

# Brilliant: Vector Calculus

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Note: Latex reference: <http://tug.ctan.org/info/undergradmath/undergradmath.pdf>

## 1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against  $t$  of the form  $\vec{x}(t) = \langle x(t), y(t), \dots \rangle$ .

- A **line** through  $p = (a, b, c)$  parallel to  $\vec{v} = \langle v_x, v_y, v_z \rangle$  is  $\vec{x}(t) = \vec{p} + t\vec{v}$
- **velocity** is characterized completely by  $\vec{v}(t) = \vec{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ .
- The **speed** of an object along that line versus  $t$  is the length of  $v$  ( $\|v\|$ )
- Therefore, the speed of an object along line

$$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

is

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that  $\vec{v}$  need not be constant. The speed of

$$\vec{x}(t) = \vec{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

would then be

$$\|6\pi \cos(2\pi t)\hat{u}\| = |6\pi \cos(2\pi t)|$$

- **Acceleration**  $a(t) = v'(t) = x''(t)$  is straightforward. Acceleration of

$$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance  $r$  from the sun could be  $\langle r \cos(t), r \sin(t) \rangle$ . The acceleration vector points in the opposite direction:  $\langle -r \cos(t), -r \sin(t) \rangle$ . In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by  $F = ma$  is proportional to acceleration) points towards the sun.

- A **helix** could be a 3D extension like  $\langle r \cos(t), r \sin(t), b \cdot t \rangle$ .

## 2 Chapter 2.2: Space Curves

- TODO: Problem 5 - rotating ellipses and solving intersections with planes
- Note that while  $\vec{x}(t) = \langle \cos(t), \sin(t), 5 \rangle$  and  $\vec{x}(t) = \langle \cos(2t), \sin(2t), 5 \rangle$  describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If  $\vec{x}(t) = t \vec{v}$ , then speed is  $\frac{\|\vec{x}(t+\Delta t) - \vec{x}(t)\|}{\Delta t} = \|\vec{v}\|$ , direction is  $\frac{\vec{v}}{\|\vec{v}\|}$ , and velocity  $\vec{v}$  is the product of speed and direction.
- So  $\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} = \vec{x}'(t) = \frac{d\vec{x}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$
- Neat conceptual result: any  $y = f(x)$  can be made into  $x(t) = \langle t, f(t), 0 \rangle$ , and then  $v(t) = \langle 1, f'(t), 0 \rangle$ , which points along the tangent line at  $\langle t, f(t), 0 \rangle$ .
- Note that dot product derivatives work like regular product:  $[\vec{a}(t) \cdot \vec{b}(t)]' = \vec{a}'(t) \cdot \vec{b}(t) + \vec{a}(t) \cdot \vec{b}'(t)$ , but the cross product does not work the same since  $\frac{d}{dt}[a \times b] = a' \times b + a \times b'$ , but since  $a \times b' = -b' \times a$ , can't switch the order to  $a' \times b + b' \times a$  due to this non-commutativity.
- If

$$\vec{x}(t) = \vec{p} + t \vec{v},$$

calculating velocity with respect to origin becomes

$$\frac{d}{dt} \|\vec{x}(t)\| = \frac{\vec{x}(t) \cdot \vec{x}'(t)}{\|\vec{x}(t)\|} = \frac{\vec{x}}{\|\vec{x}\|} \cdot \vec{v},$$

after rewriting the distance formula and chugging through the chain rule.

