Brilliant: Vector Calculus

Dave Fetterman

6/21/22

Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against t of the form $\vec{x}(t) = \langle x(t), y(t), \ldots \rangle$.

- A **line** through $p = (a, b, c)$ parallel to $\vec{v} = \langle v_x, v_y, v_z \rangle$ is $\vec{x}(t) = \vec{p} + t\vec{v}$
- **velocity** is characterized completely by $\vec{v}(t) = \vec{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- The **speed** of an object along that line versus t is the length of v ($\|v\|$)
- Therefore, the speed of an object along line

$$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

  is

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that $\vec{v}$ need not be constant. The speed of

$$\vec{x}(t) = \vec{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

  would then be

$$\|6\pi\cos(2\pi t)\hat{u}\| = |6\pi\cos(2\pi t)|$$

- **Acceleration** $a(t) = v'(t) = x''(t)$ is straightforward. Acceleration of

$$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance r from the sun could be $\langle r\cos(t), r\sin(t) \rangle$. The acceleration vector points in the opposite direction: $\langle -r\cos(t), -r\sin(t) \rangle$. In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by $F = ma$ is proportional to acceleration) points towards the sun, *with acceleration perpendicular to velocity.*

- A **helix** could be a 3D extension like $\langle r\cos(t), r\sin(t), b \cdot t\rangle$.

2 Chapter 2.2: Space Curves

- Note that while $\vec{x}(t) = \langle \cos(t), \sin(t), 5\rangle$ and $\vec{x}(t) = \langle \cos(2t), \sin(2t), 5\rangle$ describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If $\vec{x}(t) = t\vec{v}$, then speed is $\frac{\|\vec{x}(t+\Delta t)-\vec{t}\|}{\Delta t} = \|\vec{v}\|$, direction is $\frac{\vec{v}}{\|\vec{v}\|}$, and velocity $\vec{v}$ is the product of speed and direction.
- So $\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{x}(t+\Delta t)-\vec{x}(t)}{\Delta t} = \vec{x}'(t) = \frac{d\vec{x}}{dt} = \langle x'(t), y'(t), z'(t)\rangle$
- Neat conceptual result: any $y = f(x)$ can be made into $x(t) = \langle t, f(t)\rangle$, and then $v(t) = \langle 1, f'(t)\rangle$, which points along the tangent line at $\langle t, f(t)\rangle$.
- Note that dot product derivatives work like regular product: $[\vec{a}(t)\cdot\vec{b}(t)]' = \vec{a}'(t)\cdot\vec{b}(t) + \vec{a}(t)\cdot\vec{b}'(t)$, but the cross product does not work the same since $\frac{d}{dt}[a \times b] = a' \times b + a \times b'$, but since $a \times b' = -b' \times a$, can't switch the order to $a' \times b + b' \times a$ due to this non-commutativity.
- If
$$\vec{x}(t) = \vec{p} + t\vec{v},$$
calculating velocity with respect to origin becomes
$$\frac{d}{dt}\|\vec{x}(t)\| = \frac{\vec{x}(t)\cdot\vec{x}'(t)}{\|\vec{x}(t)\|} = \frac{\vec{x}}{\|\vec{x}\|}\cdot\vec{v},$$
after rewriting the distance formula and chugging through the chain rule.
- However, it becomes more clear when considering that $(\vec{v}\cdot\hat{x})\hat{x}$ is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way: $\int_a^b \vec{x}(t) = \langle \int_a^b x(t), \int_a^b y(t), \int_a^b z(t)\rangle$
- Example: if ball launched from origin with velocity $\langle 1, 2, 3\rangle$ and acceleration $\langle 0, 0, -1\rangle$,

it lands at

$$\frac{dv}{dt}dt = \langle 0, 0, -1 \rangle \tag{1}$$

$$\int \frac{dv}{dt}dt = v = \langle C, D, -t + F \rangle = \langle 1, 2, 3 \rangle = \langle 1, 2, -t + 3 \rangle, t = 0 \tag{2}$$

$$x = \int v = \langle t + K, 2t + M, -\frac{1}{2}t^2 + 3t + N \rangle, x(\vec{0}) = \langle 0, 0, 0 \rangle \tag{3}$$

$$\vec{x}(t) = \langle t, 2t, 3t - \frac{1}{2}t^2 \rangle \tag{4}$$

$$z(t) = 0 \to t = 6 \to \vec{x}(6) = \langle 6, 12, 0 \rangle \tag{5}$$

$$\tag{6}$$

- Also, generalizing $ds = \sqrt{(dx)^2 + (dy)^2}$, the length of an arc from point a to b approaches $\int_a^b ds = \int_{t_a}^{t_b} \|x'(t)\|dt$
- Example: a helix $\langle a\cos(\omega t), a\sin(\omega t), b\omega t \rangle$, parametrized by time t can be rewritten in terms of s, the arc length:

$$s = \int \|x'(t)\|dt \tag{7}$$

$$s = \int \sqrt{(-\omega a\sin(\omega t))^2 + (\omega a\cos(\omega t))^2 + (b\omega)^2}dt \tag{8}$$

$$s = |\omega| \int \sqrt{(a^2 + b^2)}dt \tag{9}$$

$$s = |\omega|t\sqrt{a^2 + b^2} \tag{10}$$

- *Note: It's weird to think of time in terms of length.* Could be analytically useful?

4 Chapter 2.4: Frenet Formulae

Main idea: Establish three new vectors $\hat{T}(s), \hat{N}(s), \hat{B}(s)$ that change as we move along a space curve, instead of $\vec{x}(t)$ that changes over an external "time" idea.

Remember that $s = \int_0^t \|\vec{x}'(\tilde{t})\|d\tilde{t}$, so $\frac{ds}{dt} = \|\vec{x}'(t)\|$.

