Spotting Graph Theory Problems in Spot It

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[1]Obviously Unemployed
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4/16/23

Abstract

The card game "Spot It" supports a unique mechanic: every card of the deck has eight different symbols, and shares exactly one symbol of some kind with every other card. The obvious game play ("spot the match') works for children as young as two; the intricacies of deck construction astound these children of forty-two. In hypothetically constructing our own deck, we run across interesting problems in graph theory, number theory, and abstract algebra [1]. Some are solved, some remain unsolved. Until now.[2]

Notably, starting with $g = p^k + 1, p \in \mathbb{P}, k \in \mathbb{N}$ and $n = g^2 + g + 1$, we prove the following four constructions are equivalent:

1. (The Children's Game) A deck of n "Spot It" Cards with $s = g$ symbol slots, where each of $m = n$ symbols occurs exactly g times,
2. (Graph Theory) An edge partition of the complete graph K_n into complete subgraphs K_g,
3. (Abstract Algebra) A finite (Galois) field of order $g - 1$, and
4. (Number Theory) A Perfect Difference Set[**1**] on n elements.

We then examine other configurations of s and g, as well as comment on other reasonable constructions of a Spot It Card deck.

1 The game and the problem

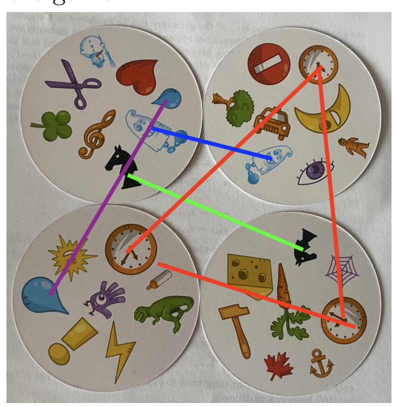
Introduced to us by Ari Steinberg, "Spot It" is a children's game of 55 cards as shown in Figure 1a, featuring eight colorful symbols on each. Though gameplay comes with a few variants in its tiny rulebook, the primary mechanic when presented two cards is simply *spot single the common symbol first*. The game is simple enough for a two- or three-year

[1]There are also representations in projective geometry, but they're left alone.
[2]Not really. We're just putting unsolved problems into new boxes

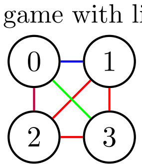
old to grasp (and win!), but this poses the question: *just how did they construct such a deck?*

*Note: There have been some other investigations here[**2**], but for the purposes of enjoyment, everything in this paper was researched without reference to prior (Spot It) work.*

(a) Four cards in the game



(b) Four cards in the game with links



(c) Four cards graph

Naturally, there are trivial constructions: every symbol occurs only twice, or once, or the count of symbols is so varied that a deck can be constructed almost greedily. However, the

Spot It game has uniformly 8 symbol "slots" on each card, which each appear 8 times [3] across the different cards.

This is what we examine in this paper:

The Core Question: For what choices of g and s can we construct a "Spot It" deck where each of the s symbols on each card appears exactly g times throughout the deck?

1.1 Reframing as a graph

Noticing that every card has a relationship to every other card (notably, the identity of the single symbol shared between them) as in Fig. 1b, we take our first step by reconstructing this problem as an undirected graph as in Fig. 1c.

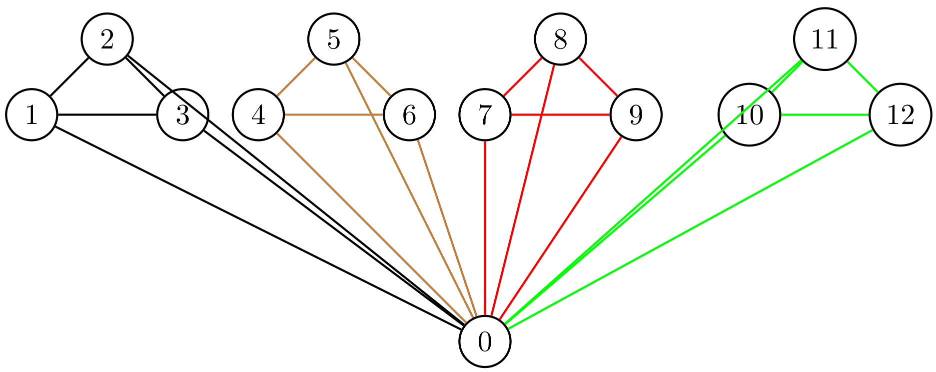


Figure 2: $n = s(g-1)+1$. Here, $s = 4, g = 3$

The Graph Representation: A deck of Spot It Cards each with s symbol slots, where each symbol appears g times c can be represented by a graph G:

1. With n nodes, where $n = s(g-1)+1$
2. With m unique edge colors, where m denotes the number of symbols,
3. Where all edges of color m_i form a complete subgraph on g nodes, s, and $m = \binom{n}{2}/\binom{g}{2}$.

Note: Singletons (groups where $g = 1$) would be rendered as self-edges. These are uninteresting and are generally ignored in this paper

Proof:

[3] As we will see later, there should be 57 cards for this to be true; it's likely two cards were removed

1. As in Fig. 2, node n_0's adjacencies are exactly s monocolor cliques of size $g-1$ (excluding n_0 itself). In a complete graph, these adjacencies comprise the total node set, so $n = (g-1)s+1$ when adding n_0 back in. Using any other node is equivalent.
2. As in Fig. 1b, while card 1 and card 2 having the relationship "clock", node 1 and node 2 instead share an edge with the color red. This is the same relationship between nodes 2 and 3, and nodes 1 and 3. "Drop", "knight", and "ghost" would be colors purple, green, and blue, respectively. This works because every edge has exactly one color (corresponding 1:1 with a symbol) in this formulation, and every card pair has exactly one symbol shared.
3. All cards with a given symbol (say, "clock") must correspond to nodes be linked with the color red to all other nodes whose card has a clock; this is a complete subgraph. A complete graph K_n has $\binom{n}{2}$ edges. A monocolor clique of size g is a complete graph as well, with $\binom{g}{2}$ edges. K_n's edges are exactly these equal-sized cliques, so there are therefore $m = \frac{\binom{n}{2}}{\binom{g}{2}}$ of them, corresponding to colors.

