1 Turan's Theorem Original Proof

Set

$$f(n,k) = \frac{1}{2}\,n^2\Big(1 - \frac{1}{k}\Big).$$

Turan's theorem

Theorem 1.1 (Turan)**.** *If a graph $G = (V, E)$ on n vertices satisfies $|E| > f(n, k)$, then G contains a $(k + 1)$-clique.*

Contrapositive. If G has no $(k + 1)$-clique, then

$$|E| \le f(n, k).$$

Proof Plan: If we can transform any graph $G = (V, E)$ without a $(k + 1)$-clique into a graph $G^* = (V, E^*)$ where G^* has no $(k + 1)$-clique, and $|E| = |E^*| \le f(n, k)$, the contrapositive of Turan's theorem follows directly.

We consider such a set of graphs next: k-partite graphs.

Lemma 1.2 (k-partite bound)**.** *A k-partite graph on n vertices contains no $(k+1)$-clique and satisfies*

$$|E| \le f(n, k).$$

(The proof of the k-partite bound lemma is in a later section.)

Lemma 1.3 (Transformation lemma)**.** *Let G be a graph on n vertices with no $(k + 1)$-clique. Then there exists a k-partite graph on the same vertex set with the same number of edges, in particular:*

$$|E| \le \frac{1}{2}n^2\Big(1 - \frac{1}{k}\Big),$$

Additionally, G possesses a vertex of degree at most $n\big(1 - \frac{1}{k}\big)$.

Base case

For $n \le k + 1$ the claim is immediate: the complete graph K_{k+1} contains $\frac{1}{2}(k + 1)k$ edges. If we have less than or equal to $\frac{1}{2}(k + 1)k - 1$ edges, we cannot form a $k + 1$-clique. We show that this edge count is less than the bound.

- $\frac{1}{2}(k + 1)k - 1 \le \frac{1}{2}(k + 1)^2 \frac{k-1}{k}$
- $\frac{1}{2}k^3 + \frac{1}{2}k^2 - k \le \frac{1}{2}(k^2 + 2k + 1)(k - 1)$
- $\frac{1}{2}k^3 + \frac{1}{2}k^2 - k \le \frac{1}{2}k^3 + k^2 - \frac{1}{2}k - \frac{1}{2}$

- $-\frac{1}{2}k^2 - \frac{k}{2} + \frac{1}{2} \leq 0$

which is clear since $k > 1$.

Then, any graph avoiding a $(k+1)$-clique already satisfies the bound and is itself k-partite (some parts may be empty).

Average degree observation

Because $|E| \leq \frac{1}{2}n^2(1 - \frac{1}{k})$, the average degree is at most $n(1 - \frac{1}{k})$; hence some vertex has degree at most $\lfloor (n(1 - \frac{1}{k}) \rfloor$.

Inductive step

Assume $n > k+1$ and that the lemma holds for all graph of node count n or fewer. Let G be a graph on $n+1$ vertices with no $(k+1)$-clique.

Every induced subgraph of size n also avoids $(k+1)$-cliques and so meets the edge bound $f(n, k)$.

Goal: find a vertex v of degree

$$\deg(v) \leq D := n - \left\lfloor \frac{n}{k} \right\rfloor.$$

With such a vertex we delete v, apply the induction hypothesis to $G - v$ to obtain a k-partite graph, then assign v to a smallest partition (necessarily of size $\leq \lfloor \frac{n}{k} \rfloor$), rewiring its at most D edges across the remaining partitions. The resulting graph is k-partite with the same edge count as G.

Lemma 1.4 (Degree bound lemma)**.** *The degree bound $\lfloor n(1 - \frac{1}{k}) \rfloor$ will be equal to the bound D if $k \mid n$ and equal to $D - 1$ if $k \nmid n$.*

If $k \mid n$ then

$$n - \left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor n\Big(1 - \frac{1}{k}\Big) \right\rfloor,$$

the floor operation becomes the identity and equality follows.

If $n = qk + r$, $\lfloor n(1 - \frac{1}{k}) \rfloor = \lfloor (qk + r) - q - \frac{r}{k} \rfloor$, and $n - \lfloor \frac{n}{k} \rfloor = qk + r - \lfloor (q + \frac{r}{k}) \rfloor$.