- However, it becomes more clear when considering that  $(\vec{v} \cdot \hat{x})\hat{x}$  is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

## 3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way:  
 $\int_a^b \vec{x}(t) = \langle \int_a^b x(t), \int_a^b y(t), \int_a^b z(t) \rangle$

- Example: if ball launched from origin with velocity  $\langle 1, 2, 3 \rangle$  and acceleration  $\langle 0, 0, -1 \rangle$ , it lands at

$$\frac{dv}{dt}dt = \langle 0, 0, -1 \rangle \quad (1)$$

$$\int \frac{dv}{dt}dt = v = \langle C, D, -t + F \rangle = \langle 1, 2, 3 \rangle = \langle 1, 2, -t + 3 \rangle, t = 0 \quad (2)$$

$$x = \int v = \langle t + K, 2t + M, -\frac{1}{2}t^2 + 3t + N \rangle, x(\vec{0}) = \langle 0, 0, 0 \rangle \quad (3)$$

$$\vec{x}(t) = \langle t, 2t, 3t - \frac{1}{2}t^2 \rangle \quad (4)$$

$$z(t) = 0 \rightarrow t = 6 \rightarrow \vec{x}(6) = \langle 6, 12, 0 \rangle \quad (5)$$

$$(6)$$

- Also, generalizing  $ds = \sqrt{(dx)^2 + (dy)^2}$ , the length of an arc from point  $a$  to  $b$  approaches  $\int_a^b \|x'(t)\|dt$
- Example: a helix  $\langle a \cos(\omega t), a \sin(\omega t), b\omega t \rangle$ , parametrized by time  $t$  can be rewritten in terms of  $s$ , the arc length:

$$s = \int \|x'(t)\|dt \quad (7)$$

$$s = \int \sqrt{(-\omega a \sin(\omega t))^2 + (\omega a \cos(\omega t))^2 + (b\omega)^2}dt \quad (8)$$

$$s = |\omega| \int \sqrt{a^2 + b^2}dt \quad (9)$$

$$s = |\omega|t\sqrt{a^2 + b^2} \quad (10)$$

- Note: It's weird to think of time in terms of length. Could be analytically useful?

## 4 Chapter 2.4: Frenet Formulae

Main idea: Establish three new vectors  $\hat{T}(s), \hat{N}(s), \hat{B}(s)$  that change as we move along a space curve, instead of  $\vec{x}(t)$  that changes over an external "time" idea.

Remember that  $s = \int_0^t \|\vec{x}'(\tilde{t})\|d\tilde{t}$ , so  $\frac{ds}{dt} = \|\vec{x}'(t)\|$ .

### 4.1 $\hat{T}$ : Vector tangent to space curve

- Remember arc length is  $s = \int_0^t \|\vec{x}'(\tilde{t})\|d\tilde{t}$
- $\hat{T}$  is just normalized grad:  $\frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$

- This implies  $\frac{d\vec{x}}{ds} = \hat{T}$  since

$$s = \int_0^t \|\vec{x}'(\tilde{t})\| d\tilde{t} \quad (11)$$

$$\frac{ds}{dt} = \|\vec{x}'(t)\| \quad (12)$$

$$\hat{T} = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \frac{d\vec{x}}{dt} \cdot \frac{dt}{ds} \quad (13)$$

$$\hat{T} = \frac{d\vec{x}}{ds} \quad (14)$$

$$(15)$$

## 4.2 $\hat{N}$ : Vector normal to space curve and also in the direction of acceleration

Normal vectors include

- $\frac{\hat{T}(t)}{\|\hat{T}(t)\|}$  since, as  $\|\hat{T}(t)\|$  is just 1:

$$d(\|\hat{T}\|^2) = 0 \quad (16)$$

$$d(\|\hat{T}\|^2) = d(\hat{T} \cdot \hat{T}) = \hat{T}(t) \cdot 2\hat{T}'(t) \quad (17)$$

$$\hat{T}(t) \cdot \hat{T}'(t) = 0 \quad (18)$$

- $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|}$  since it's the same as the above, but parametrized over  $s$  instead of  $t$ . Doesn't change the direction of the vector!

Example: if  $\vec{x}(t) = \langle R \cos(\omega t), R \sin(\omega t), 0 \rangle$ , then acceleration  $\vec{a}(t)$  is

- $\vec{a} = \frac{d^2\vec{x}}{dt^2}$  just by definition
- $\vec{a} = -\omega^2 \vec{x}$  just by calculation
- $\hat{T}(t) = \langle -\sin(\omega t), \cos(\omega t), 0 \rangle$
- $\|\hat{T}(t)\| = 1$
- $\hat{N} = \frac{\hat{T}(t)}{\|\hat{T}(t)\|} = \langle -\cos(\omega t), -\sin(\omega t), 0 \rangle$
- So  $\vec{a} = R\omega^2 \hat{N}$  by these formulae.

