Polynomial Uniqueness via Tournaments

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Abstract

In 2D space, two points $(x_1, y_1), (x_2, y_2), x_1 \neq x_2$ define a line, a polynomial of degree 1. Three distinct points $(x_1, y_1), (x_2, y_2), (x_3, y_3) x_1 \neq x_2 \neq x_3 \neq x_1$ define a parabola, a polynomial of degree 2. In general, for finite univariate polynomials of nonnegative, whole degree, $n + 1$ such points uniquely specify a polynomial of degree n. Why?

This is not a new result. This is a paper is simply a thoroughly awkward trip through a few mathematical domains to arrive at a well known destination. Helicopters and cars both have their uses. But you wouldn't build a car by turning a helicopter on its side and adding wheels.

Metaphorically, I do, so you don't have to.

1 Setup

If we have points $f(x_0) = y_0, f(x_1) = y_1, \ldots f(x_n) = y_n$, how can we determine the coefficients a_i of the polynomial $f(x) = a_0 x^0 + a_1 x^1 + \ldots + a_n x^n$?

This square matrix of width $n + 1$, which I'll denote X_n, is known as a Vandermonde matrix[**1**], and models this set of $n + 1$ equations as $X \cdot \vec{a} = \vec{y}$:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Therefore, we can find our unique coefficient vector A if and only if we can solve $X \cdot \vec{a} = \vec{y}$, or $\vec{a} = X^{-1}\vec{y}$. This has a unique solution if and only if $\det(X) \neq 0$. The rest of this paper tries to find this determinant through all the wrong ways.

2 Finding the Vandermonde determinant

It should be noted that there are other, clearer methods of finding this determinant[**1**] either starting with polynomial unqiueness (basically, going the "other" direction), abstract algebra, direct linear algebra, vector maps, and likely others. These, however, were not the ones I stumbled on.

First, we know that if any $x_i = x_j$ for distinct i, j, we have no solution, and a zero determinant. If $f(x_i) = f(x_j), x_i = x_j$, then we are simply underdetermined (not enough points for a unique polynomial). If $f(x_i) = f(x_j), x_i \neq x_j$, then we have a impossible vertical section of our graph. Otherwise, we are in good shape.

This suggests that every pair $(x_i, x_j), i < j$ corresponds to a factor $(x_j - x_i)$ in the determinant, and that the determinant is then some multiple of $D = \prod_{0 \leq i < j \leq n}(x_j - x_i)$.

Taking $n = 2$ as a base case ($n = 1$ produces a boring constant $f(x)$), we see that $\det \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} = (x_1 - x_0)$, suggesting the determinant of a Vandermonde matrix is exactly D.

Theorem: The determinant of an X_n with generating coefficients $x_0, x_1 ... x_n$ is $\prod_{0 \leq i < j \leq n}(x_j - x_i)$.

With the base case $n = 2$ in hand, the rest of the paper handles the inductive step of proving the main theorem.

Inductive Step of Proof of Theorem: If $\det(X_n) = \prod_{0 \leq i < j \leq n}(x_j - x_i)$ for all X_n, then $\det(X_{n+1}) = \prod_{0 \leq}$

I will do this in the most roundabout way possible.

2.1 Setup: Vandermonde inductive step and main theorem

2.1.1 Definitions

Let's create a few definitions:

- Denote by $M_{n,k}$ the Vandermonde matrix X_n with row k and last column excluded, often called a "matrix minor".
- Given an ordered set of indices $I = [0, n]$, denote by P_I the product of all factors the form $(x_j - x_i)$, given $i < j$ and $i, j \in I$. So $P_{[0,2]} = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1)$.
- Given an ordered set of indices $I = [0, n]$, denote by S_I the sum over all $n + 1$-sized permutations σ on I , of all terms of form $sgn(\sigma) x_{\sigma(n)}^n x_{\sigma(n-1)}^{n-1} ... x_{\sigma(0)}^0$. So $S_{[0,2]} = x_2^2 x_1^1 x_0^0 - x_2^2 x_0^1 x_1^0 - x_1^2 x_2^1 x_0^0 + x_1^2 x_0^1 x_2^0 + x_0^2 x_2^1 x_1^0 - x_0^2 x_1^1 x_2^0$.

The rest of the proof of the inductive step above follows from showing:

TODO: Make sure I have the exponents right for the minor-based determinant formula.

