

Brilliant: Vector Calculus

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Note: Latex reference: <http://tug.ctan.org/info/undergradmath/undergradmath.pdf>

1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against t of the form $\vec{x}(t) = \langle x(t), y(t), \dots \rangle$.

- A **line** through $p = (a, b, c)$ parallel to $\vec{v} = \langle v_x, v_y, v_z \rangle$ is $\vec{x}(t) = \vec{p} + t\vec{v}$
- **velocity** is characterized completely by $\vec{v}(t) = \vec{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- The **speed** of an object along that line versus t is the length of v ($\|v\|$)
- Therefore, the speed of an object along line

$$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

is

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that \vec{v} need not be constant. The speed of

$$\vec{x}(t) = \vec{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

would then be

$$\|6\pi \cos(2\pi t)\hat{u}\| = |6\pi \cos(2\pi t)|$$

- **Acceleration** $a(t) = v'(t) = x''(t)$ is straightforward. Acceleration of

$$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance r from the sun could be $\langle r \cos(t), r \sin(t) \rangle$. The acceleration vector points in the opposite direction: $\langle -r \cos(t), -r \sin(t) \rangle$. In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by $F = ma$ is proportional to acceleration) points towards the sun, *with acceleration perpendicular to velocity*.

- A **helix** could be a 3D extension like $\langle r \cos(t), r \sin(t), b \cdot t \rangle$.

2 Chapter 2.2: Space Curves

- Note that while $\vec{x}(t) = \langle \cos(t), \sin(t), 5 \rangle$ and $\vec{x}(t) = \langle \cos(2t), \sin(2t), 5 \rangle$ describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If $\vec{x}(t) = t\vec{v}$, then speed is $\frac{\|\vec{x}(t+\Delta t) - \vec{x}(t)\|}{\Delta t} = \|\vec{v}\|$, direction is $\frac{\vec{v}}{\|\vec{v}\|}$, and velocity \vec{v} is the product of speed and direction.
- So $\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} = \vec{x}'(t) = \frac{d\vec{x}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$
- Neat conceptual result: any $y = f(x)$ can be made into $x(t) = \langle t, f(t) \rangle$, and then $v(t) = \langle 1, f'(t) \rangle$, which points along the tangent line at $\langle t, f(t) \rangle$.
- Note that dot product derivatives work like regular product: $[\vec{a}(t) \cdot \vec{b}(t)]' = \vec{a}'(t) \cdot \vec{b}(t) + \vec{a}(t) \cdot \vec{b}'(t)$, but the cross product does not work the same since $\frac{d}{dt}[a \times b] = a' \times b + a \times b'$, but since $a \times b' = -b' \times a$, can't switch the order to $a' \times b + b' \times a$ due to this non-commutativity.
- If

$$\vec{x}(t) = \vec{p} + t\vec{v},$$

calculating velocity with respect to origin becomes

$$\frac{d}{dt} \|\vec{x}(t)\| = \frac{\vec{x}(t) \cdot \vec{x}'(t)}{\|\vec{x}(t)\|} = \frac{\vec{x}}{\|\vec{x}\|} \cdot \vec{v},$$

after rewriting the distance formula and chugging through the chain rule.

- However, it becomes more clear when considering that $(\vec{v} \cdot \hat{x})\hat{x}$ is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way:
 $\int_a^b \vec{x}(t) dt = \langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \rangle$
- Example: if ball launched from origin with velocity $\langle 1, 2, 3 \rangle$ and acceleration $\langle 0, 0, -1 \rangle$,

it lands at

$$\frac{dv}{dt}dt = \langle 0, 0, -1 \rangle \quad (1)$$

$$\int \frac{dv}{dt}dt = v = \langle C, D, -t + F \rangle = \langle 1, 2, 3 \rangle = \langle 1, 2, -t + 3 \rangle, t = 0 \quad (2)$$

$$x = \int v = \langle t + K, 2t + M, -\frac{1}{2}t^2 + 3t + N \rangle, x(\vec{0}) = \langle 0, 0, 0 \rangle \quad (3)$$

$$\vec{x}(t) = \langle t, 2t, 3t - \frac{1}{2}t^2 \rangle \quad (4)$$

$$z(t) = 0 \rightarrow t = 6 \rightarrow \vec{x}(6) = \langle 6, 12, 0 \rangle \quad (5)$$

