

# Brilliant: Vector Calculus

Dave Fetterman

6/21/22

Note: Latex reference: <http://tug.ctan.org/info/undergradmath/undergradmath.pdf>

## 1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against  $t$  of the form  $\vec{x}(t) = \langle x(t), y(t), \dots \rangle$ .

- A **line** through  $p = (a, b, c)$  parallel to  $\vec{v} = \langle v_x, v_y, v_z \rangle$  is  $\vec{x}(t) = \vec{p} + t\vec{v}$
- **velocity** is characterized completely by  $\vec{v}(t) = \vec{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ .
- The **speed** of an object along that line versus  $t$  is the length of  $v$  ( $\|v\|$ )
- Therefore, the speed of an object along line

$$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

is

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that  $\vec{v}$  need not be constant. The speed of

$$\vec{x}(t) = \vec{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

would then be

$$\|6\pi \cos(2\pi t)\hat{u}\| = |6\pi \cos(2\pi t)|$$

- **Acceleration**  $a(t) = v'(t) = x''(t)$  is straightforward. Acceleration of

$$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance  $r$  from the sun could be  $\langle r \cos(t), r \sin(t) \rangle$ . The acceleration vector points in the opposite direction:  $\langle -r \cos(t), -r \sin(t) \rangle$ . In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by  $F = ma$  is proportional to acceleration) points towards the sun, *with acceleration perpendicular to velocity*.

- A **helix** could be a 3D extension like  $\langle r \cos(t), r \sin(t), b \cdot t \rangle$ .

## 2 Chapter 2.2: Space Curves

- Note that while  $\vec{x}(t) = \langle \cos(t), \sin(t), 5 \rangle$  and  $\vec{x}(t) = \langle \cos(2t), \sin(2t), 5 \rangle$  describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If  $\vec{x}(t) = t\vec{v}$ , then speed is  $\frac{\|\vec{x}(t+\Delta t) - \vec{x}(t)\|}{\Delta t} = \|\vec{v}\|$ , direction is  $\frac{\vec{v}}{\|\vec{v}\|}$ , and velocity  $\vec{v}$  is the product of speed and direction.
- So  $\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} = \vec{x}'(t) = \frac{d\vec{x}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$
- Neat conceptual result: any  $y = f(x)$  can be made into  $x(t) = \langle t, f(t) \rangle$ , and then  $v(t) = \langle 1, f'(t) \rangle$ , which points along the tangent line at  $\langle t, f(t) \rangle$ .
- Note that dot product derivatives work like regular product:  $[\vec{a}(t) \cdot \vec{b}(t)]' = \vec{a}'(t) \cdot \vec{b}(t) + \vec{a}(t) \cdot \vec{b}'(t)$ , but the cross product does not work the same since  $\frac{d}{dt}[a \times b] = a' \times b + a \times b'$ , but since  $a \times b' = -b' \times a$ , can't switch the order to  $a' \times b + b' \times a$  due to this non-commutativity.
- If

$$\vec{x}(t) = \vec{p} + t\vec{v},$$

calculating velocity with respect to origin becomes

$$\frac{d}{dt} \|\vec{x}(t)\| = \frac{\vec{x}(t) \cdot \vec{x}'(t)}{\|\vec{x}(t)\|} = \frac{\vec{x}}{\|\vec{x}\|} \cdot \vec{v},$$

after rewriting the distance formula and chugging through the chain rule.

- However, it becomes more clear when considering that  $(\vec{v} \cdot \hat{x})\hat{x}$  is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

## 3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way:  
 $\int_a^b \vec{x}(t) dt = \langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \rangle$
- Example: if ball launched from origin with velocity  $\langle 1, 2, 3 \rangle$  and acceleration  $\langle 0, 0, -1 \rangle$ ,

it lands at

$$\frac{dv}{dt}dt = \langle 0, 0, -1 \rangle \quad (1)$$

$$\int \frac{dv}{dt}dt = v = \langle C, D, -t + F \rangle = \langle 1, 2, 3 \rangle = \langle 1, 2, -t + 3 \rangle, t = 0 \quad (2)$$

$$x = \int v = \langle t + K, 2t + M, -\frac{1}{2}t^2 + 3t + N \rangle, x(\vec{0}) = \langle 0, 0, 0 \rangle \quad (3)$$

$$\vec{x}(t) = \langle t, 2t, 3t - \frac{1}{2}t^2 \rangle \quad (4)$$

$$z(t) = 0 \rightarrow t = 6 \rightarrow \vec{x}(6) = \langle 6, 12, 0 \rangle \quad (5)$$

$$(6)$$

- Also, generalizing  $ds = \sqrt{(dx)^2 + (dy)^2}$ , the length of an arc from point  $a$  to  $b$  approaches  $\int_a^b ds = \int_{t_a}^{t_b} \|x'(t)\| dt$
- Example: a helix  $\langle a \cos(\omega t), a \sin(\omega t), b\omega t \rangle$ , parametrized by time  $t$  can be rewritten in terms of  $s$ , the arc length:

$$s = \int \|x'(t)\| dt \quad (7)$$

$$s = \int \sqrt{(-\omega a \sin(\omega t))^2 + (\omega a \cos(\omega t))^2 + (b\omega)^2} dt \quad (8)$$

$$s = |\omega| \int \sqrt{a^2 + b^2} dt \quad (9)$$

$$s = |\omega| t \sqrt{a^2 + b^2} \quad (10)$$

- *Note: It's weird to think of time in terms of length. Could be analytically useful?*

## 4 Chapter 2.4: Frenet Formulae

Main idea: Establish three new vectors  $\hat{T}(s), \hat{N}(s), \hat{B}(s)$  that change as we move along a space curve, instead of  $\vec{x}(t)$  that changes over an external “time” idea.