4.1 $\hat{T}$: Vector tangent to space curve

- Remember arc length is $s = \int_0^t \|\vec{x}'(\tilde{t})d\tilde{t}\|$
- $\hat{T}$ is just normalized grad: $\frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$

- This implies $\frac{d\vec{x}}{ds} = \hat{T}$ since

$$s = \int_0^t \|\vec{x}'(\tilde{t})d\tilde{t}\| \tag{11}$$

$$\frac{ds}{dt} = \|\vec{x}(t)\| \tag{12}$$

$$\hat{T} = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \frac{d\vec{x}}{dt} \cdot \frac{dt}{ds} \tag{13}$$

$$\hat{T} = \frac{d\vec{x}}{ds} \tag{14}$$

$$\tag{15}$$

- So this is how the space curve $\vec{x}$ changes as it moves along s.
- It's normalized, so it's the same whether parameterized by t, s, or whatever.

4.2 $\hat{N}$: Normal to curve (perpendicular to $\hat{T}$)

Normal vectors are:

- $\vec{x}''(t)$ normalized as $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} = \hat{N}$
- The normal vector to the curve
- $\perp$ to $\hat{T}$ in direction of acceleration. So a multiple of acceleration vector.
- $\frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$. The following sequence shows any unit length vector is perpendicular to its derivative.

$$\|\hat{T}\| = 1 \tag{16}$$

$$d(\|\hat{T}\|^2) = 0 \tag{17}$$

$$d(\|\hat{T}\|^2) = d(\hat{T} \cdot \hat{T}) = \hat{T}(t) \cdot 2\hat{T}'(t) \tag{18}$$

$$\hat{T}(t) \cdot \hat{T}'(t) = 0 \tag{19}$$

- $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|}$ since it's the same as the above, but parametrized over s instead of t. Doesn't change the direction of the vector!

Example:if $\vec{x}(t) = \langle R\cos(\omega t), R\sin(\omega t), 0\rangle$, then:

- $\vec{a} = \frac{d^2\vec{x}}{dt^2}$ just by definition
- $\vec{a} = -\omega^2\vec{x}$ just by calculation

- $\hat{T}(t) = \langle -\sin(\omega t), \cos(\omega t), 0\rangle$
- $\|\hat{T}(t)\| = 1$
- $\hat{N} = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = \langle -\cos(\omega t), -\sin(\omega t), 0\rangle$
- So $\vec{a} = R\omega^2\hat{N}$ by these formulae.

This leads us to believe acceleration and $\hat{N}$, the normed derivative of $\hat{T}$ are related.

The part of acceleration $\vec{a}$ parallel to $\hat{T}$ is the projection $(\vec{a}\cdot\hat{T})\hat{T}$

The perpendicular part is then $\vec{a}$ minus that: $\vec{a} - (\vec{a}\cdot\hat{T})\hat{T}$

This also equals $(\frac{ds}{dt})^2\|\frac{d\hat{T}}{ds}\|\hat{N}$ because

$$\vec{x}' = \frac{dx}{dt} = T = \hat{T}\cdot\|\frac{dx}{dt}\| \tag{20}$$
$$s = \int_0^t \|\vec{x}'(t)\| \to \frac{ds}{dt} = \|\vec{x}'(t)\| \tag{21}$$

$\hat{N} = \frac{d\hat{T}}{ds}$ normalized, so

$$\vec{a} = \frac{d^2\vec{x}}{dt^2} = \frac{d}{dt}(\|\vec{x}'(t)\hat{T}(t)\|) = \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + \|\vec{x}'(t)\|\frac{d\hat{T}}{dt} \tag{22}$$
$$= \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{ds}\frac{ds}{dt} \tag{23}$$
$$= \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + (\frac{ds}{dt})^2\|\frac{d\hat{T}}{ds}\|\hat{N} \tag{24}$$

This is “a = parallel part plus perpendicular (N) part”, so the second term is $a_\perp$

4.3 $\hat{T}$ and $\hat{N}$

- Form a plane, since first, any normal vector’s derivative is perpendicular to the vector
- κ is curvature: how much we’re curving in that $T \times N$ plane.
- $\kappa = \|\frac{d\hat{T}}{ds}\|$
- Therefore, by above, $\frac{d\hat{T}}{ds} = \kappa\hat{N}$ **(Frenet equation 1)**

Note that curvature $\kappa(s) = \|\frac{d\hat{T}}{ds}\|$ is geometric (depends on s, not time) and changes as $\hat{T}$ changes.

Example: Curvature of $\vec{x}(t) = \langle \cos(t), \sin(t), bt \rangle$

$$x'(t) = \langle -\sin(t), \cos(t), b \rangle \tag{25}$$

$$\|x'(t)\| = \sqrt{1+b^2} \tag{26}$$

$$s = \int_0^t \|x'(\tilde{t})\| d\tilde{t} = \int_0^t \sqrt{(1+b^2)} = t\sqrt{(1+b^2)} \to t = \frac{s}{\sqrt{1+b^2}} \tag{27}$$

Do the substitution of $\frac{s}{\sqrt{1+b^2}}$ for t above to get $x'(s)$, and from there, you can figure out $\frac{d\hat{T}}{ds}$ and normalize to get $\kappa = \frac{1}{1+b^2}$

4.4 $\hat{B}$ is binormal: perpendicular to both

- defined as $\hat{B} = \hat{T} \times \hat{N}$
- Therefore, by derivative

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \tag{28}$$

$$\frac{d\hat{B}}{ds} = \kappa\hat{N} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \tag{29}$$

$$\frac{d\hat{B}}{ds} = \hat{T} \times \frac{d\hat{N}}{ds} \tag{30}$$

$$\tag{31}$$

  but this means T is orthogonal to dB, and we already know B and dB are orthogonal. We're working in 3d with the cross product, so dB is parallel to N.