This leads us to believe acceleration and  $\hat{N}$ , the normed derivative of  $\hat{T}$  are related.

The part of acceleration  $\vec{a}$  parallel to  $\hat{T}$  is the projection  $(\vec{a} \cdot \hat{T})\hat{T}$

The perpendicular part is then  $\vec{a}$  minus that:  $\vec{a} - (\vec{a} \cdot \hat{T})\hat{T}$

This also equals  $(\frac{ds}{dt})^2 \|\frac{d\hat{T}}{ds}\| \hat{N}$  because

$$\vec{x} = \frac{dx}{dt} = T = \hat{T} \cdot \|\frac{dx}{dt}\| \quad (19)$$

$$s = \int_0^t \|\vec{x}'(t)\| dt \rightarrow \frac{ds}{dt} = \|\vec{x}'(t)\| \quad (20)$$

$$\hat{N} = \frac{d\hat{T}}{ds} \text{ normalized, so} \quad (21)$$

$$\vec{a} = \frac{d^2 \vec{x}}{dt^2} = \frac{d}{dt}(\|\vec{x}'(t)\|\hat{T}(t)) = \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + \|\vec{x}'(t)\|\frac{d\hat{T}}{dt} \quad (22)$$

$$= \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + \frac{ds}{dt} \frac{d\hat{T}}{ds} \frac{ds}{dt} \quad (23)$$

$$= \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + (\frac{ds}{dt})^2 \|\frac{d\hat{T}}{ds}\| \hat{N} \quad (24)$$

This is “a = parallel part plus perpendicular (N) part”, so the second term is  $a_{\perp}$

### 4.3 Binormal vector $\hat{B}$

Note that curvature  $\kappa(s) = \|\frac{d\hat{T}}{ds}\|$  is geometric (depends on s, not time) and changes as  $\hat{T}$  changes.

Example: Curvature of  $\vec{x}(t) = \langle \cos(t), \sin(t), bt \rangle$

$$x'(t) = \langle -\sin(t), \cos(t), b \rangle \quad (25)$$

$$\|x'(t)\| = \sqrt{1 + b^2} \quad (26)$$

$$s = \int_0^t \|x'(t)\| dt = \int_0^t \sqrt{1 + b^2} dt = t\sqrt{1 + b^2} \rightarrow t = \frac{s}{\sqrt{1 + b^2}} \quad (27)$$

### 4.4 $\hat{T}$ is:

- $\vec{x}'(t)$  normalized
- The tangent vector to the curve
- The same whether parametrized by  $\hat{T}'(t)$  or  $\frac{dx}{ds}$

### 4.5 $\hat{N}$ is:

- $\vec{x}''(t)$  normalized as  $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} = \hat{N}$

- The normal vector to the curve
- $\perp$  to  $\hat{T}$  in direction of acceleration. So a multiple of acceleration vector.
- The same whether parametrized by  $\hat{T}'(t)$  or  $\frac{dx}{ds}$

#### 4.6 $\hat{T}$ and $\hat{N}$

:

- Form a plane, since first, any normal vector's derivative is perpendicular to the vector

$$\frac{d}{ds} \|\hat{T}\|^2 = \frac{d}{ds} \hat{T} \cdot \hat{T} \quad (28)$$

$$= 2\hat{T} \cdot \hat{T}' \quad (29)$$

$$\frac{d}{ds} \|\hat{T}\|^2 = \frac{d}{ds} 1 = 0 \quad (30)$$

$$(31)$$

and

$$\hat{T} \cdot \hat{N} = \hat{T} \cdot \frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} \quad (32)$$

$$= \hat{T} \cdot \frac{\hat{T}'(s)}{\|\frac{d\hat{T}}{ds}\|} = 0 \quad (33)$$

- $\kappa$  is curvature: how much we're curving in that  $T \times N$  plane.
- $\kappa = \|\frac{d\hat{T}}{ds}\|$
- Therefore, by above,  $\frac{d\hat{T}}{ds} = \kappa \hat{N}$  (**Frenet equation 1**)

