Brilliant: Differential Equations II

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Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 1: Basics

1.1 Chapter 1: Nonlinear Equations

The two types of problems in this course are:

- Nonlinear equations (several equations on one independent variable)
- Partial differential equations (single equation with several independent variables)

Linear equations have solutions like y_1, y_2 that can be combined using any $c \in \mathbb{R}$ like $y_1 + cy_2$.

Example: Bacteria in a dish with a lot of food, no deaths

- $b'(t) = r_b b(t), r_b > 0. r_b$ would be the rate of growth.
- This is linear. Reason 1: $\frac{d}{dt}(y_1 + cy_2) = y_1' + cy_2' = r_b(y_1 + c_y 2)$ since $y' = r_b y(t)$, and same for y2.
- Also, this works because the solution is $b(t) = b(0)e^{r_b t}$, so $b_1(t) + cb_2(t) = b_1(0)e^{r_b t} + cb_2(0)e^{r_b t} = (b_1(0) + cb_2(0))e^{r_b t}$

Example: **Logistic** equation: Bacteria in a dish with a lot of food, limited by carrying capacity M.

- $b'(t) = r_b b(t)[M - b(t)]$.
- This is nonlinear. Reason: $\frac{d}{dt}(y_1' + cy_2') = y_1' + cy_2' = r_b[y_1 + cy_2][M - y_1 - cy_2] = My_1 + Mcy_2 - y_1^2 - 2cy_1y_2 - cy_1^2y_2^2$
- $\neq My_1 - y_1^2 + Mcy_2 - c^2y_2^2$ because of the extra $-2cy_1y_2$ term.

Sidebar: Note that this equation $b' = r_b b[M - b]$ is *separable*, so it can be solved.

- $\frac{db}{dt} = rb[M - b]$
- $\frac{db}{b(M-b)} = rdt$
- $\frac{1}{M}(\frac{1}{b} + \frac{1}{M-b})db = rdt$ after partial fractions work
- $(\ln(b) - \ln(M - b)) = Mrt + C \Rightarrow \ln(\frac{b}{M-b}) = Mrt + C$
- $\frac{b}{M-b} = e^{Mrt}e^{C}$
- Initial conditions $b = b(0), t = 0 \Rightarrow \frac{b}{M-b} = \frac{b(0)}{M-b(0)}e^{Mrt}$
- $b(1 + \frac{b(0)}{M-b(0)}e^{Mrt}) = M\frac{b(0)}{M-b(0)}e^{Mrt}$
- $b(M - b(0) + b(0)e^{Mrt}) = Mb(0)e^{Mrt}$
- $b = \frac{Mb(0)e^{Mrt}}{M+b(0)[e^{Mrt}-1]}$

This logistic solution will taper off to M at some point. Note that $\lim_{t\to\infty} b(t) = M$ since the non-exponential terms stop mattering. Also $b(t) = M$ sticks as a constant solution or **equilibrium** immediately. *These equilibria tell us what matters - the long-term behavior of solutions!*

Another **Example**: Lotka-Volterra equation pairs: Bacteria (b) and bacteria-killing phages (p), with kill rate k.

- The "product" $kb(t)p(t)$ measures the interactions and kills resulting from this.
- $b'(t) = r_b b(t) - kp(t)b(t)$, or the normal growht rate minus kill rate
- $p'(t) = kp(t)b(t)$ since its population grows as it kills bacteria.
- Equilibria include $b = 0, p = 0$ and $b = 0, p > 0$, since these are *constant* solutions, or places where $b'(t) = 0, p'(t) = 0$.

Direction fields, with vector pointing towards $\langle b'(t), p'(t)\rangle$ (TODO - I think) let us follow the arrows to determine the curve over time. In this case, the bacteria will always go extinct.

However, if we add a new death rate term $-d_p p(t)$ so $p'(t) = -d_p p(t) + kp(t)b(t)$:

- We get an equilibrium at $b = \frac{d_p}{k}, p = \frac{r_b}{k}$. (Since $0 = b'(t) = rb - kpb, (\Rightarrow pk = r), 0 = p'(t) - dp + kpb, (\Rightarrow bk = d)$)
- But otherwise the solutions swirl around this point. This is called a **cycle**. TODO What is a **limit cycle**?

Note that there are systems where the "solution particle" neither reaches an equilibrium or cycles around one point. The **Lorenz system** famously has this owl-eye shaped double attractor (an example of **strange sets**) where initially close particles diverge unpredictably if the constants ρ, σ, b are chosen right:

- $x'(t) = \sigma(y - x)$
- $y'(t) = x(\rho - z) - y$
- $z'(t) = xy - bz$
- TODO

1.2 Chapter 1.2: PDEs

Many methods of attack for PDEs

- Separation of variables
- Power series (Note: did we actually touch on this?)
- Fourier Transform

Example: Standing wave, where one end of a rope is fixed.

- Vertical displacement from a line of rope: $u(x, t)$ depends on where (x) and when (tt).
- Rope's **wave equation** is $u_{tt} = v^2 u_{xx}$, where v is the "constant wave speed", and the others are the space, time partials.
- Note that $u = \cos(vt)\sin(x)$ and $u = \sin(vt)cos(x)$ both work.
- If you guess the solution has split variables like $u = X(x)Y(y)T(t)$, then, upon substitution and division by $X(x)Y(y)T(t)$, $\frac{\delta^2 u}{\delta t^2} = v^2[\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}]$ yields $\frac{T''(t)}{T(t)} = v^2[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}]$
- This method may or may not work. But if it does, it means that since x, y, and t are independent variables, each individual piece must be constant.
- So, for example, if we know $\frac{X''(x)}{X(x)} = -4\pi^2$, we can get to $X(x) = \sin(2\pi x)$
- The wave equation is similar in 3D: $u_{tt} = v^2[u_{xx} + u_{yy} + u_{zz}]$, or using the Laplacian, $u_{tt} = v^2\nabla^2 u$. Here, u measures not displacement but expansion/compression of air at (x, y, z), time t.

Using Fourier transforms helps turn difficult PDEs into an easier problem like an ODE. *Fourier transforms work best* when

- The domain is all of $\mathbb{R}^n$
- The function u vanishes at infinity.

The Fourier transform changes the domain of x to that of ω. It comes with the (highly simplified) rule (see Vector Calculus course): $F[\frac{\delta f}{\delta x}] = i\omega F[f]$. **Example**: Drunkard's walk.

- One dimensional: moves left or right in a random way. Starts at $x = 0, t = 0$.
- $u(x,t)$ is probability of being at point x at time t. Naturally, $\int_{x=-\infty}^{x=\infty} u(x,t)dx = 1$.
- Also, it obeys the 1-dD diffusion equation $\frac{\delta u}{\delta t} = \frac{\delta^2 u}{\delta x^2}$
- The Fourier transform doesn't affect t at all.
- So by taking Fourier transform of both sides of diffusion equation we get
  - $F(u_t) = \frac{\delta}{\delta t}F(u)$ since F doesn't care about t.
  - $\frac{\delta^2 u}{\delta x^2} = i\omega F(\frac{\delta u}{\delta x}) = -\omega^2 F(u)$
  - So $\frac{\delta}{\delta t}F(u) = -\omega^2 F(u)$
  - This is solvable as $F(u) = ce^{-\omega^2 t}$. Take it on faith that $c = \frac{1}{2\pi}$ for now. TODO
  - Known fact: $F[Ae^{-\frac{ax^2}{2}}] = \sqrt{\frac{1}{2\pi a}}Ae^{\frac{-\omega^2}{2a}}$
  - This means $t = \frac{1}{2a}$ and $a = \frac{1}{2t}$
  - $F(u) = \frac{1}{2\pi}e^{-\omega^2 t}, F[Ae^{-\frac{ax^2}{2}}] = \sqrt{\frac{1}{2\pi a}}Ae^{\frac{-\omega^2}{2a}}$ so $u = Ae^{\frac{-ax^2}{2}}$
  - Solving, you get $A = \sqrt{\frac{1}{4\pi t}}, a = \frac{1}{2t}$, so $u(x,t) = \sqrt{\frac{1}{4\pi t}}e^{-\frac{x^2}{4t}}$

2 Chapter 2: Nonlinear Equations

2.1 2.1: Lotka-Volterra I

Major ideas:

