Brilliant: Vector Calculus

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6/21/22

Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against t of the form $\vec{x}(t) = \langle x(t), y(t), \ldots \rangle$.

- A **line** through $p = (a, b, c)$ parallel to $\vec{v} = \langle v_x, v_y, v_z \rangle$ is $\vec{x}(t) = \vec{p} + t\vec{v}$
- **velocity** is characterized completely by $\vec{v}(t) = \vec{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- The **speed** of an object along that line versus t is the length of v ($\|v\|$)
- Therefore, the speed of an object along line
$$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$
is
$$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$
- Note that $\vec{v}$ need not be constant. The speed of
$$\vec{x}(t) = \vec{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$
would then be
$$\|6\pi\cos(2\pi t)\hat{u}\| = |6\pi\cos(2\pi t)|$$
- **Acceleration** $a(t) = v'(t) = x''(t)$ is straightforward. Acceleration of
$$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$
- An example position vector for a planet of distance r from the sun could be $\langle r\cos(t), r\sin(t) \rangle$. The acceleration vector points in the opposite direction: $\langle -r\cos(t), -r\sin(t) \rangle$. In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by $F = ma$ is proportional to acceleration) points towards the sun, *with acceleration perpendicular to velocity.*

- A **helix** could be a 3D extension like $\langle r\cos(t), r\sin(t), b\cdot t\rangle$.

2 Chapter 2.2: Space Curves

- Note that while $\vec{x}(t) = \langle \cos(t), \sin(t), 5\rangle$ and $\vec{x}(t) = \langle \cos(2t), \sin(2t), 5\rangle$ describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If $\vec{x}(t) = t\vec{v}$, then speed is $\frac{\|\vec{x}(t+\Delta t)-\vec{t}\|}{\Delta t} = \|\vec{v}\|$, direction is $\frac{\vec{v}}{\|\vec{v}\|}$, and velocity $\vec{v}$ is the product of speed and direction.
- So $\vec{v}(t) = \lim_{\Delta t\to 0} \frac{\vec{x}(t+\Delta t)-\vec{x}(t)}{\Delta t} = \vec{x}'(t) = \frac{d\vec{x}}{dt} = \langle x'(t), y'(t), z'(t)\rangle$
- Neat conceptual result: any $y = f(x)$ can be made into $x(t) = \langle t, f(t)\rangle$, and then $v(t) = \langle 1, f'(t)\rangle$, which points along the tangent line at $\langle t, f(t)\rangle$.
- Note that dot product derivatives work like regular product: $[\vec{a}(t)\cdot\vec{b}(t)]' = \vec{a}'(t)\cdot\vec{b}(t) + \vec{a}(t)\cdot\vec{b}'(t)$, but the cross product does not work the same since $\frac{d}{dt}[a\times b] = a'\times b + a\times b'$, but since $a\times b' = -b'\times a$, can't switch the order to $a'\times b + b'\times a$ due to this non-commutativity.
- If
$$\vec{x}(t) = \vec{p} + t\vec{v},$$
calculating velocity with respect to origin becomes
$$\frac{d}{dt}\|\vec{x}(t)\| = \frac{\vec{x}(t)\cdot\vec{x}'(t)}{\|\vec{x}(t)\|} = \frac{\vec{x}}{\|\vec{x}\|}\cdot\vec{v},$$
after rewriting the distance formula and chugging through the chain rule.
- However, it becomes more clear when considering that $(\vec{v}\cdot\hat{x})\hat{x}$ is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way: $\int_a^b \vec{x}(t) = \langle \int_a^b x(t), \int_a^b y(t), \int_a^b z(t)\rangle$
- Example: if ball launched from origin with velocity $\langle 1, 2, 3\rangle$ and acceleration $\langle 0, 0, -1\rangle$,

it lands at

$$\frac{dv}{dt}dt = \langle 0, 0, -1 \rangle \tag{1}$$

$$\int \frac{dv}{dt}dt = v = \langle C, D, -t + F \rangle = \langle 1, 2, 3 \rangle = \langle 1, 2, -t + 3 \rangle, t = 0 \tag{2}$$

$$x = \int v = \langle t + K, 2t + M, -\frac{1}{2}t^2 + 3t + N \rangle, x(\vec{0}) = \langle 0, 0, 0 \rangle \tag{3}$$

$$\vec{x}(t) = \langle t, 2t, 3t - \frac{1}{2}t^2 \rangle \tag{4}$$

$$z(t) = 0 \to t = 6 \to \vec{x}(6) = \langle 6, 12, 0 \rangle \tag{5}$$

$$\tag{6}$$

- Also, generalizing $ds = \sqrt{(dx)^2 + (dy)^2}$, the length of an arc from point a to b approaches $\boxed{\int_a^b ds = \int_{t_a}^{t_b} \|x'(t)\| dt}$
- Example: a helix $\langle a\cos(\omega t), a\sin(\omega t), b\omega t \rangle$, parametrized by time t can be rewritten in terms of s, the arc length:

$$s = \int \|x'(t)\| dt \tag{7}$$

$$s = \int \sqrt{(-\omega a \sin(\omega t))^2 + (\omega a \cos(\omega t))^2 + (b\omega)^2} dt \tag{8}$$

$$s = |\omega| \int \sqrt{(a^2 + b^2)} dt \tag{9}$$

$$s = |\omega| t \sqrt{a^2 + b^2} \tag{10}$$

- *Note: It's weird to think of time in terms of length.* Could be analytically useful?

4 Chapter 2.4: Frenet Formulae

Main idea: Establish three new vectors $\hat{T}(s), \hat{N}(s), \hat{B}(s)$ that change as we move along a space curve, instead of $\vec{x}(t)$ that changes over an external "time" idea.

Remember that $s = \int_0^t \|\vec{x}'(\tilde{t})\| d\tilde{t}$, so $\frac{ds}{dt} = \|\vec{x}'(t)\|$.

4.1 $\hat{T}$: Vector tangent to space curve

- Remember arc length is $s = \int_0^t \|\vec{x}'(\tilde{t}) d\tilde{t}\|$
- $\hat{T}$ is just normalized grad: $\frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$

- This implies $\boxed{\frac{d\vec{x}}{ds} = \hat{T}}$ since

$$s = \int_0^t \|\vec{x}'(\tilde{t})d\tilde{t}\| \tag{11}$$
$$\frac{ds}{dt} = \|\vec{x}(t)\| \tag{12}$$
$$\hat{T} = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \frac{d\vec{x}}{dt} \cdot \frac{dt}{ds} \tag{13}$$
$$\hat{T} = \frac{d\vec{x}}{ds} \tag{14}$$
$$\tag{15}$$

- So this is how the space curve $\vec{x}$ changes as it moves along the curve at length s.
- It's normalized, so it's the same whether parameterized by t, s, or whatever.

4.2 $\hat{N}$: Normal to curve (perpendicular to $\hat{T}$)

Normal vectors are:

- $\vec{x}''(t)$ normalized as $\boxed{\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} = \hat{N}}$
- The normal vector to the curve
- $\perp$ to $\hat{T}$ in direction of acceleration. So a multiple of acceleration vector.
- $\frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$. The following sequence shows any unit length vector is perpendicular to its derivative.

$$\|\hat{T}\| = 1 \tag{16}$$
$$d(\|\hat{T}\|^2) = 0 \tag{17}$$
$$d(\|\hat{T}\|^2) = d(\hat{T} \cdot \hat{T}) = \hat{T}(t) \cdot 2\hat{T}'(t) \tag{18}$$
$$\hat{T}(t) \cdot \hat{T}'(t) = 0 \tag{19}$$

- $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|}$ since it's the same as the above, but parametrized over s instead of t. Doesn't change the direction of the vector!