$$(6)$$

- Also, generalizing $ds = \sqrt{(dx)^2 + (dy)^2}$, the length of an arc from point a to b approaches $\int_a^b ds = \int_{t_a}^{t_b} \|x'(t)\|dt$
- Example: a helix $\langle a \cos(\omega t), a \sin(\omega t), b\omega t \rangle$, parametrized by time t can be rewritten in terms of s , the arc length:

$$s = \int \|x'(t)\|dt \quad (7)$$

$$s = \int \sqrt{(-\omega a \sin(\omega t))^2 + (\omega a \cos(\omega t))^2 + (b\omega)^2}dt \quad (8)$$

$$s = |\omega| \int \sqrt{a^2 + b^2}dt \quad (9)$$

$$s = |\omega|t\sqrt{a^2 + b^2} \quad (10)$$

- *Note: It's weird to think of time in terms of length. Could be analytically useful?*

4 Chapter 2.4: Frenet Formulae

Main idea: Establish three new vectors $\hat{T}(s), \hat{N}(s), \hat{B}(s)$ that change as we move along a space curve, instead of $\vec{x}(t)$ that changes over an external “time” idea.

Remember that $s = \int_0^t \|\vec{x}'(\tilde{t})\|d\tilde{t}$, so $\frac{ds}{dt} = \|\vec{x}'(t)\|$.

4.1 \hat{T} : Vector tangent to space curve

- Remember arc length is $s = \int_0^t \|\vec{x}'(\tilde{t})\|d\tilde{t}$
- \hat{T} is just normalized grad: $\frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$

- This implies $\frac{d\vec{x}}{ds} = \hat{T}$ since

$$s = \int_0^t \|\vec{x}'(\tilde{t})\| d\tilde{t} \quad (11)$$

$$\frac{ds}{dt} = \|\vec{x}'(t)\| \quad (12)$$

$$\hat{T} = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \frac{d\vec{x}}{dt} \cdot \frac{dt}{ds} \quad (13)$$

$$\hat{T} = \frac{d\vec{x}}{ds} \quad (14)$$

$$(15)$$

- So this is how the space curve \vec{x} changes as it moves along s .
- It's normalized, so it's the same whether parameterized by t , s , or whatever.

4.2 \hat{N} : Normal to curve (perpendicular to \hat{T})

Normal vectors are:

- $\vec{x}''(t)$ normalized as $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} = \hat{N}$
- The normal vector to the curve
- \perp to \hat{T} in direction of acceleration. So a multiple of acceleration vector.
- $\frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$. The following sequence shows any unit length vector is perpendicular to its derivative.

$$\|\hat{T}\| = 1 \quad (16)$$

$$d(\|\hat{T}\|^2) = 0 \quad (17)$$

$$d(\|\hat{T}\|^2) = d(\hat{T} \cdot \hat{T}) = \hat{T}'(t) \cdot 2\hat{T}(t) \quad (18)$$

$$\hat{T}'(t) \cdot \hat{T}(t) = 0 \quad (19)$$

- $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|}$ since it's the same as the above, but parametrized over s instead of t . Doesn't change the direction of the vector!

Example: if $\vec{x}(t) = \langle R \cos(\omega t), R \sin(\omega t), 0 \rangle$, then:

- $\vec{a} = \frac{d^2\vec{x}}{dt^2}$ just by definition
- $\vec{a} = -\omega^2 \vec{x}$ just by calculation

- $\hat{T}(t) = \langle -\sin(\omega t), \cos(\omega t), 0 \rangle$
- $\|\hat{T}(t)\| = 1$
- $\hat{N} = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = \langle -\cos(\omega t), -\sin(\omega t), 0 \rangle$
- So $\vec{a} = R\omega^2 \hat{N}$ by these formulae.

This leads us to believe acceleration and \hat{N} , the normed derivative of \hat{T} are related.

