Brilliant: Differential Equations II

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Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 1: Basics

1.1 Chapter 1: Nonlinear Equations

The two types of problems in this course are:

- Nonlinear equations (several equations on one independent variable)
- Partial differential equations (single equation with several independent variables)

Linear equations have solutions like y_1, y_2 that can be combined using any $c \in \mathbb{R}$ like $y_1 + cy_2$.

Example: Bacteria in a dish with a lot of food, no deaths

- $b'(t) = r_b b(t), r_b > 0. r_b$ would be the rate of growth.
- This is linear. Reason 1: $\frac{d}{dt}(y_1 + cy_2) = y_1' + cy_2' = r_b(y_1 + c_y 2)$ since $y' = r_b y(t)$, and same for y2.
- Also, this works because the solution is $b(t) = b(0)e^{r_b t}$, so $b_1(t) + cb_2(t) = b_1(0)e^{r_b t} + cb_2(0)e^{r_b t} = (b_1(0) + cb_2(0))e^{r_b t}$

Example: **Logistic** equation: Bacteria in a dish with a lot of food, limited by carrying capacity M.

- $b'(t) = r_b b(t)[M - b(t)]$.
- This is nonlinear. Reason: $\frac{d}{dt}(y_1' + cy_2') = y_1' + cy_2' = r_b[y_1 + cy_2][M - y_1 - cy_2] = My_1 + Mcy_2 - y_1^2 - 2cy_1y_2 - cy_1^2y_2^2$
- $\neq My_1 - y_1^2 + Mcy_2 - c^2y_2^2$ because of the extra $-2cy_1y_2$ term.

Sidebar: Note that this equation $b' = r_b b[M - b]$ is *separable*, so it can be solved.

- $\frac{db}{dt} = rb[M-b]$
- $\frac{db}{b(M-b)} = rdt$
- $\frac{1}{M}(\frac{1}{b} + \frac{1}{M-b})db = rdt$ after partial fractions work
- $(\ln(b) - \ln(M-b)) = Mrt + C \Rightarrow \ln(\frac{b}{M-b}) = Mrt + C$
- $\frac{b}{M-b} = e^{Mrt}e^{C}$
- Initial conditions $b = b(0), t = 0 \Rightarrow \frac{b}{M-b} = \frac{b(0)}{M-b(0)}e^{Mrt}$
- $b(1 + \frac{b(0)}{M-b(0)}e^{Mrt}) = M\frac{b(0)}{M-b(0)}e^{Mrt}$
- $b(M - b(0) + b(0)e^{Mrt}) = Mb(0)e^{Mrt}$
- $b = \frac{Mb(0)e^{Mrt}}{M+b(0)[e^{Mrt}-1]}$

This logistic solution will taper off to M at some point. Note that $\lim_{t\to\infty} b(t) = M$ since the non-exponential terms stop mattering. Also $b(t) = M$ sticks as a constant solution or **equilibrium** immediately. *These equilibria tell us what matters - the long-term behavior of solutions!*

Another **Example**: Lotka-Volterra equation pairs: Bacteria (b) and bacteria-killing phages (p), with kill rate k.

- The "product" $kb(t)p(t)$ measures the interactions and kills resulting from this.
- $b'(t) = r_b b(t) - kp(t)b(t)$, or the normal growht rate minus kill rate
- $p'(t) = kp(t)b(t)$ since its population grows as it kills bacteria.
- Equilibria include $b = 0, p = 0$ and $b = 0, p > 0$, since these are *constant* solutions, or places where $b'(t) = 0, p'(t) = 0$.

Direction fields, with vector pointing towards $\langle b'(t), p'(t)\rangle$ (TODO - I think) let us follow the arrows to determine the curve over time. In this case, the bacteria will always go extinct.

However, if we add a new death rate term $-d_p p(t)$ so $p'(t) = -d_p p(t) + kp(t)b(t)$:

- We get an equilibrium at $b = \frac{d_p}{k}, p = \frac{r_b}{k}$. (Since $0 = b'(t) = rb - kpb, (\Rightarrow pk = r), 0 = p'(t) - dp + kpb, (\Rightarrow bk = d)$)
- But otherwise the solutions swirl around this point. This is called a **cycle**. TODO What is a **limit cycle**?

Note that there are systems where the "solution particle" neither reaches an equilibrium or cycles around one point. The **Lorenz system** famously has this owl-eye shaped double attractor (an example of **strange sets**) where initially close particles diverge unpredictably if the constants ρ, σ, b are chosen right:

- $x'(t) = \sigma(y - x)$
- $y'(t) = x(\rho - z) - y$
- $z'(t) = xy - bz$
- TODO

1.2 Chapter 1.2: PDEs

Many methods of attack for PDEs

- Separation of variables
- Power series (Note: did we actually touch on this?)
- Fourier Transform

Example: Standing wave, where one end of a rope is fixed.

- Vertical displacement from a line of rope: $u(x, t)$ depends on where (x) and when (tt).
- Rope's **wave equation** is $u_{tt} = v^2 u_{xx}$, where v is the "constant wave speed", and the others are the space, time partials.
- Note that $u = \cos(vt)\sin(x)$ and $u = \sin(vt)cos(x)$ both work.
- If you guess the solution has split variables like $u = X(x)Y(y)T(t)$, then, upon substitution and division by $X(x)Y(y)T(t)$, $\frac{\delta^2 u}{\delta t^2} = v^2[\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}]$ yields $\frac{T''(t)}{T(t)} = v^2[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}]$
- This method may or may not work. But if it does, it means that since x, y, and t are independent variables, each individual piece must be constant.
- So, for example, if we know $\frac{X''(x)}{X(x)} = -4\pi^2$, we can get to $X(x) = \sin(2\pi x)$
- The wave equation is similar in 3D: $u_{tt} = v^2[u_{xx} + u_{yy} + u_{zz}]$, or using the Laplacian, $u_{tt} = v^2\nabla^2 u$. Here, u measures not displacement but expansion/compression of air at (x, y, z), time t.

Using Fourier transforms helps turn difficult PDEs into an easier problem like an ODE. *Fourier transforms work best* when

- The domain is all of $\mathbb{R}^n$
- The function u vanishes at infinity.

The Fourier transform changes the domain of x to that of ω. It comes with the (highly simplified) rule (see Vector Calculus course): $F[\frac{\delta f}{\delta x}] = i\omega F[f]$. **Example**: Drunkard's walk.

- One dimensional: moves left or right in a random way. Starts at $x = 0, t = 0$.
- $u(x,t)$ is probability of being at point x at time t. Naturally, $\int_{x=-\infty}^{x=\infty} u(x,t)dx = 1$.
- Also, it obeys the 1-dD diffusion equation $\frac{\delta u}{\delta t} = \frac{\delta^2 u}{\delta x^2}$
- The Fourier transform doesn't affect t at all.
- So by taking Fourier transform of both sides of diffusion equation we get
  - $F(u_t) = \frac{\delta}{\delta t}F(u)$ since F doesn't care about t.
  - $\frac{\delta^2 u}{\delta x^2} = i\omega F(\frac{\delta u}{\delta x}) = -\omega^2 F(u)$
  - So $\frac{\delta}{\delta t}F(u) = -\omega^2 F(u)$
  - This is solvable as $F(u) = ce^{-\omega^2 t}$. Take it on faith that $c = \frac{1}{2\pi}$ for now. TODO
  - Known fact: $F[Ae^{-\frac{ax^2}{2}}] = \sqrt{\frac{1}{2\pi a}}Ae^{\frac{-\omega^2}{2a}}$
  - This means $t = \frac{1}{2a}$ and $a = \frac{1}{2t}$
  - $F(u) = \frac{1}{2\pi}e^{-\omega^2 t}, F[Ae^{-\frac{ax^2}{2}}] = \sqrt{\frac{1}{2\pi a}}Ae^{\frac{-\omega^2}{2a}}$ so $u = Ae^{\frac{-ax^2}{2}}$
  - Solving, you get $A = \sqrt{\frac{1}{4\pi t}}, a = \frac{1}{2t}$, so $u(x,t) = \sqrt{\frac{1}{4\pi t}}e^{-\frac{x^2}{4t}}$

2 Chapter 2: Nonlinear Equations

2.1 2.1: Lotka-Volterra I

Major ideas:

- **phase plane**: TODO
- **nullcline**: TODO
- **direction field**: TODO
- **equilibria**: TODO

Example: Bacteria vs. phages (again)

- Bacteria unrestrained grow in proportion to their population, so $\frac{db}{dt} = r_b b(t)$ (solved: $b(t) = b(0)e^{r_b t}$)
- Phages unfed decrease in proportion to current size, so $\frac{dp}{dt} = -d_p p(t)$ (solved: $p(t) = p(0)e^{-d_p t}$)
- Bacteria die with likelihood of meeting a phage, and phages increase with likelihood of meeting a bacterium. So the set of equations, for constant k, becomes:
  - $b'(t) = r_b b(t) - kb(t)p(t)$
  - $p'(t) = -d_p p(t) + kb(t)p(t)$
  - *The product of p and b makes our equations nonlinear* (WHY?)
  - I guess, very generally, $b_1p_1 = k, b_2p_2 = k$, but $(b_1 + b_2)(p_1 + p_2) = b_1p_1 + b_2p_2 + b_1p_2 + b_2p_1 = 2k + b_1p_2 + b_2p_1 \neq 2k$, so the last two "mixed" terms mean you can't just add solutions (b_1, p_1) and (b_2, p_2).

General thoughts on this solution:

- So a solution $(b(t), p(t))$, traces out a curve on the bp-phase plane (b is x-axis, p is y-axis) as time (unrepresented in the plane) continues.
- If we add a unit tangent vector at every point (B, P) aligned with $(b'(t), p'(t)) = (r_bB - kBP, -d_pP + kBP)$, we can follow the arrows to see the solution over time.
- The above is called a **direction field**
- This is sometimes hard to sketch analytically, so we can look to the **nullclines**: places where one of the components of the direction field is zero.
- In this case, $r_bB - kBP = (r_b - kP)B = 0$ when $P = 0$ or $P = \frac{r_b}{k}$, and $-d_pP + kBP = (kB - d_p)P = 0$ when $P = 0$ or $B = \frac{d_p}{k}$.
- The **upshot of nullclines** (since we don't care about $P, B \leq 0$): The lines $B = \frac{d_p}{k}, P = \frac{r_b}{k}$ *divide the plane into pieces where the components of this (continuous) function pair can't change sign.*
- For instance, $B > \frac{d_p}{k}, P < \frac{r_b}{k}$ means $r_bb - kbp > 0, -d_pp + kdp > 0$, so both populations are growing here. This helps to sketch the curve.
- The curve looks like a counterclockwise whirlpool around the $(B, P) = (\frac{d_p}{k}, \frac{r_b}{k})$. (bacteria grow with low but growing phages; bacteria decrease as phages overwhelm; both decrease as phages starve; bacteria start coming back)
- The center point is a (constant **equilibrium**) solution, and other solutions swirl around it but don't get attracted or repelled.

There are a few types of equilibria:

- This one is a **center** around which solutions circle.
- A **stable equilibrum** would see small upsets come back to an unchanging state.
- An **unstable equilibrum** would see small upsets create wildly divergent paths.

2.2 2.2: Lotka-Volterra II

In the Bacteria-Phage system, we can't yet prove everything rotates around the **center**. Let's do that.

Developing a **conserved quantity** will help to do that. **Example**: Block on a horizontal spring with mass m, spring constant k_s:

- $x(t)$: Displacement from rest position.
- $v(t) = \frac{dx}{dt}$: Horizontal velocity
- $\frac{dv}{dt} = -\frac{k_s}{m}x(t)$ by Hooke's law, I think.
- Suppose there's some Energy function $E(x, v)$. By chain rule $\frac{d}{dt}E(x(t), v(t)) = \frac{dE}{dx}\frac{dx}{dt} + \frac{dE}{dv}\frac{dv}{dt}$
- $= \frac{dE}{dx}v - \frac{k_s}{m}\frac{dE}{dv}x$. If we set E as conserved, as in $E'(t) = 0$, then $\frac{dE}{dx}v = \frac{k_s}{m}\frac{dE}{dv}x$
- We can eyeball and see that $E = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ solves this equation, or we can assume $E(x, v) = F(x) + G(v) \Rightarrow 0 = E'(t) = F'(x)v - \frac{k_s}{m}G'(v)x = 0$ from the above equations and guess from there.
- This means in the xv phase space, that there's a fixed E such that the particle follows the ellipse $E = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ in phase space around the solution point (0,0).

Extended Example: Continuing on finding a conserved quantity for Bacteria / Phage:

- We need to find $U(b(t), p(t))$ such that $U'(t) = 0$, or by chain rule $\frac{\delta U}{\delta b}\frac{\delta b}{\delta t} + \frac{\delta U}{\delta p}\frac{\delta p}{\delta t} = 0$
- Subbing in, $\frac{\delta U}{\delta b}[r_b b - kbp] + \frac{\delta U}{\delta p}[-d_p p + kbp] = 0$
- A hint suggests finding U such that $\frac{\delta U}{\delta b} = -\frac{d_p}{b} + k, \frac{\delta U}{\delta p} = -\frac{r_b}{p} + k$ to make terms cancel.
- Integrating these gives us U as both $-d_p \ln(b) + kb + Q(p)$ and $-r_b \ln(p) + kp + R(b)$ so $U = -d_p \ln(b) - r_b \ln(p) + kb + kp$. This weird curve consistutes a level set in pb-space upon which a solution sits.
- The spring example has an elliptic paraboloid solution. There's an absolute minimum ($E = 0$ at $(0, 0)$) but level sets become closed loops away from it.

- For the Lotka example, there is a critical point ($\nabla U = \vec{0}$) when $\nabla U(b,p) = (\frac{\delta U}{\delta b}, \frac{\delta U}{\delta p}) = (k - \frac{d_p}{b}, k - \frac{r_b}{p})$, which is $(0,0)$ at our known center $(\frac{d_p}{k}, \frac{r_b}{k})$
- Showing that we always increase gong away from the point $(\frac{d_p}{k}, \frac{r_b}{k})$ should guarantee us closed level sets.
- One method: Assume we're picking a unit vector $\vec{v} = \langle \hat{v_b}, \hat{v_p} \rangle$ so that our line from our center is $\vec{v} = \langle \frac{d_p}{k} + tv_b, \frac{r_b}{k} + tv_b \rangle$. $U = F(b) + G(p)$ in this case, so sub the b part into F to get $F(\frac{d_p}{k} + t\hat{v_b}) = d_p[1 - \ln(\frac{d_p}{k} + t\hat{v_b})] + kt\vec{v_b}$. Taking derivative of that w.r.t t shows it is always positive. Same goes for the G(p) portion of U.
- Another (DF) method: Note that $\nabla U = (k - \frac{d_p}{b}, k - \frac{r_b}{p})$'s grad (second derivative) is always positive. So derivative always has positive curvature (maybe using that term wrong), and we'll always increase around this point.
- Also, we know that the particle travels around the level set (loop) and doesn't reverse course, because then, $b'(t) = p'(t) = 0$, and we only have that at the center point (nullcline intersection).

2.3 2.3: Linearization

Extended Example: Suppose there's a limit to bacterial growth, so we cap our population at M_b.

- If $b(t) << M_b$, things should be similar. If $b(t)$ is nearly M_b, then growth should approach 0. So, this implies $\frac{db}{dt} = r_b b(t) \to \frac{db}{dt} = r_b b(t)(1 - \frac{b(t)}{M_b})$. Note: This isn't the only possibility but we'll use it.
- This updates our Lotka-Volterra model to something more complicated:
  - $b'(t) = r_b b(t)(1 - \frac{b(t)}{M_b}) - kb(t)p(t)$
  - $p'(t) = -d_p p(t) + kb(t)p(t)$
- Other than $b = 0, p = 0$, the meaningful nullclines are solved by setting $b'(t) = 0$ (yielding $r_b(1 - \frac{b}{M_b}) - kp = 0$) and $p'(t) = 0$ (yielding $b = \frac{d_p}{k}$)
- Note: We'll clean up through some MAGIC non-dimensionalization (how to derive?) to simplify:
  - $x(t) = \frac{1}{M_b} b(\frac{t}{r_b}), y(t) = \frac{k}{r_b}(\frac{t}{r_b}), \alpha = \frac{d_p}{r_b}, \beta = \frac{kM_b}{r_b}$
  - Gives us new equations: $\frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \frac{dy}{dt} = -\alpha y(t) + \beta x(t)y(t)$
  - And new nullclines: $x + y = 1, x = \frac{a}{b}$

- So there's an equilibrium point in the positive xy quadrant if: $y = 1 - x = 1 - \frac{\alpha}{\beta}$ and $y > 0$ implies $1 - \frac{\alpha}{\beta} > 0 \Rightarrow \frac{\alpha}{\beta} < 1$
- Looking at the direction field, it appears solutions swirl around and are attracted *into* the center point $(\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta})$, making it a **stable equilibrum**

This is similar to the block-spring example, if a damping term $-\frac{\gamma}{m}v$ is added.

- $\frac{dx}{dt} = v, \frac{dv}{dt} = -\frac{k_s}{m}x - \frac{\gamma}{m}v$
- This can be thought of in matrix terms: $\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{\gamma}{m} \end{pmatrix}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ Call the matrix A.
- From Diff Eq I, the solution is $exp(tA)$ (matrix exponential), making x(t) a linear combination of $e^{\lambda t}$ or possibly $te^{\lambda t}$ terms, with the eigenvalues as λs.
- The eigenvalues in this case, using the quadratic formula, could be:
  - Two real, distinct, negative roots. So, these $e^{\lambda t}$ terms decay, and x(t) levels off.
  - Two distinct complex roots with real part $-\frac{\gamma}{2m} < 0$. This ends up being some sines and cosines multiplied by $e^{-\frac{\gamma t}{2m}}$, which decays too.
  - Finally, if we have a repeated negative real eigenvalue, we have solution $x(t) = Ae^{-\frac{\gamma t}{2m}} + Bte^{-\frac{\gamma t}{2m}}$, also decaying.
  - So any disturbance in the spring will oscillate and come to rest at $x(t) = v(t) = 0$ quickly.

So with linear systems $\vec{x}'(t) = A\vec{x}(t)$, the eigenvalues determine what happens around the equilibrium point. However, the **bacteria-phage model is non-linear**. Here is **how we linearize** for nearby solutions in a nonlinear system:

- Set small disturbance $\delta x(t) << 1, \delta y(t) <<$ so $x(t) = \frac{\alpha}{\beta} + \delta x(t), y(t) = 1 - \frac{\alpha}{\beta} + \delta y(t)$
- Since they're small, all powers like $\delta x(t)^2$ and $\delta x(t)\delta y(t)$ are considered basically zero.
- So substitute $x(t) \to \frac{\alpha}{\beta} + \delta x(t), y(t) \to= 1 - \frac{\alpha}{\beta} + \delta y(t)$ into our $\frac{dx}{dt}$ and $\frac{dy}{dt}$ equations.
- This gives us the A solving $\frac{d}{dt}\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = A\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$, which is $A = \begin{pmatrix} -\frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \\ \beta - \alpha & 0 \end{pmatrix}$ after working through the substitution.

- Finding the eigenvalues here yields the same situation as the block-spring example: decays in all situations.

It turns out through the **Hartman-Grobman Theorem** that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$, for some continuously differential vector field F, if we linearize near equilibrium x_0, then what falls out of this A approach works if the eigenbalues *aren't all purely imaginary.*

It turns out the uncapped bacteria system from before looks like $\frac{d}{dt}\begin{pmatrix}\delta x(t)\\ \delta y(t)\end{pmatrix} = \begin{pmatrix}0 & -1\\ \alpha & 0\end{pmatrix}\begin{pmatrix}\delta x(t)\\ \delta y(t)\end{pmatrix}$, with characteristic equation $\lambda^2 + \alpha = 0, \alpha > 0$. This means both values are imaginary, and we had to use the conserved quantity appraoach!

2.4 2.4: Hartman-Grobman Theorem

Extended Example: Consider a phage that dies off quicky:

- $\frac{db}{dt} = r_b b(t) - k_b b(t)p(t), \frac{dp}{dt} = -r_p p(t) = 0 \cdot b(t)p(t)$, where k_p is the zero (phages don't increase), and k_b is still the kill factor for the bacteria.
- In this base, $b(t) = p(t) = 0$ is the only equilibrium.
- Non-dimensionalize as $x(t) = b(\frac{t}{r_b}), y(t) = \frac{k_b}{r_b}p(\frac{t}{r_b}), \alpha = \frac{r_p}{r_b}$
- This makes the equations $x'(t) = x(t) - x(t)y(t), y'(t) = -\alpha y(t)$, and the nullclines therefore $x(t) = 0, y(t) = 1, y(t) = 0$
- Looking at this six-section dierection field, we see that solutions exactly on the y-axis are attracted to equilibrium $(0,0)$, and other are repelled.
- This makes sense since if the bacteria is 0, the phage die and approach $(0,0)$, otherwise the bacteria multiply and win (so it's a *saddle point*)
- The way to tell: linearize the equations. $\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} \approx \begin{pmatrix}1 & 0\\ 0 & -\alpha\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$ since, if $x(t), y(t) << 1, x(y)t(t) = 0$.
- Then the eigenvalues are $\lambda = 1, -\alpha$ so the solution is Ae^t, $Be^{-\alpha t}$ for $x(t), y(t)$ (TODO respectively?) **Hartman-Grobman ensures this is the general solution**.

However, let's solve directly and see if we come to the same result.

- $y'(t) = -\alpha y(t), y(0) = y_0 \Rightarrow y(t) = y_0 e^{-\alpha t}$
- With this in hand, $\frac{dx}{dt} = x(t) - = x(y)y(t) = x(t)[1 - y_0 e^{-\alpha t}]], x(0) = x_0$ separates out to
  - $\frac{dx}{x} = [1 - y_0 e^{-\alpha t}]dt$

- $\ln(x) = [t + \frac{y_0}{\alpha} e^{-\alpha t}] + C$
- $x = e^C e^t \exp(\frac{y_0}{\alpha} e^{-\alpha t})$
- $x(0) = x_0 \Rightarrow e^C = x_0 e^{-\frac{y_0}{\alpha}}$
- $\Rightarrow x(t) = x_0 e^t \exp(\frac{y_0}{\alpha}(e^{-\alpha t} - 1))$

But how do we deform the phase plane so this looks linear? We need some mapping $\vec{h}(x, y) = \langle u(x, y), v(x, y)\rangle$ that is continuous and invertible (so we don't "damage" the phase plane). This is called a **homeomorphism**.

- So near the equilibrium $(0, 0)$, the equations $y'(t) = -\alpha y(t), y(0) = y_0 \Rightarrow y(t) = y_0 e^{-\alpha t}$ linearized for $\delta x, \delta y$ must be similar to those for $u(x(t), y(t), v(x(t), y(t))$
- This means we need $\frac{du}{dt} = u$, $\frac{dv}{dt} = -\alpha v$
- After doing the substitution, we see that $v = v_0 e^{-\alpha t}$ exactly mimics $y(t) = y_0 e^{-\alpha t}$ for the phage solution. So we take $v = y$.
- Therefore, we know that since $u = u_0 e^t$ and $x(ty) = x_0 \exp(t + \frac{y_0}{\alpha}(e^{-\alpha t} - 1))$, that we need $u(x(t), y(t) = u(x_0, y_0)e^t$
- And this is satisfied if we guess $u(x, y) = xe^{-y}\alpha$ and work it out.
- This function $\vec{h}(x, y) = (u, v) = \langle xe^{-\frac{y}{\alpha}}, y\rangle$ is invertible by $(x, y) = \langle ue^{\frac{v}{\alpha}}, v\rangle$, which is continuous.

2.5 2.5: Application - Lasers

Lasers create excited atoms, which then emit photos while transitioning to an unexcited state. This system has a close analogue with the previous phages (like photons) and bacteria (like atoms) model.