And since every edge in our complete graph K_n is in exactly one monocolor clique of size K_g, this becomes a crisp graph theory problem.

The Core Question in Graph Terms: Given s and g as before, can we construct an edge partition (colloquially here, "coloring")[4] of K_n into a set of complete subgraphs of size g (denoted K_g)?

Though exhaustive research wasn't done, this graph problem does not appear to have a clear analytical solution out there.[5]

Since m and n are determined from s and g, we'll start by looking at possibly candidate configurations of s and g.

2 The Candidate Theorem: $g|s(s-1), g \leq s$

Suppose that that every symbol s has exactly g cards containing it [6]. Then

1. $g|s(s-1)$.
2. If $s > 1$ and $g > 1$ then $g \leq s$
3. Corollary to (2): $m = (\frac{s}{g})n$ and therefore $m \geq n$

[5] Or people who care about publishing it within the reach of lazy hobbyists, anyway!
[6] for example, all $s = 7$ symbols in Fig. **??** correspond to cliques of size $g = 3$

4. All candidate configurations of g, s are $g \leq s$, $g|s(s-1)$.

Proof:

1.

$$\binom{g}{2}\Bigg|\binom{n}{2} \Rightarrow \frac{n(n-1)}{g(g-1)} \in \mathbb{N} \Rightarrow g(g-1)|n(n-1) \quad (1)$$

$$n = (g-1)s+1 \Rightarrow g(g-1)\Big|(sg-s+1)(sg-s) = (sg-s+1)s(g-1) \quad (2)$$

$$\Rightarrow g|s^2g - s^2 + s \Rightarrow g|(1-s)s \Rightarrow g|s(s-1) \quad (3)$$

2. Any node n_i is adjacent to s monocolor cliques of size g. These cliques $C_1 ... C_s$, containing non-n_i nodes if $g > 1$, comprise all nodes, and any other cliques can contain no more than one of each K_i. This means that clique of size g greater than s cannot be formed, since the only place to find nodes are these $C_1 ... C_s$. The other trivial case, $s = 1$, means there is only one color in the whole graph.
   This means we need not consider configurations like $g = 6, s = 3$ even though $6|3(3-1)$.

3. Another corollary here is that $\boxed{m \geq n}$, since:

$$n = (sg - s + 1) \quad (4)$$

$$m = \frac{\binom{\frac{(sg-s+1)(sg-s)}{2}}{}}{\binom{g}{2}} = \frac{(sg-s+1)(sg-s)}{g(g-1)} = \frac{(sg-s+1)s}{g} \quad (5)$$

$$\Rightarrow m = \left(\frac{s}{g}\right)n \quad (6)$$

$$s \geq g \Rightarrow m \geq n \quad (7)$$

4. This is just a combination of (1) and (2). But for example, a tiling of triangles ($g = 3$) means that either $s \equiv 0 \mod 3$ (see Fig. **??**) or $s \equiv 1 \mod 3$ (see Fig, **??**).

3 Constructing $g = s - 1$ over a field

Though we presented a few legitimate examples of complete graphs tiled by uniformly sized complete subgraphs in Fig. **??** and Fig. **??**, these are not easy to find by hand once s becomes much larger. The whole problem of graph partitioning admits many algorithms, most approximations[**3**], though usually over an arbitrary graph instead of a relatively simple complete graph, and many referring to separating actual *nodes* into partitions, rather than edges.

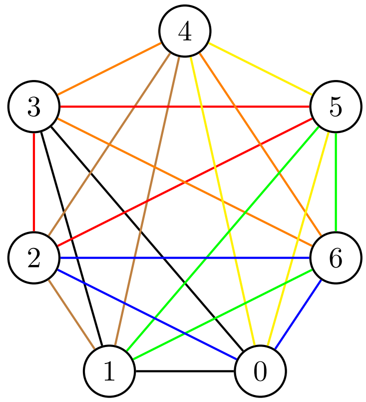


Figure 3: s=3, g=3, n=7, m=7. Cliques of form (n_i, n_{i+1}, n_{i+3}). for all i

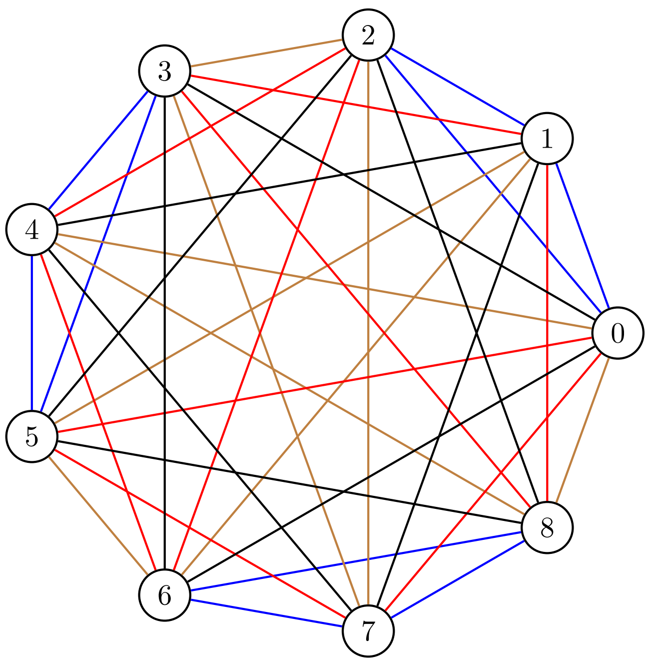


Figure 4: s=4, g=3, n=9, m=12.
Cliques: $(n_i, n_{i+3}, n_{i+6}), (n_i, n_{i+1}, n_{i+2}), (n_i, n_{i+4}, n_{i+8}), (n_i, n_{i+5}, n_{i+7}), 3|i$

