

Brilliant: Differential Equations II

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Note: Latex reference: <http://tug.ctan.org/info/undergradmath/undergradmath.pdf>

1 Chapter 1: Basics

1.1 Chapter 1: Nonlinear Equations

The two types of problems in this course are:

- Nonlinear equations (several equations on one independent variable)
- Partial differential equations (single equation with several independent variables)

Linear equations have solutions like y_1, y_2 that can be combined using any $c \in \mathbb{R}$ like $y_1 + cy_2$.

Example: Bacteria in a dish with a lot of food, no deaths

- $b'(t) = r_b b(t), r_b > 0$ would be the rate of growth.
- This is linear. Reason 1: $\frac{d}{dt}(y_1 + cy_2) = y_1' + cy_2' = r_b(y_1 + cy_2)$ since $y' = r_b y(t)$, and same for y_2 .
- Also, this works because the solution is $b(t) = b(0)e^{r_b t}$, so $b_1(t) + cb_2(t) = b_1(0)e^{r_b t} + cb_2(0)e^{r_b t} = (b_1(0) + cb_2(0))e^{r_b t}$

Example: Logistic equation: Bacteria in a dish with a lot of food, limited by carrying capacity M .

- $b'(t) = r_b b(t)[M - b(t)]$.
- This is nonlinear. Reason: $\frac{d}{dt}(y_1 + cy_2) = y_1' + cy_2' = r_b[y_1 + cy_2][M - y_1 - cy_2] = My_1 + Mcy_2 - y_1^2 - 2cy_1y_2 - cy_1^2y_2^2$
- $\neq My_1 - y_1^2 + Mcy_2 - c^2y_2^2$ because of the extra $-2cy_1y_2$ term.

Sidebar: Note that this equation $b' = r_b b[M - b]$ is *separable*, so it can be solved.

- $\frac{db}{dt} = rb[M - b]$
- $\frac{db}{b(M-b)} = rdt$
- $\frac{1}{M}(\frac{1}{b} + \frac{1}{M-b})db = rdt$ after partial fractions work
- $(\ln(b) - \ln(M - b)) = Mrt + C \Rightarrow \ln(\frac{b}{M-b}) = Mrt + C$
- $\frac{b}{M-b} = e^{Mrt} e^C$
- Initial conditions $b = b(0), t = 0 \Rightarrow \frac{b}{M-b} = \frac{b(0)}{M-b(0)} e^{Mrt}$
- $b(1 + \frac{b(0)}{M-b(0)} e^{Mrt}) = M \frac{b(0)}{M-b(0)} e^{Mrt}$
- $b(M - b(0) + b(0)e^{Mrt}) = Mb(0)e^{Mrt}$
- $b = \frac{Mb(0)e^{Mrt}}{M+b(0)[e^{Mrt}-1]}$

This logistic solution will taper off to M at some point. Note that $\lim_{t \rightarrow \infty} b(t) = M$ since the non-exponential terms stop mattering. Also $b(t) = M$ sticks as a constant solution or **equilibrium** immediately. *These equilibria tell us what matters - the long-term behavior of solutions!*

Another **Example**: Lotka-Volterra equation pairs: Bacteria (b) and bacteria-killing phages (p), with kill rate k .

- The “product” $kb(t)p(t)$ measures the interactions and kills resulting from this.
- $b'(t) = r_b b(t) - kp(t)b(t)$, or the normal growth rate minus kill rate
- $p'(t) = kp(t)b(t)$ since its population grows as it kills bacteria.
- Equilibria include $b = 0, p = 0$ and $b = 0, p > 0$, since these are *constant* solutions, or places where $b'(t) = 0, p'(t) = 0$.

Direction fields, with vector pointing towards $\langle b'(t), p'(t) \rangle$ (TODO - I think) let us follow the arrows to determine the curve over time. In this case, the bacteria will always go extinct.

However, if we add a new death rate term $-d_p p(t)$ so $p'(t) = -d_p p(t) + kp(t)b(t)$:

- We get an equilibrium at $b = \frac{d_p}{k}, p = \frac{r_b}{k}$. (Since $0 = b'(t) = r_b b - kpb, (\Rightarrow pk = r_b), 0 = p'(t) = -d_p p + kpb, (\Rightarrow bk = d_p)$)
- But otherwise the solutions swirl around this point. This is called a **cycle**. TODO What is a **limit cycle**?

Note that there are systems where the “solution particle” neither reaches an equilibrium or cycles around one point. The **Lorenz system** famously has this owl-eye shaped double attractor (an example of **strange sets**) where initially close particles diverge unpredictably if the constants ρ, σ, b are chosen right:

- $x'(t) = \sigma(y - x)$
- $y'(t) = x(\rho - z) - y$
- $z'(t) = xy - bz$
- TODO

1.2 Chapter 1.2: PDEs

Many methods of attack for PDEs

- Separation of variables
- Power series (Note: did we actually touch on this?)
- Fourier Transform

Example: Standing wave, where one end of a rope is fixed.

- Vertical displacement from a line of rope: $u(x, t)$ depends on where (x) and when (t).
- Rope’s **wave equation** is $u_{tt} = v^2 u_{xx}$, where v is the “constant wave speed”, and the others are the space, time partials.
- Note that $u = \cos(vt)\sin(x)$ and $u = \sin(vt)\cos(x)$ both work.
- If you guess the solution has split variables like $u = X(x)Y(y)T(t)$, then, upon substitution and division by $X(x)Y(y)T(t)$, $\frac{\delta^2 u}{\delta t^2} = v^2[\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}]$ yields $\frac{T''(t)}{T(t)} = v^2[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}]$
- This method may or may not work. But if it does, it means that since x, y , and t are independent variables, each individual piece must be constant.
- So, for example, if we know $\frac{X''(x)}{X(x)} = -4\pi^2$, we can get to $X(x) = \sin(2\pi x)$
- The wave equation is similar in 3D: $u_{tt} = v^2[u_{xx} + u_{yy} + u_{zz}]$, or using the Laplacian, $u_{tt} = v^2 \nabla^2 u$. Here, u measures not displacement but expansion/compression of air at (x, y, z) , time t .

Using Fourier transforms helps turn difficult PDEs into an easier problem like an ODE. *Fourier transforms work best when*

- The domain is all of \mathbb{R}^n
- The function u vanishes at infinity.

The Fourier transform changes the domain of x to that of ω . It comes with the (highly simplified) rule (see Vector Calculus course): $F[\frac{\delta f}{\delta x}] = i\omega F[f]$. **Example:** Drunkard's walk.

- One dimensional: moves left or right in a random way. Starts at $x = 0, t = 0$.
- $u(x, t)$ is probability of being at point x at time t . Naturally, $\int_{x=-\infty}^{x=\infty} u(x, t) dx = 1$.
- Also, it obeys the 1-dD diffusion equation $\frac{\delta u}{\delta t} = \frac{\delta^2 u}{\delta x^2}$
- The Fourier transform doesn't affect t at all.
- So by taking Fourier transform of both sides of diffusion equation we get
 - $F(u_t) = \frac{\delta}{\delta t} F(u)$ since F doesn't care about t .
 - $\frac{\delta^2 u}{\delta x^2} = i\omega F(\frac{\delta u}{\delta x}) = -\omega^2 F(u)$
 - So $\frac{\delta}{\delta t} F(u) = -\omega^2 F(u)$
 - This is solvable as $F(u) = ce^{-\omega^2 t}$. Take it on faith that $c = \frac{1}{2\pi}$ for now. TODO
 - Known fact: $F[Ae^{-\frac{ax^2}{2}}] = \sqrt{\frac{1}{2\pi a}} Ae^{-\frac{\omega^2}{2a}}$
 - This means $t = \frac{1}{2a}$ and $a = \frac{1}{2t}$
 - $F(u) = \frac{1}{2\pi} e^{-\omega^2 t}, F[Ae^{-\frac{ax^2}{2}}] = \sqrt{\frac{1}{2\pi a}} Ae^{-\frac{\omega^2}{2a}}$ so $u = Ae^{-\frac{ax^2}{2}}$
 - Solving, you get $A = \sqrt{\frac{1}{4\pi t}}, a = \frac{1}{2t}$, so $u(x, t) = \sqrt{\frac{1}{4\pi t}} e^{-\frac{x^2}{4t}}$

2 Chapter 2: Nonlinear Equations

2.1 2.1: Lotka-Volterra I

Major ideas:

- **phase plane:** TODO
- **nullcline:** TODO
- **direction field:** TODO
- **equilibria:** TODO

Example: Bacteria vs. phages (again)

- Bacteria unrestrained grow in proportion to their population, so $\frac{db}{dt} = r_b b(t)$ (solved: $b(t) = b(0)e^{r_b t}$)
- Phages unfed decrease in proportion to current size, so $\frac{dp}{dt} = -d_p p(t)$ (solved: $p(t) = p(0)e^{-d_p t}$)
- Bacteria die with likelihood of meeting a phage, and phages increase with likelihood of meeting a bacterium. So the set of equations, for constant k , becomes:
 - $b'(t) = r_b b(t) - kb(t)p(t)$
 - $p'(t) = -d_p p(t) + kb(t)p(t)$
 - *The product of p and b makes our equations nonlinear (WHY?)*
 - I guess, very generally, $b_1 p_1 = k, b_2 p_2 = k$, but $(b_1 + b_2)(p_1 + p_2) = b_1 p_1 + b_2 p_2 + b_1 p_2 + b_2 p_1 = 2k + b_1 p_2 + b_2 p_1 \neq 2k$, so the last two “mixed” terms mean you can’t just add solutions (b_1, p_1) and (b_2, p_2) .

General thoughts on this solution:

- So a solution $(b(t), p(t))$, traces out a curve on the bp-phase plane (b is x-axis, p is y-axis) as time (unrepresented in the plane) continues.
- If we add a unit tangent vector at every point (B, P) aligned with $(b'(t), p'(t)) = (r_b B - kBP, -d_p P + kBP)$, we can follow the arrows to see the solution over time.
- The above is called a **direction field**
- This is sometimes hard to sketch analytically, so we can look to the **nullclines**: places where one of the components of the direction field is zero.
- In this case, $r_b B - kBP = (r_b - kP)B = 0$ when $P = 0$ or $P = \frac{r_b}{k}$, and $-d_p P + kBP = (kB - d_p)P = 0$ when $P = 0$ or $B = \frac{d_p}{k}$.
- The **upshot of nullclines** (since we don’t care about $P, B \leq 0$): The lines $B = \frac{d_p}{k}, P = \frac{r_b}{k}$ divide the plane into pieces where the components of this (continuous) function pair can’t change sign.
- For instance, $B > \frac{d_p}{k}, P < \frac{r_b}{k}$ means $r_b b - kbp > 0, -d_p p + kdp > 0$, so both populations are growing here. This helps to sketch the curve.
- The curve looks like a counterclockwise whirlpool around the $(B, P) = (\frac{d_p}{k}, \frac{r_b}{k})$. (bacteria grow with low but growing phages; bacteria decrease as phages overwhelm; both decrease as phages starve; bacteria start coming back)
- The center point is a (constant **equilibrium**) solution, and other solutions swirl around it but don’t get attracted or repelled.