Remember that  $s = \int_0^t \|\vec{x}'(\tilde{t})\| d\tilde{t}$ , so  $\frac{ds}{dt} = \|\vec{x}'(t)\|$ .

### 4.1 $\hat{T}$ : Vector tangent to space curve

- Remember arc length is  $s = \int_0^t \|\vec{x}'(\tilde{t})\| d\tilde{t}$
- $\hat{T}$  is just normalized grad:  $\frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$

- This implies  $\frac{d\vec{x}}{ds} = \hat{T}$  since

$$s = \int_0^t \|\vec{x}'(\tilde{t})\| d\tilde{t} \quad (11)$$

$$\frac{ds}{dt} = \|\vec{x}'(t)\| \quad (12)$$

$$\hat{T} = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \frac{d\vec{x}}{dt} \cdot \frac{dt}{ds} \quad (13)$$

$$\hat{T} = \frac{d\vec{x}}{ds} \quad (14)$$

$$(15)$$

- So this is how the space curve  $\vec{x}$  changes as it moves along  $s$ .
- It's normalized, so it's the same whether parameterized by  $t$ ,  $s$ , or whatever.

## 4.2 $\hat{N}$ : Normal to curve (perpendicular to $\hat{T}$ )

Normal vectors are:

- $\vec{x}''(t)$  normalized as  $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} = \hat{N}$
- The normal vector to the curve
- $\perp$  to  $\hat{T}$  in direction of acceleration. So a multiple of acceleration vector.
- $\frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$ . The following sequence shows any unit length vector is perpendicular to its derivative.

$$\|\hat{T}\| = 1 \quad (16)$$

$$d(\|\hat{T}\|^2) = 0 \quad (17)$$

$$d(\|\hat{T}\|^2) = d(\hat{T} \cdot \hat{T}) = \hat{T}'(t) \cdot 2\hat{T}(t) \quad (18)$$

$$\hat{T}'(t) \cdot \hat{T}(t) = 0 \quad (19)$$

- $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|}$  since it's the same as the above, but parametrized over  $s$  instead of  $t$ . Doesn't change the direction of the vector!

Example: if  $\vec{x}(t) = \langle R \cos(\omega t), R \sin(\omega t), 0 \rangle$ , then:

- $\vec{a} = \frac{d^2\vec{x}}{dt^2}$  just by definition
- $\vec{a} = -\omega^2 \vec{x}$  just by calculation

- $\hat{T}(t) = \langle -\sin(\omega t), \cos(\omega t), 0 \rangle$
- $\|\hat{T}(t)\| = 1$
- $\hat{N} = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = \langle -\cos(\omega t), -\sin(\omega t), 0 \rangle$
- So  $\vec{a} = R\omega^2 \hat{N}$  by these formulae.

This leads us to believe acceleration and  $\hat{N}$ , the normed derivative of  $\hat{T}$  are related.

The part of acceleration  $\vec{a}$  parallel to  $\hat{T}$  is the projection  $(\vec{a} \cdot \hat{T})\hat{T}$

The perpendicular part is then  $\vec{a}$  minus that:  $\vec{a} - (\vec{a} \cdot \hat{T})\hat{T}$

This also equals  $(\frac{ds}{dt})^2 \|\frac{d\hat{T}}{ds}\| \hat{N}$  because

$$\vec{x}' = \frac{dx}{dt} = T = \hat{T} \cdot \|\frac{dx}{dt}\| \quad (20)$$

$$s = \int_0^t \|\vec{x}'(t)\| \rightarrow \frac{ds}{dt} = \|\vec{x}'(t)\| \quad (21)$$

$\hat{N} = \frac{d\hat{T}}{ds}$  normalized, so

$$\vec{a} = \frac{d^2 \vec{x}}{dt^2} = \frac{d}{dt}(\|\vec{x}'(t)\| \hat{T}(t)) = \frac{d\|\vec{x}'(t)\|}{dt} \hat{T} + \|\vec{x}'(t)\| \frac{d\hat{T}}{dt} \quad (22)$$

$$= \frac{d\|\vec{x}'(t)\|}{dt} \hat{T} + \frac{ds}{dt} \frac{d\hat{T}}{ds} \frac{ds}{dt} \quad (23)$$

$$= \frac{d\|\vec{x}'(t)\|}{dt} \hat{T} + (\frac{ds}{dt})^2 \|\frac{d\hat{T}}{ds}\| \hat{N} \quad (24)$$

This is “a = parallel part plus perpendicular (N) part”, so the second term is  $a_{\perp}$

### 4.3 $\hat{T}$ and $\hat{N}$

- Form a plane, since first, any normal vector's derivative is perpendicular to the vector
- $\kappa$  is curvature: how much we're curving in that  $T \times N$  plane.
- $\kappa = \|\frac{d\hat{T}}{ds}\|$
- Therefore, by above,  $\frac{d\hat{T}}{ds} = \kappa \hat{N}$  (**Frenet equation 1**)

Note that curvature  $\kappa(s) = \|\frac{d\hat{T}}{ds}\|$  is geometric (depends on s, not time) and changes as  $\hat{T}$  changes.