The part of acceleration \vec{a} parallel to \hat{T} is the projection $(\vec{a} \cdot \hat{T})\hat{T}$

The perpendicular part is then \vec{a} minus that: $\vec{a} - (\vec{a} \cdot \hat{T})\hat{T}$

This also equals $(\frac{ds}{dt})^2 \|\frac{d\hat{T}}{ds}\| \hat{N}$ because

$$\vec{x}' = \frac{dx}{dt} = T = \hat{T} \cdot \|\frac{dx}{dt}\| \quad (20)$$

$$s = \int_0^t \|\vec{x}'(t)\| \rightarrow \frac{ds}{dt} = \|\vec{x}'(t)\| \quad (21)$$

$\hat{N} = \frac{d\hat{T}}{ds}$ normalized, so

$$\vec{a} = \frac{d^2 \vec{x}}{dt^2} = \frac{d}{dt}(\|\vec{x}'(t)\| \hat{T}(t)) = \frac{d\|\vec{x}'(t)\|}{dt} \hat{T} + \|\vec{x}'(t)\| \frac{d\hat{T}}{dt} \quad (22)$$

$$= \frac{d\|\vec{x}'(t)\|}{dt} \hat{T} + \frac{ds}{dt} \frac{d\hat{T}}{ds} \frac{ds}{dt} \quad (23)$$

$$= \frac{d\|\vec{x}'(t)\|}{dt} \hat{T} + (\frac{ds}{dt})^2 \|\frac{d\hat{T}}{ds}\| \hat{N} \quad (24)$$

This is “a = parallel part plus perpendicular (N) part”, so the second term is a_{\perp}

4.3 \hat{T} and \hat{N}

- Form a plane, since first, any normal vector's derivative is perpendicular to the vector
- κ is curvature: how much we're curving in that $T \times N$ plane.
- $\kappa = \|\frac{d\hat{T}}{ds}\|$
- Therefore, by above, $\frac{d\hat{T}}{ds} = \kappa \hat{N}$ (**Frenet equation 1**)

Note that curvature $\kappa(s) = \|\frac{d\hat{T}}{ds}\|$ is geometric (depends on s, not time) and changes as \hat{T} changes.

Example: Curvature of $\vec{x}(t) = \langle \cos(t), \sin(t), bt \rangle$

$$x'(t) = \langle -\sin(t), \cos(t), b \rangle \quad (25)$$

$$\|x'(t)\| = \sqrt{1 + b^2} \quad (26)$$

$$s = \int_0^t \|x'(\tilde{t})\| d\tilde{t} = \int_0^t \sqrt{1 + b^2} = t\sqrt{1 + b^2} \rightarrow t = \frac{s}{\sqrt{1 + b^2}} \quad (27)$$

Do the substitution of $\frac{s}{\sqrt{1+b^2}}$ for t above to get $x'(s)$, and from there, you can figure out $\frac{d\hat{T}}{ds}$ and normalize to get $\kappa = \frac{1}{1+b^2}$

4.4 \hat{B} is binormal: perpendicular to both

- defined as $\hat{B} = \hat{T} \times \hat{N}$
- Therefore, by derivative

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \quad (28)$$

$$\frac{d\hat{B}}{ds} = \kappa \hat{N} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \quad (29)$$

$$\frac{d\hat{B}}{ds} = \hat{T} \times \frac{d\hat{N}}{ds} \quad (30)$$

$$(31)$$

but this means \hat{T} is orthogonal to $d\hat{B}$, and we already know \hat{B} and $d\hat{B}$ are orthogonal. We're working in 3d with the cross product, so $d\hat{B}$ is parallel to \hat{N} .

- Therefore, we define **torsion** τ so that $-\frac{d\hat{B}}{ds} = \tau \hat{N}$ (**Frenet equation 2**). Negative sign by convention.
- Can also cross by \hat{N} on both sides to get $-\frac{d\hat{B}}{ds} \times \hat{N} = \tau$
- \hat{B} measures how the plane defined by \hat{T}, \hat{N} twists around. On a circle, \hat{B} wouldn't change, so the derivative would be zero.
- **Final Frenet equation.** Prereq: $\hat{B} = \hat{T} \times \hat{N} \rightarrow \hat{N} = \hat{B} \times \hat{T} \rightarrow \hat{T} = \hat{N} \times \hat{B}$

$$\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} \quad (32)$$

$$\frac{d\hat{N}}{ds} = -\tau \hat{N} \times \hat{T} + \hat{B} \times \kappa \hat{N} \quad (33)$$

$$\frac{d\hat{N}}{ds} = \tau \hat{B} - \kappa \hat{T} \quad (34)$$

5 Chapter 2.5: Parametrized Surfaces

Main ideas:

- Can parameterize by $\vec{x}(u, v) = x(u, v), y(u, v), z(u, v)$
- Can perhaps parameterize $f(x, y, z) = c$ by $z = g(x, y)$
- Can also use ideas like $\nabla f = 0$ to find a normal.

