

Polynomial Uniqueness via Tournaments

Dave Fetterman

Obviously Unemployed

2/10/23

Abstract

In 2D space, two points $(x_1, y_1), (x_2, y_2), x_1 \neq x_2$ define a line, a polynomial of degree 1. Three distinct points $(x_1, y_1), (x_2, y_2), (x_3, y_3), x_1 \neq x_2 \neq x_3 \neq x_1$ define a parabola, a polynomial of degree 2. In general, for finite univariate polynomials of nonnegative, whole degree, $n + 1$ such points uniquely specify a polynomial of degree n . Why?

This is not a new result. This is a paper is simply a thoroughly awkward trip through a few mathematical domains to arrive at a well known destination. Helicopters and cars both have their uses. But you wouldn't build a car by turning a helicopter on its side and adding wheels.

Metaphorically, I do, so you don't have to.

1 Setup

If we have points $f(x_0) = y_0, f(x_1) = y_1, \dots, f(x_n) = y_n$, how can we determine the coefficients a_i of the polynomial $f(x) = a_0x^0 + a_1x^1 + \dots + a_nx^n$?

This matrix X , known as a Vandermonde matrix[1], models this set of equations as $X \cdot \vec{a} = \vec{y}$:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Therefore, we can find our unique coefficient vector

\vec{a} if and only if we can solve $X \cdot \vec{a} = \vec{y}$, or $\vec{a} = X^{-1}\vec{y}$. This has a unique solution if and only if $\det(X) \neq 0$. The rest of this paper tries to find this determinant through all the wrong ways.

2 Finding the Vandermonde determinant

It should be noted that there are other, clearer methods of finding this determinant[1] either starting with polynomial uniqueness (basically, going the “other” direction), abstract algebra, direct linear algebra, vector maps, and likely others. These, however, were not the ones I stumbled on.

First, we know that if any $x_i = x_j$ for distinct i, j , we have no solution, and a zero determinant. If $f(x_i) = f(x_j), x_i = x_j$, then we are simply underdetermined (not enough points for a unique polynomial). If $f(x_i) = f(x_j), x_i \neq x_j$, then we have a impossible vertical section of our graph. Otherwise, we are in good shape.

This suggests that every pair $(x_i, x_j), i < j$ corresponds to a factor $(x_j - x_i)$ in the determinant, and that the determinant is then some multiple of $D = \prod_{0 \leq i < j \leq n} (x_j - x_i)$.

Taking $n = 2$ as a base case ($n = 1$ produces a constant $f(x)$), we see that $\det \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} = (x_1 - x_0)$, suggesting the determinant of a Vandermonde matrix is exactly D .

The rest of the paper will be handling the inductive step of proving $D = \prod_{0 \leq i < j \leq n} (x_j - x_i)$ in the most roundabout way possible. This requires that we prove this product $P_{[0,n]}$ equal to $S_{[0,n]}$, a yet-undefined sum of products generated by permutations of the nodes.

2.1 Setup: Vandermonde inductive step and main theorem

2.1.1 Definitions

Let's create a few definitions:

- Denote by $M_{n,k}$ the Vandermonde matrix X with row k and column n excluded as a “matrix minor”.
- Assuming an ordered set of indices $I = [0, n]$, denote by P_I the ordered products of the form $(x_j - x_i); i < j; i, j \in I$. So $P_{[0,2]} = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1)$.
- Assuming an ordered set of indices $I = [0, n]$, denote by S_I the sum of over σ , all terms of form $\text{sgn}(\sigma)x_{\sigma(n)}^n x_{\sigma(n-1)}^{n-1} \dots x_{\sigma(0)}^0$, with σ a permutation on the node set I . So $S_{[0,2]} = x_2^2 x_1^1 x_0^0 - x_2^2 x_0^1 x_1^0 - x_1^2 x_2^1 x_0^0 + x_1^2 x_0^1 x_2^0 + x_0^2 x_2^1 x_1^0 - x_0^2 x_1^1 x_2^0$

2.1.2 Vandermonde Determinant

The determinant of $X = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix}$ can be calculated down the rightmost column as

$$\det(X) = x_0^n \det(M_{n,0}) - x_1^n \det(M_{n,1}) + \dots + (-1)^n x_n^n \det(M_{n,n}).$$