We can, however, systematically find an edge partition if g is a prime power.

g=s-1 construction: If g is a prime power p^k, we can explicitly construct a graph that satisfies our game with $g = s - 1$.

As the combination of s, g determine the shape of the graph entirely, $g = s-1$ implies:

- The graph has $n = s(g-1)+1 = (g+1)(g-1)+1 = g^2$ nodes.
- Those nodes can be grouped into g groups of size g.
- There are $m = (\frac{s}{g})n = (\frac{s}{g})g^2 = sg = (g+1)g$ colors in the graph.

We can start with the easiest way to see this: $g = p, p \in \mathbb{P}$.

Construct the multiplication table for the finite field on $p = 3$, as in Fig. 5.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	1	0

(a) Addition table GF(3)

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

(b) Multiplication table GF(3)

Figure 5: Field tables for GF(3)

To construct the $m = g(g-1)$ colors (symbol cliques) in the graph:

- Construct a finite field $\mathcal{GF}(\})$, which is of size g.
- Divide the g^2 nodes into g groups $(G_0, G_1, G_2 ... G_{g-1})$, where $[0...g-1] \in \mathcal{F}$, of g nodes each $((G_{0,0}, G_{0,1} ... G_{0,g-1}) ... G_{g-1,0}, G_{g-1,1} ... G_{g-1,g-1})$
- For all $c \in [0, g-1]$:
  - For all $y \in [0, g-1]$:
    * Set $C_{c,y}$ to the empty set.
    * For all $x \in [0, g-1]$, add $G_{x, G_x \cdot G_y + c}$ to clique $C_{c,y}$, where $+$ and $\cdot$ refer to addition and multiplication rules for $\mathcal{F}$.
- The g cliques formed from the node groups $(G_0, G_1, G_2 ... G_{g-1})$, plus the g^2 cliques like C_c, y form the $(g+1)g$ cliques or "colors".

For an example, compare identical graphs Fig. **??** (generated from the algorithm) and Fig. **??**. The mapping between node (i, j) in Fig. **??** and node k in Fig. **??** is $(i, j) \rightarrow k = i + 3j$, with inverse $k \rightarrow (k \mod 3, \lfloor k/3 \rfloor)$.

1. Cliques of form $G_i : G_0, G_1, G_2$: these are the triangles in black. These correspond to nodes $n \mod i$ in Fig. **??**; for example $G_0 = \{0, 3, 6\}$.

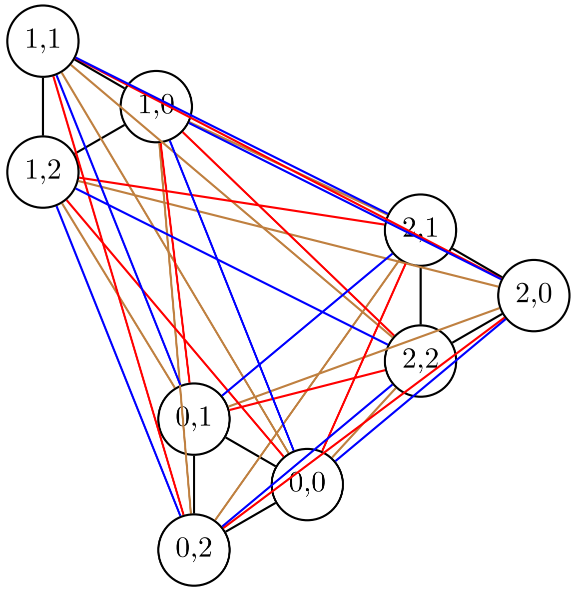


Figure 6: whole K_9: $s = 4, g = 3, n = 9, m = 12$

2. Cliques of form $C_{j,0}$: $C_{0,0}, C_{1,0}, C_{2,0}$: these are the triangles in blue. Fig. **??** sees these connect between nodes with the same $n \mod i$ in Fig. **??**; for example $C_{1,0} = \{1, 4, 7\}$.
3. Cliques of form $C_{j,1}$: $C_{0,1}, C_{1,1}, C_{2,1}$: these are the triangles in brown. Fig. **??** sees these connect between nodes that "advance one" with each hop between black cliques in **??**; for example $C_{0,1} = \{0, 4, 8\}$.
4. Cliques of form $C_{j,2}$: $C_{0,1}, C_{1,1}, C_{2,1}$: these are the triangles in red. Fig. **??** sees these connect between nodes that "go back one" with each hop between black cliques in **??**; for example $C_{0,1} = \{0, 5, 7\}$.

Every node is connected to every other node once in Fig. **??**, since for nodes x, y and z, w:

- $x = z$ and and they're in the same "black" clique G_x.
- $y = w$ and they're in a "blue" clique $C_{0,y}$
- There is some "increment" a where starting from node x, y, $z - x$ "hops" away lands you at node w, like the brown ($a = 1$) or red ($a = 2 \equiv -1 \mod 3$) cliques.

