

Strategy for bitches (a dice game)

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Abstract

This paper details a decision framework for playing well at bitches (a dice game), introduced to me by Larry Waldman. Mr. Waldman asked for a truly *optimal* strategy but, like blackjack, the state of the game is too large for a human to remember during gameplay. With some simplifications, we can produce a memorable “blackjack table” that performs reasonably well.

1 Introduction

Following is a series of desperately important text messages from Larry “F.” Waldman about the game “bitches (a dice game)”, buyable here (<https://gluebunnygames.com/products/bitches-a-dice-game>), or easy to assemble from fifteen dice and the frantic instructions below in Figs. 1 and 2.

Gotta tell you about this game. I need a paper on the game theory optimized strategy

Figure 1: Larry has to talk to me

I find the use of “regular dice” to be six-normative but agree to proceed with the following two missions:

- Determine the strategy for getting the best expected score (“optimal strategy”).
- Determine the expected number of points from playing that strategy.

Note: I generally proceed with solving an equivalent game: picking up a die gives you as many points as the face value. So, instead of getting one (bad) point when picking up a six-sided die showing five, you receive five (good) points. A score minimizing strategy using the “bitches” scoring system and one maximizing this scoring system are equivalent;

12 regular dice + 1 8 sided die, 1 10 sided die, 1 12 sided die.

Roll all the dice. You must take at least one die out each roll. If the die (or dice) you remove are not their max value (eg 6 for a regular die), you get max - <die value> amount of points. So for example if you take away a regular die showing four, you get two points.

Points are bad.

After all the dice are gone, add up your points and that is your score.

Figure 2: Larry is unhelpful after this point

accepting a die with s sides gives you $s - p$ (bad) points in the first system and p in the second. We can convert “expected face value” to “expected ‘b (a. d. g.)’ points” trivially when we’re done.

The proof/algorithm sketch:

- Section 2: We can solve the game definitively for one die of any size.
- Section 3: Solving the game perfectly for the standard “bitches” setup (15 dice of sizes indicated in Fig. 2) requires evaluating way too many states and transitions to be feasible, fun, or useful in play.
- Section 4: We can reduce the game by considering 15 dice as each playing parallel and separate single-die games and bear the cost when this ends up being untrue.
- Section 5: The expected score at the end can then be approximated readily.

2 Solving the one die case

As the old saying goes, “if you cannot play chess well with 3 pieces, you cannot play well with 32.” Therefore, in chess, we study endgames first and here, first solve the game with played with a single die with unique incremental pip counts 1, 2, 3... s up to some arbitrary s .

Though we make a misstep in solving this next (the Aside section), the following principle is rock solid.

Principle: Know when to quit

- We establish an optimal strategy for the base case $r = 1$ (one roll remaining).
- With $r > 1$ rolls remaining, if accepting the current value of the die has a higher expected value than the expected value using the optimal strategy going forward, accept the die value. Else roll.

If¹ we have an optimal strategy for $r = 1$, then we necessarily have one for $r = 2, 3, \dots$ and so on.

2.1 Aside: (Incorrectly) using the expected maximum future roll

We will now take a misstep in finding the optimal strategy, an example of an intuitive but wrong move in probabilistic thinking. It helps illustrate why the best strategy is best, and so we include it.

The logic: We, of course, want to choose the best future roll from this die, so the **expected max strategy** is *if the current roll is worse than the expected future maximum among the r rolls remaining, keep rolling. Otherwise, pick up the die and accept the points.*

This *seems* like the right idea. After all, if we can expect our future rolls to include a better roll on average, we should keep rolling, right? This is true, but not the whole story.

At least it's easy to calculate. Given s (a single die's number of sides) and r rolls remaining, the expected value of the max roll value m is straightforward:

$$p_{m \leq k}(r) = \left(\frac{k}{s}\right)^r \quad (1)$$

$$p_{m \leq k-1}(r) = \left(\frac{k-1}{s}\right)^r \quad (2)$$

$$p_{m=k}(r) = \left(\frac{k}{s}\right)^r - \left(\frac{k-1}{s}\right)^r \quad (3)$$

$$E_r = \sum_{k=1}^s k \left[\frac{k^r - (k-1)^r}{s} \right] \quad (4)$$

For any given number of pips k between 1 and s :

- (1) is the probability that all of the r rolls are between 1 and k .
- (2) is the probability that all of the r rolls are between 1 and $k-1$.
- (3) subtracts these two for the probability that the max roll is exactly k (as in, there's at least one k roll in the space of (1)).

¹We do: we only have the option of accepting the roll. Expected face value is clearly $\frac{s+1}{2}$.