- (1) $det(X_n) = \sum_{k=0}^{n}(-1)^k x_k^n \det(M_{n,k})$
- (2) For our base base, $\det(X_2) = P_{[0,1]}$
- (3) By inductive hypothesis $det(X_n) = \sum_{k=0}^{n}(-1)^k x_k^n P_{[0,n]-\{k\}}$
- (4) $\sum_{k=0}^{n}(-1)^k x_k^n P_{[0,n]-\{k\}} = S_[0,n]$
- **Lemma**: For a set of indices I, $P_I = S_I$.
- Therefore, transitively, $det(X_n) = P_{[0,n]}$.

NOTE : TODO - There is an order problem here. (1) is shown readily.

The determinant of $X = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix}$ can be calculated down the rightmost column as

$\det(X) = x_0^n \det(M_{n,0}) - x_1^n \det(M_{n,1}) + ... + (-1)^n x^n \det(M_{n,n})$.

(2) is clear, with $\det(\begin{bmatrix} 1 & x_0^1 \\ 1 & x_1^1 \end{bmatrix}) = -1 \cdot (1 * M_{2,1} - 1 * M_{2,0}) = (x_1 - x_0) = P_{[0,1]}$.

(3) says inductively, we can presuppose that for any $M_{n,k}$, *which is itself a Vandermonde matrix*, $\det(M_{n,k})$ can be expressed as $P_{[0,n]-\{k\}}$

(4) Assuming the theorem true, the above bullet sequence should be clear with some checking. On $\{c, b, a\}$, for example, the terms split out exactly into $c^2(b^1a^0 - b^0a^1) - b^2(c^1a^0 - a^1c^0) + a^2(c^1b^0 - b^1c^0) = c^2b^1a^0 - c^2a^0b^1 - b^2c^1a^0 + b^2a^1c^0 + a^2c^1b^0 - a^2b^1c^0$

2.2 Proof of the main theorem

Base Case: We've shown this is true for $n = 2 \Rightarrow S_{[0,1]} = (x_1^1x_0^0 - x_0^1x_1^0)$. So our inductive step supposes that all terms of (3) for ranges $[0, n-1]$ are of the form $sgn(\sigma)x_{\sigma(n-1)}^{n-1}x_{\sigma(n-2)}^{n-2}...x_{\sigma(0)}^0$ for some permutation σ on the node set $[0, n-1]$.

The proof that $S_[0,n] = \prod_{0\le i<j\le n}(x_j - x_i)$ requires adding a new node x_{n+1} to the left side and a multiplying new set of factors $\prod_{0\le i<\le n}(x_{n+1} - x_i)$ by the right side and showing they are equal.

- Lemma 1: Show an isomorphism between products of the form (3) and tournament graphs on $n+1$ nodes.
- Lemma 2: Show that terms of the form $sgn(\sigma)x^n_{\sigma(n)}x^{n-1}_{\sigma(n-1)}...x^0_{\sigma(0)}$ remain in (3) after expansion. These correspond to acyclic tournaments on $n+1$ nodes.
- Lemma 3: Show that all other terms in the expansion of (3), which correspond to tournaments with a cycle, can be paired 1:1 with a identical but inverted term, corresponding to an identical graph with *one 3-cycle reversed.*
- Thus, the sum of the terms addressed

3 Prove : VanDerMonde matrix determinant is prod $(x_i - x_j), 1 <= i < j <= n$

This is the determinant of the van der modne matrix

3.1 Base cane: n = 2

3.2 Inductive case

This equalsx^n (product without x), $+y^n$(product without y)...

4 Pieceyard

By inductive hypothesis, each of the terms $x^n_k \det(M_{n,k})$ becomes:

$x^n_k \prod_{0\leq i<j\leq n; i\neq k, j\neq k}(x_j - x_i)$ or $x^n_k \prod_{0\leq i<j\leq n; i,j\in I_{n-k}}(x_j - x_i)$

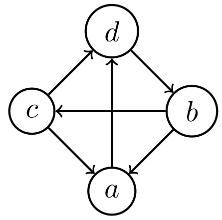
Therefore, we need to prove that $0 \neq \det(X)$

$$= \sum_{k=0}^{n}(-1)^k x^n_k \det(M_{n,k}) \tag{1}$$

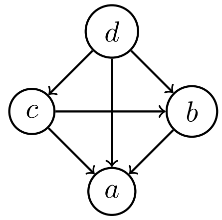
$$= \sum_{k=0}^{n}(-1)^k x^n_k \Big[\prod_{0\leq i<j\leq n; i,j\in[0,n]-\{k\}} (x_j - x_i)\Big] \tag{2}$$

$$= \prod_{0\leq i<j\leq n; i,j\in[0,n]} (x_j - x_i) \tag{3}$$

(1) is a determinant expansion. The *det* term equals the bracketed term of (2) by inductive hypothesis.