Example: Curvature of  $\vec{x}(t) = \langle \cos(t), \sin(t), bt \rangle$

$$x'(t) = \langle -\sin(t), \cos(t), b \rangle \quad (25)$$

$$\|x'(t)\| = \sqrt{1 + b^2} \quad (26)$$

$$s = \int_0^t \|x'(\tilde{t})\| d\tilde{t} = \int_0^t \sqrt{1 + b^2} = t\sqrt{1 + b^2} \rightarrow t = \frac{s}{\sqrt{1 + b^2}} \quad (27)$$

Do the substitution of  $\frac{s}{\sqrt{1+b^2}}$  for  $t$  above to get  $x'(s)$ , and from there, you can figure out  $\frac{d\hat{T}}{ds}$  and normalize to get  $\kappa = \frac{1}{1+b^2}$

#### 4.4 $\hat{B}$ is binormal: perpendicular to both

- defined as  $\hat{B} = \hat{T} \times \hat{N}$
- Therefore, by derivative

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \quad (28)$$

$$\frac{d\hat{B}}{ds} = \kappa \hat{N} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \quad (29)$$

$$\frac{d\hat{B}}{ds} = \hat{T} \times \frac{d\hat{N}}{ds} \quad (30)$$

$$(31)$$

but this means  $\hat{T}$  is orthogonal to  $d\hat{B}$ , and we already know  $\hat{B}$  and  $d\hat{B}$  are orthogonal. We're working in 3d with the cross product, so  $d\hat{B}$  is parallel to  $\hat{N}$ .

- Therefore, we define **torsion**  $\tau$  so that  $-\frac{d\hat{B}}{ds} = \tau \hat{N}$  (**Frenet equation 2**). Negative sign by convention.
- Can also cross by  $\hat{N}$  on both sides to get  $-\frac{d\hat{B}}{ds} \times \hat{N} = \tau$
- $\hat{B}$  measures how the plane defined by  $\hat{T}, \hat{N}$  twists around. On a circle,  $\hat{B}$  wouldn't change, so the derivative would be zero.
- **Final Frenet equation.** Prereq:  $\hat{B} = \hat{T} \times \hat{N} \rightarrow \hat{N} = \hat{B} \times \hat{T} \rightarrow \hat{T} = \hat{N} \times \hat{B}$

$$\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} \quad (32)$$

$$\frac{d\hat{N}}{ds} = -\tau \hat{N} \times \hat{T} + \hat{B} \times \kappa \hat{N} \quad (33)$$

$$\frac{d\hat{N}}{ds} = \tau \hat{B} - \kappa \hat{T} \quad (34)$$

## 5 Chapter 2.5: Parametrized Surfaces

Main ideas:

- Can parameterize by  $\vec{x}(u, v) = x(u, v), y(u, v), z(u, v)$
- Can perhaps parameterize  $f(x, y, z) = c$  by  $z = g(x, y)$
- Can also use ideas like  $\nabla f = 0$  to find a normal.

There are many out-of-the-box parametrizations including:

- Sphere at 0,0,0:  $\vec{x}(u, v) = \langle R \cos(u) \sin(v), R \sin(u) \sin(v), R \cos(v) \rangle$ , where  $u \in [0, 2\pi), v \in [0, \pi]$
- Rotate function  $y = f(x)$  around the x-axis:  $\vec{x}(u, v) = \langle u, f(u) \cos(v), f(u) \sin(v) \rangle$ , where  $u \in D, v \in [0, 2\pi]$

Tangent vectors to  $\vec{x}(u, v)$  are  $\frac{\delta \vec{x}}{\delta u}, \frac{\delta \vec{x}}{\delta v}$ , so unit normal is  $\pm \frac{\frac{d\vec{x}}{du} \times \frac{\delta \vec{x}}{dv}}{\|\frac{d\vec{x}}{du} \times \frac{\delta \vec{x}}{dv}\|}$

Example: Torus  $\vec{x} = \langle [2 + \cos(v)] \cos(u), [2 + \cos(v)] \sin(u), \sin(v) \rangle, u, v \in [0, 2\pi)$ . What's the tangent plane at  $u = \frac{\pi}{4}, v = 0$ ?

$$d\vec{x}/du = \langle -\sin(u)(2 + \cos(v)), \cos(u)(2 + \cos(v)), 0 \rangle \quad (35)$$

$$d\vec{x}/dv = \langle -\sin(v) \cos(u), -\sin(v) \sin(u), \cos(v) \rangle \quad (36)$$

$$u = \frac{\pi}{4}, v = 0 \rightarrow \quad (37)$$

$$d\vec{x}/du = \langle -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0 \rangle \quad (38)$$

$$d\vec{x}/dv = \langle 0, 0, 1 \rangle \quad (39)$$

$$dx/du \times dx/dv = \langle \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0 \rangle \quad (40)$$

$$\hat{n} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle \quad (41)$$

$$\hat{n} \cdot \vec{x} = 0 \rightarrow \hat{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \quad (42)$$

$$\rightarrow \dots \rightarrow x + y = 3\sqrt{2} \quad (43)$$

$$(44)$$

### 5.1 Example: Ellipsoid $x^2 + 2y^2 + z^2 = 4$

What's the normal at  $(1, \frac{1}{\sqrt{2}}, \sqrt{2})$ ?