- $n(t)$: number of photons in the laser; r_g: rate of photons gained (created by excited atoms transitioning to unexcited state); r_l: rate of photons lost (emitted)
- $\Rightarrow \frac{dn}{dt} = r_g - r_l$ by definition.
- We can assume we're losing a constant k (kill?) portion of photons per unit time, so $\frac{dn}{dt} = r_g - kn(t)$
- $e(t)$: number of excited atoms (that will maybe create photons). Atoms are excited by external energy pump.
- Excited atoms radiate when meeting a photon (which survives the meeting)
- So we can use the same setup from the bacteria: with I the constant of meeting (intersect?), $r_g = Ie(t)n(t) \Rightarrow n'(t) = Ie(t)n(t) - kn(t)$

Mini example: Assume no photons leave (cap the end of the laser)

- $k = 0$ in this scenario.
- So every meeting creates one more photon ($n \to n+1$) while enervating one excited atom ($e \to e-1$). This implies, equivalently:
  - $e + n$ is a conserved quantity,
  - $e(t) + n(t) = e(0) + n(0)$,
  - $[e(t) + n(t)]' = 0$
  - Then, if$k = 0, n'(t) = Ie(t)n(t) - kn(t)$, and coupled with $e'(t) + n'(t) = 0$ above, we have $e'(t) = -Ie(t)n(t)$

Extended example: Atoms spontaneously lose energy. This is actually what happens

- From quantum physics, we have a rate s of atoms just (s)pontaneously losing energty.
- We also have an energy (p)ump that energizes atoms with quantity p.
- Then, our change in (e)xcited atoms is $e'(t) = p - s - Ie(n)(t)$
- So our **final laser equations** are $e'(t) = p - s - Ie(n)(t), n'(t) = Ie(t)n(t) - kn(t)$
- If we want to find the smallest p guaranteeing $n \geq 1$ (there's at least one photo) at equilibrium ($e'(t) = n'(t) = 0$):
  - $n'(t) = 0 \Rightarrow Ien = kn \Rightarrow n(Ie - k) = 0$. If $n \neq 0, \Rightarrow e = \frac{k}{I}$
  - $e'(t) = 0 \Rightarrow Ien = p - se$
  - Together, $p - se = Ien = kn \Rightarrow kn + se = p \Rightarrow kn + s\frac{k}{I} = p$
  - $n \geq 1 \Rightarrow p \leq k + \frac{ks}{I}$
  - **Another tactic**: We could also assume we *start out at equilibrium*, so n_0, e_0 are constant solutions.
  - Solving $n' = 0 = Ie_0n_0 - kn_0, e' = 0 = Ie_0n_0 - se_0 + p$, we find equilibria $n_0 = \frac{p}{k} - \frac{s}{I}, e_0 = \frac{k}{I}$
  - Then, $n_0 \geq 1 \Rightarrow \frac{p}{k} - \frac{s}{I} \geq 1 \Rightarrow p \geq k + \frac{ks}{I}$

Non-dimensionalization time:

- Scale against $e_0(= \frac{k}{I}), n_0(= \frac{p}{k} - \frac{s}{I})$ like this: $x(t) = \frac{n(\alpha t)}{n_0}, y(t) = \frac{e(\alpha t)}{e_0}$

- NOTE: What does this do? This makes (1,1) the equilibrium, as $x(t) = \frac{n_0}{n_0} = 1, y(t) = \frac{e_0}{e_0} = 1$!
- What α lets us tke $n' = Ien - kn, e' = -Ien - se + p$ and write
  - $\frac{dx}{dt} = x(t)y(t) - x(t)$
  - $\frac{dy}{dt} = \frac{1}{k}(\frac{pI}{k} - s)[1 - x(t)y(t)] + \frac{s}{k}[1 - y(t)]$
  - $x' = \frac{\alpha n'(\alpha t)}{n_0} = xy - x = \frac{Ie(\alpha t)n(\alpha t)}{kn_0} - \frac{n(\alpha t)}{n_0}$
  - $\frac{\alpha Ien - \alpha kn(\alpha t)}{n_0} = \frac{Ie(\alpha t)n(\alpha t)}{kn_0} - \frac{n(\alpha t)}{n_0}$
  - $\alpha Ie - \alpha k = \frac{Ie(\alpha t)}{k} - 1 \Rightarrow \alpha(Ie - k) = \frac{Ie-k}{k} \Rightarrow \alpha = \frac{1}{k}$
  - This solves the x equation, and I suppose it can be validated in the y equation (tediously).
  - If we chunk up our (somehow positive?) constants as $c = \frac{1}{k}(\frac{pI}{k} - s), d = \frac{s}{k}$, we end up with $y' = c[1 - xy] + d[1 - y]$
  - We only care about $x, y > 0$, so $x' = 0 = xy - x = x(y - 1)$ implies $y = 1$ is a nullcline
  - $y' = 0 = c[1 - xy] + d[1 - y] = c - cxy + d - dy \Rightarrow c + d = y(d + cx) \Rightarrow y = \frac{c+d}{d+cx}$, a scaled and shifted hyperbola.

Look at the solutions:

  - It appears we have a counterclockwise swirl around $(1, 1)$, and nearby solutions tend toward this equilibrium.
  - Hartman-Grobman: rewrite our linearized solution in neighborhood of $(1, 1)$ as $x(t) = 1 + \delta x(t), y(t) = 1 + \delta y(t)$
  - Using $x' = xy - x, y' = c[1 - xy] + d[1 - y]$ and $\frac{d}{dt}\begin{pmatrix}\delta x(t) \\ \delta y(t)\end{pmatrix} = A\begin{pmatrix}\delta x(t) \\ \delta y(t)\end{pmatrix}$, we can solve and write $A = \begin{pmatrix}0 & 1 \\ -c & -c-d\end{pmatrix}$
  - Eigenvalues: $\lambda = \frac{1}{2}(-c - d \pm \sqrt{(c + d)^2 - 4c})$
    * If square root term is zero, we have repeated eigenvalue, so $\delta x(t), \delta y(t)$ are combos of $e^{-\frac{c+d}{2}}, te^{-\frac{c+d}{2}}$, which decays
    * If square root term is greater than zero, we have two distinct real, negative eigenvalues (since c, d are positive), so this decays.

* If square root term is less than zero, we have distinct complex eigenvalues, but combos of $e^{-\frac{c+d}{2}}\cos(\frac{1}{2}\sqrt{-(c+d)^2+4c}), e^{-\frac{c+d}{2}}\sin(\frac{1}{2}\sqrt{-(c+d)^2+4c})$ decay too
* Note : I suppose Hartman-Grobman can't work in purely imaginary eigenvalue scenario, because these kinds of functions don't converge or diverge without a term outside the sin or cos
* And in any case, since these lambdas aren't strictly imaginary, Hartman-Grobman works.

2.6 2.6: Liapunov Equations

We had some intuition that "nearby" solutions would fall into an equilibrium, but what does "nearby" mean? **Liapunov Equations** help us here. What is the "basin of attraction"?

- Suppose we turn the pump off ($p = 0$), and set spontaneous enervation equal to photon leak $s = k$.
- (TODO?) Somehow we can rescale to $\frac{dx}{dt} = Ie(t)n(t) - kn(t)$, $\frac{dy}{dt} = -Ie(t)n(t) - kn(t)$ which (TODO??) gives us $\frac{dx}{dt} = xy - x$, $\frac{dy}{dt} = xy - y$
- This means equilibria ($x' = y' = 0$) exist at $(0,0), (1,1)$
- If we're turning the pump off, we're looking at equilibrium (0,0). Linearizing, we get $x' = -\delta x, y' = -\delta y$, so a matrix of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
- With repeated non-imaginary (H-G applies!) eigenvalues $-1, -1$, we can see that both e^{-t}, te^{-t} decay, and we get sucked into the origin.

But how do we prove this? Let's find a conserved quantity $U'(x(t), y(t)) = 0$

- $U'(x(t), y(t)) = \frac{\delta U}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta U}{\delta y}\frac{\delta y}{\delta t} = \frac{\delta U}{\delta x}x(y-1) + \frac{\delta U}{\delta y}y(x-1)$
- Setting $\frac{\delta U}{\delta x}x = -x+1, \frac{\delta U}{\delta y}y = y-1$ makes this zero
- Solving those two by separating variables and combining, we get $U = -x + y + \ln(|\frac{x}{y}|)$
- So if we're stabiling $f = (x-y)$ (why?), we see $(x-y)' = x' - y' = (xy-x) - (xy-y) = x - y = f \Rightarrow f = e^{-t}$
- With $x(0) = x_0, y(0) = y_0 \Rightarrow f(0) = x_0 - y_0, f = x(t) - y(t) = (x_0 - y_0)e^{-t}$
- How to express $y(t)$ while eliminating $x(t)$, knowing $x(y) - y(t) = (x_0 - y_0)e^{-t}$ and $U(x,y) = y - x + \ln(|\frac{x}{y}|)$ is conserved? **The trick:** $U(x_0, y_0) = U(x,y)$ since *it doesn't change*!

- $y_0 - x_0 + \ln(|\frac{x}{y}|) = y - x + \ln(|\frac{x}{y}|) = -(x_0 - y_0)e^{-t} + \ln(|\frac{x}{y}|)$
- $(1 - e^{-t})(y_0 - x_0) = \ln(\frac{x/y}{x_0/y_0})$
- Defining for convenience, $f = \exp((1 - e^{-t})(y_0 - x_0))$, then $f\frac{y}{y_0} = \frac{x}{x_0}$
- Sub in to $x - y = (x_0 - y_0)e^{-t} : y[\frac{x_0}{y_0}f - 1] = (x_0 - y_0)e^{-t}$
- Solve for $y : y = \frac{y_0(x_0 - y_0)e^{-t}}{x_0 f(t) - y_0}$
- Combine with above to get $x = \frac{x_0(x_0 - y_0)e^{-t}f(t)}{x_0 f(t) - y_0}$

- So with equilibria $(0, 0), (1, 1)$, the direction field computer plot shows us attracted to $(0, 0)$ (no laser action) pretty much anywhere left and down from $(1, 1)$ in the x, y phase plane.
- Apparently the linearized solutions near $0, 0$ are $x_{lin} = x_0 e^{-t}, y_{lin} = y_0 e^{-t}$ (WHY?)
- Looking above, if $(x_0 - y_0) \approx 0$, then $f(t) \approx 1$, and $x, y \to x_{lin}, y_{lin}$

On to **Liapunov** functions, which will tell us perhaps the size of the "basin of convergence", unlike Hartman-Grobman, which just says there is a neighborhood.

A **Liapunov** function $U(x, y)$ is

- Continuously differentiable
- With a unique minimum (x_0, y_0), usually aligned to be U's only zero.
- $U'(x(t), y(t)) \leq 0$. Everything "flows downhill";
- Tailor made for the problem, hard to find.

Back to the rescaled laser example

- $x'(t) = x(t)y(t) - x(t)$
- $y'(t) = c[1 - x(t)y(t)] + d[1 - y(t)], c, d > 0$
  - **Analogy: The damped-block spring system** $\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{\gamma}{m} \end{pmatrix}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$
  - When $\gamma = 0$, we know $E(x, v) = \frac{1}{2}k_s x^2 + \frac{1}{2}mv^2$ is conserved when looking at E'
  - $\gamma = 0 \Rightarrow x' = v, v' = -\frac{k_s}{m}x$
  - $\frac{dE}{dt} = (\frac{1}{2}k_s x^2 + \frac{1}{2}mv^2)' = 0$ since $\frac{1}{2}(k_s x x' + mvv') = \frac{1}{2}(kxv + mv\frac{-k}{m}x) = 0$
  - But if $\gamma \neq 0$, $\frac{d}{dt}E(x(t), v(t)) = \frac{d}{dt}[\frac{1}{2}k_s x^2 + \frac{1}{2}mv^2] = kxx' + mvv'$

- $= kx(v) + mv(\frac{-k_s}{m}x - \frac{\gamma}{m}v) = -\gamma v(t)^2 = \frac{dE}{dt}$
- Total spring energy is then decreasing in the fluid.
- Brilliant has Cool visualization of spiraling down into the "bowl" of x, y with E as the z dimension, equilibrium $(0, 0, 0)$
- We need to choose a γ-fied $E-$like function that decreases for pairs $\delta x(t), \delta y(t)$. We can choose, like E, some $u(\delta x, \delta y) = \frac{1}{2}C_1[\delta x]^2 + C_2[\delta y]^2$.
- Choosing $C_1 = c, C_2 = 1$ gives us $\frac{d}{dt}u(\delta x(t), \delta y(t)) = \frac{d}{dt}(\frac{c}{2}[\delta x(t)]^2 + \frac{1}{2}[\delta y(t)]^2)$
- $= c(dx)(dx)' + dy(dy)' = c(dx)(dy) + dy(-c(dx) - (c+d)(dy)) = -(c+d)[\delta y(t)]^2$
- So $u = \frac{d}{dt}(\frac{c}{2}[\delta x(t)]^2 + \frac{1}{2}[\delta y(t)]^2)$ is an energy function that could work for the laser.

Finally, we want to construct a function that

- Doesn't increase (derivative negative) on any pairs $x, y > 0$ (pulls down)
- Is near equal to $u = \frac{c}{2}(x-1)^2 + \frac{1}{2}(y-1)^2$ near $(1, 1)$. (the energy function for block-spring above)
- With $x' - = xy - x, y' = c - cxy + d - dy$, plus the identity near $z \approx 1$ of $\ln(z) \approx (z-1) - \frac{1}{2}(z-1)^2$...
- You can find $U(x, y) = c(x-1) + (y-1) - c\ln(x) - \ln(y)$ that satisfies all of these
- It therefore shows that pumped laser solutions tend to equilibrium $(1, 1)$ in the long term.

TODO: So this is enough to establish a convergence to an equilibrium?

- Find an equilibrium (x_0, y_0)
- Find an energy function u that decreases for all pairs $(\delta x(t)\delta y(t))$ near the minimum.
- Find a Liapunov function U function that decreases EVERYWHERE along $x(t), y(t)$ (in our domain, like $x, y > 0$)
- Ensure that $U = u$ in the neighborhood of the equilibrium.
- Then Liapunov's theorem somehow makes this work (TODO)?

2.7 2.7: Dog chasing a duck (Limit Cycles)

This is a pair of nonlinear equations to determine if a dog in the pond's interior catches a duck who skates along the border.

- Variables:
  - r_p: Radius of pond.
  - $\vec{r_H}$: Distance of duck to center (always the radius of the pond)
  - $\vec{l}$: Displacement of dog from duck, which is of some length R at any point.
  - θ: Duck's position in the lake (think polar coordinates)
  - ϕ: Angle between $\vec{r_H}$ and $\vec{l}$.
  - Duck always swims at speed $r_p\theta'(t)$, and dog swims at $k > 0$ times this, or $kr_p\theta'(t)$.
- Therefore $\vec{r_H} = \langle r_p \cos(\theta), r_p \sin(\theta)\rangle$. It's just the polar coordinates.
- Doing some geometry gets you $\vec{l} = R\langle \cos(\theta+\phi, \sin(\theta+\phi)\rangle$
- We can establish $\vec{T} = \vec{r_H} - \vec{l}$ and dog's speed squared $\|T'(t)\|^2 = (\vec{r_H}' - \vec{l'})\cdot(\vec{r_H}' - \vec{l'}) = \|\vec{r_H}'\|^2 + \|\vec{l'}\|^2 - 2\vec{r_H}'\vec{l'}$
- Naturally, this $\|T'\|^2$ is also equal to the constant $(kr_p\theta')^2$. Our diff equations will fall out of these.
- $\vec{r_H}^2 = r_p^2[\theta'(t)]^2$ since duck's speed is constant. $\vec{l} = (R')^2 + R^2[\theta' + \phi']^2$ after working it out.
- Finally, after using identities $\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi), \sin(\theta+\phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)$, we can work out $-2\vec{r_H}'\vec{l'} = -2r_p\theta'[R\cos(\phi)(\theta'+\phi') + \sin(\phi)R']$
- After rescaling R to ρ such that $\frac{R}{r_p} = \rho$ and diving our speed equation by constant $r_p\theta'$, we end up with speed equation $k^2 = [\rho(1 + \frac{d\phi}{d\theta} - \cos(\phi)]^2 + (\frac{d\rho}{d\theta} - \sin(\phi))^2$
- We propose that there are some solutions here for the **pursuit equations**. We'll ignore the generalized form an focus on one set
  - $\rho(1 + \frac{d\phi}{d\theta}) - \cos(\phi) = 0, \frac{d\rho}{d\theta} - \sin(\phi) = -k$ do work in the above. (Doesn't prove others don't work)
  - This leaves our equations as $\frac{d\phi}{d\theta} = \frac{\cos(\phi)}{\rho} - 1, \frac{d\rho}{d\theta} = -k + \sin(\phi)$
  - However, there *aren't simple equilibria here.* In no world with $k \neq 0$ does the dog sit still (or the duck).

- Supposing $k < 1$ and R, ϕ are fixed (dog never gets closer and just loops), this means he's going in a circle, since the two legs of a triangle $(\vec{l}, \vec{r_p})$ and the interior angle (ϕ) are fixed, so this fixes length of the third leg, which is a radius
- You can also use dog's position vectors $x(t) = r_p \cos(\theta) - R\cos(\theta + \phi), y(t) = r_p \sin(\theta) - R\sin(\theta + \phi)$ and trig identities to prove $x(t)^2 + y(t)^2 = r_p^2 + R^2 - 2r_p R\cos(\phi)$
- If $k < 1$, then solving $\frac{d\rho}{d\theta} = 0 = -k + \sin(\phi) \Rightarrow \sin(\phi) = k \Rightarrow \phi = \sin^{-1}(k)$ and $\rho = \cos(\phi) = \cos(\sin^{-1}(k)) = \sqrt{1-k^2}$
  * Quick proof of $\cos(\sin^{-1}(k)) = \sqrt{1-k^2}$:
  * $\cos^2(\sin^{-1}(k)) + \sin^2(\sin^{-1}(k)) = 1 \Rightarrow \cos(\sin^{-1}(k)) = 1 - \sin^2(\sin^{-1}(k))$
  * $= 1 - k^2 \Rightarrow \cos(\sin^{-1}(k)) = \sqrt{1-k^2}$
- When $k < 1$, the direction field seems to have attractive equilibriua but **GOTCHA**: there are ϕ angles that differ by 2π units, so they're the same. The direction field is a cylinder with circumference 2π, and there are other solutions tracking toward $(\sin^{-1}(k), \sqrt{1-k^2})$
- Linearizing, assume we are near our equilibirum point and $\phi = \sin^{-1} k + \delta\phi, \rho = \sqrt{1-k^2} + \delta\rho$.
- We can also remember that $f(x + \delta x) \approx f(x) + f'(x)\delta x$ from calculus.
- $\frac{d}{d\theta}[\delta\rho] = \frac{d}{d\theta}[\rho - \sqrt{1-k^2}] = \frac{d\rho}{d\theta} - \frac{d}{d\theta}\sqrt{1-k^2} = -k + \sin(\phi)$
- $= -k + \sin(\sin^{-1}(k) + \delta\phi)$ and by the calculus rule $\frac{d}{d\theta}[\delta\rho] = -k + \sin(\sin^{-1}(k)) + \cos(\sin^{-1}(k))\delta\phi = \sqrt{1-k^2}\delta\phi$
- And for $\frac{d}{d\theta}[\delta\phi] = \frac{d}{d\theta}\phi - \frac{d}{d\theta}(\sin^{-1}(k)) = \frac{\cos(\phi)}{\rho} - 1$
- Using multivariable hint $f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y$,
- $f = \frac{\cos(\sin^{-1}(k)+\delta\phi)}{\sqrt{1-k^2}+\delta\rho} - 1 \approx \frac{\sqrt{1-k^2}}{\sqrt{1-k^2}} - 1 + \frac{-\sin(\sin^{-1}(k)}{\sqrt{1-k^2}}\delta\phi - \frac{\cos(\sin^{-1}(k))}{1-k^2}\delta\rho$
- $= -\frac{k\delta\phi + \delta\rho}{\sqrt{1-k^2}}$
- So $\frac{d}{d\theta}\begin{pmatrix}\delta\phi \\ \delta\rho\end{pmatrix} \begin{pmatrix} -\frac{k}{\sqrt{1-k^2}} & -\frac{1}{\sqrt{1-k^2}} \\ \sqrt{1-k^2} & 0 \end{pmatrix} \begin{pmatrix}\delta\phi \\ \delta\rho\end{pmatrix}$, and the eigenvalues aren't purely imeginary, and the real part is negative, so all decay. Therefore, the equilibrium at $(\sin^{-1}(k), \sqrt{1-k^2})$ attracts nearby solutions.

- There aren't solutions (for $K < 1$?), but numerically solved, the dog catches at $k > 1$, and for $k \leq 1$, swims out to a path approaching a circle. This is a **limit cycle**, an isolated trajectory that closes on itself.

2.8 Poincare-Bendixson Theorem

Limit cycles in the real world: a chemical reaction in perpetual osciallation!

Key concept - **trapping region**: a region in phase plane on some region D, with differential solutions touching every point, where the direction field sees every boundary arrow point IN. This means:

- The solution has to stay in D.
- Any solution that self-intersects forms a cycle in the phase plane.
- The three conceivable ways a solution can "snake" around forever (the **Poincare-Bendixson theorem** says it):
  - Approaches a closed loop in D.
  - Approaches a fixed point in D (possibly a special case of the last bullet)
  - Cycle: Snake eats its own tail
- A non-cycling solution is the only other possibility - just a point equilibirum.

Example: Chemical oscillatory reaction.

- x is concentration of I^-, y is concentration of ClO_2^- ions in some reaction.
- a is positive, and clearly $x, y \geq 0$ in the physical world.
- Otherwise meaningless equations: $\frac{dx}{dt} = 5a - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(\frac{4y}{1+x^2})$
- Solve for equilibria by setting $\frac{dx}{dt} = \frac{dy}{dt} = 0$
  - Denote $Q = \frac{y}{1+x^2}$
  - First equation implies $x(1 + 4Q) = 5a$
  - Second euqation, plus knowing $x \neq 0$, $\Rightarrow x(1 - Q) = 0 \Rightarrow Q = 1$
  - $Q = 1 \Rightarrow 5x = 5a \Rightarrow x = a$
  - $\Rightarrow 1 = \frac{y}{1+x^2} \Rightarrow y = 1 + a^2$
  - Only solution pair is $(a, 1 + a^2)$

Linearizing the solution around $(a, 1 + a^2)$

- $x = a + \delta x, y = 1 + a^2 + \delta y \Rightarrow \frac{dx}{dt} = \frac{d[\delta x]}{dt}, \frac{dy}{dt} = \frac{d[\delta y]}{dt}$
- Call $f = \frac{d[\delta x]}{dt} = 5a - x - \frac{4xy}{1+x^2}$,
- Approximate $f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y$
- $f(x, y)(a, 1 + a^2) = 5a - x - \frac{4xy}{1+x^2}(a, 1 + a^2) = 5a - a - 4(a\frac{1+a^2}{1+a^2}) = 0$
- $\frac{\delta f}{\delta x}\delta x(a, 1+a^2) = (-1-\frac{(1+x^2)(4y-2x4xy}{(1+x^2)^2})\delta x(a, 1+a^2)) = (-1-4-\frac{8a^2}{1+a^2})\delta x = \frac{-5+3a^2}{1+a^2}\delta x$
- $\frac{\delta f}{\delta y}\delta y(a, 1 + a^2) = \frac{-4x}{1+x^2}\delta y(a, 1 + a^2)) = \frac{-4a}{1+a^2}\delta y$
- Call $g = \frac{d[\delta y]}{dt} = x - \frac{xy}{1+x^2}$
- Approximate $g(x + \delta x, y + \delta y) \approx g(x, y) + \frac{\delta g}{\delta x}\delta x + \frac{\delta g}{\delta y}\delta y$
- $g(x, y)(a, 1 + a^2) = x - \frac{xy}{1+x^2}(a, 1 + a^2) = a - a\frac{1+a^2}{1+a^2} = 0$
- $\frac{\delta g}{\delta x}\delta x(a, 1 + a^2) = (1 - \frac{(1+x^2)y-xy2x}{(1+x^2)^2})\delta x(= 1 - \frac{(1+a^2)^2-2a^2(1+a^2)}{(1+a^2)^2})\delta x = 2a^2\delta x$
- $\frac{\delta g}{\delta y}\delta y(a, 1 + a^2) = \frac{-x}{1+x^2}\delta y = \frac{-a}{1+a^2}\delta y$
- $\Rightarrow \frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \frac{1}{1+a^2}\begin{pmatrix}3a^2-5 & -4a\\ 2a^2 & -a\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$
- Let's arbitrarily choose $a = 2 \Rightarrow (a, 1 + a^2) = (2, 5)$. The coefficient matrix ends up being $\frac{1}{5}\begin{pmatrix}7 & -8\\ 8 & -2\end{pmatrix}$, which has eigenvalues with a positive real $\pm$ some i component. So, Hartman-Grobman applies and we don't decay into our point but push away.

We want to **build the trapping region**.

- Remember, $\frac{dx}{dt} = 10 - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(1 - \frac{y}{1+x^2})$ subbing in 2 for a)
- On the left, if $x = 0$ we see $\frac{dx}{dt} = 10, \frac{dy}{dt} = 0$. So we're pointing right (into the first quadrant region)
- On the bottom, if $y = 0$, we're pointing at $\langle 10 - x, x\rangle$ (into the region).
- On the right, for some $x = b$, $10 - b - \frac{4b}{1+b^2}y$ will make sure we point left.
- On the top, for some $y = c$, $x(1 - \frac{c}{1+x^2} < 0$ makes sure we point down.
- Assume, since we're encircling $(2, 5)$, that $b \geq 3, c \geq 6$ for comfort.
- To satisfy all of these, note $x(1 - \frac{c}{1+x^2}) < 0 \Rightarrow 1 - \frac{c}{1+x^2} < 0 \Rightarrow c > 1 + x^2, 0 < x < b \Rightarrow c > 1 + b^2 \Rightarrow \sqrt{c-1} > b$

- And for $0 < y < c$, note that $10 - x - \frac{4xy}{1+x^2} < 10 - b < 0$.
- Pick $b = 11$, say, implying $11 < \sqrt{c-1}$, so then $123 < c$. So $(b, c) = (11, 124)$ ensures oscillation around $(2, 5)$ without leaving that region.

