

# Spotting Graph Theory Problems in Spot It

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## Abstract

The card game “Spot It” supports a unique mechanic: every card of the deck has eight different symbols, and shares exactly one symbol of some kind with every other card. The obvious game play (“spot the match”) works for children as young as two; the intricacies of deck construction astound these children of forty-two. In hypothetically constructing our own deck, we run across interesting problems in graph theory, number theory, and abstract algebra. Some are solved, some remain unsolved. Until now. Not really. We just prove hard things are hard.

Notably, starting with  $g = p^k, p \in \mathbb{P}, k \in \mathbb{N}$  and  $n = g^2 + g + 1$ , we prove the following four constructions are equivalent:

1. (The Children’s Game) A deck of  $n$  “Spot It” Cards with  $s = g + 1$  symbol slots, where each of  $m = n$  symbols occurs exactly  $g + 1$  times,
2. (Graph Theory) An edge partition of the complete graph  $K_n$  into complete subgraphs  $K_g$ ,
3. (Abstract Algebra) A finite (Galois) field of order  $g$ , and
4. (Number Theory) A Perfect Difference Set[1] on  $n$  elements.

We then examine other configurations of  $s$  and  $g$ , as well as comment on other reasonable constructions of a Spot It Card deck.

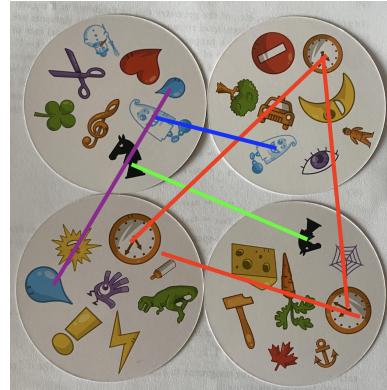
## 1 The game and the problem

Introduced to us by Ari Steinberg, “Spot It” is a children’s game of 55 cards as shown in Figure 1a, featuring eight colorful symbols on each. Though gameplay comes with a few variants in its tiny rulebook, the primary mechanic when presented two cards is simply *spot single the common symbol first*. The game is simple enough for a two- or three-year old to grasp (and win!), but this poses the question: *just how did they construct such a deck?*

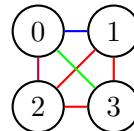
*Note: There have been some other investigations here[2], but for the purposes of enjoyment, everything in this paper was researched without reference to prior (Spot It) work.*



(a) Four cards in the game



(b) Four cards in the game with links



(c) Four cards graph

Naturally, there are trivial constructions: every symbol occurs only twice, or once, or the count of symbols is so varied that a deck can be constructed almost greedily. However, the Spot It game has uniformly 8 symbol “slots” on each card, which each appear 8 times<sup>1</sup> across the different cards.

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<sup>1</sup>As we will see later, there should be 57 cards for this to be true; it's likely two cards were removed

This is what we examine in this paper:

**The Core Question:** For what choices of  $g$  and  $s$  can we construct a “Spot It” deck where each of the  $s$  symbols on each card appears exactly  $g$  times throughout the deck?

### 1.1 Reframing as a graph

Noticing that every card has a relationship to every other card (notably, the identity of the single symbol shared between them) as in Fig. 1b, we take our first step by reconstructing this problem as an undirected graph as in Fig. 1c.

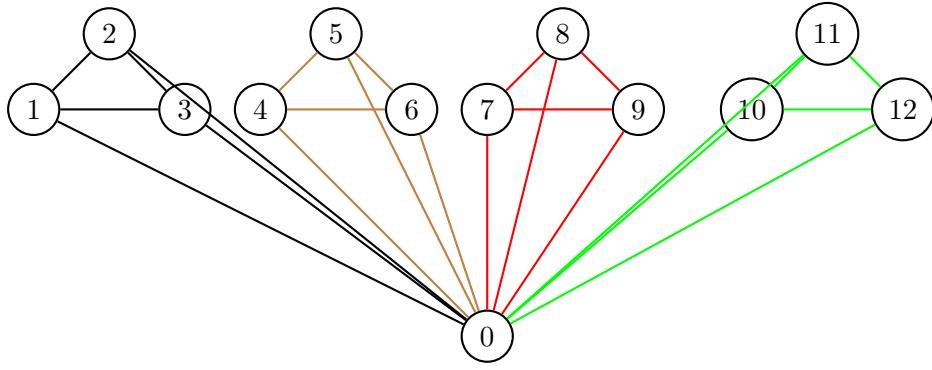


Figure 2:  $n = s(g - 1) + 1$ . Here,  $s = 4, g = 3$

**The Graph Representation:** A deck of Spot It Cards each with  $s$  symbol slots, where each symbol appears  $g$  times can be represented by a graph  $G$ :