There are many out-of-the-box parametrizations including:

- Sphere at 0,0,0: $\vec{x}(u, v) = \langle R \cos(u) \sin(v), R \sin(u) \sin(v), R \cos(v) \rangle$, where $u \in [0, 2\pi), v \in [0, \pi]$
- Rotate function $y = f(x)$ around the x-axis: $\vec{x}(u, v) = \langle u, f(u) \cos(v), f(u) \sin(v) \rangle$, where $u \in D, v \in [0, 2\pi]$

Tangent vectors to $\vec{x}(u, v)$ are $\frac{\delta \vec{x}}{\delta u}, \frac{\delta \vec{x}}{\delta v}$, so unit normal is $\pm \frac{\frac{d\vec{x}}{du} \times \frac{\delta \vec{x}}{dv}}{\|\frac{d\vec{x}}{du} \times \frac{\delta \vec{x}}{dv}\|}$

Example: Torus $\vec{x} = \langle [2 + \cos(v)] \cos(u), [2 + \cos(v)] \sin(u), \sin(v) \rangle, u, v \in [0, 2\pi)$. What's the tangent plane at $u = \frac{\pi}{4}, v = 0$?

$$d\vec{x}/du = \langle -\sin(u)(2 + \cos(v)), \cos(u)(2 + \cos(v)), 0 \rangle \quad (35)$$

$$d\vec{x}/dv = \langle -\sin(v) \cos(u), -\sin(v) \sin(u), \cos(v) \rangle \quad (36)$$

$$u = \frac{\pi}{4}, v = 0 \rightarrow \quad (37)$$

$$d\vec{x}/du = \langle -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0 \rangle \quad (38)$$

$$d\vec{x}/dv = \langle 0, 0, 1 \rangle \quad (39)$$

$$dx/du \times dx/dv = \langle \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0 \rangle \quad (40)$$

$$\hat{n} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle \quad (41)$$

$$\hat{n} \cdot \vec{x} = 0 \rightarrow \hat{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \quad (42)$$

$$\rightarrow \dots \rightarrow x + y = 3\sqrt{2} \quad (43)$$

$$(44)$$

5.1 Example: Ellipsoid $x^2 + 2y^2 + z^2 = 4$

What's the normal at $(1, \frac{1}{\sqrt{2}}, \sqrt{2})$?

Method 1: parametrize with spherical \mathbf{u}, \mathbf{v} First, transform to sphere with change of coordinates, then flip to speherical coordinates.

$$x^2 + 2y^2 + z^2 = 4 \quad (45)$$

$$X = x/2, Y = \frac{Y}{\sqrt{2}}, Z = z/2 \quad (46)$$

$$X^2 + Y^2 + Z^2 = 1 \quad (47)$$

$$X = \cos(u) \sin(v), Y = \sin(u) \sin(v), Z = \cos(v) \quad (48)$$

$$p = (1, \frac{1}{\sqrt{2}}, \sqrt{2}) \rightarrow u = v = \frac{\pi}{4} \quad (49)$$

$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle -1, \frac{1}{\sqrt{2}}, 0 \rangle \quad (50)$$

$$\frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \frac{1}{\sqrt{2}}, -\sqrt{2} \rangle \quad (51)$$

$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) \times \frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \sqrt{2}, \sqrt{2} \rangle \quad (52)$$

$$\hat{n}_{out} = \frac{\langle -1, -\sqrt{2}, -\sqrt{2} \rangle}{\sqrt{5}} \quad (53)$$

