Brilliant: Vector Calculus

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Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against t of the form $\overrightarrow{x}(t) = \langle x(t), y(t), \ldots \rangle$.

- A **line** through $p = (a, b, c)$ parallel to $\overrightarrow{v} = \langle v_x, v_y, v_z \rangle$ is $\overrightarrow{x}(t) = \overrightarrow{p} + t\overrightarrow{v}$
- **velocity** is characterized completely by $\overrightarrow{v}(t) = \overrightarrow{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- The **speed** of an object along that line versus t is the length of v ($\|v\|$)
- Therefore, the speed of an object along line

$$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

  is

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that $\overrightarrow{v}$ need not be constant. The speed of

$$\overrightarrow{x}(t) = \overrightarrow{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

  would then be

$$\|6\pi\cos(2\pi t)\hat{u}\| = |6\pi\cos(2\pi t)|$$

- **Acceleration** $a(t) = v'(t) = x''(t)$ is straightforward. Acceleration of

$$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance r from the sun could be $\langle r\cos(t), r\sin(t) \rangle$. The acceleration vector points in the opposite direction: $\langle -r\cos(t), -r\sin(t) \rangle$. In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by $F = ma$ is proportional to acceleration) points towards the sun.

- A **helix** could be a 3D extension like $\langle r\cos(t), r\sin(t), b\cdot t\rangle$.

2 Chapter 2.2: Space Curves

- TODO: Problem 5 - rotating ellipses and solving intersections with planes
- Note that while $\overrightarrow{x}(t) = \langle \cos(t), \sin(t), 5\rangle$ and $\overrightarrow{x}(t) = \langle \cos(2t), \sin(2t), 5\rangle$ describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If $\overrightarrow{x}(t) = t\overrightarrow{v}$, then speed is $\frac{\|\overrightarrow{x}(t+\Delta t)-\overrightarrow{t}\|}{\Delta t} = \|\overrightarrow{v}\|$, direction is $\frac{\overrightarrow{v}}{\|\overrightarrow{v}\|}$, and velocity $\overrightarrow{v}$ is the product of speed and direction.
- So $\overrightarrow{v}(t) = \lim_{\Delta t\to 0}\frac{\overrightarrow{x}(t+\Delta t)-\overrightarrow{x}(t)}{\Delta t} = \overrightarrow{x}'(t) = \frac{d\overrightarrow{x}}{dt} = \langle x'(t), y'(t), z'(t)\rangle$
- Neat conceptual result: any $y = f(x)$ can be made into $x(t) = \langle t, f(t), 0\rangle$, and then $v(t) = \langle 1, f'(t), 0\rangle$, which points along the tangent line at $\langle t, f(t), 0\rangle$.
- Note that dot product derivatives work like regular product: $[\overrightarrow{a}(t)\cdot\overrightarrow{b}(t)]' = \overrightarrow{a}'(t)\cdot\overrightarrow{b}(t) + \overrightarrow{a}(t)\cdot\overrightarrow{b}'(t)$, but the cross product does not work the same since $\frac{d}{dt}[a\times b] = a'\times b + a\times b'$, but since $a\times b' = -b'\times a$, can't switch the order to $a'\times b + b'\times a$ due to this non-commutativity.
- If

  $$\overrightarrow{x}(t) = \overrightarrow{p} + t\overrightarrow{v},$$

  calculating velocity with respect to origin becomes

  $$\frac{d}{dt}\|\overrightarrow{x}(t)\| = \frac{\overrightarrow{x}(t)\cdot\overrightarrow{x}'(t)}{\|\overrightarrow{x}(t)\|} = \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}\cdot\overrightarrow{v},$$

  after rewriting the distance formula and chugging through the chain rule.
- However, it becomes more clear when considering that $(\overrightarrow{v}\cdot\hat{x})\hat{x}$ is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way: $\int_a^b \overrightarrow{x}(t) = \langle \int_a^b x(t), \int_a^b y(t), \int_a^b z(t)\rangle$