There are a few types of equilibria:

- This one is a **center** around which solutions circle.
- A **stable equilibrium** would see small upsets come back to an unchanging state.
- An **unstable equilibrium** would see small upsets create wildly divergent paths.

2.2 2.2: Lotka-Volterra II

In the Bacteria-Phage system, we can't yet prove everything rotates around the **center**. Let's do that.

Developing a **conserved quantity** will help to do that. **Example:** Block on a horizontal spring with mass m , spring constant k_s :

- $x(t)$: Displacement from rest position.
- $v(t) = \frac{dx}{dt}$: Horizontal velocity
- $\frac{dv}{dt} = -\frac{k_s}{m}x(t)$ by Hooke's law, I think.
- Suppose there's some Energy function $E(x, v)$. By chain rule $\frac{d}{dt}E(x(t), v(t)) = \frac{dE}{dx}\frac{dx}{dt} + \frac{dE}{dv}\frac{dv}{dt}$
- $= \frac{dE}{dx}v - \frac{k_s}{m}\frac{dE}{dv}x$. If we set E as conserved, as in $E'(t) = 0$, then $\frac{dE}{dx}v = \frac{k_s}{m}\frac{dE}{dv}x$
- We can eyeball and see that $E = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ solves this equation, or we can assume $E(x, v) = F(x) + G(v) \Rightarrow 0 = E'(t) = F'(x)v - \frac{k_s}{m}G'(v)x = 0$ from the above equations and guess from there.
- This means in the xv phase space, that there's a fixed E such that the particle follows the ellipse $E = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ in phase space around the solution point $(0,0)$.

Extended Example: Continuing on finding a conserved quantity for Bacteria / Phage:

- We need to find $U(b(t), p(t))$ such that $U'(t) = 0$, or by chain rule $\frac{\delta U}{\delta b}\frac{\delta b}{\delta t} + \frac{\delta U}{\delta p}\frac{\delta p}{\delta t} = 0$
- Subbing in, $\frac{\delta U}{\delta b}[r_b b - kbp] + \frac{\delta U}{\delta p}[-d_p p + kbp] = 0$
- A hint suggests finding U such that $\frac{\delta U}{\delta b} = -\frac{d_p}{b} + k$, $\frac{\delta U}{\delta p} = -\frac{r_b}{p} + k$ to make terms cancel.
- Integrating these gives us U as both $-d_p \ln(b) + kb + Q(p)$ and $-r_b \ln(p) + kp + R(b)$ so $U = -d_p \ln(b) - r_b \ln(p) + kb + kp$. This weird curve constitutes a level set in pb -space upon which a solution sits.
- The spring example has an elliptic paraboloid solution. There's an absolute minimum ($E = 0$ at $(0,0)$) but level sets become closed loops away from it.

- For the Lotka example, there is a critical point ($\nabla U = \vec{0}$) when $\nabla U(b, p) = (\frac{\delta U}{\delta b}, \frac{\delta U}{\delta p}) = (k - \frac{d_p}{b}, k - \frac{r_b}{p})$, which is $(0, 0)$ at our known center $(\frac{d_p}{k}, \frac{r_b}{k})$
- Showing that we always increase going away from the point $(\frac{d_p}{k}, \frac{r_b}{k})$ should guarantee us closed level sets.
- One method: Assume we're picking a unit vector $\vec{v} = \langle \hat{v}_b, \hat{v}_p \rangle$ so that our line from our center is $\vec{v} = \langle \frac{d_p}{k} + tv_b, \frac{r_b}{k} + tv_p \rangle$. $U = F(b) + G(p)$ in this case, so sub the b part into F to get $F(\frac{d_p}{k} + tv_b) = d_p[1 - \ln(\frac{d_p}{k} + tv_b)] + kt\vec{v}$. Taking derivative of that w.r.t t shows it is always positive. Same goes for the $G(p)$ portion of U .
- Another (DF) method: Note that $\nabla U = (k - \frac{d_p}{b}, k - \frac{r_b}{p})$'s grad (second derivative) is always positive. So derivative always has positive curvature (maybe using that term wrong), and we'll always increase around this point.
- Also, we know that the particle travels around the level set (loop) and doesn't reverse course, because then, $b'(t) = p'(t) = 0$, and we only have that at the center point (nullcline intersection).

2.3 2.3: Linearization

Extended Example: Suppose there's a limit to bacterial growth, so we cap our population at M_b .

- If $b(t) \ll M_b$, things should be similar. If $b(t)$ is nearly M_b , then growth should approach 0. So, this implies $\frac{db}{dt} = r_b b(t) \rightarrow \frac{db}{dt} = r_b b(t)(1 - \frac{b(t)}{M_b})$. Note: This isn't the only possibility but we'll use it.
- This updates our Lotka-Volterra model to something more complicated:
 - $b'(t) = r_b b(t)(1 - \frac{b(t)}{M_b}) - kb(t)p(t)$
 - $p'(t) = -d_p p(t) + kb(t)p(t)$
- Other than $b = 0, p = 0$, the meaningful nullclines are solved by setting $b'(t) = 0$ (yielding $r_b(1 - \frac{b}{M_b}) - kp = 0$) and $p'(t) = 0$ (yielding $b = \frac{d_p}{k}$)
- Note: We'll clean up through some MAGIC non-dimensionalization (how to derive?) to simplify:
 - $x(t) = \frac{1}{M_b} b(\frac{t}{r_b}), y(t) = \frac{k}{r_b} (\frac{t}{r_b}), \alpha = \frac{d_p}{r_b}, \beta = \frac{kM_b}{r_b}$
 - Gives us new equations: $\frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \frac{dy}{dt} = -\alpha y(t) + \beta x(t)y(t)$
 - And new nullclines: $x + y = 1, x = \frac{\alpha}{\beta}$

- So there's an equilibrium point in the positive xy quadrant if: $y = 1 - x = 1 - \frac{\alpha}{\beta}$ and $y > 0$ implies $1 - \frac{\alpha}{\beta} > 0 \Rightarrow \frac{\alpha}{\beta} < 1$
- Looking at the direction field, it appears solutions swirl around and are attracted *into* the center point $(\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta})$, making it a **stable equilibrium**

This is similar to the block-spring example, if a damping term $-\frac{\gamma}{m}v$ is added.

- $\frac{dx}{dt} = v, \frac{dv}{dt} = -\frac{k_s}{m}x - \frac{\gamma}{m}v$
- This can be thought of in matrix terms: $\frac{d}{dt} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{\gamma}{m} \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ Call the matrix A .
- From Diff Eq I, the solution is $\exp(tA)$ (matrix exponential), making $\mathbf{x}(t)$ a linear combination of $e^{\lambda t}$ or possibly $te^{\lambda t}$ terms, with the eigenvalues as λ s.
- The eigenvalues in this case, using the quadratic formula, could be:
 - Two real, distinct, negative roots. So, these $e^{\lambda t}$ terms decay, and $\mathbf{x}(t)$ levels off.
 - Two distinct complex roots with real part $-\frac{\gamma}{2m} < 0$. This ends up being some sines and cosines multiplied by $e^{-\frac{\gamma t}{2m}}$, which decays too.
 - Finally, if we have a repeated negative real eigenvalue, we have solution $x(t) = Ae^{-\frac{\gamma t}{2m}} + Bte^{-\frac{\gamma t}{2m}}$, also decaying.
 - So any disturbance in the spring will oscillate and come to rest at $x(t) = v(t) = 0$ quickly.

So with linear systems $\vec{x}'(t) = A\vec{x}(t)$, the eigenvalues determine what happens around the equilibrium point. However, the **bacteria-phage model is non-linear**. Here is **how we linearize** for nearby solutions in a nonlinear system:

- Set small disturbance $\delta x(t) \ll 1, \delta y(t) \ll 1$ so $x(t) = \frac{\alpha}{\beta} + \delta x(t), y(t) = 1 - \frac{\alpha}{\beta} + \delta y(t)$
- Since they're small, all powers like $\delta x(t)^2$ and $\delta x(t)\delta y(t)$ are considered basically zero.
- So substitute $x(t) \rightarrow \frac{\alpha}{\beta} + \delta x(t), y(t) \rightarrow 1 - \frac{\alpha}{\beta} + \delta y(t)$ into our $\frac{dx}{dt}$ and $\frac{dy}{dt}$ equations.
- This gives us the A solving $\frac{d}{dt} \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = A \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$, which is $A = \begin{pmatrix} -\frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \\ \beta - \alpha & 0 \end{pmatrix}$ after working through the substitution.

- Finding the eigenvalues here yields the same situation as the block-spring example: decays in all situations.

It turns out through the **Hartman-Grobman Theorem** that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$, for some continuously differential vector field F , if we linearize near equilibrium x_0 , then what falls out of this A approach works if the eigenvalues *aren't all purely imaginary*.

It turns out the uncapped bacteria system from before looks like $\frac{d}{dt} \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$, with characteristic equation $\lambda^2 + \alpha = 0, \alpha > 0$. This means both values are imaginary, and we had to use the conserved quantity approach!

2.4 2.4: Hartman-Grobman Theorem

Extended Example: Consider a phage that dies off quickly:

- $\frac{db}{dt} = r_b b(t) - k_b b(t)p(t), \frac{dp}{dt} = -r_p p(t) = 0 \cdot b(t)p(t)$, where k_p is the zero (phages don't increase), and k_b is still the kill factor for the bacteria.
- In this base, $b(t) = p(t) = 0$ is the only equilibrium.
- Non-dimensionalize as $x(t) = b(\frac{t}{r_b}), y(t) = \frac{k_b}{r_b} p(\frac{t}{r_b}), \alpha = \frac{r_p}{r_b}$
- This makes the equations $x'(t) = x(t) - x(t)y(t), y'(t) = -\alpha y(t)$, and the nullclines therefore $x(t) = 0, y(t) = 1, y(t) = 0$
- Looking at this six-section direction field, we see that solutions exactly on the y-axis are attracted to equilibrium $(0, 0)$, and other are repelled.
- This makes sense since if the bacteria is 0, the phage die and approach $(0, 0)$, otherwise the bacteria multiply and win (so it's a *saddle point*)
- The way to tell: linearize the equations. $\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ since, if $x(t), y(t) \ll 1, x(y)t(t) = 0$.
- Then the eigenvalues are $\lambda = 1, -\alpha$ so the solution is $Ae^t, Be^{-\alpha t}$ for $x(t), y(t)$ (TODO respectively?) **Hartman-Grobman ensures this is the general solution.**

However, let's solve directly and see if we come to the same result.

- $y'(t) = -\alpha y(t), y(0) = y_0 \Rightarrow y(t) = y_0 e^{-\alpha t}$
- With this in hand, $\frac{dx}{dt} = x(t) - x(t)y(t) = x(t)[1 - y_0 e^{-\alpha t}], x(0) = x_0$ separates out to
 - $\frac{dx}{x} = [1 - y_0 e^{-\alpha t}] dt$

- $\ln(x) = [t + \frac{y_0}{\alpha} e^{-\alpha t}] + C$
- $x = e^C e^t \exp(\frac{y_0}{\alpha} e^{-\alpha t})$
- $x(0) = x_0 \Rightarrow e^C = x_0 e^{-\frac{y_0}{\alpha}}$
- $\Rightarrow x(t) = x_0 e^t \exp(\frac{y_0}{\alpha} (e^{-\alpha t} - 1))$

But how do we deform the phase plane so this looks linear? We need some mapping $\vec{h}(x, y) = \langle u(x, y), v(x, y) \rangle$ that is continuous and invertible (so we don't "damage" the phase plane). This is called a **homeomorphism**.