Tricky: How to reduce this region? No real way except simulation or some tricks. If we PRESUME a cycle, we can prove the cycle extens to the left of $x = 3$ or $x_{min} < 3$

- **META trick**: Don't worry if you have unsolvable integrals - maybe you can cancel them out. **Run with what you have.**
- Trick: Assume $x(t+T) = x(t), y(t+T) = y(t)$ for some $T > 0$, or that there's a PERIOD T.
- $\int_0^T \frac{dx}{dt} dt = x(T) - x(0) = 0, \int_0^T \frac{dy}{dt} dt = y(T) - y(0) = 0$ by fundamental theorem.
- Our equations again: $\frac{dx}{dt} = 10 - x - \frac{4xy}{1+x^2}, \frac{dy}{dt} = x(1 - \frac{y}{1+x^2})$
- So $0 = \int_0^T [10 - \int x(t) - 4 \int \frac{x(t)y(t)}{1+x(t)^2}] dt$ by the first equation
- $0 = \int x(t) - \int \frac{x(t)y(t)}{1+x(t)^2}] dt$ by the second.
- Subtract four times the second from the first to get $0 = 10T - 5\int_0^T x(t) \Rightarrow 2T = \int_0^T x(t) dt \geq int_0^T x_{min} dt = T x_{min}$
- So $2 \geq x_{min}$

2.9 Chaos and the Lorenz Equation

What enabled mathematical **chaos** (unpredictability in nonlinear differential equations) was really computers and seeing simulated solutions.

The (simplified) **Lorenz system** are these equations

- $\frac{dx}{dt} = \sigma(y - x)$
- $\frac{dy}{dt} = x(\rho - z) - y$
- $\frac{dx}{dt} = xy - bz$
- All with $\sigma, \rho, b > 0$

Solving the equations, we see equilibria for these are:

- $(0, 0)$ always
- The two solutions $(\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1)$ when $\rho > 1$.

Looking at $0 < \rho < 1$ specifically:

- Linearaizing is simple, with $x(t) = \delta x(t), y(t) = \delta y(t), z(t) = \delta z(t)$ and linearized system:
- $\frac{d[\delta x]}{dt} = \sigma(\delta y - \delta x)$
- $\frac{d[\delta y]}{dt} = \rho\delta x - \delta y$
- $\frac{d[\delta z]}{dt} = -b\delta z$
- $\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix} \approx \begin{pmatrix}-\sigma & \sigma & 0\\ \rho & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix}$
- Characteristic equation is $(-b-\lambda)[(1+\lambda)(\sigma+\lambda) - \sigma\rho] = 0$
- Eigenvalues are $-b < 0$ and $\lambda = \frac{1}{2}[-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}]$
- If $\rho < 1$, we have distinct, real, negative eigenvalues, and a locally attractive equilibrium by Hartman-Grobman.

But if $\rho < 1$ globally attactive? Find a Liapunov function.

- Requirement is that the function $U(x(t), y(t), z(t))$ is minimized at the equilibrium, and that as time progresses, U decreases (so we're sucked into the bowl)
- We suppose that $U(x, y, z) = ax^2 + y^2 + z^2$ and using $xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$:
  - Identity derivation: $0 \leq (x-y)^2 \Rightarrow 0 \leq x^2 - 2xy + y^2 \Rightarrow xy \leq \frac{1}{2}(x^2 + y^2)$
- $\frac{\delta U}{\delta x}x'(t) + \frac{\delta U}{\delta y}y'(t) + \frac{\delta U}{\delta z}z'(t) = 2ax\sigma(y-x) + 2yx(\rho - z) - 2y^2x + 2zxy - 2bz^2$
- $= 2(a\sigma + \rho)xy - 2a\sigma x^2 - 2y^2 - 2bz^2$
- **GOTHCA**: We can't choose $a = -\frac{\rho}{\sigma}$ since then $U = -\frac{\rho}{\sigma}x^2 + y^2 + z^2$ isn't minimized at $(0, 0, 0)$! So a needs to be positive.
- Choosing $a = \frac{1}{\sigma} \Rightarrow a\sigma = 1$, with $\rho < 1 \Rightarrow 2(a\sigma + \rho)xy - 2a\sigma x^2 - 2y^2 - 2bz^2 < 2a\sigma(2xy - x^2 - y^2) - 2bz^2 \leq -2bz^2$ by the identity above.
- Then $U = \frac{1}{\sigma}x^2 + y^2 + z^2$ decreases as $t \to \infty$ and is minimized at the globally attractive $(0, 0, 0)$

If $\rho > 1$ things get chaotic. Instead of one equilibrium, we have two new ones at $(\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho-1)$. Everything **bifurcates**, or qualitatively shifts when inching past $\rho = 1$:

- We have three equilibria.
- The origin turns into a saddle equilibrium.

- Linearizing around $(\alpha, \alpha, \rho-1)$ with α denoting $\sqrt{b(\rho-1)}$ (pretty straightforward), we get characteristic equation for A of $-\lambda^3-(\sigma+b+1)\lambda^2-b(\sigma+\rho)\lambda-2\sigma b(\rho-1)=0$
- Problem is, setting $\rho=1$ drops the $(\rho-1)$ term and we have $-\lambda(\lambda^2+(\sigma+1+b)\lambda+b(\sigma+1))=0$, with solutions $\lambda=0,-b,-\sigma-1$.
- The last two solutions are attractive, but the zero doesn't work for Hartman-Grobman.
- If we set $\lambda=(\rho-1)\Delta r$ when nudging ρ just over 1, we ignore all λ^2, λ^3... as negligible and get $-b(\sigma+\rho)(\rho-1)\Delta r-2\rho b(\rho-1)\approx 0$
- This means $\Delta r \approx -\frac{2\rho}{\rho+\sigma}$, or that this nudged root has to be negative when ρ is near 1.
- More rigorously, we could have proven the roots of the equation are negative for small $\rho-1>0$
- In any case, this means that the near-zero root is negative, so $(\alpha, \alpha, \rho-1)$ attracts locally.
- We can show that this applies the same for $(-\alpha,-\alpha, \rho-1)$

How do equilibria change as we change ρ?

- We saw the What about as we dial past $\rho=1$, our origin equilibrium changes from globally attractive to saddle point.
- In going from a stable equilibrium with negative real-part eigenvalues (attractors) to $(0,0,0)$ as a saddle (mix of negative and positive real parts), we necessarily have a point where the eigenvalues' real parts are zero.
- In other words, $\lambda=ia$ for some real a.
- Subbing ia into our $-\lambda^3-(\sigma+b+1)\lambda^2 b(\sigma+\rho)\lambda-2\sigma b(\rho-1)=0$, we end up getting $[(\sigma+b+1)a^2-2\sigma b(\rho-1)]+i[a^3-(b(\sigma+\rho)a]=0$
- Then we need $a^3-b(\sigma+\rho)a=0 \Rightarrow a=0, a=\pm, \sqrt{b(\sigma+\rho)}$
- If $a=0$. the real part isn't zero. But subbing $a=\pm\sqrt{b(\sigma+\rho)}$ gives us solutions for a set of $\rho=\frac{\sigma(\sigma+b+3)}{\sigma-b-1}$
- So, when moving past this value, our two new equilibria change from locally attractive to saddles too.

Can we create a trapping region?

- The hint: The solutions have to pass through every ellipsoid of form $R^2=\rho x^2+]sigmay^2+\sigma(z-2p)^2$

- What we need to prove: At every point on the boundary, the direction field points "in", or more specificlally, *the angle between inward normal and direction field is acute.*
- This also means that the gradient ∇g of the level set $R^2 = \rho x^2 + \sigma y^2 + \sigma(z - 2r\rho)^2$ is the normal. This is $\langle 2\rho, 2\sigma, 2(z - 2\rho)\rangle$
- So $-\nabla g \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle > 0 \Rightarrow ... \Rightarrow 2\rho\sigma x^2 + 2\sigma y^2 + 2\sigma z^2 - 4\rho\sigma z > 0$
- Use $R^2 - \rho x^2 - \sigma(z - 2p)^2 = \sigma y^2 \Rightarrow \rho\sigma x^2 + 2\sigma y^2 + 2\sigma z^2 - 4\rho\sigma z > 0$
- This simplifies the dot product to $2R^2 - 8\rho^2\sigma + 4\sigma\rho z + 2\rho(\sigma - 1)x^2 > 0$
- Since the x^2 term is always positive, we just need to set R to clear zero when z is its most negative (x = 0, y = 0, z = $2\rho - \frac{R}{\sqrt{\sigma}}$). If we churn a little more we can see that setting $R > 2\sqrt{\sigma}\rho$ will provide a trapping region.

Question: Do the confined solutions fill up the whole (ellipsoid) container?

- Looking at the divergence (volume change of a cube over time) of the solution will tell us.
- For unspecified reasons, $\frac{1}{v(t)}v'(t) = \nabla \cdot \langle \sigma(y - x), x(\rho - z) - y, xy - bz\rangle = -\sigma - 1 - b$
- Solving, $v(t) = v(0)e^{-t(\sigma+1+b)}$
- This means the volume decays to 0, so therefore, our line is confined to a smaller and smaller space (but not just a point, I guess?)

3 Partial Differential Equations

3.1 1D Wave Equation and D'Lambert's Formula

General set up: A rope with a fixed right end (boundary condition and a moving left end), moving up and down.

Start out with special case: no boundary condition (infinite rope, pulse in the middle)

- $u(x, t)$ measures the vertical displacement form the x-axis of the rope at point x, time t
- Physical observation gives us the PDE rule $\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 u}{\delta t^2}$ (or $u_{xx} = u_{tt}$)
- $g(x) = u(x, 0)$ is the initial shape of the rope.
- It's assumed that the rope is not moving initally, so $u_t(x, 0) = 0$

Beginning to solve this:

- $u_{tt} = u_{xx} \Rightarrow u_{tt} - u_{xx} = 0$
- Sort of like $a^2 - b^2 = 0 \Rightarrow (a+b)(a-b) = 0$, we have $0 = (\frac{\delta}{\delta t} \pm \frac{\delta}{\delta x})(u_t \mp u_x) = u_{tt} - u_{xt} + u_{tx} - u_{xx} = u_{tt} - u_{xx}$
- This means the solution is either $u_+ = u_t + u_x$ or $u_- = u_t - u_x$. Note - we don't solve these simultaneously, since that just gives us $u(x,t) = 0$.
- These can be written as, e.g. $0 = u_t + u_x = \langle 1, 1 \rangle \cdot \langle u_x, u_t \rangle = \langle 1, 1 \rangle \cdot \nabla u$
- TRICK: This is a directional derviative along $\langle 1, 1 \rangle$. Introducing a variable like s (accelerant along $\langle 1, 1 \rangle$?) below does nothing interesting:
  - $\frac{d}{ds}[u(x+sb, t+sc)] = \frac{\delta u}{\delta x}(x+sb, t+sc)b + \frac{\delta u}{\delta t}(x+sb, t+sc)c = \langle b, c \rangle \cdot \nabla u(x+sb, t+sc)$
  - So if we set $b = c = 1$, we see that $\frac{d}{ds}[u(x+s, t+s)] = \langle 1, 1 \rangle \cdot \nabla u(x+s, t+s)$
  - However, since in our world, $u_x + u_t = 0$, then this dot product is zero, and $\frac{d}{ds}u = 0$. This is then *constant in s.*
  - So then $u(x,t) = u(x+s, t+s)$, and shifting x forward by s (seconds?) and t by the same changes nothing. *Interpretation: $u(x,t) = u(x+s, t+s)$ means that s ("shift") seconds later, the point $x+s$ will see the same displacement as x. The wave goes "right" down the line.*
  - From this, we see that $u_+(x,t) = u_+(x-t, 0)$ as well. So, our function at t is what happened t seconds ago at the origin.
- Note: We can't have one solution satisfy both conditions $u_+ = g(x), (u_+)_t = 0$, since then $g'(x) = 0$ which only works if g is a constant.
- Also, $u_{tt} - u_{xx} = 0$ is a linear PDE, in that solutons $u_1(x,t), u_2(x,t)$ see that $\frac{\delta^2}{\delta t^2}[c_1 u_1(x,t) + c_2 u_2(x,t)] - \frac{\delta^2}{\delta x^2}[c_1 u_1(x,t) + c_2 u_2(x,t)] = 0$. Multiply by a constant or add solutions together and it's still zero.
- If we set $t = 0$, we get $u(x,t) = c_+ g(x+t) + c_- g(x-t) \Rightarrow u(x,0) = c_+ g(x+0) + c_-(x-0) = (c_+ + c_-)g(x) = u(x,0)$,so $(c_+ + c_-) = 1$
- Differentiating by t, $u_t(x,t) = c_+ g'(x+t) - c_- g'(x-t)$ so $u_t(x,0) = (c_+ - c_-)g'(x) \Rightarrow (c_+ - c_-) = 0$. So $c_+ = c_- = \frac{1}{2}$, and **our solution with initial shape $g(x)$ with** $g'(x) = u_t(x,0) = 0$ is $u = \frac{1}{2}g(x+t) - \frac{1}{2}g(x-t)$
- **This no-initial-velocity wave function translates** into "my displacement at time 3, say, is the average of the initial displacements 3 to my left and 3 to my right" (as those urges meet at "me" 3 seconds from the start). Conceptually, along the fixed initial curve $g(x)$, each point sends out two sensors, one left, one right, and averages

the initial values at those points to find itself at time t. So the top of a hill will start dipping down, becoming two hills pushing out, for example.

With inverted conditions $u(x,0) = 0, u_t(x,0) = f(x)$, we can use the fact that $u(x,t)$ solving the wave equation implies u_t solves it as well!

- $\frac{\delta^2 u}{\delta t^2} - \frac{\delta^2 u}{\delta x^2} = 0 \Rightarrow \frac{\delta}{\delta t}[\frac{\delta^2 u}{\delta t^2} - \frac{\delta^2 u}{\delta x^2}] = 0 \Rightarrow \frac{\delta^2 u_t}{\delta t^2} - \frac{\delta^2 u_t}{\delta x^2} = 0.$
- Therefore, $u_t(x,0) = f(x)$ admits the same solution $u_t(x,t) = \frac{1}{2}[f(x+t) - f(x-t)]$
- Since $u(x,t) - u(x,0) = \int \frac{1}{2}[f(x+t) - f(x-t)]dt$, and $u(x,0) = 0$ by assumption in this setup, $u(x,t) = \int_{s=0}^{s=t}[f(x+s) - f(x-s)]ds$, which is $\frac{1}{2}\int f(s)$ from $x-t$ to $x+t$ $= \frac{1}{2}\int_{s=x-t}^{s=x+t} f(s)ds$

And because the region of the intergral for a point x gets wider as $t \to \infty$, on a flat rope with a pulse in the middle at $x = 0$, we see $u(x,t)$ sitting at 0 until the wave meets it, at which point it rises and then stays at the peak (integral of the whole thing).

So **d'Alembert's formula** is the superposition of the initially flat wave with the initially still wave, which accomodates *all* solutions:

$$\boxed{u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{s=x-t}^{s=x+t} f(s)ds}$$

For the case of no boundary conditions, this solves $u_{tt} = u_{xx}$ for $u(x,0) = g(x), u_t(x,0) = f(t)$. In this instance, the propagation speed is clearly finite.

Note: This complete of a PDE solution is unusual.

3.2 Sources and Boundary conditions

Scenario 1: Here, we fix the infinite rope at the origin, with the wave coming in from the negative x-axis.

Looking at **boundary conditions**, or constraints on spatial edges of a PDE problem:

- A free boundary (a loop that can shift up and down a pole) will cause a reflected wave to travel backwards.
- A fixed boundary (setting $u(0,t) = 0, t \geq 0$) will cause an inverted pulse backwards.

We set up a function $\tilde{u}(x,t) = \{u(x,t), x \leq 0; = -u(-x,t), x \geq 0\}$ using **extension by odd reflection**. So an inverted ghost rope exists to the right of the origin.

Note: This seems to be more about cleverly encoding a boundary behavior (we will invert our wave) with this ghost rope than proving we'll have that behavior with math.

- And if $u_t(x,0) = 0, g(0) = 0, u(x,0) = g(x)$ extended to $x > 0$ as $\tilde{g}(x)$, then d'Alembert's applies: $\tilde{u}(x,t) = \frac{1}{2}[\tilde{g}(x+t) + \tilde{g}(x-t)]$

- So when $x \leq 0, x \leq t \Rightarrow -t \leq x \leq 0$: (meaning, negative x, close enough to the origin to be affected by time t)
  - $\tilde{u}(x,t) = u(x,t)$ here, since there's no inversion on the left side.
  - $(x+t)$ is positive , so $\tilde{g}(x+t) = -g(-(x+t))$ by definition of $\tilde{u}$.
  - $(x-t)$ is negative , so $\tilde{g}(x-t) = g(x-t)$ by definition of $\tilde{u}$.
  - So d'Alembert's reduces to $u(x,t) = \frac{1}{2}[-g(-(x+t)) + g(x-t)]$. This means *I'm the average of the starting position to my left t seconds ago, and the inverted right-of-origin ghost position to my right t seconds ago*

This means that for the part of the rope we care about, $x \leq 0$:

- For $x \leq -t$ (parts of the line unaffected by the reflection so far), $u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)]$
- For $x \geq -t$ (parts affected by the reflection) $u(x,t) = \frac{1}{2}[-g(-(x+t)) + g(x-t)]$
- The intuition, still hard to visualize: if I'm zero at point -10, and wave crests at -11, then
  - First my left sensor will eat the left wave and I'll go up and over.
  - Then much later my right sensor will eat the right shadow wave and I'll do the inverted behavior.
  - These in total mean I'll get a reflection.
  - For the intuition, keep moving my point closer to the origin - nothing changes.

Scenario 2: Here, we let the rope slide up and down at the origin, but bound the total energy:

- One hand on the rope at $x = -L$, very far away:
- Our energy is the sum of kinetic (change in u based on time?) and elastic (change in u based on x?) energies.
- $E = \int_{x=-L}^{x=0} [(\frac{\delta u}{\delta t})^2 + (\frac{\delta u}{\delta x})^2] dx$
- We can't gain or lose energy. This means $\frac{dE}{dt} = 0$. Solving that:
  - $0 = \frac{d}{dt}(\frac{1}{2}\int_{x=-L}^{x=0} [(\frac{\delta u}{\delta t})^2 + (\frac{\delta u}{\delta x})^2] dx) = \frac{1}{2}\int_{x=-L}^{x=0} [\frac{\delta}{\delta t}(\frac{\delta u}{\delta t})^2 + \frac{\delta}{\delta t}(\frac{\delta u}{\delta x})^2] dx)$
  - $= \int_{x=-L}^{x=0} [u_t u_{tt} + u_x \frac{\delta u_t}{\delta x}] dx)$.
  - (Do integration by parts on the second term with $U = u_x, dV = \frac{\delta u_t}{\delta x}$): $0 = \int_{x=-L}^{x=0} [u_t u_{tt}] dx + u_x u_t - \int_{x=-L}^{x=0} [u_t u_{xx}] dx = \int_{x=-L}^{x=0} u_t [u_{tt} - u_{xx}] dx + u_x u_t$

- Since $u_{tt} - u_{xx} = 0$ (REMEMBER YOUR PROBLEM-SPECIFIC IDENTITIES!), $u_x(0,t)u_t(0,t) = 0$
- Saying the displacement can't change with respect to t there gives us the fixed rope case above, so that's uninteresting.
- Therefore, if there's no energy change as the rope vibrates, we know $u_x(0,t) = 0$

Note: Dirichlet conditions are constraints on the value of the function at the boundary (like $u(0,t) = 0$). Neumann constraints are on the derivatives at the boundary.

So redoing d'Alembert with the energy conservation, and therefore the "Neumann" condition $u_x(0,t) = 0$:

- We know if u solves $u_{tt} - u_{xx} = 0$, then u_x does too, since $0 = u_{tt} - u_{xx} \implies 0 = \frac{d}{dx}[u_{tt} - u_{xx}] = [[u_x]_{tt} - [u_x]_{xx}] = 0$.
- We know $u_x(0,t) = 0$ by given constaints, so then we enforce this through odd reflection on u_x as well: $\tilde{u}_x = \{u_x(x,t), x \leq 0; -u_x(-x,t), x \geq 0\}$
- By D'Alembert, this solves the wave equation with $u_x(x,0) = g'(x)$, so $\tilde{u}_x = \frac{1}{2}[\tilde{g}'(x+t) + \tilde{g}'(x-t)]$
- Therefore at $-t \leq x \leq 0 =, \tilde{u}_x(x,t) = \frac{1}{2}[\tilde{g}'(x+t) + \tilde{g}'(x-t)] = \frac{1}{2}[-g'(-x-t) + g'(x-t)]$
- Then integrating, we drop the minus sign in the first term! $u(x,t) = \frac{1}{2}[g(-x-t) + g(x-t)] + C$. Note that $u(x,0) = \frac{1}{2}[g(x) + g'(x))] \Rightarrow C = 0$!

(Note: a nonzero initial velocity profile $u_t(x,0) = f(x)$ can be handled as well. We skip it).

Remember the 1D springs Wave Equation, where springs are initially l apart, have displacement from this measured by $u(x,t)$, have Hooke's coefficient k?

- Force pushing from the left on ball x: $F_L = k[u(x-l,t) - u(x,t)]$
- Force pushing from the right on ball x: $F_R = k[u(x+l,t) - u(x,t)]$
- Additional "source" force $F(x,t)$ means total force $F_{tot} = F_L(x,t) + F_R(x,t) + F(x,t)$
- $F_{tot} = ma = mu_{tt}$
- The Taylor-ish formula $f(x+\delta x) \approx f(x) + f'(x)(\delta x) + f''(x)(\delta x)^2$ means $F_L + F_R \approx kl^2 f''(x) = kl^2 u_{xx}$
- Therefore, $mu_{tt} = kl^2 u_{xx} + F(x,t) \Rightarrow F(x,t) = u_{tt} - \frac{kl^2}{m} u_{xx}$. Set $1 = v = \frac{kl^2}{m}$, $f(x,t) = \frac{F(x,t)}{m}$ to get a simplified all-purpose wave equation. $f(x,t) = u_{tt} - u_{xx}$, with f as the source force-per-unit-mass.

New setup: Source force $f(x,t)$, ignore boundary conditions, and set $u(x,0) = 0, u_t(x,0) = 0$ (still, flat (infinite) rope).

- Part 1: We can relate $f(x,t)$ to a made-up intermediate function $I(x,t)$ which has properties motivated by $u_{tt} - u_{xx} = (\frac{\delta}{\delta t} - \frac{\delta}{\delta x})(\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$
- $I(x,t) = (\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$
- $u_{tt} - u_{xx} = f(x,t) \Rightarrow (\frac{\delta I}{\delta t} - \frac{\delta I}{\delta x}) = f(x,t)$
- We can derive that $u(x,0) = 0, u_t(x,0) = 0$ means that at $x = 0$, $I(x,0) = u_t(x,0) + u_x(x,0) = u_x(x,0)$
- Since $u(x,0) = 0$ and $I(x,0) = u_x(x,0)$, $I(x,0) = 0$.

We can relate $f(x,t)$ and $I(x,t)$:

- Use the dummy variable trick, and look at $f(x-s, t+s)$. We know also that $\frac{\delta I}{\delta t} - \frac{\delta I}{\delta x} = f(x,t)$
- $f(x-s,t+s) = \frac{\delta I}{\delta t}(x-s,t+s) - \frac{\delta I}{\delta x}(x-s,t+s) = \frac{d}{ds}[I(x-s,t+s)]$ by chain rule.
- Integrating both sides: $\int_{s=-t}^{s=0} f(x-s,t+s)ds = I(x,t) - I(x+t,0) = I(x,t)$
- We can rewrite, using $k = -s$, as $I(x,t) = \int_{k=t}^{k=0} f(x+k,t-k)d(-k) = \int_{s=0}^{t} f(x+s,t-s)ds$

Using the same technique, we can relate $I(x,t)$ and $u(x,t)$ since $I(x,t) = (\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x})$

- Use the dummy variable trick with variable s' and look at $f(x-s', t-s')$. We know also that $\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x} = I(x,t)$
- $I(x-s',t-s') = \frac{\delta u}{\delta t}(x-s,t-s) - \frac{\delta u}{\delta x}(x-s',t-s') = \frac{d}{ds'}[u(x-s',t-s)']$ by chain rule.
- Integrating both sides: $\int_{s'=-t}^{s'=0} f(x-s',t-s')ds = u(x,t) - I(x-t,0) = u(x,t)$
- We can rewrite, using $j = -s'$, as $u(x,t) = \int_{j=t}^{j=0} f(x-j,t-j)d(-j) = \int_{j=0}^{t} I(x-j,t-j)dj$
- So, $u(x,t) = \int_{s'=t}^{s'=0} f(x-s',t-s')ds' = \int_{s'=0}^{t} I(x-s',t-s')ds'$

Combining these, $u(x,t) = \int_{s'=0}^{t} I(x-s',t-s')ds'$, and $I(x,t) = \int_{s=0}^{t} f(x+s,t-s)ds$:

- $I(x-s',t-s') = \int_{s=0}^{t} f(x+s-s',t-s-s')ds$:
- So $u(x,t) = \int_{s'=0}^{t} \int_{s=0}^{t} f(x+s-s',t-s-s')dsds'$

- Change of variables, $\boxed{y = x + s - s', w = s' + s \Rightarrow u(x,t) = \frac{1}{2}\int_{w=0}^{t}\int_{y=x-w}^{y=x+w} f(y, t-w)dydw}$
  - TODO: The $\frac{1}{2}$ term apparently comes from the Jacobian (TODO) $\|\frac{\delta(s',s)}{\delta(w,y)}\|$
- This together means that the *points that can influence* $u(x,t)$ in the $xt-$plane are a triangle with (x,t) as the top, reaching down to $t=0$, slope 1. So the "wave speed" in this setup is 1.

3.3 2D and 3D (Compression) Waves

(Note: The 2D equation will fall out of the 3D one).