- Therefore, we define **torsion** τ so that $-\frac{d\hat{B}}{ds} = \tau\hat{N}$ **(Frenet equation 2)**. Negative sign by convention.
- Can also cross by N on both sides to get $-\frac{d\hat{B}}{ds} \times \hat{N} = \tau$
- $\hat{B}$ measures how the plane defined by $\hat{T}, \hat{N}$ twists around. On a circle, $\hat{B}$ wouldn't change, so the derivative would be zero.
- **Final Frenet equation**. Prereq: $\hat{B} = \hat{T} \times \hat{N} \to \hat{N} = \hat{B} \times \hat{T} \to \hat{T} = \hat{N} \times \hat{B}$

$$\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} \tag{32}$$

$$\frac{d\hat{N}}{ds} = -\tau\hat{N} \times \hat{T} + \hat{B} \times \kappa\hat{N} \tag{33}$$

$$\frac{d\hat{N}}{ds} = \tau\hat{B} - \kappa\hat{T} \tag{34}$$

5 Chapter 2.5: Parametrized Surfaces

Main ideas:

- Can parameterize by $\vec{x}(u,v) = x(u,v), y(u,v), z(u,v)$
- Can perhaps parameterize $f(x,y,z) = c$ by $z = g(x,y)$
- Can also use ideas like $\nabla f = 0$ to find a normal.

There are many out-of-the-box paremetrizations including:

- Sphere at 0,0,0: $\vec{x}(u,v) = \langle R\cos(u)\sin(v), Rsin(u)\sin(v), R\cos(v)\rangle$, where $u \in [0, 2\pi), v \in [0, \pi]$
- Rotate function $y = f(x)$ around the x-axis: $\vec{x}(u,v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$, where $u \in D, v \in [0, 2\pi]$

Tangent vectors to $\vec{x}(u,v)$ are $\frac{\delta\vec{x}}{du}, \frac{\delta\vec{x}}{dv}$, so unit normal is $\pm\frac{\frac{d\vec{x}}{du}\times\frac{\delta\vec{x}}{dv}}{\|\frac{d\vec{x}}{du}\times\frac{\delta\vec{x}}{dv}\|}$

Example: Torus $\vec{x} = \langle[2+\cos(v)]\cos(u), [2+\cos(v)]\sin(u), \sin(v)\rangle, u, v \in [0, 2\pi)$. What's the tangent plane at $u = \frac{\pi}{4}, v = 0$?

$$d\vec{x}/du = \langle -\sin(u)(2+\cos(v)), \cos(u)(2+\cos(v)), 0\rangle \quad (35)$$

$$d\vec{x}/dv = \langle -\sin(v)\cos(u), -\sin(v)\sin(u), \cos(v)\rangle \quad (36)$$

$$u = \frac{\pi}{4}, v = 0 \rightarrow \quad (37)$$

$$d\vec{x}/du = \langle -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0, \rangle \quad (38)$$

$$d\vec{x}/dv = \langle 0, 0, 1\rangle \quad (39)$$

$$dx/du \times dx/dv = \langle \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0\rangle \quad (40)$$

$$\hat{n} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle \quad (41)$$

$$\hat{n}\cdot\vec{x} = 0 \rightarrow \hat{n}\cdot(x - x_0, y - y_0, z - z_0) = 0 \quad (42)$$

$$\rightarrow \ldots \rightarrow x + y = 3\sqrt{2} \quad (43)$$

$$(44)$$

5.1 Example: Ellipsoid $x^2 + 2y^2 + z^2 = 4$

What's the normal at $(1, \frac{1}{\sqrt{2}}, \sqrt{2})$?

Method 1: parametrize with spherical u, v First, transform to sphere with change of coordinates, then flip to speherical coordinates.

$$x^2 + 2y^2 + z^2 = 4 \tag{45}$$
$$X = x/2, Y = \frac{Y}{\sqrt{2}}, Z = z/2 \tag{46}$$
$$X^2 + Y^2 + Z^2 = 1 \tag{47}$$
$$X = \cos(u)\sin(v), Y = \sin(u)\sin(v), Z = \cos(v) \tag{48}$$
$$p = (1, \frac{1}{\sqrt{2}}, \sqrt{2}) \to u = v = \frac{\pi}{4} \tag{49}$$
$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle -1, \frac{1}{\sqrt{2}}, 0 \rangle \tag{50}$$
$$\frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \frac{1}{\sqrt{2}}, -\sqrt{2} \rangle \tag{51}$$
$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) \times \frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \sqrt{2}, \sqrt{2} \rangle \tag{52}$$
$$\hat{n}_{out} = \frac{\langle -1, -\sqrt{2}, -\sqrt{2} \rangle}{\sqrt{5}} \tag{53}$$

Method 2: rewrite as z = g(x,y)

$$x^2 + 2y^2 + z^2 = 4 \tag{54}$$
$$z = (4 - x^2 - 2y^2)^{\frac{1}{2}} \tag{55}$$
$$dz/dx = \frac{1}{2} \times -2x(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -\frac{1}{\sqrt{2}} \tag{56}$$
$$dz/dy = \frac{1}{2} \times -4y(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -2\sqrt{2}/\sqrt{2} = -1 \tag{57}$$
$$f \approx \sqrt{2} + dz/dx(1, \frac{1}{\sqrt{2}})(x-1) + dz/dy(1, \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) \tag{58}$$
$$\to ... \to \frac{1}{\sqrt{2}}x + y + z = 2\sqrt{2} \tag{59}$$
$$\tag{60}$$

giving us normal vector $\langle \frac{1}{\sqrt{2}}, 1, 1 \rangle = \frac{\langle 1, \sqrt{2}, \sqrt{2} \rangle}{\sqrt{5}}$ after normalization.