1. With  $n$  nodes, where  $n = s(g - 1) + 1$
2. With  $m$  unique edge colors, where  $m$  denotes the number of symbols,
3. Where all edges of color  $m_i$  form a complete subgraph on  $g$  nodes,  $s$ , and  $m = \binom{n}{2} / \binom{g}{2}$ .

*Note: Singletons (groups where  $g = 1$ ) would be rendered as self-edges. These are uninteresting and are generally ignored in this paper*

*Proof:*

1. As in Fig. 2, node  $n_0$ 's adjacencies are exactly  $s$  monochromatic cliques of size  $g - 1$  (excluding  $n_0$  itself). In a complete graph, these adjacencies comprise the total node set, so  $n = (g - 1)s + 1$  when adding  $n_0$  back in. Using any other node is equivalent.

2. As in Fig. 1b, while card 1 and card 2 having the relationship “clock”, node 1 and node 2 instead share an edge with the color red. This is the same relationship between nodes 2 and 3, and nodes 1 and 3. “Drop”, “knight”, and “ghost” would be colors purple, green, and blue, respectively. This works because every edge has exactly one color (corresponding 1:1 with a symbol) in this formulation, and every card pair has exactly one symbol shared.
3. All cards with a given symbol (say, “clock”) must correspond to nodes be linked with the color red to all other nodes whose card has a clock; this is a complete subgraph. A complete graph  $K_n$  has  $\binom{n}{2}$  edges. A monocolored clique of size  $g$  is a complete graph as well, with  $\binom{g}{2}$  edges.  $K_n$ ’s edges are exactly these equal-sized cliques, so there are therefore  $m = \frac{\binom{n}{2}}{\binom{g}{2}}$  of them, corresponding to colors.

And since every edge in our complete graph  $K_n$  is in exactly one monocolored clique of size  $K_g$ , this becomes a crisp graph theory problem.

**The Core Question in Graph Terms:** Given  $s$  and  $g$  as before, can we construct an edge partition (colloquially here, “coloring”)<sup>2</sup> of  $K_n$  into a set of complete subgraphs of size  $g$  (denoted  $K_g$ )?

**Though exhaustive research wasn’t done, this graph problem does not appear to have a clear analytical solution out there.<sup>3</sup>**

Since  $m$  and  $n$  are determined from  $s$  and  $g$ , we’ll start by looking at possibly candidate configurations of  $s$  and  $g$ .

## 2 The Candidate Theorem: $g|s(s - 1)$ , $g \leq s$

Suppose that every symbol  $s$  has exactly  $g$  cards containing it<sup>4</sup>. Then

1.  $g|s(s - 1)$ .
2. If  $s > 1$  and  $g > 1$  then  $g \leq s$
3. Corollary to (2):  $m = \binom{s}{g}n$  and therefore  $m \geq n$
4. All candidate configurations of  $g, s$  are  $g \leq s$ ,  $g|s(s - 1)$ .

*Proof:*

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<sup>3</sup>Or people who care about publishing it within the reach of lazy hobbyists, anyway!

<sup>4</sup>for example, all  $s = 7$  symbols in Fig. ?? correspond to cliques of size  $g = 3$

1.

$$\binom{g}{2} \left| \binom{n}{2} \right. \Rightarrow \frac{n(n-1)}{g(g-1)} \in \mathbb{N} \Rightarrow g(g-1) | n(n-1) \quad (1)$$

$$n = (g-1)s + 1 \Rightarrow g(g-1) | (sg-s+1)(sg-s) = (sg-s+1)s(g-1) \quad (2)$$

$$\Rightarrow g|s^2g - s^2 + s \Rightarrow g|(1-s)s \Rightarrow g|s(s-1) \quad (3)$$

2. Any node  $n_i$  is adjacent to  $s$  monochromatic cliques of size  $g$ . These cliques  $C_1 \dots C_s$ , containing non- $n_i$  nodes if  $g > 1$ , comprise all nodes, and any other cliques can contain no more than one of each  $K_i$ . This means that clique of size  $g$  greater than  $s$  cannot be formed, since the only place to find nodes are these  $C_1 \dots C_s$ . The other trivial case,  $s = 1$ , means there is only one color in the whole graph.