**Method 1: parametrize with spherical  $\mathbf{u}, \mathbf{v}$**  First, transform to sphere with change of coordinates, then flip to spherical coordinates.

$$x^2 + 2y^2 + z^2 = 4 \quad (45)$$

$$X = x/2, Y = \frac{Y}{\sqrt{2}}, Z = z/2 \quad (46)$$

$$X^2 + Y^2 + Z^2 = 1 \quad (47)$$

$$X = \cos(u) \sin(v), Y = \sin(u) \sin(v), Z = \cos(v) \quad (48)$$

$$p = (1, \frac{1}{\sqrt{2}}, \sqrt{2}) \rightarrow u = v = \frac{\pi}{4} \quad (49)$$

$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle -1, \frac{1}{\sqrt{2}}, 0 \rangle \quad (50)$$

$$\frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \frac{1}{\sqrt{2}}, -\sqrt{2} \rangle \quad (51)$$

$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) \times \frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \sqrt{2}, \sqrt{2} \rangle \quad (52)$$

$$\hat{n}_{out} = \frac{\langle -1, -\sqrt{2}, -\sqrt{2} \rangle}{\sqrt{5}} \quad (53)$$

**Method 2: rewrite as  $z = g(x,y)$**

$$x^2 + 2y^2 + z^2 = 4 \quad (54)$$

$$z = (4 - x^2 - 2y^2)^{\frac{1}{2}} \quad (55)$$

$$dz/dx = \frac{1}{2} \times -2x(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -\frac{1}{\sqrt{2}} \quad (56)$$

$$dz/dy = \frac{1}{2} \times -4y(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -2\sqrt{2}/\sqrt{2} = -1 \quad (57)$$

$$f \approx \sqrt{2} + dz/dx(1, \frac{1}{\sqrt{2}})(x - 1) + dz/dy(1, \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) \quad (58)$$

$$\rightarrow \dots \rightarrow \frac{1}{\sqrt{2}}x + y + z = 2\sqrt{2} \quad (59)$$

$$(60)$$

giving us normal vector  $\langle \frac{1}{\sqrt{2}}, 1, 1 \rangle = \frac{\langle 1, \sqrt{2}, \sqrt{2} \rangle}{\sqrt{5}}$  after normalization.

**Method 3: gradient**

Gradient is always normal to the tangent plane. Recognize level set of  $f(x, y, z) = x^2 + 2y^2 + z^2$ .



$$\nabla f = \langle 2x, 4y, 2z \rangle \rightarrow \nabla f(1, \frac{1}{\sqrt{2}}, \sqrt{2}) = \langle 2, 2\sqrt{2}, 2\sqrt{2} \rangle$$

Then normalize.

## 5.2 Mobius strip and “outward direction”

Mobius strip is

- $x = 2 \cos(u) + v \cos(\frac{u}{2})$
- $y = 2 \sin(u) + v \cos(\frac{u}{2})$
- $z = v \sin(\frac{u}{2})$
- $u \in [0, 2\pi], v \in [-\frac{1}{2}, \frac{1}{2}]$

$$\hat{n} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} \text{ at } (0,0) \text{ is } \langle 0, 0, -1 \rangle,$$

but at  $(2\pi, 0)$  is the same point, but  $\hat{n} = \langle 0, 0, 1 \rangle!!$

## 6 Chapter 2.6: Vector Fields

(Lots of intuition questions here...)

One nugget: using **gradient vector fields**: Suppose  $\vec{F}(x, y) = \langle 2, -4y^3 \rangle$ . If  $F = \nabla f$  for some  $f$ , then  $F$ 's arrows are perpendicular to a level set  $f = c$ . So look at  $f = 2x - y^4$  and find perpendicular arrows to these. That's actually F!

**Linear approximation for  $\vec{F} : D \in \mathbf{R}^n \rightarrow \mathbf{R}^m$**

$$\text{Main idea: } \vec{F}(\vec{x}) = \vec{F}(\vec{a}) + A(\vec{a})(\vec{x} - \vec{a})$$

Note that  $A$  takes in vectors of size  $n$  (so it has that many columns), and has  $m$  functions (rows) that operate on it. So the Jacobian,  $A$ , has as row  $i$ , column  $j$ , the quantity  $\frac{dF_i}{dx_j}(\vec{a})$ .

$dF_i/d\vec{x}$  extends across row  $i$ .

## 7 Chapter 2.7: Jack and the Beanstalk (Newton's method)

**Basis for Newton's:**

If we're estimating  $x_1$  by following the derivative at  $x_0$ , this means we're looking at the line with x-intercept  $x_1$ , with slope  $f'(x_0)$ .

So instead of  $y = mx + b$ , we'll flip the two and use

$$x = y/m + x_{int}$$

$$\text{or } x_0 = f(x_0) \frac{1}{f'(x_0)} + x_1,$$

$$\text{or } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Note that, under Newton's something like  $|x|$  will converge immediately,  $x^3$  will converge moderately, and a S-curve might barely converge if at all.

The extension of this with the Jacobian matrix  $A = DF'(x_0)$  is  $\vec{x}_1 = \vec{x}_0 - (D\vec{F}(\vec{x}_0))^{-1} \vec{F}(x_0)$

## 8 Chapter 2.8: Electrostatic bootcamp

Electric charge radiates out equally in all directions, and is inversely proportional to distance.

Formula, with  $Q$  as the charge,  $\epsilon_0$  is a constant:  $\vec{E}(x, y, z) = \frac{Q}{4\pi\epsilon_0\|x\|^2} \hat{x}$

A field line is a special case of a **flow line** - the space curve that follows  $\vec{F}$ 's arrows. The tangent vector to the flow line is  $\vec{F}(\vec{x}(\tilde{t}))$  ( $\tilde{t}$  is not time here), so  $\frac{d\vec{x}}{d\tilde{t}} = \vec{F}(\vec{x}(\tilde{t}))$

Example: Vector field  $\vec{F}(x, y) = \langle -2y, 3x \rangle$ . What's the flow line through (2,0)?

