Brilliant: Differential Equations II

Dave Fetterman

9/2/22

Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 1: Basics

1.1 Chapter 1: Nonlinear Equations

The two types of problems in this course are:

- Nonlinear equations (several equations on one independent variable)
- Partial differential equations (single equation with several independent variables)

Linear equations have solutions like y_1, y_2 that can be combined using any $c \in \mathbb{R}$ like $y_1 + cy_2$.

Example: Bacteria in a dish with a lot of food, no deaths

- $b'(t) = r_b b(t), r_b > 0.r_b$ would be the rate of growth.
- This is linear. Reason 1: $\frac{d}{dt}(y_1 + cy_2) = y_1' + cy_2' = r_b(y_1 + c_y 2)$ since $y' = r_b y(t)$, and same for y2.
- Also, this works because the solution is $b(t) = b(0)e^{r_b t}$, so $b_1(t) + cb_2(t) = b_1(0)e^{r_b t} + cb_2(0)e^{r_b t} = (b_1(0) + cb_2(0))e^{r_b t}$

Example: **Logistic** equation: Bacteria in a dish with a lot of food, limited by carrying capacity M.

- $b'(t) = r_b b(t)[M - b(t)]$.
- This is nonlinear. Reason: $\frac{d}{dt}(y_1' + cy_2') = y_1' + cy_2' = r_b[y_1 + cy_2][M - y_1 - cy_2] = My_1 + Mcy_2 - y_1^2 - 2cy_1 y_2 - cy_1^2 y_2^2$
- $\neq My_1 - y_1^2 + Mcy_2 - c^2 y_2^2$ because of the extra $-2cy_1y_2$ term.

Sidebar: Note that this equation $b' = r_b b[M - b]$ is *separable*, so it can be solved.

- $\frac{db}{dt} = rb[M-b]$
- $\frac{db}{b(M-b)} = rdt$
- $\frac{1}{M}(\frac{1}{b} + \frac{1}{M-b})db = rdt$ after partial fractions work
- $\frac{1}{M}(\ln(b) - \ln(M-b)) = rt + C \Rightarrow \ln(\frac{b}{M-b}) = Mrt + CM$
- $\frac{b}{M-b} = e^{Mrt}e^{CM} \Rightarrow b(1 + e^{Mrt}e^{CM}) = Me^{Mrt}e^{CM}$
- $b = \frac{Mb(0)e^{Mrt}}{M+b(0)[e^{Mrt}-1]}$ after more manipulation

This logistic solution will taper off to M at some point. Note that $\lim_{t\to\infty} b(t) = M$ since the non-exponential terms stop mattering. Also $b(t) = M$ sticks as a constant solution or **equilibrium** immediately. *These equilibria tell us what matters - the long-term behavior of solutions!*

Another **Example**: Lotka-Volterra equation pairs: Bacteria (b) and bacteria-killing phages (p), with kill rate k.

- The "product" $kb(t)p(t)$ measures the interactions and kills resulting from this.
- $b'(t) = r_b b(t) - kp(t)b(t)$, or the normal growht rate minus kill rate
- $p'(t) = kp(t)b(t)$ since its population grows as it kills bacteria.
- Equilibria include $b = 0, p = 0$ and $b = 0, p > 0$, since these are *constant* solutions, or places where $b'(t) = 0, p'(t) = 0$.

Direction fields, with vector pointing towards $\langle b'(t), p'(t)\rangle$ (TODO - I think) let us follow the arrows to determine the curve over time. In this case, the bacteria will always go extinct.

However, if we add a new death rate term $-d_p p(t)$ so $p'(t) = -d_p p(t) + kp(t)b(t)$:

- We get an equilibrium at $b = \frac{d_p}{k}, p = \frac{r_b}{k}$. (Since $0 = b'(t) = rb - kpb, (\Rightarrow pk = r), 0 = p'(t) - dp + kpb, (\Rightarrow bk = d))$
- But otherwise the solutions swirl around this point. This is called a **cycle**. TODO What is a **limit cycle**?

Note that there are systems where the "solution particle" neither reaches an equilibrium or cycles around one point. The **Lorenz system** famously has this owl-eye shaped double attractor (an example of **strange sets**) where initially close particles diverge unpredictably if the constants ρ, σ, b are chosen right:

- $x'(t) = \sigma(y - x)$

- $y'(t) = x(\rho - z) - y$
- $z'(t) = xy - bz$
- TODO