- So near the equilibrium $(0, 0)$, the equations $y'(t) = -\alpha y(t), y(0) = y_0 \Rightarrow y(t) = y_0 e^{-\alpha t}$ linearized for $\delta x, \delta y$ must be similar to those for $u(x(t), y(t), v(x(t), y(t)))$
- This means we need $\frac{du}{dt} = u, \frac{dv}{dt} = -\alpha v$
- After doing the substitution, we see that $v = v_0 e^{-\alpha t}$ exactly mimics $y(t) = y_0 e^{-\alpha t}$ for the phage solution. So we take $v = y$.
- Therefore, we know that since $u = u_0 e^t$ and $x(ty) = x_0 \exp(t + \frac{y_0}{\alpha} (e^{-\alpha t} - 1))$, that we need $u(x(t), y(t)) = u(x_0, y_0) e^t$
- And this is satisfied if we guess $u(x, y) = x e^{-y} \alpha$ and work it out.
- This function $\vec{h}(x, y) = (u, v) = \langle x e^{-\frac{y}{\alpha}}, y \rangle$ is invertible by $(x, y) = \langle u e^{\frac{v}{\alpha}}, v \rangle$, which is continuous.

2.5 2.5: Application - Lasers

Lasers create excited atoms, which then emit photons while transitioning to an unexcited state. This system has a close analogue with the previous phages (like photons) and bacteria (like atoms) model.

- $n(t)$: number of photons in the laser; r_g : rate of photons gained (created by excited atoms transitioning to unexcited state); r_l : rate of photons lost (emitted)
- $\Rightarrow \frac{dn}{dt} = r_g - r_l$ by definition.
- We can assume we're losing a constant k (kill?) portion of photons per unit time, so $\frac{dn}{dt} = r_g - kn(t)$
- $e(t)$: number of excited atoms (that will maybe create photons). Atoms are excited by external energy pump.
- Excited atoms radiate when meeting a photon (which survives the meeting)
- So we can use the same setup from the bacteria: with I the constant of meeting (intersect?), $r_g = Ie(t)n(t) \Rightarrow n'(t) = Ie(t)n(t) - kn(t)$

Mini example: Assume no photons leave (cap the end of the laser)

- $k = 0$ in this scenario.
- So every meeting creates one more photon ($n \rightarrow n + 1$) while enervating one excited atom ($e \rightarrow e - 1$). This implies, equivalently:
 - $e + n$ is a conserved quantity,
 - $e(t) + n(t) = e(0) + n(0)$,
 - $[e(t) + n(t)]' = 0$
 - Then, if $k = 0$, $n'(t) = Ie(t)n(t) - kn(t)$, and coupled with $e'(t) + n'(t) = 0$ above, we have $e'(t) = -Ie(t)n(t)$

Extended example: Atoms spontaneously lose energy. This is actually what happens

- From quantum physics, we have a rate s of atoms just (s)pontaneously losing energy.
- We also have an energy (p)ump that energizes atoms with quantity p .
- Then, our change in (e)xcited atoms is $e'(t) = p - s - Ie(n)(t)$
- So our **final laser equations** are $e'(t) = p - s - Ie(n)(t)$, $n'(t) = Ie(t)n(t) - kn(t)$
- If we want to find the smallest p guaranteeing $n \geq 1$ (there's at least one photo) at equilibrium ($e'(t) = n'(t) = 0$):
 - $n'(t) = 0 \Rightarrow Ien = kn \Rightarrow n(Ie - k) = 0$. If $n \neq 0$, $\Rightarrow e = \frac{k}{I}$
 - $e'(t) = 0 \Rightarrow Ien = p - se$
 - Together, $p - se = Ien = kn \Rightarrow kn + se = p \Rightarrow kn + s\frac{k}{I} = p$
 - $n \geq 1 \Rightarrow p \leq k + \frac{ks}{I}$
 - **Another tactic:** We could also assume we *start out at equilibrium*, so n_0, e_0 are constant solutions.
 - Solving $n' = 0 = Ie_0n_0 - kn_0$, $e' = 0 = Ie_0n_0 - se_0 + p$, we find equilibria $n_0 = \frac{p}{k} - \frac{s}{I}$, $e_0 = \frac{k}{I}$
 - Then, $n_0 \geq 1 \Rightarrow \frac{p}{k} - \frac{s}{I} \geq 1 \Rightarrow p \geq k + \frac{ks}{I}$

Non-dimensionalization time:

- Scale against $e_0 (= \frac{k}{I})$, $n_0 (= \frac{p}{k} - \frac{s}{I})$ like this: $x(t) = \frac{n(\alpha t)}{n_0}$, $y(t) = \frac{e(\alpha t)}{e_0}$

- NOTE: What does this do? This makes (1,1) the equilibrium, as $x(t) = \frac{n_0}{e_0} = 1, y(t) = \frac{e_0}{e_0} = 1$!
- What α lets us take $n' = Ien - kn, e' = -Ien - se + p$ and write
 - $\frac{dx}{dt} = x(t)y(t) - x(t)$
 - $\frac{dy}{dt} = \frac{1}{k}(\frac{pI}{k} - s)[1 - x(t)y(t)] + \frac{s}{k}[1 - y(t)]$
 - $x' = \frac{\alpha n'(\alpha t)}{n_0} = xy - x = \frac{Ie(\alpha t)n(\alpha t)}{kn_0} - \frac{n(\alpha t)}{n_0}$
 - $\frac{\alpha Ien - \alpha kn(\alpha t)}{n_0} = \frac{Ie(\alpha t)n(\alpha t)}{kn_0} - \frac{n(\alpha t)}{n_0}$
 - $\alpha Ie - \alpha k = \frac{Ie(\alpha t)}{k} - 1 \Rightarrow \alpha(Ie - k) = \frac{Ie - k}{k} \Rightarrow \alpha = \frac{1}{k}$
 - This solves the x equation, and I suppose it can be validated in the y equation (tediously).
 - If we chunk up our (somehow positive?) constants as $c = \frac{1}{k}(\frac{pI}{k} - s), d = \frac{s}{k}$, we end up with $y' = c[1 - xy] + d[1 - y]$
 - We only care about $x, y > 0$, so $x' = 0 = xy - x = x(y - 1)$ implies $y = 1$ is a nullcline
 - $y' = 0 = c[1 - xy] + d[1 - y] = c - cxy + d - dy \Rightarrow c + d = y(d + cx) \Rightarrow y = \frac{c+d}{d+cx}$, a scaled and shifted hyperbola.

Look at the solutions:

- It appears we have a counterclockwise swirl around (1,1), and nearby solutions tend toward this equilibrium.
- Hartman-Grobman: rewrite our linearized solution in neighborhood of (1,1) as $x(t) = 1 + \delta x(t), y(t) = 1 + \delta y(t)$
- Using $x' = xy - x, y' = c[1 - xy] + d[1 - y]$ and $\frac{d}{dt} \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = A \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$, we can solve and write $A = \begin{pmatrix} 0 & 1 \\ -c & -c - d \end{pmatrix}$
- Eigenvalues: $\lambda = \frac{1}{2}(-c - d \pm \sqrt{(c + d)^2 - 4c})$
 - * If square root term is zero, we have repeated eigenvalue, so $\delta x(t), \delta y(t)$ are combos of $e^{-\frac{c+d}{2}}, te^{-\frac{c+d}{2}}$, which decays
 - * If square root term is greater than zero, we have two distinct real, negative eigenvalues (since c, d are positive), so this decays.

- * If square root term is less than zero, we have distinct complex eigenvalues, but combos of $e^{-\frac{c+d}{2}} \cos(\frac{1}{2}\sqrt{-(c+d)^2+4c})$, $e^{-\frac{c+d}{2}} \sin(\frac{1}{2}\sqrt{-(c+d)^2+4c})$ decay too
- * Note : I suppose Hartman-Grobman can't work in purely imaginary eigenvalue scenario, because these kinds of functions don't converge or diverge without a term outside the sin or cos
- * And in any case, since these lambdas aren't strictly imaginary, Hartman-Grobman works.

2.6 2.6: Liapunov Equations

We had some intuition that “nearby” solutions would fall into an equilibrium, but what does “nearby” mean? **Liapunov Equations** help us here. What is the “basin of attraction”?

- Suppose we turn the pump off ($p = 0$), and set spontaneous enervation equal to photon leak $s = k$.
- (TODO?) Somehow we can rescale to $\frac{dx}{dt} = Ie(t)n(t) - kn(t)$, $\frac{dy}{dt} = -Ie(t)n(t) - kn(t)$ which (TODO??) gives us $\frac{dx}{dt} = xy - x$, $\frac{dy}{dt} = xy - y$
- This means equilibria ($x' = y' = 0$) exist at $(0, 0)$, $(1, 1)$
- If we're turning the pump off, we're looking at equilibrium $(0, 0)$. Linearizing, we get $x' = -\delta x$, $y' = -\delta y$, so a matrix of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
- With repeated non-imaginary (H-G applies!) eigenvalues $-1, -1$, we can see that both e^{-t} , te^{-t} decay, and we get sucked into the origin.

But how do we prove this? Let's find a conserved quantity $U'(x(t), y(t)) = 0$

- $U'(x(t), y(t)) = \frac{\delta U}{\delta x} \frac{\delta x}{\delta t} + \frac{\delta U}{\delta y} \frac{\delta y}{\delta t} = \frac{\delta U}{\delta x} x(y - 1) + \frac{\delta U}{\delta y} y(x - 1)$
- Setting $\frac{\delta U}{\delta x} x = -x + 1$, $\frac{\delta U}{\delta y} y = y - 1$ makes this zero
- Solving those two by separating variables and combining, we get $U = -x + y + \ln(|\frac{x}{y}|)$
- So if we're stabilizing $f = (x - y)$ (why?), we see $(x - y)' = x' - y' = (xy - x) - (xy - y) = x - y = f \Rightarrow f = e^{-t}$
- With $x(0) = x_0$, $y(0) = y_0 \Rightarrow f(0) = x_0 - y_0$, $f = x(t) - y(t) = (x_0 - y_0)e^{-t}$
- How to express $y(t)$ while eliminating $x(t)$, knowing $x(y) - y(t) = (x_0 - y_0)e^{-t}$ and $U(x, y) = y - x + \ln(|\frac{x}{y}|)$ is conserved? **The trick:** $U(x_0, y_0) = U(x, y)$ since it doesn't change!