Solution: Need to solve  $dx/dt = -2y, dy/dt = 3x$ . Key is "separating the equations". Remember  $x$  and  $y$  are functions of  $t$ !

$$\frac{d^2x}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6x. \quad (61)$$

$$\frac{d^2y}{dt^2} = -2\frac{dx}{dt} = -2 \times 3y = -6y. \quad (62)$$

$$x(t) = -6x''(t), y(t) = -6y''(t) \quad (63)$$

$$\rightarrow x = A \cos(\sqrt{6}t) + B \sin(\sqrt{6}t), y = C \cos(\sqrt{6}t) + D \sin(\sqrt{6}t) \quad (64)$$

$$\frac{dx}{dt} = -2y(t) \rightarrow \frac{\sqrt{6}}{2} A \sin(\sqrt{6}t) - \frac{\sqrt{6}}{2} B \cos(\sqrt{6}t) = y(t) \quad (65)$$

$$x(t=0) = 2 \rightarrow A = 2 \quad (66)$$

$$y(t=0) = 0 \rightarrow B = 0 \quad (67)$$

$$\vec{F}(t) = \langle 2 \cos(\sqrt{6}t), \sqrt{6} \sin(\sqrt{6}t) \rangle \quad (68)$$

$$(69)$$

Note: **Field lines** follow rules:

- Go from positive charges to negative
- Density of lines directly relates to how much charge a point has
- Lines don't intersect.
- Corollary: If count of out equals count of in, point has zero charge
- “Number” (to be defined) of field lines in and out of a *surface* related to the charge inside. Upcoming.

## 9 3.1: Surface Integrals

Example: Fluid pressure in a tank is:

- Proportional (via some weight constant  $p_{fluid}$ ) to depth of the point
- Pushes out via the normal  $\hat{n}$
- So, for the  $x = l$  side of a cube of length  $l$ , this would be

$$\vec{F}_{x=l} = (\iint_{[0,l] \times [0,l]} p_{fluid} [1 - \frac{z}{l}] dy dz) \hat{i}$$

**Example:** Hemisphere of size  $l$ , sitting at  $(0, 0, 0)$

Finding the out pointing unit normal of hemisphere at point  $(x, y, -\sqrt{l^2 - x^2 - y^2})$

Note: Can just eyeball this, but one way is the **gradient**.

First, the relation is  $x^2 + y^2 + (z - l)^2 = l^2$ . Make it a function  $g$  and take the level set at  $l^2$ :

$$g(x, y, z) = x^2 + y^2 + (z - l)^2 = l^2 \quad (70)$$

$$\nabla g(x, y, z) = \langle 2x, 2y, 2(z - l) \rangle \quad (71)$$

$$\hat{n} = \pm \frac{\nabla g(x, y, z)}{\|\nabla g(x, y, z)\|} \quad (72)$$

$$\hat{n} = \pm \frac{\langle x, y, (z - l) \rangle}{\sqrt{x^2 + y^2 + (z - l)^2}} \quad (73)$$

$$\hat{n} = \pm \frac{\langle x, y, (z - l) \rangle}{\sqrt{l^2}} \quad (74)$$

$$\hat{n} = \pm \langle \frac{x}{l}, \frac{y}{l}, \frac{z}{l} - 1 \rangle \quad (75)$$

$$(76)$$

Note: Integrating over a patch  $dA$  on the surface means finding the area of micro-patches  $\Delta A_{ij}$ , which is the parallelogram defined by

$$s_1 = \langle \Delta x_i, 0, \Delta x_i f_x(x_i^*, y_j^*) \rangle \quad (77)$$

$$s_2 = \langle 0, \Delta y_j, \Delta y_j f_y(x_i^*, y_j^*) \rangle \quad (78)$$

$$\Delta A_{ij} \approx \|s_1 \times s_2\| \quad (79)$$

$$= \sqrt{(1 + [f_x(x_i^*, y_j^*)]^2 + [f_y(x_i^*, y_j^*)]^2) \Delta x_i \Delta y_j} \quad (80)$$

$$(81)$$

So the total pressure ends up being  $\vec{F}_{tot} = p_{fluid} \iint (p \cdot \hat{n}) dA$

$$= p_{fluid} \iint_{x^2+y^2 \leq l^2} [1 - \frac{f(x, y)}{l}] \hat{n} \sqrt{1 + [f_x]^2 + [f_y]^2} dx dy \quad (82)$$

$$f(x, y) = l - \sqrt{l^2 - x^2 - y^2} \quad (83)$$

$$\hat{n} = \langle \frac{x}{l}, \frac{y}{l}, \frac{f(x, y)}{l} - 1 \rangle \quad (84)$$

$$f_x = \frac{x}{\sqrt{l^2 - x^2 - y^2}}, f_y = \frac{y}{\sqrt{l^2 - x^2 - y^2}} \quad (85)$$

And for the only-nonzero component,  $\hat{k}$ , this simplifies after a lot of hand-math to  $F_{tot} = -p_{fluid} (\iint_{x^2+y^2 \leq l^2} \sqrt{1 - (\frac{x^2+y^2}{l^2})} dx dy) \hat{k}$

Side Note during solving:  $dx dy \rightarrow r dr d\theta$ .

- TODO: This looks to be something to do with the determinant of the Jacobian matrix  $F_i/x_j$ .
- Intuitively, consider that a patch  $dx \cdot dy$  is a slice of a big disk which has dimensions  $dr$  on the ray,  $r d\theta$  on the arc.

## 10 3.2: Flux Part I

Main idea: Field lines are innumerable, so counting them through a surface makes no sense. Instead, we'll use **flux** to help us measure charge pushed through a surface per unit time.

