The Office DVD Problem

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Screensavers have captivated [this] man since the 1990s. If watched long enough, what will the spirits of the machine tell us?

Specifically, the question of whether a bouncing rectangle will slide *exactly* into the corner of the screen, for a satisfying, perfectly diametric rebound, was even addressed on *The Office*: `https://www.youtube.com/watch?v=QOtuX0jL85Y`

However, though these characters reportedly watched this sleep-mode drama play out for years until payoff, we ask - under what conditions *will* the rectangle perfectly bounce into the screen's corner?

0.1 Statement

Suppose we have a screen of length l, height h, containing an axis-aligned rectangle of length j and height k centered at point (x, y).

Suppose this rectangle is launched at direction $\langle 1, m \rangle$ [1] and "bounces" according to billiard

[1] Think of this as slope m

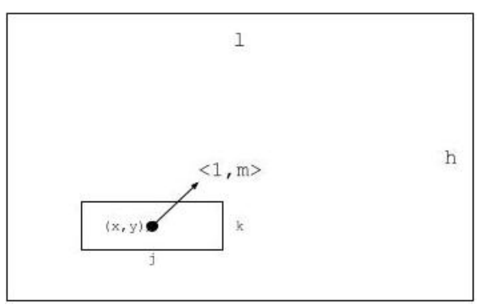


Figure 1: The Office DVD problem's most generic setup

rules [2].

Given $l, h, j, k, m \in \mathbb{R}$, can we tell whether the rectangle ever bounce perfectly into a corner?

We can approach this problem from the simplest version to the most complex.

0.2 Problem 1

Suppose $j = k = 0$ and $x = y = 0$. In other words, suppose we have a *point* starting at the corner (origin). Under what conditions does this bounce into a corner?

0.3 Problem 2

Suppose $j, k > 0, x = \frac{j}{2}, y = \frac{k}{2}$. In other words, suppose we have a rectangle starting at the corner. Under what conditions does this bounce into a corner?

0.4 Problem 3

Suppose we have maximally open (reasonble) conditions, with $x \in [\frac{j}{2}, l - \frac{j}{2}], y \in [\frac{k}{2}, h - \frac{k}{2}]$. Under what conditions does this bounce into a corner?

1 Solutions

Note: If our initial slope is zero $m = \langle 1, 0 \rangle$ or "infinite" (sort of disallowed in setup), the solution is trivial: if we're in a corner now, we'll be in one shortly, otherwise we never will.s

Note also that, for the sake of simplicitly, we can treat m as always positive (going up and to the right). If not, inverting the problem ($m \to -m, y = h - y$) yields the same answer. The box initially moving leftwards (disallowed in the problem) submits to the same kind of reduction by the same sort of symmetry.

1.1 Problem 1 solution

The key insight here is that though the point bounces "within a box" until meeting $(0, 0), (0, h), (l, 0)$, or (l, h), (as in Figure 2) we can instead look at the path of the point in an unconstrained space, seeing if we hit a point of the form $a \cdot l, b \cdot h$ with $a, b \in \mathbb{N}$.

Consider that, on meeting the point $(0, \frac{2}{3})$, we can either consider what happens if we reflect "back" as in Figure 2, or, equivalently, if we pass "through" as in Figure 3. We quickly see that:

[2]Glancing off a horizontal boundary, our trajectory goes from $\langle 1, m \rangle$ to $\langle 1, -m \rangle$, with $\langle \pm 1, m \rangle$ to $\langle \mp 1, m \rangle$ for horizontal

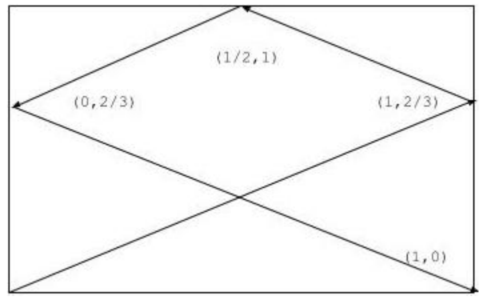


Figure 2: Sol 1:Success for $m = \frac{2}{3}, h, l = 1, j, k = 0$

- If the left-hand side meets a corner, the mirror-image on the right-hand side will meet a corner.
- Likewise for the converse: the right meeting a corner means the left has as well.
- If the left-hand side does *not* meet a corner, the right-hand side cannot.
- Likewise for the converse.