Method 3: gradient

Gradient is always normal to the tangent plane. Recognize level set of $f(x, y, z) = x^2 + 2y^2 + z^2$.

$\nabla f = \langle 2x, 4y, 2z\rangle \to \nabla f(1, \frac{1}{\sqrt{2}}, \sqrt{2}) = \langle 2, 2\sqrt{2}, 2\sqrt{2}\rangle$

Then normalize.

5.2 Mobius strip and "outward direction"

Mobius strip is

- $x = 2\cos(u) + v\cos(\frac{u}{2})$
- $y = 2\sin(u) + v\cos(\frac{u}{2})$
- $z = v\sin(\frac{u}{2})$
- $u \in [0, 2\pi], v \in [-\frac{1}{2}, \frac{1}{2}]$

$\hat{n} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$ at (0,0) is $\langle 0, 0, -1\rangle$,

but at $(2\pi, 0)$ is the same point, but $\hat{n} = \langle 0, 0, 1\rangle$!!

6 Chapter 2.6: Vector Fields

(Lots of intuition questions here...)

One nugget: using **gradient vector fields**: Suppose $\vec{F}(x, y) = \langle 2, -4y^3\rangle$. If $F = \nabla f$ for some f, then F's arrows are perpendicular to a level set $f = c$. So look at $f = 2x - y^4$ and find perpendicular arrows to these. That's actually F!

Linear approximation for $\vec{F} : D \in \mathbf{R}^n \to \mathbf{R}^m$

Main idea: $\vec{F}(\vec{x}) = \vec{F}(\vec{a}) + A(\vec{a})(\vec{x} - \vec{a})$

Note that A takes in vectors of size n (so it has that many columns), and has m functions (rows) that operate on it. So the Jacobian, A, has as row i, column j, the quantity $\frac{dF_i}{dx_j}(\vec{a})$.

$dF_i/d\vec{x}$ extends across row i.

7 Chapter 2.7: Jack and the Beanstalk (Newton's method)

Basis for Newton's:

If we're estimating x_1 by following the derivative at x_0, this means we're looking at the line with x-intercept x_1, with slope $f'(x_0)$.

So instead of $y = mx + b$, we'll flip the two and use

$x = y/m + x_{int}$

or $x_0 = f(x_0)\frac{1}{f'(x_0)} + x_1$,

or $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

Note that, under Newton's something like $|x|$ will converge immediately, x^3 will converge moderately, and a S-curve might barely converge if at all.

The extension of this with the Jacobian matrix $A = DF'(x_0)$ is $\vec{x}_1 = \vec{x}_0 - (D\vec{F}(\vec{x}_0))^{-1}\vec{F}(x_0)$

8 Chapter 2.8: Electrostatic bootcamp

Electric charge radiates out equally in all directions, and is inversely proportional to distance.

Formula, with Q as the charge, ϵ_0 is a constant: $\vec{E}(x, y, z) = \frac{Q}{4\pi\epsilon_0\|x\|^2}\hat{x}$

A field line is a special case of a **flow line** - the space curve that follows $\vec{F}$'s arrows. The tangent vector to the flow line is $\vec{F}(\vec{x}(\tilde{t}))$ ($\tilde{t}$ is not time here), so $\frac{d\vec{x}}{d\tilde{t}} = \vec{F}(\vec{x}(\tilde{t}))$

Example: Vector field $\vec{F}(x, y) = \langle -2y, 3x\rangle$. What's the flow line through (2,0)?

Solution: Need to solve $dx/dt = -2y, dy/dt = 3x$. Key is "separating the equations". Remember x and y are functions of t!

$$\frac{d^2x}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6x. \tag{61}$$

$$\frac{d^2y}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6y. \tag{62}$$

$$x(t) = -6x''(t), y(t) = -6y''(t) \tag{63}$$

$$\rightarrow x = A\cos(\sqrt{6}t) + B\sin(\sqrt{6}t), y = C\cos(\sqrt{6}t) + D\sin(\sqrt{6}t) \tag{64}$$

$$\frac{dx}{dt} = -2y(t) \rightarrow \frac{\sqrt{6}}{2}A\sin(\sqrt{6}t) - \frac{\sqrt{6}}{2}B\cos(\sqrt{6}t) = y(t) \tag{65}$$

$$x(t = 0) = 2 \rightarrow A = 2 \tag{66}$$

$$y(t = 0) = 0 \rightarrow B = 0 \tag{67}$$

$$\vec{F}(t) = \langle 2\cos(\sqrt{6}t), \sqrt{6}\sin(\sqrt{6}t)\rangle \tag{68}$$

$$\tag{69}$$

Note: **Field lines** follow rules:

- Go from positive charges to negative
- Density of lines directly relates to how much charge a point has
- Lines don't intersect.
- Corollary: If count of out equals count of in, point has zero charge
- "Number" (to be defined) of field lines in and out of a *surface* related to the charge inside. Upcoming.

9 3.1: Surface Integrals

Example: Fluid pressure in a tank is:

- Proportional (via some weight constant p_{fluid}) to depth of the point
- Pushes out via the normal $\hat{n}$
- So, for the $x = l$ side of a cube of length l, this would be

  $\vec{F}_{x=l} = (\iint_{[0,l]\times[0,l]} p_{fluid}[1 - \frac{z}{l}]dydz)\hat{i}$

Example: Hemisphere of size l, sitting at (0, 0, 0)

Finding the out pointing unit normal of hemisphere at point $(x, y, -\sqrt{l^2 - x^2 - y^2})$

Note: Can just eyeball this, but one way is the **gradient**.