- **phase plane**: TODO
- **nullcline**: TODO
- **direction field**: TODO
- **equilibria**: TODO

Example: Bacteria vs. phages (again)

- Bacteria unrestrained grow in proportion to their population, so $\frac{db}{dt} = r_b b(t)$ (solved: $b(t) = b(0)e^{r_b t}$)
- Phages unfed decrease in proportion to current size, so $\frac{dp}{dt} = -d_p p(t)$ (solved: $p(t) = p(0)e^{-d_p t}$)
- Bacteria die with likelihood of meeting a phage, and phages increase with likelihood of meeting a bacterium. So the set of equations, for constant k, becomes:
  - $b'(t) = r_b b(t) - kb(t)p(t)$
  - $p'(t) = -d_p p(t) + kb(t)p(t)$
  - *The product of p and b makes our equations nonlinear* (WHY?)
  - I guess, very generally, $b_1 p_1 = k, b_2 p_2 = k$, but $(b_1 + b_2)(p_1 + p_2) = b_1 p_1 + b_2 p_2 + b_1 p_2 + b_2 p_1 = 2k + b_1 p_2 + b_2 p_1 \neq 2k$, so the last two "mixed" terms mean you can't just add solutions (b_1, p_1) and (b_2, p_2).

General thoughts on this solution:

- So a solution $(b(t), p(t))$, traces out a curve on the bp-phase plane (b is x-axis, p is y-axis) as time (unrepresented in the plane) continues.
- If we add a unit tangent vector at every point (B, P) aligned with $(b'(t), p'(t)) = (r_b B - kBP, -d_p P + kBP)$, we can follow the arrows to see the solution over time.
- The above is called a **direction field**
- This is sometimes hard to sketch analytically, so we can look to the **nullclines**: places where one of the components of the direction field is zero.
- In this case, $r_b B - kBP = (r_b - kP)B = 0$ when $P = 0$ or $P = \frac{r_b}{k}$, and $-d_p P + kBP = (kB - d_p)P = 0$ when $P = 0$ or $B = \frac{d_p}{k}$.
- The **upshot of nullclines** (since we don't care about $P, B \leq 0$): The lines $B = \frac{d_p}{k}, P = \frac{r_b}{k}$ *divide the plane into pieces where the components of this (continuous) function pair can't change sign.*
- For instance, $B > \frac{d_p}{k}, P < \frac{r_b}{k}$ means $r_b b - kbp > 0, -d_p p + kdp > 0$, so both populations are growing here. This helps to sketch the curve.
- The curve looks like a counterclockwise whirlpool around the $(B, P) = (\frac{d_p}{k}, \frac{r_b}{k})$. (bacteria grow with low but growing phages; bacteria decrease as phages overwhelm; both decrease as phages starve; bacteria start coming back)
- The center point is a (constant **equilibrium**) solution, and other solutions swirl around it but don't get attracted or repelled.

There are a few types of equilibria:

- This one is a **center** around which solutions circle.
- A **stable equilibrum** would see small upsets come back to an unchanging state.
- An **unstable equilibrum** would see small upsets create wildly divergent paths.

2.2 2.2: Lotka-Volterra II

In the Bacteria-Phage system, we can't yet prove everything rotates around the **center**. Let's do that.

Developing a **conserved quantity** will help to do that. **Example**: Block on a horizontal spring with mass m, spring constant k_s:

- $x(t)$: Displacement from rest position.
- $v(t) = \frac{dx}{dt}$: Horizontal velocity
- $\frac{dv}{dt} = -\frac{k_s}{m}x(t)$ by Hooke's law, I think.
- Suppose there's some Energy function $E(x,v)$. By chain rule $\frac{d}{dt}E(x(t),v(t)) = \frac{dE}{dx}\frac{dx}{dt} + \frac{dE}{dv}\frac{dv}{dt}$
- $= \frac{dE}{dx}v - \frac{k_s}{m}\frac{dE}{dv}x$. If we set E as conserved, as in $E'(t) = 0$, then $\frac{dE}{dx}v = \frac{k_s}{m}\frac{dE}{dv}x$
- We can eyeball and see that $E = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ solves this equation, or we can assume $E(x,v) = F(x) + G(v) \Rightarrow 0 = E'(t) = F'(x)v - \frac{k_s}{m}G'(v)x = 0$ from the above equations and guess from there.
- This means in the xv phase space, that there's a fixed E such that the particle follows the ellipse $E = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ in phase space around the solution point (0,0).

Extended Example: Continuing on finding a conserved quantity for Bacteria / Phage:

- We need to find $U(b(t),p(t))$ such that $U'(t) = 0$, or by chain rule $\frac{\delta U}{\delta b}\frac{\delta b}{\delta t} + \frac{\delta U}{\delta p}\frac{\delta p}{\delta t} = 0$
- Subbing in, $\frac{\delta U}{\delta b}[r_b b - kbp] + \frac{\delta U}{\delta p}[-d_p p + kbp] = 0$
- A hint suggests finding U such that $\frac{\delta U}{\delta b} = -\frac{d_p}{b} + k, \frac{\delta U}{\delta p} = -\frac{r_b}{p} + k$ to make terms cancel.
- Integrating these gives us U as both $-d_p \ln(b) + kb + Q(p)$ and $-r_b \ln(p) + kp + R(b)$ so $U = -d_p \ln(b) - r_b \ln(p) + kb + kp$. This weird curve consistutes a level set in pb-space upon which a solution sits.
- The spring example has an elliptic paraboloid solution. There's an absolute minimum ($E = 0$ at $(0,0)$) but level sets become closed loops away from it.

- For the Lotka example, there is a critical point ($\nabla U = \vec{0}$) when $\nabla U(b,p) = (\frac{\delta U}{\delta b}, \frac{\delta U}{\delta p}) = (k - \frac{d_p}{b}, k - \frac{r_b}{p})$, which is $(0,0)$ at our known center $(\frac{d_p}{k}, \frac{r_b}{k})$
- Showing that we always increase gong away from the point $(\frac{d_p}{k}, \frac{r_b}{k})$ should guarantee us closed level sets.
- One method: Assume we're picking a unit vector $\vec{v} = \langle \hat{v_b}, \hat{v_p} \rangle$ so that our line from our center is $\vec{v} = \langle \frac{d_p}{k} + tv_b, \frac{r_b}{k} + tv_b \rangle$. $U = F(b) + G(p)$ in this case, so sub the b part into F to get $F(\frac{d_p}{k} + t\hat{v_b}) = d_p[1 - \ln(\frac{d_p}{k} + t\hat{v_b})] + kt\vec{v_b}$. Taking derivative of that w.r.t t shows it is always positive. Same goes for the G(p) portion of U.
- Another (DF) method: Note that $\nabla U = (k - \frac{d_p}{b}, k - \frac{r_b}{p})$'s grad (second derivative) is always positive. So derivative always has positive curvature (maybe using that term wrong), and we'll always increase around this point.
- Also, we know that the particle travels around the level set (loop) and doesn't reverse course, because then, $b'(t) = p'(t) = 0$, and we only have that at the center point (nullcline intersection).

2.3 2.3: Linearization

Extended Example: Suppose there's a limit to bacterial growth, so we cap our population at M_b.

- If $b(t) << M_b$, things should be similar. If $b(t)$ is nearly M_b, then growth should approach 0. So, this implies $\frac{db}{dt} = r_b b(t) \rightarrow \frac{db}{dt} = r_b b(t)(1 - \frac{b(t)}{M_b})$. Note: This isn't the only possibility but we'll use it.
- This updates our Lotka-Volterra model to something more complicated:
  - $b'(t) = r_b b(t)(1 - \frac{b(t)}{M_b}) - kb(t)p(t)$
  - $p'(t) = -d_p p(t) + kb(t)p(t)$
- Other than $b = 0, p = 0$, the meaningful nullclines are solved by setting $b'(t) = 0$ (yielding $r_b(1 - \frac{b}{M_b}) - kp = 0$) and $p'(t) = 0$ (yielding $b = \frac{d_p}{k}$)
- Note: We'll clean up through some MAGIC non-dimensionalization (how to derive?) to simplify:
  - $x(t) = \frac{1}{M_b} b(\frac{t}{r_b}), y(t) = \frac{k}{r_b}(\frac{t}{r_b}), \alpha = \frac{d_p}{r_b}, \beta = \frac{kM_b}{r_b}$
  - Gives us new equations: $\frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t)$, $\frac{dy}{dt} = -\alpha y(t) + \beta x(t)y(t)$
  - And new nullclines: $x + y = 1, x = \frac{a}{b}$

- So there's an equilibrium point in the positive xy quadrant if: $y = 1 - x = 1 - \frac{\alpha}{\beta}$ and $y > 0$ implies $1 - \frac{\alpha}{\beta} > 0 \Rightarrow \frac{\alpha}{\beta} < 1$
- Looking at the direction field, it appears solutions swirl around and are attracted *into* the center point $(\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta})$, making it a **stable equilibrum**

This is similar to the block-spring example, if a damping term $-\frac{\gamma}{m}v$ is added.