Major setup for 3D compression waves:

- Air molecules compress together from sound, so $u(x,y,z,y)$ measures the density of air at that point.
- Let's assume $g(x \pm t)$ plays the same role as last time: the initial wave state. (Note: The setup implies we're looking at waves that propagate at "one x per t".)
- Sound can come from multiple directions, so the expanded version should look like $u(\vec{x},t) = g(\hat{n}\cdot\vec{x} \pm t)$, with $\hat{n}$ some fixed direction in $\mathbb{R}^3$.
- The equation is $\boxed{u_{tt} = u_{xx} + u_{yy} + u_{zz}}$ since:
  - Setting $\hat{n} = \hat{i}$ or any other basis vector, $g(\hat{i}\cdot\vec{x}\pm) = g(x \pm t)$, which means if the other dims are zero, then $u_{tt} = u_{xx}$ (reduce to 1D case). That checks out (necessary, not sufficient)
  - $\frac{\delta}{\delta t}[g(\hat{n}\cdot\vec{x} \pm t)] = \pm g'(\hat{n}\cdot\vec{x} \pm t)$, same for g'' and $\frac{\delta^2}{\delta t^2}$
  - $\frac{\delta^2}{\delta x^2}[g(\hat{n}\cdot\vec{x} \pm t)] = \hat{n}_x^2 g''(\hat{n}\cdot\vec{x} \pm t)$, same for y, z
  - Since $[\hat{n}_x^2 + hatn_y^2 + hatn_z^2 = 1], u_{tt} = u_{xx} + u_{yy} + u_{zz}$ works out.
  - This can also be written $u_{tt} - \nabla^2 u$

Setup with a forcing function:

- $u_{tt} - \nabla^2 u = f(\vec{x},t), u(\vec{x},0) = 0, u_t(\vec{x},0) = 0.$
- So with a still, blank initial state f is going to be a POP at the origin for a brief time.
- Taking our experience from actual sound, we expect it to decrease away from the origin, and for there to be a finite propagation speed.
- It should also be spherically symmetric.

Switching to spherical coordinates (u depends on r, θ, ϕ), and using the multivariable chain rule from vecctor calculus, we get

- $\nabla^2 u = \frac{1}{r^2}\frac{\delta}{\delta r}[r^2 \frac{\delta u}{\delta r}] + \frac{1}{r^2 \sin^2(\phi)}\frac{\delta^2 u}{\delta \theta^2} + \frac{1}{r^2 \sin(\phi)}\frac{\delta}{\delta \phi}[\sin(\phi)\frac{\delta u}{\delta \phi}]$
- If we're taking this to be spherically symmetric, then we can zero out ϕ, θ terms: $\nabla^2 u = \frac{1}{r^2}\frac{\delta}{\delta r}[r^2 \frac{\delta u}{\delta r}]$
- Expanding this out, this means that $u_{tt} - \frac{1}{r^2}\frac{\delta}{\delta r}[r^2 \frac{\delta u}{\delta r}] = u_{tt} - \frac{2}{r}u_r - u_{rr} = f(r,t)$
- If we set $U = ru$, and churn through with e.g. $u_r = \frac{\delta}{\delta r}[\frac{U}{r}] = -\frac{U}{r^2} + \frac{U_r}{r}$, etc., we end up with $U_{tt} - U_{rr} = rf(r,t)$. Note that $\frac{U}{r} = u$ means that the solution diminishes u with distance.
- So now we're solving with $U(r,0) = U_t(r,0) = 0$ since $U = ru$.
- Though only $r \geq 0$ matters, we need to keep $U(0,t)$ at zero through odd reflection. Note that if f is even, $rf(r,t)$ is odd.
- The result from last quiz implies: $U(x,t) = \frac{1}{2}\int_{s=0}^{t}\int_{\rho=r-s}^{\rho=r+s} \rho f(\rho, t-s) d\rho ds$, with s subbing for dummy w and ρ being the distance instead of y. Note that our function is really ρf now instead of f.
- Building a "Dirac delta snap" for a symmetric pop at the origin, set $f(r,t) = \frac{1}{\epsilon}e^{-\pi^2 \frac{r^2}{\epsilon^2}}\chi(t)$ for tiny ϵ
- Define $\delta(\rho) = \frac{\exp(-\frac{\pi^2 \rho^2}{\epsilon^2})}{\epsilon^2}$, change $s' = t - s$
- Eventually the math reduces to a delta pop at $r - t + s'$ (in range) and $r + t - s'$ (outside the interval)
- The math reduces to $\int_{s'=0}^{s'=t} \chi(s')\delta(r - t + s')ds'$, or just $\chi(t - r)$ (due to the integral of δ being one exactly at $t - r$.
- Therefore, $\boxed{U(r,t) = \{\frac{\epsilon^3}{(2\pi)^2}\chi(t-r), t - r > 0; 0, t - r \leq 0\}}$
- Looking at this, we confirm that disturbance diminishes with distance, and has a finite propagation speed.
- *TODO: So I guess χ is the actual initial wave function of time at the origin?* The delta was I suppose there to "center" it?

What if we don't have spherical symmetry?

- in the general case, all points $\vec{x}$ influence fixed point $\vec{P}$ through
  - Distance separating points $r = \|\vec{x} - \vec{P}\|$

- Normalized direction $\frac{\vec{x}-\vec{P}}{\|\vec{x}-\vec{P}\|}$

- However, we can *average* u over all points r away: $U(r,t;P) = \frac{1}{4\pi r^2}\iint_{S(\vec{P},r} u(\vec{x},t)d\sigma(\vec{x},$ with S being the r-sphere around P
- Getting the r-partials requires writing each $\vec{x}$ as some $\vec{P}+r\hat{n}$ over all directions, and using the divergence theorem.
- LOTS OF ALGEBGRA IN HERE to get $U_{tt}-U_{rr}-\frac{2}{r}U_r = F(r,t)$ with a dependence on $r,t.\vec{P}$. This F is an even function, and it approaches $u(x,t)$ as r approaches zero.
- Our equation ens up being $u(\vec{P},t) = \frac{1}{4\pi}\iiint_{B(\vec{P},t)} \frac{f(\vec{y},t-\|\vec{y}-\vec{P}\|)}{\|\vec{y}-\vec{P}\|}d\vec{y}$
- Like the other case, we see that we points in space affecting U are a "4D cone" with vertex at $(\vec{P},t)$
- Also, if we consider that f doesn't depend on z, we can flatten this spherical integral to a 2D one by looking at columns of z over the disc of radius t
- This ends up being $u(\vec{P},t) = \frac{1}{2\pi}\iint_{B_2(\vec{P},t)} f(\vec{y})\ln(\sqrt{(\frac{t}{\|\vec{y}-\vec{P}\|})^2-1}+\frac{t}{\|\vec{y}-\vec{P}\|})d\vec{y}$
- This **method of descent** is really just "reducing" our 2D case from a 3D one.
- Also, the **spherical averages** let us reduce a 3D problem to a 1D problem, given the assumptions of the problem.

3.4 2D waves (boundary constrained): Separation of Variables

Main idea: "Guess" that a function like $u(x,t)$ can be factored into $u(x,t) = X(x)T(t)$ and work from there. You can do this recursively as well like $u(x,y,t) = S(x,y)T(t), S(x,y) = X(x)Y(y)$.

Main Setup:

- Rectangular drumhead from $[0,0]$ to $[w,l]$
- Vertical (z) displacement is $u(x,y,t)$, with Dirichlet condition $u(x,y,t) = 0$ enforced on the boundary.
- Known that $u_{tt} = u_{xx} + u_{yy}$.

Solving for u by guessing that there's a split solution $u(x,y,t) = S(x,y)T(t)$.

- $u_{tt} = u_{xx} + u_{yy}$.
- So $S(x,y)T''(t) = \frac{d^2S(x,y)}{dx^2}T(t) + \frac{d^2S(x,y)}{dy^2}T(t)$
- $\Rightarrow \frac{T''(t)}{T(t)} = \frac{\nabla^2 S(x,y)}{S(x,y)}$

- **Big A-ha**: left hand side is a function of t, and right hand of x,y. If they are to be equal, they must both be consta

To solve for T:

- We know $\frac{T''(t)}{T(t)}$ is a constant, so equate it to $-k^2$ for some constant k.
- Supposing the solution $T(t) = e^{rt} \Rightarrow T'' = r^2 e^{rt}$, we know $r^2 = -k^2 \Rightarrow r = \pm ik$.
- The solution $T(t) = Me^{ikt} + Ne^{-ikt}$ is equivalent to $T(t) = A\cos(kt) + B\sin(kt)$ since you can express $\sin, \cos$ as linear combos of $e^{ikt} = \cos(kt) + i\sin(kt)$ and $e^{-ikt} = \cos(kt) - i\sin(kt)$

To solve for X, Y:

- $\frac{\nabla^2 S(x,y)}{S(x,y)} = -k^2$ by its equality with $\frac{T''(t)}{T(t)}$.
- Suppose we can formulate a solution so that $S(x, y) = X(x)Y(y) \Rightarrow -k^2 X(x)Y(y) = Y(y)X''(x) + Y''(y)X(x) \Rightarrow -k^2 = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}$.
- So, for some j, $\frac{X''(x)}{X(x)} = -j^2 \Rightarrow X(x) = C\cos(jx) + D\sin(jx)$ by the T solution above.
- However, $X(x) = 0$ and $X \neq 0$ (not a constant function) forces us to conclude $C = 0 \Rightarrow X(x) = D\sin(jx)$.
- Also, the boundary condition $X(w) = 0$ also means for every $j = \frac{n\pi}{w}$, $n \in \mathbb{N}$, $X(x) = D\sin(\frac{\pi n}{w}x)$
- Following this identical logic for Y over length l, y $q = \frac{m\pi}{l}$,$Y(y) = F\sin(\frac{\pi m}{l}x)$
- Considering that $-k^2 = -j^2 - q^2$, this means $k = \sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}$
- Then, $u = T(t)X(x)Y(y)$
- $\Rightarrow u = (A_{mn}\cos(\sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}) + B_{mn}\sin(\sqrt{(\frac{\pi n}{w}x)^2 + (\frac{\pi m}{l}x)^2}))(\sin(\frac{\pi n}{w}x))(\sin(\frac{\pi m}{l}x))$
- This runs over all $m, n \in \mathbb{N}$

Turning to a **circular membrane** with radius r_0, displacement described by $z = u(r, \theta, t)$:

- Boundary condition is then Dirichlet condition $u(r_0, \theta, t) = 0$
- $u_{tt} = \nabla^2 u \implies u_{tt} = u_r r + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$
- (Note: I suppose this magic (forgotten from vector calculus) is because $u_{tt} = u_{xx} + u_{yy}$, and we're converting between x, y and r, θ)
- $T(t) = A\sin(kt) + B\cos(kt)$ by identical logic to the rectangular drum, where $\frac{T''(t)}{T(t)}$ also was a constant.

- Assuming similarly that $S(r,\theta) = R(r)\Theta(\theta)$, you end up with $R(r)\Theta(\theta)[\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2] = 0$
- Finally, if $\frac{\Theta''}{\Theta} = \kappa$ for $\kappa \in \mathbb{R}$, then $\Theta(\theta) = Ce^{\sqrt{\kappa}\theta} + De^{-\sqrt{\kappa}\theta}$. H
- However, we have an additional condition that we have to be able to rotate the whole scene by $\theta = m2\pi$ radians and have it remain the same, or $\Theta(\theta + 2\pi) = \Theta(\theta)$. This means $\kappa < 0$ since we're in "imaginary exponents yielding sign an cosine" territory.
- This implies $\kappa = -m^2$ for an integer m, and therefore, $\boxed{\Theta(\theta) = C\cos(m\theta) + D\sin(m\theta)}$.
- We know also that $\boxed{T(t) = A\cos(kt) + B\sin(kt)}$
- Setting $\frac{\Theta''}{\Theta} = \kappa = -m^2$, and with $R(r)\Theta(\theta)[\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2] = 0 \Rightarrow \boxed{r^2R'' + rR' + [k^2r^2 - m^2]R}$ (a solution we'll look for later), we have the circular drum solution.

Notes to self:

- Need a clear intuition for a lot of things. What do the variables and their derivatives physically mean?
- Need more symbolic comfort with how integration works.

3.5 Fundamental Solutions

Main idea: The fundamental solution for the problem seems to:

- Solve the differential equation. Here, it is $u_t = \nabla^2 u$ or, specifically, $u_t = u_{xx} + u_{yy}$
- Solve the initial conditions
- More specifically, as $t \to 0_+, u(x,y,t) \to g(x,y)$, or we can "rewind back" to the initial displacement setup.
- Somehow , this is the function from which all heat equations (presumably specified as $g(x,y)$?) are built.

Motivation for heat equation (random 1D walk)

- Drunkard starts at lamppost (position 0) and walks left or right every Δt, each with probability $\frac{1}{2}$
- Probability of being i steps away from lamppost at time $n\Delta t$ is $p(i, n\Delta t)$.
- This depends only on $p(i-1, n\Delta t), p(i+1, n\Delta t)$ as $p(i,(n+1)\Delta t) = \frac{1}{2}p(i-1, n\Delta t) + \frac{1}{2}p(i+1, n\Delta t)$

- We can do the same thing we did the the 1D spring equation and approximate p_x, p_{xx} by means of its relation to small perturbances. Then we can relate this to p_t.
  - Assume we're going to shrink these moves to ΔX instead of 1.
  - We know $p(x,(n+1)\Delta t) = \frac{1}{2}p(x+\Delta x, n\Delta t) + \frac{1}{2}p(x-\Delta x, n\Delta t)$.
  - We know the calculus rule for smooth f, small Δx: $f(x+\Delta x) \approx f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2$
  - Applying this, this means $\frac{1}{2}p(x,(n+1)\Delta t) + \frac{1}{2}p(x-\Delta x, n\Delta t) = \frac{1}{2}[p(x,n\Delta t) + p_x(x,n\Delta t)\Delta x + \frac{1}{2}p_{xx}(x,n\Delta t)(\Delta x)^2] + \frac{1}{2}[p(x,n\Delta t) + p_x(x,n\Delta t)\Delta - x + \frac{1}{2}p_{xx}(x,n\Delta t)(\Delta - x)^2] = p(x,n\Delta t) + \frac{1}{2}[\Delta x]^2 p_{xx}(x,n\Delta t)$
  - So for small steps, $p(x,(n+1)\Delta t) \approx p(x,n\Delta t) + \frac{1}{2}[\Delta x]^2 p_{xx}(x,n\Delta t)]$. *So we have this rule for p relating to derivatives along x.*
  - Now let's relate to derivatives along t! For small Δt, the definition of $\frac{\delta p}{\delta t} = \frac{p(x,(n+1)\Delta t) - p(x,n\Delta t)}{\Delta t} \approx \frac{[\Delta x]^2}{2\Delta t}\frac{\delta^2 p}{\delta x^2}(x,n\Delta t)$
  - Special case: If we set $t = n\Delta t$, then in this case, we can say $\frac{\delta p}{\delta t} = D\nabla^2 p, D > 0$ if $D = \frac{[\delta x]^2}{2\Delta t}$, the **diffusion equation**. Note: Since we're pushing $\Delta x, \Delta t \to 0$, is this, uh, $D \approx \frac{[\Delta x]^2}{2\Delta t}$?

We consider that p should be peaked (sharply) around 0, since it's just as likely to go left as right. We hypothesize that we should use a (sharp) bell curve to make that happen. (Note: Is this sort of like the Dirac Delta function?)

- Looking at $D = \frac{[\Delta x]^2}{2\Delta t}$, *we hypothesize that the operative units must be* $\frac{x^2}{t}$
- Therefor rour bell curve looks like $p(x,t) = C(t)\exp\{-\frac{x^2}{\sigma^2 t}\}$
- Since it's a probability measure along the real line, $\int_{x=-\infty}^{x=\infty} p(x,t)dx = 1$. At time t, the particle is at *some* x.
- Using identity $\int_{u=-\infty}^{u=\infty} e^{-au^2}du = \sqrt{\frac{\pi}{a}}, a > 0 \Rightarrow a = \frac{1}{\sigma^2 t} \Rightarrow C(t) = \frac{1}{\sigma\sqrt{\pi t}}$
- Even though our guess was unsubstantiated, we see $p(x,t) = \frac{1}{\sigma\sqrt{\pi t}}\exp\{-\frac{x^2}{\sigma^2 t}\}$ solves $p_t = Dp_{xx}$. To solve for D:
- To solve for D, find $p_t = \frac{\delta}{\delta t}[\frac{1}{\sigma\sqrt{\pi t}}\exp\{-\frac{x^2}{\sigma^2 t}\}]$, same for p_{xx}. Through lots of chain rule churning, we see $p_t = \frac{\sigma^2}{4}p_{xx} \Rightarrow D = \frac{\sigma^2}{4} \Rightarrow \sigma = 2\sqrt{D}$. *Note that p can't be negative, so ignore* -2$\sqrt{D}$.
- This implies our solution is $p(x,t) = [\frac{1}{\sqrt{4\pi Dt}}\exp\{-\frac{1}{4D}\frac{x^2}{t}\}], t > 0$

To say our solution is a **fundamental diffusion solution** means, further, that we can start with any initial conditions $p(x,0) = g(x)$, watch the equation unfold over t, and we'll still have $p_t = Dp_{xx}$.

- A **convolution** $(\Phi \star g)(x,t)$ of (probability function) Φ and initial state g is the function that combines the two at point x, or $(f \star g)(x) \int_{y=-\infty}^{y=\infty} f(x-y)g(y)dy = \int_{y=-\infty}^{y=\infty} f(y)g(x-y)dy = ...$ (TODO)
- In this case, $(\Phi \star g) = p(x,t) = \int_{y=-\infty}^{y=\infty} \frac{1}{\sqrt{4\pi Dt}} \exp\{-\frac{1}{4D}\frac{(x-y)^2}{t}\}g(y)dy, t > 0$
- It has the properties:
  - $\lim_{t\to 0^+} p(x,t) = g(x)$ for every reasonable choice of $p(\vec{x},0) = g(\vec{x})$.
  - $p_t = Dp_{xx}$
- Note we can find a fundamental solution for the heat equation $u_t = \nabla^2 u$ as well
  - For some v, we can hypothesize a distribution function $\Phi(\vec{x},t) = t^{-a}v(\frac{\|x\|^2}{t})$
  - Taking derivates $\frac{\delta}{\delta t}, \frac{\delta^2}{\delta x_i^2}$ lets us use $u_t = ux_0x_0 + u_{x_1x_1} + ...$
  - Calling the argument $z = \frac{\|x\|^2}{t}$, we take derivatives and find $0 = av(z) + (z + 2n)v'(z) + 4zv''(z)$. Note that n is from $\mathbb{R}^n$, since $\nabla(\|\vec{x}\|^2) = 2\vec{x}$ and $\nabla(2\vec{x}) = 2n$.
  - How do we choose a so that the equation can be solved as $[\frac{1}{4}v(z) + v'(z)$?
  - Note that if $[\frac{1}{4}v(z) + v'(z)] = 0$, then its derivative $\frac{d}{dz}[\frac{1}{4}v(z) + v'(z)] = [\frac{1}{4}v'(z) + v''(z)] = 0$
  - Setting $a = \frac{n}{2}$, we get $0 = 2n[\frac{1}{4}v(z) + v'(z)] + \frac{z}{4}\frac{d}{dz}[\frac{1}{4}v'(z) + v''(z)]$
  - We can rearrange to see $[av(z) = 2nv'(z)] + z[v'(z) + 4v''(z)] = 0$.
  - Therefore the solution is $\Phi(\vec{x},t) = t^{-\frac{n}{2}}v(\frac{\|x\|^2}{t})$. To find v, solve $[\frac{1}{4}v(z) + v'(z)] = 0$
  - This is obviously $v(z) = Ce^{-frac14z}$, or $\Phi(\vec{x},t) = Ct^{-\frac{n}{2}}v(\frac{\|x\|^2}{t})$, but we must find C so that $\int \Phi = 1$.
  - The identity $\int_{\mathbb{R}^n} e^{-a\|x\|^2} = (\frac{\pi}{a})^{\frac{n}{2}}$ helps us find $\lim_{t\to 0^+} \frac{C}{t^{-\frac{n}{2}}} \int e^{-\frac{\|x\|^2}{4t}} d\vec{x} = \lim_{t\to 0^+} \frac{C}{t^{-\frac{n}{2}}}(4\pi t)^{\frac{n}{2}} = C(4\pi)^{\frac{n}{2}}$
  - So the fundamental heat solution is $\Phi(\vec{x},t) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{\|x\|^2}{4t}}$
  - As an example, take $g(x,y) = u(x,y,0) = u_0e^{-x^2+y^2}$, like a candle had heated the origin.

- Solving $u(x,y) = (\Phi \star g)(\vec{x},t) = \frac{u_0}{4\pi t}\int_{\mathbb{R}^2} e^{\frac{-\|x-y\|^2}{4t}} e^{-\|y\|^2} d\vec{y}$ basically requires completing the square to get rid of the $2\vec{x}\cdot\vec{y}$ components, then changing variables $\vec{v} = \vec{y} - \frac{1}{1+4t}\vec{x}$, which, over the whole plane, is the same integral.
- You end up with $u(x,y,t) = \frac{u_0}{1+4t} e^{-\frac{x^2+y^2}{1+4t}}$
- Computing partial derivatives u_t, u_{xx}, u_{yy}, checking $u_t = u_{xx} + u_{yy}$ and seeing that $\lim_{t\to 0^+} u(x,y,t) = g(x,y)$ validates it.
- And it turns out, using identity $\int_{\mathbb{R}^2} e^{-a(x^2+y^2)} dxdy = \frac{\pi}{a}$ (the square of the 1d case), that $u_tot = \frac{u_0}{1+4t}\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{1+4t}} dxdy = \frac{u_0}{1+4t}(\pi(1+4t)) = \pi u_0$, so the total energy doesn't depend on time. It is conserved.

3.6 Fun with Functionals

Note that the heat solution ($u_t = nabla^2 u$) and the diffusion equation ($p_t = D\nabla^2 p$) are very similar. The first measures how much heat (density) $u(x,y,z,t)$ exists at a spot u in time; the second measures the probability density of a (particular) particle being in that spot at a time. This means *we can think about one in terms of the other.*

If, for example, we confined our space G to an ellipsoid (or really, any closed region) in $\mathbb{R}^3$, then saying "it's insulated to any heat going in our out" and "particles are confined to not leave or enter" is the same thing. They would have these things in common:

- The solutions *feel* like they should settle down to a constant, uniform value as $t \to \infty$
- The transfer at the "skin" δG is zero so on the boundary, for normal $\hat{n}, D_{\hat{n}}|_{\delta G} = \nabla u \cdot n|_{\delta G} = 0$. Otherwise, the heat transfer, say, would be nonzero. (Note that this is an example of a Neumann condition.)
- This means that $\frac{d}{dt}\iiint_G u d\vec{x} = 0$. For one, no heat entering or leaving means the total has to stay the same. Also consider that $\frac{d}{dt}\iiint_G u d\vec{x} = \iiint_G u_t d\vec{x} = \iiint_G \nabla^2 u d\vec{x}$ (by heat equation definition) $= \iiint_G \nabla \cdot (\nabla u) d\vec{x} = \iint_{\delta G} \nabla u \cdot \hat{n} d\sigma(\vec{x})$ (by divergence them on function ∇u) $= 0$.
- We can define a cost function $C = \frac{1}{2}\iiint_G [(\nabla u)\cdot(\nabla u)] d\vec{x} = \frac{1}{2}\iiint_G [(\nabla u)]^2 d\vec{x}$, since this is zero exactly when $\nabla u \cdot \nabla u = 0$ all over the space.

The idea of **functionals** is that these setups for "energy" ($E[u] = \iiint_G u d\vec{x}$) and "cost" ($C[u] = \frac{1}{2}\iiint_G [\nabla u]^2 d\vec{x}$) take in *functions* to become scalar-producing functions themselves.

We can prove a few things: the cost decreases over time, TODO

Cost decreases over time:

- $\frac{d}{dt}[\frac{1}{2}\iiint_G [\nabla u]^2 dx]$
- $= \frac{1}{2}[\iiint_G \frac{\delta}{\delta t}[\nabla u]^2 dx]$ by... Fubini's maybe ?
- $=\iiint_G [\nabla u \cdot \nabla u_t] dx$ by the chain rule (applied to each dimension, basically)
- Do a multivariable product rule $\nabla \cdot [fV] = \nabla f + V \cdot f \nabla \cdot V \Rightarrow \nabla f \cdot V = \nabla(fV) - f\nabla V$, setting $f = u_t, V = \nabla u$
- This gives $C[u] = \frac{1}{2}\iiint_G [\nabla u]^2 dx = \iiint_G \{\nabla \cdot [u_t \cdot \nabla u] - u_t \nabla \cdot \nabla u\} dx$
- Using divergence on the first term gives $= \iiint_G \{\nabla \cdot [u_t \cdot \nabla u] = \iint_{\delta G} [u_t \cdot \nabla u] = 0$ by the boundary Neumann conditions
- The second term $-\iiint_G (u_t \nabla \cdot \nabla u) dx = -\iiint_G (u_t \nabla^2 u) dx = -\iint_G [\nabla^2 u]^2 dx$ by definotion of heat equation. The integrand must be non-negative, so the whole thing has to decrease or stay constant.

Note that if $\iint_G [\nabla^2 u]^2 dx = 0$ (exact equality), then u must be constant.