Method 2: rewrite as $z = g(\mathbf{x}, \mathbf{y})$

$$x^2 + 2y^2 + z^2 = 4 \quad (54)$$

$$z = (4 - x^2 - 2y^2)^{\frac{1}{2}} \quad (55)$$

$$dz/dx = \frac{1}{2} \times -2x(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -\frac{1}{\sqrt{2}} \quad (56)$$

$$dz/dy = \frac{1}{2} \times -4y(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -2\sqrt{2}/\sqrt{2} = -1 \quad (57)$$

$$f \approx \sqrt{2} + dz/dx(1, \frac{1}{\sqrt{2}})(x - 1) + dz/dy(1, \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) \quad (58)$$

$$\rightarrow \dots \rightarrow \frac{1}{\sqrt{2}}x + y + z = 2\sqrt{2} \quad (59)$$

$$(60)$$

giving us normal vector $\langle \frac{1}{\sqrt{2}}, 1, 1 \rangle = \frac{\langle 1, \sqrt{2}, \sqrt{2} \rangle}{\sqrt{5}}$ after normalization.

Method 3: gradient

Gradient is always normal to the tangent plane. Recognize level set of $f(x, y, z) = x^2 + 2y^2 + z^2$.

$$\nabla f = \langle 2x, 4y, 2z \rangle \rightarrow \nabla f(1, \frac{1}{\sqrt{2}}, \sqrt{2}) = \langle 2, 2\sqrt{2}, 2\sqrt{2} \rangle$$

Then normalize.

5.2 Mobius strip and “outward direction”

Mobius strip is

- $x = 2 \cos(u) + v \cos(\frac{u}{2})$
- $y = 2 \sin(u) + v \cos(\frac{u}{2})$
- $z = v \sin(\frac{u}{2})$
- $u \in [0, 2\pi], v \in [-\frac{1}{2}, \frac{1}{2}]$

$$\hat{n} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} \text{ at } (0,0) \text{ is } \langle 0, 0, -1 \rangle,$$

but at $(2\pi, 0)$ is the same point, but $\hat{n} = \langle 0, 0, 1 \rangle$!!

6 Chapter 2.6: Vector Fields

(Lots of intuition questions here...)

One nugget: using **gradient vector fields**: Suppose $\vec{F}(x, y) = \langle 2, -4y^3 \rangle$. If $F = \nabla f$ for some f , then F 's arrows are perpendicular to a level set $f = c$. So look at $f = 2x - y^4$ and find perpendicular arrows to these. That's actually F!

Linear approximation for $\vec{F} : D \in \mathbf{R}^n \rightarrow \mathbf{R}^m$

$$\text{Main idea: } \vec{F}(\vec{x}) = \vec{F}(\vec{a}) + A(\vec{a})(\vec{x} - \vec{a})$$

Note that A takes in vectors of size n (so it has that many columns), and has m functions (rows) that operate on it. So the Jacobian, A , has as row i , column j , the quantity $\frac{dF_i}{dx_j}(\vec{a})$.

$dF_i/d\vec{x}$ extends across row i .

7 Chapter 2.7: Jack and the Beanstalk (Newton's method)

Basis for Newton's:

If we're estimating x_1 by following the derivative at x_0 , this means we're looking at the line with x-intercept x_1 , with slope $f'(x_0)$.

So instead of $y = mx + b$, we'll flip the two and use

$$x = y/m + x_{int}$$

$$\text{or } x_0 = f(x_0) \frac{1}{f'(x_0)} + x_1,$$

$$\text{or } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Note that, under Newton's something like $|x|$ will converge immediately, x^3 will converge moderately, and a S-curve might barely converge if at all.

The extension of this with the Jacobian matrix $A = DF'(x_0)$ is $\vec{x}_1 = \vec{x}_0 - (D\vec{F}(\vec{x}_0))^{-1} \vec{F}(x_0)$

8 Chapter 2.8: Electrostatic bootcamp

Electric charge radiates out equally in all directions, and is inversely proportional to distance.

Formula, with Q as the charge, ϵ_0 is a constant: $\vec{E}(x, y, z) = \frac{Q}{4\pi\epsilon_0\|x\|^2} \hat{x}$

A field line is a special case of a **flow line** - the space curve that follows \vec{F} 's arrows. The tangent vector to the flow line is $\vec{F}(\vec{x}(\tilde{t}))$ (\tilde{t} is not time here), so $\frac{d\vec{x}}{d\tilde{t}} = \vec{F}(\vec{x}(\tilde{t}))$

Example: Vector field $\vec{F}(x, y) = \langle -2y, 3x \rangle$. What's the flow line through (2,0)?