- Example: if ball launched from origin with velocity $\langle 1,2,3\rangle$ and acceleration $\langle 0,0,-1\rangle$, it lands at

$$\frac{dv}{dt}dt = \langle 0,0,-1\rangle \tag{1}$$

$$\int \frac{dv}{dt}dt = v = \langle C, D, -t+F\rangle = \langle 1,2,3\rangle = \langle 1,2,-t+3\rangle, t=0 \tag{2}$$

$$x = \int v = \langle t+K, 2t+M, -\frac{1}{2}t^2+3t+N\rangle, x(\overrightarrow{0}) = \langle 0,0,0\rangle \tag{3}$$

$$\overrightarrow{x}(t) = \langle t, 2t, 3t-\frac{1}{2}t^2\rangle \tag{4}$$

$$z(t) = 0 \to t = 6 \to \overrightarrow{x}(6) = \langle 6, 12, 0\rangle \tag{5}$$

$$\tag{6}$$

- Also, generalizing $ds = \sqrt{(dx)^2+(dy)^2}$, the length of an arc from point a to b aporoaches $\int_a^b \|x'(t)\|dt$
- Example: a helix $\langle a\cos(\omega t), a\sin(\omega t), b\omega t\rangle$, parametrized by time t can be rewritten in terms of s, the arc length:

$$s = \int \|x'(t)\|dt \tag{7}$$

$$s = \int \sqrt{(-\omega a\sin(\omega t))^2 + (\omega a\cos(\omega t))^2 + (b\omega)^2}dt \tag{8}$$

$$s = |\omega| \int \sqrt{(a^2+b^2)}dt \tag{9}$$

$$s = |\omega| t\sqrt{a^2+b^2} \tag{10}$$

- *Note: It's weird to think of time in terms of length.* Could be analytically useful?

4 Chapter 2.4: Frenet Formulae

Main idea: Establish three new vectors $\hat{T}(s), \hat{N}(s), \hat{B}(s)$ that change as we move along a space curve, instead of $\overrightarrow{x}(t)$ that changes over an external "time" idea.

Remember that $s = \int_0^t \|\overrightarrow{x}'(\tilde{t})\|d\tilde{t}$, so $\frac{ds}{dt} = \|\overrightarrow{x}'(t)\|$.

4.1 $\hat{T}$: Vector tangent to space curve

- Remember arc length is $s = \int_0^t \|\overrightarrow{x}'(\tilde{t})d\tilde{t}\|$
- $\hat{T}$ is just normalized grad: $\frac{\overrightarrow{x}'(t)}{\|\overrightarrow{x}'(t)\|}$

- This implies $\frac{d\overrightarrow{x}}{ds} = \hat{T}$ since

$$s = \int_0^t \|\overrightarrow{x}'(\tilde{t})d\tilde{t}\| \tag{11}$$
$$\frac{ds}{dt} = \|\overrightarrow{x}(t)\| \tag{12}$$
$$\hat{T} = \frac{\overrightarrow{x}'(t)}{\|\overrightarrow{x}'(t)\|} = \frac{d\overrightarrow{x}}{dt} \cdot \frac{dt}{ds} \tag{13}$$
$$\hat{T} = \frac{d\overrightarrow{x}}{ds} \tag{14}$$
$$\tag{15}$$

4.2 $\hat{N}$: Vector normal to space curve and also in the direction of acceleration

Normal vectors include

- $\frac{\hat{T}(t)}{\|\hat{T}(t)\|}$ since, as $\|\hat{(T)}\|$ is just 1:

$$d(\|\hat{T}\|^2) = 0 \tag{16}$$
$$d(\|\hat{T}\|^2) = d(\hat{T} \cdot \hat{T}) = \hat{T}(t) \cdot 2\hat{T}'(t) \tag{17}$$
$$\hat{T}(t) \cdot \hat{T}'(t) = 0 \tag{18}$$

- $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|}$ since it's the same as the above, but parametrized over s instead of t. Doesn't change the direction of the vector!

Example:if $\overrightarrow{x}(t) = \langle R\cos(\omega t), R\sin(\omega t), 0\rangle$, then acceleration $\overrightarrow{a}(t)$ is

- $\overrightarrow{a} = \frac{d^2\overrightarrow{x}}{dt^2}$ just by definition
- $\overrightarrow{a} = -\omega^2\overrightarrow{x}$ just by calculation
- $\hat{T}(t) = \langle -\sin(\omega t), \cos(\omega t), 0\rangle$
- $\|\hat{T}(t)\| = 1$
- $\hat{N} = \frac{\hat{T}(t)}{\|\hat{T}(t)\|} = \langle -cos(\omega t), -\sin(\omega t), 0\rangle$
- So $\overrightarrow{a} = R\omega^2\hat{N}$ by these formulae.

This leads us to believe acceleration and $\hat{N}$, the normed derivative of $\hat{T}$ are related.