This means we need not consider configurations like  $g = 6, s = 3$  even though  $6|3(3-1)$ .

3. Another corollary here is that  $\boxed{m \geq n}$ , since:

$$n = (sg - s + 1) \quad (4)$$

$$m = \frac{\binom{(sg-s+1)(sg-s)}{2}}{\binom{g}{2}} = \frac{(sg-s+1)(sg-s)}{g(g-1)} = \frac{(sg-s+1)s}{g} \quad (5)$$

$$\Rightarrow m = \left(\frac{s}{g}\right)n \quad (6)$$

$$s \geq g \Rightarrow m \geq n \quad (7)$$

4. This is just a combination of (1) and (2). But for example, a tiling of triangles ( $g = 3$ ) means that either  $s \equiv 0 \pmod{3}$  (see Fig. ??) or  $s \equiv 1 \pmod{3}$  (see Fig. ??).

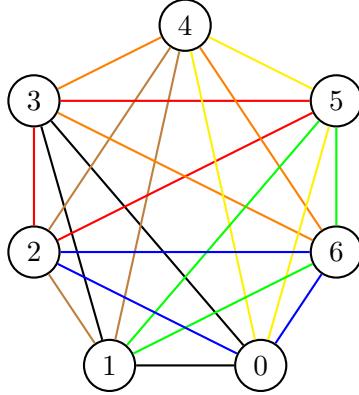


Figure 3:  $s=3, g=3, n=7, m=7$ . Cliques of form  $(n_i, n_{i+1}, n_{i+3})$ . for all  $i$

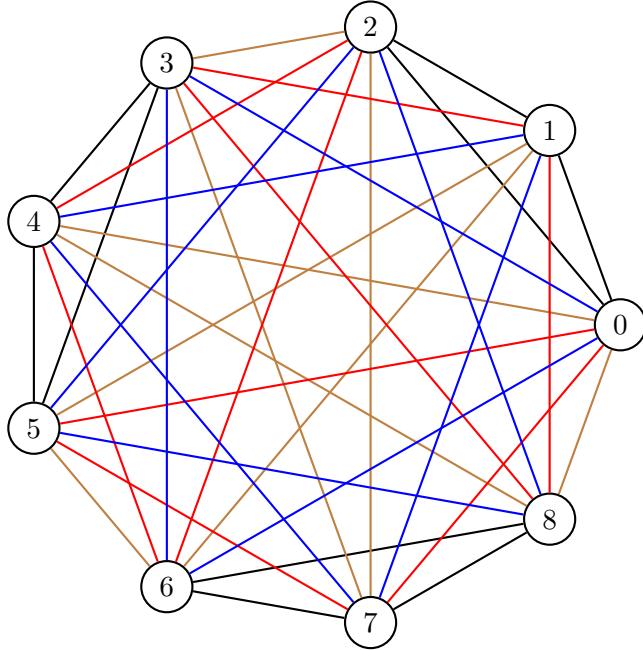


Figure 4:  $s=4$ ,  $g=3$ ,  $n=9$ ,  $m=12$ .

Cliques:  $(n_i, n_{i+3}, n_{i+6}), (n_i, n_{i+1}, n_{i+2}), (n_i, n_{i+4}, n_{i+8}), (n_i, n_{i+5}, n_{i+7}), 3|i$

### 3 Constructing $g = s - 1$ over a field

Though we presented a few legitimate examples of complete graphs tiled by uniformly sized complete subgraphs in Fig. ?? and Fig. ??, these are not easy to find by hand once  $s$  becomes much larger. The whole problem of graph partitioning admits many algorithms, most approximations[3], though usually over an arbitrary graph instead of a relatively simple complete graph.

