Dinosaur War: A Strategic Game of Utter Chance

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3/2/23

Abstract

We present a modified version of the simple game *War* with an equal deck set, no replacement, and dealer choice, invented in large part and played by my preschool children. Unlike *War*, there are choices that must be made by the players. But like *War*, the outcome of the game, when played by rational agents, remains 100 percent chance. This new game, *Dinosaur War*, resembles something more akin to *Rock-Scissors-Paper*; knowing an opponent's guess can guarantee a win, but like *Rock-Scissors-Paper*, we show a Nash Equilibirum occurs if both players randomize their guesses uniformly across their options. This result is intuitive but non-obvious.

Therefore:

- You can play optimally against your child by paying no attention at all.
- Expect a Pokemon-branded version to hit the shelves soon.

1 The Game

Children's games need to be simple. The game *Memory* has seen innumerable rebranded recreations, not least because the mechanic is approachable (and nominally educational) but because it can be sold repeatedly, with cartoon characters, animals, or whatever to engage a short attention span. A set of*Memory* comes with matched pairs of cards with identical backs. Once the main mechanic is exhausted, the enterprising child will find some other game to create with them. Here is that game, *Dinosaur War*, created with the cards like those in Figure 1.

1.1 Rules of Dinosaur War

- Players establish a ranking of cards. Those might be the commonly-accepted Ace-to-2 of a deck of playing cards, or "Baryonyx beats Mosasaurus beats T-Rex... beats Apatosaurus" in Figure 1.
- Two players get each get an identical deck of these cards. They were unique in this

Figure 1: Dinosaur Cards

set but need not be. Players conceal their hand (though the content of the hands is well-known to those tracking it).

- At each turn:
  - Each player simultaneously plays a card face up.
  - The player whose card outranks the others gets one point. If there is a tie, no points are awarded. The two played cards are set aside.
  - Play continues until the cards are exhausted.
- The player with the most points at the end wins.

The maximum score individual score is 9 (since your opponent's 10 cannot be beat, only tied). Ties are relatively common.

1.2 "Strategy" in Dinosaur War

Intuitively, your hand has a certain amount of "power" that you deploy to beat an opponent; spending the minimum amount of "power" to win preserves better cards for later.

Imagine on the first turn of a 10 card deck $\{1, 2, ...10\}$, players (P_A, P_B) play respective cards $(9, 10)$. This means:

- P_B takes a one-point lead.
- 10 is preserved for P_A. They will necessarily win one hand in the future.
- The powerful card 9 is lost for P_A.

Alternatively, imagine the first move is $(1, 10)$. This means:

- P_B takes a one-point lead.
- 10 is preserved for P_A. They will necessarily win one hand in the future.
- 1 is lost for P_A, the worst card in the hand.
- Each card of P_A's hand beats at least one card in P_B's hand.

The second scenario *seems* better [1]. But how much better? And how can one strategically strive to lose bad cards and win "by just enough" to take tricks? This is the focus of the paper.

1.3 A reduced example

Throughout, we'll use the following conventions:

- P_A's available options are listed in bold down the left column of the payoff matrix (Fig 2a, 2b).
- P_B's available options are listed in bold across the top row.
- A trick has a payoff of 1 if P_A wins, and -1 if P_B wins. P_A is trying to get the total score as high above zero as possible, P_B below.
- For these payoff matrices M, the cell at row i, column j is the value of that trick, plus the expected value of the remaining game.

This is easy to see in Fig 2a, where the hands are identical. There are only four games of (P_A, P_B) move pairs:

- $(1, 1)$ means the first trick payoff is zero, and the rest of the game (necessarily $(3, 3)$) is determined, also of payoff zero.
- $(3, 3)$ follows similarly.
- $(3, 1)$ means the first trick payoff is 1, and the rest of the game (necessarily $(1, 3)$) pays off -1, for a total of zero.
- $(1, 3)$ follows in reverse, with another time game.

It's clear that *any strategy* is equivalent in this very boring small game. P_A *could even announce his moves before* P_B *selects a card, and the result of the game is still determined.* The expected value of the game is zero.

Observe in figure 2b that starting sets of $P_A : \{1, 5\}, P_B : \{3, 4\}$, while not identical, also yield this result; the choices don't matter in the end.

But some uneven sets of cards, like in Fig 2c, are different.

[1] In this paper, we measure goodness by expected tricks taken by the hand.

