homework2

March 17, 2025

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
import math
from pprint import pp
```

1 Problem 9

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Let $f_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Then we have the following relation:

$$f_{n+1} = -f_n \frac{x^2}{(2n+2)(2n+3)}$$

This way we can efficiently calculate the Taylor series. We can use a similar result for the cosine.

Also, we only have to consider the interval $[-\pi/4, \pi/4]$, because we can use periodicity and (a)symmetry of the sine and cosine.

[3]:
```
def sine_taylor_generator(x):
    term = x  # first term is x (when n=0)
    n = 0
    while True:
        yield term
        # Compute next term using the recurrence relation:
        term = -term * x * x / ((2 * n + 2) * (2 * n + 3))
        n += 1

def cosine_taylor_generator(x):
    term = 1.0  # The first term: cos(x) starts with 1 (when n = 0)
    n = 0
    while True:
        yield term
        # Recurrence relation: t(n+1) = -t(n) * x^2 / ((2n+1)*(2n+2))
        term = -term * x * x / ((2 * n + 1) * (2 * n + 2))
        n += 1
```

1.1 Reducing Angle

This method computes $\sin(x)$ for any angle x by reducing it to a base interval $[-\pi/4, \pi/4]$, where the sine and cosine functions can be well approximated.

1. Reduce Modulo 2π

- Reduce x modulo 2π so that the equivalent angle x_{mod} lies within $(-\pi, \pi]$.

2. Express in Terms of a Base Interval

- Express the reduced angle as:

$$x_{\text{mod}} = k\frac{\pi}{2} + r,$$

  where r is chosen to be in $[-\pi/4, \pi/4]$.

- Compute k as:

$$k = \text{round}\left(\frac{2x_{\text{mod}}}{\pi}\right).$$

- Compute r as:

$$r = x_{\text{mod}} - k\frac{\pi}{2}.$$

  Since

$$\frac{2x_{\text{mod}}}{\pi} - \frac{2r}{\pi} = k$$

  and

$$\frac{2r}{\pi} \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

  rounding gives the correct expression.

3. Compute $\sin(x)$ Based on $k \mod 4$

- If $k \equiv 0 \pmod 4$:

$$\sin(x) = \sin(r)$$

- If $k \equiv 1 \pmod 4$:

$$\sin(x) = \cos(r)$$

- If $k \equiv 2 \pmod 4$:

$$\sin(x) = -\sin(r)$$

- If $k \equiv 3 \pmod 4$:

$$\sin(x) = -\cos(r)$$

[4]:
```
def base_sin(r, order):
    gen = sine_taylor_generator(r)
    approx = 0.
    for _ in range(order):
        approx += next(gen)
    return approx

def base_cos(r, order):
    gen = cosine_taylor_generator(r)
    approx = 0.
    for _ in range(order):
        approx += next(gen)
    return approx


def custom_sin(x, order=2):
    # Step 1: Reduce the angle to (-pi, pi]
    x_mod = x % (2 * math.pi)
    if x_mod > math.pi:
        x_mod -= 2 * math.pi
    # Step 2: Find k such that r = x_mod - k*(pi/2) lies in [-pi/4, pi/4]
    k = round(2 * x_mod / math.pi)
    r = x_mod - k * (math.pi / 2)