We can, however, systematically find a graph partition (edge coloring)

*We find that if  $g$  is a prime power, we can explicitly construct a graph of any size that satisfies our game.*

This is most obvious for primes, but can be constructed from, say,  $GF(4)$ :

- Assume  $g_y g_{x_1} + c = g_y^* g_{x_1} + c^*$ , and  $g_y g_{x_2} + c = g_y^* g_{x_2} + c^*$  for  $g_{x_1} \neq g_{x_2}$
- Subtract the two to get  $g_y(g_{x_1} - g_{x_2}) = g_y^*(g_{x_1} - g_{x_2})$
- The field  $\mathcal{F}$  requires the nonzero  $(g_{x_1} - g_{x_2}) \in \mathcal{F}$  to have an inverse.
- Multiplying both sides by that inverse, we have  $g_y = g_y^*$

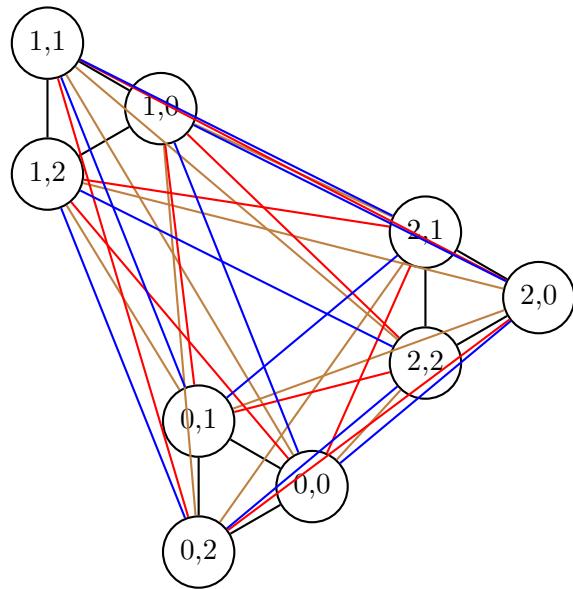


Figure 5: whole  $K_9$ :  $s = 4, g = 3, n = 9, m = 12$

$+$	0	1	B	D
0	0	1	B	D
1	1	0	D	B
B	B	D	0	1
D	D	B	1	0

(a) Addition table GF(4)

$\cdot$	0	1	B	D
0	0	0	0	0
1	0	1	B	D
B	0	B	D	1
D	0	D	1	B

(b) Multiplication table GF(4)

#### 4 Constructing $g = s$ from $g = s - 1$

With the previous construction in hand, we can easily construct a  $g = s$  graph with the same restrictions on  $g$ .

$G_0$	$G_1$	$G_2$	$G_3$
0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3

(a)  $I_0$ 

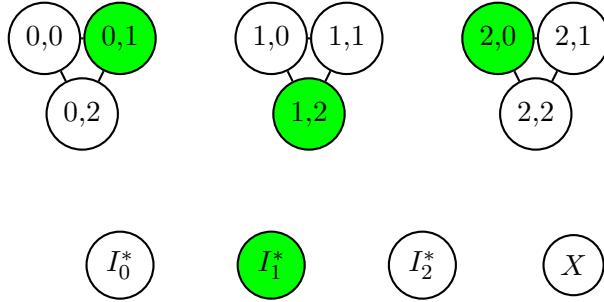
$G_0$	$G_1$	$G_2$	$G_3$
0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

(b)  $I_1$ 

$G_0$	$G_1$	$G_2$	$G_3$
0	2	3	1
1	3	0	2
2	0	1	3
3	1	2	0

(c)  $I_2$ 

$G_0$	$G_1$	$G_2$	$G_3$
0	3	1	2
1	0	2	3
2	1	3	0
3	2	0	1

(d)  $I_3$ Figure 7:  $s=5, g=4$  adjacency tablesFigure 8:  $s=4, g=4$  from  $s=4, g=3$ 

## 5 Alternative: Constructing $g = s$ with perfect difference sets

Though it will be shown equivalent to the last construction, we can use another concept to build these graphs: perfect difference sets. Notably, these are proven to exist for  $g = p^k$  by Singer. TODO Cite

## 6 Interlude: Graph Equivalence up to relabeling

proving all  $g = s$  graph colorings are the same up to relabeling. This means that the graphs constructed in sections X and Y are the same.