#### 4.7 $\hat{B}$ is binormal: perpendicular to both

- defined as  $\hat{B} = \hat{T} \times \hat{N}$
- Therefore, by derivative

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \quad (34)$$

$$\frac{d\hat{B}}{ds} = \kappa \hat{N} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \quad (35)$$

$$\frac{d\hat{B}}{ds} = \hat{T} \times \frac{d\hat{N}}{ds} \quad (36)$$

$$(37)$$

but this means  $T$  is orthogonal to  $dB$ , and we already know  $B$  and  $dB$  are orthogonal. We're working in 3d with the cross product, so  $dB$  is parallel to  $N$ .

- Therefore, we define "torsion"  $\tau$  so that  $-\frac{d\hat{B}}{ds} = \tau\hat{N}$  (**Frenet equation 2**). Negative sign by convention.
- Can also cross by  $N$  on both sides to get  $-\frac{d\hat{B}}{ds} \times \hat{N} = \tau$
- $\hat{B}$  measures how the plane defined by  $\hat{T}, \hat{N}$  twists around. On a circle,  $\hat{B}$  wouldn't change, so the derivative would be zero.
- **Final Frenet equation.** Prereq:  $\hat{B} = \hat{T} \times \hat{N} \rightarrow \hat{N} = \hat{B} \times \hat{T} \rightarrow \hat{T} = \hat{N} \times \hat{B}$

$$\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} \quad (38)$$

$$\frac{d\hat{N}}{ds} = -\tau\hat{N} \times \hat{T} + \hat{B} \times \kappa\hat{N} \quad (39)$$

$$\frac{d\hat{N}}{ds} = \tau\hat{B} - \kappa\hat{T} \quad (40)$$

## 5 Chapter 2.5: Parametrized Surfaces

Main ideas:

- Can parameterize by  $\vec{x}(u, v) = x(u, v), y(u, v), z(u, v)$
- Can perhaps parameterize  $f(x, y, z) = c$  by  $z = g(x, y)$
- Can also use ideas like  $\nabla f = 0$  to find a normal.

There are many out-of-the-box parametrizations including:

- Sphere at 0,0,0:  $\vec{x}(u, v) = \langle R \cos(u) \sin(v), R \sin(u) \sin(v), R \cos(v) \rangle$ , where  $u \in [0, 2\pi), v \in [0, \pi]$
- Rotate function  $y = f(x)$  around the x-axis:  $\vec{x}(u, v) = \langle u, f(u) \cos(v), f(u) \sin(v) \rangle$ , where  $u \in D, v \in [0, 2\pi]$

Tangent vectors to  $\vec{x}(u, v)$  are  $\frac{\delta \vec{x}}{du}, \frac{\delta \vec{x}}{dv}$ , so unit normal is  $\pm \frac{\frac{d\vec{x}}{du} \times \frac{\delta \vec{x}}{dv}}{\|\frac{d\vec{x}}{du} \times \frac{\delta \vec{x}}{dv}\|}$

Example: Torus  $\vec{x} = \langle [2 + \cos(v)] \cos(u), [2 + \cos(v)] \sin(u), \sin(v) \rangle, u, v \in [0, 2\pi)$ . What's

the tangent plane at  $u = \frac{\pi}{4}, v = 0$ ?

$$dx/du = \langle -\sin(u)(2 + \cos(v)), \cos(u)(2 + \cos(v)), 0 \rangle \quad (41)$$

$$dx/dv = \langle -\sin(v) \cos(u), -\sin(v) \sin(u), \cos(v) \rangle \quad (42)$$

$$u = \frac{\pi}{4}, v = 0 \rightarrow dx/du \quad (43)$$

$$= -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0, dx/dv = 0, 0, 1 \quad (44)$$

$$dx/du \times dx/dv = \langle \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0 \rangle \quad (45)$$

$$\hat{n} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle \quad (46)$$

$$\hat{n} \cdot \vec{x} = 0 \rightarrow \hat{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \quad (47)$$

$$\rightarrow \dots \rightarrow x + y = 3\sqrt{2} \quad (48)$$

$$(49)$$

### 5.1 Example: Ellipsoid $x^2 + 2y^2 + z^2 = 4$

What's the normal at  $(1, \frac{1}{\sqrt{2}}, \sqrt{2})$ ?