The last statement is deliberately informal. For prime numbers, it's clear that every increment in $[1, p - 1]$ traverses a different, non-overlapping path $a, 2a, 3a...(a - 1)p$.

The leap comes in realizing that these are not additive increments, but journeys through a field's multiplication table; the brown groups correspond to walking through the second row of Fig. **??** and adding each to either $c = 0, 1, or 2$; the reds correspond to the third. The blues (0 "increment") are actually a walk through the top row!

This is most obvious for primes, but can be constructed from, say, $GF(4)$:

+	0	1	B	D
0	0	1	B	D
1	1	0	D	B
B	B	D	0	1
D	D	B	1	0

(a) Addition table GF(4)

·	0	1	B	D
0	0	0	0	0
1	0	1	B	D
B	0	B	D	1
D	0	D	1	B

(b) Multiplication table GF(4)

Unlike a prime-order finite field, the addition table is not cyclic, so in a sense, our indices for $G_{0,i}$ say are not $i \in [0, 3]$ but $i \in \{0, 1, B, D\}$! This is why the phrase "where $[0...g-1] \in \mathcal{F}$" is important in the algorithm.

Here are the resulting tables for $s = 5, g = 4$. For example, in Fig. **??**, consider the third row as saying "clique $C_{0,B}$ contains nodes $G_{0,0}, G_{1,B}, G_{B,D}, G_{D,1}$".

c	y	0	1	B	D
0	0	0	0	0	0
0	1	0	1	B	D
0	B	0	B	D	1
0	D	0	D	1	B

Figure 8: Groups $G_{0,i}$

c	y	0	1	B	D
1	0	1	1	1	1
1	1	1	0	D	B
1	B	1	D	B	0
1	D	0	B	0	D

Figure 9: Groups $G_{1,i}$

(Of course, once the multiplication is defined, feel free to substitute 2 for B and 3 for D.)

Including groups like $G_B = \{G_{0,B}, G_{1,B}, G_{2,B}, G_{3,B}\}$, we see that there are no repeated edges, and all edges are accounted for.

c	y	0	1	B	D
B	0	B	D	0	1
B	1	B	0	1	D
B	B	B	1	D	0
B	D	B	B	B	B

Figure 10: Groups $G_{B,i}$

c	y	0	1	B	D
D	0	D	D	D	D
D	1	D	B	1	0
D	B	D	1	0	B
D	D	D	0	B	1

Figure 11: Groups $G_{D,i}$

In general, to prove that this table exactly represents the "color" of every edge, we need to assert that for every pair of elements, say, $(B, 1)$, and every pair of columns, say 0 and B, (see Fig. 12c) that B and 1 appear in the same row in columns 0 and B exactly once.

To ensure there are no duplicates, consider two columns x_1 and x_2 that have a repeated value pair in some table, once on row g_y, once on g_y^*:

- Assume $g_y g_{x_1} + c = g_y^* g_{x_1} + c^*$, and $g_y g_{x_2} + c = g_y^* g_{x_2} + c^*$ for $g_{x_1} \neq g_{x_2}$
- Subtract the two to get $g_y(g_{x_1} - g_{x_2}) = g_y^*(g_{x_1} - g_{x_2})$
- The field $\mathcal{F}$ requires the nonzero $(g_{x_1} - g_{x_2}) \in \mathcal{F}$ to have an inverse.
- Multiplying both sides by that inverse, we have $g_y = g_y^*$

Similar field-based arguments can be made to ensure that every pair is represented. Since everything we're dealing with is finite, however, we can also use a pigeonhole approach:

- For any pair of columns, there are g^2 pairs of values to account for.
- Across the tables above, there are g^2 rows (combinations of c and y that generate the cliques).
- From the previous arugment, no two rows can be duplicated.
- Therefore, by the pigeonhole principle, every combination is represented, and therefore, between any two G_* cliques, every node has an edge with every other.

4 Constructing $g = s$ from $g = s - 1$

With the previous construction ($g = s - 1$) in hand, we can easily construct a $g = s$ graph (adding one to g, keeping s the same), with the same restrictions on g. See Fig. 13 for a visual on this.

- Create node n_y for $y \in \mathcal{F}$
- Add n_y to all cliques $C_{c,y}$. For a given y, these cliques share no nodes.
- Create node n^*. Add this to every G_i clique.
- Create clique of all $n_y, y \in [0, g]$ plus n^*.

We see then that:

- Every new node n_y gets added to the new "n" clique plus $s - 1$ cliques, all of size s.
- n^* is added to $s - 1$ cliques plus the bottom clique.
- Every G-style clique gets n^* added to it.
- Every $C_{c,y}$ clique adds node n_y.
- So, the new partition is one where $g = s$, built on the previous where $g = s - 1$.

