Brilliant: Vector Calculus

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Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against t of the form $\overrightarrow{x}(t) = \langle x(t), y(t), \ldots \rangle$.

- A **line** through $p = (a, b, c)$ parallel to $\overrightarrow{v} = \langle v_x, v_y, v_z \rangle$ is $\overrightarrow{x}(t) = \overrightarrow{p} + t\overrightarrow{v}$
- **velocity** is characterized completely by $\overrightarrow{v}(t) = \overrightarrow{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- The **speed** of an object along that line versus t is the length of v ($\|v\|$)
- Therefore, the speed of an object along line

  $$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

  is

  $$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that $\overrightarrow{v}$ need not be constant. The speed of

  $$\overrightarrow{x}(t) = \overrightarrow{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

  would then be

  $$\|6\pi\cos(2\pi t)\hat{u}\| = |6\pi\cos(2\pi t)|$$

- **Acceleration** $a(t) = v'(t) = x''(t)$ is straightforward. Acceleration of

  $$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance r from the sun could be $\langle r\cos(t), r\sin(t) \rangle$. The acceleration vector points in the opposite direction: $\langle -r\cos(t), -r\sin(t) \rangle$. In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by $F = ma$ is proportional to acceleration) points towards the sun.

- A **helix** could be a 3D extension like $\langle r\cos(t), r\sin(t), b\cdot t\rangle$.

2 Chapter 2.2: Space Curves

- TODO: Problem 5 - rotating ellipses and solving intersections with planes
- Note that while $\vec{x}(t) = \langle\cos(t), \sin(t), 5\rangle$ and $\vec{x}(t) = \langle\cos(2t), \sin(2t), 5\rangle$ describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If $\vec{x}(t) = t\vec{v}$, then speed is $\frac{\|\vec{x}(t+\Delta t)-\vec{t}\|}{\Delta t} = \|\vec{v}\|$, direction is $\frac{\vec{v}}{\|\vec{v}\|}$, and velocity $\vec{v}$ is the product of speed and direction.
- So $\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\vec{x}(t+\Delta t)-\vec{x}(t)}{\Delta t} = \vec{x}'(t) = \frac{d\vec{x}}{dt} = \langle x'(t), y'(t), z'(t)\rangle$
- Neat conceptual result: any $y = f(x)$ can be made into $x(t) = \langle t, f(t), 0\rangle$, and then $v(t) = \langle 1, f'(t), 0\rangle$, which points along the tangent line at $\langle t, f(t), 0\rangle$.
- Note that dot product derivatives work like regular product: $[\vec{a}(t)\cdot\vec{b}(t)]' = \vec{a}'(t)\cdot\vec{b}(t) + \vec{a}(t)\cdot\vec{b}'(t)$, but the cross product does not work the same since $\frac{d}{dt}[a\times b] = a'\times b + a\times b'$, but since $a\times b' = -b'\times a$, can't switch the order to $a'\times b + b'\times a$ due to this non-commutativity.
- If
$$\vec{x}(t) = \vec{p} + t\vec{v},$$
calculating velocity with respect to origin becomes
$$\frac{d}{dt}\|\vec{x}(t)\| = \frac{\vec{x}(t)\cdot\vec{x}'(t)}{\|\vec{x}(t)\|} = \frac{\vec{x}}{\|\vec{x}\|}\cdot\vec{v},$$
after rewriting the distance formula and chugging through the chain rule.
- However, it becomes more clear when considering that $(\vec{v}\cdot\hat{x})\hat{x}$ is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way: $\int_a^b \vec{x}(t) = \langle\int_a^b x(t), \int_a^b y(t), \int_a^b z(t)\rangle$

- Example: if ball launched from origin with velocity $\langle 1,2,3\rangle$ and acceleration $\langle 0,0,-1\rangle$, it lands at

$$\frac{dv}{dt}dt = \langle 0,0,-1\rangle \tag{1}$$
$$\int \frac{dv}{dt}dt = v = \langle C, D, -t+F\rangle = \langle 1,2,3\rangle = \langle 1,2,-t+3\rangle, t=0 \tag{2}$$
$$x = \int v = \langle t+K, 2t+M, -\frac{1}{2}t^2+3t+N\rangle, x(\overrightarrow{0}) = \langle 0,0,0\rangle \tag{3}$$
$$\overrightarrow{x}(t) = \langle t, 2t, 3t-\frac{1}{2}t^2\rangle \tag{4}$$
$$z(t) = 0 \to t = 6 \to \overrightarrow{x}(6) = \langle 6,12,0\rangle \tag{5}$$
$$\tag{6}$$

- Also, generalizing $ds = \sqrt{(dx)^2+(dy)^2}$, the length of an arc from point a to b aporoaches $\int_a^b \|x'(t)\|dt$
- Example: a helix $\langle a\cos(\omega t), a\sin(\omega t), b\omega t\rangle$, parametrized by time t can be rewritten in terms of s, the arc length:

$$s = \int \|x'(t)\|dt \tag{7}$$
$$s = \int \sqrt{(-\omega a\sin(\omega t))^2 + (\omega a\cos(\omega t))^2 + (b\omega)^2}dt \tag{8}$$
$$s = |\omega| \int \sqrt{(a^2+b^2)}dt \tag{9}$$
$$s = |\omega| t\sqrt{a^2+b^2} \tag{10}$$

- *Note: It's weird to think of time in terms of length.* Could be analytically useful?