Example:if $\vec{x}(t) = \langle R\cos(\omega t), R\sin(\omega t), 0\rangle$, then:

- $\vec{a} = \frac{d^2\vec{x}}{dt^2}$ just by definition

- $\vec{a} = -\omega^2\vec{x}$ just by calculation
- $\hat{T}(t) = \langle -\sin(\omega t), \cos(\omega t), 0\rangle$
- $\|\hat{T}(t)\| = 1$
- $\hat{N} = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = \langle -\cos(\omega t), -\sin(\omega t), 0\rangle$
- So $\vec{a} = R\omega^2\hat{N}$ by these formulae.

This leads us to believe acceleration and $\hat{N}$, the normed derivative of $\hat{T}$ are related.

The part of acceleration $\vec{a}$ parallel to $\hat{T}$ is the projection $(\vec{a}\cdot\hat{T})\hat{T}$

The perpendicular part is then $\vec{a}$ minus that: $\vec{a} - (\vec{a}\cdot\hat{T})\hat{T}$

This also equals $(\frac{ds}{dt})^2\|\frac{d\hat{T}}{ds}\|\hat{N}$ because

$$\vec{x}' = \frac{dx}{dt} = T = \hat{T}\cdot\|\frac{dx}{dt}\| \tag{20}$$

$$s = \int_0^t \|\vec{x}'(t)\| \rightarrow \frac{ds}{dt} = \|\vec{x}'(t)\| \tag{21}$$

$\hat{N} = \frac{d\hat{T}}{ds}$ normalized, so

$$\vec{a} = \frac{d^2\vec{x}}{dt^2} = \frac{d}{dt}(\|\frac{dx}{dt}\|\frac{\frac{dx}{dt}}{\|\frac{dx}{dt}\|}) = \frac{d}{dt}(\|\vec{x}'(t)\|\hat{T}(t)) = \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + \|\vec{x}'(t)\|\frac{d\hat{T}}{dt} \tag{22}$$

$$= \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{ds}\frac{ds}{dt} \tag{23}$$

$$= \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + (\frac{ds}{dt})^2\|\frac{d\hat{T}}{ds}\|\hat{N} \tag{24}$$

This is "a = $\hat{T}$'s parallel part plus $\hat{T}$'s perpendicular (N) part", so the second term is $a_\perp$

4.3 $\hat{T}$ and $\hat{N}$

- Form a plane, since first, any normal vector's derivative is perpendicular to the vector
- κ is curvature: how much we're curving in that $T\times N$ plane.
- $\boxed{\kappa = \|\frac{d\hat{T}}{ds}\|}$
- Therefore, by the definition of $\hat{N} = \frac{d\hat{T}/ds}{\|d\hat{T}/ds\|}$, $\boxed{\frac{d\hat{T}}{ds} = \kappa\hat{N}}$ **(Frenet equation 1)**

Note that curvature $\kappa(s) = \|\frac{d\hat{T}}{ds}\|$ is geometric (depends on s, not time) and changes as $\hat{T}$ changes.

Example: Curvature of $\vec{x}(t) = \langle \cos(t), \sin(t), bt \rangle$

$$x'(t) = \langle -\sin(t), \cos(t), b \rangle \tag{25}$$

$$\|x'(t)\| = \sqrt{1+b^2} \tag{26}$$

$$s = \int_0^t \|x'(\tilde{t})\| d\tilde{t} = \int_0^t \sqrt{(1+b^2)} = t\sqrt{(1+b^2)} \rightarrow t = \frac{s}{\sqrt{1+b^2}} \tag{27}$$

Do the substitution of $\frac{s}{\sqrt{1+b^2}}$ for t above to get $x'(s)$, and from there, you can figure out $\frac{d\hat{T}}{ds}$ and normalize to get $\kappa = \frac{1}{1+b^2}$

4.4 $\hat{B}$ is binormal: perpendicular to both

- defined as $\boxed{\hat{B} = \hat{T} \times \hat{N}}$
- Therefore, by derivative

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \tag{28}$$

$$\frac{d\hat{B}}{ds} = \kappa\hat{N} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \tag{29}$$

$$\frac{d\hat{B}}{ds} = \hat{T} \times \frac{d\hat{N}}{ds} \tag{30}$$

  This means T is orthogonal to dB, and we already know B and dB are orthogonal. We're working in 3D with the cross product, so dB is parallel to N.

- Therefore, we define **torsion** τ so that $\boxed{-\frac{d\hat{B}}{ds} = \tau\hat{N}}$ **(Frenet equation 2)**. Negative sign by convention.
- Can also cross by N on both sides to get $-\frac{d\hat{B}}{ds} \times \hat{N} = \tau$
- τ measures how the plane defined by $\hat{T}, \hat{N}$ twists around. On a circle, $\hat{B}$ wouldn't change, so the derivative would be zero.

- **Final Frenet equation**. Prereq: $\hat{B} = \hat{T} \times \hat{N} \to \hat{N} = \hat{B} \times \hat{T} \to \hat{T} = \hat{N} \times \hat{B}$

$$\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} \tag{31}$$

$$\frac{d\hat{N}}{ds} = -\tau\hat{N} \times \hat{T} + \hat{B} \times \kappa\hat{N} \tag{32}$$

$$\frac{d\hat{N}}{ds} = \tau\hat{B} - \kappa\hat{T} \tag{33}$$

5 Chapter 2.5: Parametrized Surfaces

Main approaches to describing a surface:

- Can parameterize by $\vec{x}(u, v) = x(u, v), y(u, v), z(u, v)$
- Can perhaps parameterize $f(x, y, z) = c$ by $z = g(x, y)$
- Can also use ideas like $\nabla f = 0$ to find a normal.

There are many out-of-the-box paremetrizations including:

- Sphere at (0,0,0): $\vec{x}(u, v) = \langle R\cos(u)\sin(v), Rsin(u)\sin(v), R\cos(v)\rangle$, where $u \in [0, 2\pi), v \in [0, \pi]$
- Rotate function $y = f(x)$ around the x-axis: $\vec{x}(u, v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$, where $u \in D, v \in [0, 2\pi]$

Tangent vectors to $\vec{x}(u, v)$ are $\frac{\delta\vec{x}}{du}, \frac{\delta\vec{x}}{dv}$, so unit normal $\boxed{\hat{n} = \pm\frac{\frac{d\vec{x}}{du} \times \frac{\delta\vec{x}}{dv}}{\|\frac{d\vec{x}}{du} \times \frac{\delta\vec{x}}{dv}\|}}$

Example: Torus $\vec{x} = \langle [2 + \cos(v)]\cos(u), [2 + \cos(v)]\sin(u), \sin(v)\rangle, u, v \in [0, 2\pi)$. What's the tangent plane at $u = \frac{\pi}{4}, v = 0$?

$$d\vec{x}/du = \langle -\sin(u)(2+\cos(v)), \cos(u)(2+\cos(v)), 0\rangle \tag{34}$$

$$d\vec{x}/dv = \langle -\sin(v)\cos(u), -\sin(v)\sin(u), \cos(v)\rangle \tag{35}$$

$$u = \frac{\pi}{4}, v = 0 : \tag{36}$$

$$d\vec{x}/du = \langle -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0, \rangle \tag{37}$$

$$d\vec{x}/dv = \langle 0, 0, 1\rangle \tag{38}$$

$$dx/du \times dx/dv = \langle \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0\rangle \tag{39}$$

$$\hat{n} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle \tag{40}$$

$$\hat{n} \cdot \vec{x} = 0 \to \hat{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \tag{41}$$

$$\to ... \to x + y = 3\sqrt{2} \tag{42}$$

$$\tag{43}$$

5.1 Example: Ellipsoid $x^2+2y^2+z^2 = 4$ What's the normal at $(1, \frac{1}{\sqrt{2}}, \sqrt{2})$?

Method 1: parametrize with spherical u, v First, transform to sphere with change of coordinates, then flip to speherical coordinates.