**Method 1: parametrize with spherical  $u, v$**  First, transform to sphere with change of coordinates

$$x^2 + 2y^2 + z^2 = 4 \quad (50)$$

$$X = x/2, Y = \frac{Y}{\sqrt{2}}, Z = z/2 \quad (51)$$

$$X^2 + Y^2 + Z^2 = 1 \quad (52)$$

$$X = \cos(u) \sin(v), Y = \sin(u) \sin(v), Z = \cos(v) \quad (53)$$

$$p = (1, \frac{1}{\sqrt{2}}, \sqrt{2}) \rightarrow u = v = \frac{\pi}{4} \quad (54)$$

$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle -1, \frac{1}{\sqrt{2}}, 0 \rangle \quad (55)$$

$$\frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \frac{1}{\sqrt{2}}, -\sqrt{2} \rangle \quad (56)$$

$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) \times \frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \sqrt{2}, \sqrt{2} \rangle \quad (57)$$

$$\hat{n}_{out} = \frac{\langle -1, -\sqrt{2}, -\sqrt{2} \rangle}{\sqrt{5}} \quad (58)$$

**Method 2: rewrite as  $z = g(x, y)$**



$$x^2 + 2y^2 + z^2 = 4 \quad (59)$$

$$z = (4 - x^2 - 2y^2)^{\frac{1}{2}} \quad (60)$$

$$dz/dx = \frac{1}{2} \times -2x(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -\frac{1}{\sqrt{2}} \quad (61)$$

$$dz/dy = \frac{1}{2} \times -4y(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -2\sqrt{2}/\sqrt{2} = -1 \quad (62)$$

$$f \approx \sqrt{2} + dz/dx(1, \frac{1}{\sqrt{2}})(x - 1) + dz/dy(1, \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) \quad (63)$$

$$\rightarrow \dots \rightarrow \frac{1}{\sqrt{2}}x + y + z = 2\sqrt{2} \quad (64)$$

$$(65)$$

giving us normal vector  $\langle \frac{1}{\sqrt{2}}, 1, 1 \rangle = \frac{\langle 1, \sqrt{2}, \sqrt{2} \rangle}{\sqrt{5}}$  after normalization.

### Method 3: gradient

Gradient is always normal to the tangent plane. Recognize level set of  $f(x, y, z) = x^2 + 2y^2 + z^2$ .

$$\nabla f = \langle 2x, 4y, 2z \rangle \rightarrow \nabla f(1, \frac{1}{\sqrt{2}}, \sqrt{2}) = \langle 2, 2\sqrt{2}, 2\sqrt{2} \rangle$$

Then normalize.

## 5.2 Mobius strip and "direction of out"

Mobius strip is

- $x = 2 \cos(u) + v \cos(\frac{u}{2})$
- $y = 2 \sin(u) + v \cos(\frac{u}{2})$
- $z = v \sin(\frac{u}{2})$
- $u \in [0, 2\pi], v \in [-\frac{1}{2}, \frac{1}{2}]$

$$\hat{n} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} \text{ at } (0,0) \text{ is } \langle 0, 0, -1 \rangle,$$

but at  $(2\pi, 0)$  is the same point, but  $\hat{n} = \langle 0, 0, 1 \rangle!!$

## 6 Chapter 2.6: Vector Fields

(Lots of intuition questions here...)