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{1} & \mathbf{3} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{3} \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

(a) Even 2x2 game

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{3} & \mathbf{4} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{5} \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

(b) Another even 2x2 game

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{1} & \mathbf{3} \end{array} \\ \begin{array}{c} \mathbf{2} \\ \mathbf{4} \end{array} & \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{array}$$

(c) Uneven 2x2 game

Figure 2: Simple 2x2 games

- If P_A is able to play his 2 against a 1 (on either first or second trick), he wins both tricks for a score of 2.
- If P_A plays his 2 against a 3, this trick score is -1, but guaranteed to balance by the imminent (or recently played) $(4, 1)$ trick, for a total of zero.

This is more like *Rock-Scissors-Paper*: knowing your opponent's choice wins you the game. However, like RSP, the existence of better choices does not mean that there exists a perfect-information strategy that benefits one player.

How can we quantify the goodness of one hand versus another? We introduce a metric for this particular game [2] called the *Dominance Score* and use this to compute the expected value of more complicated (larger) games.

2 Dominance Score

The dominance score of two equal-sized sets (hands) $A = \{a_1, a_2, ...a_n\}, B = \{a_1, a_2, ...a_n\}$ is $\boxed{\text{D(A,B)} = \sum_{i=1}^{n} \sum_{j=1}^{n} T(a_i, b_j)}$, where

$$T(a,b) = \begin{cases} 1 & a > b \\ 0 & a = b \\ -1 & a < b \end{cases}$$

[2] which should not be conflated with a *dominant* Nash strategy

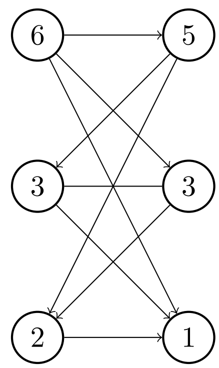

Figure 3: Dominance graph of $\{2, 3, 6\}$ vs. $\{1, 3, 5\}$

This is just adding up all possible wins for A and subtracting all possible wins for B, ignoring ties. For example, if $A = \{2, 3, 6\}, B = \{1, 3, 5\}$, then $D(A, B) = [T(2, 1) + T(3, 1) + T(6, 1) + T(6, 3) + T(6, 5)] + [T(2, 3) + T(2, 5) + T(3, 5)] = 5 - 3 = 2$.

- The identical hands in figure 2a necessarily have a dominance score of zero.
- Figure 2b's pair of a hand with the 2nd- and 3rd-highest cards versus one with the lowest and highest rank, also has a dominance score of zero.
- Figure 2c has a dominance score of two, so it's not surprising that the expected value of the game is in player A's favor (+2).

Another way to visualize hand A against hand B is a bipartite graph like Figure 3, counting "right-pointing" edges as +1, and "left-pointing" edges as -1. This shows that D($\{2, 3, 6\}$, $\{1, 3, 5\}$) = 2.

3 Main Theorem and Proof Layout

The main theorem we wish to prove states:

Given rational players, an optimal score is achieved in Dinosaur War by playing options with uniform randomness.

The steps to proving this are:

- Lemma 1: Across every row and column of the payoff matrix determined by hands A, B, elements sum to the graph's domination score $D(A, B)$.
- Lemma 2: A payoff matrix of such a form of width n has a Nash equilibrium of $p(a_i) = \frac{1}{n}, p(b_j) = \frac{1}{n}$ for all $i, j \in [1, n]$.

$$\begin{array}{cc} & \mathbf{5} \\ \mathbf{6} & \begin{bmatrix} 1 \end{bmatrix} \end{array}$$

(a) 1x1 base case

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{3} & \mathbf{5} \end{array} \\ \begin{array}{c} \mathbf{3} \\ \mathbf{6} \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

(b) 2x2 base case

Figure 4: Base cases

- Corollary: The expected value of the game A, B is then $\frac{D(A,B)}{n}$.
- Corollary: There are no equilibria with a higher expected value in the game.

Finally, we write some code to compute our payoff matrix for a game of $[1, 10]$ vs. $[1, 10]$.

3.1 Lemma 1: Rows and Columns sum to Domination Score

Across every row and column of the payoff matrix determined by hands A, B, elements sum to the graph's domination score $D(A, B)$.

We prove this inductively.

Base case, n=1: We see in Figure 4a that D is equal to function H at $n = 1$: a 1 if player A's single card outranks B's, a 0 for a tie, and a -1 if B's outranks. With a single element and therefore single row and column, the Lemma is clearly true.

For the *inductive case*, consider the arrays figure 5a and figure 5b. In each, the vertical axis is the set of moves A for player P_A, and the horizontal the move set B for player P_B.

In Figure 5a, the element (i, j) represents the game continuations should the plays be (a_i, b_j) (so, excluding cards a_i and b_j).