The part of acceleration $\overrightarrow{a}$ parallel to $\hat{T}$ is the projection $(\overrightarrow{a} \cdot \hat{T})\hat{T}$

The perpendicular part is then $\overrightarrow{a}$ minus that: $\overrightarrow{a} - (\overrightarrow{a} \cdot \hat{T})\hat{T}$

This also equals $(\frac{ds}{dt})^2 \| \frac{d\hat{T}}{ds} \| \hat{N}$ because

$$\overrightarrow{x} = \frac{dx}{dt} = T = \hat{T} \cdot \| \frac{dx}{dt} \| \tag{19}$$

$$s = \int_0^t \| \overrightarrow{x}'(t) \rightarrow \frac{ds}{dt} = \| \overrightarrow{x}'(t) \| \tag{20}$$

$$\hat{N} = \frac{d\hat{T}}{ds} normalized, so \tag{21}$$

$$\overrightarrow{a} = \frac{d^2 \overrightarrow{x}}{dt^2} = \frac{d}{dt}(\| \overrightarrow{x}'(t) \hat{T}(t) \|) = \frac{d \| \overrightarrow{x}'(t) \|}{dt} \hat{T} + \| \overrightarrow{x}'(t) \| \frac{d\hat{T}}{dt} \tag{22}$$

$$= \frac{d \| \overrightarrow{x}'(t) \|}{dt} \hat{T} + \frac{ds}{dt} \frac{d\hat{T}}{ds} \frac{ds}{dt} \tag{23}$$

$$= \frac{d \| \overrightarrow{x}'(t) \|}{dt} \hat{T} + (\frac{ds}{dt})^2 \| \frac{d\hat{T}}{ds} \| \hat{N} \tag{24}$$

This is "a = parallel part plus perpendicular (N) part", so the second term is $a_\perp$

4.3 Binormal vector $\hat{B}$

Note that curvature $\kappa(s) = \| \frac{d\hat{T}}{ds} \|$ is geometric (depends on s, not time) and changes as $\hat{T}$ changes.

Example: Curvature of $\overrightarrow{x}(t) = \langle \cos(t), \sin(t), bt \rangle$

$$x'(t) = \langle -\sin(t), \cos(t), b \rangle \tag{25}$$

$$\| x'(t) \| = \sqrt(1 + b^2) \tag{26}$$

$$s = \int_0^t \| x'(t) \| = \int_0^t \sqrt{(1 + b^2)} = t\sqrt{(1 + b^2)} \rightarrow t = \frac{s}{\sqrt{1 + b^2}} \tag{27}$$

4.4 $\hat{T}$ is:

- $\overrightarrow{x}'(t)$ normalized
- The tangent vector to the curve
- The same whether paremetrized by $\hat{T}'(t)$ or $\frac{dx}{ds}$

4.5 $\hat{N}$ is:

- $\overrightarrow{x}''(t)$ normalized as $\frac{\frac{d\hat{T}}{ds}}{\| \frac{d\hat{T}}{ds} \|} = \hat{N}$

- The normal vector to the curve
- $\perp$ to $\hat{T}$ in direction of acceleration. So a multiple of acceleration vector.
- The same whether paremetrized by $\hat{T}'(t)$ or $\frac{dx}{ds}$

4.6 $\hat{T}$ and $\hat{N}$

:

- Form a plane, since first, any normal vector's derivative is perpendicular to the vector

$$\frac{d}{ds}\|\hat{T}\|^2 = \frac{d}{ds}\hat{T}\cdot\hat{T} \tag{28}$$
$$= 2\hat{T}\cdot\hat{T}' \tag{29}$$
$$\frac{d}{ds}\|\hat{T}\|^2 = \frac{d}{ds}1 = 0 \tag{30}$$
$$\tag{31}$$

and

$$\hat{T}\cdot\hat{N} = \hat{T}\cdot\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} \tag{32}$$
$$= \hat{T}\cdot\frac{\hat{T}'(s)}{\|\frac{d\hat{T}}{ds}\|} = 0 \tag{33}$$

- κ is curvature: how much we're curving in that $T \times N$ plane.
- $\kappa = \|\frac{d\hat{T}}{ds}\|$
- Therefore, by above, $\frac{d\hat{T}}{ds} = \kappa\hat{N}$ **(Frenet equation 1)**

4.7 $\hat{B}$ is binormal: perpendicular to both

- defined as $\hat{B} = \hat{T}\times\hat{N}$
- Therefore, by derivative

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds}\times\hat{N} + \hat{T}\times\frac{d\hat{N}}{ds} \tag{34}$$
$$\frac{d\hat{B}}{ds} = \kappa\hat{N}\times\hat{N} + \hat{T}\times\frac{d\hat{N}}{ds} \tag{35}$$
$$\frac{d\hat{B}}{ds} = \hat{T}\times\frac{d\hat{N}}{ds} \tag{36}$$
$$\tag{37}$$

but this means T is orthogonal to dB, and we already know B and dB are orthogonal. We're working in 3d with the cross product, so dB is parallel to N.