- $\frac{dx}{dt} = v, \frac{dv}{dt} = -\frac{k_s}{m}x - \frac{\gamma}{m}v$
- This can be thought of in matrix terms: $\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{\gamma}{m} \end{pmatrix}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ Call the matrix A.
- From Diff Eq I, the solution is $exp(tA)$ (matrix exponential), making x(t) a linear combination of $e^{\lambda t}$ or possibly $te^{\lambda t}$ terms, with the eigenvalues as λs.
- The eigenvalues in this case, using the quadratic formula, could be:
  - Two real, distinct, negative roots. So, these $e^{\lambda t}$ terms decay, and x(t) levels off.
  - Two distinct complex roots with real part $-\frac{\gamma}{2m} < 0$. This ends up being some sines and cosines multiplied by $e^{-\frac{\gamma t}{2m}}$, which decays too.
  - Finally, if we have a repeated negative real eigenvalue, we have solution $x(t) = Ae^{-\frac{\gamma t}{2m}} + Bte^{-\frac{\gamma t}{2m}}$, also decaying.
  - So any disturbance in the spring will oscillate and come to rest at $x(t) = v(t) = 0$ quickly.

  So with linear systems $\vec{x}'(t) = A\vec{x}(t)$, the eigenvalues determine what happens around the equilibrium point. However, the **bacteria-phage model is non-linear**. Here is **how we linearize** for nearby solutions in a nonlinear system:

  - Set small disturbance $\delta x(t) << 1, \delta y(t) <<$ so $x(t) = \frac{\alpha}{\beta} + \delta x(t), y(t) = 1 - \frac{\alpha}{\beta} + \delta y(t)$
  - Since they're small, all powers like $\delta x(t)^2$ and $\delta x(t)\delta y(t)$ are considered basically zero.
  - So substitute $x(t) \to \frac{\alpha}{\beta} + \delta x(t), y(t) \to= 1 - \frac{\alpha}{\beta} + \delta y(t)$ into our $\frac{dx}{dt}$ and $\frac{dy}{dt}$ equations.
  - This gives us the A solving $\frac{d}{dt}\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = A\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$, which is $A = \begin{pmatrix} -\frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \\ \beta - \alpha & 0 \end{pmatrix}$ after working through the substitution.

- Finding the eigenvalues here yields the same situation as the block-spring example: decays in all situations.

It turns out through the **Hartman-Grobman Theorem** that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$, for some continuously differential vector field F, if we linearize near equilibrium x_0, then what falls out of this A approach works if the eigenbalues *aren't all purely imaginary.*

It turns out the uncapped bacteria system from before looks like $\frac{d}{dt}\begin{pmatrix}\delta x(t)\\ \delta y(t)\end{pmatrix} = \begin{pmatrix}0 & -1\\ \alpha & 0\end{pmatrix}\begin{pmatrix}\delta x(t)\\ \delta y(t)\end{pmatrix}$, with characteristic equation $\lambda^2 + \alpha = 0, \alpha > 0$. This means both values are imaginary, and we had to use the conserved quantity appraoach!

2.4 2.4: Hartman-Grobman Theorem

Extended Example: Consider a phage that dies off quicky:

- $\frac{db}{dt} = r_b b(t) - k_b b(t)p(t)$, $\frac{dp}{dt} = -r_p p(t) = 0 \cdot b(t)p(t)$, where k_p is the zero (phages don't increase), and k_b is still the kill factor for the bacteria.
- In this base, $b(t) = p(t) = 0$ is the only equilibrium.
- Non-dimensionalize as $x(t) = b(\frac{t}{r_b}), y(t) = \frac{k_b}{r_b}p(\frac{t}{r_b}), \alpha = \frac{r_p}{r_b}$
- This makes the equations $x'(t) = x(t) - x(t)y(t), y'(t) = -\alpha y(t)$, and the nullclines therefore $x(t) = 0, y(t) = 1, y(t) = 0$
- Looking at this six-section dierection field, we see that solutions exactly on the y-axis are attracted to equilibrium $(0, 0)$, and other are repelled.
- This makes sense since if the bacteria is 0, the phage die and approach $(0, 0)$, otherwise the bacteria multiply and win (so it's a *saddle point*)
- The way to tell: linearize the equations. $\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} \approx \begin{pmatrix}1 & 0\\ 0 & -\alpha\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$ since, if $x(t), y(t) << 1, x(y)t(t) = 0$.
- Then the eigenvalues are $\lambda = 1, -\alpha$ so the solution is Ae^t, $Be^{-\alpha t}$ for $x(t), y(t)$ (TODO respectively?) **Hartman-Grobman ensures this is the general solution**.

However, let's solve directly and see if we come to the same result.

- $y'(t) = -\alpha y(t), y(0) = y_0 \Rightarrow y(t) = y_0 e^{-\alpha t}$
- With this in hand, $\frac{dx}{dt} = x(t) - = x(y)y(t) = x(t)[1 - y_0 e^{-\alpha t}]], x(0) = x_0$ separates out to
  - $\frac{dx}{x} = [1 - y_0 e^{-\alpha t}]dt$

- $\ln(x) = [t + \frac{y_0}{\alpha} e^{-\alpha t}] + C$
- $x = e^C e^t \exp(\frac{y_0}{\alpha} e^{-\alpha t})$
- $x(0) = x_0 \Rightarrow e^C = x_0 e^{-\frac{y_0}{\alpha}}$
- $\Rightarrow x(t) = x_0 e^t \exp(\frac{y_0}{\alpha}(e^{-\alpha t} - 1))$

But how do we deform the phase plane so this looks linear? We need some mapping $\vec{h}(x, y) = \langle u(x, y), v(x, y)\rangle$ that is continuous and invertible (so we don't "damage" the phase plane). This is called a **homeomorphism**.

- So near the equilibrium $(0, 0)$, the equations $y'(t) = -\alpha y(t), y(0) = y_0 \Rightarrow y(t) = y_0 e^{-\alpha t}$ linearized for $\delta x, \delta y$ must be similar to those for $u(x(t), y(t), v(x(t), y(t))$
- This means we need $\frac{du}{dt} = u$, $\frac{dv}{dt} = -\alpha v$
- After doing the substitution, we see that $v = v_0 e^{-\alpha t}$ exactly mimics $y(t) = y_0 e^{-\alpha t}$ for the phage solution. So we take $v = y$.
- Therefore, we know that since $u = u_0 e^t$ and $x(ty) = x_0 \exp(t + \frac{y_0}{\alpha}(e^{-\alpha t} - 1))$, that we need $u(x(t), y(t) = u(x_0, y_0)e^t$
- And this is satisfied if we guess $u(x, y) = xe^{-y}\alpha$ and work it out.
- This function $\vec{h}(x, y) = (u, v) = \langle xe^{-\frac{y}{\alpha}}, y\rangle$ is invertible by $(x, y) = \langle ue^{\frac{v}{\alpha}}, v\rangle$, which is continuous.

2.5 2.5: Application - Lasers

Lasers create excited atoms, which then emit photos while transitioning to an unexcited state. This system has a close analogue with the previous phages (like photons) and bacteria (like atoms) model.

