

Brilliant: Differential Equations II

Dave Fetterman

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Note: Latex reference: <http://tug.ctan.org/info/undergradmath/undergradmath.pdf>

1 Chapter 1: Basics

1.1 Chapter 1: Nonlinear Equations

The two types of problems in this course are:

- Nonlinear equations (several equations on one independent variable)
- Partial differential equations (single equation with several independent variables)

Linear equations have solutions like y_1, y_2 that can be combined using any $c \in \mathbb{R}$ like $y_1 + cy_2$.

Example: Bacteria in a dish with a lot of food, no deaths

- $b'(t) = r_b b(t), r_b > 0$. r_b would be the rate of growth.
- This is linear. Reason 1: $\frac{d}{dt}(y_1 + cy_2) = y_1' + cy_2' = r_b(y_1 + cy_2)$ since $y' = r_b y(t)$, and same for y_2 .
- Also, this works because the solution is $b(t) = b(0)e^{r_b t}$, so $b_1(t) + cb_2(t) = b_1(0)e^{r_b t} + cb_2(0)e^{r_b t} = (b_1(0) + cb_2(0))e^{r_b t}$

Example: Logistic equation: Bacteria in a dish with a lot of food, limited by carrying capacity M .

- $b'(t) = r_b b(t)[M - b(t)]$.
- This is nonlinear. Reason: $\frac{d}{dt}(y_1 + cy_2) = y_1' + cy_2' = r_b[y_1 + cy_2][M - y_1 - cy_2] = My_1 + Mcy_2 - y_1^2 - 2cy_1y_2 - cy_1^2y_2^2$
- $\neq My_1 - y_1^2 + Mcy_2 - c^2y_2^2$ because of the extra $-2cy_1y_2$ term.

Sidebar: Note that this equation $b' = r_b b[M - b]$ is *separable*, so it can be solved.

- $\frac{db}{dt} = rb[M - b]$
- $\frac{db}{b(M-b)} = rdt$
- $\frac{1}{M}(\frac{1}{b} + \frac{1}{M-b})db = rdt$ after partial fractions work
- $(\ln(b) - \ln(M - b)) = Mrt + C \Rightarrow \ln(\frac{b}{M-b}) = Mrt + C$
- $\frac{b}{M-b} = e^{Mrt} e^C$
- Initial conditions $b = b(0), t = 0 \Rightarrow \frac{b}{M-b} = \frac{b(0)}{M-b(0)} e^{Mrt}$
- $b(1 + \frac{b(0)}{M-b(0)} e^{Mrt}) = M \frac{b(0)}{M-b(0)} e^{Mrt}$
- $b(M - b(0) + b(0)e^{Mrt}) = Mb(0)e^{Mrt}$
- $b = \frac{Mb(0)e^{Mrt}}{M+b(0)[e^{Mrt}-1]}$

This logistic solution will taper off to M at some point. Note that $\lim_{t \rightarrow \infty} b(t) = M$ since the non-exponential terms stop mattering. Also $b(t) = M$ sticks as a constant solution or **equilibrium** immediately. *These equilibria tell us what matters - the long-term behavior of solutions!*

Another **Example**: Lotka-Volterra equation pairs: Bacteria (b) and bacteria-killing phages (p), with kill rate k .

- The “product” $kb(t)p(t)$ measures the interactions and kills resulting from this.
- $b'(t) = r_b b(t) - kp(t)b(t)$, or the normal growth rate minus kill rate
- $p'(t) = kp(t)b(t)$ since its population grows as it kills bacteria.
- Equilibria include $b = 0, p = 0$ and $b = 0, p > 0$, since these are *constant* solutions, or places where $b'(t) = 0, p'(t) = 0$.

Direction fields, with vector pointing towards $\langle b'(t), p'(t) \rangle$ (TODO - I think) let us follow the arrows to determine the curve over time. In this case, the bacteria will always go extinct.

However, if we add a new death rate term $-d_p p(t)$ so $p'(t) = -d_p p(t) + kp(t)b(t)$:

- We get an equilibrium at $b = \frac{d_p}{k}, p = \frac{r_b}{k}$. (Since $0 = b'(t) = r_b b - kpb, (\Rightarrow pk = r_b), 0 = p'(t) = -d_p p + kpb, (\Rightarrow bk = d_p)$)
- But otherwise the solutions swirl around this point. This is called a **cycle**. TODO What is a **limit cycle**?

Note that there are systems where the “solution particle” neither reaches an equilibrium or cycles around one point. The **Lorenz system** famously has this owl-eye shaped double attractor (an example of **strange sets**) where initially close particles diverge unpredictably if the constants ρ, σ, b are chosen right:

- $x'(t) = \sigma(y - x)$
- $y'(t) = x(\rho - z) - y$
- $z'(t) = xy - bz$
- TODO

1.2 Chapter 1.2: PDEs

Example: Standing wave, where one end of a rope is fixed.

- Vertical displacement from a line of rope: $u(x, t)$ depends on where (x) and when (tt).
- Rope’s **wave equation** is $u_{tt} = v^2 u_{xx}$, where v is the “constant wave speed”, and the others are the space, time partials.
- Note that $u = \cos(vt) \sin(x)$ and $u = \sin(vt) \cos(x)$ both work.
- If you guess the solution has split variables like $u = X(x)Y(y)T(t)$, then, upon substitution and division by $X(x)Y(y)T(t)$, $\frac{\delta^2 u}{\delta t^2} = v^2 [\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}]$ yields $\frac{T''(t)}{T(t)} = v^2 [\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}]$
- This method may or may not work. But if it does, it means that since x, y , and t are independent variables, each individual piece must be constant.
- So, for example, if we know $\frac{X''(x)}{X(x)} = -4\pi^2$, we can get to $X(x) = \sin(2\pi x)$
- The wave equation is similar in 3D: $u_{tt} = v^2 [u_{xx} + u_{yy} + u_{zz}]$, or using the Laplacian, $u_{tt} = v^2 \nabla^2 u$. Here, u measures not displacement but expansion/compression of air at (x, y, z) , time t .

Using Fourier transforms helps turn difficult PDEs into an easier problem like an ODE. *Fourier transforms work best* when

- The domain is all of \mathbb{R}^n
- The function u vanishes at infinity.

The Fourier transform changes the domain of x to that of ω . It comes with the (highly simplified) rule (see Vector Calculus course): $F[\frac{\delta f}{\delta x}] = i\omega F[f]$. **Example:** Drunkard’s walk.

- One dimensional: moves left or right in a random way. Starts at $x = 0, t = 0$.
 - $u(x, t)$ is probability of being at point x at time t . Naturally, $\int_{x=-\infty}^{x=\infty} u(x, t) dx = 1$.
 - Also, it obeys the 1-dD diffusion equation $\frac{\delta u}{\delta t} = \frac{\delta^2 u}{\delta x^2}$
 - So by taking Fourier transform of both sides of diffusion equation
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- TODO