- Therefore, we define "torsion" τ so that $-\frac{d\hat{B}}{ds} = \tau\hat{N}$ **(Frenet equation 2)**. Negative sign by convention.
- Can also cross by N on both sides to get $-\frac{d\hat{B}}{ds} \times \hat{N} = \tau$
- $\hat{B}$ measures how the plane defined by $\hat{T}, \hat{N}$ twists around. On a circle, $\hat{B}$ wouldn't change, so the derivative would be zero.
- **Final Frenet equation**. Prereq: $\hat{B} = \hat{T} \times \hat{N} \rightarrow \hat{N} = \hat{B} \times \hat{T} \rightarrow \hat{T} = \hat{N} \times \hat{B}$

$$\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} \tag{38}$$

$$\frac{d\hat{N}}{ds} = -\tau\hat{N} \times \hat{T} + \hat{B} \times \kappa\hat{N} \tag{39}$$

$$\frac{d\hat{N}}{ds} = \tau\hat{B} - \kappa\hat{T} \tag{40}$$

5 Chapter 2.5: Parametrized Surfaces

Main ideas:

- Can parameterize by $\overrightarrow{x}(u,v) = x(u,v), y(u,v), z(u,v)$
- Can perhaps parameterize $f(x,y,z) = c$ by $z = g(x,y)$
- Can also use ideas like $\nabla f = 0$ to find a normal.

There are many out-of-the-box paremetrizations including:

- Sphere at 0,0,0: $\overrightarrow{x}(u,v) = \langle R\cos(u)\sin(v), Rsin(u)\sin(v), R\cos(v)\rangle$, where $u \in [0, 2\pi), v \in [0, \pi]$
- Rotate function $y = f(x)$ around the x-axis: $\overrightarrow{x}(u,v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$, where $u \in D, v \in [0, 2\pi]$

Tangent vectors to $\overrightarrow{x}(u,v)$ are $\frac{\delta\overrightarrow{x}}{du}, \frac{\delta\overrightarrow{x}}{dv}$, so unit normal is $\pm\frac{\frac{d\overrightarrow{x}}{du} \times \frac{\delta\overrightarrow{x}}{dv}}{\|\frac{d\overrightarrow{x}}{du} \times \frac{\delta\overrightarrow{x}}{dv}\|}$

Example: Torus $\overrightarrow{x} = \langle[2+\cos(v)]\cos(u), [2+\cos(v)]\sin(u), \sin(v)\rangle, u, v \in [0, 2\pi)$. What's the tangent plane at $u = \frac{\pi}{4}, v = 0$?

$$dx/du = \langle -\sin(u)(2 + \cos(v)), \cos(u)(2 + \cos(v)), 0\rangle \tag{41}$$
$$dx/dv = \langle -\sin(v)\cos(u), -\sin(v)\sin(u), \cos(v)\rangle \tag{42}$$
$$u = \frac{\pi}{4}, v = 0 \to dx/du \tag{43}$$
$$= -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0, dx/dv = 0, 0, 1 \tag{44}$$
$$dx/du \times dx/dv = \langle \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0\rangle \tag{45}$$
$$\hat{n} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle \tag{46}$$
$$\hat{n} \cdot \overrightarrow{x} = 0 \to \hat{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \tag{47}$$
$$\to ... \to x + y = 3\sqrt{2} \tag{48}$$
$$\tag{49}$$

5.1 Example: Ellipsoid $x^2 + 2y^2 + z^2 = 4$

What's the normal at $(1, \frac{1}{\sqrt{2}}, \sqrt{2})$?