By inductive hypothesis, each of the terms $x_k^n \det(M_{n,k})$ becomes:

$$x_k^n \prod_{0 \leq i < j \leq n; i \neq k, j \neq k} (x_j - x_i) \text{ or } x_k^n \prod_{0 \leq i < j \leq n; i, j \in I_{n-k}} (x_j - x_i)$$

Therefore, we need to prove that $0 \neq \det(X)$

$$= \sum_{k=0}^n (-1)^k x_k^n \det(M_{n,k}) \quad (1)$$

$$= \sum_{k=0}^n (-1)^k x_k^n \left[\prod_{0 \leq i < j \leq n; i, j \in [0, n] - \{k\}} (x_j - x_i) \right] \quad (2)$$

$$= \prod_{0 \leq i < j \leq n; i, j \in [0, n]} (x_j - x_i) \quad (3)$$

(1) is a determinant expansion. The *det* term equals the bracketed term of (2) by inductive hypothesis.

We seek to prove this main theorem:

Theorem: The expansion of (3) is exactly the sum of all possible terms of the form $\text{sgn}(\sigma) x_{\sigma(n-1)}^{n-1} x_{\sigma(n-2)}^{n-2} \dots x_{\sigma(0)}^0$ for some permutation σ on the node set $[0, n-1]$. Call this $S_{[0, n]}$. So, for example $S_{\{d, c, b, a\}}$, would be exactly all terms like $d^3 c^2 b^1 a^0$, $-c^3 d^2 b^1 a^0$ or $-c^3 a^2 d^1 b^0$.

If we have this theorem proven, then:

- For $n = 2$, the determinant of X_2 is $1 \cdot x_1 - 1 \cdot x_0 = (x_1^1 x_0^0 - x_0^1 x_1^0) = S_{[0, 1]}$
- By inductive hypothesis, the expansion of the bracketed term of (2), $S_{[0, n] - \{k\}}$ yields the same set of sums except each sum excludes all use of x_k .
- The sum of all terms $(-1)^k x_k^n S_{[0, n] - k}$ is exactly $S_{[0, n]}$, meaning (3).

- Therefore, (2) = (3) and we have our Vandermonde determinant (and thus our proof of polynomial uniqueness).

Assuming the theorem true, the above bullet sequence should be clear with some checking. On $\{c, b, a\}$, for example, the terms split out exactly into $c^2(b^1a^0 - b^0a^1) - b^2(c^1a^0 - a^1c^0) + a^2(c^1b^0 - b^1c^0) = c^2b^1a^0 - c^2a^0b^1 - b^2c^1a^0 + b^2a^1c^0 + a^2c^1b^0 - a^2b^1c^0$

2.2 Proof of the main theorem

Base Case: We've shown this is true for $n = 2 \Rightarrow S_{[0,1]} = (x_1^1x_0^0 - x_0^1x_1^0)$. So our inductive step supposes that all terms of (3) for ranges $[0, n-1]$ are of the form $sgn(\sigma)x_{\sigma(n-1)}^{n-1}x_{\sigma(n-2)}^{n-2}\dots x_{\sigma(0)}^0$ for some permutation σ on the node set $[0, n-1]$.

The proof that $S_{[0,n]} = \prod_{0 \leq i < j \leq n} (x_j - x_i)$ requires adding a new node x_{n+1} to the left side and a multiplying new set of factors $\prod_{0 \leq i < n} (x_{n+1} - x_i)$ by the right side and showing they are equal.

- Lemma 1: Show an isomorphism between products of the form (3) and tournament graphs on $n + 1$ nodes.
- Lemma 2: Show that terms of the form $sgn(\sigma)x_{\sigma(n)}^nx_{\sigma(n-1)}^{n-1}\dots x_{\sigma(0)}^0$ remain in (3) after expansion. These correspond to acyclic tournaments on $n + 1$ nodes.
- Lemma 3: Show that all other terms in the expansion of (3), which correspond to tournaments with a cycle, can be paired 1:1 with a identical but inverted term, corresponding to an identical graph with *one 3-cycle reversed*.
- Thus, the sum of the terms addressed

3 Prove : VanDerMonde matrix determinant is $\prod (x_i - x_j), 1 \leq i < j \leq n$

This is the determinant of the van der Monde matrix

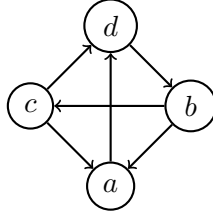
3.1 Base case: $n = 2$

3.2 Inductive case

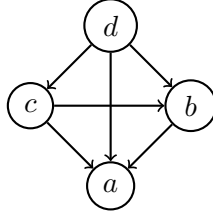
This equals x^n (product without x), $+y^n$ (product without y)...