- $n(t)$: number of photons in the laser; r_g: rate of photons gained (created by excited atoms transitioning to unexcited state); r_l: rate of photons lost (emitted)
- $\Rightarrow \frac{dn}{dt} = r_g - r_l$ by definition.
- We can assume we're losing a constant k (kill?) portion of photons per unit time, so $\frac{dn}{dt} = r_g - kn(t)$
- $e(t)$: number of excited atoms (that will maybe create photons). Atoms are excited by external energy pump.
- Excited atoms radiate when meeting a photon (which survives the meeting)
- So we can use the same setup from the bacteria: with I the constant of meeting (intersect?), $r_g = Ie(t)n(t) \Rightarrow n'(t) = Ie(t)n(t) - kn(t)$

Mini example: Assume no photons leave (cap the end of the laser)

- $k = 0$ in this scenario.
- So every meeting creates one more photon ($n \to n+1$) while enervating one excited atom ($e \to e-1$). This implies, equivalently:
  - $e + n$ is a conserved quantity,
  - $e(t) + n(t) = e(0) + n(0)$,
  - $[e(t) + n(t)]' = 0$
  - Then, if$k = 0, n'(t) = Ie(t)n(t) - kn(t)$, and coupled with $e'(t) + n'(t) = 0$ above, we have $e'(t) = -Ie(t)n(t)$

Extended example: Atoms spontaneously lose energy. This is actually what happens

- From quantum physics, we have a rate s of atoms just (s)pontaneously losing energty.
- We also have an energy (p)ump that energizes atoms with quantity p.
- Then, our change in (e)xcited atoms is $e'(t) = p - s - Ie(n)(t)$
- So our **final laser equations** are $e'(t) = p - s - Ie(n)(t), n'(t) = Ie(t)n(t) - kn(t)$
- If we want to find the smallest p guaranteeing $n \geq 1$ (there's at least one photo) at equilibrium ($e'(t) = n'(t) = 0$):
  - $n'(t) = 0 \Rightarrow Ien = kn \Rightarrow n(Ie - k) = 0$. If $n \neq 0, \Rightarrow e = \frac{k}{I}$
  - $e'(t) = 0 \Rightarrow Ien = p - se$
  - Together, $p - se = Ien = kn \Rightarrow kn + se = p \Rightarrow kn + s\frac{k}{I} = p$
  - $n \geq 1 \Rightarrow p \leq k + \frac{ks}{I}$
  - **Another tactic**: We could also assume we *start out at equilibrium*, so n_0, e_0 are constant solutions.
  - Solving $n' = 0 = Ie_0n_0 - kn_0, e' = 0 = Ie_0n_0 - se_0 + p$, we find equilibria $n_0 = \frac{p}{k} - \frac{s}{I}, e_0 = \frac{k}{I}$
  - Then, $n_0 \geq 1 \Rightarrow \frac{p}{k} - \frac{s}{I} \geq 1 \Rightarrow p \geq k + \frac{ks}{I}$

Non-dimensionalization time:

- Scale against $e_0(= \frac{k}{I}), n_0(= \frac{p}{k} - \frac{s}{I})$ like this: $x(t) = \frac{n(\alpha t)}{n_0}, y(t) = \frac{e(\alpha t)}{e_0}$

- NOTE: What does this do? This makes (1,1) the equilibrium, as $x(t) = \frac{n_0}{n_0} = 1, y(t) = \frac{e_0}{e_0} = 1$!
- What α lets us tke $n' = Ien - kn, e' = -Ien - se + p$ and write
  - $\frac{dx}{dt} = x(t)y(t) - x(t)$
  - $\frac{dy}{dt} = \frac{1}{k}(\frac{pI}{k} - s)[1 - x(t)y(t)] + \frac{s}{k}[1 - y(t)]$
  - $x' = \frac{\alpha n'(\alpha t)}{n_0} = xy - x = \frac{Ie(\alpha t)n(\alpha t)}{kn_0} - \frac{n(\alpha t)}{n_0}$
  - $\frac{\alpha Ien - \alpha kn(\alpha t)}{n_0} = \frac{Ie(\alpha t)n(\alpha t)}{kn_0} - \frac{n(\alpha t)}{n_0}$
  - $\alpha Ie - \alpha k = \frac{Ie(\alpha t)}{k} - 1 \Rightarrow \alpha(Ie - k) = \frac{Ie-k}{k} \Rightarrow \alpha = \frac{1}{k}$
  - This solves the x equation, and I suppose it can be validated in the y equation (tediously).
  - If we chunk up our (somehow positive?) constants as $c = \frac{1}{k}(\frac{pI}{k} - s), d = \frac{s}{k}$, we end up with $y' = c[1 - xy] + d[1 - y]$
  - We only care about $x, y > 0$, so $x' = 0 = xy - x = x(y - 1)$ implies $y = 1$ is a nullcline
  - $y' = 0 = c[1 - xy] + d[1 - y] = c - cxy + d - dy \Rightarrow c + d = y(d + cx) \Rightarrow y = \frac{c+d}{d+cx}$, a scaled and shifted hyperbola.

  Look at the solutions:

  - It appears we have a counterclockwise swirl around $(1, 1)$, and nearby solutions tend toward this equilibrium.
  - Hartman-Grobman: rewrite our linearized solution in neighborhood of $(1, 1)$ as $x(t) = 1 + \delta x(t), y(t) = 1 + \delta y(t)$
  - Using $x' = xy - x, y' = c[1 - xy] + d[1 - y]$ and $\frac{d}{dt}\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = A \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$, we can solve and write $A = \begin{pmatrix} 0 & 1 \\ -c & -c - d \end{pmatrix}$
  - Eigenvalues: $\lambda = \frac{1}{2}(-c - d \pm \sqrt{(c + d)^2 - 4c})$
    * If square root term is zero, we have repeated eigenvalue, so $\delta x(t), \delta y(t)$ are combos of $e^{-\frac{c+d}{2}}, te^{-\frac{c+d}{2}}$, which decays
    * If square root term is greater than zero, we have two distinct real, negative eigenvalues (since c, d are positive), so this decays.

    * If square root term is less than zero, we have distinct complex eigenvalues, but combos of $e^{-\frac{c+d}{2}}\cos(\frac{1}{2}\sqrt{-(c+d)^2+4c}), e^{-\frac{c+d}{2}}\sin(\frac{1}{2}\sqrt{-(c+d)^2+4c})$ decay too
    * Note : I suppose Hartman-Grobman can't work in purely imaginary eigenvalue scenario, because these kinds of functions don't converge or diverge without a term outside the sin or cos
    * And in any case, since these lambdas aren't strictly imaginary, Hartman-Grobman works.

2.6 2.6: Liapunov Equations

We had some intuition that "nearby" solutions would fall into an equilibrium, but what does "nearby" mean? **Liapunov Equations** help us here. What is the "basin of attraction"?

- Suppose we turn the pump off ($p = 0$), and set spontaneous enervation equal to photon leak $s = k$.
- (TODO?) Somehow we can rescale to $\frac{dx}{dt} = Ie(t)n(t) - kn(t)$, $\frac{dy}{dt} = -Ie(t)n(t) - kn(t)$ which (TODO??) gives us $\frac{dx}{dt} = xy - x$, $\frac{dy}{dt} = xy - y$
- This means equilibria ($x' = y' = 0$) exist at $(0,0), (1,1)$
- If we're turning the pump off, we're looking at equilibrium (0,0). Linearizing, we get $x' = -\delta x, y' = -\delta y$, so a matrix of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
- With repeated non-imaginary (H-G applies!) eigenvalues $-1, -1$, we can see that both e^{-t}, te^{-t} decay, and we get sucked into the origin.

But how do we prove this? Let's find a conserved quantity $U'(x(t), y(t)) = 0$

- $U'(x(t), y(t)) = \frac{\delta U}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta U}{\delta y}\frac{\delta y}{\delta t} = \frac{\delta U}{\delta x}x(y-1) + \frac{\delta U}{\delta y}y(x-1)$
- Setting $\frac{\delta U}{\delta x}x = -x + 1$, $\frac{\delta U}{\delta y}y = y - 1$ makes this zero
- Solving those two by separating variables and combining, we get $U = -x + y + \ln(|\frac{x}{y}|)$
- So if we're stabiling $f = (x-y)$ (why?), we see $(x-y)' = x' - y' = (xy-x) - (xy-y) = x - y = f \Rightarrow f = e^{-t}$
- With $x(0) = x_0, y(0) = y_0 \Rightarrow f(0) = x_0 - y_0, f = x(t) - y(t) = (x_0 - y_0)e^{-t}$
- How to express $y(t)$ while eliminating $x(t)$, knowing $x(y) - y(t) = (x_0 - y_0)e^{-t}$ and $U(x, y) = y - x + \ln(|\frac{x}{y}|)$ is conserved? **The trick:** $U(x_0, y_0) = U(x, y)$ since *it doesn't change*!