Fig: 5 nodes each with four neighbors

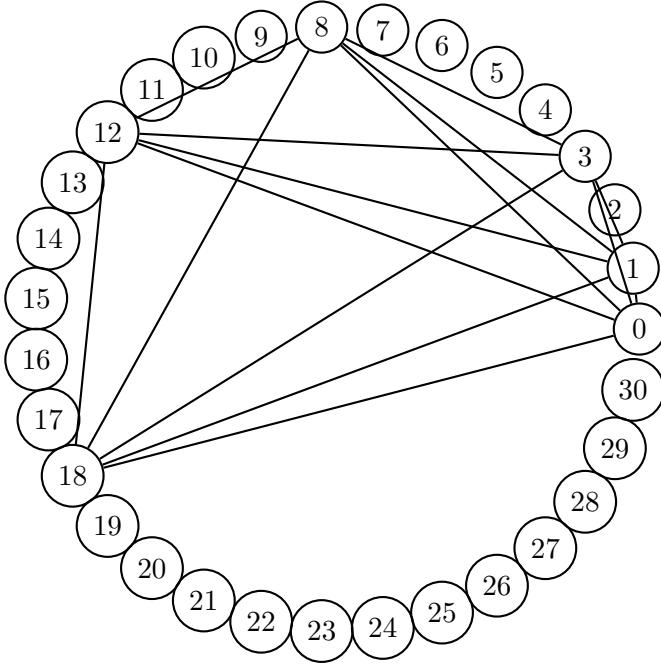


Figure 9: perfect Set difference on  $g=6, s=6, n=m=31$

## 7 Constructing $g = s - 1$ from the previous

*Removing one clique and all edges should suffice.*

Todo: are these all the same, looking at complement.color graph  $g - 1$  cliques left behind?

## 8 Considering wider $g|s$ and $g|s - 1$ : Inception

Rule:  $g = 3, s = 4, n = 9, m = 12$

Note that since  $81 = 3^4, 3 \in \mathbb{P}$ ,  $K_{81}$  can be partitioned into 9-cliques (complete graphs of size  $K_9$ ). Because we can recursively construct  $K_9$ s from the rule below (or “incept” the graph), we can turn the graph  $n = 81, s = 10, g = 9, m = 90$  into one of  $n = 81, s = 40, g = 3, m = 1080$ , in which every node still has a uniform number of attached colors, and every color has the same number of node members. This graph has not been attempted.

*Figure:  $s=4, g=3$  has 9 nodes.  $S=g=9$  has 9-cliques which could be broken down*

Note: You can take the  $s=7, g=3, n=15, m=35$  and change the  $n, n+5, n+10$  triangles into unique colors for  $s=8, m=45, n=15$  and  $g$  in 2,3 for example.

## 9 Considering wider $g|s$ : Kirkman's schoolgirl problem

NOTE: We have the rotator of size  $g$  iff  $g|s - 1$ , since  $gk = s - 1 \Rightarrow s = gk + 1 \Rightarrow m = \frac{gk+1}{s}(sg - s + 1)$ . This means ( I think ) that there are  $k(sg - s + 1)$  cliques, or  $k$  rooted at each node, plus  $\frac{sg-s+1}{g}$  other rotator cliques, being  $s - \frac{s-1}{g}$  of size  $g$  that are like the island triangles

