Polynomial Uniqueness by way of Tournament Graphs

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Abstract

In 2D space, two points $(x_1, y_1), (x_2, y_2), x_1 \neq x_2$ define a line, a polynomial of degree 1. Three distinct points $(x_1, y_1), (x_2, y_2), (x_3, y_3) x_1 \neq x_2 \neq x_3 \neq x_1$ define a parabola, a polynomial of degree 2. In general, for finite univariate polynomials of nonnegative, whole degree, $n + 1$ such points uniquely specify a polynomial of degree n. Why?

This is not a new result. This is a paper is simply a thoroughly awkward trip through a few mathematical domains to arrive at a well known destination. Helicopters and cars both have their uses. But you wouldn't build a car by turning a helicopter on its side and adding wheels.

Metaphorically, I do, so you don't have to.

1 Setup

If we have points $f(x_1) = y_1, f(x_2) = y_2, \ldots f(x_{n+1}) = y_{n+1}$, how can we determine the coefficients a_i of the polynomial $f(x) = a_0 x^0 + a_1 x^1 + \ldots + a_n x^n$?

This matrix X, known as a Vandermonde matrix[**1**], models this set of equations as $X \cdot \vec{a} = \vec{y}$:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_{n+1} & x_{n+1}^2 & \ldots & x_{n+1}^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{bmatrix}$$

Therefore, we can find our unique coefficient vector A if and only if we can solve $X \cdot \vec{a} = \vec{y}$, or $\vec{a} = X^{-1}\vec{y}$. This has a unique solution if and only if $\det(X) \neq 0$. The rest of this paper tries to find this determinant through all the wrong ways.

2 Finding the Vandermonde determinant

It should be noted that there are other, clearer methods of finding this determinant[**1**] either starting with polynomial unqiueness (basically, going the "other" direction), abstract algebra, direct linear algebra, vector maps, and likely others. These, however, were not the ones I stumbled on.

First, we know that if any $x_i = x_j$ for distinct i, j, we have no solution, and a zero determinant. If $f(x_i) = f(x_j), x_i = x_j$, then we are simply underdetermined (not enough points for a unique polynomial). If $f(x_i) = f(x_j), x_i \neq x_j$, then we have a impossible vertical section of our graph. Otherwise, we are in good shape.

This suggests that every pair $(x_i, x_j), i < j$ corresponds to a factor $(x_j - x_i)$ in the determinant, and that the determinant is then some multiple of $D = \prod_{0 \leq i < j \leq n}(x_j - x_i)$.

Taking $n = 2$ as a base case ($n = 1$ produces a constant $f(x)$), we see that $\det \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} = (x_1 - x_0)$, suggesting our determinant is exactly D.

The rest of the paper will be handling the inductive step in the most roundabout way possible.

3 Prove : VanDerMonde matrix determinant is prod $(x_i - x_j), 1 <= i < j <= n$

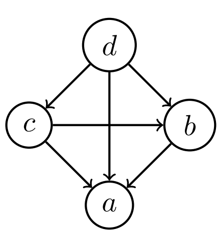
This is the determinant of the van der modne matrix

3.1 Base cane: n = 2

3.2 Inductive case

This equalsx^n (product without x), $+y^n$(product without y)...

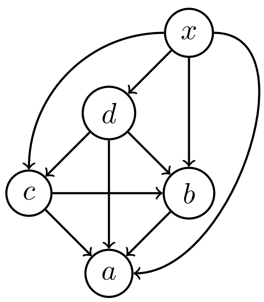
4 Pieceyard



(a) $-x^3a \cdot d^3c^2b^1a^0$, with cycle (xba) (b) $-x^3b \cdot -d^3c^2a^1b^0$, with cycle (xab)

Figure 1: Terms in expanded $\prod(x_j - x_i)$ are inverses with inverted 3-cycles

The sorted tournament $d^3c^2b^1a^0$



The sorted tournament $x^4d^3c^2b^1a^0$

Factors of $(x-a)(x-b)(x-c)(x-d)$ multiplied by $\sigma = d^3c^2b^1a^0$

Factor	Product	Matching Factor	Matching σ	Critical pair
x^4	$x^4d^3c^2b^1a^0$	none	none	none
$-x^3a$	$-x^3d^3c^2b^1a^1$	$-x^3b$	$-d^3c^2a^1b^0$	ba
$-x^3b$	$-x^3d^3c^2b^2a^0$	$-x^3c$	$-d^3b^2c^1a^0$	cb
$-x^3c$	$-x^3d^3c^3b^1a^0$	$-x^3d$	$-c^3d^2b^1a^0$	dc
$-x^3d$	$-x^3d^4c^2b^1a^0$	none	none	none
x^2ba	$x^2d^3c^2b^2a^1$	x^2ca	$-d^3b^2c^1a^0$	cb
x^2ca	$x^2d^3c^3b^1a^1$	x^2da	$-c^3d^2b^1a^0$	dc
x^2da	$x^2d^4c^2b^1a^1$	x^2db	$-d^3c^2a^1b^0$	ba
x^2cb	$x^2d^3c^3b^2a^0$	x^2db	$-c^3d^2b^1a^0$	dc
x^2db	$x^2d^4c^2b^2a^0$	x^2dc	$-d^3b^2c^1a^0$	dc
x^2dc	$x^2d^4c^3b^1a^0$	none	none	none
$-xcba$	$-xd^3c^3b^2a^1$	$-xdba$	$-c^3d^2b^1a^0$	dc
$-xdba$	$-xd^4c^2b^2a^1$	$-xcba$	$-d^3b^2c^1a^0$	cb
$-xdca$	$-xd^4c^3b^1a^1$	$-xdcb$	$-d^3c^2a^1b^0$	ba
$-xdcb$	$-xd^4c^3b^2a^0$	none	none	none
$dcba$	$d^4c^3b^2a^1$	none	none	none

5 TODO

5.1 TODO

References

[1] Wikipedia: `https://en.wikipedia.org/wiki/Vandermonde_matrix`