- $y_0 - x_0 + \ln(|\frac{x}{y}|) = y - x + \ln(|\frac{x}{y}|) = -(x_0 - y_0)e^{-t} + \ln(|\frac{x}{y}|)$
- $(1 - e^{-t})(y_0 - x_0) = \ln(\frac{x/y}{x_0/y_0})$
- Defining for convenience, $f = \exp((1 - e^{-t})(y_0 - x_0))$, then $f \frac{y}{y_0} = \frac{x}{x_0}$
- Sub in to $x - y = (x_0 - y_0)e^{-t} : y[\frac{x_0}{y_0}f - 1] = (x_0 - y_0)e^{-t}$
- Solve for $y : y = \frac{y_0(x_0 - y_0)e^{-t}}{x_0f(t) - y_0}$
- Combine with above to get $x = \frac{x_0(x_0 - y_0)e^{-t}f(t)}{x_0f(t) - y_0}$
- So with equilibria $(0, 0), (1, 1)$, the direction field computer plot shows us attracted to $(0, 0)$ (no laser action) pretty much anywhere left and down from $(1, 1)$ in the x, y phase plane.
- Apparently the linearized solutions near $0, 0$ are $x_{lin} = x_0e^{-t}, y_{lin} = y_0e^{-t}$ (WHY?)
- Looking above, if $(x_0 - y_0) \approx 0$, then $f(t) \approx 1$, and $x, y \rightarrow x_{lin}, y_{lin}$

On to **Liapunov** functions, which will tell us perhaps the size of the “basin of convergence”, unlike Hartman-Grobman, which just says there is a neighborhood.

A **Liapunov** function $U(x, y)$ is

- Continuously differentiable
- With a unique minimum (x_0, y_0) , usually aligned to be U ’s only zero.
- $U'(x(t), y(t)) \leq 0$. Everything “flows downhill”;
- Tailor made for the problem, hard to find.

Back to the rescaled laser example

- $x'(t) = x(t)y(t) - x(t)$
- $y'(t) = c[1 - x(t)y(t)] + d[1 - y(t)], c, d > 0$
 - **Analogy: The damped-block spring system** $\frac{d}{dt} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{\gamma}{m} \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$
 - When $\gamma = 0$, we know $E(x, v) = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ is conserved when looking at E'
 - $\gamma = 0 \Rightarrow x' = v, v' = -\frac{k_s}{m}x$
 - $\frac{dE}{dt} = (\frac{1}{2}k_s x^2 + \frac{1}{2}mv^2)' = 0$ since $\frac{1}{2}(k_s x x' + mv v') = \frac{1}{2}(k_s x v + mv \frac{-k}{m}x) = 0$
 - But if $\gamma \neq 0$, $\frac{d}{dt}E(x(t), v(t)) = \frac{d}{dt}[\frac{1}{2}k_s x^2 + \frac{1}{2}mv^2] = k_s x x' + mv v'$

- $= kx(v) + mv(\frac{-k_s}{m}x - \frac{\gamma}{m}v) = -\gamma v(t)^2 = \frac{dE}{dt}$
- Total spring energy is then decreasing in the fluid.
- Brilliant has Cool visualization of spiraling down into the "bowl" of x, y with E as the z dimension, equilibrium $(0, 0, 0)$
- We need to choose a γ -fied E -like function that decreases for pairs $\delta x(t), \delta y(t)$. We can choose, like E , some $u(\delta x, \delta y) = \frac{1}{2}C_1[\delta x]^2 + C_2[\delta y]^2$.
- Choosing $C_1 = c, C_2 = 1$ gives us $\frac{d}{dt}u(\delta x(t), \delta y(t)) = \frac{d}{dt}(\frac{c}{2}[\delta x(t)]^2 + \frac{1}{2}[\delta y(t)]^2)$
- $= c(dx)(dx)' + dy(dy)' = c(dx)(dy) + dy(-c(dx) - (c+d)(dy)) = -(c+d)[\delta y(t)]^2$
- So $u = \frac{d}{dt}(\frac{c}{2}[\delta x(t)]^2 + \frac{1}{2}[\delta y(t)]^2)$ is an energy function that could work for the laser.

Finally, we want to construct a function that

- Doesn't increase (derivative negative) on any pairs $x, y > 0$ (pulls down)
- Is near equal to $u = \frac{c}{2}(x-1)^2 + \frac{1}{2}(y-1)^2$ near $(1, 1)$. (the energy function for block-spring above)
- With $x' = xy - x, y' = c - cxy + d - dy$, plus the identity near $z \approx 1$ of $\ln(z) \approx (z-1) - \frac{1}{2}(z-1)^2 \dots$
- You can find $U(x, y) = c(x-1) + (y-1) - c\ln(x) - \ln(y)$ that satisfies all of these
- It therefore shows that pumped laser solutions tend to equilibrium $(1, 1)$ in the long term.

TODO: So this is enough to establish a convergence to an equilibrium?

- Find an equilibrium (x_0, y_0)
- Find an energy function u that decreases for all pairs $(\delta x(t), \delta y(t))$ near the minimum.
- Find a Liapunov function U function that decreases EVERYWHERE along $x(t), y(t)$ (in our domain, like $x, y > 0$)
- Ensure that $U = u$ in the neighborhood of the equilibrium.
- Then Liapunov's theorem somehow makes this work (TODO)?

2.7 2.7: Dog chasing a duck (Limit Cycles)

This is a pair of nonlinear equations to determine if a dog in the pond's interior catches a duck who skates along the border.

- Variables:
 - r_p : Radius of pond.
 - r_H : Distance of duck to center (always the radius of the pond)
 - \vec{l} : Displacement of dog from duck, which is of some length R at any point.
 - θ : Duck's position in the lake (think polar coordinates)
 - ϕ : Angle between r_H and \vec{l} .
 - Duck always swims at speed $r_p\theta'(t)$, and dog swims at $k > 0$ times this, or $kr_p\theta'(t)$.
- Therefore $r_H = \langle r_p \cos(\theta), r_p \sin(\theta) \rangle$. It's just the polar coordinates.
- Doing some geometry gets you $\vec{l} = R\langle \cos(\theta + \phi), \sin(\theta + \phi) \rangle$
- We can establish $\vec{T} = r_H - \vec{l}$ and dog's speed squared $\|T'(t)\|^2 = (r_H' - \vec{l}') \cdot (r_H' - \vec{l}') = \|r_H'\|^2 + \|\vec{l}'\|^2 - 2r_H'\vec{l}'$
- Naturally, this $\|T'\|^2$ is also equal to the constant $(kr_p\theta')^2$. Our diff equations will fall out of these.
- $r_H'^2 = r_p^2[\theta'(t)]^2$ since duck's speed is constant. $\vec{l}' = (R')^2 + R^2[\theta' + \phi']^2$ after working it out.
- Finally, after using identities $\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$, $\sin(\theta + \phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)$, we can work out $-2r_H'\vec{l}' = -2r_p\theta'[R\cos(\phi)(\theta' + \phi') + \sin(\phi)R']$
- After rescaling R to ρ such that $\frac{R}{r_p} = \rho$ and diving our speed equation by constant $r_p\theta'$, we end up with speed equation $k^2 = [\rho(1 + \frac{d\phi}{d\theta} - \cos(\phi))]^2 + (\frac{d\rho}{d\theta} - \sin(\phi))^2$
- We propose that there are some solutions here for the **pursuit equations**. We'll ignore the generalized form and focus on one set
 - $\rho(1 + \frac{d\phi}{d\theta}) - \cos(\phi) = 0$, $\frac{d\rho}{d\theta} - \sin(\phi) = -k$ do work in the above. (Doesn't prove others don't work)
 - This leaves our equations as $\frac{d\phi}{d\theta} = \frac{\cos(\phi)}{\rho} - 1$, $\frac{d\rho}{d\theta} = -k + \sin(\phi)$
 - However, there *aren't simple equilibria here*. In no world with $k \neq 0$ does the dog sit still (or the duck).

- Supposing $k < 1$ and R, ϕ are fixed (dog never gets closer and just loops), this means he's going in a circle, since the two legs of a triangle (\vec{l}, \vec{r}_p) and the interior angle (ϕ) are fixed, so this fixes length of the third leg, which is a radius
- You can also use dog's position vectors $x(t) = r_p \cos(\theta) - R \cos(\theta + \phi), y(t) = r_p \sin(\theta) - R \sin(\theta + \phi)$ and trig identities to prove $x(t)^2 + y(t)^2 = r_p^2 + R^2 - 2r_p R \cos(\phi)$
- If $k < 1$, then solving $\frac{d\rho}{d\theta} = 0 = -k + \sin(\phi) \Rightarrow \sin(\phi) = k \Rightarrow \phi = \sin^{-1}(k)$ and $\rho = \cos(\phi) = \cos(\sin^{-1}(k)) = \sqrt{1 - k^2}$
 - * Quick proof of $\cos(\sin^{-1}(k)) = \sqrt{1 - k^2}$:
 - * $\cos^2(\sin^{-1}(k)) + \sin^2(\sin^{-1}(k)) = 1 \Rightarrow \cos^2(\sin^{-1}(k)) = 1 - \sin^2(\sin^{-1}(k))$
 - * $= 1 - k^2 \Rightarrow \cos(\sin^{-1}(k)) = \sqrt{1 - k^2}$
- When $k < 1$, the direction field seems to have attractive equilibria but **GOTCHA**: there are ϕ angles that differ by 2π units, so they're the same. The direction field is a cylinder with circumference 2π , and there are other solutions tracking toward $(\sin^{-1}(k), \sqrt{1 - k^2})$
- Linearizing, assume we are near our equilibrium point and $\phi = \sin^{-1} k + \delta\phi, \rho = \sqrt{1 - k^2} + \delta\rho$.
- We can also remember that $f(x + \delta x) \approx f(x) + f'(x)\delta x$ from calculus.
- $\frac{d}{d\theta}[\delta\rho] = \frac{d}{d\theta}[\rho - \sqrt{1 - k^2}] = \frac{d\rho}{d\theta} - \frac{d}{d\theta}\sqrt{1 - k^2} = -k + \sin(\phi)$
- $= -k + \sin(\sin^{-1}(k) + \delta\phi)$ and by the calculus rule $\frac{d}{d\theta}[\delta\rho] = -k + \sin(\sin^{-1}(k)) + \cos(\sin^{-1}(k))\delta\phi = \sqrt{1 - k^2}\delta\phi$
- And for $\frac{d}{d\theta}[\delta\phi] = \frac{d}{d\theta}\phi - \frac{d}{d\theta}(\sin^{-1}(k)) = \frac{\cos(\phi)}{\rho} - 1$
- Using multivariable hint $f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y$,
- $f = \frac{\cos(\sin^{-1}(k) + \delta\phi)}{\sqrt{1 - k^2} + \delta\rho} - 1 \approx \frac{\sqrt{1 - k^2}}{\sqrt{1 - k^2}} - 1 + \frac{-\sin(\sin^{-1}(k))\delta\phi}{\sqrt{1 - k^2}} - \frac{\cos(\sin^{-1}(k))\delta\rho}{1 - k^2}$
- $= -\frac{k\delta\phi + \delta\rho}{\sqrt{1 - k^2}}$
- So $\frac{d}{d\theta} \begin{pmatrix} \delta\phi \\ \delta\rho \end{pmatrix} \begin{pmatrix} -\frac{k}{\sqrt{1 - k^2}} & -\frac{1}{\sqrt{1 - k^2}} \\ \sqrt{1 - k^2} & 0 \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta\rho \end{pmatrix}$, and the eigenvalues aren't purely imaginary, and the real part is negative, so all decay. Therefore, the equilibrium at $(\sin^{-1}(k), \sqrt{1 - k^2})$ attracts nearby solutions.