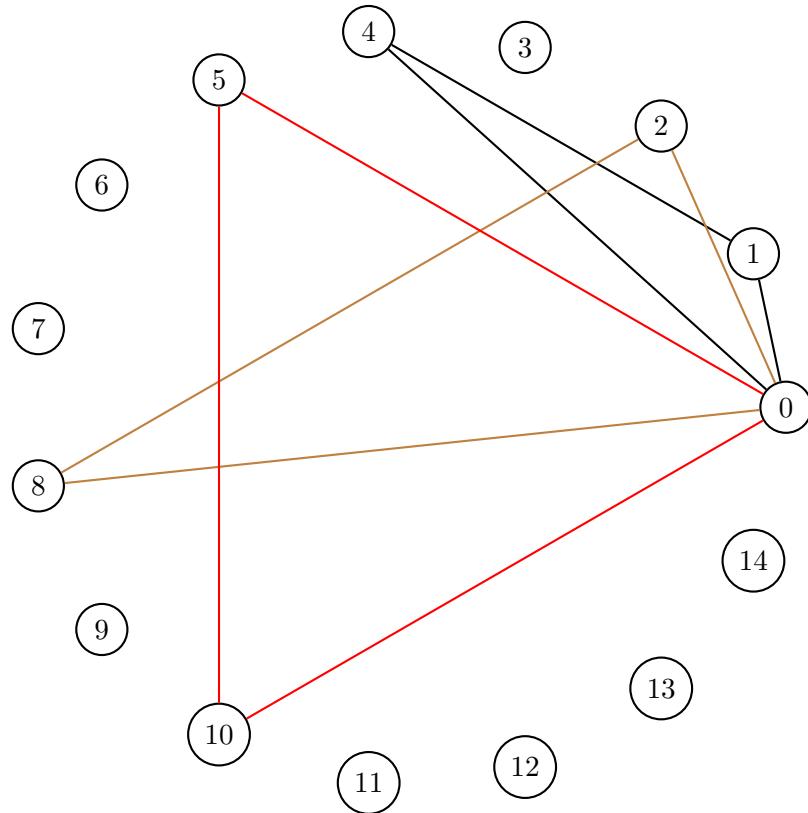


Figure 10:  $s=7$ ,  $g=3$ ,  $n=15$ ,  $m=35$ , node 0 adjacencies. Rule:  $(i, i+5, i+10) \times 3$ ,  $(i, i+1, i+4)$  and  $(i, i+2, i+8)$