Solution: Need to solve $dx/dt = -2y, dy/dt = 3x$. Key is "separating the equations". Remember x and y are functions of t !

$$\frac{d^2x}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6x. \quad (61)$$

$$\frac{d^2y}{dt^2} = -2\frac{dx}{dt} = -2 \times 3y = -6y. \quad (62)$$

$$x(t) = -6x''(t), y(t) = -6y''(t) \quad (63)$$

$$\rightarrow x = A \cos(\sqrt{6}t) + B \sin(\sqrt{6}t), y = C \cos(\sqrt{6}t) + D \sin(\sqrt{6}t) \quad (64)$$

$$\frac{dx}{dt} = -2y(t) \rightarrow \frac{\sqrt{6}}{2}A \sin(\sqrt{6}t) - \frac{\sqrt{6}}{2}B \cos(\sqrt{6}t) = y(t) \quad (65)$$

$$x(t=0) = 2 \rightarrow A = 2 \quad (66)$$

$$y(t=0) = 0 \rightarrow B = 0 \quad (67)$$

$$\vec{F}(t) = \langle 2 \cos(\sqrt{6}t), \sqrt{6} \sin(\sqrt{6}t) \rangle \quad (68)$$

$$(69)$$

Note: **Field lines** follow rules:

- Go from positive charges to negative
- Density of lines directly relates to how much charge a point has
- Lines don't intersect.
- Corollary: If count of out equals count of in, point has zero charge
- “Number” (to be defined) of field lines in and out of a *surface* related to the charge inside. Upcoming.

9 3.1: Surface Integrals

Example: Fluid pressure in a tank is:

- Proportional (via some weight constant p_{fluid}) to depth of the point
- Pushes out via the normal \hat{n}
- So, for the $x = l$ side of a cube of length l , this would be

$$\vec{F}_{x=l} = (\iint_{[0,l] \times [0,l]} p_{fluid} [1 - \frac{z}{l}] dy dz) \hat{i}$$

Example: Hemisphere of size l , sitting at $(0, 0, 0)$

Finding the out pointing unit normal of hemisphere at point $(x, y, -\sqrt{l^2 - x^2 - y^2})$

Note: Can just eyeball this, but one way is the **gradient**.

First, the relation is $x^2 + y^2 + (z - l)^2 = l^2$. Make it a function g and take the level set at l^2 :

$$g(x, y, z) = x^2 + y^2 + (z - l)^2 = l^2 \quad (70)$$

$$\nabla g(x, y, z) = \langle 2x, 2y, 2(z - l) \rangle \quad (71)$$

$$\hat{n} = \pm \frac{\nabla g(x, y, z)}{\|\nabla g(x, y, z)\|} \quad (72)$$

$$\hat{n} = \pm \frac{\langle x, y, (z - l) \rangle}{\sqrt{x^2 + y^2 + (z - l)^2}} \quad (73)$$

$$\hat{n} = \pm \frac{\langle x, y, (z - l) \rangle}{\sqrt{l^2}} \quad (74)$$

$$\hat{n} = \pm \langle \frac{x}{l}, \frac{y}{l}, \frac{z}{l} - 1 \rangle \quad (75)$$

$$(76)$$

Note: Integrating over a patch dA on the surface means finding the area of micro-patches ΔA_{ij} , which is the parallelogram defined by

$$s_1 = \langle \Delta x_i, 0, \Delta x_i f_x(x_i^*, y_j^*) \rangle \quad (77)$$

$$s_2 = \langle 0, \Delta y_j, \Delta y_j f_y(x_i^*, y_j^*) \rangle \quad (78)$$

$$\Delta A_{ij} \approx \|s_1 \times s_2\| \quad (79)$$

$$= \sqrt{(1 + [f_x(x_i^*, y_j^*)]^2 + [f_y(x_i^*, y_j^*)]^2) \Delta x_i \Delta y_j} \quad (80)$$