- There aren't solutions (for $K < 1$?), but numerically solved, the dog catches at $k > 1$, and for $k \leq 1$, swims out to a path approaching a circle. This is a **limit cycle**, an isolated trajectory that closes on itself.

2.8 Poincare-Bendixson Theorem

Limit cycles in the real world: a chemical reaction in perpetual oscillation!

Key concept - **trapping region**: a region in phase plane on some region D , with differential solutions touching every point, where the direction field sees every boundary arrow point IN. This means:

- The solution has to stay in D .
- Any solution that self-intersects forms a cycle in the phase plane.
- The three conceivable ways a solution can “snake” around forever (the **Poincare-Bendixson theorem** says it):
 - Approaches a closed loop in D .
 - Approaches a fixed point in D (possibly a special case of the last bullet)
 - Cycle: Snake eats its own tail
- A non-cycling solution is the only other possibility - just a point equilibrium.

Example: Chemical oscillatory reaction.

- x is concentration of I^- , y is concentration of ClO_2^- ions in some reaction.
- a is positive, and clearly $x, y \geq 0$ in the physical world.
- Otherwise meaningless equations: $\frac{dx}{dt} = 5a - x - \frac{4xy}{1+x^2}$, $\frac{dy}{dt} = x(\frac{4y}{1+x^2})$
- Solve for equilibria by setting $\frac{dx}{dt} = \frac{dy}{dt} = 0$
 - Denote $Q = \frac{y}{1+x^2}$
 - First equation implies $x(1 + 4Q) = 5a$
 - Second equation, plus knowing $x \neq 0$, $\Rightarrow x(1 - Q) = 0 \Rightarrow Q = 1$
 - $Q = 1 \Rightarrow 5x = 5a \Rightarrow x = a$
 - $\Rightarrow 1 = \frac{y}{1+x^2} \Rightarrow y = 1 + a^2$
 - Only solution pair is $(a, 1 + a^2)$

Linearizing the solution around $(a, 1 + a^2)$

- $x = a + \delta x, y = 1 + a^2 + \delta y \Rightarrow \frac{dx}{dt} = \frac{d[\delta x]}{dt}, \frac{dy}{dt} = \frac{d[\delta y]}{dt}$
- Call $f = \frac{d[\delta x]}{dt} = 5a - x - \frac{4xy}{1+x^2}$,
- Approximate $f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\delta f}{\delta x} \delta x + \frac{\delta f}{\delta y} \delta y$
- $f(x, y)(a, 1 + a^2) = 5a - x - \frac{4xy}{1+x^2}(a, 1 + a^2) = 5a - a - 4(a \frac{1+a^2}{1+a^2}) = 0$
- $\frac{\delta f}{\delta x} \delta x(a, 1 + a^2) = (-1 - \frac{(1+x^2)(4y-2x4xy)}{(1+x^2)^2})\delta x(a, 1 + a^2) = (-1 - 4 - \frac{8a^2}{1+a^2})\delta x = \frac{-5+3a^2}{1+a^2} \delta x$
- $\frac{\delta f}{\delta y} \delta y(a, 1 + a^2) = \frac{-4x}{1+x^2} \delta y(a, 1 + a^2) = \frac{-4a}{1+a^2} \delta y$
- Call $g = \frac{d[\delta y]}{dt} = x - \frac{xy}{1+x^2}$
- Approximate $g(x + \delta x, y + \delta y) \approx g(x, y) + \frac{\delta g}{\delta x} \delta x + \frac{\delta g}{\delta y} \delta y$
- $g(x, y)(a, 1 + a^2) = x - \frac{xy}{1+x^2}(a, 1 + a^2) = a - a \frac{1+a^2}{1+a^2} = 0$
- $\frac{\delta g}{\delta x} \delta x(a, 1 + a^2) = (1 - \frac{(1+x^2)y-xy2x}{(1+x^2)^2})\delta x = (1 - \frac{(1+a^2)^2-2a^2(1+a^2)}{(1+a^2)^2})\delta x = 2a^2 \delta x$
- $\frac{\delta g}{\delta y} \delta y(a, 1 + a^2) = \frac{-x}{1+x^2} \delta y = \frac{-a}{1+a^2} \delta y$
- $\Rightarrow \frac{d}{dt} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \frac{1}{1+a^2} \begin{pmatrix} 3a^2 - 5 & -4a \\ 2a^2 & -a \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$
- Let's arbitrarily choose $a = 2 \Rightarrow (a, 1 + a^2) = (2, 5)$. The coefficient matrix ends up being $\frac{1}{5} \begin{pmatrix} 7 & -8 \\ 8 & -2 \end{pmatrix}$, which has eigenvalues with a positive real \pm some i component. So, Hartman-Grobman applies and we don't decay into our point but push away.

We want to **build the trapping region**.

- Remember, $\frac{dx}{dt} = 10 - x - \frac{4xy}{1+x^2}$, $\frac{dy}{dt} = x(1 - \frac{y}{1+x^2})$ subbing in 2 for a)
- On the left, if $x = 0$ we see $\frac{dx}{dt} = 10$, $\frac{dy}{dt} = 0$. So we're pointing right (into the first quadrant region)
- On the bottom, if $y = 0$, we're pointing at $\langle 10 - x, x \rangle$ (into the region).
- On the right, for some $x = b$, $10 - b - \frac{4b}{1+b^2}y$ will make sure we point left.
- On the top, for some $y = c$, $x(1 - \frac{c}{1+x^2}) < 0$ makes sure we point down.
- Assume, since we're encircling $(2, 5)$, that $b \geq 3, c \geq 6$ for comfort.
- To satisfy all of these, note $x(1 - \frac{c}{1+x^2}) < 0 \Rightarrow 1 - \frac{c}{1+x^2} < 0 \Rightarrow c > 1 + x^2, 0 < x < b \Rightarrow c > 1 + b^2 \Rightarrow \sqrt{c-1} > b$

- And for $0 < y < c$, note that $10 - x - \frac{4xy}{1+x^2} < 10 - b < 0$.
- Pick $b = 11$, say, implying $11 < \sqrt{c-1}$, so then $123 < c$. So $(b, c) = (11, 124)$ ensures oscillation around $(2, 5)$ without leaving that region.

Tricky: How to reduce this region? No real way except simulation or some tricks. If we PRESUME a cycle, we can prove the cycle extends to the left of $x = 3$ or $x_{min} < 3$

- **META trick:** Don't worry if you have unsolvable integrals - maybe you can cancel them out. **Run with what you have.**
- Trick: Assume $x(t+T) = x(t), y(t+T) = y(t)$ for some $T > 0$, or that there's a PERIOD T .
- $\int_0^T \frac{dx}{dt} dt = x(T) - x(0) = 0, \int_0^T \frac{dy}{dt} dt = y(T) - y(0) = 0$ by fundamental theorem.
- Our equations again: $\frac{dx}{dt} = 10 - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(1 - \frac{y}{1+x^2})$
- So $0 = \int_0^T [10 - \int x(t) - 4 \int \frac{x(t)y(t)}{1+x(t)^2}] dt$ by the first equation
- $0 = \int x(t) - \int \frac{x(t)y(t)}{1+x(t)^2} dt$ by the second.
- Subtract four times the second from the first to get $0 = 10T - 5 \int_0^T x(t) \Rightarrow 2T = \int_0^T x(t) dt \geq \int_0^T x_{min} dt = Tx_{min}$
- So $2 \geq x_{min}$

2.9 Chaos and the Lorenz Equation

What enabled mathematical **chaos** (unpredictability in nonlinear differential equations) was really computers and seeing simulated solutions.

The (simplified) **Lorenz system** are these equations

- $\frac{dx}{dt} = \sigma(y - x)$
- $\frac{dy}{dt} = x(\rho - z) - y$
- $\frac{dz}{dt} = xy - bz$
- All with $\sigma, \rho, b > 0$

Solving the equations, we see equilibria for these are:

- $(0, 0)$ always
- The two solutions $(\pm \sqrt{b(\rho-1)}, \pm \sqrt{b(\rho-1)}, \rho-1)$ when $\rho > 1$.

Looking at $0 < \rho < 1$ specifically:

- Linearizing is simple, with $x(t) = \delta x(t), y(t) = \delta y(t), z(t) = \delta z(t)$ and linearized system:
- $\frac{d[\delta x]}{dt} = \sigma(\delta y - \delta x)$
- $\frac{d[\delta y]}{dt} = \rho\delta x - \delta y$
- $\frac{d[\delta z]}{dt} = -b\delta z$
- $\frac{d}{dt} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} \approx \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$
- Characteristic equation is $(-b - \lambda)[(1 + \lambda)(\sigma + \lambda) - \sigma\rho] = 0$
- Eigenvalues are $-b < 0$ and $\lambda = \frac{1}{2}[-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4\sigma(1 - \rho)}]$
- If $\rho < 1$, we have distinct, real, negative eigenvalues, and a locally attractive equilibrium by Hartman-Grobman.

But if $\rho < 1$ globally attractive? Find a Liapunov function.

- Requirement is that the function $U(x(t), y(t), z(t))$ is minimized at the equilibrium, and that as time progresses, U decreases (so we're sucked into the bowl)
- We suppose that $U(x, y, z) = ax^2 + y^2 + z^2$ and using $xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$:
 - Identity derivation: $0 \leq (x - y)^2 \Rightarrow 0 \leq x^2 - 2xy + y^2 \Rightarrow xy \leq \frac{1}{2}(x^2 + y^2)$
- $\frac{\delta U}{\delta x} x'(t) + \frac{\delta U}{\delta y} y'(t) + \frac{\delta U}{\delta z} z'(t) = 2a\sigma\sigma(y - x) + 2yx(\rho - z) - 2y^2x + 2zxy - 2bz^2$
- $= 2(a\sigma + \rho)xy - 2a\sigma x^2 - 2y^2 - 2bz^2$
- **GOTHCA:** We can't choose $a = -\frac{\rho}{\sigma}$ since then $U = -\frac{\rho}{\sigma}x^2 + y^2 + z^2$ isn't minimized at $(0, 0, 0)$! So a needs to be positive.
- Choosing $a = \frac{1}{\sigma} \Rightarrow a\sigma = 1$, with $\rho < 1 \Rightarrow 2(a\sigma + \rho)xy - 2a\sigma x^2 - 2y^2 - 2bz^2 < 2a\sigma(2xy - x^2 - y^2) - 2bz^2 \leq -2bz^2$ by the identity above.
- Then $U = \frac{1}{\sigma}x^2 + y^2 + z^2$ decreases as $t \rightarrow \infty$ and is minimized at the globally attractive $(0, 0, 0)$

If $\rho > 1$ things get chaotic. Instead of one equilibrium, we have two new ones at $(\pm\sqrt{b(\rho - 1)}, \pm\sqrt{b(\rho - 1)}, \rho - 1)$. Everything **bifurcates**, or qualitatively shifts when inching past $\rho = 1$:

- We have three equilibria.
- The origin turns into a saddle equilibrium.