    # Step 3: Use the identity based on k modulo 4
    mod = k % 4
    if mod == 0:
        return base_sin(r, order)
    elif mod == 1:
        return base_cos(r, order)
    elif mod == 2:
        return -base_sin(r, order)
    elif mod == 3:
        return -base_cos(r, order)
```

Lagrange form of the remainder

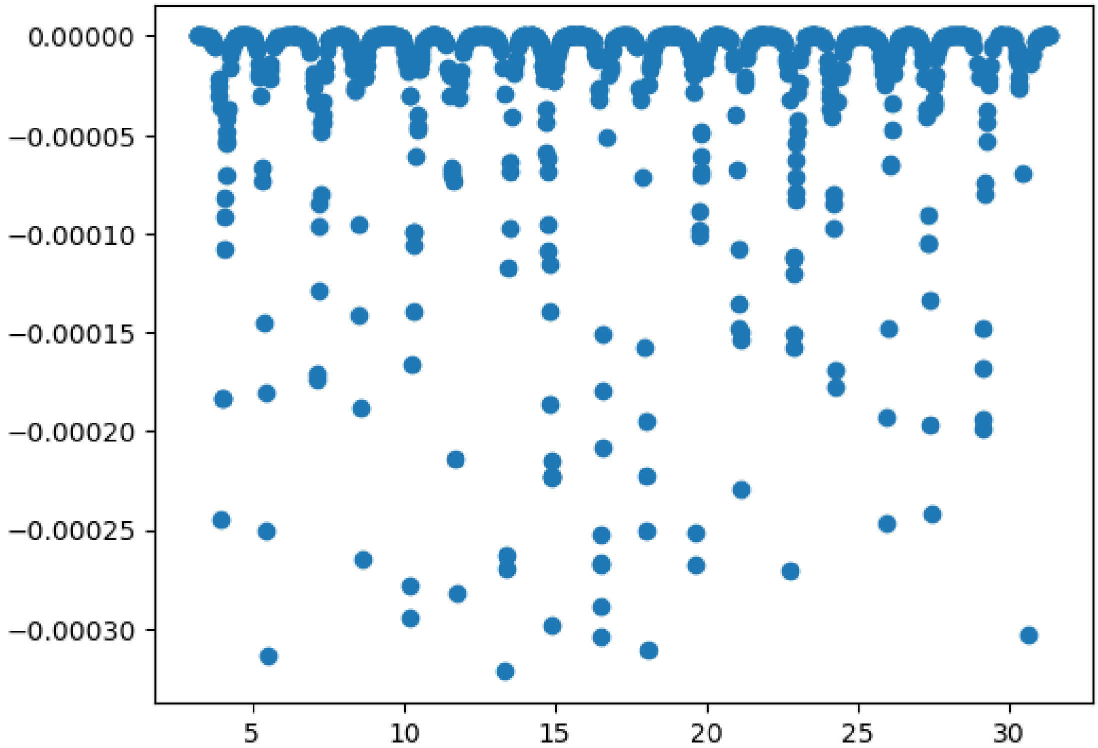
There exists c in $[a, x]$ such that

$$f(x) - \left(\sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j\right) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

So to get an approximation of the error we just calculate one more Taylor series term evaluated in x.

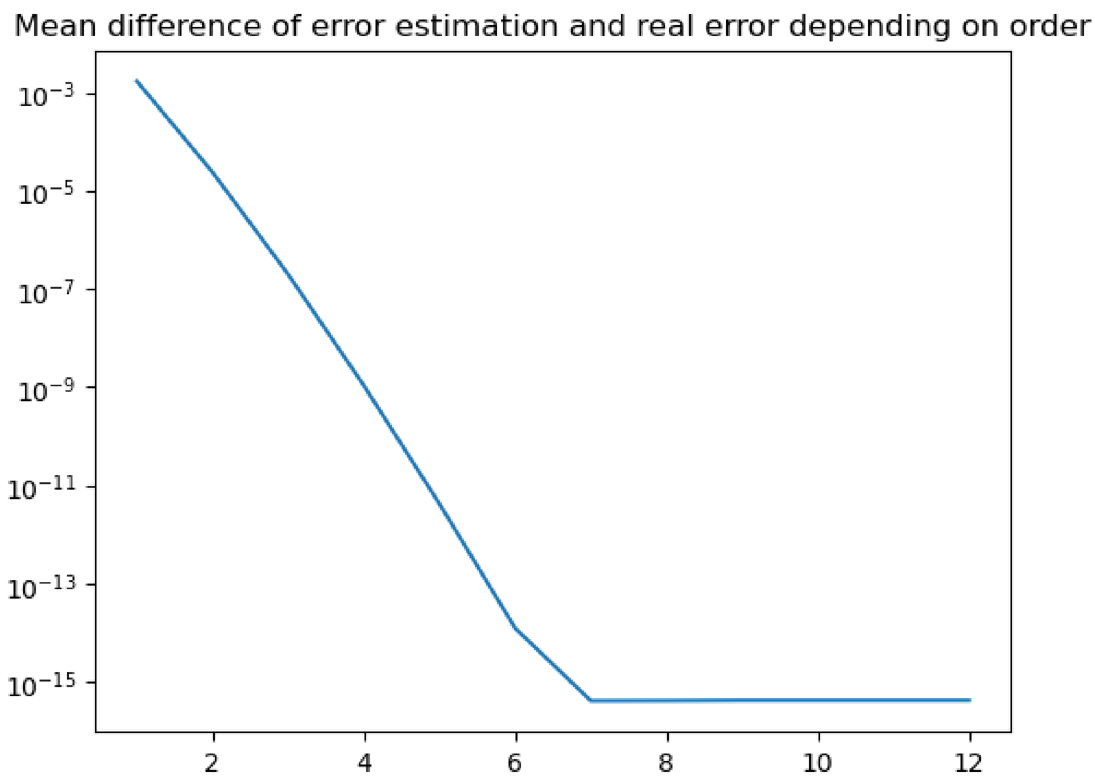
[135]:
```python
test_values = np.random.uniform(math.pi, 10*math.pi, 1000)