$$x^2 + 2y^2 + z^2 = 4 \tag{44}$$

$$X = x/2, Y = \frac{Y}{\sqrt{2}}, Z = z/2 \tag{45}$$

$$X^2 + Y^2 + Z^2 = 1 \tag{46}$$

$$X = \cos(u)\sin(v), Y = \sin(u)\sin(v), Z = \cos(v) \tag{47}$$

$$p = (1, \frac{1}{\sqrt{2}}, \sqrt{2}) \to u = v = \frac{\pi}{4} \tag{48}$$

$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle -1, \frac{1}{\sqrt{2}}, 0\rangle \tag{49}$$

$$\frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \frac{1}{\sqrt{2}}, -\sqrt{2}\rangle \tag{50}$$

$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) \times \frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \sqrt{2}, \sqrt{2}\rangle \tag{51}$$

$$\hat{n}_{out} = \frac{\langle -1, -\sqrt{2}, -\sqrt{2}\rangle}{\sqrt{5}} \tag{52}$$

Method 2: rewrite as z = g(x,y)

$$x^2 + 2y^2 + z^2 = 4 \tag{53}$$
$$z = (4 - x^2 - 2y^2)^{\frac{1}{2}} \tag{54}$$
$$dz/dx = \frac{1}{2} \times -2x(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -\frac{1}{\sqrt{2}} \tag{55}$$
$$dz/dy = \frac{1}{2} \times -4y(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -2\sqrt{2}/\sqrt{2} = -1 \tag{56}$$
$$f \approx \sqrt{2} + dz/dx(1, \frac{1}{\sqrt{2}})(x-1) + dz/dy(1, \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) \tag{57}$$
$$\to ... \to \frac{1}{\sqrt{2}}x + y + z = 2\sqrt{2} \tag{58}$$
$$\tag{59}$$

giving us normal vector $\langle \frac{1}{\sqrt{2}}, 1, 1\rangle = \frac{\langle 1, \sqrt{2}, \sqrt{2}\rangle}{\sqrt{5}}$ after normalization.

Method 3: gradient

Gradient is always normal to the tangent plane. Recognize level set of $f(x, y, z) = x^2 + 2y^2 + z^2$.

$\nabla f = \langle 2x, 4y, 2z\rangle \to \nabla f(1, \frac{1}{\sqrt{2}}, \sqrt{2}) = \langle 2, 2\sqrt{2}, 2\sqrt{2}\rangle$

Then normalize.

5.2 Mobius strip and "outward direction"

Mobius strip is

- $x = 2\cos(u) + v\cos(\frac{u}{2})$
- $y = 2\sin(u) + v\cos(\frac{u}{2})$
- $z = v\sin(\frac{u}{2})$
- $u \in [0, 2\pi], v \in [-\frac{1}{2}, \frac{1}{2}]$

$\hat{n} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$ at (0,0) is $\langle 0, 0, -1\rangle$,

but at the same point $(2\pi, 0)$ $\hat{n} = \langle 0, 0, 1\rangle$!!

6 Chapter 2.6: Vector Fields

(Lots of intuition questions here...)

One nugget: using **gradient vector fields**: Suppose $\vec{F}(x,y) = \langle 2, -4y^3 \rangle$. If $\vec{F} = \nabla f$ for some (single value function) f, then F's arrows are perpendicular to a level set $f = c$. So look at $f = 2x - y^4$ and find perpendicular arrows to these. That's actually F!

Linear approximation for $\vec{F} : D \in \mathbf{R}^n \to \mathbf{R}^m$

Main idea: $\vec{F}(\vec{x}) = \vec{F}(\vec{a}) + A(\vec{a})(\vec{x} - \vec{a})$

Note that A takes in vectors of size n (so it has as many columns as the input space), and has m functions (rows) that operate on it. So the Jacobian, A, has as row i, column j, the quantity $\frac{dF_i}{dx_j}(\vec{a})$.

$dF_i/d\vec{x}$ extends across row i.

7 Chapter 2.7: Jack and the Beanstalk (Newton's method)

Basis for Newton's:

If we're estimating x_1 by following the derivative at x_0, this means we're looking at the line with x-intercept x_1, with slope $f'(x_0)$.

So instead of $y = mx + b$, we'll flip the two and use

$x = y/m + x_{int}$

or $x_0 = f(x_0)\frac{1}{f'(x_0)} + x_1$,

or $\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$

Note that, under Newton's something like $|x|$ will converge immediately, x^3 will converge moderately, and a S-curve might barely converge if at all.

The extension of this with the Jacobian matrix $A = DF'(x_0)$ is $\vec{x}_1 = \vec{x}_0 - (D\vec{F}(\vec{x}_0))^{-1}\vec{F}(x_0)$

8 Chapter 2.8: Electrostatic bootcamp

Electric charge radiates out equally in all directions, and is inversely proportional to distance.

Formula, with Q as the charge, ϵ_0 is a constant: $\vec{E}(x,y,z) = \frac{Q}{4\pi\epsilon_0 \|x\|^2}\hat{x}$

A field line is a special case of a **flow line** - the space curve that follows $\vec{F}$'s arrows. The tangent vector to the flow line is $\vec{F}(\vec{x}(\tilde{t}))$ ($\tilde{t}$ is not time here), so $\frac{d\vec{x}}{d\tilde{t}} = \vec{F}(\vec{x}(\tilde{t}))$

Example: Vector field $\vec{F}(x,y) = \langle -2y, 3x \rangle$. What's the flow line through (2,0)?

Solution: Need to solve $dx/dt = -2y, dy/dt = 3x$. Key is "separating the equations". Remember x and y are functions of t!

$$\frac{d^2x}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6x. \tag{60}$$
$$\frac{d^2y}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6y. \tag{61}$$
$$x(t) = -6x''(t), y(t) = -6y''(t) \tag{62}$$
$$\to x = A\cos(\sqrt{6}t) + B\sin(\sqrt{6}t), y = C\cos(\sqrt{6}t) + D\sin(\sqrt{6}t) \tag{63}$$
$$\frac{dx}{dt} = -2y(t) \to \frac{\sqrt{6}}{2}A\sin(\sqrt{6}t) - \frac{\sqrt{6}}{2}B\cos(\sqrt{6}t) = y(t) \tag{64}$$
$$x(t=0) = 2 \to A = 2 \tag{65}$$
$$y(t=0) = 0 \to B = 0 \tag{66}$$
$$\vec{F}(t) = \langle 2\cos(\sqrt{6}t), \sqrt{6}\sin(\sqrt{6}t)\rangle \tag{67}$$
$$\tag{68}$$

Note: **Field lines** follow rules:

- Go from positive charges to negative
- Density of lines directly relates to how much charge a point has
- Lines don't intersect.
- Corollary: If count of out equals count of in, point has zero charge
- "Number" (to be defined) of field lines in and out of a *surface* related to the charge inside. Upcoming.

9 3.1: Surface Integrals

Example: Fluid pressure in a tank is:

- Proportional (via some weight constant p_{fluid}) to depth of the point
- Pushes out via the normal $\hat{n}$
- So, for the $x = l$ side of a cube of length l, this would be

  $\vec{F}_{x=l} = (\int\int_{[0,l]\times[0,l]} p_{fluid}[1 - \frac{z}{l}]dydz)\hat{i}$

Example: Hemisphere of size l, sitting at (0, 0, 0)

Finding the out pointing unit normal of hemisphere at point $(x, y, \sqrt{l^2 - x^2 - y^2})$

Note: Can just eyeball this, but one way is the **gradient**.