Method 1: parametrize with spherical u, v First, transform to sphere with change of coordinates

$$x^2 + 2y^2 + z^2 = 4 \tag{50}$$
$$X = x/2, Y = \frac{Y}{\sqrt{2}}, Z = z/2 \tag{51}$$
$$X^2 + Y^2 + Z^2 = 1 \tag{52}$$
$$X = \cos(u)\sin(v), Y = \sin(u)\sin(v), Z = \cos(v) \tag{53}$$
$$p = (1, \frac{1}{\sqrt{2}}, \sqrt{2}) \to u = v = \frac{\pi}{4} \tag{54}$$
$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle -1, \frac{1}{\sqrt{2}}, 0\rangle \tag{55}$$
$$\frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \frac{1}{\sqrt{2}}, -\sqrt{2}\rangle \tag{56}$$
$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) \times \frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \sqrt{2}, \sqrt{2}\rangle \tag{57}$$
$$\hat{n}_{out} = \frac{\langle -1, -\sqrt{2}, -\sqrt{2}\rangle}{\sqrt{5}} \tag{58}$$

Method 2: rewrite as z = g(x,y)

$$x^2 + 2y^2 + z^2 = 4 \quad (59)$$
$$z = (4 - x^2 - 2y^2)^{\frac{1}{2}} \quad (60)$$
$$dz/dx = \frac{1}{2} \times -2x(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -\frac{1}{\sqrt{2}} \quad (61)$$
$$dz/dy = \frac{1}{2} \times -4y(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -2\sqrt{2}/\sqrt{2} = -1 \quad (62)$$
$$f \approx \sqrt{2} + dz/dx(1, \frac{1}{\sqrt{2}})(x-1) + dz/dy(1, \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) \quad (63)$$
$$\to ... \to \frac{1}{\sqrt{2}}x + y + z = 2\sqrt{2} \quad (64)$$
$$(65)$$

giving us normal vector $\langle \frac{1}{\sqrt{2}}, 1, 1\rangle = \frac{\langle 1, \sqrt{2}, \sqrt{2}\rangle}{\sqrt{5}}$ after normalization.

Method 3: gradient

Gradient is always normal to the tangent plane. Recognize level set of $f(x, y, z) = x^2 + 2y^2 + z^2$.

$\nabla f = \langle 2x, 4y, 2z\rangle \to \nabla f(1, \frac{1}{\sqrt{2}}, \sqrt{2}) = \langle 2, 2\sqrt{2}, 2\sqrt{2}\rangle$

Then normalize.

5.2 Mobius strip and "direction of out"

Mobius strip is

- $x = 2\cos(u) + v\cos(\frac{u}{2})$
- $y = 2\sin(u) + v\cos(\frac{u}{2})$
- $z = v\sin(\frac{u}{2})$
- $u \in [0, 2\pi], v \in [-\frac{1}{2}, \frac{1}{2}]$

$\hat{n} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$ at (0,0) is $\langle 0, 0, -1\rangle$,

but at $(2\pi, 0)$ is the same point, but $\hat{n} = \langle 0, 0, 1\rangle$!!

6 Chapter 2.6: Vector Fields

(Lots of intuition questions here...)

One nugget: using **gradient vector fields**: Suppose $\overrightarrow{F}(x,y) = \langle 2, -4y^3 \rangle$. Then if F = ∇f, then F's arrows are perpendicular to a level set $f = c$. So look at $f = 2x - y^4$ and find perpendicular arrows to these. That's actually F!

Linear approximation for $\overrightarrow{F} : D \in \mathbf{R}^n \to \mathbf{R}^m$

Main idea: $\overrightarrow{F}(\overrightarrow{x}) = \overrightarrow{F}(\overrightarrow{a}) + A(\overrightarrow{a})(\overrightarrow{x} - \overrightarrow{a})$

Note that A takes in vectors of size n (so it has that many columns), and has m functions (rows) that operate on it. So the Jacobian, A, has as row i, column j, the quantity $\frac{dF_i}{dx_j}(\overrightarrow{a})$.

$dF_i/d\overrightarrow{x}$ extends across row i.

7 Chapter 2.7: Jack and the Beanstalk (Newton's method)

Basis for Newton's:

If we're estimating x_1 by following the derivative at x_0, this means we're looking at the line with x-intercept x_1, with slope $f'(x_0)$.

So instead of $y = mx + b$, we'll flip the two and use

$x = y/m + x_{int}$

or $x_0 = f(x_0)\frac{1}{f'(x_0)} + x_1$,

or $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

Note that, under Newton's something like $|x|$ will converge immediately, x^3 will converge moderately, and a S-curve might barely converge if at all.

The extension of this with the Jacobian matrix $A = DF'(x_0)$ is $\overrightarrow{x}_1 = \overrightarrow{x}_0 - (D\overrightarrow{F}(\overrightarrow{x}_0))^{-1}\overrightarrow{F}(x_0)$

8 Chapter 2.8: Electrostatic bootcamp

Electric charge radiates out equally in all directions, and is inversely proportional to distance.