- $y_0 - x_0 + \ln(|\frac{x}{y}|) = y - x + \ln(|\frac{x}{y}|) = -(x_0 - y_0)e^{-t} + \ln(|\frac{x}{y}|)$
- $(1 - e^{-t})(y_0 - x_0) = \ln(\frac{x/y}{x_0/y_0})$
- Defining for convenience, $f = \exp((1 - e^{-t})(y_0 - x_0))$, then $f\frac{y}{y_0} = \frac{x}{x_0}$
- Sub in to $x - y = (x_0 - y_0)e^{-t} : y[\frac{x_0}{y_0}f - 1] = (x_0 - y_0)e^{-t}$
- Solve for $y : y = \frac{y_0(x_0 - y_0)e^{-t}}{x_0 f(t) - y_0}$
- Combine with above to get $x = \frac{x_0(x_0 - y_0)e^{-t} f(t)}{x_0 f(t) - y_0}$

- So with equilibria $(0, 0), (1, 1)$, the direction field computer plot shows us attracted to $(0, 0)$ (no laser action) pretty much anywhere left and down from $(1, 1)$ in the x, y phase plane.
- Apparently the linearized solutions near $0, 0$ are $x_{lin} = x_0 e^{-t}, y_{lin} = y_0 e^{-t}$ (WHY?)
- Looking above, if $(x_0 - y_0) \approx 0$, then $f(t) \approx 1$, and $x, y \to x_{lin}, y_{lin}$

On to **Liapunov** functions, which will tell us perhaps the size of the "basin of convergence", unlike Hartman-Grobman, which just says there is a neighborhood.

A **Liapunov** function $U(x, y)$ is

- Continuously differentiable
- With a unique minimum (x_0, y_0), usually aligned to be U's only zero.
- $U'(x(t), y(t)) \leq 0$. Everything "flows downhill";
- Tailor made for the problem, hard to find.

Back to the rescaled laser example

- $x'(t) = x(t)y(t) - x(t)$
- $y'(t) = c[1 - x(t)y(t)] + d[1 - y(t)], c, d > 0$
  - **Analogy: The damped-block spring system** $\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{\gamma}{m} \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$
  - When $\gamma = 0$, we know $E(x, v) = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ is conserved when looking at E'
  - $\gamma = 0 \Rightarrow x' = v, v' = -\frac{k_s}{m}x$
  - $\frac{dE}{dt} = (\frac{1}{2}k_s x^2 + \frac{1}{2}mv^2)' = 0$ since $\frac{1}{2}(k_s x x' + mvv') = \frac{1}{2}(kxv + mv\frac{-k}{m}x) = 0$
  - But if $\gamma \neq 0$, $\frac{d}{dt}E(x(t), v(t)) = \frac{d}{dt}[\frac{1}{2}k_s x^2 + \frac{1}{2}mv^2] = kxx' + mvv'$

- $= kx(v) + mv(\frac{-k_s}{m}x - \frac{\gamma}{m}v) = -\gamma v(t)^2 = \frac{dE}{dt}$
- Total spring energy is then decreasing in the fluid.
- Brilliant has Cool visualization of spiraling down into the "bowl" of x, y with E as the z dimension, equilibrium $(0, 0, 0)$
- We need to choose a γ-fied $E-$like function that decreases for pairs $\delta x(t), \delta y(t)$. We can choose, like E, some $u(\delta x, \delta y) = \frac{1}{2}C_1[\delta x]^2 + C_2[\delta y]^2$.
- Choosing $C_1 = c, C_2 = 1$ gives us $\frac{d}{dt}u(\delta x(t), \delta y(t)) = \frac{d}{dt}(\frac{c}{2}[\delta x(t)]^2 + \frac{1}{2}[\delta y(t)]^2)$
- $= c(dx)(dx)' + dy(dy)' = c(dx)(dy) + dy(-c(dx) - (c+d)(dy)) = -(c+d)[\delta y(t)]^2$
- So $u = \frac{d}{dt}(\frac{c}{2}[\delta x(t)]^2 + \frac{1}{2}[\delta y(t)]^2)$ is an energy function that could work for the laser.

Finally, we want to construct a function that

- Doesn't increase (derivative negative) on any pairs $x, y > 0$ (pulls down)
- Is near equal to $u = \frac{c}{2}(x-1)^2 + \frac{1}{2}(y-1)^2$ near $(1, 1)$. (the energy function for block-spring above)
- With $x' - = xy - x, y' = c - cxy + d - dy$, plus the identity near $z \approx 1$ of $\ln(z) \approx (z-1) - \frac{1}{2}(z-1)^2 ...$
- You can find $U(x, y) = c(x-1) + (y-1) - c\ln(x) - \ln(y)$ that satisfies all of these
- It therefore shows that pumped laser solutions tend to equilibrium $(1, 1)$ in the long term.

TODO: So this is enough to establish a convergence to an equilibrium?

- Find an equilibrium (x_0, y_0)
- Find an energy function u that decreases for all pairs $(\delta x(t)\delta y(t))$ near the minimum.
- Find a Liapunov function U function that decreases EVERYWHERE along $x(t), y(t)$ (in our domain, like $x, y > 0$)
- Ensure that $U = u$ in the neighborhood of the equilibrium.
- Then Liapunov's theorem somehow makes this work (TODO)?

2.7 2.7: Dog chasing a duck (Limit Cycles)

This is a pair of nonlinear equations to determine if a dog in the pond's interior catches a duck who skates along the border.

- Variables:
  - r_p: Radius of pond.
  - $\vec{r_H}$: Distance of duck to center (always the radius of the pond)
  - $\vec{l}$: Displacement of dog from duck, which is of some length R at any point.
  - θ: Duck's position in the lake (think polar coordinates)
  - ϕ: Angle between $\vec{r_H}$ and $\vec{l}$.
  - Duck always swims at speed $r_p\theta'(t)$, and dog swims at $k > 0$ times this, or $kr_p\theta'(t)$.
- Therefore $\vec{r_H} = \langle r_p\cos(\theta), r_p\sin(\theta)\rangle$. It's just the polar coordinates.
- Doing some geometry gets you $\vec{l} = R\langle\cos(\theta+\phi, \sin(\theta+\phi)\rangle$
- We can establish $\vec{T} = \vec{r_H} - \vec{l}$ and dog's speed squared $\|T'(t)\|^2 = (\vec{r_H}' - \vec{l}') \cdot (\vec{r_H}' - \vec{l}') = \|\vec{r_H}'\|^2 + \|\vec{l}'\|^2 - 2\vec{r_H}'\vec{l}'$
- Naturally, this $\|T'\|^2$ is also equal to the constant $(kr_p\theta')^2$. Our diff equations will fall out of these.
- $\vec{r_H}^2 = r_p^2[\theta'(t)]^2$ since duck's speed is constant. $\vec{l} = (R')^2 + R^2[\theta' + \phi']^2$ after working it out.
- Finally, after using identities $\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi), \sin(\theta+\phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)$, we can work out $-2\vec{r_H}'\vec{l}' = -2r_p\theta'[R\cos(\phi)(\theta'+\phi') + \sin(\phi)R']$
- After rescaling R to ρ such that $\frac{R}{r_p} = \rho$ and diving our speed equation by constant $r_p\theta'$, we end up with speed equation $k^2 = [\rho(1+\frac{d\phi}{d\theta} - \cos(\phi)]^2 + (\frac{d\rho}{d\theta} - \sin(\phi))^2$
- We propose that there are some solutions here for the **pursuit equations**. We'll ignore the generalized form an focus on one set
  - $\rho(1+\frac{d\phi}{d\theta}) - \cos(\phi) = 0, \frac{d\rho}{d\theta} - \sin(\phi) = -k$ do work in the above. (Doesn't prove others don't work)
  - This leaves our equations as $\frac{d\phi}{d\theta} = \frac{\cos(\phi)}{\rho} - 1, \frac{d\rho}{d\theta} = -k + \sin(\phi)$
  - However, there *aren't simple equilibria here.* In no world with $k \neq 0$ does the dog sit still (or the duck).