First, the relation is $x^2 + y^2 + (z - l)^2 = l^2$. Make it a function g and take the level set at l^2:

$$
\begin{align}
g(x, y, z) = x^2 + y^2 + (z - l)^2 = l^2 \tag{69} \\
\nabla g(x, y, z) = \langle 2x, 2y, 2(z - l) \rangle \tag{70} \\
\hat{n} = \pm \frac{\nabla g(x, y, z)}{\|\nabla g(x, y, z)\|} \tag{71} \\
\hat{n} = \pm \frac{\langle x, y, (z - l) \rangle}{\sqrt{x^2 + y^2 + (z - l)^2}} \tag{72} \\
\hat{n} = \pm \frac{\langle x, y, (z - l) \rangle}{\sqrt{l^2}} \tag{73} \\
\hat{n} = \pm \langle \frac{x}{l}, \frac{y}{l}, \frac{z}{l} - 1 \rangle \tag{74} \\
\tag{75}
\end{align}
$$

Note: Integrating over a patch dA on the surface means finding the area of micro-patches ΔA_{ij}, which is the parallelogram defined by

$$
\begin{align}
s_1 = \langle \Delta x_i, 0, \Delta x_i f_x(x_i^*, y_j^*) \rangle \tag{76} \\
s_2 = \langle 0, \Delta y_j, \Delta y_j f_y(x_i^*, y_j^*) \rangle \tag{77} \\
\Delta A_{ij} \approx \|s_1 \times s_2\| \tag{78} \\
= \sqrt{(1 + [f_x(x_i^*, y_j^*)]^2 + [f_y(x_i^*, y_j^*)]^2} \Delta x_i \Delta y_j \tag{79} \\
\tag{80}
\end{align}
$$

So if $\boxed{z = f(x, y), dA = \sqrt{1 + f_x^2 + f_y^2}}$.

So the total pressure ends up being $\vec{F}_{tot} = p_{fluid} \iint (p \cdot \hat{n}) dA$

$$= p_{fluid} \iint_{x^2+y^2 \leq l^2} [1 - \frac{f(x,y)}{l}] \hat{n} \sqrt{1 + [f_x]^2 + [f_y]^2} dxdy \tag{81}$$

$$f(x,y) = l - \sqrt{l^2 - x^2 - y^2} \tag{82}$$

$$\hat{n} = \langle \frac{x}{l}, \frac{t}{l}, \frac{f(x,y)}{l} - 1 \rangle \tag{83}$$

$$f_x = \frac{x}{\sqrt{l^2 - x^2 - y^2}}, f_y = \frac{y}{\sqrt{l^2 - x^2 - y^2}} \tag{84}$$

And for the only-nonzero component, $\hat{k}$, this simplifies after a lot of hand-math to $F_{tot} = -p_{fluid} (\iint_{x^2+y^2 \leq l^2} \sqrt{1 - (\frac{x^2+y^2}{l^2})} dxdy) \hat{k}$

Side Note during solving: $dxdy \to rdrd\theta$.

- TODO: This looks to be something to do with the determinant of the Jacobian matrix F_i / x_j.
- Intuitively, consider that a patch $dx \cdot dy$ is a slice of a big disk which has dimensions dr on the ray, $rd\theta$ on the arc.

10 3.2: Flux Part I

Main idea: Field lines are innumerable, so counting them through a surface makes no sense. Instead, we'll use **flux** to help us measure charge pushed through a surface per unit time.

Example: If charge q of mass m in a field of $\vec{E} = E_0 \hat{i}$ moves from origin along x towards R according to $\frac{d^2x}{dt^2} = \frac{q}{m} E_0$, then solving the diff eq. means that $x = \frac{q}{2m} E_0 (\Delta t)^2 = R$. This means we're pushing all charges within $\frac{q}{2m} E_0 (\Delta t)^2$ to the left of the disk through it.

Then, if we're considering a cylinder of base area A, mass density δ, charge density ρ:

- Every test charge chunk ΔV within $\frac{\rho \Delta V}{2 \delta \Delta V} E_0 (\Delta t)^2$ passes through. That's the height.
- Area is A, so total volume is $\frac{\rho (\Delta t)^2}{2\delta} E_0 A$
- Density of charge per volume is ρ, so total is $\frac{\rho^2 (\Delta t)^2}{2\delta} E_0 A$

Note: Tilting this forward from the z-axis by θ multiplies the cross-section area of the cylinder (now an ellipse) by $\cos(\theta)$. Can work out the ellipse volume, or just note that each "Riemann bar" orthogonal to x-axis just got squished by $\cos(\theta)$.

So we define **flux** as amount of charge through a closed surface. $\boxed{\Phi = (\vec{E} \cdot \hat{n})A}$ if $\vec{E}$ is a constant field. (Units: joules/second/m^2, or watts/m^2), and $\boxed{\Phi = \iint_S (\vec{E} \cdot \hat{n}) dA}$ generally.

We can further note $(\vec{E} \cdot \hat{n}) = \|\vec{E}\| \cos(\theta)$ by last problem.

Example: Flux through an empty cube from the origin is necessarily 0 since every face cancels the other.

Another example: A square pyramind with top at $(0, 0, 1)$, sides at 1 on each axis:

- All the triangles will cancel in the x, y directions.
- A triangle $(1, 0, 0)(0, 1, 0), (0, 0, 1)$ has two displacement vectors $P_1P_3 = P_3 - P_1 = (-1, 0, 1), P_2P_3 = (0, -1, 1)$.
- $P_1P_3 \times P_2P_3 = (1, 1, 1) \to \hat{n} = \frac{(1,1,1)}{\sqrt{3}}$
- $A = \frac{1}{2}\|P_1P_3 \times P_2P_3\| = \frac{\sqrt{3}}{2}$
- $\Phi = (\vec{E} \cdot \hat{n})A = (E_0 \frac{1}{\sqrt{3}})\frac{\sqrt{3}}{2} = \frac{E_0}{2}$
- So total flux through these is $4 \cdot \frac{1}{2} E_0 = 2E_0$
- However, the bottom has area $\sqrt{2}^2 = 2$ and flux E_0 , so total is 0!

11 3.3: Flux Part II

Note:

- Charge (q) is the volts of the point charge. Total charge Q_{tot} is total charge inside some surface.
- Electric field is sum of those point charges acting at a distance, and a is a single vector.
- Flux is the sum of the electric field flowing through a surface.
- Total charge Q_{tot} of a surface is basically the sum of all the flux going in/out, except that it's that divided by some constant ϵ_0.

Note: $\vec{E}$ isn't usually constant, and the surface S is usually curved. So we need calculus to break up surface S into small pieces ΔA_i and evaluate $\vec{E}_i$ there at that normal $\hat{n}_i$. So

$\sum_{patches} (\vec{E}_i \cdot \hat{n}_i) \Delta A_i = \iint_S (\vec{E} \cdot \hat{n})) dA = \Phi$

Easy Example: If, say, $(\vec{E} \cdot \hat{n}) = 1$ everywhere, we're just looking at $\iint_S dA$, or the total surface area.

Another example. Given:

- Real electric field law: $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3}$
- Real observation: Total electric flux through a surface (Φ) is proportional to total charge inside (Q_{tot}). $\Phi = \iint_S (\vec{E} \cdot \hat{n}) \Delta A \propto Q_{tot}$
- Then constant must be $\frac{1}{\epsilon_0}$. Why?
  - On unit sphere, $\hat{n} = \frac{\vec{x}}{\|\vec{x}\|}$
  - So $\vec{E} \cdot \hat{n} = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3} \cdot \hat{n}$
  - $= \frac{q}{4\pi\epsilon_0}$ since $\|\vec{x}\| = 1$ on unit sphere
  - Then $\Phi = \iint_S \frac{q}{4\pi\epsilon_0} dA$
  - $= \frac{q}{4\pi\epsilon_0} 4\pi$ by surface area of unit sphere
  - $= \frac{q}{\epsilon_0}$
- Therefore, because all of the field goes through the surface (no matter the shape), $\boxed{\textbf{Gauss's law} \text{ says } \iint_S (\vec{E} \cdot \vec{n}) dA = \frac{Q_{tot}}{\epsilon_0}}$

Note: Because (UNEXPLAINED!) symmetry of a contained *ball* implies that, for distance ρ from origin, $\vec{E} = E(\rho)\hat{\rho}$, the above works the same for a point charge or a uniform (contained) ball.