ANOTHER ONE

## 10 Nonuniform $g$ : deletion and partial inception

### 11 Another question

*Can it be true that  $g|s(s - 1)$  but not true that  $g|s$  or  $g|s - 1$ ?—*

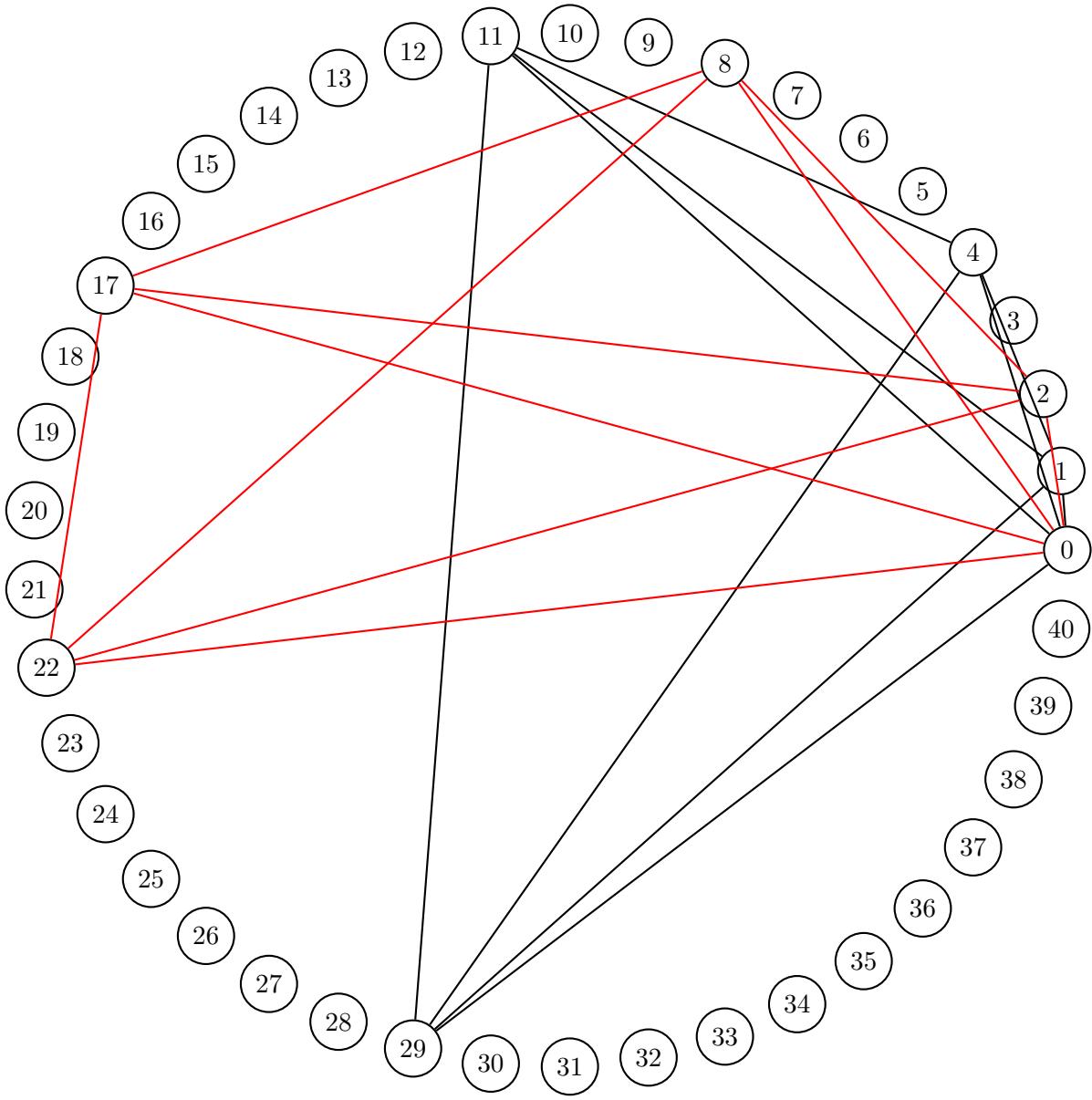


Figure 11: Perfect Difference Set on  $s=10$ ,  $g=5$ ,  $n=41$ ,  $m=82$ :  $(0\ 1\ 4\ 11\ 29)$ ,  $(0\ 2\ 8\ 17\ 22)$

## 12 THIS IS THE END

## 13 Boneyard: Some examples with $g = 3$

- Rule:  $g = 3, s = 3, n = 7, m = 7 : (0, 1, 3)$

- Rule:  $g = 3, s = 4, n = 9, m = 12 : (0, 3, 6) \cdot 3, (0, 1, 2) \cdot 3, (0, 4, 8) \cdot 3, (0, 5, 7) \cdot 3$  (this is 3 separate  $K_3$ , then circuit inc 0, inc1, inc 2)
- Rule: Another example:  $g = 3, s = 6, n = 13, m = 26$ . 3-graphs are at  $(i, i + 2, i + 8)$  and  $(i, i + 1, i + 4)$ , addition being  $\mod 13..$  NOTE: Is this a subset of  $s = 6, g = 6?$
- Rule:  $g = 3, s = 7, n = 15, m = 35, (i, i + 5, i + 10) \cdot 3, (i, i + 1, i + 4), (i, i + 2, i + 8)$
- Rule:  $g = 3, s = 9, n = 19, m = 57 : (0, 1, 6), (0, 2, 10), (0, 3, 7)$
- Rule:  $g = 3, s = 10, n = 21, m = 70 : (0, 7, 14) \cdot 3, (0, 2, 10), (0, 1, 5), (0, 3, 9)$
- Rule:  $s = g = 4, n = m = 13 : (0, 1, 3, 9)$
- Rule:  $(0, 1, 4, 14, 16) \text{ ons } s = g = 5, n = 21$
- Rule:  $s = g = 6, n = m = 31 : (0, 1, 3, 8, 12, 18)$
- NONE on  $s = 7 = g$
- Rule:  $s = g = 8, n = m = 57 - (0, 1, 3, 13, 32, 36, 43, 52)$
- Rule:  $s = g = 9, n = m = 73 - (0, 1, 3, 7, 15, 31, 36, 54, 63) ;$  NOTE - these are all  $2^{n-1}$  for a bit
- Rule:  $s = g = 10, n = m = 91 - (0, 1, 3, 9, 27, 49, 56, 61, 77, 81)$
- NONE on  $s = g = 11$
- Rule:  $g = s = 12, m = n = 133 - (01, 3, 12, 20, 34, 38, 81, 88, 94, 104, 109)$
- NOTE All  $(g, s=g+1)$  don't seem to work with the rotators
- TODO Build something for the 2-rotators

## References

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