First, the relation is $x^2 + y^2 + (z - l)^2 = l^2$. Make it a function g and take the level set at l^2:

$$g(x, y, z) = x^2 + y^2 + (z - l)^2 = l^2 \tag{70}$$
$$\nabla g(x, y, z) = \langle 2x, 2y, 2(z - l)\rangle \tag{71}$$
$$\hat{n} = \pm\frac{\nabla g(x, y, z)}{\|\nabla g(x, y, z)\|} \tag{72}$$
$$\hat{n} = \pm\frac{\langle x, y, (z - l)\rangle}{\sqrt{x^2 + y^2 + (z - l)^2}} \tag{73}$$
$$\hat{n} = \pm\frac{\langle x, y, (z - l)\rangle}{\sqrt{l^2}} \tag{74}$$
$$\hat{n} = \pm\langle \frac{x}{l}, \frac{y}{l}, \frac{z}{l} - 1\rangle \tag{75}$$
$$\tag{76}$$

Note: Integrating over a patch dA on the surface means finding the area of micro-patches ΔA_{ij}, which is the parallelogram defined by

$$s_1 = \langle \Delta x_i, 0, \Delta x_i f_x(x_i^*, y_j^*) \rangle \tag{77}$$
$$s_2 = \langle 0, \Delta y_j, \Delta y_j f_y(x_i^*, y_j^*) \rangle \tag{78}$$
$$\Delta A_{ij} \approx \|s_1 \times s_2\| \tag{79}$$
$$= \sqrt{(1 + [f_x(x_i^*, y_j^*)]^2 + [f_y(x_i^*, y_j^*)]^2} \Delta x_i \Delta y_j \tag{80}$$
$$\tag{81}$$

So the total pressure ends up being $\vec{F}_{tot} = p_{fluid} \iint (p \cdot \hat{n}) dA$

$$= p_{fluid} \iint_{x^2+y^2 \leq l^2} [1 - \frac{f(x,y)}{l}] \hat{n} \sqrt{1 + [f_x]^2 + [f_y]^2} dxdy \tag{82}$$
$$f(x,y) = l - \sqrt{l^2 - x^2 - y^2} \tag{83}$$
$$\hat{n} = \langle \frac{x}{l}, \frac{t}{l}, \frac{f(x,y)}{l} - 1 \rangle \tag{84}$$
$$f_x = \frac{x}{\sqrt{l^2 - x^2 - y^2}}, f_y = \frac{y}{\sqrt{l^2 - x^2 - y^2}} \tag{85}$$

And for the only-nonzero component, $\hat{k}$, this simplifies after a lot of hand-math to $F_{tot} = -p_{fluid}(\iint_{x^2+y^2 \leq l^2} \sqrt{1 - (\frac{x^2+y^2}{l^2})} dxdy)\hat{k}$

Side Note during solving: $dxdy \to rdrd\theta$.

- TODO: This looks to be something to do with the determinant of the Jacobian matrix F_i/x_j.
- Intuitively, consider that a patch $dx \cdot dy$ is a slice of a big disk which has dimensions dr on the ray, $rd\theta$ on the arc.

10 3.2: Flux Part I

Main idea: Field lines are innumerable, so counting them through a surface makes no sense. Instead, we'll use **flux** to help us measure charge pushed through a surface per unit time.

Example: If charge q of mass m in a field of $\vec{E} = E_0 \hat{i}$ moves from origin along x towards R according to $\frac{d^2x}{dt^2} = \frac{q}{m} E_0$, then solving the diff eq. means that $x = \frac{q}{2m} E_0 (\Delta t)^2 = R$. This means we're pushing all charges within $\frac{q}{2m} E_0 (\Delta t)^2$ to the left of the disk through it.

Then, if we're considering a cylinder of base area A, mass density δ, charge density ρ:

- Every test charge chunk ΔV within $\frac{\rho\Delta V}{2\delta\Delta V}E_0(\Delta t)^2$ passes through. That's the height.
- Area is A, so total volume is $\frac{\rho(\Delta t)^2}{2\delta}E_0 A$
- Density of charge per volume is ρ, so total is $\frac{\rho^2(\Delta t)^2}{2\delta}E_0 A$

Note: Tilting this forward from the z-axis by θ multiplies the cross-section area of the cylinder (now an ellipse) by $\cos(\theta)$. Can work out the ellipse volume, or just note that each "Riemann bar" orthogonal to x-axis just got squished by $\cos(\theta)$.

So we define **flux** as amount of charge through a closed surface. $\Phi = (\vec{E} \cdot \hat{n})A$ where $\vec{E}$ is a constant field. (Units: joules/second/m^2, or watts/m^2)

We can further note $(\vec{E} \cdot \hat{n}) = \|\vec{E}\| \cos(\theta)$ by last problem.

Example: Flux through an empty cube from the origin is necessarily 0 since every face cancels the other.

Another example: A square pyramind with top at $(0, 0, 1)$, sides at 1 on each axis:

- All the triangles will cancel in the x, y directions.
- A triangle $(1, 0, 0)(0, 1, 0), (0, 0, 1)$ has two displacement vectors $P_1P_3 = P_3 - P_1 = (-1, 0, 1), P_2P_3 = (0, -1, 1)$.
- $P_1P_3 \times P_2P_3 = (1, 1, 1) \to \hat{n} = \frac{(1,1,1)}{\sqrt{3}}$
- $A = \frac{1}{2}\|P_1P_3 \times P_2P_3\| = \frac{\sqrt{3}}{2}$
- $\Phi = (\vec{E} \cdot \hat{n})A = (E_0 \frac{1}{\sqrt{3}})\frac{\sqrt{3}}{2} = \frac{E_0}{2}$
- So total flux through these is $4 \cdot \frac{1}{2}E_0 = 2E_0$
- However, the bottom has area $\sqrt{2}^2 = 2$ and flux E_0 , so total is 0!