4 Pieceyard

The sorted tournament $d^3c^2b^1a^0$

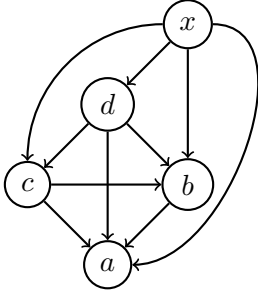


(a) An arbitrary tournament on 4 nodes

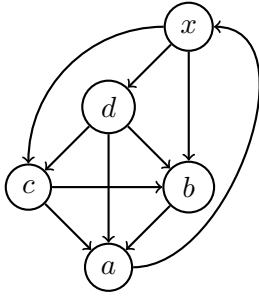


(b) An (acyclic) tournament $d^3c^2b^1a^0$

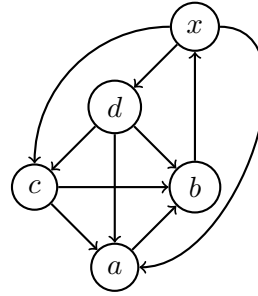
Figure 1: Tournaments



The sorted tournament $x^4d^3c^2b^1a^0$



(a) $-x^3a \cdot d^3c^2b^1a^0$, with cycle (xba)



(b) $-x^3b \cdot -d^3c^2a^1b^0$, with cycle (xab)

Figure 2: Terms in expanded $\prod(x_j - x_i)$ are inverses with inverted 3-cycles

Factors of $(x - a)(x - b)(x - c)(x - d)$ multiplied by $\sigma = d^3 c^2 b^1 a^0$

Factor	Product	Matching Factor	Matching σ	Critical pair
x^4	$x^4 d^3 c^2 b^1 a^0$	none	none	none
$-x^3 a$	$-x^3 d^3 c^2 b^1 a^1$	$-x^3 b$	$-d^3 c^2 a^1 b^0$	ba
$-x^3 b$	$-x^3 d^3 c^2 b^2 a^0$	$-x^3 c$	$-d^3 b^2 c^1 a^0$	cb
$-x^3 c$	$-x^3 d^3 c^3 b^1 a^0$	$-x^3 d$	$-c^3 d^2 b^1 a^0$	dc
$-x^3 d$	$-x^3 d^4 c^2 b^1 a^0$	none	none	none
$x^2 ba$	$x^2 d^3 c^2 b^2 a^1$	$x^2 ca$	$-d^3 b^2 c^1 a^0$	cb
$x^2 ca$	$x^2 d^3 c^3 b^1 a^1$	$x^2 da$	$-c^3 d^2 b^1 a^0$	dc
$x^2 da$	$x^2 d^4 c^2 b^1 a^1$	$x^2 db$	$-d^3 c^2 a^1 b^0$	ba
$x^2 cb$	$x^2 d^3 c^3 b^2 a^0$	$x^2 db$	$-c^3 d^2 b^1 a^0$	dc
$x^2 db$	$x^2 d^4 c^2 b^2 a^0$	$x^2 dc$	$-d^3 b^2 c^1 a^0$	dc
$x^2 dc$	$x^2 d^4 c^3 b^1 a^0$	none	none	none
$-xcba$	$-xd^3 c^3 b^2 a^1$	$-xdba$	$-c^3 d^2 b^1 a^0$	dc
$-xdba$	$-xd^4 c^2 b^2 a^1$	$-xcba$	$-d^3 b^2 c^1 a^0$	cb
$-xdca$	$-xd^4 c^3 b^1 a^1$	$-xdcb$	$-d^3 c^2 a^1 b^0$	ba
$-xdcb$	$-xd^4 c^3 b^2 a^0$	none	none	none
$dcba$	$d^4 c^3 b^2 a^1$	none	none	none

5 TODO

5.1 TODO

References

- [1] Wikipedia: https://en.wikipedia.org/wiki/Vandermonde_matrix