- Linearizing around $(\alpha, \alpha, \rho - 1)$ with α denoting $\sqrt{b(\rho - 1)}$ (pretty straightforward), we get characteristic equation for A of $-\lambda^3 - (\sigma + b + 1)\lambda^2 - b(\sigma + \rho)\lambda - 2\sigma b(\rho - 1) = 0$
- Problem is, setting $\rho = 1$ drops the $(\rho - 1)$ term and we have $-\lambda(\lambda^2 + (\sigma + 1 + b)\lambda + b(\sigma + 1)) = 0$, with solutions $\lambda = 0, -b, -\sigma - 1$.
- The last two solutions are attractive, but the zero doesn't work for Hartman-Grobman.
- If we set $\lambda = (\rho - 1)\Delta r$ when nudging ρ just over 1, we ignore all $\lambda^2, \lambda^3 \dots$ as negligible and get $-b(\sigma + \rho)(\rho - 1)\Delta r - 2\rho b(\rho - 1) \approx 0$
- This means $\Delta r \approx -\frac{2\rho}{\rho + \sigma}$, or that this nudged root has to be negative when ρ is near 1.
- More rigorously, we could have proven the roots of the equation are negative for small $\rho - 1 > 0$
- In any case, this means that the near-zero root is negative, so $(\alpha, \alpha, \rho - 1)$ attracts locally.
- We can show that this applies the same for $(-\alpha, -\alpha, \rho - 1)$

How do equilibria change as we change ρ ?

- We saw the What about as we dial past $\rho = 1$, our origin equilibrium changes from globally attractive to saddle point.
- In going from a stable equilibrium with negative real-part eigenvalues (attractors) to $(0, 0, 0)$ as a saddle (mix of negative and positive real parts), we necessarily have a point where the eigenvalues' real parts are zero.
- In other words, $\lambda = ia$ for some real a .
- Subbing ia into our $-\lambda^3 - (\sigma + b + 1)\lambda^2 - b(\sigma + \rho)\lambda - 2\sigma b(\rho - 1) = 0$, we end up getting $[(\sigma + b + 1)a^2 - 2\sigma b(\rho - 1)] + i[a^3 - (b(\sigma + \rho)a)] = 0$
- Then we need $a^3 - b(\sigma + \rho)a = 0 \Rightarrow a = 0, a = \pm \sqrt{b(\sigma + \rho)}$
- If $a = 0$. the real part isn't zero. But subbing $a = \pm \sqrt{b(\sigma + \rho)}$ gives us solutions for a set of $\rho = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$
- So, when moving past this value, our two new equilibria change from locally attractive to saddles too.

Can we create a trapping region?

- The hint: The solutions have to pass through every ellipsoid of form $R^2 = \rho x^2 + \sigma y^2 + \sigma(z - 2p)^2$

- What we need to prove: At every point on the boundary, the direction field points “in”, or more specifically, *the angle between inward normal and direction field is acute.*
- This also means that the gradient ∇g of the level set $R^2 = \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2$ is the normal. This is $\langle 2\rho, 2\sigma, 2(z - 2\rho) \rangle$
- So $-\nabla g \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle > 0 \Rightarrow \dots \Rightarrow 2\rho\sigma x^2 + 2\sigma y^2 + 2\sigma z^2 - 4\rho\sigma z > 0$
- Use $R^2 - \rho x^2 - \sigma(z - 2\rho)^2 = \sigma y^2 \Rightarrow \rho\sigma x^2 + 2\sigma y^2 + 2\sigma z^2 - 4\rho\sigma z > 0$
- This simplifies the dot product to $2R^2 - 8\rho^2\sigma + 4\sigma\rho z + 2\rho(\sigma - 1)x^2 > 0$
- Since the x^2 term is always positive, we just need to set R to clear zero when z is its most negative ($x = 0, y = 0, z = 2\rho - \frac{R}{\sqrt{\sigma}}$). If we churn a little more we can see that setting $R > 2\sqrt{\sigma}\rho$ will provide a trapping region.

Question: Do the confined solutions fill up the whole (ellipsoid) container?

- Looking at the divergence (volume change of a cube over time) of the solution will tell us.
- For unspecified reasons, $\frac{1}{v(t)}v'(t) = \nabla \cdot \langle \sigma(y - x), x(\rho - z) - y, xy - bz \rangle = -\sigma - 1 - b$
- Solving, $v(t) = v(0)e^{-t(\sigma+1+b)}$
- This means the volume decays to 0, so therefore, our line is confined to a smaller and smaller space (but not just a point, I guess?)

3 Partial Differential Equations

3.1 1D Wave Equation and D’Lambert’s Formula

General set up: A rope with a fixed right end (boundary condition and a moving left end), moving up and down.

Start out with special case: no boundary condition (infinite rope, pulse in the middle)

- $u(x, t)$ measures the vertical displacement from the x-axis of the rope at point x , time t
- Physical observation gives us the PDE rule $\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 u}{\delta t^2}$ (or $u_{xx} = u_{tt}$)
- $g(x) = u(x, 0)$ is the initial shape of the rope.
- It’s assumed that the rope is not moving initially, so $u_t(x, 0) = 0$

Beginning to solve this:

- $u_{tt} = u_{xx} \Rightarrow u_{tt} - u_{xx} = 0$
- Sort of like $a^2 - b^2 = 0 \Rightarrow (a + b)(a - b) = 0$, we have $0 = (\frac{\delta}{\delta t} \pm \frac{\delta}{\delta x})(u_t \mp u_x) = u_{tt} - u_{xt} + u_{tx} - u_{xx} = u_{tt} - u_{xx}$
- This means the solution is either $u_+ = u_t + u_x$ or $u_- = u_t - u_x$. Note - we don't solve these simultaneously, since that just gives us $u(x, t) = 0$.
- These can be written as, e.g. $0 = u_t + u_x = \langle 1, 1 \rangle \cdot \langle u_x, u_t \rangle = \langle 1, 1 \rangle \cdot \nabla u$
- TRICK: This is a directional derivative along $\langle 1, 1 \rangle$. Introducing a variable like s (accelerant along $\langle 1, 1 \rangle$?) below does nothing interesting:
 - $\frac{d}{ds}[u(x+sb, t+sc)] = \frac{\delta u}{\delta x}(x+sb, t+sc)b + \frac{\delta u}{\delta t}(x+sb, t+sc)c = \langle b, c \rangle \cdot \nabla u(x+sb, t+sc)$
 - So if we set $b = c = 1$, we see that $\frac{d}{ds}[u(x+s, t+s)] = \langle 1, 1 \rangle \cdot \nabla u(x+s, t+s)$
 - However, since in our world, $u_x + u_t = 0$, then this dot product is zero, and $\frac{d}{ds}u = 0$. This is then *constant in s*.
 - So then $u(x, t) = u(x+s, t+s)$, and shifting x forward by s (seconds?) and t by the same changes nothing. *Interpretation: $u(x, t) = u(x+s, t+s)$ means that s (“shift”) seconds later, the point $x+s$ will see the same displacement as x . The wave goes “right” down the line.*
 - From this, we see that $u_+(x, t) = u_+(x-t, 0)$ as well. So, our function at t is what happened t seconds ago at the origin.
- Note: We can't have one solution satisfy both conditions $u_+ = g(x)$, $(u_+)_t = 0$, since then $g'(x) = 0$ which only works if g is a constant.
- Also, $u_{tt} - u_{xx} = 0$ is a linear PDE, in that solutions $u_1(x, t), u_2(x, t)$ see that $\frac{\delta^2}{\delta t^2}[c_1 u_1(x, t) + c_2 u_2(x, t)] - \frac{\delta^2}{\delta x^2}[c_1 u_1(x, t) + c_2 u_2(x, t)] = 0$. Multiply by a constant or add solutions together and it's still zero.
- If we set $t = 0$, we get $u(x, t) = c_+ g(x+t) + c_- g(x-t) \Rightarrow u(x, 0) = c_+ g(x+0) + c_- g(x-0) = (c_+ + c_-)g(x) = u(x, 0)$, so $(c_+ + c_-) = 1$
- Differentiating by t , $u_t(x, t) = c_+ g'(x+t) - c_- g'(x-t)$ so $u_t(x, 0) = (c_+ - c_-)g'(x) \Rightarrow (c_+ - c_-) = 0$. So $c_+ = c_- = \frac{1}{2}$, and **our solution with initial shape $g(x)$ with $g'(x) = u_t(x, 0) = 0$ is $u = \frac{1}{2}g(x+t) - \frac{1}{2}g(x-t)$**
- **This no-initial-velocity wave function translates** into “my displacement at time 3, say, is the average of the initial displacements 3 to my left and 3 to my right” (as those urges meet at “me” 3 seconds from the start). Conceptually, along the fixed initial curve $g(x)$, each point sends out two sensors, one left, one right, and averages

the initial values at those points to find itself at time t . So the top of a hill will start dipping down, becoming two hills pushing out, for example.

With inverted conditions $u(x, 0) = 0, u_t(x, 0) = f(x)$, we can use the fact that $u(x, t)$ solving the wave equation implies u_t solves it as well!

- $\frac{\delta^2 u}{\delta t^2} - \frac{\delta^2 u}{\delta x^2} = 0 \Rightarrow \frac{\delta}{\delta t} [\frac{\delta^2 u}{\delta t^2} - \frac{\delta^2 u}{\delta x^2}] = 0 \Rightarrow \frac{\delta^2 u_t}{\delta t^2} - \frac{\delta^2 u_t}{\delta x^2} = 0.$
- Therefore, $u_t(x, 0) = f(x)$ admits the same solution $u_t(x, t) = \frac{1}{2}[f(x+t) - f(x-t)]$
- Since $u(x, t) - u(x, 0) = \int \frac{1}{2}[f(x+t) - f(x-t)]dt$, and $u(x, 0) = 0$ by assumption in this setup, $u(x, t) = \int_{s=0}^{s=t} [f(x+s) - f(x-s)]ds$, which is $\frac{1}{2} \int f(s)$ from $x-t$ to $x+t$
 $= \frac{1}{2} \int_{s=x-t}^{s=x+t} f(s)ds$

And because the region of the integral for a point x gets wider as $t \rightarrow \infty$, on a flat rope with a pulse in the middle at $x = 0$, we see $u(x, t)$ sitting at 0 until the wave meets it, at which point it rises and then stays at the peak (integral of the whole thing).

So **d'Alembert's formula** is the superposition of the initially flat wave with the initially still wave, which accomodates *all* solutions:

$$u(x, t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2} \int_{s=x-t}^{s=x+t} f(s)ds$$

For the case of no boundary conditions, this solves $u_{tt} = u_{xx}$ for $u(x, 0) = g(x), u_t(x, 0) = f(t)$. In this instance, the propagation speed is clearly finite.

Note: This complete of a PDE solution is unusual.

3.2 Sources and Boundary conditions

Scenario 1: Here, we fix the infinite rope at the origin, with the wave coming in from the negative x-axis.

Looking at **boundary conditions**, or constraints on spatial edges of a PDE problem:

- A free boundary (a loop that can shift up and down a pole) will cause a reflected wave to travel backwards.
- A fixed boundary (setting $u(0, t) = 0, t \geq 0$) will cause an inverted pulse backwards.

