

# Brilliant: Vector Calculus

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Note: Latex reference: <http://tug.ctan.org/info/undergradmath/undergradmath.pdf>

## 1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against  $t$  of the form  $\vec{x}(t) = \langle x(t), y(t), \dots \rangle$ .

- A **line** through  $p = (a, b, c)$  parallel to  $\vec{v} = \langle v_x, v_y, v_z \rangle$  is  $\vec{x}(t) = \vec{p} + t\vec{v}$
- **velocity** is characterized completely by  $\vec{v}(t) = \vec{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ .
- The **speed** of an object along that line versus  $t$  is the length of  $v$  ( $\|v\|$ )
- Therefore, the speed of an object along line

$$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

is

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that  $\vec{v}$  need not be constant. The speed of

$$\vec{x}(t) = \vec{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

would then be

$$\|6\pi \cos(2\pi t)\hat{u}\| = |6\pi \cos(2\pi t)|$$

- **Acceleration**  $a(t) = v'(t) = x''(t)$  is straightforward. Acceleration of

$$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance  $r$  from the sun could be  $\langle r \cos(t), r \sin(t) \rangle$ . The acceleration vector points in the opposite direction:  $\langle -r \cos(t), -r \sin(t) \rangle$ . In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by  $F = ma$  is proportional to acceleration) points towards the sun.

- A **helix** could be a 3D extension like  $\langle r \cos(t), r \sin(t), b \cdot t \rangle$ .

## 2 Chapter 2.2: Space Curves

- TODO: Problem 5 - rotating ellipses and solving intersections with planes
- Note that while  $\vec{x}(t) = \langle \cos(t), \sin(t), 5 \rangle$  and  $\vec{x}(t) = \langle \cos(2t), \sin(2t), 5 \rangle$  describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If  $\vec{x}(t) = t \vec{v}$ , then speed is  $\frac{\|\vec{x}(t+\Delta t) - \vec{x}(t)\|}{\Delta t} = \|\vec{v}\|$ , direction is  $\frac{\vec{v}}{\|\vec{v}\|}$ , and velocity  $\vec{v}$  is the product of speed and direction.
- So  $\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} = \vec{x}'(t) = \frac{d\vec{x}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$
- Neat conceptual result: any  $y = f(x)$  can be made into  $x(t) = \langle t, f(t), 0 \rangle$ , and then  $v(t) = \langle 1, f'(t), 0 \rangle$ , which points along the tangent line at  $\langle t, f(t), 0 \rangle$ .
- Note that dot product derivatives work like regular product:  $[\vec{a}(t) \cdot \vec{b}(t)]' = \vec{a}'(t) \cdot \vec{b}(t) + \vec{a}(t) \cdot \vec{b}'(t)$ , but the cross product does not work the same since  $\frac{d}{dt}[a \times b] = a' \times b + a \times b'$ , but since  $a \times b' = -b' \times a$ , can't switch the order to  $a' \times b + b' \times a$  due to this non-commutativity.
- If

$$\vec{x}(t) = \vec{p} + t \vec{v},$$

calculating velocity with respect to origin becomes

$$\frac{d}{dt} \|\vec{x}(t)\| = \frac{\vec{x}(t) \cdot \vec{x}'(t)}{\|\vec{x}(t)\|} = \frac{\vec{x}}{\|\vec{x}\|} \cdot \vec{v},$$

after rewriting the distance formula and chugging through the chain rule.

- However, it becomes more clear when considering that  $(\vec{v} \cdot \hat{x})\hat{x}$  is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!