Example: For a big radius R ball of charge Q containing a small ball of radius ρ with charge Q_{tot}, what must the charge $E(\rho)$ at any point be?

- Small charge Q_{tot} is proportional to volume of the big charge Q by $Q_{tot} = Q \frac{V_{small}}{V_{big}} = Q \frac{\rho^3}{R^3}$
- $\frac{Q_{tot}}{\epsilon_0}$ = total charge $= \iint_S E(\rho)(\|\hat{\rho}\|) dA = E(\rho) \iint_S 1 dA = E(\rho) 4\pi\rho^2$
- So $\frac{Q_{tot}}{\epsilon_0} = Q \frac{\rho^3}{R^3 \epsilon_0} = E(\rho) 4\pi\rho^2$
- So $E(\rho) = \frac{Q}{4\pi\epsilon_0} \frac{\rho}{R^3}$

Example: Infinite wire, x=y=0, charge per length is λ. What's the magnitude of the field r units away?

- Use a cylinder.

- What's the total charge of the cylinder? Top and bottom are perpendicular to the field so can be ignored.
- There's some function $E(r)$ which, time $\hat{r}$, is the field by symmetry.
- $\Phi = \iint_{cylinder}(E(r) \cdot \hat{r})dA = E(r)\iint_{cylinder} 1dA = E(r)2\pi rh.$
- $\frac{Q_{tot}}{\epsilon_0} = E(r)2\pi rh \Rightarrow E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$

Example: Infinite plane, x=y=0, charge per area is σ. What's the mangitude of the field at height h?

- Use a cylinder again
- What's the total charge of the cylinder? Side is perpendicular to the field so can be ignored. Looking at top and bottom, $\phi = 2EA + 2EA.$, where E is charge through the top.
- $2EA = \frac{\sigma A}{\epsilon_0} \rightarrow E = \frac{\sigma}{2\epsilon_0}$
- Note: It appears it's height-invariant!

12 3.4: Surface Integrals

- Flux is a specific form of the general $\iint_S F da$.
- dA is a patch of a parallelogram on the surface. This is defined by corners $\vec{x}(u_0, v_0), \vec{x}(u_0, v_0) + \delta_u\vec{x}(u_0, v_0)$, and $\vec{x}(u_0, v_0) + \delta_v\vec{x}(u_0, v_0)$
- Therefore, using the parallelogram area formula,$dA = \Delta_u\Delta_v\|\vec{x}_u \times \vec{x}_v\|$
- Taking to the limit, this means the area is $\iint_D F(\vec{x}(u, v))\|\vec{x}_u \times \vec{x}_v\|dudv$

Example: Sphere $x^2 + y^2 + z^2 = R^2$ surface area. Take θ as angle around ϕ as angle from top of z axis.

- Parametrization $x = R\sin\phi\cos\theta, y = R\sin\phi\sin\theta, z = R\cos\phi$
- $dx/d_\theta = -R\sin\phi\sin\theta, dy/d_\theta = R\sin\phi\cos\theta, dz/d_\theta = 0$
- $dx/d_\phi = R\cos\phi\cos\theta, dy/d_\phi = R\cos\phi\cos\theta, dz/d_\phi = -R\sin\phi$
- After working it out, $dx/d_\theta \times dx/d_\phi = R^2\sin\phi\langle-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi\rangle$
- Doing the math, $\|dx/d_\theta \times dx/d_\phi\| = R^2\sin\phi$
- So $\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 1 \cdot R^2\sin\phi = 2\pi\int_{\phi=0}\pi R^2\sin\phi = 2\pi R^2[-\cos\phi]_0^\pi = 4\pi R^2$

Example: Parabaloid $z = 1 - x^2 - y^2, x^2 + y^2 \leq 1$

- Parametrization $x = R\sin\phi\cos\theta, y = R\sin\phi\sin\theta, z = R\cos\phi$
- $dz/dx = \langle 1, 0, -2x\rangle, dz/dy = \langle 0, -1, -2y\rangle$
- $\|dz/dx \times dz/dy\| = 1 + 4x^2 + 4y^2$
- Area $= \iint_D 1 \cdot dA = \iint_D \sqrt{1 + 4x^2 + 4y^2} dxdy$
- Change to polar, remembering this square depends on r: $\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{1 + 4r^2\cos^2\pi 4r^2\sin^2\pi} rdrd\theta = 2\pi \int_0^1 \sqrt{1 + 4r^2} rdr$
- After working it out, this ends up being $[\frac{2}{3} \cdot \frac{1}{8}(4r^2 + 1)^{\frac{3}{2}}]_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1)$

Example: Torus $x(u, v) = [R+r\cos(u)]\sin(v)], y(u, v) = [R+r\cos(u)]\cos(v)], z = r\sin(u), u, v \in [0, 2\pi)$

- Already parametrized in polar, basically,
- $d\vec{x}/du = \langle -r\sin(u)\sin(v), -r\sin(u)\cos(v), r\cos(u)\rangle$
- $d\vec{x}/dv = \langle Rcos(v) + r\cos(u)\cos(v), -R\sin(v) - r\cos(v)\sin(v), 0\rangle$
- After lots of math, $\|d\vec{x}/du \times \vec{x}/dv\| = r(R + r\cos(u))$
- $\int_{u=0}^{2\pi} \int_{v=0}^{2\pi} r(R + r\cos(u)du = 2\pi r \int_{u=0}^{2\pi} r(R + r\cos(u))du$
- $= 2\pi r[2\pi R] = 4\pi^2 Rr$

Example: Center of mass of unit (hollow?) hemisphere sitting on origin.

- Center of mass for density ρ is $\frac{\iint_S \vec{x}\rho dA}{\iint_S \rho dA}$
- Obvious that x, y center at zero.
- For denominator, $\iint_S dA$ is just surface area, or half of $4\pi 1^2 = 2\pi$.
- For numerator:
  - Do typical θ, ϕ parametrization.
  - $\vec{x}_\theta \times \vec{x}_\phi = \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi)\rangle$
  - Pull out the $\sin(\phi)$ and the remaining norm is one, so $\|\vec{x}_\theta \times \vec{x}_\phi\| = \sin(\phi)$
  - $\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} z \cdot dA = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \cos(\phi)\sin(\phi) = \frac{1}{2}$

Example: Moment of inertia

- Formula: $I_z = M \iint_S (x^2 + y^2) dA$.
- Object to spin: helicoid $\vec{x}(\theta, v) = \langle \theta\cos(v), \theta\sin(v), v\rangle \theta \in [0, R], v \in [0, 2\pi]$

- Assumption for the problem: $\int_{\theta=0}^{\theta=R} \theta^2\sqrt{1+\theta^2}d\theta = 2$
- Center of mass for density ρ is $\frac{\iint_S \vec{x}\rho dA}{\iint_S \rho dA}$
- Use polar coordinates r, θ.
- After computation, $\|\vec{x}_r \times \vec{x}_\theta\| = \sqrt{1+r^2}$
- $M \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \sqrt{1+r^2}(r^2\cos^2(\theta)+r^2\cos^2(\theta))dA = M\iint \sqrt{1+r^2}r^2 = 2\pi M \iint \sqrt{1+r^2}r^2 = 4\pi$ by hint

Example: Flux through unit hemisphere

- Formula: $\Phi = \iint_S (\vec{E}\cdot\vec{n})dA = \iint_S F dA$
- Field: $\vec{E} = \langle yz, xz, xy\rangle$
- Use polar coordinates
- **Base**: $\hat{n} = -\hat{k}$ so $\langle yz, xz, xy\rangle \cdot \langle 0, 0, -1\rangle = -xy$ It's clear by symmetry that $\iint_{u^2+v^2\leq 1} -xydxdy = 0$
- **Top**: Set $u = \theta \in [0, 2\pi), v = \phi \in [0, \frac{\pi}{2}]$.
- As usual, $dA = \|\vec{x}_u \times \vec{x}_v\| = \sin(v)$.
- Norm just points out from the center: $\hat{n} = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle$
- $\vec{E} = \langle \sin(u)\sin(v)\cos(v), \cos(u)\sin(v)\cos(v), \cos(u)\sin^2(v)\sin(u)\rangle$
- So $\vec{E}\cdot\hat{n} = 3\cos(u)\sin(u)\cos(v)\sin^2(v)$
- Looking at this, this is really $\int_{u=0}^{u=2\pi} k(v)\sin^2(u)$ for some $k(v)$, so this will be 0.
- Therefore, total flux is zero, and by Gauss's law, total field contained inside has to be 0 too.