One nugget: using **gradient vector fields**: Suppose  $\vec{F}(x, y) = \langle 2, -4y^3 \rangle$ . Then if  $F = \nabla f$ , then  $F$ 's arrows are perpendicular to a level set  $f = c$ . So look at  $f = 2x - y^4$  and find perpendicular arrows to these. That's actually  $F$ !

**Linear approximation for  $\vec{F} : D \in \mathbf{R}^n \rightarrow \mathbf{R}^m$**

Main idea:  $\vec{F}(\vec{x}) = \vec{F}(\vec{a}) + A(\vec{a})(\vec{x} - \vec{a})$

Note that  $A$  takes in vectors of size  $n$  (so it has that many columns), and has  $m$  functions (rows) that operate on it. So the Jacobian,  $A$ , has as row  $i$ , column  $j$ , the quantity  $\frac{dF_i}{dx_j}(\vec{a})$ .

$dF_i/d\vec{x}$  extends across row  $i$ .

## 7 Chapter 2.7: Jack and the Beanstalk (Newton's method)

**Basis for Newton's:**

If we're estimating  $x_1$  by following the derivative at  $x_0$ , this means we're looking at the line with x-intercept  $x_1$ , with slope  $f'(x_0)$ .

So instead of  $y = mx + b$ , we'll flip the two and use

$$x = y/m + x_{int}$$

$$\text{or } x_0 = f(x_0) \frac{1}{f'(x_0)} + x_1,$$

$$\text{or } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Note that, under Newton's something like  $|x|$  will converge immediately,  $x^3$  will converge moderately, and a S-curve might barely converge if at all.

The extension of this with the Jacobian matrix  $A = DF'(x_0)$  is  $\vec{x}_1 = \vec{x}_0 - (D\vec{F}(\vec{x}_0))^{-1} \vec{F}(x_0)$

## 8 Chapter 2.8: Electrostatic bootcamp

Electric charge radiates out equally in all directions, and is inversely proportional to distance.

Formula, with  $Q$  as the charge,  $\epsilon_0$  is a constant:  $\vec{E}(x, y, z) = \frac{Q}{4\pi\epsilon_0\|x\|^2} \hat{x}$

**Field line** is a special case of a flow line - the space curve that follows  $\vec{F}$ 's arrows. The tangent vector to the flow line is  $\vec{F}(\vec{x}(t))$  ( $t$  is not time here), so  $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}(t))$

Example: Vector field  $\vec{F}(x, y) = \langle -2y, 3x \rangle$ . What's the flow line through  $(2, 0)$ ?

Solution: Need to solve  $dx/dt = -2y, dy/dt = 3x$ . Key is "separating the equations". Remember  $x$  and  $y$  are functions of  $t$ !

$$\frac{d^2x}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6x. \quad (66)$$

$$\frac{d^2y}{dt^2} = -2\frac{dx}{dt} = -2 \times 3y = -6y. \quad (67)$$

$$x(t) = -6x''(t), y(t) = -6y''(t) \quad (68)$$

$$\rightarrow x = A \cos(\sqrt{6}t) + B \sin(\sqrt{6}t), y = C \cos(\sqrt{6}t) + D \sin(\sqrt{6}t) \quad (69)$$

$$\frac{dx}{dt} = -2y(t) \rightarrow \frac{\sqrt{6}}{2}A \sin(\sqrt{6}t) - \frac{\sqrt{6}}{2}B \cos(\sqrt{6}t) = y(t) \quad (70)$$

$$x(t=0) = 2 \rightarrow A = 2 \quad (71)$$

$$y(t=0) = 0 \rightarrow B = 0 \quad (72)$$

$$\vec{F}(t) = \langle 2 \cos(\sqrt{6}t), \sqrt{6} \sin(\sqrt{6}t), \rangle \quad (73)$$

$$(74)$$

Note: **Field lines** follow rules:

- Go from positive charges to negative
- Density of lines directly relates to how much charge a point has
- Lines don't intersect.
- Corollary: If count of out equals count of in, point has zero charge
- "Number" (to be defined) of field lines in and out of a *surface* related to the charge inside. Upcoming.