- The above equation implies that $\nabla^2 u = 0$ throughout G.
- Use the product rule in reverse again, with $f = u, V = \nabla u : \nabla \cdot (u \nabla u) = \nabla u \cdot \nabla u + u \nabla^2 u$
- We proved the second term is zero. (The first term is equivalent to $\|\nabla u\|^2$, by the way.)
- The divergence theorem say $\iiint_G \nabla \cdot (u \nabla u) dx = \iint_{\delta G} u \nabla u \cdot \hat{n} d\sigma = 0$.
- Therefore the cost $C[u]$ is 0, therefore u is constant.

If it's an inequality, $C[u]$ must decrease forever.

Example: Unit cube $[0,1]\times[0,1]\times[0,1]$, intial condition $u(x,y,z,0) = u_1 \cos(\pi x)\cos(2\pi y)\cos(3\pi z) + u_0$

- Note that the boundary does satisfy $\nabla u \cdot \hat{n}|_{\delta G} = 0$:
- Notice that $\nabla u = \langle K \sin(\pi x), L \sin(2\pi y), M \sin(3\pi z)\rangle$ for some messy constants, and those are all 0 at $x, y, z \in \{0, 1\}$ (the boundaries)
- Also, we can solve this equation $u_t = \nabla u$:
  - Note that $u_x x = -\pi^2(u - u_0), u_y y = -4\pi^2(u - u_0), u_y y = -9\pi^2(u - u_0)$, so $\nabla^2 u = -14\pi^2(u - u_0)$
  - $\frac{du}{dt} = 1 - 4\pi^2(u - u_0)$
  - $\frac{du}{(u-u_0)} = -14\pi^2 dt$

  - $\ln(u - u_0) = -14\pi^2 t + C$
  - $u = u_0 + De^{-14\pi^2 t}$. Only $D = u_1 \cos(\pi x)\cos(2\pi y)\cos(3\pi z)$ satisfies initial conditions.
  - So $u = u_0 + u_1 \cos(\pi x)\cos(2\pi y)\cos(3\pi z)e^{-14\pi^2 t}$

- We see also that energy is conserved, or that $E[u] = \iiint_G u(x,y,z,t) = \iiint_G u_0 + u_1[\int_0^1 \cos(\pi x)] \times [\int_0^1 \cos(2\pi y)] \times [\int_0^1 \cos(3\pi z)] = u_0$, since the factors (we can separate into them easily) are zero on this $[0,1]$ interval.
- Finally $C[u] = \frac{1}{2}\iiint_G [\nabla u]^2 dx$ for $u = u_0 + u_1 \cos(\pi x)\cos(2\pi y)\cos(3\pi z)e^{-14\pi^2 t}$ decreases over time approaching (but in general not hitting) 0, since $[\nabla u]$ is some function $u_1 e^{-14\pi^2 t}[-\pi \sin(...)\cos(...)\cos(...) + ..]$, and $[\nabla u]^2$ is $u_1^2 e^{-28\pi^2 t}$ times something less than one. The integral is therefore less than $u_1^2 e^{-28\pi^2 t}$, and trends toward zero.

3.7 Laplace Equation

Motivation: Soap bubble created over an (arbitrary) wire loop. Physics dictates this is of smallest area (**minimal area surfaaces**).

Laplaces's equation $\nabla^2 u = 0$

Setup:

- Disk $D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq \mathbb{R}^2\}$
- Function $h(x,y) > 0$ gives height of the wire at any point on δD.
- So what is the function $u(x,y)$ with $u|_{\delta D}$ that describes the soap bubble?
- Note that area is $A[u] = \iint_D \sqrt{1 + (\frac{\delta u}{\delta x})^2 + (\frac{\delta u}{\delta y})^2} dxdy$
  - TODO: Re-learn surface integrals again!
- *Lower bound* on $A[u]$ is necessarily πR^2, since $\sqrt{1 + (\frac{\delta u}{\delta x})^2 + (\frac{\delta u}{\delta y})^2} \geq \sqrt{1}$, and D is of area πR^2.
- *Upper bound* on $\sqrt{1 + (\frac{\delta u}{\delta x})^2 + (\frac{\delta u}{\delta y})^2}$, if u is super spiky, I guess:
  - Start with known theorem: *Holder inequality* : $|\iint_D f(x,y)g(x,y)dxdy| \leq \sqrt{\iint_D f(x,y)^2 dxdy}\sqrt{\iint_D g(x,y)^2 dxdy}$
  - Set $f = \sqrt{1 + (\frac{\delta u}{\delta x})^2 + (\frac{\delta u}{\delta y})^2} \equiv \sqrt{1 + \langle \frac{\delta u}{\delta x}, \frac{\delta u}{\delta y}\rangle \cdot \langle \frac{\delta u}{\delta x}, \frac{\delta u}{\delta y}\rangle}$
    $\equiv \sqrt{1 + \nabla u \cdot \nabla u}$ (written as $\sqrt{1 + [\nabla u]^2}$)

- Set $g = 1$
- Then $A[u] = \iint_D fg dxdy \leq \sqrt{\iint_D (\sqrt{1+[\nabla u]^2})^2 dxdy}\sqrt{\iint_D (1)^2} dxdy$
  $= \sqrt{\iint_D (1+[\nabla u]^2) dxdy}\sqrt{\pi R^2}$
- Square both sides and divide by πR^2 to get $\frac{A[u]^2}{\pi R^2} \leq \iint_D [1+[\nabla u]^2] dxdy = \pi R^2 + \iint_D [\nabla u]^2 dxdy$
- So we find our upper bound: $0 \leq \frac{A[u]^2 - \pi R^2}{\pi R^2} \leq \iint_D [\nabla u]^2 dxdy$

- The upshot: We have to choose a u bounded by the wire that makes $A[u]$ as close to πR^2 as possible.
- We can make last quiz's cost functional $C[u] = \frac{1}{2}\iint_D [\nabla u]^2 dxdy$ (look familiar?) as small as possible.
- The main idea of *nudges*: a 1D function $f(x)$ is at a minimum at x_0 if $f(x_0+\eta) > f(x_0)$ for small x_0.
- Instead of a bump η, we'll use a function bump (in display, looks almost like a Dirac delta) that distorts our function u, but still $\eta = 0$ around δD to maintain our conditions.
- So a "minimizing" function (? that minimizes $C[u]$ on this loop?) u, with a minimum at $u_0(x,y)$ will necessarily be a minimum if $M = 0$ for $C[u_0(x,y) + M\eta(x,y)]$. This means $\frac{d}{dM}|_{M=0} C[u_0(x,y) + M\eta(x,y)] = 0$
- Since $C[u] = \frac{1}{2}\iint_D [\nabla u]^2 dxdy$ We can find $\frac{dC}{dM}$ at 0:
  - $\nabla[u_0(x,y) + M\eta(x,y)] = \nabla u_0(x,y) + M\nabla\eta(x,y)$
  - $\nabla[u_0(x,y)+M\eta(x,y)]^2 = \|\nabla u_0(x,y)\|^2 + M^2\|\nabla\eta(x,y)\|^2 + 2M\nabla u_0(x,y)\cdot\nabla\eta(x,y)$
  - Integrating the above and dividing by 2 we we $C[u_0(x,y) + M\eta(x,y)] = C[u] + M^2 C[\eta] + M\iint_D \nabla u_0(x,y)\cdot\nabla\eta(x,y)$
  - At $M = 0$, $\frac{dC}{dM} = \iint_D \nabla u_0(x,y)\cdot\eta(x,y)$
  - Since we know this is 0, then we know $\iint_D \nabla u_0 \cdot \nabla\eta = 0$
  - So at $M = 0$, the derivative w.r.t. M is 0, and therefore $\iint_D \nabla u_0 \cdot \nabla\eta = 0$
- Now we use this to prove that our minimizing function u_0 obeys Laplace: $\nabla^2 u_0 = 0$
  - If we have a mix of various ∇ equations, it's often useful to see what we pieces we can play with by using the product rule for gradients: $\nabla\cdot(\eta\nabla u_0) = \eta\nabla^2 u_0 + \nabla u_0 \cdot \nabla\eta$.

- Also, we have the divergence theorem $\iint_D \nabla V = \int_{\delta D} V_r$. (I suppose V_r substitutes for $V \cdot \hat{n}$).
- So integrate everything over D: $\iint_D \nabla \cdot (\eta \nabla u_0) = \iint_D \eta \nabla^2 u_0 + \iint_D \nabla u_0 \cdot \nabla \eta$.
- First term is $\int_{\delta D} \eta \nabla (u_0)_r dl$ by Divergence, but $\eta \to 0$ near δD, so it's 0. Third is 0 by previous result.
- So $\iint_D \eta \nabla^2 u_0 dxdy = 0$ for ANY bump function η. Then if $\nabla^2 u_0 \neq 0$ at some point, we can design an η, say a Dirac delta, that makes $\eta \nabla^2 u_0 \neq 0$ there, and 0 elsewhere. But since $\iint_D \eta \nabla^2 u_0 dxdy = 0$ always, this means that $\nabla^2 u_0 = 0$ everywhere!

• Just like making infinitesimal changes to functions in calcuus, this whole idea of "bumping" a functional forms the basis of **calculus of variations**, which is calculus on functionals. We learned that any function that minimizes $C[u]$ with fixed boundary values $h(x, y)$ on δD, that $\boxed{\text{Laplace's equation } \nabla^2 u = 0}$ holds. Solutions of Laplace's equation are called **harmonic** functions.

To find a solution for Laplace, use technique of *spherical averages*:

• Around any point P, fix an r and the corresponding disk $D = \{(x, y) \in \mathbb{R}^2 | (x - P_x)^2 + (y - P_y)^2 \leq r^2\}$
• The average of u on that boundary δD is then $U(r; P) = \frac{1}{2\pi r} \int_{\delta D(P,r)} udl$
• How do we find how this function changes (derivative) with respect to r? **GOTCHA**: Can't crank through using r directly since r is in the limits of integration! Solution: fix r and integrate over θ.
  - Set $\delta D(P, r) = \{(P_x + r\cos(\theta), P_y + r\sin(\theta)\}$.
  - Then $U(r; P) = \frac{1}{2\pi r} \int_{\theta=0}^{\theta=2\pi} u(P_x + rcos(\theta), P_y + r\sin(\theta))rd\theta$.
  - (Note: $dl = rd\theta$, so the r sneaks in there.
  - Since the integral is over θ, cancel r:$U(r; P) = \frac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} u(P_x + rcos(\theta), P_y + r\sin(\theta))d\theta$.
  - So we want $U'(r; P) = \frac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} \frac{d}{dr} u(P_x + rcos(\theta), P_y + r\sin(\theta))d\theta$.
  - Notice this is $\nabla u(P_x + rcos(\theta), P_y + r\sin(\theta)) \cdot \langle \cos(\theta), \sin(\theta) \rangle$
  - That means we can use the divergence theorem as $U'(r; P) = \frac{1}{2\pi r} \int_{\delta D(P,r)} \nabla [u]_r dl = \frac{1}{2\pi r} \iint_{D(P,r)} \nabla^2 udxdy$
  - The last term is 0 by supposition of the Laplace condition, so $U'(r; P) = 0$

- So the average is constant in r, which means by shrinking $\lim_{r \to 0^+} U(r; P)$, we have to get $u(P)$. The **mean value property** of a harmonic ($\nabla^2 u = 0$) function u says that the point's value is the average of the values around it.
- Logically, this means that either the whole function is constant, or that the max (min) has to occur on a boundary.
- This "average of the surrounding circle" has to extend to surrounding disks too:
  - $\frac{1}{2\pi}_{\delta D(P,r)} \int udl = ru(P)$ is the integral over a circle of radius r.
  - Integrating all of those from $r = 0$ to $r = r_0$: $u(P) \int_{r=0}^{r=r_0} rdr = u(P)\frac{r_0^2}{2}$
  - But also equals $\frac{1}{2\pi} \int_{r=0}^{r=r_0} [\int_{\delta D(P,r)} udl]dr = \frac{1}{2\pi} \iint_{D(P,r_0)} udxdy$
  - So $u(P) = \frac{1}{\pi r_0^2} \iint_{D(P,r_0)} udxdy$. So $u(P)$ is the average of the u-values of the disk too.

Example: Using mean value of $z = \frac{1}{2}\sin(2\theta) + 2$ around the circle of radius 1:

- By mean value theorem, $u(0,0) = 2$ (substitute directly) should be the same as $\frac{1}{2\pi} \int_{\delta D} udl$
- $\frac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} [\frac{1}{2}\sin(2\theta) + 2]d\theta = \frac{1}{2\pi}[-\frac{1}{4}cos(2\theta) + 2\theta]|_{\theta=0}^{\theta=2\pi} = 2.$

3.8 Approximating Laplace

Main idea: Most PDEs don't have exact solutions. Even approximate solutions often need to be tailored to the particular PDE. Two numerical solutions in the toolkit include:

- **finite difference method**: Discretize the space and use the spherical (surroundg points) average property of harmonic functions.
- **Rayleigh-Ritz variational method**: Guess a solution with unspecified parameters and minimize the cost function over those parameters

Example for Finite differences: Imagine we're on a lattice of square size h with P_1, P_2, P_3, P_4 in the N, W, S, E positions around point (x, y). What is u?

- Tool: use Taylor approximation $f(x + h) \approx f(x) + hf'(x) + h^2\frac{f''(x)}{2}, h \neq 0$
- This means $u(P_1) = u(x_0, y_0+h) \approx u(x_0, y_0)+hu_y(x_0, y_0+h)+h^2\frac{u_{yy}(x_0,y_0+h)}{2}, u(P_3) = u(x_0, y_0 - h) \approx u(x_0, y_0) - hu_y f(x_0, y_0 - h) + h^2\frac{u_{yy}(x_0,y_0-h)}{2}$, similar for P_2, P_4.
- Adding these together yields $\frac{1}{4}[u(P_1) + u(P_2) + u(P_3) + u(P_4)] = u(x_0, y_0) + \frac{2}{4}h^2\nabla^2 u$, and since $\nabla^2 u = 0$ in this setup, $u(x_0, y_0)$ is the average of its neighbors.

- This should remind us of the mean-value property over circles and disks for these Laplace setups: $u(x,y) = \frac{1}{2\pi r}\int_{C((x,y),r)\subset D} udl = \frac{1}{\pi r^2}\int_{D((x,y),r)\subset D} ud\sigma$
- But *big idea*: We aren't limited to immediately surrounding points. We can extend this over the whole grid (reaching, say, known Dirichlet boundary conditions) and solve for the unknowns, if we have enough information! (matrix algebra).

Example:If our grid looks like

$$\begin{bmatrix} h_4 & h_3 & h_2 & h_1 \\ h_5 & u_2 & u_1 & h_{12} \\ h_6 & u_3 & u_4 & h_{11} \\ h_7 & h_8 & h_9 & h_{10} \end{bmatrix}$$

- We can, express, e.g. $u_3 = \frac{h_8+u_2+u_4+h_6}{4}$ and similar for other u_1, u_2, u_4
- And multiply to $4u_3 = u_4 + u_2 + h_6 + h_8$ and similar
- Define $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$, $\vec{h} = \begin{pmatrix} h_{12}+h_2 \\ h_5+h_3 \\ h_6+h_8 \\ h_{11}+h_9 \end{pmatrix}$, write $A\vec{u} = \vec{h}$, with $A = \begin{pmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{pmatrix}$
- With boundary conditions in the unit square $u(x,0) = u(0,y) = 0, u(x,1) = 3x, u(1,y) = 3y$ this lets us get the value of all the h_i and plug into $\vec{h}$ to get $= \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$
- Find the matrix inverse and solve to get $\vec{u} = \frac{1}{3}\begin{pmatrix} 4 \\ 2 \\ 1 \\ 2 \end{pmatrix}$
- This suggests $u(\frac{2}{3},\frac{2}{3}) \approx u_1 = \frac{4}{3}, u(\frac{1}{3},\frac{2}{3}) \approx u_2 = \frac{2}{3}$...
- From there, we hand wave to get exact solution $u(x,y) = 3xy$. *(Presumably, a computer interpolates this?)*
- Without an exact solution, you can shrink h and solve bigger and bigger matrix problems.

Finite differences work well for a square D, though it can be extended to rectangles and disks. But for irregular shapes, we use Rayleigh-Ritz technique.

Example for Rayleigh-Ritz technique.

- Setup, circle interior: $\nabla^2 u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0, x^2 + y^2 < 1$. "u is harmonic inside disk D"
- Setup, circle boundary: $u(x, y) = x^2, x^2 + y^2 = 1$. "u equals x^2 on δD"
- *Main idea*: We want to find the u that minimzes $C[u] = \frac{1}{2}\iint_D [\nabla u]^2 dxdy$ while maintaining $u = x^2$ on δD.
- Restated: Among set of D-situated differentiate functions F, find $u \in F$ that has the smallest cost.
- Tactic: Pick a parametrized function that fits, and tweak the parameters to minimize cost.
- So, guess u looks like $v(x, y) = x^2 + a(x^2 + y^2 - 1)$.
- Then $C[v] = C[x^2 + a(x^2 + y^2 - 1)] = \frac{1}{2}\iint_D [x^2 + a(x^2 + y^2 - 1)]^2 dxdy$
- Call the integrand $[\nabla v]^2$. Since $\nabla v = \langle 2x(1 + a), 2ya\rangle, [\nabla v]^2 = \nabla v \cdot \nabla v = 4x^2(1 + a) + 4y^2 a^2$
- To integrate over the circle $D = \{x^2 + y^2 \leq 1\}$, switch to polar $x = r\cos(\theta), y = r\sin(\theta)$, and churn through to get $C[v] = \frac{\pi}{2}((1 + a)^2 + a^2)$ This is minimized at $a = -\frac{1}{2}$
- Checking $v(x, y) = x^2 - \frac{1}{2}(x^2 + y^2 - 1)$, we see that:
  - $\nabla^2 v = 2 + -2 = 0$ everywhere.
  - $x^2 + y^2 = 1 \Rightarrow v = x^2$
  - Pretend we know u and minimize $\frac{1}{2}\iint_D [\nabla(v - u)]^2 dxdy$:
  - $= \frac{1}{2}\iint_D [\nabla(v - u)] \cdot [\nabla(v - u)] dxdy$
  - With product rule $\nabla(f\nabla g) - f\nabla^2 g = \nabla f \cdot \nabla g$
  - And divergence rule $\iint_D \nabla(f\nabla g) = \int_{\delta D} f\nabla g \cdot \hat{n} dl$
  - We get $= \frac{1}{2}\int_{\delta D}(v - u)\nabla(v - u) \cdot \hat{n} dl - \frac{1}{2}\iint_D \nabla(v - u)\nabla^2(v - u)$
  - Since $v - u = 0$ on δD, the first term is 0.
  - Since $\nabla^2(v - u) = \nabla^2 v - \nabla^2 u = 0$ inside the disk, the second term is zero.
- Therefore the cost is zero, which can only be true if $v = u$!

4 Chapter 4 - Transform methods

4.1 Fourier Transforms

Main idea:

- We use identity $e^{int} = i\sin(nt) + \cos(nt)$.
- We have a signal composed of amplitudes c_n at frequencies n: $s(t) = \sum_{n=-\infty}^{n=\infty} a_n i \sin(nt) + b_n \cos(nt) = \sum_{n=0}^{n=\infty} c_n e^{int}$ for a c_n composed of a_n, b_n (the e-based format is equivalent but easier to integrate). This coefficients of $\sin, \cos$ are called a **Fourier series**
- **Fourier's trick**: If you multiply signal $s(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ by a particular e^{-imt} and integrate, you extract c_m since $c_n = \frac{1}{2\pi}\int_{t=0}^{t=2\pi} s(t)e^{-imt}dt = \int_{t=0}^{t=2\pi} e^{i(n-m)t} = 1$ if $m = n$ and 0 otherwise (since $e^{int} = i\sin(nt) + \cos(nt)$ and those each integrate to 0 over a full period).
- This means we can extract frequencies c_n or ("source"?) $\hat{s}_n$ as $\hat{s}_n(t) = \frac{1}{2\pi}\int_{t=0}^{t=2\pi} s(t)e^{-int}$, with $s(t) = \sum_{n=-\infty}^{\infty} \hat{s_n} e^{int}$ as the full signal.
- And since n gives us integer frequencies, our **Fourier Transform** says "given a signal, integrate over the whole time period to get the source amplitude at a given frequency": $\hat{s}(\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{t=\infty} s(t)e^{-iwt}dt$. TODO: Why did we switch from $t \in [0, 2\pi]$ to $t \in \mathbb{R}$ here? *Note: We only use $\mathbb{R}$ as our domain from here on out, not $t \in [0, 2\pi]$.*
- The inverse is just the definition of $s(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ expressed over a continuum of frequencies: $s(t) = \int_{w=-\infty}^{\infty} \hat{s}(\omega)e^{iwt}d\omega$: "given a source set of amplitudes, integrate over the whole frequency spectrum to get our result signal"

Example: Contribution of amplitude, frequency to signal strength

- If we're looking for the strength of a signal based on sin or cos, integrating over 2π doesn't do since that would be zero.
- Therefore, we use "root mean square average strength" measure $\boxed{s_{rms} = \sqrt{\frac{1}{2\pi}\int_{t=0}^{t=2\pi}[s(t)]^2 dt}}$
  - Say there's a signal $s(t) = 2\sin(3t) - \cos(t)$. Which part contributes more to the strength?
  - Temporarily relabel amplitudes as w_1, w_2 so $s(t) = \sqrt{\frac{1}{2\pi}\int_{t=0}^{t=2\pi}[w_1\sin(3t) - w_2\cos(t)]^2 dt}$.
  - Rewrite using identities $\sin(at) = \frac{e^{iat} - e^{-iat}}{2i}, \cos(at) = \frac{e^{iat} + e^{-iat}}{2}$
  - ...(churn through) Everything ends up being paired up into sines and cosine terms (eliminated over 0 to 2π integral) except $(w_1\frac{-2}{2i})^2 + (w_2\frac{2}{2})^2$, and we end up with $s_{rms} = \sqrt{\frac{1}{2}(w_1^2 + w_2^2)}$

  - Takeaway: *Only amplitude contributes to signal strength, not frequency*

- So signal stregth of $s(t)$ is $\sqrt{\frac{1}{2\pi}\int_{t=0}^{t=2\pi}[s(t)]^2 dt} = \sqrt{\frac{1}{2\pi}\int_{t=0}^{t=2\pi}(\sum_{m=-\infty}^{\infty} c_m)(\sum_{k=-\infty}^{\infty} c_k)} = \sqrt{\sum_{k=-\infty}^{\infty} c_n c_{-n}}$ due to integral being zero for $e^{i(m+k)t}, m+k \neq 0$ terms

Using this to actually in solving PDEs: apparently for one example, we can use the Fourier transform of a function, find out its solution, then transform that back to the original domain. (Does this work just some of the time?)

- Setup: The wave equation $u_{tt} = \nabla^2 u$ actually does apply to (I suppose?) an infinitely long string over x.
- Other parameters: Intial shape $u(x,0) = g(x)$ on a still $u_t(x,0) = 0$ wire that tapers to zero on either end: $lim_{x\to\pm\infty} g(x) = 0$.
- (Note that this only looks like Fourier if x is like time.)
- $\hat{u}(\omega, t) = \frac{1}{2\pi}\int_{\mathbb{R}} u(x,t)e^{-i\omega x}dx$
- To find out what $\widehat{u_{tt}}$ is, first consider $\frac{\delta^2}{\delta t^2}\hat{u} = \frac{1}{2\pi}\int_{\mathbb{R}} u_{tt}(x,t)e^{-i\omega x}dx$. (Note: This also equals $\hat{u}_{tt}$, the transform of u_{tt}, since the variable t isn't used elsewhere.)
- And (the trick!) $\widehat{u_{tt}} = \widehat{u_{xx}}$ by $u_{tt} = u_{xx}$, so this $= \frac{1}{2\pi}\int_{\mathbb{R}} u_{xx}(x,t)e^{-i\omega x}dx$
- And by integration by parts, with $U = e^{-i\omega x}, V = u_{xx}$, we have $= \frac{1}{2\pi}[e^{-i\omega x}u_x|_{-\infty}^{\infty} - (-i\omega)\int_{\mathbb{R}} u_x(x,t)e^{-i\omega x}dx]$. The first term is zero from the boundary conditions (I guess if $u \to 0$ then $u_x \to 0$ too? Did I miss something?)
- Repeating the process on the second term, we get $\hat{u}_{tt} = \frac{1}{2\pi}\omega^2\int_{\mathbb{R}}(x,t)e^{-i\omega x}dx$, which suggests $\frac{\delta^2\hat{u}}{\delta t^2} = -\omega^2\hat{u}$
- WIth the boundary conditions $\hat{u}(\omega, 0) = \hat{g}(\omega)$ (the transform of the initial $g(x)$ state on u, and similarly $\hat{u}(\omega, t) = 0$ implies a solution of $\hat{u}(\omega,t) = \hat{g}(\omega)\cos(|\omega|t)$. TODO: Why the absolute value?

So, we reduced a second-order PDE with two variables (t, x) to an ordinary one with one variable ω. The wave solution then has a Fourier transform $\hat{u}(\omega,t) = \hat{g}(\omega)\cos(|\omega|t)$. With the initial condition set as, say, $g(x) = 2\pi u_0 e^{-\frac{x^2}{2}}$, we can transform back.