Note that you can create $s = g = 8$ this way, with $n = m = 57$. The Spot It card deck comes with 55 cards with $s = 8$ "slots", so our suspicion is that two cards were simply dropped from the set.

c	y	G_0	G_1	G_B	G_D
0	0	0	0	0	0
0	1	0	1	B	D
0	B	0	B	D	1
0	D	0	D	1	B

(a) Groups $G_{0,i}$

c	y	G_0	G_1	G_B	G_D
1	0	1	1	1	1
1	1	1	0	D	B
1	B	1	D	B	0
1	D	0	B	0	D

(b) Groups $G_{1,i}$

c	y	G_0	G_1	G_B	G_D
B	0	B	D	0	1
B	1	B	0	1	D
B	B	B	1	D	0
B	D	B	B	B	B

(c) Groups $G_{B,i}$

c	y	G_0	G_1	G_B	G_D
D	0	D	D	D	D
D	1	D	B	1	0
D	B	D	1	0	B
D	D	D	0	B	1

(d) Groups $G_{D,i}$

Figure 12: s=5, g=4 adjacency tables

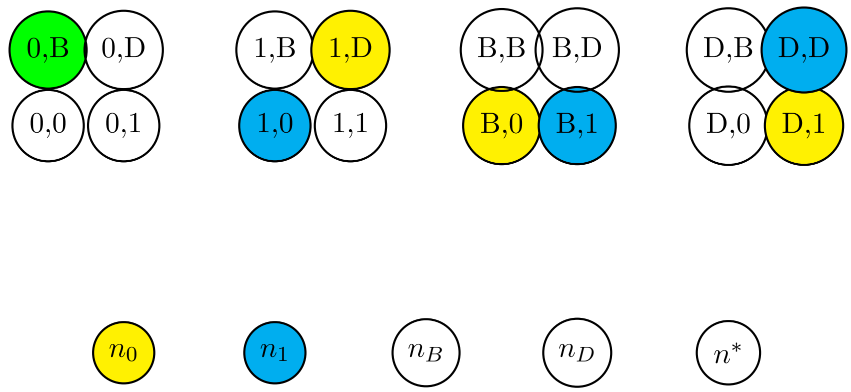


Figure 13: s=5, g=5 from s=5, g=4 with no edges. The blue is the $C_{B,1}$ clique newly adding in n_1. The yellow is the $C_{0,1}$ clique newly adding in n_0. Node $0, B$ is in both.

5 Alternative: Constructing $g = s$ with perfect difference sets

*Though it will be shown equivalent to the last construction, we can use another concept to build these graphs when $g = s$: perfect difference sets. Notably, these are proven to exist for $g = p^k - 1$ by Singer[**1**]*

Searching for complete graph partitions by brute force is difficult. Even a graph of size 7 like Fig. **??** requires sorting through putting 21 distinct objects into 7 distinct bins[7], and the numbers get worse from there.

There is some hope that a partition or coloring, should it exist, would exhibit some regularity; after all, every node has an equal configuration of edges and adjacent nodes in a complete graph.

Additionally, when $g = s$, we've seen that $n = m$, so finding a unique complete subgraph K_g for each node which, in total, partition the complete graph, seems like a good strategy.

Looking at Fig. **??**, we see that every node at index i can be mapped to a K_3 (color) whose vertices are i, $i + 1$, and $i + 3$, with addition being modulo 7. (Take a look at node 0's black triangle $(0, 1, 3)$. After this, node i needs to find an adjacency with node $i + 2$, $i + 4$, $i + 5$, and $i + 6$, or remapped over modulo 7, $i + 2$, $i - 3$, $i - 2$, and $i - 1$. However, we can find these latter four by simply repeating triangles $(j, j + 1, j + 3)$ over the rest of the graph; in particular, the triangles at (i-1, i, i+2) (here, blue if $i = 0$) and $(i - 3, i - 2, i)$ (yellow) take care of i's six adjancies.

[7] $(21!)/((3!)^7)$, though with some symmetries

This works because among the $n = 7$ nodes in this graph $g = s = 3$, we found a complete graph K_3(in this case among nodes $\{0, 1, 3\}$) such that every chord length in $[1, \frac{n-1}{2}] \mod n$ is represented exactly once. Thinking of creating adjacencies for node 0, this means that connections to $\{1, 3\}$ will be taken care of by the associated triangle, that a 2-chord exists by virtue of the triangle, and that $\{-1, -2, -3\}$ will be symmetrically connected *to* 0 by the same means.

So, if for a given n, we can find a set $a_i \in [1, n-1]$ of size $g-1$ such that $\bigcup_{i>j}(a_i - a_j \mod n) = [1, \frac{n-1}{2}]$, then this can be repeated for every node to cover the graph.

That such a set exists is not obvious. However, through some basic search algorithms at `https://github.com/fettermania/mathnotes/tree/main/spotit/code`, we can sniff out a few of them:

g,s	n	0 adjacencies
3	7	{1, 3}
4	13	{1, 3, 9}
5	21	{1,4,14,16}
6	31	{1, 3, 8, 12, 18}
8	57	{1, 3, 13, 32, 36, 43, 52}
9	73	{1, 3, 7, 15, 31, 36, 54, 63}
10	91	{1, 3, 9, 27, 49, 56, 61, 77, 81}
12	133	{1, 3, 12, 20, 34, 38, 81, 88, 94, 104, 109}

Figure 14: Perfect Difference Sets up to $g = s = 12$

Here is an example of 0's adjacencies for $s = g = 6$:

It turns out that in the 1930s, James Singer found that these configurations, termed *perfect difference sets*, exist for $g^* = p^k, p \in \mathbb{P}, k \in \mathbb{N}$ and, setting $g = g^* + 12$, and $n = g^2 + g + 1$[**1**]. This result was not connected to graph edge partitioning in his paper. The existence for some of the other types of g have been disproven, but this appears to be the main theorem in the area.

Singer's Theorem on Perfect Difference Sets: If $g = s$, then $n = g^2 - g + 1$ by the Graph Representation Theorem and $n = m$ by the Candidate Theorem. Thus, we're looking for a PDS over $n = g^2 - g + 1$. This has been proven to exist if $g = p^k, p \in \mathbb{P}, k \in \mathbb{N}$.

Singer's proof relies also on finding a finite field $\mathcal{F}$ of size g. There are some deterministic, non-brute-force methods of doing this[**4**] though they seem quite intricate.