Formula, with Q as the charge, ϵ_0 is a constant: $\overrightarrow{E}(x,y,z) = \frac{Q}{4\pi\epsilon_0\|x\|^2}\hat{x}$

Field line is a special case of a flow line - the space cure that follows $\overrightarrow{F}$'s arrows. The tangent vector to the flow line is $\overrightarrow{F}(\overrightarrow{x}(t))$ (t is not time here), so $\frac{d\overrightarrow{x}}{dt} = \overrightarrow{F}(\overrightarrow{x}(t))$

Example: Vector field $\overrightarrow{F}(x,y) = \langle -2y, 3x \rangle$. What's the flow line through (2,0)?

Solution: Need to solve $dx/dt = -2y, dy/dt = 3x$. Key is "separating the equations". Remember x and y are functions of t!

$$\frac{d^2x}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6x. \tag{66}$$
$$\frac{d^2y}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6y. \tag{67}$$
$$x(t) = -6x''(t), y(t) = -6y''(t) \tag{68}$$
$$\to x = A\cos(\sqrt{6}t) + B\sin(\sqrt{6}t), y = C\cos(\sqrt{6}t) + D\sin(\sqrt{6}t) \tag{69}$$
$$\frac{dx}{dt} = -2y(t) \to \frac{\sqrt{6}}{2}A\sin(\sqrt{6}t) - \frac{\sqrt{6}}{2}B\cos(\sqrt{6}t) = y(t) \tag{70}$$
$$x(t=0) = 2 \to A = 2 \tag{71}$$
$$y(t=0) = 0 \to B = 0 \tag{72}$$
$$\overrightarrow{F}(t) = \langle 2\cos(\sqrt{6}t), \sqrt{6}\sin(\sqrt{6}t), \rangle \tag{73}$$
$$\tag{74}$$

Note: **Field lines** follow rules:

- Go from positive charges to negative
- Density of lines directly relates to how much charge a point has
- Lines don't intersect.
- Corollary: If count of out equals count of in, point has zero charge
- "Number" (to be defined) of field lines in and out of a *surface* related to the charge inside. Upcoming.

9 3.1: Surface Integrals

Example: Fluid pressure in a tank is:

- Proportional (via some weight constant p_{fluid}) to depth of the point
- Pushes out via the normal $\hat{n}$
- So, for the x side of a cube of length l, this would be

  $\overrightarrow{F}_{x=l} = (\iint_{[0,l]\times[0,l]} p_{fluid}[1 - \frac{z}{l}]dydz)\hat{i}$

Example: Hemisphere of size l, sitting at (0, 0, 0)

Finding the out pointing unit normal of hemisphere at point $(x, y, -\sqrt{l^2 - x^2 - y^2})$

Note: Can just eyeball this, but one way is the **gradient**. Take the level set

$$g(x, y, z) = x^2 + y^2 + (z - l)^2 - l^2 = 0 \tag{75}$$
$$\nabla g(x, y, z) = \langle 2x, 2y, 2(z - l)\rangle \tag{76}$$
$$\hat{n} = \pm\frac{\nabla g(x, y, z)}{\|\nabla g(x, y, z)\|} \tag{77}$$
$$\hat{n} = \pm\frac{\langle x, y, (z - l)\rangle}{\sqrt{x^2 + y^2 + (z - l)^2}} \tag{78}$$
$$\hat{n} = \pm\frac{\langle x, y, (z - l)\rangle}{\sqrt{l^2}} \tag{79}$$
$$\hat{n} = \pm\langle \frac{x}{l}, \frac{y}{l}, \frac{z}{l} - 1\rangle \tag{80}$$
$$\tag{81}$$

Note: Integrating over a patch dA on the surface means finding the area of micro-patches δA_{ij}, which is the parallelogram defined by

$$s_1 = \langle \Delta x_i, 0, \Delta x_i f_x(x_i^*, y_j^*)\rangle \tag{82}$$
$$s_2 = \langle 0, \Delta y_j, \Delta y_j f_y(x_i^*, y_j^*)\rangle \tag{83}$$
$$\Delta A_{ij} \approx \|s_1 \times s_2\| \tag{84}$$
$$= \sqrt{(1 + [f_x(x_i^*, y_j^*)]^2 + [f_y(x_i^*, y_j^*)]^2}\Delta x_i \Delta y)_j \tag{85}$$
$$\tag{86}$$

So the total pressure ends up being $\overrightarrow{F}_{tot} = p_{fluid} \iint p\hat{n} dA$