- With the identity $\int_{\mathbb{R}} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$, and knowing $\hat{g}(\omega) = \int_{\mathbb{R}} g(x)e^{-i\omega x}dx$,, we can complete the square to $\hat{g}(\omega) = u_0\int_{\mathbb{R}} e^{-\frac{1}{2}(x+i\omega)^2 - \frac{1}{2}\omega^2}dx$
- We can sub $y = x + i\omega$ and use our identity well enough, to get $\hat{g}(\omega) = u_0\sqrt{2\pi}e^{-\frac{\omega^2}{2}}$

- Then, since $\hat{u}(\omega,t) = u_0\sqrt{2\pi}e^{-\frac{\omega^2}{2}}\cos(|\omega|t)$, we can use $u(x,t) = \int_{\mathbb{R}} \hat{u}(\omega,t)e^{ix\omega}d\omega$, knowing $\cos(|\omega|t) = \frac{e^{i|\omega|t}+e^{-|\omega|t}}{2}$
- After splitting the integrals over the positive and negative domains for the absolute values and completing the square, we get $\pi u_0[e^{-\frac{(x+t)^2}{2}} + e^{-\frac{(x-t)^2}{2}}]$
- Apparently this is the same solution as with d'Alembert's formula (the two sensors going out from x at time 0 and querying at t): $u(x,t) = \frac{1}{2}[g(x+t) - g(x-t)]$, since $g(x) = 2\pi u_0 e^{-\frac{x^2}{2}}$.

4.2 Fourier and the Heat Equation

Main ideas:

- The last section was solving $u_{tt} = u_{xx}$, the 1D wave equation.
- This section solves $u_t = \nabla^2 u$, the heat equation of on a 1D rod, with initial temp distribution $g(x)$.
- Solution to this in a previous section, assuming a Dirac heat spike at $x = 0$ was $C\exp\{-\frac{x^2}{\sigma^2 t}\}$
- Main tactic again is to take a PDE, Fourier transform into a simpler ODE, solve, and transform back.
- Like the radio frequency waves of the previous chapter, these Fourier transforms go over all reals $x \in \mathbb{R}$. So we can't do it over time ($t \geq 0$) but can over x.

OK, let's get to it. How do the derivatives of the transformed function relate to the transform of the function's derivatives?

- Main Fourier transform from signal domain to frequency domain, just like before: $\hat{u}(\omega,t) = \frac{1}{2\pi}\int_{x=-\infty}^{x=\infty} u(x,t)e^{-i\omega x}dx$.
- Then $\widehat{u_x}(\omega,t) = \frac{1}{2\pi}\int_{x=-\infty}^{x=\infty} u_x(x,t)e^{-i\omega x}dx = \frac{1}{2\pi}u(x,t)e^{-i\omega x}|_{x=-\infty}^{x=\infty} - (-i\omega)\int_{x=-\infty}^{x=\infty} u(x,y)e^{-i\omega x}dx$ by integration by parts. Assuming $u \to 0$ when approaching either infinity, the first term is zero, and the second is just $\hat{u}$, so:
  $\boxed{\text{taking a FT of a derivative is just } (i\omega) \text{ times the FT of the original: } \widehat{u_x} = (i\omega)\hat{u}}$
- This can be repeated easily, e.g. $\widehat{u_{xx}} = -\omega^2\hat{u}$, or $\widehat{\frac{\delta^n u}{\delta x^n}}(\omega,t) = (i\omega)^n\hat{u}(\omega,t)$

Solving our Heat Equation

- A-HA: And since, in this Laplace setup, $u_t = \nabla^2 u \equiv u_{xx}$, then $\widehat{u_t} = -\omega^2\hat{u}$ since transforms preserve equality *(TODO: did we prove they were 1-1 before?) I guess if inv(trans(a)) = a, then they must be...*

- Boundary: $u(x,0) = g(x) \Rightarrow \hat{u}(\omega, 0) = \hat{g}(\omega)$.
- $\widehat{u_t} = \frac{d\hat{u}}{dt} = -\omega^2 \hat{u} \Rightarrow \hat{u} = Ce^{-\omega^2 t}$
- $\hat{u}(\omega, t=0) = \hat{g}(\omega) \Rightarrow \hat{u} = \hat{g}(\omega)e^{-\omega^2 t}$
- We then need to "undo" the transform to get our actual u in the signal domain. We need to know g to do this.
- Assume g is a Dirac Delta at a: $g(x) = \delta_a(x)$.
- Then, by the nature of the delta function, then $\hat{g}(\omega) = \frac{1}{2\pi}\int_{x=-\infty}^{x=\infty} \delta_a(x)e^{-i\omega x}dx = \frac{1}{2\pi}e^{-i\omega a}$, just picking out the one value where the integral is nonzero, and $dx = 1$ for this infinitesimal slice, basically.
- So to reverse $\hat{u} = \hat{g}(\omega)e^{-\omega^2 t}$, we transform back $\int_{x=-\infty}^{x=\infty}(\frac{1}{2\pi}e^{-i\omega a})e^{-\omega^2 t}e^{i\omega x}d\omega$
- Complete the square so the exponent is $-t(\omega + \frac{[x-a]}{2ti})^2 - (\frac{[x-a]^2}{4t})$
- Use the identity $\int_{k=-\infty}^{k=\infty} e^{-ak^2} = \sqrt{\frac{\pi}{a}}$ with $k = (\omega + \frac{[x-a]}{2ti})$ to get $u(x,t) = \sqrt{\frac{1}{4\pi t}}e^{-\frac{[x-a]^2}{4t}}$

This $F(x,t;a) = \sqrt{\frac{1}{4\pi t}}[e^{-\frac{[x-a]^2}{4t}}]$ turns out to be a *fundamental solution*, in that if we have any initial "data" $g(x) = u(x,0)$, then we can sum over all points a where the unit of heat is found:
More concretely, $u(x,t) = \int_{a=-\infty}^{a=\infty} F(x,t;a)u(a,0)da = \sqrt{\frac{1}{4\pi t}}\int_{a=-\infty}^{a=\infty}[e^{-\frac{[x-a]^2}{4t}}u(a,0)]da$ says

"Set a as the point where all the heat is concentrated, in amount $u(a,0)$. This is the initial condition for the fundamental solution. Integrate over all of these".
This (integral sum) obeys $u_t = u_{xx}$ since F does!

- Note: this requires that $\lim_{t\to 0^+}\sqrt{\frac{1}{4\pi t}}\int_{a=-\infty}^{a=\infty}[e^{-\frac{[x-a]^2}{4t}}u(a,0)]da = u(x,0)$

Example:

- If we set $u(x,0) = e^{-\beta x^2}$, then we have to simply evaluate $\sqrt{\frac{1}{4\pi t}}\int_{a=-\infty}^{a=\infty} u(a,0)e^{-\frac{[x-a]^2}{4t}}da = \sqrt{\frac{1}{4\pi t}}\int_{a=-\infty}^{a=\infty} e^{-\frac{[x-a]^2}{4t}-\beta a^2}da$
- This requires completing the square, with steps $-\frac{1}{4t}[(1+4\beta t)a^2 - 2ax + x^2]$ and eventually $-\frac{1+4\beta t}{4t}[a - \frac{x}{1+4\beta t}]^2 - \frac{\beta x^2}{1+4\beta t}$
- Applying the shift and the lemma from above, we get $u(x,t) = \sqrt{\frac{1}{1+4\beta t}}e^{-\frac{\beta x^2}{1+4\beta t}}$
- APPARENTLY (unverified) this solves $u_t = u_{xx}$ (lots of bad looking derivatives), and $u(x,0) = e^{-\beta x^2}$ (L' Hopital's Rule?

4.3 Practice: Fourier and Laplace

Usually, doing the FT is not hard. The inverse is hard. // Theme of this section is the Laplace equation: $\nabla^2 u = u_{xx} + u_{yy} = 0$

Setup: Heat equation on the positive y half-plane

- Some $g(x)$ heat distribution on the x-axis
- Note: There's not a unique solution to $u_{xx} + u_{yy} = 0$ in general.
  - Suppose there is a solution $f(x, y)$. Then, since this is linear, if there's a solution u with $u(x, 0) = 0$, then the solution $f + u$ will hold.
  - If $u''(y) = 0$, then $u = ay + b$ has $u_{yy} = 0$ so $u + f$ is a solution.
  - However, *if u is bounded (that is, $u(x, y) < B$ for all x, y)*, then there is a unique solution (why?)
- If $u_{xx} + u_{yy} = 0, u(x, 0) = g(x)$, and $u, u_x \to 0$ as $x \to \pm\infty$, then we can solve the FT $\hat{u}(\omega, dy) = \frac{1}{2\pi} \int_{x=-\infty}^{x=\infty} u(x, y)e^{-i\omega x}dx$
  - $\widehat{u_{yy}} = \hat{u}_{yy} = \frac{1}{2\pi} \int_{x=-\infty}^{x=\infty} u_{yy}e^{-i\omega x}dw$
  - $\hat{u}_{yy} = -\frac{1}{2\pi} \int_{x=-\infty}^{x=\infty} u_{xx}e^{-i\omega x}dw$ by $u_{xx} + u_{yy} = 0$
  - IBP: $dV = u_{xx}, U = e^{-i\omega x}$: $= -\frac{1}{2\pi}[e^{-i\omega x}u_x|_{-\infty}^{\infty} + (i\omega)\int u_x e^{-i\omega x}]$
  - $= -\frac{1}{2\pi}[0 + -i\omega(e^{-i\omega x}u)|_{-\infty}^{\infty} + (i\omega)^2 \int ue^{-i\omega x}]$
  - So $\frac{d^2}{dy^2}\hat{u} = \omega^2\hat{u}$
  - And if we assume a solution of form $\hat{u}(\omega, y) = e^{ry}$, then $\frac{d\hat{u}}{dy^2} = r^2e^{ry} = \omega^2e^{ry} \Rightarrow r = \pm|\omega|$
  - So, we have combinations of two possible solutions: $A(\omega)e^{|\omega|y} + B(\omega)e^{-|\omega|y}$
  - However, if u is bounded as $y \to \infty$, then $A(\omega) = 0$ by necessity.

  To recap for the purpose of solving:
  - $\hat{g}(\omega) = \frac{1}{2\pi} \int_{x=-\infty}^{x=\infty} g(x)e^{-i\omega x}$ by general FT.
  - $\hat{u}(\omega, y) = \hat{g}(\omega)e^{-|\omega|y}$ by the solution above.
  - $u(x, y) = \int_{\omega=-\infty}^{\omega=\infty} \hat{u}(\omega, y)e^{i\omega x}d\omega$ by general inverse FT.
  - So $u(x, y) = \int_{s=-\infty}^{s=\infty} [\hat{g}(\omega)e^{-|\omega|y}]e^{i\omega x}d\omega$ by substitution of $u(x, y)$

- And $u(x,y) = \int_{\omega=-\infty}^{\omega=\infty}(\frac{1}{2\pi}\int_{s=-\infty}^{s=\infty} g(s)e^{-i\omega s})e^{-|\omega|y}e^{i\omega x}d\omega = \frac{1}{2\pi}\int g(s)[\int e^{-i\omega s-|\omega|y+i\omega x}d\omega]ds$ (So there's no need to find $\hat{g}(\omega)$ explicitly.)
- The innermost integral evaluates to $\frac{2y}{(x-s)^2+y^2}$, so we're solving $u(x,y) = \frac{1}{\pi}\int_{s=-\infty}^{s=\infty} g(s)\frac{y}{(x-s)^2+y^2}ds, y > 0$, which obeys $\nabla^2 u = 0$ on the half-plane.
- We also need $\lim_{y\to 0^+} u(x,y) = g(x)$ for all real x, which,since $g(s) \approx g(x) = g(x)(s-x) + \frac{1}{2}g''(x)(s-x)^2 ...$, reduces to understanding what happens to $\int_{u=-\epsilon}^{u=\epsilon} \frac{u^n}{u^2+y^2}du$ for $u = s - x$. $n \geq 2$ has no singularities, so this is all about solving $n \in \{0,1\}$, which ends up evaluating to $g(x)$

- To solve $u(x,y) = \frac{1}{\pi}\int_{s=-\infty}^{s=\infty} g(s) = \frac{y}{(x-s)^2+y^2}ds$, with $g(s) = 1+\cos(x)$, using $cos(A+B) = \cos(A)\cos(B)-\sin(A)\sin(B)$, we end up with $u = 1+\cos(x)e^{-y}$ after expanding and using some integral identities.

Note that for Laplace solutions, the maximum value is achieved on the boundary, at $y = 0$, where e^{-y} is maximized.

4.4 Challenge: Fourier and 3D Waves

We generalize from the 1D Fourier wave equations (relating x and some ω thing) (TODO Read Fourier Transforms: https://blog.endaq.com/fourier-transform-basics) to $n-dimensional$ in trying to solve 3D (or n-D) Laplace: $u_{tt} = \nabla^2 u$.

An example would be compression waves (density at 3D point $\vec{x}$).

The core equations then become:

- $s(\vec{x}) = \int_{\vec{\omega}\in\mathbb{R}^n} \hat{s}(\vec{\omega})\exp\{i\vec{\omega}\cdot\vec{x}\}d\vec{\omega}$
- $\hat{s}(\vec{\omega}) = \frac{1}{(2\pi)^n}\int_{\vec{x}\in\mathbb{R}^n} s(\vec{x})\exp\{-i\vec{\omega}\cdot\vec{x}\}d\vec{x}$

First, develop the differential equation on the other side of the transform (meaning, of $\hat{u}$):

- Assume: $u(\vec{x},0) = g(\vec{x})$ goes to 0 at infinity, and $u_t(\vec{x},0) = 0$.
- This means we can use the divergence theorem, I guess since the balls near infinity are all zero flux? $\int_{\vec{x}\in\mathbb{R}^3} \nabla\cdot\vec{u}\ d\vec{x} = 0 \Leftrightarrow \lim_{\vec{x}\to\infty}\vec{u}(\vec{x}) = \vec{0}$
- Also, we have an (actually obvious) identity in hand that helps: $\nabla\cdot[f\nabla h - h\nabla f] = f\nabla^2 h - h\nabla^2 f]$
- And, remember that $u_t t = \nabla^2 u$ in our setup.
- $\frac{d^2\hat{u}}{dt^2} = \widehat{u_{tt}} = \frac{1}{(2\pi)^3}\int_{\vec{x}\in\mathbb{R}^n} u_{tt}\exp\{-i\vec{\omega}\cdot\vec{x}\}d\vec{x}$

- This equals $\frac{1}{(2\pi)^3}\int_{\vec{x}\in\mathbb{R}^n}\nabla^2 u\exp\{-i\vec{\omega}\cdot\vec{x}\}d\vec{x}$ by the Laplace setup.
- Set $f=\exp\{-i\vec{\omega}\vec{x}\}, h=u$, so $\nabla\cdot[e^{-i\vec{\omega}\vec{x}}\nabla u-u\nabla e^{-i\vec{\omega}\vec{x}}]=e^{-i\vec{\omega}\vec{x}}\nabla^2 u-u\nabla^2 e^{-i\vec{\omega}\vec{x}}]$
- This means $\frac{1}{(2\pi)^3}\int_{\vec{x}\in\mathbb{R}^n}[\nabla^2 ue^{-i\vec{\omega}\vec{x}}d\vec{x}-u\nabla^2 e^{-i\vec{\omega}\vec{x}}]+u\nabla^2 e^{-i\vec{\omega}\vec{x}}$, and the bracketed part is $\nabla\cdot[e^{-i\vec{\omega}\vec{x}}\nabla u-u\nabla e^{-i\vec{\omega}\vec{x}}]$ by the last line. This will be zero as $\vec{x}\to\infty$ by Divergence theorem.
- This also means, in this particular setup, that $\int_{\vec{x}\in\mathbb{R}^n}(\nabla^2 u)e^{-i\vec{\omega}\cdot\vec{x}}d\vec{x}=\int_{\vec{x}\in\mathbb{R}^n}u\nabla^2 e^{-i\vec{\omega}\cdot\vec{x}}d\vec{x}$
- Since $\nabla^2 e^{-i\vec{\omega}\cdot\vec{x}}=(-i\vec{\omega})^2=-\|\vec{\omega}\|^2$, we get $\boxed{\frac{d^2\hat{u}}{dt^2}=-\|\vec{\omega}\|^2\hat{u} \text{ as the diff eq on the omega side.}}$
- Then $\boxed{u=\int_{\mathbb{R}^3}\hat{g}\cos(\|\omega\|t)e^{i\vec{\omega}\cdot\vec{x}}d\vec{\omega} \text{ is the solution}}$ since anything of the form $A(\omega)e^{i\vec{\omega}\cdot\vec{x}}+B(\omega)e^{-i\vec{\omega}\cdot\vec{x}}$ is also expressable as $C(\omega)\cos(\|\omega\|t)+D(\omega)\sin(\|\omega\|t)$ and our inital conditions force C to be g and thus D to be zero.

If we have an initial density distribution g, we can transform it to the omega domain. Here, $\boxed{g(x,y,z)=u_0\exp\{-\frac{1}{2}(x^2+y^2+z^2)\} \text{ is given}}$ so $\hat{g}(\vec{\omega})=\frac{1}{(2\pi)^n}\int_{\vec{x}\in\mathbb{R}^n}g(\vec{x})\exp\{-i\vec{\omega}\cdot\vec{x}\}d\vec{x}$

- We're given identity $\int_{x\in\mathbb{R}^n}\exp\{-a\|x\|^2\}dx=(\frac{\pi}{a})^{\frac{n}{2}}$
- Recognize $\|\vec{\omega}\|^2$ in the exponent of g: $e^{-\frac{1}{2}\|\vec{x}\|^2}$
- Looks like we're completing squares again. The exponent of the product becomes $-\frac{1}{2}(x+i\omega)^2-\frac{1}{2}\|\vec{\omega}\|^2$
- Set $q=x+iw, dq=dx$ and with the identity in hand, the integral becomes $\frac{u_0}{(2\pi)^3}e^{-\frac{1}{2}\|\vec{\omega}\|^2}\int e^{-\frac{1}{2}q^2}dq=\boxed{\frac{u_0}{(2\pi)^{\frac{3}{2}}}e^{-\frac{1}{2}\|\vec{\omega}\|^2}=\hat{g}(\omega)}$

Putting $\hat{g}(\omega)$ into the inverse transform, we have $u(x,y,z,t)=\int_{\vec{\omega}\in\mathbb{R}^3}\hat{g}(\vec{\omega})\cos(\|\vec{\omega}\|t)e^{i\vec{\omega}\cdot\vec{x}}d\vec{\omega}$

So our inverse FT ends up with the usual pattern - integral of: $\hat{g}$ times "$\hat{u}$ diff eq solution" times $e^{i\omega x}$

Combining $\hat{g}$ with u equations gives us

$u=\frac{u_0}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}e^{-\frac{1}{2}\|\vec{\omega}\|^2}\cos(\|\omega\|t)e^{i\vec{\omega}\cdot\vec{x}}d\vec{\omega}$

$\equiv\frac{u_0}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\cos(\|\omega\|t)\exp\{-\frac{1}{2}\|\vec{\omega}\|^2+i\|\vec{\omega}\|\|\vec{x}\|\cos(\phi)\}d\vec{\omega}$

Solving this is tricky and involves moving to spherical coordinates.

- Big trick: change $\vec{x}\cdot\vec{\omega}$ to $\|\vec{\omega}\|\|\vec{x}\|\cos(\phi)$ and use spherical coordinates to define ω.

- ω still integrates over all $\mathbb{R}^3$ but with ϕ declination from $\vec{x}$ and θ "around" $\vec{x}$. (Think of $\vec{x}$ pointing up like z).
- TODO: I think the Jacobian implies $d\vec{\omega} = \|\vec{\omega}\|^2 \sin(\phi) d\phi d\theta d(\|\vec{\omega}\|)$
- The whole thing is then $2\pi \int_{\|\vec{\omega}\|=0}^{\|\vec{\omega}\|=\infty} \cos(\|\vec{\omega}\|t)\|\vec{\omega}\|^2 e^{-\frac{\|\omega\|^2}{2}} (\int_{\phi=0}^{\phi=\pi} \exp\{i\|\vec{x}\|\|\vec{\omega}\|\cos(\phi)\}\sin(\phi)d\phi)d(\|\vec{\omega}\|)$
- Subbing $u = \cos(\phi)$, the inner integral is:
  * $\int -\exp\{i\|\vec{x}\|\|\vec{\omega}\|u\}du = -\frac{1}{i\|\vec{x}\|\|\vec{\omega}\|}e^{i\|\vec{x}\|\|\vec{\omega}\|\cos(\phi)}|_{\phi=0}^{\phi=\pi} = -\frac{1}{i\|\vec{x}\|\|\vec{\omega}\|}[e^{-ixw} - e^{ixw}] = \frac{2\sin(\|\vec{x}\|\|\vec{\omega}\|)}{\|\vec{x}\|\|\vec{\omega}\|}$
  * So the whole intergral is now $u(\vec{x},t) = \frac{u_0}{\|\vec{x}\|}\sqrt{\frac{2}{\pi}} \int_{\|\vec{\omega}\|=0}^{\|\vec{\omega}\|=\infty} \cos(\|\vec{\omega}\|t)\|\vec{\omega}\| e^{-\frac{\|\vec{\omega}\|^2}{2}} \sin(\|\vec{x}\|\|\vec{\omega}\|)d(\|\vec{\omega}\|)$
  * Replace $\|\vec{\omega}\| = v$. calculate $\cos(vt)\sin(\|\vec{x}\|v) = \frac{e^{vt}+e^{-vt}}{2}\frac{e^{\|\vec{x}\|v}-e^{-\|\vec{x}\|v}}{2i}$ to get $\frac{1}{2}[\sin(v\|x\| + t) + \sin(v\|x\| - t)]$
  * Now evaluate $\int_{v=0}^{v=\infty} ve^{-\frac{v^2}{2}}\sin(av)dv$
    $= a\int_{v=0}^{v=\infty} e^{-\frac{v^2}{2}}\cos(av)dv$ (IBP)
    $= \frac{a}{2}\int_{v=0}^{v=\infty} e^{-\frac{v^2}{2}}\cos(av)dv$ (cosine is even)
    $= \frac{a}{2}\int_{v=0}^{v=\infty} e^{-\frac{v^2}{2}}\frac{1}{2}(e^{av} + e^{-av})dv$ (Euler cosine)
    $=$ ... (Completing squares, $\int e^{-a^2}2 = \sqrt{\frac{\pi}{a}}$ theorem) ... $= \sqrt{\frac{\pi}{2}}ae^{\frac{-a^2}{2}}$
  * Finally, we get $\boxed{u(x,t) = \frac{u_0}{2\|\vec{x}\|}[(\|\vec{x}\| + t)\exp\{-\frac{(\|\vec{x}\|+t)^2}{2}\} + (\|\vec{x}\| - t)\exp\{-\frac{(\|\vec{x}\|-t)^2}{2}\}]}$
- We can confirm $u(x,0) = u_0\exp\{-\frac{\|x\|^2}{2}\}$ directly, and $u_t(x,0) = 0$ by a TRICK: note that u is symmetric in t, so $u_t(x,-t) = u_t(x,t)$, implying $u_t(x,0) = 0$!
- To confirm $u_{tt} = \nabla^2 u$, we define $h(z) = u_0 ze^{-\frac{z^2}{2}}$, making $u = \frac{h(\|x\|+t)+h(\|x\|-t)}{2\|x\|}$
- Since $u(\vec{x},t)$ depends only on $\|x\| = r$, spherical coordinates' formula can be used: $\nabla^2 u = \frac{1}{r^2}\frac{\delta}{\delta r}[r^2\frac{\delta u}{\delta r}]$
- Remember h is a one-dimensional function.
- $u_t = \frac{h'(r+t)-h'(r-t)}{2r}, u_{tt} = \frac{h''(r+t)+h''(r-t)}{2r}$
- $u_r = \frac{rh'(r+t)+rh'(r-t)-h(r+t)-h(r-t)}{2r^2}$
- Don't go for u_{rr}! Instead: $\frac{\delta}{\delta r}[r^2 u_r] = \frac{r(h''(r+t)+h''(r-t))}{2}$
- Multiplying by $\frac{1}{r^2}$ shows they are equal.

Results:

- 1D waves ($u(x,t) = \frac{g(x+t)+g(x-t)}{2}$)
- 3D (compression) waves ($u(\vec{x},t) = \frac{u_0}{2\|x\|}[(\|x\|+t)\exp\{\frac{(\|x\|+t)^2}{2}\}+(\|x\|-t)\exp\{\frac{(\|x\|-t)^2}{2}\}]$

Starting with bump $g(x) = u_0 e^{-\frac{x^2}{2}}$ and similar in 3D, there are some similarities:

- Both are bounded between 0 and 1. (Use L'Hopital's for the 3D case to be sure)
- Both have disturbances travel at finite speed t

Differences:

- The 1D wave is a translation of the original function. The 3D wave has $\|x\|$ in the denominator, so it flattens out.
- The 1D wave with this setup is always positive. The 3D wave can be negative (rarefied, in compression terms)!

4.5 Schroedinger's Equation

This uses the Fourier transform to make isomorphic a probability density over position (x) with that over v velocity (p), and shows that the other side of the FT can have intrinsic meaning.

Note: Much of this requires complex conjugation ($(a+bi)^* = a - bi$).

4.5.1 Complex Conjugation tips:

- $(a+b)^* = a^* + b^*$. Separate the real and imaginary and it's clear.
- $(a \times b)^* = a^* \times b^*$. Mulitply it out and it's clewar.
- $(e^{it}){*} = (e^{-it})^*$. Change e^{it} to $\cos(t) + i\sin(t)$ and it falls right out.
- $\|a+bi\|^2$, the squared "length" of tjhe complex number, is a^2+b^2 or $(a+bi)(a-bi) = (a+bi)(a+bi)^*$
- $(f'(t))^* = ((f^*)'(t))$. Substitute f^* for f in the limit definition and it's clear.
- This makes sense since it looks like any linear transform (incl. derivative) of conjugation looks to be the conjugate of the transform.
- FT $F(\omega)$ of $(f^*)(x)$ ends up being $F^*(-\omega)$.