Example: Field $\vec{E} = \ln(x^2+y^2)\langle x, y, 0\rangle$ through R-wide cylinder, height h

- Parameterize: $x = r\cos\theta, y = r\sin\theta, z = z$
- **Top/Bottom**: $\hat{n} = \langle 0, 0, 1\rangle, \vec{E} = f(x, y)\langle x, y, 0\rangle \to \hat{n}\cdot\vec{E} = 0$
- **Side**: $\hat{n} = \frac{1}{R}\langle R\cos(\theta), R\sin(\theta), 0\rangle$
- $\Phi = \iint_{cylinder} \frac{1}{R}\langle R\cos(\theta), R\sin(\theta), 0\rangle \cdot \langle R\cos(\theta), R\sin(\theta), 0\rangle \ln(R^2\cos^2(\theta)+R^2\sin^2(\theta))$
- $= R\iint_{cylinder} \ln(R^2) + \ln(\cos^2(\theta)+\sin^2(\theta)) = R\cdot 2\ln(R)\cdot h\cdot 2\pi R = 4\pi R^2\ln(R)h$

Example: Field $\vec{E} = e^{-x^2-y^2-z^2}\vec{x}$ with sphere S at radius R, setting $\epsilon_0 = 1$

- Parameterize: $x = R\cos(\theta)\sin(\phi), y = R\sin(\theta)\sin(\phi), z = R\cos(phi)$
- $\hat{n} = \langle \cos(\theta)\sin(\phi), y = \sin(\theta)\sin(\phi), z = \cos(phi) \rangle$
- $\vec{x} = R\hat{n}$, so $\vec{E} \cdot n = Re^{-R^2}$
- $R \iint_{sphere} e^{-R^2} = 4\pi R^3 e^{-R^2}$

Note: In the future we write $\boxed{\hat{n} dA = \vec{dA}}$

12.1 3.5: Divergence part I

Main idea: Last chapter was all about having field $\vec{E}$ and wanting to figure out Q_{tot} (or $\frac{\phi}{\epsilon_0}$). Usually, we have the charge distribution Q and want to figure out $\vec{E}$. Most of the field derivation from 3.3 was through tricks for highly symmetric spaces (infinite line, infinite plane, uniform ball, etc.)

Point: The flux through a sphere in a uniform field is zero. Why? Move the center point to the origin, rotate so field is $\hat{k}$ (both don't change the flux), and consider that what goes out at $\langle x, y, z \rangle$ comes in at $\langle x, y, -z \rangle$. This same argument applies for $\iint_{S=sphere} \hat{n}_i \hat{n}_j dA$, where i, j are components in $\{x, y, z\}$.

However, if $i = j$, then $\iint_S \hat{n}_i \hat{n}_j dA = \iint_S \hat{n}_i^2 = \frac{4}{3}\pi R^2$, since $\iint_S (\hat{n}_x^2 + \hat{n}_y^2 + \hat{n}_z^2) dA = \iint_S 1 dA = 4\pi R^2$, so each of the components must be a third of that.

12.1.1 Defining Divergence

Remember that, in Gauss's law $\frac{Q}{\epsilon_0} = \iint_S \vec{E} \cdot \vec{dA}$, we're using information about $\vec{E}$ spread out over surface S. We can also shrink this to a smaller surface.

Shrinking to a point $\vec{P}$, $\lim_{R \to 0} \frac{1}{4\pi R^3} \iint_S \vec{E} \cdot \vec{dA} = \frac{Q_{tot}}{\epsilon_0 4\pi R^3} = \frac{\rho(\vec{P})}{\epsilon_0}$. (This works by dividing both sides by volume of a sphere)

Deriving Divergence: Computing $\lim_{R \to 0} \frac{1}{4\pi R^3} \iint_S \vec{E} \cdot \vec{dA}$

- $\iint_S \hat{n}_i \hat{n}_j dA = 0$ if $i \neq j$
- $\iint_S \hat{n}_i \hat{n}_j dA = \frac{4}{3}\pi R^3$ if $i = j$
- Use linear approximation with Jacobian $D = \frac{\delta E_i}{\delta x_j}$, $\vec{E}(\vec{x}) = \vec{E}(\vec{P}) + D\vec{E}(\vec{P})(\vec{x} - \vec{P})$
- $\iint_S \vec{E}(\vec{P}) = 0$ for any constant. (think of the flux of a sphere in a constant field as above)
- This leaves $D\vec{E}(\vec{P})(\vec{x} - \vec{P}) \cdot \hat{n} = \sum_{i,j} \hat{n}_i [\vec{x} - \vec{P}]_j D\vec{E}(\vec{P})_{ij}$
- Since it's a sphere, the normal $\hat{n} = \frac{\vec{x} - \vec{P}}{R}$

- Therefore $D\vec{E}(\vec{P})(\vec{x}-\vec{P}) \cdot \hat{n} = R \sum \hat{n}_i \hat{n}_j D\vec{E}(\vec{P})_{ij}$ (swap $R\hat{n}_j$ for $[\vec{x}-\vec{P}]_j$)
- These terms are all 0 except where $i = j$, so $D\vec{E}(\vec{P})(\vec{x}-\vec{P}) \cdot \hat{n} = \frac{4}{3}\pi R^2 \times R \times [\frac{\delta E_x}{\delta x} + \frac{\delta E_y}{\delta y} + \frac{\delta E_z}{\delta z}]$
- This equals $\lim_{R\to 0} \frac{1}{4\pi R^3} \iint_S \vec{E} \cdot d\vec{A}$ so eliminating the sphere volume gives us $\boxed{\frac{\rho(\vec{P})}{\epsilon_0} = [\frac{\delta E_x}{\delta x} + \frac{\delta E_y}{\delta y} + \frac{\delta E_z}{\delta z}] = \nabla \cdot \vec{E}}$

We can think of the divergence ∇, also like an operator:
$\boxed{\nabla \cdot \vec{F} = \nabla \cdot (F_x\hat{\imath} + F_x\hat{\jmath} + F_x\hat{k}) = (\frac{\delta}{\delta x}\hat{\imath} + \frac{\delta}{\delta y}\hat{\jmath} + \frac{\delta}{\delta z}\hat{k}) \cdot (F_x\hat{\imath} + F_x\hat{\jmath} + F_x\hat{k})}$

Shifting to Cylindrical Coordinates: If instead we want to describe $\vec{F} = \vec{F}_r\hat{r} + \vec{F}_\theta\hat{\theta} + \vec{F}_z\hat{z}$, we have $\boxed{\nabla \cdot \vec{F} = \frac{1}{r}\frac{\delta r F_R}{\delta r} + \frac{1}{r}\frac{\delta F_\theta}{\delta\theta} + F_z\frac{\delta F_z}{\delta z}}$. How to derive?