# Plot the custom_sin function
# Create a list of orders to test
orders = list(range(1, 13))  # from 2 to 10



# Calculate true sine values using numpy's sin function
true_sin_values = np.sin(test_values)
custom_sin_values_of_order = lambda order: np.array([custom_sin(x, order) for x␣
 ↪in test_values])

# Calculate errors for each order
lagrangian_error_of_order = lambda order: np.
 ↪abs(custom_sin_values_of_order(order+1) - custom_sin_values_of_order(order))
real_error_of_order = lambda order: np.abs(true_sin_values -␣
 ↪custom_sin_values_of_order(order))

plt.scatter(test_values, real_error_of_order(2) - lagrangian_error_of_order(2))
```

[135]:
```
<matplotlib.collections.PathCollection at 0x1219ecd40>
```



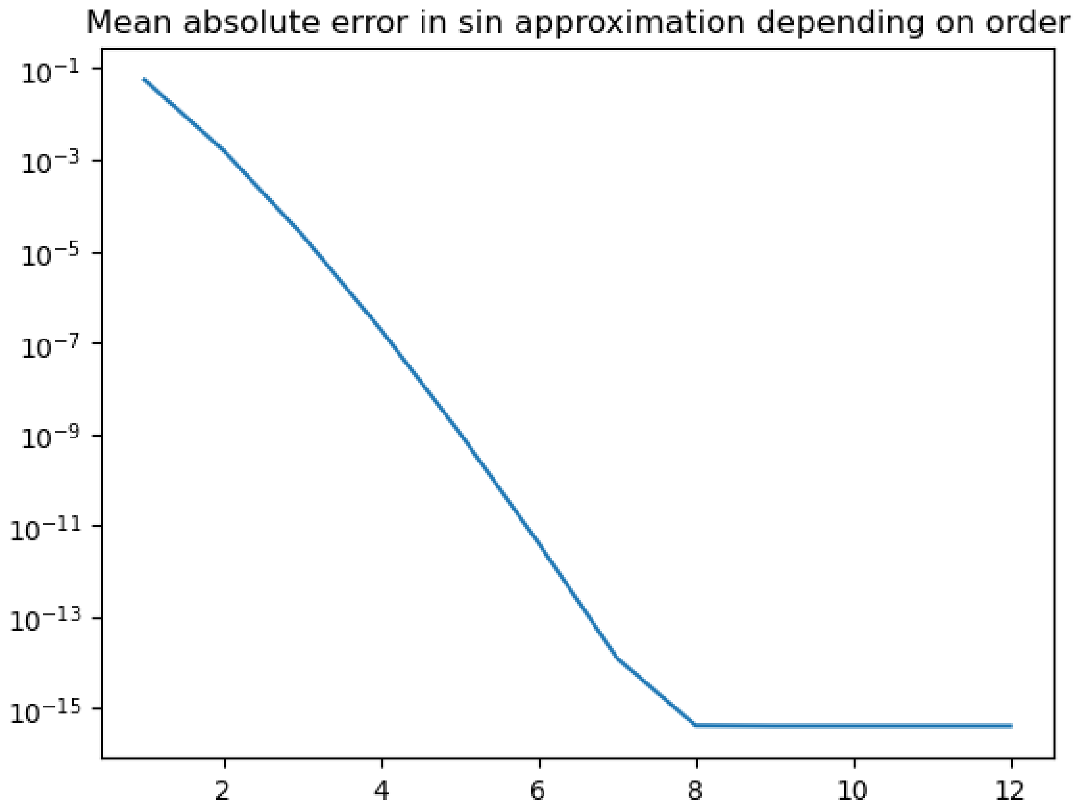
[142]:
```
mean_difference_of_error_estimations = [np.mean(np.
    ↪abs(real_error_of_order(order) - lagrangian_error_of_order(order))) for␣
    ↪order in orders]
plt.title("Mean difference of error estimation and real error depending on␣
    ↪order")
plt.semilogy(orders, mean_difference_of_error_estimations)
```

[142]:
```
[<matplotlib.lines.Line2D at 0x121df78f0>]
```



[115]:
```
# Plot the errors
mean_errors = [np.mean(real_error_of_order(order)) for order in orders]
# errors = [np.mean(np.abs(true_sin_values -␣
 ↪custom_sin_values_of_order(order))) for order in orders]
plt.title("Mean absolute error in sin approximation depending on order")
plt.semilogy(orders, mean_errors)
```

[115]:
```