4.5.2 Main ideas of Schroedinger's equation

- Equation is $u_t = \frac{i}{2}\nabla^2 u$.
- The "signal" (u) side sees $\|u(x,t)\|^2$ as probability particle is near x at time t.
- The transformed "motion" ($\hat{u}$) sees $\|\hat{u}(p,t)\|^2$ as probability particle's *velocity vector* is near $\vec{pd}$ at time t.
- This works because change in expected position (expected velocity) $\frac{dX}{dt} = \iiint_p \vec{p}(2\pi)^3|\hat{u}(p,t)|^2 d\vec{p}$, where X is $\iiint_x x\|u(x,t)\|^2$

Initial conditions $u(x,0) = g(x)$ require.

- $\iiint_{\mathbb{R}^3} |g(\vec{x})|^2 d\vec{x} = 1$, to be a legit probability distribution at time 0.
- Knowing $u_t = \frac{i}{2}\nabla * 2u$ means $u_t^* = -\frac{i}{2}\nabla^2 u$. Then $\frac{d}{dt}[\iiint_{\mathbb{R}^3} |u|^2 d\vec{x}] = \iiint u_t u^* + u u_t^* = \frac{i}{2}\iiint u^*\nabla^2 u - u\nabla^2 u^* = \nabla\cdot(u^*\nabla u - u\nabla u^*)$ (Note: This is an identity - take $\nabla(f\nabla g - g\nabla f)$ and see!)
- So If $u, \nabla u \to 0$ go to zero near infinity, then we can use divergence theorem (since any ball would be zero on the surface). Those are the three conditions for using $u(x,0) = g(x)$ as an initial distribution.

Step one: Solve the diff eq on the $\hat{u}$ side.

- $\hat{u}(p,t) = \frac{1}{(2\pi^3}\iiint_{\mathbb{R}^3} u(\vec{x},t)\exp\{-i\vec{p}\cdot\vec{x}\}d\vec{x}$ by standard FT.
- Taking $\frac{d}{dt}$ of both sides and doing the $u_t = \frac{i}{2}\nabla^2 u$ substitution, cranking through ∇s, yields $\boxed{\frac{d\hat{u}}{dt} = -\frac{i}{2}\|\vec{p}\|^2\hat{u}(\vec{p},t)}$, since the double derivative of the exp spits out $(-i\vec{p}\cdot -i\vec{p})$
- So $\hat{u}(\vec{p},t) = \hat{g}(\vec{p})e^{-\frac{i}{2}\|\vec{p}\|^2 t}$

Aside: The Dirac Delta function δ

- Definition : (Also works in $\mathbb{R}^3$: $\int_{x=-\infty}^{x=\infty} f(x)\delta(x-a) = f(a)$
- Fourier transform is $\hat{\delta}(\vec{p}) = \frac{1}{(2\pi)^3}\iiint_{\vec{x}\in\mathbb{R}^3}\delta(\vec{x}-\vec{a})\exp\{-i\vec{p}\cdot\vec{x}\}d\vec{x} = \frac{1}{(2\pi)^3}e^{-i\vec{p}\cdot\vec{a}}$. Straightforward use of delta's main feature.
- Inverse of the transform is the $\delta(x-a) = \frac{1}{(2\pi)^3}\iiint_{\vec{p}\in\mathbb{R}^3}\exp\{-i\vec{p}\cdot(\vec{x}-\vec{a})\}d\vec{p}$
- This creates **Plancherel's Theorem**: $\iiint_{\mathbb{R}^3}|g(x)|^2 = (2\pi)^3\iiint_{\mathbb{R}^3}|\hat{g}(p)|^2$, since
  - FT of g is $\hat{g}(\vec{p}) = \frac{1}{(2\pi)^3}\iiint_{\vec{x}\in\mathbb{R}^3} g(p)exp\{-i\vec{p}\cdot\vec{x}\}d\vec{x}$
  - $|g(x)|^2 = g(x)(g^(x))^*$
  - So $(2\pi)^3\iiint_{\mathbb{R}^3}|\hat{g}(p)|^2 = \int(\int FT(g))(\int FT(g))^*$

– $(FT(g))^* = \int g^* e^{iyt} dy$, so $\int (\int g(x) g^*(y) (\int \exp\{ip \cdot (y-x)) = \int \int g(x) g^*(y) \delta(y-x) dy dx = \int g(x) g^*(x)$ There were some $(2\pi)^3$s here too.

– This shows that $(2\pi)^3 \iiint \|\hat{u}(p,t)\|^2 dp = 1$, and it's something of a probability density too.

Expected position is then $\vec{X}(t) = \iiint \vec{x} |u(x,t)|^2 d\vec{x}$

- Used together with $u_t = \frac{i}{2} \nabla^2 u$
- and a divergence theorem (when u and ∇u are 0 at inifity): $\iiint \vec{x} \nabla \cdot \vec{V} d\vec{x} = -\iiint \vec{V} \vec{dx}$
- Gives us that $\frac{dX}{dt} = \frac{d}{dt}[\iiint \vec{x} |u(x,t)|^2 d\vec{x}] = \iiint_p \vec{p} (2\pi)^3 |\hat{u}(p,t)|^2 d\vec{p}$, or that the change in expected position is the expected value of the velocity, and the transform, $\iiint_p \vec{p} (2\pi)^3 |\hat{u}(p,t)|^2 d\vec{p}$ measures the likelihood of finding the particle with *velocity* near p at time t.

Example: If we start=with a known velocity distribution, $\hat{g}(p) = \frac{1}{(2\pi\sigma)^{\frac{3}{2}}} \exp\{-\frac{\pi \|p-p_0\|^2}{2\sigma^2}\}, \sigma << 1$, whcih looks like a "normal" about point p_0 with some σ^2 variance:

- Glue it on to $\hat{u} = e^{-\frac{i}{2}\|p\|^2 t}$
- Calculate expected position $\hat{P} = \iiint_p p(2\pi)^3 |\hat{u}(p,t)|^2 dp$
- Calculate variance $Var(\hat{p}) = \iiint_p (p-P)^2 (2\pi)^3 |\hat{u}(p,t)|^2 dp = \frac{3}{2}\frac{\sigma^2}{\pi^2}$ (requires some identities)
- Transform $\hat{u}$ back to get Schroedinger wave function $u(x,t) = \frac{1}{(2\pi\sigma)^{\frac{3}{2}}} \iiint_p \exp\{-\frac{\pi\|p-p_0\|^2}{2\sigma^2} - \frac{i}{2}\|p\|^2\}$
- For this one, take $p_0 = \vec{0}, a(t) = \frac{\pi}{a^2} + it$
- Complete the square to get $u(x,t) = \frac{1}{[\sigma a(t)]^{\frac{3}{2}}} \exp\{-\frac{\|x\|^2}{2a(t)}\}$
- The variance of the position is then $\frac{3}{2}\frac{\sigma^2 |a(t)|^2}{\pi^2}$ by our result before (replacing σ with $\sigma a(t)$
- And the product of the two variances $(\frac{3}{2}\frac{\sigma^2|a(t)|^2}{\pi^2})(\frac{3}{2}\frac{\sigma^2}{\pi^2})$ ends up as $\frac{9}{4}(1 + \frac{\sigma^4}{\pi^2}t^2)$
- This is always positive and goes up with time. Therefore, the **Heisenberg uncertainty principle** says the product of the variances in position and velocity is alwways positive.
- Perhaps this says "you can't know both position and velocity with surety at the same time"?

4.6 Conformal Maps

Visual: a conformal map is like flattening the Earth map out into a circle with a pole at the center.

Main ideas:

- Solving Laplace on a disk $D = \{(u, v) : u^2 + v^2 \leq 1\}$ can't use Fourier transform, since D has an edge.
- Also, we can find $f(0, 0)$ for any harmonic (mean-value) function with $f(0, 0) = \frac{1}{2\pi} \int_{\delta D} g dl$, if g is the boundary of the disk D.
- So if we rearrange points using **conformal maps** (maps that preserve angles) and **stereographic projection** (move points in a figure along lines mapping to a single focus, which I think makes a conformal map) then we can reshape D and point (x_0, y_0) to the center of a similar disk.
- The process for this, starting from unit disk D:
  - First, a translated disk lying parallel to $y = 0$, axis, centered at $(0, -1, 1)$
  - Then, a hemisphere via a stereographic projection.
  - Then, the half plane H via projecting through the north pole and the point to the plane
  - Then, the half plane to itself, to align our (x_0, y_0) to something like $(0, 2)$
  - Then, reversing all these to have a disk with our point at the center
- We map from (u, v) to (x, y) like $(x(u, v), y(u, v))$.
- We have some function on each, representable as either $f(x(u, v), y(u, v))$ or $\tilde{f}(u, v)$

Main setup:

- We map from (u, v) to (x, y) like $(x(u, v), y(u, v))$. (Note that this chapter has a lot of mapping to e.g. $(x, y, 0)$ or $(x, 0, z)$, but that's still 2d-to-2d.)
- We have some function on each, representable as either $f(x(u, v), y(u, v))$ or $\tilde{f}(u, v)$
- We need to make sure that f is harmonic ($f_{xx} + f_{yy} = 0$) implies that $\tilde{f}$ is too.
- We can do this by ensuring $x_u = y_v$ and $x_v = -y_u$. Why?
  - First, if $\tilde{f}(u, v) = f(x(u, v), y(u, v))$, then $\nabla^2 \tilde{f} = \tilde{f_{uu}} + \tilde{f_{vv}}$.
  - Take the tricky two-step derivatives along u *through* x and y: $\tilde{f}_u = (f_x x_u + f_y y_u)$, so $\tilde{f}_{uu} = \frac{d}{du} \tilde{f}_u = \frac{d}{du}(f_x x_u) + \frac{d}{du}(f_y y_u) = (\frac{d}{du} f_x) x_u + f_x(\frac{d}{du} x_u) + (\frac{d}{du} f_y) y_u + f_y(\frac{d}{du} y_u) = (f_{xx} x_u + f_{xy} y_u) x_u + f_x x_{uu} + (f_{yy} y_u + f_{yx} x_u) y_u + f_y y_{uu}$.

- Similarly, $\tilde{f}_{vv} = (f_{xx}x_v + f_{xy}y_v)x_v + f_x x_{vv} + (f_{yy}y_v + f_{yx}x_v)y_v + f_y y_{vv}$.
- So $\nabla^2 \tilde{f} = [x_u^2 + x_v^2]f_{xx} + [y_u^2 + y_v^2]f_{yy} + 2f_{xy}[x_u y_u + x_v y_v] + f_x[x_{uu} + x_{vv}] + f_y[y_{uu} + y_{vv}]$
- If $x_u = y_v$ and $x_v = -y_u$ and $f_{xx} + f_{yy} = 0$ then this is $([y_u^2 + (-y_v)^2][f_{xx} + f_{yy}] + (2f_{xy}[y_v y_u + -y_u y_v]) + (f_x[y_{vu} + -y_{uv}]) + (f_y[-x_{vu} + x_{uv}]) = 0 + 0 + 0 + 0$.
- Notice also that f is harmonic, since $x_{uu} + x_{vv} = y_{vu} - y_{uv} = 0$, similar for $y_{uu} + y_{vv}$

An example harmonic function would be $\tilde{f} = (x(u,v), y(u,v)) = (e^u \cos(v), e^u \sin(v))$, where it can be easily validated that $x_u = y_v, x_v = -y_u$.

You can prove the angles are the same before and after the map by:

- Starting with two curves $(u_1(t), v_1(t)), (u_2(t), v_2(t))$ meeting in the plane at t=0, with their tangent vectors (pointwise derivatives $\vec{t_1}, \vec{t_2}$.
- Defining $\cos(\theta)$ with the dot poduct formula.
- Define $\cos(\theta')$ as meeting of image $(x(u_1(t), v_1(t)), y(u_1(t), v_1(t))$ and its u_2, v_2 mate.
- We can see these are equal if we build a matrix. We see $\frac{d}{dt}|_{t=0}[x(u_1(t), v_1(t))] = x_u u_1'(0) + x_v v_1'(0)$ and the like to a derivative matrix $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \langle u_j'(0), v_j'(0) \rangle$
- Note that this matrix A also equals $\begin{pmatrix} y_v & -y_u \\ y_u & y_v \end{pmatrix}$, and tehrefore $A^T A = I(y_u^2 + y_v^2)$
- After a bit more linear algebra, the $A-$transformed θ' in the dot product formula ends up equaling the original $\cos(\theta)$

So the derivative conditions mean we preserve angles under that kind of map. There's a simple (but not only?) way to make such a map by sliding any three points along lines with $t = 0$ (starting point) to a common focus terminus at $t = 1$. The proof ends up being:

- Set each of three points $\vec{p_j}(t) = (1-t)\vec{p_j} + t\vec{c}$
- Use the dot product formula $\cos(\theta(t)) = ...$ using vectors $\vec{p_2}(t) - \vec{p_0}(t)$ and $\vec{p_1}(t) - \vec{p_0}(t)$
- Show that this equals $\cos(\theta(0))$, or the same setup with $\vec{p_2} - \vec{p_0}$ and $\vec{p_1} - \vec{p_0}$

The sequence of mappings from initial disk to elevated disk D to unit hemisphere S^+ (sitting on origin, top at $(0,0,2)$ to half plane H and back basically are just algebra.

Example: H to $S^+ : x^2 + y^2 + (z-1)^2 = 1$

- Line between $(x, y, 0)$ $(0, 0, 2)$ is $t\langle x, y, 2(1-t)\rangle, t \in [0, 1]$
- The point touching S^+ satisfies $(tx)^2 + (ty)^2 + (2(1-t) - 1)^2 = 1$
- Simplify to find $t = \frac{4}{x^2+y^2+4}$, and substitute into line $t\langle x, y, 2(1-t)\rangle$
- Thus (x, y, z) on S^+ maps to $\frac{1}{x^2+y^2+4}\langle 4x, 4y, 2x^2 + 2y^2\rangle$
- A similar stragegy gives us inverse $(\frac{2x}{2-z}, \frac{2y}{2-z}, 0)$ going from S^+ to H.

Idea for mapping from S^+ to a disk $(x, 0, z) : x^2 + (z - 1)^2 \leq 1$ is to draw a line from $(0, -1, 1)$ the sphere, and see where we hit the y-parallel disk centered at $(0, 0, 1)$.

- A similar strategy: Create the line parametrized by t, see where t satisfies the intersection condition.
- We end up with map from S^+ to disk as $(x, y, z) \to (\frac{x}{y+1}, y, \frac{z+y}{y+1})$
- The inverse, from a similar path, is $\frac{1}{x^2+(z-1)^2+1}\langle 2x, 1 - x^2 - (z - 1)^2, x^2 + z^2\rangle$

Combining all of these maps, including the map that elevates and rotates a unit disk to $(x, 0, z)$ in $D : (x = u, z = 1 + v)$, maps (u, v) on the unit disk to $(x, y, z) = \frac{1}{u^2+(v-1)^2}\langle 4u, 2(1 - u^2 - v^2), 0\rangle$, with an inverse of $(u, v) = \frac{1}{x^2+(y+2)^2}\langle 4x, x^2 + y^2 - 4\rangle$

One final map before we can go from unit disk to itself fully: moving $(x_0, y_0) \to (0, 2)$ on H:

- Note: You can shift left/right without changing any angles. (Can't shift along y, since 0 is the edge of the universe)
- Note: You can scale without changing any angles. Can probably confirm this very easily in cross product theorem, but it's clear.
- So, shifting the coordinate over for x and scaling it for y yields (verifiably) conformal $(x, y) \to (\frac{2(x-x_0)}{y_0}, \frac{2y}{y_0})$

So if we wanted to find the value on the unit disk of some harmonic function f of $f(0, 0)$ with a known boundary condition $g(u, v) = u^2$, we could use the mean value property and average over the boundary.

Instead, if we wanted to know $f(0, \frac{1}{2})$, we could:

- Make a conformal transformation of $(0, \frac{1}{2})$ to $(0, 0)$
- Know that that will become a similar subdisk in the original disk with the mean value property still holding.
- Find the condition on the edge of that destination disk knowing the original g on the first disk.

- Integrate that around the edge of that destination disk.

Knowing the mapping $(\frac{3u}{u^2+(v+2)^2}, \frac{2u^2+2v^2+5v+2}{u^2+(v+2)^2})$ takes $(0,0)$ back to $(0, \frac{1}{2})$ means that we can integrate $r\cos^2(\theta) = [\frac{3u}{u^2+(v+2)^2}]^2$ around the disk to get $\frac{3}{8}$ on that disk, and thus the original.

4.7 The Laplace Transform

Main idea: Laplace is similar to Fourier (transform some PDEs into ODEs), but can handle a separate domain.

- Fourier transforms are from $x \in (-\infty, -\infty)$. (Note: what about the $[0, 2\pi)$ ones from the first chapter?). Laplace handles $x \in [0, \infty)$, better suited actually for time t!
- Requirements:
  - $\mathcal{L}[f](s)$ should be an integral, to handle the derivatives of diff eqs
  - Integration limits should be $0, +\infty$
  - $\mathcal{L}$ should be linear in $f(t)$.
  - Finally, we need a handy derivative mapping like Fourier's $\mathcal{F}[f'] = (i\omega)\mathcal{F}[f]$
- Laplace transform of $f(t)$, operating on s, is $\mathcal{L}[f](s) = \int_{t=0}^{t=\infty} K(t;s)f(t)dt$.
- s is a variable like Fourier's ω, and $K(t;s)$ is called the kernel.
- Suppose we want a derivative rule like : $\mathcal{L}[\frac{df}{dt}](s) = \int_{t=0}^{t=\infty} \mathcal{L}K(t;s)f'(t)dt = s\mathcal{L}[f](s) +$ ...
- If $f(t), K(t;s) \to 0$ as $t \to \infty$, then selecting $K(t;s) = e^{-st}$ and therefore $\mathcal{L}[f](s) = \int_{t=0}^{t=\infty} e^{-st}f(t)dt$ satisfies all these. (note: do integration by parts with $u = e^{-rs}, v' = f$)
- Churning the IBP gives you $\mathcal{L}[\frac{df}{dt}](s) = s\mathcal{L}[f](s) - f(0), \mathcal{L}[\frac{d^2f}{dt^2}](s) = s(\mathcal{L}[\frac{df}{dt}](s) - f'(0)) - f(0) = s(s\mathcal{L}[f](s) - f'(0)) - f(0)$...
- So ultimately, our rule is $\boxed{\mathcal{L}[\frac{d^nf}{dt^n}](s) = s^n\mathcal{L}[f](s) - \sum_{j=0}^{n-1} s^{n-j-1}f^{(j)}(0)}$
- This means we can get rid of all the derivatives on the transformed side and solve for the diff eq for $\mathcal{L}[f]$. (We can also handle weirdos like δ below)

Major Example to solve: Impulse kicking an oscillating spring with force α at time t_0.

- Diff Eq to solve: $x''(t) + x(t) = \alpha\delta(t - t_0)$, with Dirac δ.
- $\mathcal{L}[\alpha\delta_{t-t_0}](s) = \int_{t=0}^{t=\infty} e^{-st}\alpha\delta(t - t_0)dt = \alpha e^{-st_0}$

- Taking $\mathcal{L}[x''(t) + x(t)] = \mathcal{L}[\alpha\delta(t - t_0)] = \alpha e^{-st_0}$, use the derivative rule to find the left hand side is $-x'(0) - sx(0) + s^2\mathcal{L}[x(t)](s)] + \mathcal{L}[x(t)](s)$.
- Assuming $x(0) = x_0$ (stretched x_0 units) and $x'(0) = 0$ (at arest), then we see $\mathcal{L}[x(t)](s) = \frac{\alpha e^{-st_0} + sx_0}{s^2+1}$

Note: The hard part is reversing these BACK into the original domain. The integral reverse won't be detailed here (it's a lot of hard ideas, apparently.). It turns out you *generally look up the inverse Laplace transform in a table.*

The Heaviside step function $u(x) = \{1, x > 0; \frac{1}{2}, x = 0; 0, x < 0\}$ is included in the list, such that $\mathcal{L}[u(t - t_0]f(t - t_0) = e^{-st_0}F(s)$, and $\mathcal{L}[u(t)] = s^{-1}$ (Since $\int_{t=0}^{t=\infty} e^{-st}u(t)dt = \int_{t=0}^{t=\infty} e^{-st} = \frac{-1}{s} - e^{-st}|_{t=0}^{t=\infty} = [0 - \frac{-1}{s}] = s^{-1}$)

With this table, we find that $\mathcal{L} = \alpha\mathcal{L}[u(t - t_0)\sin(t - t_0)](s) + x_0\mathcal{L}[\cos(t)](s) = \mathcal{L}[\alpha u(t - t_0)\sin(t - t_0) + x_0\cos(t)](s)$, which obeys $x(0) = 0, x'(0) = 0$ for $t_0 > 0$. This is a continuous but non-differentiable function, since the "kick" changes the derivative discontinuously.

4.8 Laplace Transform applications

Main motivation: RLC Circuit

- Original equation: $\frac{V_0(t)}{L} = Q''(t) + \frac{R}{L}Q'(t) + \frac{Q(t)}{LC}$.
- After nondimensionalization, we have our main equation to solve this section: $x''(t) + 2\epsilon x'(t) + x(t) = v_0\sin(\omega t), \epsilon > 0$
- Looking up in the table, we have $\mathcal{L}[v_0\sin(\omega t)] = \frac{\omega v_0}{s^2+\omega^2}$

Main tactic: Take Laplace transform of the side with the derivaties, setting up some $\mathcal{L}[x(t)] = f(s)$, do the same for the fixed side, and figure out $x(t)$ by triangulating in the transform table.

- General rule is $\mathcal{L}[x'(t)] = s\mathcal{L}[x(t)] - x(0)$. Sub in $x''(t)$ or whatever for $x'(t)$ as needed.
- $\mathcal{L}[x''(t)] = s\mathcal{L}[x'(t)] - x'(0) = s(s\mathcal{L}[x(t)] - x(0)) - x'(0)$. Can do the same for $x'(t)$.
- Adding up, $\mathcal{L}[x''(t) + 2\epsilon x'(t) + x(t)] = \mathcal{L}[x(t)](s^2 + 2\epsilon s + 1) = \frac{\omega v_0}{s^2+\omega^2}$
- Then $\mathcal{L}[x(t)] = \frac{\omega v_0}{s^2+\omega^2}\frac{1}{s^2+2\epsilon s+1}$. Settting $\epsilon = 0$, we can get $\mathcal{L}[x(t)] = \omega v_0\frac{1}{s^2+\omega^2}\frac{1}{s^2+1}$
- Doing partial fractions gets us $= \frac{\omega v_0}{1-\omega^2}[\frac{1}{s^2+\omega^2} + \frac{1}{s^2+1}]$
- Using the table to find $\mathcal{L}[s = \sin(at)](s) = \frac{a}{s^2+a^2}$, we can work forwards to get $\frac{v_0}{1-\omega^2}\mathcal{L}[\sin(\omega t) - \omega\sin(t)](s) = \frac{v_0}{1-\omega^2}(\frac{\omega}{s^2+\omega^2} - \frac{\omega}{s^2+1})$

Now, the derivative trick on $\frac{d}{ds}\mathcal{L}[x(t)]$. Suppose we're looking for the solution to the above when $\omega = 1$ (and $\epsilon = 0$). We end up with $\mathcal{L}[x(t)] = \omega v_0 \frac{1}{s^2+\omega^2}\frac{1}{s^2+1} = v_0 \frac{1}{(s^2+1)^2}$

- We know $\mathcal{L}[\cos(t)](s) = \frac{s}{(s^2+1)}$. Guess $\cos(t)$ as our $x(t)$ and take $\frac{d}{ds}$ of both sides.
- The right side becomes $-\frac{s^2-1}{(s^2+1)^2}$
- The left side becomes $\frac{d}{ds}\int_{t=0}^{t=\infty}\cos(t)e^{-st}dt = \int_{t=0}^{t=\infty}\frac{d}{ds}e^{-st}\cos(t)dt = \int_{t=0}^{t=\infty} -te^{-st}\cos(t)dt = \mathcal{L}[-t\cos(t)]$
- The Laplace transform of sine is $\frac{1}{s^2+1} = \frac{s^2+1}{(s^2+1)^2}$
- Subtracting these two yields $x(t) = \frac{v_0}{2}[\sin(t) - t\cos(t)]$

Another example: Instead of the wall outlet, we have a battery-operated switch tso that $x''(t) + x(t) = v_0 u(t-t_0)$ (Heaviside step function)

- The Laplace transform of the right side is $\mathcal{L}[u(t-t_0)](s) = \int u(t-t_0)e^{-st}dt = -\frac{1}{s}e^{-st}|_{t=t_0}^{t=\infty} = 0 - (-\frac{1}{s}e^{-st_0}) = \frac{1}{s}e^{-st_0}$
- Since our solution of the left hand side is the same, $(s^2+1)\mathcal{L}[x(t)](s) = \frac{v_0}{s}e^{-st_0} \Rightarrow \mathcal{L}[x(t)](s) = \frac{v_0 e^{-st_0}}{s(s^2+1)}$
- Partial fractions yield that $\frac{v_0 e^{-st_0}}{s(s^2+1)} = v_0[\frac{1}{s} - \frac{s}{s^2+1}]e^{-st_0}$
- Looking this up in the table yields that $x(t) = v_0 u(t-t_0)(1-\cos(t))$. So the circuit starts at t_0 and oscillates from there.