This limitation to prime powers, just like the partitions for $g = s$ found in section 3, explains

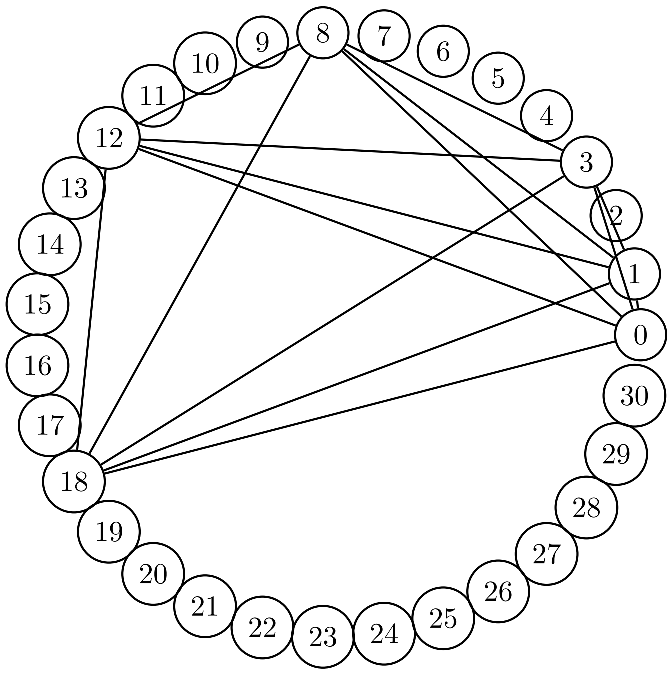


Figure 15: perfect Set difference on g=6, s=6, n=m=31

the row omissions in the brute force search table in Fig,. 14. So, if $g = p^k$ a perfect difference set representation exists , and a section 3 representation also exists.

6 Interlude: Graph Equivalence up to relabeling

g=s equivalence Theorem: Any partition on a graph where $g = s$ is equivalent up to relabeling.

Proof: If $g = s$, and therefore $m = n = g^2 - g + 1$, consider any clique.

A node like n_0 in this clique (call it the"blue" clique at the bottom of Fig. 16) is adjacent to $g - 1$ unique other colors or cliques, represented as the cliques $C_i, 0$.

No color can be shared between these color adjacency sets (like $C_{4,0}$ and $C_{3,3}$ here). Consider $if n_0$ and n_3 were both adjacent to yellow, they would have both a yellow and blue edge between them.

Therefore, this blue clique shares a with all of the other $g(g-1)$ colors. Because blue was an arbitrary selection, every color is adjacent to every other color on some single node. So the graph *where colors are nodes with edges between adjacent colors* is complete. This, up

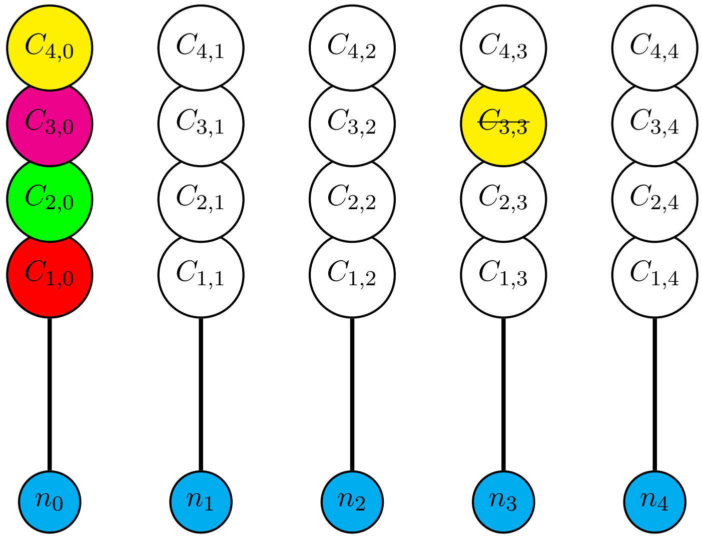


Figure 16: s=5, g=5 from s=5, g=4 with no edges. The blue n_j is an arbitrary clique, and $C_{j,*}$ are the $g-1$ cliques (colors) adjacent to each node.

to relabeling, there is only one way to partition a complete graph into complete subgraphs when $g = s$.

An important consequence: Because $g = s$ in both the cycle generation in section 3 as well as the PDS generation in section 5, these are, up to isomorphism, the same coloring.

7 Constructing a $g = s-1$ partition from a $g = s$ partition

Suppose we have constructed $g = s$ by Perfect Difference Sets or (as we have shown, ultimately identically) by the method shown in Fig. **??**. Consider removing the $I_* \cup X$ clique and all associated edges. This changes us into a partition in which:

1. We have one fewer clique (the $I_* \cup X$ clique is gone.)
2. Each remaining clique has one member removed (we removed some I_* member from each I_* clique, and node X was removed frrom each G clique.)
3. We lose g nodes.
4. s remains the same.

This matches the configuration of $g = s-1$. Though every graph constructed through this method seems like it would be isomorphic (in particular, to graphs constructed using the method in section 4), we can't categorically rule out other constructions.

8 Considering wider $g|s$ and $g|s-1$: Inception

It seems we have reached the limit of possibllities when $g = s$. In such a graph, when $g - 1 = p^k$:

- We can create a game of Spot It with s symbols, each of which appear g times (section 1)
- We can always construct $g = s - 1$ through a finite field $GF(g - 1)$ (section 3), and augment with another clique to form $g = s$ (section 4).
- We can always find a perfect difference set modulo $n = (g+1)^2 + g + 1$ and construct a graph partition where $g = s$ (section 5).
- These partitions, and any where $g = s$, are isomorphic (section 6).
- We can always reduce to a partition where $g = s - 1$ by removing a clique and associated edges. (section 7)

However, these only cover the equality cases of $g = s - 1$ and $g = s$. Section 2 showed that we possibly could accommodate $g|s$ and $g|s - 1$ more generally.