11 3.3: Flux Part II

Note:

- Charge (q) is the volts of the point charge. Total charge Q_{tot} is total charge inside some surface.
- Electric field is sum of those point charges acting at a distance, and a is a single vector.
- Flux looks like the integral of the electric field flowing through a surface.

- Total charge Q_{tot} of a surface is basically the sum of all the flux going out, except that it's that divided by some constant ϵ_0.

Note: $\vec{E}$ isn't usually constant, and the surface S is usually curved. So we need calculus to break up surface S into small pieces ΔA_i and evaluate $\vec{E_i}$ there at that normal $\hat{n}_i$. So

$\sum_{patches} = \sum_{patches}(\vec{E_i} \cdot \hat{n}_i)\Delta A_i = \iint_S(\vec{E_i} \cdot \hat{n}_i)\Delta A_i = \Phi$

Easy Example: If, say, $(\vec{E} \cdot \hat{n}) = 1$ everywhere, we're just looking at $\iint_S dA$, or the total surface area.

Another example. Given:

- Real electric field law: $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{x}}{\|\vec{x}\|^3}$
- Real observation: Total electric flux through a surface (Φ) is proportional to total charge inside (Q_{tot}). $\Phi = \iint_S(\vec{E} \cdot \hat{n})\Delta A \propto Q_{tot}$
- Then constant must be $\frac{1}{\epsilon_0}$. Why?
  - On unit sphere, $\hat{n} = \frac{\vec{x}}{\|\vec{x}\|}$
  - So $\vec{E} \cdot \hat{n} = \frac{q}{4\pi\epsilon_0}\frac{\vec{x}}{\|\vec{x}\|^3} \cdot \hat{n}$
  - $= \frac{q}{4\pi\epsilon_0}$
  - Then $\Phi = \iint_S \frac{q}{4\pi\epsilon_0} dA$
  - $= \frac{q}{4\pi\epsilon_0}4\pi$ by surface area of unit sphere
  - $= \frac{q}{\epsilon_0}$
- Therefore, because all of the field goes through the surface (no matter the shape), **Gauss's law** says $\iint_S(\vec{E} \cdot \vec{n})dA = \frac{Q_{tot}}{\epsilon_0}$

Note: Because (UNEXPLAINED!) symmetry of a contained *ball* implies that, for distance ρ from origin, $\vec{E} = E(\rho)\hat{\rho}$, the above works the same for a point charge or a uniform (contained) ball.

Example: For a big radius R ball of charge Q containing a small ball of radius ρ with charge Q_{tot}, what must the charge $E(\rho)$ at any point be?

- Small charge Q_{tot} is proportional to volume of the big charge Q by $Q_{tot} = Q\frac{V_{small}}{V_{big}} = Q\frac{\rho^3}{R^3}$
- $\frac{Q_{tot}}{\epsilon_0}$ = total charge $= \iint_S E(\rho)(\|\hat{\rho}\|)dA = E(\rho)\iint_S 1dA = E(\rho)4\pi\rho^2$
- So $\frac{Q_{tot}}{\epsilon_0} = Q\frac{\rho^3}{R^3\epsilon_0} = E(\rho)4\pi\rho^2$

- So $E(\rho) = \frac{Q}{4\pi\epsilon_0}\frac{\rho}{R^3}$

Example: Infinite wire, x=y=0, charge per length is λ. Use a cylinder.

- What's the total charge of the cylinder? Top and bottom are perpendicular to the field so can be ignored.
- There's some function $E(r)$ which, time $\hat{r}$, is the field by symmetry.
- $\Phi = \iint_{cylinder}(E(r) \cdot \hat{r})dA = E(r)\iint_{cylinder} 1dA = E(r)2\pi rh.$
- $\frac{Q_{tot}}{\epsilon_0} = E(r)2\pi rh \Rightarrow E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$

Example: Infinite plane, x=y=0, charge per area is σ. Use a cylinder again

- What's the total charge of the cylinder? Side is perpendicular to the field so can be ignored. Looking at top and bottom, $\phi = 2EA + 2EA.$, where E is charge through the top.
- $2EA = \frac{\sigma A}{\epsilon_0} \to E = \frac{\sigma}{2\epsilon_0}$

12 3.3: Surface Integrals

- Flux is a specific form of the general $\iint_S F da$.
- dA is a patch of a parallelogram on the surface. This is defined by corners $\vec{x}(u_0, v_0), \vec{x}(u_0, v_0)+ \delta_u\vec{x}(u_0, v_0)$, and $\vec{x}(u_0, v_0) + \delta_v\vec{x}(u_0, v_0)$
- Therefore, using the parallelogram area formula,$dA = \Delta_u\Delta_v\|\vec{x}_u \times \vec{x}_v\|$
- Taking to the limit, this means the area is $\iint_D F(\vec{x}(u, v))\|\vec{x}_u \times \vec{x}_v\|dudv$

Example: Sphere $x^2 + y^2 + z^2 = R^2$ surface area. Take θ as angle around ϕ as angle from top of z axis.

