Brilliant: Vector Calculus

Dave Fetterman

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Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against t of the form $\overrightarrow{x}(t) = \langle x(t), y(t), \ldots \rangle$.

- A **line** through $p = (a, b, c)$ parallel to $\overrightarrow{v} = \langle v_x, v_y, v_z \rangle$ is $\overrightarrow{x}(t) = \overrightarrow{p} + t\overrightarrow{v}$
- **velocity** is characterized completely by $\overrightarrow{v}(t) = \overrightarrow{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- The **speed** of an object along that line versus t is the length of v ($\|v\|$)
- Therefore, the speed of an object along line

  $$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

  is

  $$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that $\overrightarrow{v}$ need not be constant. The speed of

  $$\overrightarrow{x}(t) = \overrightarrow{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

  would then be

  $$\|6\pi\cos(2\pi t)\hat{u}\| = |6\pi\cos(2\pi t)|$$

- **Acceleration** $a(t) = v'(t) = x''(t)$ is straightforward. Acceleration of

  $$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance r from the sun could be $\langle r\cos(t), r\sin(t) \rangle$. The acceleration vector points in the opposite direction: $\langle -r\cos(t), -r\sin(t) \rangle$. In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by $F = ma$ is proportional to acceleration) points towards the sun.

- A **helix** could be a 3D extension like $\langle r\cos(t), r\sin(t), b\cdot t\rangle$.

2 Chapter 2.2: Space Curves

- TODO: Problem 5 - rotating ellipses and solving intersections with planes
- Note that while $\overrightarrow{x}(t) = \langle\cos(t), \sin(t), 5\rangle$ and $\overrightarrow{x}(t) = \langle\cos(2t), \sin(2t), 5\rangle$ describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If $\overrightarrow{x}(t) = t\overrightarrow{v}$, then speed is $\frac{\|\overrightarrow{x}(t+\Delta t)-\overrightarrow{t}\|}{\Delta t} = \|\overrightarrow{v}\|$, direction is $\frac{\overrightarrow{v}}{\|\overrightarrow{v}\|}$, and velocity $\overrightarrow{v}$ is the product of speed and direction.
- So $\overrightarrow{v}(t) = \lim_{\Delta t\to 0}\frac{\overrightarrow{x}(t+\Delta t)-\overrightarrow{x}(t)}{\Delta t} = \overrightarrow{x}'(t) = \frac{d\overrightarrow{x}}{dt} = \langle x'(t), y'(t), z'(t)\rangle$
- Neat conceptual result: any $y = f(x)$ can be made into $x(t) = \langle t, f(t), 0\rangle$, and then $v(t) = \langle 1, f'(t), 0\rangle$, which points along the tangent line at $\langle t, f(t), 0\rangle$.
- Note that dot product derivatives work like regular product: $[\overrightarrow{a}(t)\cdot\overrightarrow{b}(t)]' = \overrightarrow{a}'(t)\cdot\overrightarrow{b}(t) + \overrightarrow{a}(t)\cdot\overrightarrow{b}'(t)$, but the cross product does not work the same since $\frac{d}{dt}[a\times b] = a'\times b + a\times b'$, but since $a\times b' = -b'\times a$, can't switch the order to $a'\times b + b'\times a$ due to this non-commutativity.
- If
$$\overrightarrow{x}(t) = \overrightarrow{p} + t\overrightarrow{v},$$
calculating velocity with respect to origin becomes
$$\frac{d}{dt}\|\overrightarrow{x}(t)\| = \frac{\overrightarrow{x}(t)\cdot\overrightarrow{x}'(t)}{\|\overrightarrow{x}(t)\|} = \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}\cdot\overrightarrow{v},$$
after rewriting the distance formula and chugging through the chain rule.
- However, it becomes more clear when considering that $(\overrightarrow{v}\cdot\hat{x})\hat{x}$ is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way: $\int_a^b \overrightarrow{x}(t) = \langle\int_a^b x(t), \int_a^b y(t), \int_a^b z(t)\rangle$

- Example: if ball launched from origin with velocity $\langle 1,2,3\rangle$ and acceleration $\langle 0,0,-1\rangle$, it lands at

$$\frac{dv}{dt}dt = \langle 0,0,-1\rangle \tag{1}$$

$$\int \frac{dv}{dt}dt = v = \langle C, D, -t+F\rangle = \langle 1,2,3\rangle = \langle 1,2,-t+3\rangle, t=0 \tag{2}$$

$$x = \int v = \langle t+K, 2t+M, -\frac{1}{2}t^2+3t+N\rangle, x(\overrightarrow{0}) = \langle 0,0,0\rangle \tag{3}$$

$$\overrightarrow{x}(t) = \langle t, 2t, 3t-\frac{1}{2}t^2\rangle \tag{4}$$

$$z(t) = 0 \to t = 6 \to \overrightarrow{x}(6) = \langle 6, 12, 0\rangle \tag{5}$$

$$\tag{6}$$

- Also, generalizing $ds = \sqrt{(dx)^2+(dy)^2}$, the length of an arc from point a to b aporoaches $\int_a^b \|x'(t)\|dt$
- Example: a helix $\langle a\cos(\omega t), a\sin(\omega t), b\omega t\rangle$, parametrized by time t can be rewritten in terms of s, the arc length:

$$s = \int \|x'(t)\|dt \tag{7}$$

$$s = \int \sqrt{(-\omega a\sin(\omega t))^2 + (\omega a\cos(\omega t))^2 + (b\omega)^2}dt \tag{8}$$

$$s = |\omega| \int \sqrt{(a^2+b^2)}dt \tag{9}$$

$$s = |\omega|\sqrt{a^2+b^2}t \tag{10}$$

- *Note: It's weird to think of time in terms of length.* Could be analytically useful?