$$(81)$$

So the total pressure ends up being $\vec{F}_{tot} = p_{fluid} \iint (p \cdot \hat{n}) dA$

$$= p_{fluid} \iint_{x^2+y^2 \leq l^2} [1 - \frac{f(x, y)}{l}] \hat{n} \sqrt{1 + [f_x]^2 + [f_y]^2} dx dy \quad (82)$$

$$f(x, y) = l - \sqrt{l^2 - x^2 - y^2} \quad (83)$$

$$\hat{n} = \langle \frac{x}{l}, \frac{y}{l}, \frac{f(x, y)}{l} - 1 \rangle \quad (84)$$

$$f_x = \frac{x}{\sqrt{l^2 - x^2 - y^2}}, f_y = \frac{y}{\sqrt{l^2 - x^2 - y^2}} \quad (85)$$

And for the only-nonzero component, \hat{k} , this simplifies after a lot of hand-math to $F_{tot} = -p_{fluid} (\iint_{x^2+y^2 \leq l^2} \sqrt{1 - (\frac{x^2+y^2}{l^2})} dx dy) \hat{k}$

Side Note during solving: $dx dy \rightarrow r dr d\theta$.

- TODO: This looks to be something to do with the determinant of the Jacobian matrix F_i/x_j .
- Intuitively, consider that a patch $dx \cdot dy$ is a slice of a big disk which has dimensions dr on the ray, $r d\theta$ on the arc.

10 3.2: Flux Part I

Main idea: Field lines are innumerable, so counting them through a surface makes no sense. Instead, we'll use **flux** to help us measure charge pushed through a surface per unit time.

Example: If charge q of mass m in a field of $\vec{E} = E_0 \hat{i}$ moves from origin along x towards R according to $\frac{d^2 x}{dt^2} = \frac{q}{m} E_0$, then solving the diff eq. means that $x = \frac{q}{2m} E_0 (\Delta t)^2 = R$. This means we're pushing all charges within $\frac{q}{2m} E_0 (\Delta t)^2$ to the left of the disk through it.

Then, if we're considering a cylinder of base area A , mass density δ , charge density ρ :

- Every test charge chunk ΔV within $\frac{\rho \Delta V}{2\delta \Delta V} E_0 (\Delta t)^2$ passes through. That's the height.
- Area is A , so total volume is $\frac{\rho (\Delta t)^2}{2\delta} E_0 A$
- Density of charge per volume is ρ , so total is $\frac{\rho^2 (\Delta t)^2}{2\delta} E_0 A$

Note: Tilting this forward from the z-axis by θ multiplies the cross-section area of the cylinder (now an ellipse) by $\cos(\theta)$. Can work out the ellipse volume, or just note that each “Riemann bar” orthogonal to x-axis just got squished by $\cos(\theta)$.

So we define **flux** as amount of charge through a closed surface. $\Phi = (\vec{E} \cdot \hat{n})A$ where \vec{E} is a constant field. (Units: joules/second/ m^2 , or watts/ m^2)

We can further note $(\vec{E} \cdot \hat{n}) = \|\vec{E}\| \cos(\theta)$ by last problem.

Example: Flux through an empty cube from the origin is necessarily 0 since every face cancels the other.

Another example: A square pyramid with top at $(0, 0, 1)$, sides at 1 on each axis:

- All the triangles will cancel in the x, y directions.
- A triangle $(1, 0, 0)(0, 1, 0), (0, 0, 1)$ has two displacement vectors $P_1 P_3 = P_3 - P_1 = (-1, 0, 1), P_2 P_3 = (0, -1, 1)$.
- $P_1 P_3 \times P_2 P_3 = (1, 1, 1) \rightarrow \hat{n} = \frac{(1, 1, 1)}{\sqrt{3}}$
- $A = \frac{1}{2} \|P_1 P_3 \times P_2 P_3\| = \frac{\sqrt{3}}{2}$
- $\Phi = (\vec{E} \cdot \hat{n})A = (E_0 \frac{1}{\sqrt{3}}) \frac{\sqrt{3}}{2} = \frac{E_0}{2}$
- So total flux through these is $4 \cdot \frac{1}{2} E_0 = 2E_0$
- However, the bottom has area $\sqrt{2}^2 = 2$ and flux E_0 , so total is 0!