- Note identities $\hat{r} = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}, \hat{\theta} = -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}$. If $\theta = 0$, these point right and up, corresponding to $\hat{\imath}, \hat{\jmath}$. If θ rotates, these do too.
- $F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} = \vec{F} = (F_r(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}) + F_\theta(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}) + F_z\hat{k}$
- Rearrange so that $\vec{F} = F_x\hat{\imath} + F_x\hat{\jmath} + F_x\hat{k} = (F_r\cos(\theta) + F_\theta(-\sin(\theta))\hat{\imath} + (F_r\sin(\theta) + F_\theta\cos(\theta))\hat{\jmath} + F_z\hat{k}$.
- Compute $\frac{\delta}{\delta x} = \frac{\delta}{\delta r}\frac{\delta r}{\delta x} + \frac{\delta}{\delta\theta}\frac{\delta\theta}{\delta x} = \frac{\delta}{\delta x} = \cos(\theta)\frac{\delta}{\delta r} - \frac{\sin(\theta)}{r}\frac{\delta}{\delta\theta}$.
  - The second term: $\frac{d\theta}{dx} = \tan^{-1}(y/x)) = \frac{y}{1+y^2/x^2} * \frac{-1}{x^2} = -\frac{r\sin(\theta)}{r^2(\sin^2+\cos^2)} = -\frac{\sin(\theta)}{r}$
- Do something similar for similar for $\frac{d}{dy}$ in the second term.
- Combine and shake it out.

Shifting to Spherical Coordinates: Using a similar process, we get
$\boxed{\nabla \cdot \vec{F} = \frac{1}{\rho^2}\frac{\delta(\rho^2 F_\rho)}{\delta\rho} + \frac{1}{\rho\sin(\phi)}\frac{\delta}{\delta\phi}(\sin(\phi)F_\phi) + \frac{1}{\rho\sin(\phi)}\frac{\delta F_\theta}{\delta\theta}}$

12.2 3.6: Divergence Part 2

Example: Compute divergence of electric field $E = \frac{Q}{4\pi\epsilon_0}\frac{\vec{x}}{\|\vec{x}\|^3}$ outside radius R.

- $\frac{\delta E_x}{\delta x}(\frac{Q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}) = v\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$.
- Symmetrical for $\frac{\delta E_y}{\delta y}, \frac{\delta E_z}{\delta z}$
- Sums to 0.

Example: Compute divergence of electric field $E = \frac{Q}{4\pi\epsilon_0}\frac{\vec{x}}{R^3}$ inside radius R.

- $\frac{\delta E_x}{\delta x}(\frac{Q}{4\pi\epsilon_0}\frac{x}{R^3}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{R^3}$
- Symmetrical for $\frac{\delta E_y}{\delta y}, \frac{\delta E_z}{\delta z}$
- Sums to $\frac{Q}{4\pi\epsilon_0}\frac{3}{R^3}$

So, the divergence of an electric field is proportional to $\frac{Q}{R^3}$ inside the sphere, and 0 outside the sphere.

So, divergence at a point intuitively measures **how much the field spreads out** or sinks into the point. For electric charge, $\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$ means that at that point, the spready-ness is proportional to the charge.

Example: if $\epsilon_0 = 1$ and the field is $\vec{E} = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$, how much charge is in the $[0,1] \times [0,1] \times [0,1]$ box?

- Answer: $\rho = \nabla \cdot \vec{E} = 1 + 2 + 1 = 4$. So 4 units.

Another Example: if $\vec{E} = \sin(yz)\hat{\imath} + \sin(xz)\hat{\jmath} + \sin(xy)\hat{k}$ in some complicated surface, then what?

- Noticing that $\nabla \cdot \vec{E} = 0$ shows you this is 0 no matter the shape of the region. This means *the vectors pointing into the region (in fact, any part of the space) are balanced out by those pointing out from the region.*

12.3 3.7: The Divergence Theorem

The Divergence Theorem falls out of equating finding charge Q with a double integral over a bounded surface with the triple integral of the contained volume:

- $\frac{Q}{\epsilon_0} = \iint_S \vec{E} \cdot d\vec{A}$ (Gauss's law)
- $\Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ within R (Proved Divergence equivalent from last section)
- $Q = \iiint_R \rho dxdydz$ (Just integrating charge over volume)
- $\Rightarrow Q = \iiint_R \rho dxdydz = \epsilon_0 \iiint_R \nabla \cdot \vec{E} dxdydz = \epsilon_0 \iint_S \vec{E} \cdot d\vec{A}$
- $\boxed{\Rightarrow \iint_S \vec{E} \cdot d\vec{A} = \iiint_R \nabla \cdot \vec{E} dxdydz}$ (Divergence Theorem)

Smooshy thought: This looks like another version of Fundamental Theorem of Calculus. The integral of the function evaluated at the boundaries is the same as the function summed inside the boundary.

Proving Divergence Generally: We're gluing micro-cubes together and not changing the total flux. This means any surface is the flux going in and out of its "skin".

- Note that since the flux outward through a cube face is the negative of it inward, gluing two cubes together on this face means we're summing the total fluxes.
- Do this for tiny cubes approximating the surface we care about.
- In a cube centered on point $P, F \approx \vec{F}(P) + D\vec{F}(P)(\vec{x} - \vec{P})$.
- $\iint_S \vec{F}(P)d\vec{A} = 0$ since it's constant, since every face i has normal $\hat{n}_i$, and a partner of equal size with normal $-\hat{n}_i$.
- However, for a cube of side ϵ the flux through, say, Face I ($x = \epsilon + P$) is $\iint_S D\vec{F}(P)(\vec{x} - \vec{P})$ ends up being $\frac{\delta F_x}{\delta x} 4\epsilon^3$, since:
  - Consider side $x = p_x + \epsilon$
  - $D\vec{F}(P)(\vec{x} - \vec{P}) \cdot \hat{n} = [D\vec{F}(P)]_{xx}(x - p_x) + [D\vec{F}(P)]_{xy}(x - p_y) + [D\vec{F}(P)]_{xz}(z - p_z)$.
  - So, the functons that consider the inputs of y, z don't matter.
  - So $\iint_{FaceI}(y - p_y)dA = 0$ around p_y by symmetry. Same for z on that face.
  - But for x, $\iint_{FaceI}(x - p_x)dA = \int_{p_y - \epsilon}^{p_y + \epsilon} \int_{p_z - \epsilon}^{p_z + \epsilon} \epsilon dy dz = 4\epsilon^3$
  - $D_{ij}\vec{F}(P)$ is constant for all $i, j \in \{x, y, z\}$, so this face is then $\frac{\delta F_x}{\delta x} 4\epsilon^3$.
  - Summing the opposite face (with the same flux), yields $\frac{\delta F_x}{\delta x} 8\epsilon^3 = \frac{\delta F_x}{\delta x} V$.
  - Summing across the other faces yields $\frac{\delta F_x}{\delta x} V + \frac{\delta F_y}{\delta y} V + \frac{\delta F_z}{\delta z} V$.