- Parametrization $x = R\sin\phi\cos\theta, y = R\sin\phi\sin\theta, z = R\cos\phi$
- $dx/d_\theta = -R\sin\phi\sin\theta, dy/d_\theta = R\sin\phi\cos\theta, dz/d_\theta = 0$
- $dx/d_\phi = R\cos\phi\cos\theta, dy/d_\phi = R\cos\phi\cos\theta, dz/d_\phi = -R\sin\phi$
- After working it out, $dx/d_\theta \times dx/d_\phi = R^2\sin\phi\langle-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi\rangle$
- Doing the math, $\|dx/d_\theta \times dx/d_\phi\| = R^2\sin\phi$
- So $\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 1 \cdot R^2\sin\phi = 2\pi\int_{\phi=0}\pi R^2\sin\phi = 2\pi R^2[-\cos\phi]_0^\pi = 4\pi R^2$

Example: Parabaloid $z = 1 - x^2 - y^2, x^2 + y^2 \leq 1$

- Parametrization $x = R\sin\phi\cos\theta, y = R\sin\phi\sin\theta, z = R\cos\phi$

- $dz/dx = \langle 1, 0, -2x \rangle, dz/dy = \langle 0, -1, -2y \rangle$
- $\|dz/dx \times dz/dy\| = 1 + 4x^2 + 4y^2$
- Area $= \iint_D 1 \cdot dA = \iint_D \sqrt{1 + 4x^2 + 4y^2} dxdy$
- Change to polar, remembering this square depends on r: $\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{1 + 4r^2 \cos^2 \pi 4r^2 \sin^2 \pi} rdrd\theta = 2\pi \int_0^1 \sqrt{1 + 4r^2} rdr$
- After working it out, this ends up being $[\frac{2}{3} \cdot \frac{1}{8}(4r^2 + 1)^{\frac{3}{2}}]_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1)$

Example: Torus $x(u, v) = [R+r \cos(u)] \sin(v)], y(u, v) = [R+r \cos(u)] \cos(v)], z = r \sin(u), u, v \in [0, 2\pi)$

- Already parametrized in polar, basically,
- $d\vec{x}/du = \langle -r \sin(u) \sin(v), -r \sin(u) \cos(v), r \cos(u) \rangle$
- $d\vec{x}/dv = \langle Rcos(v) + r \cos(u) \cos(v), -R \sin(v) - r \cos(v) \sin(v), 0 \rangle$
- After lots of math, $\|d\vec{x}/du \times \vec{x}/dv\| = r(R + r \cos(u))$
- $\int_{u=0}^{2\pi} \int_{v=0}^{2\pi} r(R + r \cos(u) du = 2\pi r \int_{u=0}^{2\pi} r(R + r \cos(u)) du$
- $= 2\pi r[2\pi R] = 4\pi^2 Rr$

Example: Center of mass of unit (hollow?) hemisphere sitting on origin.

- Center of mass for density ρ is $\frac{\iint_S \vec{x} \rho dA}{\iint_S \rho dA}$
- Obvious that x, y center at zero.
- For denominator, $\iint_S dA$ is just surface area, or half of $4\pi 1^2 = 2\pi$.
- For numerator:
  - Do typical θ, ϕ parametrization.
  - $\vec{x}_\theta \times \vec{x}_\phi = \langle \sin^2(\phi) \cos(\theta), \sin^2(\phi) \sin(\theta), \sin(\phi) \cos(\phi) \rangle$
  - Pull out the $\sin(\phi)$ and the remaining norm is one, so $\|\vec{x}_\theta \times \vec{x}_\phi\| = \sin(\phi)$
  - $\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} z \cdot dA = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \cos(\phi) \sin(\phi) = \frac{1}{2}$

Example: Moment of inertia

- Formula: $I_z = M \iint_S (x^2 + y^2) dA$.
- Object to spin: helicoid $\vec{x}(u, v) = \langle u \cos(v), u \sin(v), v \rangle u \in [0, R], v \in [0, 2\pi]$
- Assumption for the problem: $\int_{u=0}^{u=R} u^2 \sqrt{(}1 + u^2) du = 2$

- Center of mass for density ρ is $\frac{\iint_S \vec{x}\rho dA}{\iint_S \rho dA}$
- Use polar coordinates r, θ.
- After computation, $\|\vec{x}_r \times \vec{x}_\theta\| = \sqrt{1+r^2}$
- $M \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \sqrt{1+r^2}(r^2\cos^2(\theta) + r^2\cos^2(\theta))dA = M \iint \sqrt{1+r^2}r^2 = 2\pi M \iint \sqrt{1+r^2}r^2 = 4\pi$ by hint

Example: Flux through unit hemisphere

- Formula: $\Phi = \iint_S (\vec{E}\cdot\vec{n})dA = \iint_S F dA$
- Field: $\vec{E} = \langle yz, xz, xy\rangle$
- Use polar coordinates
- **Base**: $\hat{n} = -\hat{k}$ so $\langle yz, xz, xy\rangle \cdot \langle 0, 0, -1\rangle = -xy$ It's clear by symmetry that $\iint_{u^2+v^2\leq 1} -xydxdy = 0$
- **Top**: Set $u = \theta \in [0, 2\pi), v = \phi \in [0, \frac{\pi}{2}]$.
- As usual, $dA = \|\vec{x}_u \times \vec{x}_v\| = \sin(v)$.
- Norm just points out from the center: $\hat{n} = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle$
- $\vec{E} = \langle \sin(u)\sin(v)\cos(v), \cos(u)\sin(v)\cos(v), \cos(u)\sin^2(v)\sin(u)\rangle$
- So $\vec{E}\cdot\hat{n} = 3\cos(u)\sin(u)\cos(v)\sin^2(v)$
- Looking at this, this is really $\int_{u=0}^{u=2\pi} k(v)\sin^2(u)$ for some $k(v)$, so this will be 0.
- Therefore, total flux is zero, and by Gauss's law, total field contained inside has to be 0 too.