Without loss of generality, consider row $a_1 = 2$ in Figure 5a (the subgames) and in figure

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(Bonus) base case, n=2: To see how this extends to

TODO

$$\begin{array}{cc}
 & \begin{matrix}\mathbf{1} & \qquad\qquad\mathbf{3} & \qquad\qquad\mathbf{5}\end{matrix} \\
\begin{matrix}\mathbf{2}\\[3em]\mathbf{3}\\[3em]\mathbf{6}\end{matrix} &
\begin{bmatrix}
\begin{array}{cc} & \begin{matrix}\mathbf{3}&\mathbf{5}\end{matrix}\\ \begin{matrix}\mathbf{3}\\\mathbf{6}\end{matrix} & \begin{bmatrix}1&0\\0&1\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1}&\mathbf{5}\end{matrix}\\ \begin{matrix}\mathbf{3}\\\mathbf{6}\end{matrix} & \begin{bmatrix}2&0\\0&2\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1}&\mathbf{3}\end{matrix}\\ \begin{matrix}\mathbf{3}\\\mathbf{6}\end{matrix} & \begin{bmatrix}2&1\\1&2\end{bmatrix}\end{array} \\
\begin{array}{cc} & \begin{matrix}\mathbf{3}&\mathbf{5}\end{matrix}\\ \begin{matrix}\mathbf{2}\\\mathbf{6}\end{matrix} & \begin{bmatrix}0&0\\0&0\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1}&\mathbf{5}\end{matrix}\\ \begin{matrix}\mathbf{2}\\\mathbf{6}\end{matrix} & \begin{bmatrix}2&0\\0&2\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1}&\mathbf{3}\end{matrix}\\ \begin{matrix}\mathbf{2}\\\mathbf{6}\end{matrix} & \begin{bmatrix}2&0\\0&2\end{bmatrix}\end{array} \\
\begin{array}{cc} & \begin{matrix}\mathbf{3}&\mathbf{5}\end{matrix}\\ \begin{matrix}\mathbf{2}\\\mathbf{3}\end{matrix} & \begin{bmatrix}-2&-1\\-1&-2\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1}&\mathbf{5}\end{matrix}\\ \begin{matrix}\mathbf{2}\\\mathbf{3}\end{matrix} & \begin{bmatrix}0&0\\0&0\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1}&\mathbf{3}\end{matrix}\\ \begin{matrix}\mathbf{2}\\\mathbf{3}\end{matrix} & \begin{bmatrix}1&0\\0&1\end{bmatrix}\end{array}
\end{bmatrix}
\end{array}$$

(a) Recursive game matrix

$$\begin{array}{cc}
 & \begin{matrix}\mathbf{1} & \qquad\qquad\mathbf{3} & \qquad\qquad\mathbf{5}\end{matrix} \\
\begin{matrix}\mathbf{2}\\\mathbf{3}\\\mathbf{6}\end{matrix} &
\begin{bmatrix}
(\mathit{1}+.5=1.5) & (-\mathit{1}+1=0) & (-\mathit{1}+1.5=.5) \\
(\mathit{1}+0=1) & (\mathit{0}+1=1) & (-\mathit{1}+1=0) \\
(\mathit{1}+-1.5=-.5) & (\mathit{1}+0=1) & (\mathit{1}+.5=1.5)
\end{bmatrix}
\end{array}$$

(b) Payoff matrix

Figure 5: $\{2,3,6\}$ vs. $\{1,3,5\}$

$$
\begin{array}{cc}
 & \begin{matrix} \mathbf{1} & \qquad\qquad\qquad & \mathbf{3} & \qquad\qquad\qquad & \mathbf{6} \end{matrix} \\
\begin{matrix} \mathbf{2} \\[4em] \mathbf{3} \\[4em] \mathbf{6} \end{matrix} &
\begin{bmatrix}
\begin{array}{cc} & \begin{matrix}\mathbf{3} & \mathbf{6}\end{matrix} \\ \begin{matrix}\mathbf{3}\\ \mathbf{6}\end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1} & \mathbf{6}\end{matrix} \\ \begin{matrix}\mathbf{3}\\ \mathbf{6}\end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1} & \mathbf{3}\end{matrix} \\ \begin{matrix}\mathbf{3}\\ \mathbf{6}\end{matrix} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \\
\begin{array}{cc} & \begin{matrix}\mathbf{3} & \mathbf{6}\end{matrix} \\ \begin{matrix}\mathbf{2}\\ \mathbf{6}\end{matrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1} & \mathbf{6}\end{matrix} \\ \begin{matrix}\mathbf{2}\\ \mathbf{6}\end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1} & \mathbf{3}\end{matrix} \\ \begin{matrix}\mathbf{2}\\ \mathbf{6}\end{matrix} & \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{array} \\
\begin{array}{cc} & \begin{matrix}\mathbf{3} & \mathbf{6}\end{matrix} \\ \begin{matrix}\mathbf{2}\\ \mathbf{3}\end{matrix} & \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1} & \mathbf{6}\end{matrix} \\ \begin{matrix}\mathbf{2}\\ \mathbf{3}\end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} &
\begin{array}{cc} & \begin{matrix}\mathbf{1} & \mathbf{3}\end{matrix} \\ \begin{matrix}\mathbf{2}\\ \mathbf{3}\end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}
\end{bmatrix}
\end{array}
$$