  Finally, this shows the **flux on one of these microcubes is** $\boxed{\nabla \cdot \vec{F}(P)V.}$

  In total, the **divergence theorem**: $\boxed{\iint_{\delta C} \vec{F} \cdot d\vec{A} \approx \nabla \cdot \vec{F}(P)V \approx \iiint_C \nabla \cdot \vec{F} dx dy dz}$

Example of using divergence to calculate flux: Unit hemisphere with $\vec{E} = \langle yz, xz, xy \rangle$:
Answer: $\nabla \cdot \vec{E} = \frac{\delta}{\delta x} yz + \frac{\delta}{\delta y} xz + \frac{\delta}{\delta z} xy = 0$

Example of using divergence to calculate flux: Cylinder of radius R, height h, sitting on $z = 0$ with $\vec{E} = \ln(x^2 + y^2)\langle x, y, 0 \rangle$:
Answer:

- $\frac{\delta}{\delta x} E_x = \ln(x^2 + y^2) + \frac{2x^2}{x^2 + y^2}$. Similar for E_y.
- Transform to polar: $E_x + E_y = \ln(r^2) + \frac{2r^2 \cos(\theta)^2 + r^2 \sin(\theta)^2}{r^2} = 2\ln(r) + 2$
- Set up the integral, remembering the Jacobian: $\Phi = 2\pi \int_{z=0}^{h} \int_{r=0}^{r=R} [2\ln(r) + 2] r dr d\theta$
- Working it out, with identity $\int x \ln(x) = -\frac{x^2}{4} + \frac{x^2}{2} ln(x)$, you get $\Phi = 4\phi R^2] \ln(R)h$

Example of using divergence to calculate flux: Unit sphere at origin with $\vec{E} = (x^3+y^3)\hat{i} + (z^3+y^3)\hat{j} + (x^3+z^3)\hat{k}$
Answer:

- $E_x + E_y + E_z = 3x^2 + 3y^2 + 3z^2 = 3\iiint \rho^2 dxdydz$
- Each $d\rho$ is a sphere of volume $4\pi f(\rho)^2 = 4\pi\rho^4$
- So the integral is $12\pi \int_{\rho=0}^{\rho=1} \rho^4 = \frac{12\pi}{5}$

Example of using divergence to calculate flux: $\vec{F} = (\cos(z)+x^2)\hat{i}, +(xe^{-z})\hat{j} + (\sin(y)+x^2z)\hat{k}$ on parabaloid $z = x^2+y^2, z \le 4$ with top $x^2+y^2 \le 4, z = 4$

- $\iiint_R \nabla \cdot \vec{E} = \int_{z=r^2}^{4} \int_{x^2+y^2=0}^{2} (y^2+x^2) dxdydz$
- $\iiint_R \nabla \cdot \vec{E} = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r^2}^{4} (r^2\cos^2(\theta) + r^2\sin^2(\theta)) r d\theta dr dz = r^3 d\theta dr dz$
- $= 2\pi \int_{r=0}^{2} 4r^3 - r^5 = 2\pi[r^4 - \frac{r^6}{6}]_0^2 = \frac{32}{3}\pi.$

What's crazy: Evaluating divergence of a point charge $\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3}$

- $\frac{\delta}{\delta x} E_x = \frac{Q}{4\pi\epsilon_0} \frac{\delta}{\delta x} x(x^2+y^2+z^2) = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$
- E_y, E_z follow symmetrically.
- The sum is infinite at the origin and zero everywhere else
- Therefore, they had to invent a δ function that is infinite at origin, 0 elsewhere, and $\iiint_{\mathbb{R}^3} \delta(\vec{x}) = 1$

12.4 3.8: Divergence and Fluids

Looking back to section 3.1, this hydrostatic force function should follow similar patterns to flux: $\boxed{\vec{F_{tot}} = \iint_S p\hat{n} dA}$.

Extended Example: A round ball of radius R, center at depth h , with force $\vec{F_{tot}} = p_0 \iint_S [1 - \frac{z}{h}]\hat{n} dA$.

- $\hat{n} = \frac{\langle x,y,z \rangle}{R}$
- For the integral, note that x, y are completely symmetric around z axis, so they contribute 0.
- For the integral, we're then looking at $\frac{p_0}{R} \iint_S [1 - \frac{z}{h}] z dA$
- Use spherical coordinates: $\frac{p_0}{R} \iint_S [1 - \frac{R\cos(\phi)}{h}] R\cos(\phi) dA$

- Working through $dA = \sqrt{1+f_x^2+f_y^2}dxdy$ with $f = R - \sqrt{R^2+x^2+y^2}$, we get $dA = \frac{dxdy}{\sqrt{(1-\frac{x^2+y^2}{R^2})}}$
- This dA term, in spherical coordinates, becomes $R^2\sin(\phi)d\phi d\theta$
- Combining and substituting $u = \cos(\phi)$, this integral is $-\frac{4\pi R^3}{3}\frac{p_0}{h}\hat{k}$

The neat idea: $F_{tot} = \frac{4\pi R^3}{3} \times -\frac{p_0}{h}\hat{k}$ is really "ball's volume" times a constant.

- $\frac{4\pi R^3}{3} \times -\frac{p_0}{h}\hat{k}$
- $= \iiint_B 1dxdydz \times -\frac{p_0}{h}\hat{k}$
- $= \iiint_B(-\frac{p_0}{h})\hat{k}dxdydz$, with $p = p_0[1-\frac{z}{h}]$
- $= \iiint_B(\frac{\delta p}{\delta z})\hat{k}dxdydz$
- $= \iiint_B \nabla \cdot pdxdydz$
- So the upshot is the divergence theorem again: $\boxed{\iint_S p\hat{n}dA = \iiint_B \nabla \cdot pdxdydz = \iiint_B \nabla \cdot pd\vec{x}}$

Final example three ways: "oxygen flow" (really, flux) through ball of radius R at origin, under field $J = j_0\hat{i}$.

- Intuitive: what comes in at (-x, y, z) goes out at (x, y, z), so total is zero.
- Flux integra under spherical: $\iint \vec{F}\hat{n}dA = \iint_S j_0\hat{i}\cdot\frac{\langle x,y,z\rangle}{R}dA = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} j_0\cos(\theta)\sin(\phi)R^2\sin(\theta)d\theta d\phi = 0 = \frac{\pi j_0 R^2}{2}\int_{\theta=0}^{2\pi}\cos(\theta)d\theta = 0$
- Divergence: $\nabla\cdot\vec{J} = \frac{\delta}{\delta x}j_0 + 0 + 0 = 0, so \iiint_B 0 = 0.$

12.5 3.9: Flows and Divergence

Main idea: Divergence ($\nabla\cdot\vec{V}$) measures how much the flow changes volumes at that point.

Example: What is the function described by field of velocity vectors $\vec{V}(\vec{x}) = \langle -y, x\rangle$?

- $x'(t) = -y, y'(t) = x$
- $\Rightarrow x''(t) = -y' = -x, y''(t) = x' = -y$
- $\Rightarrow x = A\cos(t)+B\sin(t), y = C\cos(t)+D\sin(t)$, work it out to $x = \cos(t), y = \sin(t)$

Idea: dump a $dA = s_1$ by $s_2 = \Delta x\hat{i} \times \Delta y\hat{j}$rectangle into the flow and see how it deforms over time. Over a long time, it'll distort a lot, but consider for Δt:

- dA has sides of length $\Delta x, \Delta y$ but area of dA: $\|s_1 \times s_2\|$ (cross product norm is parallelogram area)
- What is side s_1 after Δt? The starting point plus (how the endpoint moves minus how the start point moves): $\vec{s_1} + \Delta t[\vec{V}(x_0 + \Delta x, y_0) - \vec{V}(x_0, y_0)]$
- Expanding the iterated s_1, which we call s_1' out: $\vec{s_1}' = \Delta x[(1 + \Delta t\frac{\delta V_x}{\delta x})\hat{i} + \Delta t\frac{\delta V_y}{\delta x}\hat{j}]$. Do the same for s_2' and work out in 3D: $s_1' \times s_2' \approx \Delta x \Delta y[\hat{k} + \Delta t[(V_x)_x + (V_y)_y]\hat{k}$
- We end up with $s_1' \times s_2' \approx \Delta x \Delta y[1 + \Delta t \nabla \cdot \vec{V}]\hat{k}$, so vs. original area $\Delta x \Delta y$, the ratio is $1 + \Delta t \nabla \cdot \vec{V}$
- This means **divergence $\nabla \cdot \vec{V}$ is proportional to the change in area due to the flow**!

An **incompressible** field preserves volume under flow (so $\nabla \cdot \vec{V} = 0$), like $\langle y, z, x\rangle, \langle 0, 2\sqrt{x}, 0\rangle, \langle x, y, -2z\rangle$.

A cool interactive on the page shows how a sphere migrating its points via $\langle x, y, z\rangle$ grows and changes volume, while one under $0.3\langle y, z, x\rangle$ distorts but doesn't.

Some