- Supposing $k < 1$ and R, ϕ are fixed (dog never gets closer and just loops), this means he's going in a circle, since the two legs of a triangle $(\vec{l}, \vec{r_p})$ and the interior angle (ϕ) are fixed, so this fixes length of the third leg, which is a radius
- You can also use dog's position vectors $x(t) = r_p \cos(\theta) - R\cos(\theta + \phi), y(t) = r_p \sin(\theta) - R\sin(\theta + \phi)$ and trig identities to prove $x(t)^2 + y(t)^2 = r_p^2 + R^2 - 2r_p R\cos(\phi)$
- If $k < 1$, then solving $\frac{d\rho}{d\theta} = 0 = -k + \sin(\phi) \Rightarrow \sin(\phi) = k \Rightarrow \phi = \sin^{-1}(k)$ and $\rho = \cos(\phi) = \cos(\sin^{-1}(k)) = \sqrt{1-k^2}$
  * Quick proof of $\cos(\sin^{-1}(k)) = \sqrt{1-k^2}$:
  * $\cos^2(\sin^{-1}(k)) + \sin^2(\sin^{-1}(k)) = 1 \Rightarrow \cos(\sin^{-1}(k)) = 1 - \sin^2(\sin^{-1}(k))$
  * $= 1 - k^2 \Rightarrow \cos(\sin^{-1}(k)) = \sqrt{1-k^2}$
- When $k < 1$, the direction field seems to have attractive equilibriua but **GOTCHA**: there are ϕ angles that differ by 2π units, so they're the same. The direction field is a cylinder with circumference 2π, and there are other solutions tracking toward $(\sin^{-1}(k), \sqrt{1-k^2})$
- Linearizing, assume we are near our equilibirum point and $\phi = \sin^{-1} k + \delta\phi, \rho = \sqrt{1-k^2} + \delta\rho$.
- We can also remember that $f(x + \delta x) \approx f(x) + f'(x)\delta x$ from calculus.
- $\frac{d}{d\theta}[\delta\rho] = \frac{d}{d\theta}[\rho - \sqrt{1-k^2}] = \frac{d\rho}{d\theta} - \frac{d}{d\theta}\sqrt{1-k^2} = -k + \sin(\phi)$
- $= -k + \sin(\sin^{-1}(k) + \delta\phi)$ and by the calculus rule $\frac{d}{d\theta}[\delta\rho] = -k + \sin(\sin^{-1}(k)) + \cos(\sin^{-1}(k))\delta\phi = \sqrt{1-k^2}\delta\phi$
- And for $\frac{d}{d\theta}[\delta\phi] = \frac{d}{d\theta}\phi - \frac{d}{d\theta}(\sin^{-1}(k)) = \frac{\cos(\phi)}{\rho} - 1$
- Using multivariable hint $f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y$,
- $f = \frac{\cos(\sin^{-1}(k)+\delta\phi)}{\sqrt{1-k^2}+\delta\rho} - 1 \approx \frac{\sqrt{1-k^2}}{\sqrt{1-k^2}} - 1 + \frac{-\sin(\sin^{-1}(k)}{\sqrt{1-k^2}}\delta\phi - \frac{\cos(\sin^{-1}(k))}{1-k^2}\delta\rho$
- $= -\frac{k\delta\phi + \delta\rho}{\sqrt{1-k^2}}$
- So $\frac{d}{d\theta}\begin{pmatrix}\delta\phi \\ \delta\rho\end{pmatrix}\begin{pmatrix}-\frac{k}{\sqrt{1-k^2}} & -\frac{1}{\sqrt{1-k^2}} \\ \sqrt{1-k^2} & 0\end{pmatrix}\begin{pmatrix}\delta\phi \\ \delta\rho\end{pmatrix}$, and the eigenvalues aren't purely imeginary, and the real part is negative, so all decay. Therefore, the equilibrium at $(\sin^{-1}(k), \sqrt{1-k^2})$ attracts nearby solutions.

- There aren't solutions (for $K < 1?$), but numerically solved, the dog catches at $k > 1$, and for $k \leq 1$, swims out to a path approaching a circle. This is a **limit cycle**, an isolated trajectory that closes on itself.

2.8 Poincare-Bendixson Theorem

Limit cycles in the real world: a chemical reaction in perpetual osciallation!

Key concept - **trapping region**: a region in phase plane on some region D, with differential solutions touching every point, where the direction field sees every boundary arrow point IN. This means:

- The solution has to stay in D.
- Any solution that self-intersects forms a cycle in the phase plane.
- The three conceivable ways a solution can "snake" around forever (the **Poincare-Bendixson theorem** says it):
  - Approaches a closed loop in D.
  - Approaches a fixed point in D (possibly a special case of the last bullet)
  - Cycle: Snake eats its own tail
- A non-cycling solution is the only other possibility - just a point equilibirum.

Example: Chemical oscillatory reaction.

- x is concentration of I^-, y is concentration of ClO_2^- ions in some reaction.
- a is positive, and clearly $x, y \geq 0$ in the physical world.
- Otherwise meaningless equations: $\frac{dx}{dt} = 5a - x - \frac{4xy}{1+x^2}$, $\frac{dy}{dt} = x(\frac{4y}{1+x^2})$
- Solve for equilibria by setting $\frac{dx}{dt} = \frac{dy}{dt} = 0$
  - Denote $Q = \frac{y}{1+x^2}$
  - First equation implies $x(1 + 4Q) = 5a$
  - Second euqation, plus knowing $x \neq 0$, $\Rightarrow x(1 - Q) = 0 \Rightarrow Q = 1$
  - $Q = 1 \Rightarrow 5x = 5a \Rightarrow x = a$
  - $\Rightarrow 1 = \frac{y}{1+x^2} \Rightarrow y = 1 + a^2$
  - Only solution pair is $(a, 1 + a^2)$

Linearizing the solution around $(a, 1 + a^2)$

- $x = a + \delta x, y = 1 + a^2 + \delta y \Rightarrow \frac{dx}{dt} = \frac{d[\delta x]}{dt}, \frac{dy}{dt} = \frac{d[\delta y]}{dt}$
- Call $f = \frac{d[\delta x]}{dt} = 5a - x - \frac{4xy}{1+x^2}$,
- Approximate $f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y$
- $f(x, y)(a, 1 + a^2) = 5a - x - \frac{4xy}{1+x^2}(a, 1 + a^2) = 5a - a - 4(a\frac{1+a^2}{1+a^2}) = 0$
- $\frac{\delta f}{\delta x}\delta x(a, 1+a^2) = (-1-\frac{(1+x^2)(4y-2x4xy}{(1+x^2)^2})\delta x(a, 1+a^2)) = (-1-4-\frac{8a^2}{1+a^2})\delta x = \frac{-5+3a^2}{1+a^2}\delta x$
- $\frac{\delta f}{\delta y}\delta y(a, 1 + a^2) = \frac{-4x}{1+x^2}\delta y(a, 1 + a^2)) = \frac{-4a}{1+a^2}\delta y$
- Call $g = \frac{d[\delta y]}{dt} = x - \frac{xy}{1+x^2}$
- Approximate $g(x + \delta x, y + \delta y) \approx g(x, y) + \frac{\delta g}{\delta x}\delta x + \frac{\delta g}{\delta y}\delta y$
- $g(x, y)(a, 1 + a^2) = x - \frac{xy}{1+x^2}(a, 1 + a^2) = a - a\frac{1+a^2}{1+a^2} = 0$
- $\frac{\delta g}{\delta x}\delta x(a, 1 + a^2) = (1 - \frac{(1+x^2)y-xy2x}{(1+x^2)^2})\delta x(= 1 - \frac{(1+a^2)^2-2a^2(1+a^2)}{(1+a^2)^2})\delta x = 2a^2\delta x$
- $\frac{\delta g}{\delta y}\delta y(a, 1 + a^2) = \frac{-x}{1+x^2}\delta y = \frac{-a}{1+a^2}\delta y$
- $\Rightarrow \frac{d}{dt}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} = \frac{1}{1+a^2}\begin{pmatrix}3a^2 - 5 & -4a \\ 2a^2 & -a\end{pmatrix}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix}$
- Let's arbitrarily choose $a = 2 \Rightarrow (a, 1 + a^2) = (2, 5)$. The coefficient matrix ends up being $\frac{1}{5}\begin{pmatrix}7 & -8 \\ 8 & -2\end{pmatrix}$, which has eigenvalues with a positive real $\pm$ some i component. So, Hartman-Grobman applies and we don't decay into our point but push away.