(a) Recursive game matrix

$$
\begin{array}{cc}
 & \begin{matrix} \mathbf{1} & \qquad\qquad & \mathbf{3} & \qquad\qquad & \mathbf{6} \end{matrix} \\
\begin{matrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{6} \end{matrix} &
\begin{bmatrix}
(\mathit{1} + 0 = 1) & (-\mathit{1} + .5 = -.5) & (-\mathit{1} + 1.5 = .5) \\
(\mathit{1} - .5 = .5) & (\mathit{0} + .5 = .5) & (-\mathit{1} + 1 = 0) \\
(\mathit{1} - 1.5 = -.5) & (\mathit{1} + 0 = 1) & (\mathit{0} + .5 = .5)
\end{bmatrix}
\end{array}
$$

(b) Payoff matrix

Figure 6: $\{2,3,6\}$ vs. $\{1,3,6\}$

3.2 Lemma 2: Uniform Nash Equilibrium

A payoff matrix of such a form of width n has a Nash equilibrium of $p(a_i) = \frac{1}{n}, p(b_j) = \frac{1}{n}$ *for all* $i, j \in [1, n]$.

TODO:

- 2. On such a matrix, there is a nash equilibrium of all evens, where deviating doesn't improve prospects.
- presume all weights are even
- shifting epsilon from one of the rows can't increase your score as row player, if the col player doesn't move.
- If this changes the weighted sum of a column, then player 2 can improve his EV.
- If this changes no columns, then the two rows are identical.
- if it's 123 vs. 789, you can obvously choose whatever probabilities you like.

3.2.1 Corollary: Expected payoff of the game

The expected value of the game A, B *is then* $\frac{D(A,B)}{n}$.

TODO

3.2.2 Other Equilibiria

There are no equilibria with a higher expected value in the game.

- 3. Some theorem - nash equilibria all have the same value on a two-person zero-sum game.
- Minimax theorem, von Neumann, 1928 TODO
- Corollary of Maximin: All nash equilibira have saem value - minimax (or maximin)
- 4. Therefore, even is an optimal strategy.

  TODO

4 Considerations and Examples

- Note: Not THE optimal strategy.
- 5. Of course, knowing or guessing the opponent's actual move is an advantage.S
- Show the table for 10 v 10

5 10 v. 10 payoffs

$$
\begin{array}{c|c}
 & \begin{array}{cccccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \\ \mathbf{10} \end{array} &
\begin{bmatrix}
0 & -8/9 & -2/3 & -4/9 & -2/9 & 0 & 2/9 & 4/9 & 2/3 & 8/9 \\
8/9 & 0 & -8/9 & -2/3 & -4/9 & -2/9 & 0 & 2/9 & 4/9 & 2/3 \\
2/3 & 8/9 & 0 & -8/9 & -2/3 & -4/9 & -2/9 & 0 & 2/9 & 4/9 \\
4/9 & 2/3 & 8/9 & 0 & -8/9 & -2/3 & -4/9 & -2/9 & 0 & 2/9 \\
2/9 & 4/9 & 2/3 & 8/9 & 0 & -8/9 & -2/3 & -4/9 & -2/9 & 0 \\
0 & 2/9 & 4/9 & 2/3 & 8/9 & 0 & -8/9 & -2/3 & -4/9 & -2/9 \\
-2/9 & 0 & 2/9 & 4/9 & 2/3 & 8/9 & 0 & -8/9 & -2/3 & -4/9 \\
-4/9 & -2/9 & 0 & 2/9 & 4/9 & 2/3 & 8/9 & 0 & -8/9 & -2/3 \\
-2/3 & -4/9 & -2/9 & 0 & 2/9 & 4/9 & 2/3 & 8/9 & 0 & -8/9 \\
-8/9 & -2/3 & -4/9 & -2/9 & 0 & 2/9 & 4/9 & 2/3 & 8/9 & 0
\end{bmatrix}
\end{array}
$$

6 Pieceyard

References

[1] Wikipedia: `https://en.wikipedia.org/wiki/Minimax_theorem`