[<matplotlib.lines.Line2D at 0x1211e39b0>]
```



2 Problem 10

2.0.1 LU Factorization of a Tridiagonal Matrix

Given a tridiagonal matrix \$ A \$:

$$A = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & c_2 & a_3 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & c_{n-1} & a_n \end{bmatrix}$$

we want to find \$ L \$ and \$ U \$ such that:

$$A = LU$$

We can see that

$$
L = \begin{bmatrix}
1 & 0 & 0 & \dots & 0 & \\
\ell_1 & 1 & 0 & \dots & 0 & \\
0 & \ell_2 & 1 & \dots & 0 & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & 0 & \ell_{n-2} & 1 & \\
0 & 0 & 0 & 0 & \ell_{n-1} & 1
\end{bmatrix}
$$

$$
U = \begin{bmatrix}
u_1 & b_1 & 0 & \dots & 0 & \\
0 & u_2 & b_2 & \dots & 0 & \\
0 & 0 & u_3 & \dots & 0 & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & 0 & 0 & u_{n-1} & b_{n-1} \\
0 & 0 & 0 & 0 & 0 & u_n
\end{bmatrix}
$$

with:

$$u_1 = a_1$$

$$\ell_i = \frac{c_i}{u_i}$$

$$u_{i+1} = a_{i+1} - \ell_i b_i$$

[ ]:
```python
def tridiagonal_lu_decomp(a, b, c):
    """Compute the LU decomposition of a tridiagonal matrix.
    Args:
        a (ndarray): Diagonal elements of the matrix (length n)
        b (ndarray): Superdiagonal elements of the matrix (length n-1)
        c (ndarray): Subdiagonal elements of the matrix (length n-1)

    Returns:
        tuple: A tuple (l, u) where:
            - l is the subdiagonal of the lower triangular matrix L (length n-1)
            - u is the diagonal of the upper triangular matrix U (length n)
    """