We want to **build the trapping region**.

- Remember, $\frac{dx}{dt} = 10 - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(1 - \frac{y}{1+x^2})$ subbing in 2 for a)
- On the left, if $x = 0$ we see $\frac{dx}{dt} = 10, \frac{dy}{dt} = 0$. So we're pointing right (into the first quadrant region)
- On the bottom, if $y = 0$, we're pointing at $\langle 10 - x, x\rangle$ (into the region).
- On the right, for some $x = b$, $10 - b - \frac{4b}{1+b^2}y$ will make sure we point left.
- On the top, for some $y = c$, $x(1 - \frac{c}{1+x^2} < 0$ makes sure we point down.
- Assume, since we're encircling $(2, 5)$, that $b \geq 3, c \geq 6$ for comfort.
- To satisfy all of these, note $x(1 - \frac{c}{1+x^2}) < 0 \Rightarrow 1 - \frac{c}{1+x^2} < 0 \Rightarrow c > 1 + x^2, 0 < x < b \Rightarrow c > 1 + b^2 \Rightarrow \sqrt{c - 1} > b$

- And for $0 < y < c$, note that $10 - x - \frac{4xy}{1+x^2} < 10 - b < 0$.
- Pick $b = 11$, say, implying $11 < \sqrt{c-1}$, so then $123 < c$. So $(b, c) = (11, 124)$ ensures oscillation around $(2, 5)$ without leaving that region.

Tricky: How to reduce this region? No real way except simulation or some tricks. If we PRESUME a cycle, we can prove the cycle extens to the left of $x = 3$ or $x_{min} < 3$

- **META trick**: Don't worry if you have unsolvable integrals - maybe you can cancel them out. **Run with what you have.**
- Trick: Assume $x(t+T) = x(t), y(t+T) = y(t)$ for some $T > 0$, or that there's a PERIOD T.
- $\int_0^T \frac{dx}{dt} dt = x(T) - x(0) = 0, \int_0^T \frac{dy}{dt} dt = y(T) - y(0) = 0$ by fundamental theorem.
- Our equations again: $\frac{dx}{dt} = 10 - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(1 - \frac{y}{1+x^2})$
- So $0 = \int_0^T [10 - \int x(t) - 4 \int \frac{x(t)y(t)}{1+x(t)^2}]dt$ by the first equation
- $0 = \int x(t) - \int \frac{x(t)y(t)}{1+x(t)^2}]dt$ by the second.
- Subtract four times the second from the first to get $0 = 10T - 5\int_0^T x(t) \Rightarrow 2T = \int_0^T x(t)dt \geq int_0^T x_{min} dt = Tx_{min}$
- So $2 \geq x_{min}$

2.9 Chaos and the Lorenz Equation

What enabled mathematical **chaos** (unpredictability in nonlinear differential equations) was really computers and seeing simulated solutions.

The (simplified) **Lorenz system** are these equations

- $\frac{dx}{dt} = \sigma(y - x)$
- $\frac{dy}{dt} = x(\rho - z) - y$
- $\frac{dx}{dt} = xy - bz$
- All with $\sigma, \rho, b > 0$

Solving the equations, we see equilibria for these are:

- $(0, 0)$ always
- The two solutions $(\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1)$ when $\rho > 1$.

Looking at $0 < \rho < 1$ specifically:

- Linearaizing is simple, with $x(t) = \delta x(t), y(t) = \delta y(t), z(t) = \delta z(t)$ and linearized system:
- $\frac{d[\delta x]}{dt} = \sigma(\delta y - \delta x)$
- $\frac{d[\delta y]}{dt} = \rho\delta x - \delta y$
- $\frac{d[\delta z]}{dt} = -b\delta z$
- $\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix} \approx \begin{pmatrix}-\sigma & \sigma & 0\\ \rho & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix}$
- Characteristic equation is $(-b-\lambda)[(1+\lambda)(\sigma+\lambda)-\sigma\rho] = 0$
- Eigenvalues are $-b < 0$ and $\lambda = \frac{1}{2}[-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}]$
- If $\rho < 1$, we have distinct, real, negative eigenvalues, and a locally attractive equilibrium by Hartman-Grobman.

But if $\rho < 1$ globally attactive? Find a Liapunov function.

- Requirement is that the function $U(x(t), y(t), z(t))$ is minimized at the equilibrium, and that as time progresses, U decreases (so we're sucked into the bowl)
- We suppose that $U(x, y, z) = ax^2 + y^2 + z^2$ and using $xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$:
  - Identity derivation: $0 \leq (x-y)^2 \Rightarrow 0 \leq x^2 - 2xy + y^2 \Rightarrow xy \leq \frac{1}{2}(x^2 + y^2)$
- $\frac{\delta U}{\delta x}x'(t) + \frac{\delta U}{\delta y}y'(t) + \frac{\delta U}{\delta z}z'(t) = 2ax\sigma(y-x) + 2yx(\rho - z) - 2y^2x + 2zxy - 2bz^2$
- $= 2(a\sigma + \rho)xy - 2a\sigma x^2 - 2y^2 - 2bz^2$
- **GOTHCA**: We can't choose $a = -\frac{\rho}{\sigma}$ since then $U = -\frac{\rho}{\sigma}x^2 + y^2 + z^2$ isn't minimized at $(0, 0, 0)$! So a needs to be positive.
- Choosing $a = \frac{1}{\sigma} \Rightarrow a\sigma = 1$, with $\rho < 1 \Rightarrow 2(a\sigma + \rho)xy - 2a\sigma x^2 - 2y^2 - 2bz^2 < 2a\sigma(2xy - x^2 - y^2) - 2bz^2 \leq -2bz^2$ by the identity above.
- Then $U = \frac{1}{\sigma}x^2 + y^2 + z^2$ decreases as $t \to \infty$ and is minimized at the globally attractive $(0, 0, 0)$

If $\rho > 1$ things get chaotic. Instead of one equilibrium, we have two new ones at $(\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1)$. Everything **bifurcates**, or qualitatively shifts when inching past $\rho = 1$:

- We have three equilibria.
- The origin turns into a saddle equilibrium.

- Linearizing around $(\alpha, \alpha, \rho - 1)$ with α denoting $\sqrt{b(\rho-1)}$ (pretty straightforward), we get characteristic equation for A of $-\lambda^3 - (\sigma+b+1)\lambda^2 - b(\sigma+\rho)\lambda - 2\sigma b(\rho-1) = 0$
- Problem is, setting $\rho = 1$ drops the $(\rho - 1)$ term and we have $-\lambda(\lambda^2 + (\sigma + 1 + b)\lambda + b(\sigma + 1)) = 0$, with solutions $\lambda = 0, -b, -\sigma - 1$.
- The last two solutions are attractive, but the zero doesn't work for Hartman-Grobman.
- If we set $\lambda = (\rho-1)\Delta r$ when nudging ρ just over 1, we ignore all $\lambda^2, \lambda^3...$ as negligible and get $-b(\sigma + \rho)(\rho - 1)\Delta r - 2\rho b(\rho - 1) \approx 0$
- This means $\Delta r \approx -\frac{2\rho}{\rho+\sigma}$, or that this nudged root has to be negative when ρ is near 1.
- More rigorously, we could have proven the roots of the equation are negative for small $\rho - 1 > 0$
- In any case, this means that the near-zero root is negative, so $(\alpha, \alpha, \rho - 1)$ attracts locally.
- We can show that this applies the same for $(-\alpha, -\alpha, \rho - 1)$

How do equilibria change as we change ρ?