- (4) is the familiar expected value formula.

Fig. 3. shows the results for 6-, 8-, 10-, and 12-sided dice.

r	$s = 6$	$s = 8$	$s = 10$	$s = 12$
1	3.5	4.5	5.5	6.5
2	4.472	5.812	7.15	8.486
3	4.958	6.469	7.975	9.479
4	5.245	6.858	8.467	10.072
5	5.431	7.115	8.792	10.465
6	5.56	7.295	9.022	10.744
7	5.654	7.428	9.192	10.952
8	5.724	7.529	9.323	11.111
9	5.778	7.608	9.426	11.238
10	5.82	7.67	9.509	11.34
11	5.853	7.721	9.576	11.424
12	5.88	7.763	9.633	11.495
13	5.901	7.798	9.68	11.554
14	5.919	7.827	9.72	11.605
15	5.933	7.851	9.754	11.648

Figure 3: Expected max roll of an s -sided die over r rolls

We then, look to the expected value of the next state and pick up our die if the value equals or exceeds our expected future value. For example, if we're rolling a 6-sided die and we have three rolls left, after rolling we pick up if and only if the roll exceeds what we'd expect by taking our shot with two rolls left (4.472). This means we'd pick up a 5 or 6. If we continue, we would pick up anything equal to or exceeding 3.5 ($r = 1, s = 6$).

2.2 Why doesn't this work?

Though we would like to choose the maximum roll across our future rolls, and we can be reasonably confident some high numbers are ahead of us if we have many rolls remaining, we have a problem: *we are not guaranteed to choose the max roll when we see it*, even when appropriately we adjust our expectations of the max roll *down* as we near the end of the cliff as in Fig. 3.

Counterexample to max strategy

Consider this example with a 50-sided die and three rolls remaining, with $E_{max}(s, r)$ signifying the expected value of the **max** roll over r rolls of a s -sided die, which can be computed with formula (4) above.

- $E_{max}(50, 1) = 25.5$. You must pick up once rolled.

- $E_{max}(50, 2) = 33.83$. You should pick up if your roll equals or exceeds $\lfloor E_{max}(50, 1) \rfloor = 25$.
- $E_{max}(50, 3) = 37.995$. You should pick up if your roll equals or exceeds $\lfloor E_{max}(50, 2) \rfloor = 33$.

The reroll cutoff (inclusive) on $r = 2$ will be 25 for any optimal strategy, since no decisions are possible for $r = 1$. The EV is 31.75.

$$E(2) = \sum_{k=26}^{50} \frac{k}{50} + \frac{25}{50} * 25.5 = 31.75 \quad (5)$$

$$(6)$$

So we look at the expected value of five strategies (A, B(est), C, M(ax), D) then, which are exactly the different cutoffs when $r = 3$: (31, 32, 33, 34, 35), respectively:

$$E_A(3) = \sum_{k=31}^{50} \frac{k}{50} + \frac{30}{50}(31.75) = 35.25 \quad (7)$$

$$E_B(3) = \sum_{k=32}^{50} \frac{k}{50} + \frac{\mathbf{31}}{50}(31.75) = \mathbf{35.265} \quad (8)$$

$$E_C(3) = \sum_{k=33}^{50} \frac{k}{50} + \frac{32}{50}(31.75) = 35.260 \quad (9)$$

$$E_M(3) = \sum_{k=34}^{50} \frac{k}{50} + \frac{\mathbf{33}}{50}(31.75) = 35.235 \quad (10)$$

$$E_D(3) = \sum_{k=35}^{50} \frac{k}{50} + \frac{34}{50}(31.75) = 35.19 \quad (11)$$

$$(12)$$

We see that the expected value of our strategy peaks not at our supposed max strategy of rerolling at 33 or less, but rerolling at 31 or less (accepting 32 and 33).

This counterexample proves that max is not the optimal strategy, though not by a whole lot for this small (and easily calculable) example. The real intuition comes from considering that, for example, if we're three rolls out on a 50-sider, there is a *possibility* that there are one or even *two* fifties in my future; those will drive up the expected maximum roll.

However, the likelihood that I will encounter them is diminished by the change that I'm rolling high thirties or above before that.

Three future in-order rolls of (35, 1, 50) and (50, 1, 35) both contribute 50 equally to the expected maximum. However, I will never encounter the first case, since I will have optimally picked up the 35 by then, which 'shadows' the high rolls. Note that (35, 1, 10) doesn't 'shadow' a future low roll, since the maximum is not the third number but 35.