Example: If charge  $q$  of mass  $m$  in a field of  $\vec{E} = E_0 \hat{i}$  moves from origin along x towards R according to  $\frac{d^2 x}{dt^2} = \frac{q}{m} E_0$ , then solving the diff eq. means that  $x = \frac{q}{2m} E_0 (\Delta t)^2 = R$ . This means we're pushing all charges within  $\frac{q}{2m} E_0 (\Delta t)^2$  to the left of the disk through it.

Then, if we're considering a cylinder of base area  $A$ , mass density  $\delta$ , charge density  $\rho$ :

- Every test charge chunk  $\Delta V$  within  $\frac{\rho \Delta V}{2\delta \Delta V} E_0 (\Delta t)^2$  passes through. That's the height.
- Area is  $A$ , so total volume is  $\frac{\rho (\Delta t)^2}{2\delta} E_0 A$
- Density of charge per volume is  $\rho$ , so total is  $\frac{\rho^2 (\Delta t)^2}{2\delta} E_0 A$

Note: Tilting this forward from the z-axis by  $\theta$  multiplies the cross-section area of the cylinder (now an ellipse) by  $\cos(\theta)$ . Can work out the ellipse volume, or just note that each “Riemann bar” orthogonal to x-axis just got squished by  $\cos(\theta)$ .

So we define **flux** as amount of charge through a closed surface.  $\Phi = (\vec{E} \cdot \hat{n})A$  where  $\vec{E}$  is a constant field. (Units: joules/second/ $m^2$ , or watts/ $m^2$ )

We can further note  $(\vec{E} \cdot \hat{n}) = \|\vec{E}\| \cos(\theta)$  by last problem.

Example: Flux through an empty cube from the origin is necessarily 0 since every face cancels the other.

Another example: A square pyramid with top at  $(0, 0, 1)$ , sides at 1 on each axis:

- All the triangles will cancel in the x, y directions.
- A triangle  $(1, 0, 0)(0, 1, 0), (0, 0, 1)$  has two displacement vectors  $P_1 P_3 = P_3 - P_1 = (-1, 0, 1), P_2 P_3 = (0, -1, 1)$ .
- $P_1 P_3 \times P_2 P_3 = (1, 1, 1) \rightarrow \hat{n} = \frac{(1, 1, 1)}{\sqrt{3}}$
- $A = \frac{1}{2} \|P_1 P_3 \times P_2 P_3\| = \frac{\sqrt{3}}{2}$
- $\Phi = (\vec{E} \cdot \hat{n})A = (E_0 \frac{1}{\sqrt{3}}) \frac{\sqrt{3}}{2} = \frac{E_0}{2}$
- So total flux through these is  $4 \cdot \frac{1}{2} E_0 = 2E_0$
- However, the bottom has area  $\sqrt{2}^2 = 2$  and flux  $E_0$ , so total is 0!

## 11 3.3: Flux Part II

Note:

- Charge ( $q$ ) is the volts of the point charge. Total charge  $Q_{tot}$  is total charge inside some surface.
- Electric field is sum of those point charges acting at a distance, and  $\vec{a}$  is a single vector.
- Flux looks like the integral of the electric field flowing through a surface.

- Total charge  $Q_{tot}$  of a surface is basically the sum of all the flux going out, except that it's that divided by some constant  $\epsilon_0$ .

Note:  $\vec{E}$  isn't usually constant, and the surface  $S$  is usually curved. So we need calculus to break up surface  $S$  into small pieces  $\Delta A_i$  and evaluate  $\vec{E}_i$  there at that normal  $\hat{n}_i$ . So

$$\sum_{patches} = \sum_{patches} (\vec{E}_i \cdot \hat{n}_i) \Delta A_i = \iint_S (\vec{E} \cdot \hat{n}) \Delta A_i = \Phi$$

Easy Example: If, say,  $(\vec{E} \cdot \hat{n}) = 1$  everywhere, we're just looking at  $\iint_S dA$ , or the total surface area.

Another example. Given:

- Real electric field law:  $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3}$
- Real observation: Total electric flux through a surface ( $\Phi$ ) is proportional to total charge inside ( $Q_{tot}$ ).  $\Phi = \iint_S (\vec{E} \cdot \hat{n}) \Delta A \propto Q_{tot}$
- Then constant must be  $\frac{1}{\epsilon_0}$ . Why?
  - On unit sphere,  $\hat{n} = \frac{\vec{x}}{\|\vec{x}\|}$
  - So  $\vec{E} \cdot \hat{n} = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3} \cdot \hat{n}$
  - $= \frac{q}{4\pi\epsilon_0}$
  - Then  $\Phi = \iint_S \frac{q}{4\pi\epsilon_0} dA$
  - $= \frac{q}{4\pi\epsilon_0} 4\pi$  by surface area of unit sphere
  - $= \frac{q}{\epsilon_0}$
- Therefore, because all of the field goes through the surface (no matter the shape), **Gauss's law** says  $\iint_S (\vec{E} \cdot \vec{n}) dA = \frac{Q_{tot}}{\epsilon_0}$

**Note:** Because (UNEXPLAINED!) symmetry of a contained *ball* implies that, for distance  $\rho$  from origin,  $\vec{E} = E(\rho)\hat{\rho}$ , the above works the same for a point charge or a uniform (contained) ball.

Example: For a big radius  $R$  ball of charge  $Q$  containing a small ball of radius  $\rho$  with charge  $Q_{tot}$ , what must the charge  $E(\rho)$  at any point be?