We set up a function $\tilde{u}(x, t) = \{u(x, t), x \leq 0; = -u(-x, t), x \geq 0\}$ using **extension by odd reflection**. So an inverted ghost rope exists to the right of the origin.

Note: This seems to be more about cleverly encoding a boundary behavior (we will invert our wave) with this ghost rope than proving we'll have that behavior with math.

- And if $u_t(x, 0) = 0, g(0) = 0, u(x, 0) = g(x)$ extended to $x > 0$ as $\tilde{g}(x)$, then d'Alembert's applies: $\tilde{u}(x, t) = \frac{1}{2}[\tilde{g}(x+t) + \tilde{g}(x-t)]$

- So when $x \leq 0, x \leq t \Rightarrow -t \leq x \leq 0$: (meaning, negative x , close enough to the origin to be affected by time t)
 - $\tilde{u}(x, t) = u(x, t)$ here, since there's no inversion on the left side.
 - $(x + t)$ is positive, so $\tilde{g}(x + t) = -g(-(x + t))$ by definition of \tilde{u} .
 - $(x - t)$ is negative, so $\tilde{g}(x - t) = g(x - t)$ by definition of \tilde{u} .
 - So d'Alembert's reduces to $u(x, t) = \frac{1}{2}[-g(-(x + t)) + g(x - t)]$. This means *I'm the average of the starting position to my left t seconds ago, and the inverted right-of-origin ghost position to my right t seconds ago*

This means that for the part of the rope we care about, $x \leq 0$:

- For $x \leq -t$ (parts of the line unaffected by the reflection so far), $u(x, t) = \frac{1}{2}[g(x + t) + g(x - t)]$
- For $x \geq -t$ (parts affected by the reflection) $u(x, t) = \frac{1}{2}[-g(-(x + t)) + g(x - t)]$
- The intuition, still hard to visualize: if I'm zero at point -10, and wave crests at -11, then
 - First my left sensor will eat the left wave and I'll go up and over.
 - Then much later my right sensor will eat the right shadow wave and I'll do the inverted behavior.
 - These in total mean I'll get a reflection.
 - For the intuition, keep moving my point closer to the origin - nothing changes.

Scenario 2: Here, we let the rope slide up and down at the origin, but bound the total energy:

- One hand on the rope at $x = -L$, very far away:
- Our energy is the sum of kinetic (change in u based on time?) and elastic (change in u based on x ?) energies.
- $E = \int_{x=-L}^{x=0} [(\frac{\delta u}{\delta t})^2 + (\frac{\delta u}{\delta x})^2] dx$
- We can't gain or lose energy. This means $\frac{dE}{dt} = 0$. Solving that:
 - $0 = \frac{d}{dt} (\frac{1}{2} \int_{x=-L}^{x=0} [(\frac{\delta u}{\delta t})^2 + (\frac{\delta u}{\delta x})^2] dx) = \frac{1}{2} \int_{x=-L}^{x=0} [\frac{\delta}{\delta t} (\frac{\delta u}{\delta t})^2 + \frac{\delta}{\delta t} (\frac{\delta u}{\delta x})^2] dx$
 - $= \int_{x=-L}^{x=0} [u_t u_{tt} + u_x \frac{\delta u_t}{\delta x}] dx$.
 - (Do integration by parts on the second term with $U = u_x, dV = \frac{\delta u_t}{\delta x}$): $0 = \int_{x=-L}^{x=0} [u_t u_{tt}] dx + u_x u_t - \int_{x=-L}^{x=0} [u_t u_{xx}] dx = \int_{x=-L}^{x=0} u_t [u_{tt} - u_{xx}] dx + u_x u_t$

- Since $u_{tt} - u_{xx} = 0$ (REMEMBER YOUR PROBLEM-SPECIFIC IDENTITIES!), $u_x(0, t)u_t(0, t) = 0$
- Saying the displacement can't change with respect to t there gives us the fixed rope case above, so that's uninteresting.
- Therefore, if there's no energy change as the rope vibrates, we know $u_x(0, t) = 0$

Note: Dirichlet conditions are constraints on the value of the function at the boundary (like $u(0, t) = 0$). Neumann constraints are on the derivatives at the boundary.

So redoing d'Alembert with the energy conservation, and therefore the "Neumann" condition $u_x(0, t) = 0$:

- We know if u solves $u_{tt} - u_{xx} = 0$, then u_x does too, since $0 = u_{tt} - u_{xx} \Rightarrow 0 = \frac{d}{dx}[u_{tt} - u_{xx}] = [[u_x]_{tt} - [u_x]_{xx}] = 0$.
- We know $u_x(0, t) = 0$ by given constraints, so then we enforce this through odd reflection on u_x as well: $\tilde{u}_x = \{u_x(x, t), x \leq 0; -u_x(-x, t), x \geq 0\}$
- By D'Alembert, this solves the wave equation with $u_x(x, 0) = g'(x)$, so $\tilde{u}_x = \frac{1}{2}[\tilde{g}'(x+t) + \tilde{g}'(x-t)]$
- Therefore at $-t \leq x \leq 0$, $\tilde{u}_x(x, t) = \frac{1}{2}[\tilde{g}'(x+t) + \tilde{g}'(x-t)] = \frac{1}{2}[-g'(-x-t) + g'(x-t)]$
- Then integrating, we drop the minus sign in the first term! $u(x, t) = \frac{1}{2}[g(-x-t) + g(x-t)] + C$. Note that $u(x, 0) = \frac{1}{2}[g(x) + g'(x)] \Rightarrow C = 0$!

(Note: a nonzero initial velocity profile $u_t(x, 0) = f(x)$ can be handled as well. We skip it).

Remember the 1D springs Wave Equation, where springs are initially l apart, have displacement from this measured by $u(x, t)$, have Hooke's coefficient k ?

- Force pushing from the left on ball x : $F_L = k[u(x-l, t) - u(x, t)]$
- Force pushing from the right on ball x : $F_R = k[u(x+l, t) - u(x, t)]$
- Additional "source" force $F(x, t)$ means total force $F_{tot} = F_L(x, t) + F_R(x, t) + F(x, t)$
- $F_{tot} = ma = mu_{tt}$
- The Taylor-ish formula $f(x + \delta x) \approx f(x) + f'(x)(\delta x) + f''(x)(\delta x)^2$ means $F_L + F_R \approx kl^2 f''(x) = kl^2 u_{xx}$
- Therefore, $mu_{tt} = kl^2 u_{xx} + F(x, t) \Rightarrow F(x, t) = u_{tt} - \frac{kl^2}{m} u_{xx}$. Set $1 = v = \frac{kl^2}{m}$, $f(x, t) = \frac{F(x, t)}{m}$ to get a simplified all-purpose wave equation. $f(x, t) = u_{tt} - u_{xx}$, with f as the source force-per-unit-mass.

New setup: Source force $f(x, t)$, ignore boundary conditions, and set $u(x, 0) = 0, u_t(x, 0) = 0$ (still, flat (infinite) rope).

- Part 1: We can relate $f(x, t)$ to a made-up intermediate function $I(x, t)$ which has properties motivated by $u_{tt} - u_{xx} = (\frac{\delta}{\delta t} - \frac{\delta}{\delta x})(\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$
- $I(x, t) = (\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$
- $u_{tt} - u_{xx} = f(x, t) \Rightarrow (\frac{\delta I}{\delta t} - \frac{\delta I}{\delta x}) = f(x, t)$
- We can derive that $u(x, 0) = 0, u_t(x, 0) = 0$ means that at $x = 0, I(x, 0) = u_t(x, 0) + u_x(x, 0) = u_x(x, 0)$
- Since $u(x, 0) = 0$ and $I(x, 0) = u_x(x, 0), I(x, 0) = 0$.

We can relate $f(x, t)$ and $I(x, t)$:

- Use the dummy variable trick, and look at $f(x - s, t + s)$. We know also that $\frac{\delta I}{\delta t} - \frac{\delta I}{\delta x} = f(x, t)$
- $f(x - s, t + s) = \frac{\delta I}{\delta t}(x - s, t + s) - \frac{\delta I}{\delta x}(x - s, t + s) = \frac{d}{ds}[I(x - s, t + s)]$ by chain rule.
- Integrating both sides: $\int_{s=-t}^{s=0} f(x - s, t + s) ds = I(x, t) - I(x + t, 0) = I(x, t)$
- We can rewrite, using $k = -s$, as $I(x, t) = \int_{k=t}^{k=0} f(x + k, t - k) d(-k) = \int_{s=0}^t f(x + s, t - s) ds$

Using the same technique, we can relate $I(x, t)$ and $u(x, t)$ since $I(x, t) = (\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$

- Use the dummy variable trick with variable s' and look at $f(x - s', t - s')$. We know also that $\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x} = I(x, t)$
- $I(x - s', t - s') = \frac{\delta u}{\delta t}(x - s', t - s') - \frac{\delta u}{\delta x}(x - s', t - s') = \frac{d}{ds'}[u(x - s', t - s)']$ by chain rule.
- Integrating both sides: $\int_{s'=-t}^{s'=0} f(x - s', t - s') ds' = u(x, t) - I(x - t, 0) = u(x, t)$
- We can rewrite, using $j = -s'$, as $u(x, t) = \int_{j=t}^{j=0} f(x - j, t - j) d(-j) = \int_{j=0}^t I(x - j, t - j) dj$
- So, $u(x, t) = \int_{s'=t}^{s'=0} f(x - s', t - s') ds' = \int_{s'=0}^t I(x - s', t - s') ds'$

Combining these, $u(x, t) = \int_{s'=0}^t I(x - s', t - s') ds'$, and $I(x, t) = \int_{s=0}^t f(x + s, t - s) ds$:

- $I(x - s', t - s') = \int_{s=0}^t f(x + s - s', t - s - s') ds$:
- So $u(x, t) = \int_{s'=0}^t \int_{s=0}^t f(x + s - s', t - s - s') ds ds'$

- Change of variables, $y = x + s - s', w = s' + s \Rightarrow u(x, t) = \frac{1}{2} \int_{w=0}^t \int_{y=x-w}^{y=x+w} f(y, t-w) dy dw$
- TODO: The $\frac{1}{2}$ term apparently comes from the Jacobian (TODO) $\left\| \frac{\delta(s', s)}{\delta(w, y)} \right\|$
- This together means that the *points that can influence* $u(x, t)$ in the xt -plane are a triangle with (x, t) as the top, reaching down to $t = 0$, slope 1. So the “wave speed” in this setup is 1.

3.3 2D and 3D (Compression) Waves

(Note: The 2D equation will fall out of the 3D one).