11 3.3: Flux Part II

Note:

- Charge (q) is the volts of the point charge. Total charge Q_{tot} is total charge inside some surface.
- Electric field is sum of those point charges acting at a distance, and \vec{a} is a single vector.
- Flux looks like the integral of the electric field flowing through a surface.

- Total charge Q_{tot} of a surface is basically the sum of all the flux going out, except that it's that divided by some constant ϵ_0 .

Note: \vec{E} isn't usually constant, and the surface S is usually curved. So we need calculus to break up surface S into small pieces ΔA_i and evaluate \vec{E}_i there at that normal \hat{n}_i . So

$$\sum_{patches} = \sum_{patches} (\vec{E}_i \cdot \hat{n}_i) \Delta A_i = \iint_S (\vec{E} \cdot \hat{n}) \Delta A_i = \Phi$$

Easy Example: If, say, $(\vec{E} \cdot \hat{n}) = 1$ everywhere, we're just looking at $\iint_S dA$, or the total surface area.

Another example. Given:

- Real electric field law: $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3}$
- Real observation: Total electric flux through a surface (Φ) is proportional to total charge inside (Q_{tot}). $\Phi = \iint_S (\vec{E} \cdot \hat{n}) \Delta A \propto Q_{tot}$
- Then constant must be $\frac{1}{\epsilon_0}$. Why?
 - On unit sphere, $\hat{n} = \frac{\vec{x}}{\|\vec{x}\|}$
 - So $\vec{E} \cdot \hat{n} = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3} \cdot \hat{n}$
 - $= \frac{q}{4\pi\epsilon_0}$
 - Then $\Phi = \iint_S \frac{q}{4\pi\epsilon_0} dA$
 - $= \frac{q}{4\pi\epsilon_0} 4\pi$ by surface area of unit sphere
 - $= \frac{q}{\epsilon_0}$
- Therefore, because all of the field goes through the surface (no matter the shape), **Gauss's law** says $\iint_S (\vec{E} \cdot \vec{n}) dA = \frac{Q_{tot}}{\epsilon_0}$

Note: Because (UNEXPLAINED!) symmetry of a contained *ball* implies that, for distance ρ from origin, $\vec{E} = E(\rho)\hat{\rho}$, the above works the same for a point charge or a uniform (contained) ball.

Example: For a big radius R ball of charge Q containing a small ball of radius ρ with charge Q_{tot} , what must the charge $E(\rho)$ at any point be?

- Small charge Q_{tot} is proportional to volume of the big charge Q by $Q_{tot} = Q \frac{V_{small}}{V_{big}} = Q \frac{\rho^3}{R^3}$
- $\frac{Q_{tot}}{\epsilon_0} = \text{total charge} = \iint_S E(\rho)(\|\hat{\rho}\|) dA = E(\rho) \iint_S 1 dA = E(\rho) 4\pi\rho^2$

- So $\frac{Q_{tot}}{\epsilon_0} = Q \frac{\rho^3}{R^3 \epsilon_0} = E(\rho) 4\pi \rho^2$
- So $E(\rho) = \frac{Q}{4\pi \epsilon_0} \frac{\rho}{R^3}$

Example: Infinite wire, $x=y=0$, charge per length is λ . Use a cylinder.

- What's the total charge of the cylinder? Top and bottom are perpendicular to the field so can be ignored.
- There's some function $E(r)$ which, time \hat{r} , is the field by symmetry.
- $\Phi = \iint_{cylinder} (E(r) \cdot \hat{r}) dA = E(r) \iint_{cylinder} 1 dA = E(r) 2\pi r h.$
- $\frac{Q_{tot}}{\epsilon_0} = E(r) 2\pi r h \Rightarrow E(r) = \frac{\lambda}{2\pi \epsilon_0 r}$

Example: Infinite plane, $x=y=0$, charge per area is σ . Use a cylinder again

- What's the total charge of the cylinder? Side is perpendicular to the field so can be ignored. Looking at top and bottom, $\phi = 2EA + 2EA.$, where E is charge through the top.
- $2EA = \frac{\sigma A}{\epsilon_0} \rightarrow E = \frac{\sigma}{2\epsilon_0}$