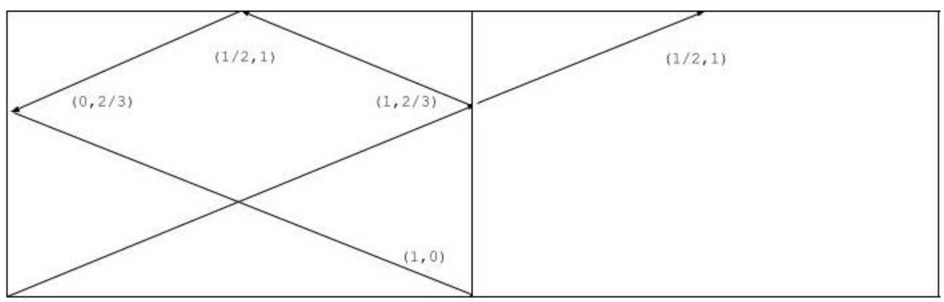


Figure 3: Sol 1: TODO

This applies for top-bottom just as easily as left-right.

Therefore, composing these two, we can cast the path of the smaller rectangle as entirely "up and to the right".

TODO TODO TODO TODOT

1.2 Problem 2 solution

The key insight here is that instead of each "frame" extending from $(0,0)$ to (l,h), with the box's center (x,y) constrained to the rectangle defined by $[\frac{j}{2}, l-\frac{j}{2}], \times[\frac{k}{2}, h-\frac{k}{2}]$, we can instead treat the *center* of that $[\frac{j}{2}, l-\frac{j}{2}], \times[\frac{k}{2}, h-\frac{k}{2}]$ box as a point like in problem 1.

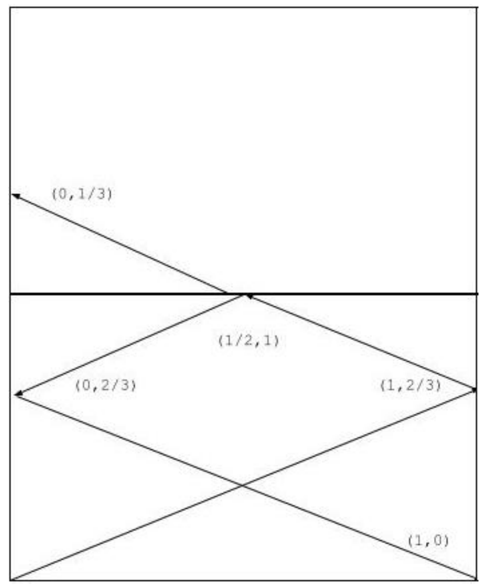


Figure 4: Sol 1: TODO

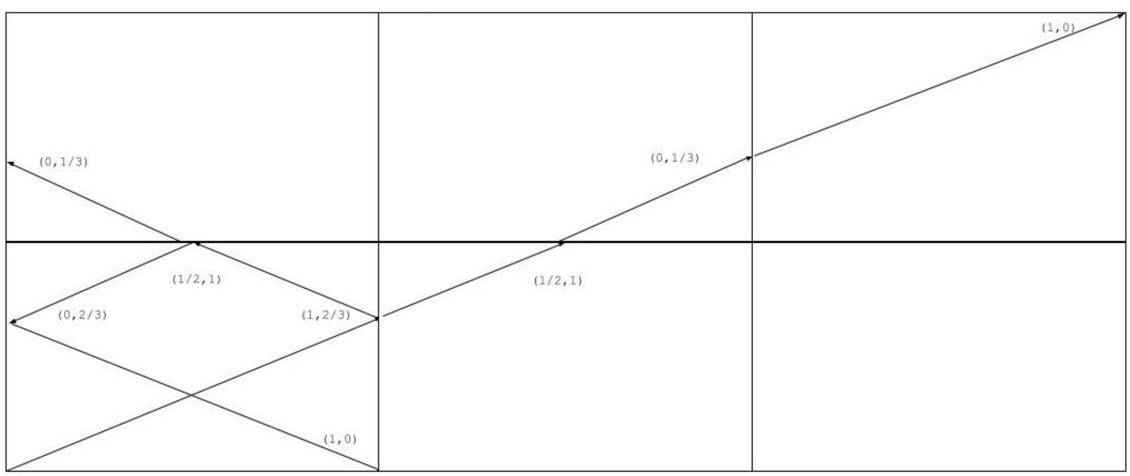
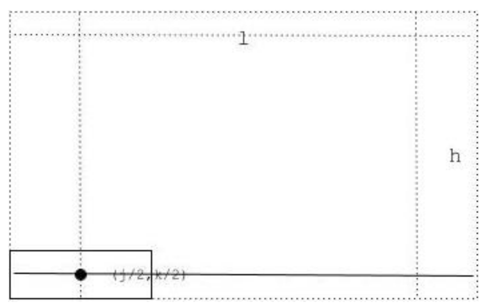


Figure 5: Sol 1: TODO

It is then clear that we can use problem 1's main insight to the center point as opposed to the small rectangle: that the small rectangle, say, $\frac{j}{2}$ left of the right border of one frame will take an equivalent trajectory to one $\frac{j}{2}$ right of the left border of the adjoining right frame.

So, restate the problem as $j, k = 0, x \to x - \frac{j}{2} y \to \frac{k}{2}$ and solve as in problem 1.


l
h
(j/2,k/2)