If this isn't intuitive enough, consider that on a 1000-sided die with three rolls left, you have to exclude the possibility of the sequence (999, *, 1000). This part of the possibility space contributes 1000 to the max strategy, but to no reasonable strategy otherwise. This is why our EVs for the optimal strategy will be lower than the intuitive "pick the expected future maximum". Instead, we need to "pick the expected future maximum *that we will select*".

2.3 Solving the one-die case optimally

Though there doesn't appear to be a closed form for the expectations and cutoffs of the optimal strategy, determining it is no big shakes. If we have r rolls left for a die of size s , and we have the optimal expected value $E(r-1, s)$ for the version of the problem with one less roll, our optimal strategy, according to the *Know when to quit principle*, is to keep rolling unless we equal or exceed the value of continuing. For ease of reading, we introduce cutoff $c(r, s)$ to mean $\lfloor E(r-1, s) \rfloor - 1$. This is the highest number we'd reject (keep rolling with) when evaluating roll r .

$$E(1, s) = \frac{s+1}{2} \tag{13}$$

$$E(r, s) = \sum_{c(r,s) < k \leq s} \frac{k}{s} + c(r, s) \frac{E(r-1, s)}{s}, r > 1. \tag{14}$$

(13) is our known only (and therefore) optimal value of a game with one roll. The first term of (14) is the expected payoff we get by picking up if we exceed our cutoff. The second term is the expected payoff we get by continuing if we don't exceed our cutoff.

So, for playing a one-die version of this game with a die of size s , start by rejecting anything except s , and follow the blue colored entries for where you should start adjusting your cutoffs down.

By building up from ground truth (one roll) to our state of r rolls remaining, we have shown that this is the optimal algorithm for one die of any size.

rolls left (6-side)	max reject	expected face	exp badg points
1		3.5	2.5
2	3	4.25	1.75
3	4	4.667	1.333
4	4	4.944	1.056
5	4	5.13	0.87
6	5	5.275	0.725
7	5	5.396	0.604
8	5	5.496	0.504
9	5	5.58	0.42
10	5	5.65	0.35
11	5	5.709	0.291
12	5	5.757	0.243
13	5	5.798	0.202
14	5	5.831	0.169
15	5	5.859	0.141

Figure 4: Optimal 6-sided die strategy

rolls left (8-side)	max reject	expected face	exp badg points
1		4.5	3.5
2	4	5.5	2.5
3	5	6.063	1.938
4	6	6.422	1.578
5	6	6.691	1.309
6	6	6.894	1.106
7	6	7.045	0.955
8	7	7.165	0.835
9	7	7.269	0.731
10	7	7.36	0.64
11	7	7.44	0.56
12	7	7.51	0.49
13	7	7.571	0.429
14	7	7.625	0.375
15	7	7.672	0.328

Figure 5: Optimal 8-sided die strategy

3 The Architecture of a perfect solution

:

rolls left (10-side)	max reject	expected face	exp badg points
1		5.5	4.5
2	5	6.75	3.25
3	6	7.45	2.55
4	7	7.915	2.085
5	7	8.241	1.76
6	8	8.492	1.508
7	8	8.694	1.306
8	8	8.855	1.145
9	8	8.984	1.016
10	8	9.087	0.913
11	9	9.179	0.821
12	9	9.261	0.739
13	9	9.335	0.665
14	9	9.401	0.599
15	9	9.461	0.539

Figure 6: Optimal 10-sided die strategy

rolls left (12-side)	max reject	expected face	exp badg points
1		6.5	5.5
2	6	8	4
3	7	8.833	3.167
4	8	9.389	2.611
5	9	9.792	2.208
6	9	10.094	1.906
7	10	10.328	1.672
8	10	10.523	1.477
9	10	10.686	1.314
10	10	10.822	1.178
11	10	10.935	1.065
12	10	11.029	0.971
13	11	11.11	0.89
14	11	11.184	0.816
15	11	11.252	0.748

Figure 7: Optimal 12-sided die strategy

The problem is that the state of the 15-die game is not quite as simple.

Whereas the one-die game's² state S can be summarized $S = [0, r] \in \mathbb{N}$, that is, a single integer between 0 and r rolls left, inclusive, we now have state $S = [0, 12] \times [0, 1] \times [0, 1] \times [0, 1] \in \mathbb{N}^4$

Solving the game perfectly for the standard “bitches” setup (15 dice of various sizes) requires evaluating the expected value of 103 different states with a max of 5,940,480 possibilities per state.

4 Splitting into parallel games

We can reduce the game by considering 15 dice as each playing separate one-die games and bear the cost when this ends up being untrue.

5 Expected Score

The expected score at the end can then be approximated readily.

²Note that we expanded the one-die game a bit with r free rolls instead of a boring one-roll game