Despite repeated attempts, there don't seem to be many surefire constructions (or reductions) from the above methods for slices like $g^* = kg, k > 1$. Conversely, proving such constructions are *impossible* also seems difficult.

However, "inception" is one very minor method of generating such a partition.

> **Inception**: If K_n can be partitioned into complete subgraphs K_g and K_g can be partitioned into complete subgraphs K_h, then K_n can be partitioned into complete subgraphs of size K_h.

Of course, by the equivalencies above, this means we can make Spot It Games of highly varied g and s, even if well outside practical possibility.

For example: We can partition of K_{81} ($s = 10, g = 9, n = 81, m = 90$) into groups of K_9, and, like in figure Fig. **??**), we can split those into ($s^* = 4, g^* = 3, n = 9, m = 12$). This gives us a new partition of configuration ($s = 40, g = 3, n = 81, m = 1080$), in which every node still has a uniform number of attached colors, and every color has the same number of node members. This graph has not been attempted.

Say $s = g$ instead, just for another example. This certainly works for turning $s = 9, g = 9, n = 73, m = 73$ into $s = 36, g = 3, n = 73, m = 876$ as well, or even any other configuration where $g \notin \{s, s-1\}$ as below.

(Note that an even more trivial example would be turning every edge into a K_2 complete graph.)

9 Considering wider $g|s$: Kirkman's schoolgirl problem

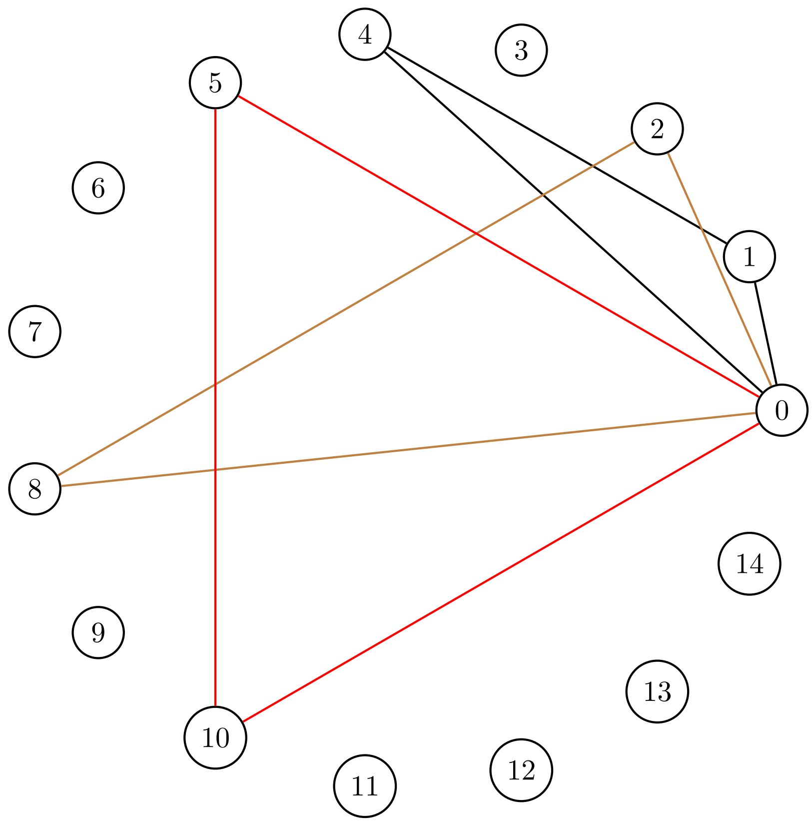


Figure 17: s=7, g=3, n=15, m=35, node 0 adjacencies. $i \in [0,4) : (i, i+5, i+10); \forall i : (i, i+1, i+4), (i, i+2, i+8)$, all $\mod 15$

Apparently some versions of this problem were studied even in the nineteenth century. Kirkman's schoolgirl problem[**5**] asks: *Can fifteen girls walk in groups of three to school for seven days, such that each pair walks together exactly once?*. With colors representing walking groups, $s = 7, g = 3, n = 15$, we have something isomorphic to a Spot It deck construction.

Through ad-hoc tinkering, we can find the graph in Fig. **??**, showing the adjacencies for a single node. These are repeated "round the horn", with the exception that the red triangle only appears five times instead of 15.