Major setup for 3D compression waves:

- Air molecules compress together from sound, so $u(x, y, z, t)$ measures the density of air at that point.
- Let's assume $g(x \pm t)$ plays the same role as last time: the initial wave state. (Note: The setup implies we're looking at waves that propagate at “one x per t”.)
- Sound can come from multiple directions, so the expanded version should look like $u(\vec{x}, t) = g(\hat{n} \cdot \vec{x} \pm t)$, with \hat{n} some fixed direction in \mathbb{R}^3 .
- The equation is $u_{tt} = u_{xx} + u_{yy} + u_{zz}$ since:
 - Setting $\hat{n} = \hat{i}$ or any other basis vector, $g(\hat{i} \cdot \vec{x} \pm t) = g(x \pm t)$, which means if the other dims are zero, then $u_{tt} = u_{xx}$ (reduce to 1D case). That checks out (necessary, not sufficient)
 - $\frac{\delta}{\delta t} [g(\hat{n} \cdot \vec{x} \pm t)] = \pm g'(\hat{n} \cdot \vec{x} \pm t)$, same for g'' and $\frac{\delta^2}{\delta t^2}$
 - $\frac{\delta^2}{\delta x^2} [g(\hat{n} \cdot \vec{x} \pm t)] = \hat{n}_x^2 g''(\hat{n} \cdot \vec{x} \pm t)$, same for y, z
 - Since $[\hat{n}_x^2 + \hat{n}_y^2 + \hat{n}_z^2 = 1]$, $u_{tt} = u_{xx} + u_{yy} + u_{zz}$ works out.
 - This can also be written $u_{tt} - \nabla^2 u$

Setup with a forcing function:

- $u_{tt} - \nabla^2 u = f(\vec{x}, t)$, $u(\vec{x}, 0) = 0$, $u_t(\vec{x}, 0) = 0$.
- So with a still, blank initial state f is going to be a POP at the origin for a brief time.
- Taking our experience from actual sound, we expect it to decrease away from the origin, and for there to be a finite propagation speed.
- It should also be spherically symmetric.

Switching to spherical coordinates (u depends on r, θ, ϕ), and using the multivariable chain rule from vector calculus, we get

- $\nabla^2 u = \frac{1}{r^2} \frac{\delta}{\delta r} [r^2 \frac{\delta u}{\delta r}] + \frac{1}{r^2 \sin^2(\phi)} \frac{\delta^2 u}{\delta \theta^2} + \frac{1}{r^2 \sin(\phi)} \frac{\delta}{\delta \phi} [\sin(\phi) \frac{\delta u}{\delta \phi}]$
- If we're taking this to be spherically symmetric, then we can zero out ϕ, θ terms: $\nabla^2 u = \frac{1}{r^2} \frac{\delta}{\delta r} [r^2 \frac{\delta u}{\delta r}]$
- Expanding this out, this means that $u_{tt} - \frac{1}{r^2} \frac{\delta}{\delta r} [r^2 \frac{\delta u}{\delta r}] = u_{tt} - \frac{2}{r} u_r - u_{rr} = f(r, t)$
- If we set $U = ru$, and churn through with e.g. $u_r = \frac{\delta}{\delta r} [\frac{U}{r}] = -\frac{U}{r^2} + \frac{U_r}{r}$, etc., we end up with $U_{tt} - U_{rr} = rf(r, t)$. Note that $\frac{U}{r} = u$ means that the solution diminishes u with distance.
- So now we're solving with $U(r, 0) = U_t(r, 0) = 0$ since $U = ru$.
- Though only $r \geq 0$ matters, we need to keep $U(0, t)$ at zero through odd reflection. Note that if f is even, $rf(r, t)$ is odd.
- The result from last quiz implies: $U(x, t) = \frac{1}{2} \int_{s=0}^t \int_{\rho=r-s}^{\rho=r+s} \rho f(\rho, t-s) d\rho ds$, with s subbing for dummy w and ρ being the distance instead of y . Note that our function is really ρf now instead of f .
- Building a “Dirac delta snap” for a symmetric pop at the origin, set $f(r, t) = \frac{1}{\epsilon} e^{-\pi^2 \frac{r^2}{\epsilon^2}} \chi(t)$ for tiny ϵ
- Define $\delta(\rho) = \frac{\exp(-\frac{\pi^2 \rho^2}{\epsilon^2})}{\epsilon^2}$, change $s' = t - s$
- Eventually the math reduces to a delta pop at $r - t + s'$ (in range) and $r + t - s'$ (outside the interval)
- The math reduces to $\int_{s'=0}^{s'=t} \chi(s') \delta(r - t + s') ds'$, or just $\chi(t - r)$ (due to the integral of δ being one exactly at $t - r$).
- Therefore, $U(r, t) = \left\{ \frac{\epsilon^3}{(2\pi)^2} \chi(t - r), t - r > 0; 0, t - r \leq 0 \right\}$
- Looking at this, we confirm that disturbance diminishes with distance, and has a finite propagation speed.
- *TODO: So I guess χ is the actual initial wave function of time at the origin? The delta was I suppose there to “center” it?*

What if we don't have spherical symmetry?

- in the general case, all points \vec{x} influence fixed point \vec{P} through
 - Distance separating points $r = \|\vec{x} - \vec{P}\|$

- Normalized direction $\frac{\vec{x}-\vec{P}}{\|\vec{x}-\vec{P}\|}$
- However, we can *average* u over all points r away: $U(r, t; P) = \frac{1}{4\pi r^2} \iint_{S(\vec{P}, r)} u(\vec{x}, t) d\sigma(\vec{x})$, with S being the r -sphere around P
- Getting the r -partials requires writing each \vec{x} as some $\vec{P} + r\hat{n}$ over all directions, and using the divergence theorem.
- LOTS OF ALGEBRA IN HERE to get $U_{tt} - U_{rr} - \frac{2}{r}U_r = F(r, t)$ with a dependence on r, t, \vec{P} . This F is an even function, and it approaches $u(x, t)$ as r approaches zero.
- Our equation ends up being $u(\vec{P}, t) = \frac{1}{4\pi} \iiint_{B(\vec{P}, t)} \frac{f(\vec{y}, t - \|\vec{y}-\vec{P}\|)}{\|\vec{y}-\vec{P}\|} d\vec{y}$
- Like the other case, we see that the points in space affecting U are a “4D cone” with vertex at (\vec{P}, t)
- Also, if we consider that f doesn’t depend on z , we can flatten this spherical integral to a 2D one by looking at columns of z over the disc of radius t
- This ends up being $u(\vec{P}, t) = \frac{1}{2\pi} \iint_{B_2(\vec{P}, t)} f(\vec{y}) \ln\left(\sqrt{\left(\frac{t}{\|\vec{y}-\vec{P}\|}\right)^2 - 1} + \frac{t}{\|\vec{y}-\vec{P}\|}\right) d\vec{y}$
- This **method of descent** is really just “reducing” our 2D case from a 3D one.
- Also, the **spherical averages** let us reduce a 3D problem to a 1D problem, given the assumptions of the problem.

3.4 2D waves (boundary constrained): Separation of Variables

Main Setup:

- Rectangular drumhead from $[0, 0]$ to $[w, l]$
- Vertical (z) displacement is $u(x, y, t)$, with Dirichlet condition $u(x, y, t) = 0$ enforced on the boundary.
- Known that $u_{tt} = u_{xx} + u_{yy}$.

Solving for u by guessing that there’s a split solution $u(x, y, t) = S(x, y)T(t)$.

- $u_{tt} = u_{xx} + u_{yy}$.
- So $S(x, y)T''(t) = \frac{d^2 S(x, y)}{dx^2} T(t) + \frac{d^2 S(x, y)}{dy^2} T(t)$
- $\Rightarrow \frac{T''(t)}{T(t)} = \frac{\nabla^2 S(x, y)}{S(x, y)}$
- **Big A-ha:** left hand side is a function of t , and right hand of x, y . If they are to be equal, they must both be constant

To solve for T :

- We know $\frac{T''(t)}{T(t)}$ is a constant, so equate it to $-k^2$ for some constant k .
- Supposing the solution $T(t) = e^{rt} \Rightarrow T'' = r^2 e^{rt}$, we know $r^2 = -k^2 \Rightarrow r = \pm ik$.
- The solution $T(t) = Me^{ikt} + Ne^{-ikt}$ is equivalent to $T(t) = A \cos(kt) + B \sin(kt)$ since you can express \sin, \cos as linear combos of $e^{ikt} = \cos(kt) + i \sin(kt)$ and $e^{-ikt} = \cos(kt) - i \sin(kt)$

To solve for X, Y :

- $\frac{\nabla^2 S(x,y)}{S(x,y)} = -k^2$ by its equality with $\frac{T''(t)}{T(t)}$.
- Suppose we can formulate a solution so that $S(x,y) = X(x)Y(y) \Rightarrow -k^2 X(x)Y(y) = Y(y)X''(x) + Y''(y)X(x) \Rightarrow -k^2 = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}$.
- So, for some j , $\frac{X''(x)}{X(x)} = -j^2 \Rightarrow X(x) = C \cos(jx) + D \sin(jx)$ by the T solution above.
- However, $X(x) = 0$ and $X \neq 0$ (not a constant function) forces us to conclude $C = 0 \Rightarrow X(x) = D \sin(jx)$.
- Also, the boundary condition $X(w) = 0$ also means for every $j = \frac{n\pi}{w}$, $n \in \mathbb{N}$, $X(x) = D \sin(\frac{\pi n}{w}x)$
- Following this identical logic for Y over length l , $Y(y) = F \sin(\frac{\pi m}{l}y)$
- Considering that $-k^2 = -j^2 - q^2$, this means $k = \sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}$
- Then, $u = T(t)X(x)Y(y)$
- $\Rightarrow u = (A_{mn} \cos(\sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}) + B_{mn} \sin(\sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}))(\sin(\frac{\pi n}{w}x))(\sin(\frac{\pi m}{l}x))$
- This runs over all $m, n \in \mathbb{N}$

Turning to a **circular membrane** with radius r_0 , displacement described by $z = u(r, \theta, t)$:

- Boundary condition is then Dirichlet condition $u(r_0, \theta, t) = 0$
- $u_{tt} = \nabla^2 u \Rightarrow u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$
- (Note: I suppose this magic (forgotten from vector calculus) is because $u_{tt} = u_{xx} + u_{yy}$, and we're converting between x, y and r, θ)
- $T(t) = A \sin(kt) + B \cos(kt)$ by identical logic to the rectangular drum, where $\frac{T''(t)}{T(t)}$ also was a constant.

- Assuming similarly that $S(r, \theta) = R(r)\Theta(\theta)$, you end up with $R(r)\Theta(\theta)[\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2] = 0$
- Finally, if $\frac{\Theta''}{\Theta} = \kappa$ for $\kappa \in \mathbb{R}$, then $\Theta(\theta) = Ce^{\sqrt{\kappa}\theta} + De^{-\sqrt{\kappa}\theta}$. H
- However, we have an additional condition that we have to be able to rotate the whole scene by $\theta = m2\pi$ radians and have it remain the same, or $\Theta(\theta + 2\pi) = \Theta(\theta)$. This means $\kappa < 0$ since we're in "imaginary exponents yielding sign an cosine" territory.
- This implies $\kappa = -m^2$ for an integer m , and therefore, $\Theta(\theta) = C \cos(m\theta) + D \sin(m\theta)$.
- We know also that $T(t) = A \cos(kt) + B \sin(kt)$
- Setting $\frac{\Theta''}{\Theta} = \kappa = -m^2$, and with $R(r)\Theta(\theta)[\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2] = 0 \Rightarrow r^2 R'' + rR' + [k^2 r^2 - m^2]R$ (a solution we'll look for later), we have the circular drum solution.

Notes to self:

- Need a clear intuition for a lot of things. What do the variables and their derivatives physically mean?
- Need more symbolic comfort with how integration works.
- TODO