Taking out the integer $qk + r$, $\lfloor -q - \frac{r}{k} \rfloor < -\lfloor q + r/k \rfloor$, and they lie on either side of integer $-q$, so $\lfloor n(1 - \frac{1}{k}) \rfloor = n - \lfloor \frac{n}{k} \rfloor - 1$.

Existence of a small degree vertex

- If $k \nmid n$, choose any subset of n vertices of G, which has $n+1$ nodes.

  By induction the subset contains a vertex w of degree $\le n(1-\frac{1}{k})$, which is $\le D-1$ by the Degree Bound Lemma, so the whole graph has such a vertex. Even if $(v,w) \in E$, w satisfies the degree bound.

- Suppose now $k \mid n$ (no more floors!) and, toward contradiction, that every vertex satisfies $\deg(v) \ge D+1$.

  Count edges in all $\binom{n+1}{n} = n+1$ induced n-vertex subgraphs. By induction each has at most

  $$f(n,k) = \frac{1}{2}n^2\Big(1-\frac{1}{k}\Big)$$

  edges. Summing and dividing by the multiplicity $n-1$ with which each edge is counted (two subgraphs will contain its endpoints), we obtain an upper bound

  $$T_{\text{upper}} = \frac{n+1}{n-1} f(n,k).$$

  On the other hand, our degree assumption forces at least

  $$T_{\text{lower}} = \frac{n+1}{2}\big(n - \tfrac{n}{k} + 1\big)$$

  edges in total. Compute

  $$T_{\text{lower}} - T_{\text{upper}} = \frac{n+1}{2}\,\frac{n-k}{k(n-1)} > 0$$

  because $n > k$. This contradiction establishes that a vertex of degree $\le D$ exists.

Completion of the inductive step

Removing the low-degree vertex and applying the rewiring argument yields a k-partite graph on $n+1$ vertices with the same edge count as G. Hence every $(n+1)$-vertex graph without a $(k+1)$-clique satisfies

$$|E| \le \frac{1}{2}n^2\Big(1-\frac{1}{k}\Big).$$

Corollary (Turan). If $|E| > \frac{1}{2}n^2(1-\frac{1}{k})$ then G must contain a $(k+1)$-clique.

2 Proof of k-partite lemma

A k-partite graph (perhaps a real term but I'll use it here) has k partitions of nodes, within which all are disconnected. The nodes from differing partitions may connect. A *perfectly balanced graph has partitions all of equal size.*

Proposition 2.1. *The edge-count formula of a perfectly balanced k-partite graph is*

$$E_p(n,k) = \frac{n^2}{2}\Big(1 - \frac{1}{k}\Big).$$

Each of n nodes connects to $n\frac{k-1}{k}$ nodes in other partitions. We divide by two to get the number of edges.

Proposition 2.2. *The edge-count formula of a perfectly balanced graph of node total n is the upper bound to any partitioning of nodes among k partitions.*

We are looking at maximizing $f(\vec{a}) = \frac{1}{2}\sum_{i\neq j;i,j<k}^{k} a_i a_j$, where a_i is the node count of partition i.

We can write this as $f(\vec{a}) = \frac{1}{2}\sum_{i\neq j;i,j<k} a_i a_j = (\sum_i^k a_i)^2 - \frac{1}{2}\sum_i^k {a_i}^2 = n^2 - \frac{1}{2}\sum_i^k {a_i}^2$. So we are looking at minimizing the final term to maximize $f(\vec{a})$

Consider the definition of statistical variance: $Var(a_1, a_2...a_k) = \sum_{i=1}^k (a_i - \bar{a})^2 = \sum_{i=1}^k (a_i^2 - 2\bar{a}a_i + \bar{a}^2)$.

So $Var(a_1, a_2...a_k) = \frac{1}{k}[\sum_{i=1}^k (a_i^2) - 2\bar{a}(\sum_{i=1}^k a_i) + k\bar{a}^2]$

Thus, ignoring constant terms, minimizing $Var(a_1, a_2...a_k)$ is the same as minimizing $\sum_{i=1}^k {a_i}^2$. But the variance reaches its minimum at zero when all values a_i are equal. This is therefore bounded by the edge count of the equal partition $E_p(n,k) = \frac{n^2}{2}\Big(1 - \frac{1}{k}\Big)$.

2.1 Conclusion

Therefore, any k-partite graph has less than or equal to $\frac{n^2}{2}\Big(1 - \frac{1}{k}\Big)$ edges.