(a) An arbitrary tournament on 4 nodes



(b) An (acyclic) tournament $d^3c^2b^1a^0$

Figure 1: Tournaments

We seek to prove this main theorem:

Theorem: The expansion of (3) is exactly the sum of all possible terms of the form $sgn(\sigma)x^{n-1}_{\sigma(n-1)}x^{n-2}_{\sigma(n-2)}...x^{0}_{\sigma(0)}$ for some permutation σ on the node set $[0, n-1]$. Call this $S_{[0,n]}$. So, for example$S_{\{d,c,b,a\}}$, would be exactly all terms like $d^3c^2b^1a^0$, $-c^3d^2b^1a^0$ or $-c^3a^2d^1b^0$.

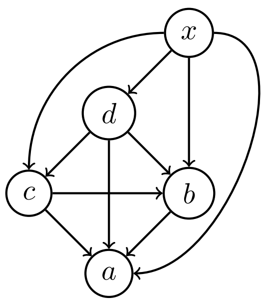
If we have this theorem proven, then:

- For $n = 2$, the determinant of X_2 is $1 \cdot x_1 - 1 \cdot x_0 = (x^1_1x^0_0 - x^1_0x^0_1) = S_{[0,1]}$
- By inductive hypothesis, the expansion of the bracketed term of (2), $S_{[0,n]-\{k\}}$ yields the same set of sums except each sum excludes all use of x_k.
- The sum of all terms $(-1)^k x^n_k S_{[0,n]-k}$ is exactly $S_{[0,n]}$, meaning (3).
- Therefore, (2) = (3) and we have our Vandermonde determinant (and thus our proof of polynomial uniquenss).

The sorted tournament $d^3c^2b^1a^0$

(a) $-x^3 a \cdot d^3 c^2 b^1 a^0$, with cycle (xba) (b) $-x^3 b \cdot -d^3 c^2 a^1 b^0$, with cycle (xab)

Figure 2: Terms in expanded $\prod(x_j - x_i)$ are inverses with inverted 3-cycles



The sorted tournament $x^4 d^3 c^2 b^1 a^0$

Factors of $(x-a)(x-b)(x-c)(x-d)$ multiplied by $\sigma = d^3 c^2 b^1 a^0$

Factor	Product	Matching Factor	Matching σ	Critical pair
x^4	$x^4d^3c^2b^1a^0$	none	none	none
$-x^3a$	$-x^3d^3c^2b^1a^1$	$-x^3b$	$-d^3c^2a^1b^0$	ba
$-x^3b$	$-x^3d^3c^2b^2a^0$	$-x^3c$	$-d^3b^2c^1a^0$	cb
$-x^3c$	$-x^3d^3c^3b^1a^0$	$-x^3d$	$-c^3d^2b^1a^0$	dc
$-x^3d$	$-x^3d^4c^2b^1a^0$	none	none	none
x^2ba	$x^2d^3c^2b^2a^1$	x^2ca	$-d^3b^2c^1a^0$	cb
x^2ca	$x^2d^3c^3b^1a^1$	x^2da	$-c^3d^2b^1a^0$	dc
x^2da	$x^2d^4c^2b^1a^1$	x^2db	$-d^3c^2a^1b^0$	ba
x^2cb	$x^2d^3c^3b^2a^0$	x^2db	$-c^3d^2b^1a^0$	dc
x^2db	$x^2d^4c^2b^2a^0$	x^2dc	$-d^3b^2c^1a^0$	dc
x^2dc	$x^2d^4c^3b^1a^0$	none	none	none
$-xcba$	$-xd^3c^3b^2a^1$	$-xdba$	$-c^3d^2b^1a^0$	dc
$-xdba$	$-xd^4c^2b^2a^1$	$-xcba$	$-d^3b^2c^1a^0$	cb
$-xdca$	$-xd^4c^3b^1a^1$	$-xdcb$	$-d^3c^2a^1b^0$	ba
$-xdcb$	$-xd^4c^3b^2a^0$	none	none	none
$dcba$	$d^4c^3b^2a^1$	none	none	none

5 TODO

5.1 TODO

References

[1] Wikipedia: `https://en.wikipedia.org/wiki/Vandermonde_matrix`