Another example: Third-order equations like $\theta^{(3)}(t) + \theta''(t) - \theta'(t) - \theta(t) = b$.

- If we assume $\theta(t) = 0, \theta'(t) = 0, \theta''(t) = 0$, then expanding quickly reveals that the left hand is $\mathcal{L}[x(t)](s)(s^3+s^2-s-1)$, exactly mirroring the θ terms in s.
- The right hand side, transforms as $\mathcal{L}[b] = \frac{b}{s}$, so the whole equation is $\mathcal{L}[x(t)] = \frac{b}{s(s^3+s^2-s-1)} = \frac{b}{s(s+1)(s^2-1)} = \frac{b}{s(s+1)^2(s-1)}$
- By Partial Fractions, this becomes $-\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{4}\frac{3s+5}{(s+1)^2}$
- We can tweak this around to $-\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{4}\frac{3(s+1)}{(s+1)^2} + \frac{2}{(s+1)^2}$ and use our previous results to get $\theta(t) = -b + \frac{b}{4}e^t + \frac{3b}{4}e^{-t} + \frac{b}{2}te^{-t}$

5 Series Solutions

5.1 Power Series (Series Solutions I)

Main idea:

- You can always render $y(x)$ as some $a_0 + a_1x + a_2x^2 + a_3x^3...$
- This means $y'(x) = a_1 + 2a_2x + 3a_3x^2 + ...$ and $y''(x) = 2a_2 + 6a_3x + 12a_4x^2...$
- When confronted with, say, $y'(x) = y(x)$, you can directly equate those and see $a_0 = a_1, a_1 = 2a_2, a_2 = 3a_3$, and thus $a_n = a_0\frac{1}{n!}$, leading to $y(x) = e^x$.
  - Note: This proves you can't solve $y'(x) = y(x)$ with a nontrivial, finite series.
  - Also, in general, we see that solution $a_0e^x = \sum_{n=0}^{\infty} \frac{a_0}{n!}$
- This works with finite sums too, like $y''(x) = 5x^3, y'(0) = 1, y(0) = 0 \Rightarrow a_0 = 0, a_1 = 1, 20a_5x^3 = 5x^3 \to a_5 = \frac{1}{4} \Rightarrow y(x) = x + \frac{1}{4}x^5$

The main idea is that $\sum_{n=0}^{N} a_nx^n$ approaches the real solution as $N \to \infty$. Obviously they are equal at $N = 0$, but as $x \to 0$, the solution seems to be more accurate.

Another idea: Using recurrence relations to solve $y'(x) = y(x)$

- For the e^x example, you can create a recurrence relation like $(n+1)(n+2)a_{n+2} = a_n$
- Note there are separate even and an odd cascades that land on a_0 and a_1 respectively.
- You can eyeball the solution as $\frac{1}{2}\frac{1}{n!}[a_0 + a_1 + (-1)^n(a_0 - a_1)]$.
- POWERFUL TRICK: OR you can use this a_n, b_n POWER RECURRENCE technique
  - Set $a_n = \frac{b_n}{n!}$ (so b_i is "Blown up")
  - Note $(n+2)(n+1)a_{n+2} = (n+2)(n+1)\frac{b_{n+2}}{(n+2)!} = \frac{b_n}{n!} = a_n = \frac{b_n}{n!} \Rightarrow b_{n+2} = b_n$. So, ignoring the factorials in the denominator, b terms are equal all the way down to a_0 or a_1
  - However, you can hypothesize that $b_n = \alpha r^n$, so $b_{n+2} = \alpha r^{n+2} = b_n = \alpha r^n \Rightarrow r^2 = 1 \Rightarrow r = \pm 1$
  - Thus the solution is any combo of $b_n = \alpha_+ 1^n + \alpha_-(-1)^n$
  - And with $a_0 = b_0, a_1 = b_1$, we can solve to get $\alpha_+ = (a_0 + a_1), \alpha_- = (a_0 - a_1)$, so $b_n = \frac{1}{2}\frac{1}{n!}[(a_0 + a_1)1^n + (-1)^n(a_0 - a_1)]$.

You can expand this to solve $y''(x) = y(x)$ most generally:

- Start with $b_n = \frac{1}{2}\frac{1}{n!}[(a_0 + a_1)1^n + (-1)^n(a_0 - a_1)]$.
- This means y(x)= $\sum_{n=0}^{\infty} \frac{1}{2}\frac{1}{n!}[(a_0 + a_1)1^n + (-1)^n(a_0 - a_1)]x_n$
- Rearrange to get $y(x) = \frac{a_0+a_1}{2}\sum_{n=0}^{\infty} \frac{1^n}{n!}x^n + \frac{a_0-a_1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n = \frac{a_0+a_1}{2}\sum_{n=0}^{\infty} \frac{1}{n!}x^n + \frac{a_0-a_1}{2}\sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n$

- This is just $\frac{a_0+a_1}{2}e^x + \frac{a_0-a_1}{2}e^{-x}$, or since $a_0 = y(0), a_1 = y'(0)$, this is $y(x) = \frac{y(0)+y'(0)}{2}e^x + \frac{y(0)-y'(0)}{2}e^{-x}$,

However, if we're solving $y''(x) = -y(x)$, neat things happen. Follow the above steps exactly, except

- $b_{n+2} = \alpha r^{n+2} = -b_n = -\alpha r^n \Rightarrow r = \pm i$
- So our combined solution is $b_n = [\alpha_+ i^n + \alpha_-(-i)^n]$
- We then end up with $a_{\pm i} = \pm\frac{1}{2i}$, and $b_n = \frac{i^n}{2i}[1^n - (-1)^n] \Rightarrow a_n = \frac{i^{n-1}}{2n!}[1-(-1)^n]$
- Noting that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we can see that the terms with i on them above are the series for $\sin(x) = (1 - \frac{x^3}{x!} + \frac{x^5}{5!} - ...)$, and the real terms are that of cos
- Strangely, sin is periodic, and the infinite sum is too! Naturally, the finite sum is a polynomial and not periodic.

5.2 Power Series (Series Solutions II)

Main idea: Centering a series at c rather than at 0 gives us flexibility (and, I think, converges faster around that point, letting us use fewer terms). The general form: $y(x) = \sum_{n=0}^{\infty} a_n[x-c]^n$

Example: $y'(x) + xy(x) = 0, y(1) = 1$

- A good choice to estimate $y(0.95)$ would be $y(x) = \sum_{n=0}^{N} a_n[x-1]^n$, since it's close to .95 and we know $y(1)$.
- Writing terms as recurrence relation:
  - Since we're looking around 1, write as $y'(x) + [x-1]y(x) + y(x) = 0, y(1) = 1$
  - $y = a_0 + a_1[x-1] + a_2[x-1]^2 + a_3[x-1]^3...$
  - $y' = a_1 + 2a_1[x-1] + 3a_2[x-1]^2 + 4a_3[x-1]^3...$
  - $[x-1]y = a_0[x-1] + a_1[x-1]^2 + a_2[x-1]^3...$
  - By writing out terms $y + [x-1]y + y = 0$ shows us $a_n + a_{n-1} + (n+1)a_{n+1} = 0, n \geq 1$
  - Dealing with the boundary cases, we can say $y(x) = a_0 + a_1 + \sum_{n=1}^{\infty}((n+1)a_{n+1} + a_n + a_{n-1})[x-1]^n = 0$
  - Solving for $y(1) = 1$, we have $a_0 = 1, a_1 = -1$

Note: At this point, solving the recurrence relation is difficult, so they embedded a small python script of the series centered around 1. $y(0.95)$ converged to 0.001% error (against real solution $e^{-\frac{1}{2}(x^2-1)}$) in 4 iterations of the sum (or $N \in [0,3]$).

We can also solve the equation analytically:

- $y' + xy = 0 \Rightarrow \frac{y'}{y} = -x \Rightarrow \int \frac{dy}{y} = \int -xdx \Rightarrow y(x) = Ce^{-\frac{1}{2}x^2}$
- $Ce^{-\frac{1}{2}} = 1 \Rightarrow C = \sqrt{e} \Rightarrow \sqrt{e}e^{-\frac{1}{2}x^2} = y(x)$
- Putting this in the form $y(x) = \sum_{n=0}^{\infty} b_n x^n$ means putting $(-\frac{1}{2}x^2)$ in for x and multiplying the result by $\sqrt{e}$. This means all odd coefficients are 0, and evens are $\sqrt{e}\frac{1}{(-2)^{n/2}}\frac{1}{(n/2)!} = b_n$

With the solution in hand we can try our sum over b_n and compare the analytical solution. If we center on 0, $y(0.95)$ converges to 0.001% error in 8 iterations to $(n = 0, 2, ...14)$. This is twice as many as we needed at 1 (4, above)!

Convergence:

- $\sum_{k=0}^{N} c_k = \lim N \to \infty c_k$ converges if and only if c_k decreases sufficiently quickly.
- If $c_{k+1}/c_k < C < 1$, for some constant C, this definitely converges.
- The **ratio test** says exactly this: if $\lim_{n\to\infty}[\frac{a_{n+1}[x-c]^{n+1}}{a_n[x-c]^n}] = \lim_{n\to\infty}[\frac{a_{n+1}}{a_n}]|x-c| < 1$, the series converges.
- For example, $a_n = \frac{i^{n-1}}{2n!}[1-(-1)^n]$ converges:
  - First, drop all the zero (even) terms by rewriting to $\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}x^{2j+1}$
  - Apply the ratio test between $c_{j+1}x^{2(j+1)+1}$ and $c_j x^{2j}$
  - $|\frac{c_{j+1}}{c_j}| = \frac{1}{(2j+3)(2j+2)}$, so this is less than one at some point regardless of x^3's value (meaning x's value).
  - Note this converges to $\sin(x)$. I suppose the series form of $\sin(\frac{\pi}{4})$ would converge faster than that of $\sin(\frac{\pi}{4} + 200\pi)$

5.3 The Airy Equation

Main motivation: The block-spring system, transferring Energy back and forth between kinetic ($\frac{1}{2}mv^2$) and potential (U(x) $= \frac{1}{2}kx^2$) to a fixed sum total $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$). The turning point x_0 is where $v = 0$ and the block switches directions. The block can never see $x > x_0$ in this case.

There's a quantum version of this in which the "block" does escape this bound.

Schroedinger's equation rewritten: $i\frac{\delta\psi}{\delta t} = -\frac{1}{2}\frac{\delta^2\psi}{\delta x^2} + U(x)\psi(x,t)$.

- Separate variables by saying $\psi(x,t) = X(x)T(t)$ take take derivatives.
- Divide by XT: $iXT' = -\frac{1}{2}X''T + U(x)XT \Rightarrow T'/T = \frac{1}{2}iX''/X + U(x)$
- Somehow (TODO) we end up with $\psi(x,t) = y(x)e^{-iEt}$, where $y(x)$, a renamed $X(x)$, has property $-\frac{1}{2}y''(x) + U(x)y(x) = Ey(x) = -\frac{1}{2}y''(x) + \frac{1}{2}x^2y(x)$
- Near x_0, we can approximate y(x) with:
  - TRICK: $\frac{1}{2}x^2 = \frac{1}{2}([x-x_0]+x_0) = \frac{1}{2}[x-x_0]^2 + x_0[x-x_0] + \frac{1}{2}x_0^2$
  - The first term is very small, and the third term is E since $v = 0$ there.
  - So we can rewrite as $-\frac{1}{2}y''(x) + \frac{1}{2}x^2y(x) =\approx -\frac{1}{2}y''(x) + x_0[x-x_0]y(x) + Ey(x) = Ey(x) \Rightarrow y''(x) - 2x_0[x-x_0]y(x) = 0$

Weird: we're defining a new variable $t = (2x_0)^{\frac{1}{3}}[x-x_0]$, which isn't time, but where $t > 0$ is a forbidden region inyto which our block tunnels. Coupled with $y'' - ty = 0$:

- Write out terms to see that $y'(t) - ty(t) = \sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} - \sum_{n=0}^{\infty} a_n t^{n+1} = 0$. It looks like the sums skip over terms at a time!
- Writing out the first few terms, the sum is $2a_2 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - a_{n-1}]t^n = 0$
- TRICK: Since this is always true, try $t = 0$ to see that a_2 must be 0.
- TRICK: Try $t \neq 0$ to see that $[(n+2)(n+1)a_{n+2} - a_{n-1}] = 0$ or that $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, n \geq 1$
- We can also write the whole thing now as $a_n = \frac{a_{n-3}}{(n)(n-1)}, n \geq 3, a_2 = 0, a_0, a_1 \in \mathbb{R}^3$
- Write out terms to see there are three strings then:
  - $n = 3k-1$, where $a_n = 0$
  - $n = 3k$, where $a_n = \frac{a_0}{3k \times 3k-1 \times \ldots \times 3 \times 2}$
  - $n = 3k+1$, where $a_n = \frac{a_1}{3k+1 \times 3k \times \ldots \times 4 \times 3}$
  - Ratio test: Dividing successive terms by each other (within each string), we see that y_0 term c_k has property $|\frac{c_{k+1}}{c_k}| = \frac{|t^3|}{(3k+3)(3k+2)}$, which, for a fixed t, will eventually be less than one, so the series converges after that point.
  - Similar for y_1 series $(3k+1)$. So, the whole sum $y(t) = \sum_{k=0}^{\infty} a_{3k-1}t^{3k-1} + \sum_{k=0}^{\infty} a_{3k}t^{3k} + \sum_{k=0}^{\infty} a_{3k+1}t^{3k+1} = a_0y_0(t) + a_1y_1(t)$ converges for all t.

  However, looking at the truncated sums, we see that while the block oscillates for $t < 0$, for $t > 0$ (block in the forbidden zone), the sum zooms to infinity. This is because the truncated sums can't be periodic - they're finite polynomials. Only infinite ones (like $\sin(t)$, for example) can oscillate.

5.4 The Wronskian (Determinant)

Main idea: We had a solution of $a_0 y_0(t) + a_1 y_1(t)$ for the last problem. How do we know that $y_0(t)$ and $y_1(t)$ are different? As in, they are not mulitples of each other or really, they are linearly independent? We use a matrix format for piles of these solutions and check if the determinant is nonzero. This can be made clear for $n = 2$, but it appears subtle for higher orders.

Note: It's not always easy to tell if two functions (esp. sums) are linearly independent, since $\sin(t)$ and $\cos(t - \frac{pi}{2})$ look different but are of course the same. Use the Wronskian determinant to tell more accurately.

- Two functions $y_0(t), y_1(t)$ are **linearly dependent** on an open interval I if there are some $a, b \neq 0$ such that $ay_0(t) + by_1(t) = 0$
- Note this implies their derivatives are linearly dependent too, by differentiating both sides: $ay_0'(t) + by_1'(t) = 0$
- Rearranging into a Wronskian matrix $W[y_0, y_1] = \begin{pmatrix} y_0(t) & y_1(t) \\ y_0'(t) & y_1'(t) \end{pmatrix}$, we see that, since $\begin{pmatrix} a \\ b \end{pmatrix} \neq 0$, that $\begin{pmatrix} y_0(t) & y_1(t) \\ y_0'(t) & y_1'(t) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ must mean the determinant of W is zero. (Otherwise we could invert $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to a nontrivial vector)
- This is the standard linear algebra test of linear independence, and works for any n functions: $W[f_1, f_n](t) = \begin{pmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ ... & ... & ... \\ f_1^{n-1} & ... & f_n^{n-1} \end{pmatrix}$ is singular if and only if the functions are linearly dependent.

Note: This works for $n = 2$ for functions a, b since $ab' - a'b \Rightarrow a'/a = b'/b \Rightarrow \ln(a) = \ln(b) + C \Rightarrow a = be^C$
Example:

- cos and sin are linearly independent since $|\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}| = 1$
- $\cosh = \frac{e^x + e^{-x}}{2}$ and $\sinh = \frac{e^x - e^{-x}}{2}$ are too since $|\begin{pmatrix} \cosh(t) & \sinh(t) \\ \frac{d}{dt}[\cosh(t)] & \frac{d}{dt}[\sinh(t)] \end{pmatrix}| = 1$

- For two functions y_0, y_1 that solve the Airy equation $y'' - ty = 0$, we can show that W is not dependent on t:
  - Starting with the diff eq,, $y_0'' - ty_0 = 0, y_01' - ty_1 = 0 \Rightarrow y_0''/y_0 = t = y_1''/y$
  - Then $y_0 y_1'' = y_0'' y_1 \Rightarrow y_0 y_1'' - y_0'' y_1 = 0$
  - Determinant $W(t) = y_0 y_1' - y_0' y_1$
  - $W'(t) = y_0 y_1'' + y_0' y_1' - y_0'' y_1 - y_0' y_1' = y_0 y_1'' - y_0'' y_1$, which is zero by line 2 above.
  - A zero derivative of $W(t)$ means that $W(t)$ does not depend on t.
  - This means that we can use ANY value of $W(t)$ for the Wronskian determinant. For the Airy equations $y_0(0)y_1'(0) - y_1(0)y_0'(0) = (1)(1) - (0)(0) = 1$

Note that Airy is a special case of $y''(t)+p_1(t)y'(t)+p_2(t)y(t) = 0$, with $p_1 = 0, p_2 = -t$. We can prove that $W = c\exp(-\int p_1(t)dt)$ is a general solution, with some unknown constant c.

- $y'' + p_1 y' + p_2 y = 0 \Rightarrow p_1 y' = -y'' - p_2 y$
- $W = y_0 y_1' - y_0' y_1$
- $p_1 W = y_0(p_1 y_1') - y_1(p_1 y_0') = y_0(-y_1'' - p_2 y_1) - y_1(-y_0'' - p_2 y_0) = y_1 y_0'' - y_0 y_1''$
- This last term is $-W'(t)$ by above results. So $p_1 W = W' \Rightarrow W = c\exp\{-\int p_1(t)dt\}$
-

This is **Abel's formula**, and (somehow) generalizes to solve any $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_n y(t) = 0$, as $W(t) = \det W[y_0, ...y_{n-1}](t) = c\exp\{-\int p_1(t)dt\}$

Appllying Abel's formula to find solutions to $y''(t) + 2y'(t) + y(t) = 0$

- Find the first (homogeneous linear) solution by hypothesizing $y = e^{rt}$ and seeing $(r^2 + 2r + 1)e^{rt} = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1 \Rightarrow y = ce^{-t}$
- (Note : Forget the c for now. It's the same but cleaner)
- Then, the Wronskian can help us find the other linearly independent solution.
- We can find the Wronksian with Abel's formula, $p_1(t) = 2 \Rightarrow W(t) = c\exp\{-\int[2]dt\} = ce^{-2t}$. Again, forget the c.
- So the determinant equation says there's a function f so that $e^{-t}f' - (-e^{-t}f) = e^{-2t}$
- Multiply all by e^t, to get $f' + f = e^{-t}$
- Solve that diff eq by using integrating factor e^t (multiply all by e^t again) to see $\frac{d}{dt}[e^t y_1(t)] = 1 \Rightarrow e^t y_1(t) = t \Rightarrow y_1(t) = te^{-t}$

- Note that if we kept constant c in there, then the solution would have been $y_1(t) = te^{-t}+ce^{-t} = te^{-t}+cy_0(t)$, but the last part isn't linearly independent of y_0's solution.
- So the whole solution set is $Ce^{-t} + Dte^{-t}$

5.5 Cauchy-Euler Equation

Main idea: Laplace equation (in this case, for disk pressure at point (x, y) on a floating air bearing disk), is $\nabla^2 p = 0$. Since it's angle-independent, in polar coordinates we have $\nabla^2 p = \frac{1}{r}\frac{\delta}{\delta r}(r\frac{\delta p}{\delta r}) + \frac{1}{r^2}\frac{\delta^2 p}{\delta \theta^2} = 0$

How to find the diff eq to solve:

- Separate variables by assuming solution is $p(r, \theta) = R(r)\Theta(\theta)$ and that $\Theta(\theta) + a\sin(mt) + b\sin(mt)$.
- The last assumption means that $\Theta'' - \Theta$
- Churning through, we have $\Theta(\theta)\{\frac{1}{r}\frac{\delta}{\delta r}(r\frac{\delta}{\delta r}[R(r)]) - \frac{m^2}{r^2 R(r)}\} = 0$
  $\Rightarrow r^2R''(r) + rR'(r) - m^2R(r) = 0$
- This is a special case of **Cauchy-Euler** problem $t^2y'' + \alpha ty' + \beta y, t > 0, \alpha, \beta \in \mathbb{R}$, with $\alpha = -t, \beta = -m^2$, which is different than Airy equation $y'' - ty = 0$ since the highest-orer derivative has a multiple of t

How to solve the diff eq attempt 1: direct power series

- Plugging $y(t) = \sum_{n=0}^{\infty} a_n t^n$ and similar for y', y'', we list terms and and see $0 = \beta a_0 + [\alpha + \beta]a_1 t_+ \sum_{n=2}^{\infty} a_n[n(n-1) + \alpha n + \beta]$
- Even though Cauchy-Euler suggests $t > 0$, we suppose the equation needs to work for all t (Ist this obvious?), so we know a_0.
- However, since the whole power series is identically zero at all times, this MUST mean all the coefficients are zero. (TODO: When does this NOT happen? When we have initial conditions?). So this method only produces $y(t) = 0$

How to solve the diff eq attempt 2: Try $y(t) = t^r$

- Plugging $y(t) = t^r$ into $y(t) = \sum_{n=0}^{\infty} a_n t^n$, we see that r must satisfy $r(r-1)+\alpha r+\beta = 0$ (this is a characteristic equation or **indicial equation**, a term used in the Frobenius method of DE solving. Apparently this is what we're doing).
- Finding two linearly indepedent solutons of form $y(t) = t^r$: $W[y_0, y_1](t) = \begin{pmatrix} t^{r_0} & t^{r_1} \\ r_0 t^{r_0-1} & r_1 t^{r_1-1} \end{pmatrix}$
- The determinant $W(t) = [r_1 - r_0]t^{r_0+r_1-1}$ which is nonzero on $t > 0$ iff $r_1 - r_0 =\neq 0$

- So we have different r_0 and r_1, which satisfy the characteristic equation $t^2y''+\alpha ty'+\beta y$ if they are basically quadratic roots: $r_\pm = \frac{1-\alpha\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$ If $(\alpha-1)^2 \neq 0$, then we should have two of them
- Case 1: Roots are distinct and real, so we have $t(t() = a_+t^{r_+} + a_-t^{r_-}$
- Case 2: Roots $r, \bar{r}$ are distinct and complex: $Re(r) \pm i \times Im(r)$.
  - $t^r = t^{Re(r)+iIm(r)} = t^{Re(r)}t^{iIm(r)} = t^{Re(r)}e^{iIm(r)ln(t)}$ (by changing t base)

    $= t^{Re(r)}[\cos(Im(r)\ln(t)) + i\sin(Im(r)\ln(t)]$ (by Euler identity)
  - Same argument gives $t^{\bar{r}} = t^{Re(r)}[\cos(Im(r)\ln(t)) - i\sin(Im(r)\ln(t)]$
  - So combinations of $at^r+bt^{\bar{r}}$ end up being $t^{Re(r)}[c_0\cos(Im(r)\ln(t))+c_1\sin(Im(r)\ln(t)]$ (TODO Verify more rgiorosuly)
- Case three: Repeated (necessarily real) root. There is a t^r but also another one out there. Use Abel's formula!
  - $W(t) = \det\begin{pmatrix} t^r & y_1(t) \\ rt^{r-1} & y_1'(t)\end{pmatrix} = t^ry_1'(t) - rt^{r-1}y_1(t)$
  - This is called **reduction of order** since with a root in hand, we are now in a first-order equation.
  - To solve $t^2y'' + \alpha ty' + \eta y = 0$, write as Cauchy-Euler form $y''(t) + p_1(t)y'(t) + p_2(t)y(t) = 0$, or $y''(t) = \frac{\alpha}{t}y'(t) + \frac{\beta}{t^2}y(t) = 0$. Use $W(t) = c\exp\{-\int p_1(t)dt\} = c\exp\{-\int \frac{\alpha}{t}dt\} \Rightarrow W(t) = ct^{-a}, t > 0$
  - Finally, solve our equation $t^ry_1'(t) - rt^{r-1}y_1(t) = ct^{-\alpha}$. Note that an integrating factor u is something that helps us turn the left side into $\frac{d}{dt}[u \times y] = uy' + u'y$. In this case, we're looking for u so $\frac{du}{dt} = -\frac{r}{t}u$, or $u = t^{-r}$ (a separable equation)
  - Setting $c = 1$ for simplicity, then $\frac{d}{dt}[t^{-r}y_1] = t^{-r}t^{-\alpha-r}$. But remember, from the quadratic equation, that $r = \frac{1-\alpha}{2}$, so solve $\frac{d}{dt}[t^{-r}y_1] = t^{-1}$
  - This is $y_1(t) = Ct^r + t^r\ln(t)$. Since $y_0(t) = Ct^r$, the combination solution is $Ct^r + Dt^r\ln(t)$

For some reason, in our air equation, $r^2R(r)'' + rR'(r) - m^2R(r) = 0$, $m = 0$, so we have a Cauchy-Euler with $\alpha = 1$. and a solution with repeated root 0. So our solution ends up being $R(r) = ar^0 + br^0\ln(r) = a + b\ln(r)$

- TODO