4 Chapter 2.4: Frenet Formulae

Main idea: Establish three new vectors $\hat{T}(s), \hat{N}(s), \hat{B}(s)$ that change as we move along a space curve, instead of $\overrightarrow{x}(t)$ that changes over an external "time" idea.

Remember that $s = \int_0^t \|\overrightarrow{x}'(\tilde{t})\|d\tilde{t}$, so $\frac{ds}{dt} = \|\overrightarrow{x}'(t)\|$.

4.1 $\hat{T}$: Vector tangent to space curve

- Remember arc length is $s = \int_0^t \|\overrightarrow{x}'(\tilde{t})d\tilde{t}\|$
- $\hat{T}$ is just normalized grad: $\frac{\overrightarrow{x}'(t)}{\|\overrightarrow{x}'(t)\|}$

- This implies $\frac{d\overrightarrow{x}}{ds} = \hat{T}$ since

$$s = \int_0^t \|\overrightarrow{x}'(\tilde{t})d\tilde{t}\| \tag{11}$$

$$\frac{ds}{dt} = \|\overrightarrow{x}(t)\| \tag{12}$$

$$\hat{T} = \frac{\overrightarrow{x}'(t)}{\|\overrightarrow{x}'(t)\|} = \frac{d\overrightarrow{x}}{dt} \cdot \frac{dt}{ds} \tag{13}$$

$$\hat{T} = \frac{d\overrightarrow{x}}{ds} \tag{14}$$

$$\tag{15}$$

4.2 $\hat{N}$: Vector normal to space curve and also in the direction of acceleration

Normal vectors include

- $\frac{\hat{T}(t)}{\|\hat{T}(t)\|}$ since, as $\|\hat{(}T)\|$ is just 1:

$$d(\|\hat{T}\|^2) = 0 \tag{16}$$

$$d(\|\hat{T}\|^2) = d(\hat{T} \cdot \hat{T}) = \hat{T}(t) \cdot 2\hat{T}'(t) \tag{17}$$

$$\hat{T}(t) \cdot \hat{T}'(t) = 0 \tag{18}$$

- $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|}$ since it's the same as the above, but parametrized over s instead of t. Doesn't change the direction of the vector!

Example:if $\overrightarrow{x}(t) = \langle R\cos(\omega t), R\sin(\omega t), 0\rangle$, then acceleration $\overrightarrow{a}(t)$ is

- $\overrightarrow{a} = \frac{d^2\overrightarrow{x}}{dt^2}$ just by definition
- $\overrightarrow{a} = -\omega^2\overrightarrow{x}$ just by calculation
- $\hat{T}(t) = \langle -\sin(\omega t), \cos(\omega t), 0\rangle$
- $\|\hat{T}(t)\| = 1$
- $\hat{N} = \frac{\hat{T}(t)}{\|\hat{T}(t)\|} = \langle -cos(\omega t), -\sin(\omega t), 0\rangle$
- So $\overrightarrow{a} = R\omega^2\hat{N}$ by these formulae.

This leads us to believe acceleration and $\hat{N}$, the normed derivative of $\hat{T}$ are related.

The part of acceleration $\overrightarrow{a}$ parallel to $\hat{T}$ is the projection $(\overrightarrow{a} \cdot \hat{T})\hat{T}$

The perpendicular part is then $\overrightarrow{a}$ minus that: $\overrightarrow{a} - (\overrightarrow{a} \cdot \hat{T})\hat{T}$

This also equals $(\frac{ds}{dt})^2 \|\frac{d\hat{T}}{ds}\| \hat{N}$ because

$$\overrightarrow{x} = \frac{dx}{dt} = T = \hat{T} \cdot \|\frac{dx}{dt}\| \tag{19}$$

$$s = \int_0^t \|\overrightarrow{x}'(t) \rightarrow \frac{ds}{dt} = \|\overrightarrow{x}'(t)\| \tag{20}$$

$$\hat{N} = \frac{d\hat{T}}{ds} normalized, so \tag{21}$$

$$\begin{aligned}
\overrightarrow{a} = \frac{d^2\overrightarrow{x}}{dt^2} = \frac{d}{dt}(\|\overrightarrow{x}'(t)\hat{T}(t)\|) = \frac{d\|\overrightarrow{x}'(t)\|}{dt}\hat{T} + \|\overrightarrow{x}'(t)\|\frac{d\hat{T}}{dt} && (22)\\
= \frac{d\|\overrightarrow{x}'(t)\|}{dt}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{ds}\frac{ds}{dt} && (23)\\
= \frac{d\|\overrightarrow{x}'(t)\|}{dt}\hat{T} + (\frac{ds}{dt})^2\|\frac{d\hat{T}}{ds}\|\hat{N} && (24)
\end{aligned}$$