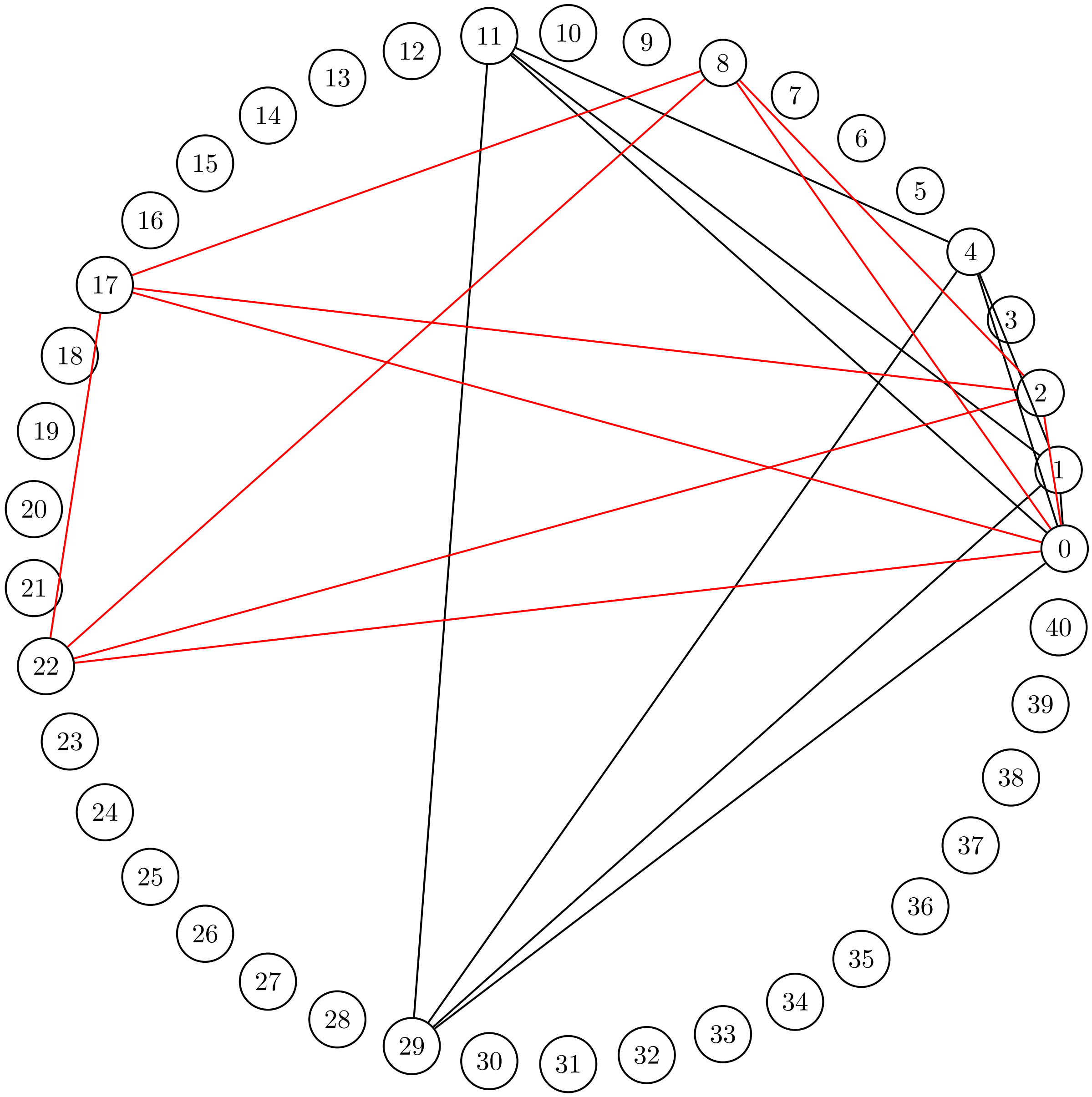


Figure 18: Perfect Difference Set on s=10, g=5, n=41, m=82: (0 1 4 11 29), (0 2 8 17 22)

These are apparently instances of a *Steiner Triple System* $S(t, k, n)$[**6**], where we look for groups of k among n total elements, which cover exactly once every set of size t. If $t = 2$ (edges in the graph, pairs of girls), and $k = 3$ (subgroups K_3, walking trios) among n, we have an instance of a Kirkman problem (and Spot It game); in fact, Kirkman problems are those where $t = 2, k = 3$.

The followups to this problem are varied (including links to projective and affine geometry) and many unsolved ($t \geq 6$). Much progress has been made on $STS(t = 2, k = 3, n)$ but results seem spotty for $k > 3$.

By some backtracking and brute force searching using our code `https://github.com/fettermania/mathnotes/tree/main/spotit/code`, we can find a few ad-hoc solutions on our own, even without the problem fully solved:

s	g	adjacencies $\forall i$	adjacencies $i \in [0, g-1]$
6	3	{{0 1, 4}, {0, 2, 7}}	{}
7	3	{{0 1 3}, {0, 4, 10}}	{{0, 5, 10}}
9	3	{{0,1,6}, {0,2,10}, {0,3,7}	{}
10	3	{{0,2,10}, {0,1,5}, {0,3,9}}	{{0, 7, 14}}
10	5	{{0, 1, 4, 11, 29}, {0, 2, 8, 17, 22}}	{}

Figure 19: Ad Hoc Steiner Solutions

The adjacencies for node 0 in $s = 10, g = 5$ are shown in Fig. **??**, obtained through computational means.

10 Nonuniform g: deletion and partial inception

All of our solutions were concerned with a Spot It deck in which every symbol appears exactly g times. The commercial deck has $s = 8$ on all cards, but contains $n = 55$ cards instead of $s(g-1)+1 = 8(7) = 57$ cards, assuming $g = 8$. The most likely explanation is that two cards were simply deleted. Deletion from a deck created with the methods above yields a deck in which s remains constant but g can vary. This could also occur if we replaced *some* of the complete subgraphs K_g with "incepted" graphs (section 8). It's also likely that we could greedily construct a deck with consistent s if we were unconcerned if g were uniform across colors.

11 Open questions

In addition to the many open questions in Steiner Systems[**6**], which would include the whether any $g|s$ or $g|s-1$ has a solution, we have other questions about our systems:

- For an it be true that $g|s(s-1)$ but not true that $g|s$ or $g|s-1$ (e.g. $g = 6, s = 9$)?
- Are all constructions where $g = s - 1$ isomorphic?

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[6] Wikipedia: https://en.wikipedia.org/wiki/Steiner_system