z, t) = \iiint_G u_0 + u_1[\int_0^1 \cos(\pi x)] \times [\int_0^1 \cos(2\pi y)] \times [\int_0^1 \cos(3\pi z)] = u_0$, since the factors (we can separate into them easily) are zero on this $[0, 1]$ interval.
* Finally $C[u] = \frac{1}{2}\iiint_G [\nabla u]^2 dx$ for $u = u_0 + u_1 \cos(\pi x)\cos(2\pi y)\cos(3\pi z)e^{-14\pi^2 t}$ decreases over time approaching (but in general not hitting) 0, since $[\nabla u]$ is some function $u_1 e^{-14\pi^2 t}[-\pi \sin(...)\cos(...)\cos(...) + ..]$, and $[\nabla u]^2$ is $u_1^2 e^{-28\pi^2 t}$ times something less than one. The integral is therefore less than $u_1^2 e^{-28\pi^2 t}$, and trends toward zero.
* TODO