- Small charge  $Q_{tot}$  is proportional to volume of the big charge  $Q$  by  $Q_{tot} = Q \frac{V_{small}}{V_{big}} = Q \frac{\rho^3}{R^3}$
- $\frac{Q_{tot}}{\epsilon_0} = \text{total charge} = \iint_S E(\rho)(\|\hat{\rho}\|) dA = E(\rho) \iint_S 1 dA = E(\rho) 4\pi\rho^2$

- So  $\frac{Q_{tot}}{\epsilon_0} = Q \frac{\rho^3}{R^3 \epsilon_0} = E(\rho) 4\pi \rho^2$
- So  $E(\rho) = \frac{Q}{4\pi \epsilon_0} \frac{\rho}{R^3}$

Example: Infinite wire,  $x=y=0$ , charge per length is  $\lambda$ . Use a cylinder.

- What's the total charge of the cylinder? Top and bottom are perpendicular to the field so can be ignored.
- There's some function  $E(r)$  which, time  $\hat{r}$ , is the field by symmetry.
- $\Phi = \iint_{cylinder} (E(r) \cdot \hat{r}) dA = E(r) \iint_{cylinder} 1 dA = E(r) 2\pi r h$ .
- $\frac{Q_{tot}}{\epsilon_0} = E(r) 2\pi r h \Rightarrow E(r) = \frac{\lambda}{2\pi \epsilon_0 r}$

Example: Infinite plane,  $x=y=0$ , charge per area is  $\sigma$ . Use a cylinder again

- What's the total charge of the cylinder? Side is perpendicular to the field so can be ignored. Looking at top and bottom,  $\phi = 2EA + 2EA$ , where E is charge through the top.
- $2EA = \frac{\sigma A}{\epsilon_0} \rightarrow E = \frac{\sigma}{2\epsilon_0}$

## 12 3.3: Surface Integrals

- Flux is a specific form of the general  $\iint_S F da$ .
- $dA$  is a patch of a parallelogram on the surface. This is defined by corners  $\vec{x}(u_0, v_0)$ ,  $\vec{x}(u_0, v_0) + \delta_u \vec{x}(u_0, v_0)$ , and  $\vec{x}(u_0, v_0) + \delta_v \vec{x}(u_0, v_0)$
- Therefore, using the parallelogram area formula,  $dA = \Delta_u \Delta_v \|\vec{x}_u \times \vec{x}_v\|$
- Taking to the limit, this means the area is  $\iint_D F(\vec{x}(u, v)) \|\vec{x}_u \times \vec{x}_v\| du dv$

Example: Sphere  $x^2 + y^2 + z^2 = R^2$  surface area. Take  $\theta$  as angle around  $\phi$  as angle from top of z axis.

- Parametrization  $x = R \sin \phi \cos \theta$ ,  $y = R \sin \phi \sin \theta$ ,  $z = R \cos \phi$
- $dx/d\theta = -R \sin \phi \sin \theta$ ,  $dy/d\theta = R \sin \phi \cos \theta$ ,  $dz/d\theta = 0$
- $dx/d\phi = R \cos \phi \cos \theta$ ,  $dy/d\phi = R \cos \phi \sin \theta$ ,  $dz/d\phi = -R \sin \phi$
- After working it out,  $dx/d\theta \times dy/d\phi = R^2 \sin \phi \langle -\sin \phi \cos \theta, -\sin \phi \sin \theta, -\cos \phi \rangle$
- Doing the math,  $\|dx/d\theta \times dy/d\phi\| = R^2 \sin \phi$
- So  $\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} 1 \cdot R^2 \sin \phi = 2\pi \int_{\phi=0}^{\pi} R^2 \sin \phi = 2\pi R^2 [-\cos \phi]_0^{\pi} = 4\pi R^2$

Example: Paraboloid  $z = 1 - x^2 - y^2$ ,  $x^2 + y^2 \leq 1$

- Parametrization  $x = R \sin \phi \cos \theta, y = R \sin \phi \sin \theta, z = R \cos \phi$
- $dz/dx = \langle 1, 0, -2x \rangle, dz/dy = \langle 0, -1, -2y \rangle$
- $\|dz/dx \times dz/dy\| = 1 + 4x^2 + 4y^2$
- Area =  $\iint_D 1 \cdot dA = \iint_D \sqrt{1 + 4x^2 + 4y^2} dx dy$
- Change to polar, remembering this square depends on r:  $\int_{\theta=0}^{2\pi} \int_{r=0}^1 \sqrt{1 + 4r^2 \cos^2 \theta} 4r^2 \sin^2 \theta r dr d\theta = 2\pi \int_0^1 \sqrt{1 + 4r^2} r dr$
- After working it out, this ends up being  $[\frac{2}{3} \cdot \frac{1}{8} (4r^2 + 1)^{\frac{3}{2}}]_0^1 = \frac{\pi}{6} (5\sqrt{5} - 1)$

Example: Torus  $x(u, v) = [R + r \cos(u)] \sin(v), y(u, v) = [R + r \cos(u)] \cos(v), z = r \sin(u), u, v \in [0, 2\pi]$

- Already parametrized in polar, basically,
- $d\vec{x}/du = \langle -r \sin(u) \sin(v), -r \sin(u) \cos(v), r \cos(u) \rangle$
- $d\vec{x}/dv = \langle R \cos(v) + r \cos(u) \cos(v), -R \sin(v) - r \cos(u) \sin(v), 0 \rangle$
- After lots of math,  $\|d\vec{x}/du \times \vec{x}/dv\| = r(R + r \cos(u))$
- $\int_{u=0}^{2\pi} \int_{v=0}^{2\pi} r(R + r \cos(u)) du = 2\pi r \int_{u=0}^{2\pi} r(R + r \cos(u)) du$
- $= 2\pi r [2\pi R] = 4\pi^2 Rr$

Example: Center of mass of unit (hollow?) hemisphere sitting on origin.