This is "a = parallel part plus perpendicular (N) part", so the second term is $a_\perp$

4.3 Binormal vector $\hat{B}$

Note that curvature $\kappa(s) = \|\frac{d\hat{T}}{ds}\|$ is geometric (depends on s, not time) and changes as $\hat{T}$ changes.

Example: Curvature of $\overrightarrow{x}(t) = \langle \cos(t), \sin(t), bt \rangle$

$$x'(t) = \langle -\sin(t), \cos(t), b \rangle \tag{25}$$

$$\|x'(t)\| = \sqrt(1 + b^2) \tag{26}$$

$$s = \int_0^t \|x'(t)\| = \int_0^t \sqrt{(1 + b^2)} = t\sqrt{(1 + b^2)} \rightarrow t = \frac{s}{\sqrt{1 + b^2}} \tag{27}$$

4.4 $\hat{T}$ is:

- $\overrightarrow{x}'(t)$ normalized
- The tangent vector to the curve
- The same whether paremetrized by $\hat{T}'(t)$ or $\frac{dx}{ds}$

4.5 $\hat{N}$ is:

- $\overrightarrow{x}''(t)$ normalized as $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} = \hat{N}$

- The normal vector to the curve
- $\perp$ to $\hat{T}$ in direction of acceleration. So a multiple of acceleration vector.
- The same whether paremetrized by $\hat{T}'(t)$ or $\frac{dx}{ds}$

4.6 $\hat{T}$ and $\hat{N}$

:

- Form a plane, since first, any normal vector's derivative is perpendicular to the vector

$$\frac{d}{ds}\|\hat{T}\|^2 = \frac{d}{ds}\hat{T}\cdot\hat{T} \tag{28}$$
$$= 2\hat{T}\cdot\hat{T}' \tag{29}$$
$$\frac{d}{ds}\|\hat{T}\|^2 = \frac{d}{ds}1 = 0 \tag{30}$$
$$\tag{31}$$

and

$$\hat{T}\cdot\hat{N} = \hat{T}\cdot\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} \tag{32}$$
$$= \hat{T}\cdot\frac{\hat{T}'(s)}{\|\frac{d\hat{T}}{ds}\|} = 0 \tag{33}$$

- κ is curvature: how much we're curving in that $T\times N$ plane.
- $\kappa = \|\frac{d\hat{T}}{ds}\|$
- Therefore, by above, $\frac{d\hat{T}}{ds} = \kappa\hat{N}$ **(Frenet equation 1)**

4.7 $\hat{B}$ is binormal: perpendicular to both

- defined as $\hat{B} = \hat{T}\times\hat{N}$
- Therefore, by derivative

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds}\times\hat{N} + \hat{T}\times\frac{d\hat{N}}{ds} \tag{34}$$
$$\frac{d\hat{B}}{ds} = \kappa\hat{N}\times\hat{N} + \hat{T}\times\frac{d\hat{N}}{ds} \tag{35}$$
$$\frac{d\hat{B}}{ds} = \hat{T}\times\frac{d\hat{N}}{ds} \tag{36}$$
$$\tag{37}$$

but this means T is orthogonal to dB, and we already know B and dB are orthogonal. We're working in 3d with the cross product, so dB is parallel to N.

- Therefore, we define "torsion" τ so that $-\frac{d\hat{B}}{ds} = \tau\hat{N}$ **(Frenet equation 2)**. Negative sign by convention.
- Can also cross by N on both sides to get $-\frac{d\hat{B}}{ds} \times \hat{N} = \tau$
- $\hat{B}$ measures how the plane defined by $\hat{T}, \hat{N}$ twists around. On a circle, $\hat{B}$ wouldn't change, so the derivative would be zero.
- **Final Frenet equation**. Prereq: $\hat{B} = \hat{T} \times \hat{N} \rightarrow \hat{N} = \hat{B} \times \hat{T} \rightarrow \hat{T} = \hat{N} \times \hat{B}$

$$\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} \tag{38}$$

$$\frac{d\hat{N}}{ds} = -\tau\hat{N} \times \hat{T} + \hat{B} \times \kappa\hat{N} \tag{39}$$

$$\frac{d\hat{N}}{ds} = \tau\hat{B} - \kappa\hat{T} \tag{40}$$