Example: Field $\vec{E} = \ln(x^2+y^2)\langle x, y, 0\rangle$ through R-wide cylinder, height h

- Parameterize: $x = r\cos\theta, y = r\sin\theta, z = z$
- **Top/Bottom**: $\hat{n} = \langle 0, 0, 1\rangle, \vec{E} = f(x, y)\langle x, y, 0\rangle \to \hat{n}\cdot\vec{E} = 0$
- **Side**: $\hat{n} = \frac{1}{R}\langle R\cos(\theta), R\sin(\theta), 0\rangle$
- $\Phi = \iint_{cylinder} \frac{1}{R}\langle R\cos(\theta), R\sin(\theta), 0\rangle \cdot \langle R\cos(\theta), R\sin(\theta), 0\rangle \ln(R^2\cos^2(\theta) + R^2\sin^2(\theta))$
- $= R\iint_{cylinder} \ln(R^2) + \ln(\cos^2(\theta) + \sin^2(\theta)) = R\cdot 2\ln(R)\cdot h\cdot 2\pi R = 4\pi R^2\ln(R)h$

Example: Field $\vec{E} = e^{-x^2-y^2-z^2}\vec{x}$ with sphere S at radius R, setting $\epsilon_0 = 1$

- Parameterize: $x = R\cos(\theta)\sin(\phi), y = R\sin(\theta)\sin(\phi), z = R\cos(phi)$
- $\hat{n} = \langle \cos(\theta)\sin(\phi), y = \sin(\theta)\sin(\phi), z = \cos(phi)\rangle$

- $\vec{x} = R\hat{n}$, so $\vec{E} \cdot n = Re^{-R^2}$
- $R \iint_{sphere} e^{-R^2} = 4\pi R^3 e^{-R^2}$

Note: In the future we write $\hat{n} dA = d\vec{A}$

12.1 3.4: Divergence

Main idea: Last chapter was all about having field $\vec{E}$ and wanting to figure out Q_{tot} (or $\phi\epsilon_0$). Usually, we have the charge distribution Q and want to figure out $\vec{E}$. Most of the field derivation from 3.3 was through tricks for highly symmetric spaces (infinite line, infinite plane, uniform ball, etc.

Point: The flux through a sphere in a uniform field is zero. Why? Move the point to the origin, rotate so field is $\hat{k}$, and consider that goes out at $\langle x, y, z\rangle$ comes in at $\langle x, y, -z\rangle$. This same argument applies for $\iint_{S=sphere} \hat{n}_i \hat{n}_j dA$, where i, j are components in $\{x, y, z\}$.

However, if $i = j$, then $\iint_S \hat{n}_i \hat{n}_j dA = \iint_S \hat{n}_i^2 = \frac{4}{3}\pi R^2$, since $\iint_S (\hat{n}_x^2 + \hat{n}_y^2 + \hat{n}_z^2) dA = \iint_S 1 dA = 4\pi R^2$, so each of the components must be a third of that.

12.1.1 Defining Divergence

Remember that in Gauss's law, $\frac{Q}{\epsilon_0} = \iint_S \vec{E} \cdot d\vec{A}$, we're using information about $\vec{E}$ spread out over surface S. We can also shrink this to a smaller surface.

Shrinking to a point $\vec{P}$, $\lim_{R\to 0} \frac{1}{4\pi R^3} \iint_S \vec{E} \cdot d\vec{A} = \frac{Q_{tot}}{\epsilon_0 4\pi R^3} = \frac{\rho(\vec{P})}{\epsilon_0}$. (This works by dividing both sides by volume of a sphere)

Deriving Divergence: Computing $\lim_{R\to 0} \frac{1}{4\pi R^3} \iint_S \vec{E} \cdot d\vec{A}$

- $\iint_S \hat{n}_i \hat{n}_j dA = 0$ if $i \neq j$
- $\iint_S \hat{n}_i \hat{n}_j dA = \frac{4}{3}\pi R^3$ if $i = j$
- Use linear approximation with Jacobian $D = \frac{\delta E_i}{\delta x_j}$, $\vec{E}(\vec{x}) = \vec{E}(\vec{P}) + D\vec{E}(\vec{P})(\vec{x} - \vec{P})$
- $\iint_S \vec{E}(\vec{P}) = 0$ for any constant. (think of the flux of a sphere in a constant field as above)
- This leaves $D\vec{E}(\vec{P})(\vec{x} - \vec{P}) \cdot \hat{n} = \sum_{i,j} \hat{n}_i [\vec{x} - \vec{P}]_j D\vec{E}(\vec{P})_{ij}$
- Since it's a sphere, the normal $\hat{n} = \frac{\vec{x} - \vec{P}}{R}$
- Therefore $D\vec{E}(\vec{P})(\vec{x} - \vec{P}) \cdot \hat{n} = R \sum \hat{n}_i \hat{n}_j D\vec{E}(\vec{P})_{ij}$ (swap $R\hat{n}_j$ for $[\vec{x} - \vec{P}]_j$)
- These terms are all 0 except where $i = j$, so $D\vec{E}(\vec{P})(\vec{x} - \vec{P}) \cdot \hat{n} = \frac{4}{3}\pi R^2 \times R \times [\frac{\delta E_x}{\delta x} + \frac{\delta E_y}{\delta y} + \frac{\delta E_z}{\delta z}]$

- This equals $\lim_{R\to 0}\frac{1}{4\pi R^3}\iint_S \vec{E}\cdot d\vec{A}$ so eliminating the sphere volume gives us $\frac{\rho(\vec{P})}{\epsilon_0} = [\frac{\delta E_x}{\delta x} + \frac{\delta E_y}{\delta y} + \frac{\delta E_z}{\delta z}] = \nabla \cdot \vec{E}$

We can think of the divergence of $\vec{F}$, $\nabla \cdot \vec{F}$ also like an operator: $\nabla \cdot (F_x\hat{i} + F_x\hat{j} + F_x\hat{k}) = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}) \cdot (F_x\hat{i} + F_x\hat{j} + F_x\hat{k})$