- Center of mass for density  $\rho$  is  $\frac{\iint_S \vec{x} \rho dA}{\iint_S \rho dA}$
- Obvious that  $x, y$  center at zero.
- For denominator,  $\iint_S dA$  is just surface area, or half of  $4\pi 1^2 = 2\pi$ .
- For numerator:
  - Do typical  $\theta, \phi$  parametrization.
  - $\vec{x}_\theta \times \vec{x}_\phi = \langle \sin^2(\phi) \cos(\theta), \sin^2(\phi) \sin(\theta), \sin(\phi) \cos(\phi) \rangle$
  - Pull out the  $\sin(\phi)$  and the remaining norm is one, so  $\|\vec{x}_\theta \times \vec{x}_\phi\| = \sin(\phi)$
  - $\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} z \cdot dA = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \cos(\phi) \sin(\phi) = \frac{1}{2}$

Example: Moment of inertia

- Formula:  $I_z = M \iint_S (x^2 + y^2) dA$ .
- Object to spin: helicoid  $\vec{x}(u, v) = \langle u \cos(v), u \sin(v), v \rangle u \in [0, R], v \in [0, 2\pi]$



- Assumption for the problem:  $\int_{u=0}^{u=R} u^2 \sqrt{1+u^2} du = 2$
- Center of mass for density  $\rho$  is  $\frac{\iint_S \vec{x} \rho dA}{\iint_S \rho dA}$
- Use polar coordinates  $r, \theta$ .
- After computation,  $\|\vec{x}_r \times \vec{x}_\theta\| = \sqrt{1+r^2}$
- $M \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \sqrt{1+r^2} (r^2 \cos^2(\theta) + r^2 \sin^2(\theta)) dA = M \iint \sqrt{1+r^2} r^2 = 2\pi M \iint \sqrt{1+r^2} r^2 = 4\pi$  by hint

Example: Flux through unit hemisphere

- Formula:  $\Phi = \iint_S (\vec{E} \cdot \vec{n}) dA = \iint_S F dA$
- Field:  $\vec{E} = \langle yz, xz, xy \rangle$
- Use polar coordinates
- **Base:**  $\hat{n} = -\hat{k}$  so  $\langle yz, xz, xy \rangle \cdot \langle 0, 0, -1 \rangle = -xy$  It's clear by symmetry that  $\iint_{u^2+v^2 \leq 1} -xy dx dy = 0$
- **Top:** Set  $u = \theta \in [0, 2\pi), v = \phi \in [0, \frac{\pi}{2}]$ .
- As usual,  $dA = \|\vec{x}_u \times \vec{x}_v\| = \sin(v)$ .
- Norm just points out from the center:  $\hat{n} = \langle \cos(u) \sin(v), \sin(u) \sin(v), \cos(v) \rangle$
- $\vec{E} = \langle \sin(u) \sin(v) \cos(v), \cos(u) \sin(v) \cos(v), \cos(u) \sin^2(v) \sin(u) \rangle$
- So  $\vec{E} \cdot \hat{n} = 3 \cos(u) \sin(u) \cos(v) \sin^2(v)$
- Looking at this, this is really  $\int_{u=0}^{u=2\pi} k(v) \sin^2(u) du$  for some  $k(v)$ , so this will be 0.
- Therefore, total flux is zero, and by Gauss's law, total field contained inside has to be 0 too.

Example: Field  $\vec{E} = \ln(x^2 + y^2) \langle x, y, 0 \rangle$  through  $R$ -wide cylinder, height  $h$

- Parameterize:  $x = r \cos \theta, y = r \sin \theta, z = z$
- **Top/Bottom:**  $\hat{n} = \langle 0, 0, 1 \rangle, \vec{E} = f(x, y) \langle x, y, 0 \rangle \rightarrow \hat{n} \cdot \vec{E} = 0$
- **Side:**  $\hat{n} = \frac{1}{R} \langle R \cos(\theta), R \sin(\theta), 0 \rangle$
- $\Phi = \iint_{cylinder} \frac{1}{R} \langle R \cos(\theta), R \sin(\theta), 0 \rangle \cdot \langle R \cos(\theta), R \sin(\theta), 0 \rangle \ln(R^2 \cos^2(\theta) + R^2 \sin^2(\theta))$
- $= R \iint_{cylinder} \ln(R^2) + \ln(\cos^2(\theta) + \sin^2(\theta)) = R \cdot 2 \ln(R) \cdot h \cdot 2\pi R = 4\pi R^2 \ln(R) h$

Example: Field  $\vec{E} = e^{-x^2-y^2-z^2} \vec{x}$  with sphere  $S$  at radius  $R$ , setting  $\epsilon_0 = 1$

- Parameterize:  $x = R \cos(\theta) \sin(\phi), y = R \sin(\theta) \sin(\phi), z = R \cos(\phi)$
- $\hat{n} = \langle \cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi) \rangle$
- $\vec{x} = R\hat{n}$ , so  $\vec{E} \cdot \hat{n} = Re^{-R^2}$
- $R \iint_{\text{sphere}} e^{-R^2} = 4\pi R^3 e^{-R^2}$

**Note:** In the future we write  $\hat{n}dA = \vec{dA}$