- We saw the What about as we dial past $\rho = 1$, our origin equilibrium changes from globally attractive to saddle point.
- In going from a stable equilibrium with negative real-part eigenvalues (attractors) to $(0, 0, 0)$ as a saddle (mix of negative and positive real parts), we necessarily have a point where the eigenvalues' real parts are zero.
- In other words, $\lambda = ia$ for some real a.
- Subbing ia into our $-\lambda^3 - (\sigma + b + 1)\lambda^2 b(\sigma + \rho)\lambda - 2\sigma b(\rho - 1) = 0$, we end up getting $[(\sigma + b + 1)a^2 - 2\sigma b(\rho - 1)] + i[a^3 - (b(\sigma + \rho)a] = 0$
- Then we need $a^3 - b(\sigma + \rho)a = 0 \Rightarrow a = 0, a = \pm, \sqrt{b(\sigma + \rho)}$
- If $a = 0$. the real part isn't zero. But subbing $a = \pm\sqrt{b(\sigma + \rho)}$ gives us solutions for a set of $\rho = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$
- So, when moving past this value, our two new equilibria change from locally attractive to saddles too.

Can we create a trapping region?

- The hint: The solutions have to pass through every ellipsoid of form $R^2 = \rho x^2 +]sigma y^2 + \sigma(z - 2p)^2$

- What we need to prove: At every point on the boundary, the direction field points "in", or more specificlally, *the angle between inward normal and direction field is acute.*
- This also means that the gradient ∇g of the level set $R^2 = \rho x^2 + \sigma y^2 + \sigma(z - 2r\rho)^2$ is the normal. This is $\langle 2\rho, 2\sigma, 2(z - 2\rho)\rangle$
- So $-\nabla g \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle > 0 \Rightarrow ... \Rightarrow 2\rho\sigma x^2 + 2\sigma y^2 + 2\sigma z^2 - 4\rho\sigma z > 0$
- Use $R^2 - \rho x^2 - \sigma(z - 2p)^2 = \sigma y^2 \Rightarrow \rho\sigma x^2 + 2\sigma y^2 + 2\sigma z^2 - 4\rho\sigma z > 0$
- This simplifies the dot product to $2R^2 - 8\rho^2\sigma + 4\sigma\rho z + 2\rho(\sigma - 1)x^2 > 0$
- Since the x^2 term is always positive, we just need to set R to clear zero when z is its most negative (x = 0, y = 0, z = $2\rho - \frac{R}{\sqrt{\sigma}}$). If we churn a little more we can see that setting $R > 2\sqrt{\sigma}\rho$ will provide a trapping region.

Question: Do the confined solutions fill up the whole (ellipsoid) container?

- Looking at the divergence (volume change of a cube over time) of the solution will tell us.
- For unspecified reasons, $\frac{1}{v(t)}v'(t) = \nabla \cdot \langle \sigma(y - x), x(\rho - z) - y, xy - bz\rangle = -\sigma - 1 - b$
- Solving, $v(t) = v(0)e^{-t(\sigma+1+b)}$
- This means the volume decays to 0, so therefore, our line is confined to a smaller and smaller space (but not just a point, I guess?)

3 Partial Differential Equations

3.1 1D Wave Equation and D'Lambert's Formula

Set up:

- A rope with a fixed right end (boundary condition and a moving left end. *Starting with no boundary condition here (infinite rope, pulse in the middle*
- $u(x, t)$ measures the vertical displacement form the x-axis of the rope at point x, time t
- Physical observation gives us the PDE rule $\frac{\delta u}{\delta x} = \frac{\delta u}{\delta t}$
- $g(x) = u(x, 0)$ is the initial shape of the rope.
- It's assumed that the rope is not moving iniitally, so $u_t(x, 0) = 0$

Beginning to solve this:

- $u_{tt} = u_{xx} \Rightarrow u_{tt} - u_{xx} = 0$
- Sort of like $a^2 - b^2 = 0 \Rightarrow (a+b)(a-b) = 0$, we have $0 = (\frac{\delta}{\delta t} \pm \frac{\delta}{\delta x})(u_t \mp u_x) = u_{tt} - u_{xt} + u_{tx} - u_{xx} = u_{tt} - u_{xx}$
- This means the solution is either $u_+ = u_t + u_x$ or $u_- = u_t - u_x$
- These can be written as, e.g. $0 = u_t + u_x = \langle 1, 1\rangle \cdot \langle u_x, u_t\rangle = \langle 1, 1\rangle \cdot \nabla u$
- TRICK: This is a directional derviative along $\langle 1, 1\rangle$. Introducing a variable like s (accelerant along $\langle 1, 1\rangle$?) below does nothing interesting:
  - $\frac{d}{ds}[u(x+sb, t+sc)] = \frac{\delta u}{\delta x}(x+sb, t+sc)b + \frac{\delta u}{\delta t}(x+sb, t+sc)c = \langle b, c\rangle \cdot \nabla u(x+sb, t+sc)$
  - So if we set $b = c = 1$, we see that $\frac{d}{ds}[u(x+s, t+s)] = \langle 1, 1\rangle \cdot \nabla u(x+s, t+s)$
  - However, since in our world, $u_x + u_t = 0$, then this dot product is zero.
  - So then $u(x, t) = u(x+s, t+s)$, and shifting x forward by s (seconds?) and t by the same changes nothing.
  - From this, we see that $u_+(x, t) = u_+(x-t, 0)$ as well. *(The height of the wave at $(10, 0)$ equals that at $(11, 1)$, $(12, 2)$ etc¿? Does this mean the wave travels "right" at 1 unit per second, say?)*
- Note: We can't have one solution satisfy both conditions $u_+ = g(x), u_{+,t} = 0$, since then $g'(x) = 0$ which only works if g is a constant.
- Also, $u_{tt} - u_{xx} = 0$ is a linear PDE, in that solutons $u_1(x, t), u_2(x, t)$ see that $\frac{\delta^2}{\delta t^2}[c_1 u_1(x, t) + c_2 u_2(x, t)] - \frac{\delta^2}{\delta x^2}[c_1 u_1(x, t) + c_2 u_2(x, t) = 0$
- If we set $t = 0$, we get $u(x, t) = c_+ g(x+t) + c_- g(x-t) \Rightarrow u(x, 0) = c_+ g(x+0) + c_-(x-0) = (c_+ + c_-)g(x) = u(x, 0)$,so $(c_+ + c_-) = 1$
- Differentiating by t, $u_t(x, t) = c_+ g'(x+t) - c_- g'(x-t)$ so $u_t(x, 0) = (c_+ - c_-)g'(x) \Rightarrow (c_+ - c_-) = 0$. So $c_+ = c_- = \frac{1}{2}$
- This translates into one "lump" of $g(x)$ around $x = 0$ starting at $t = 0$ and propagating right, and one left.

With inverted conditions $u(x, 0) = 0, u_t(x, 0) = f(x)$, we can use the fact that $u(x, t)$ solving the wave equation implies u_t solves it as well!

- $\frac{\delta^2 u_t}{\delta t^2} - \frac{\delta^2 u_t}{\delta x^2} = \frac{\delta}{\delta t}[\frac{\delta^2 u}{\delta t^2} - \frac{\delta^2 u}{\delta x^2}] = 0$, and so $\frac{\delta^2 u}{\delta t^2} - \frac{\delta^2 u}{\delta x^2} = 0$ works!
- Therefore, $u_t(x, 0) = f(x)$ admits the same solution $u_t(x, t) = \frac{1}{2}[f(x+t) - f(x-t)]$
- Since $u(x, t) - u(x, 0) = \int \frac{1}{2}[f(x+t) - f(x-t)]$, and $u(x, 0) = 0$ by assumption in this setup, $u(x, t) = \int_{s=0}^{s=t}[f(x+s) - f(x-s)]ds$, which is $\frac{1}{2}\int f(s)$ from $x-t$ to $x+t$

$= \frac{1}{2}\int_{s=x-t}^{s=x+t} f(s)ds$

And because the region of the intergral for a point x gets wider as $t \to \infty$, on a flat rope with a pulse in the middle at $x = 0$, we see $u(x,t)$ sitting at 0 until the wave, at which point it stays at the peak (integral of the whole thing).

So **d'Alembert's formula** is the superposition of the initially flat wave with the initially still wave:

$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{s=x-t}^{s=x+t} f(s)ds$, which solves at $u_{tt} = u_{xx}$ for $u(x,0) = g(x), u_t(x,0) = f(t)$. In this instance, the propagation speed is clearly finite.

- TODO
- TODO