    n = len(a)
    l = np.zeros(n-1)
    u = np.zeros(n)
    u[0] = a[0]
    for i in range(1, n):
        l[i-1] = c[i-1] / u[i-1]
        u[i] = a[i] - l[i-1] * b[i-1]
    return (l, u)
```

Now that we have the decomposition we can solve $LUx = f$ like this: 1. $Ly = f$

$$y_1 = f_1,\ y_i = f_i - \ell_i y_{i-1}$$

2. $Ux = y$

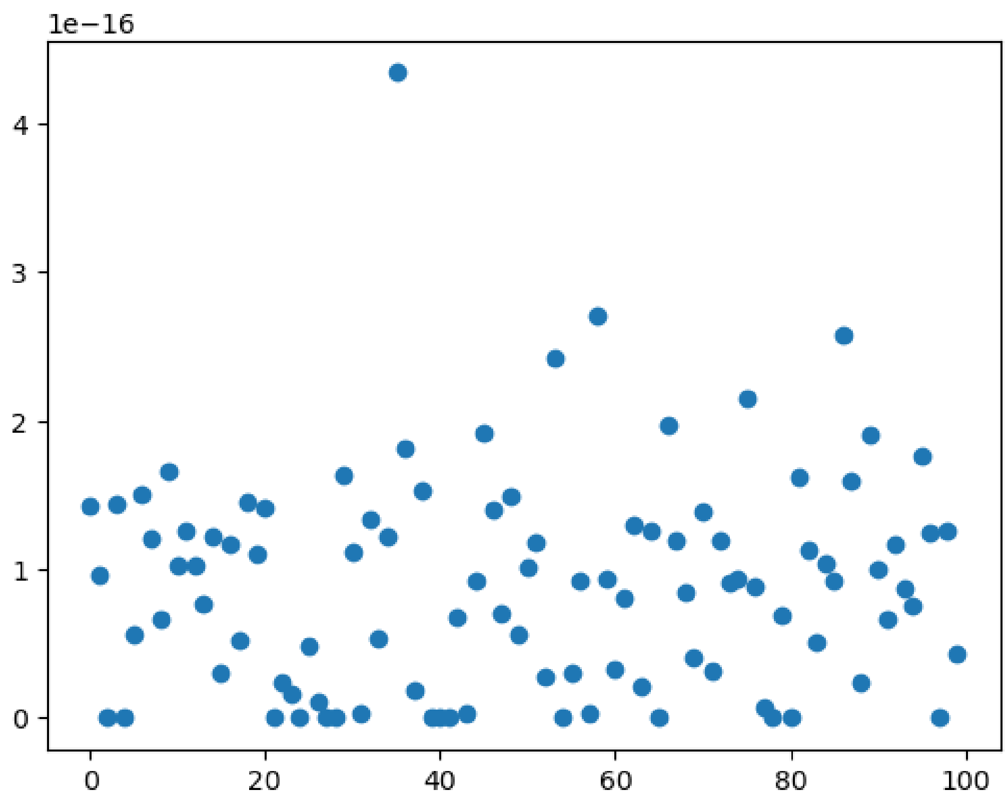
$$x_n = \frac{y_n}{u_n},\ x_i = \frac{y_i - b_i x_{i+1}}{u_i}$$

[ ]:
```
def tridiagonal_lu_solve(l, u, b, f):
    """Solves LUx = f for a tridiagonal LU decomposition.

    Args:
        l (ndarray): Subdiagonal elements of L (length n-1)
        u (ndarray): Diagonal elements of U (length n)
        b (ndarray): Superdiagonal elements of U (length n-1)
        f (ndarray): Right-hand side vector (length n)

    Returns:
        x (ndarray): Solution vector (length n)
    """
    n = len(u)

    # Forward substitution: Solve L y = f
    y = np.zeros(n)
    y[0] = f[0]  # Since L[0,0] = 1 (implicit)

    for i in range(1, n):
        y[i] = f[i] - l[i-1] * y[i-1]  # Since L[i,i] = 1

    # Backward substitution: Solve U x = y
    x = np.zeros(n)
    x[-1] = y[-1] / u[-1]

    for i in range(n-2, -1, -1):
        x[i] = (y[i] - b[i] * x[i+1]) / u[i]

    return x
```

[210]:
```
np.set_printoptions(precision=2)
np.set_printoptions(suppress=True)

def generate_tridiagonal(a,b,c):
    A = np.diag(a,k=0)
    A += np.diag(b,k=1)
    A += np.diag(c,k=-1)
    return A

n = 3
b = np.random.random(n)
c = np.random.random(n)
a = b + c + np.random.random(n)
```

```
b = b[:-1]
c = c[:-1]

l, u = tridiagonal_lu_decomp(a, b, c)


L = generate_tridiagonal(np.ones(n), np.zeros(n-1), l)
U = generate_tridiagonal(u, b, np.zeros(n-1))

A = generate_tridiagonal(a,b,c)

x_exact = np.random.random(size = (100,n))
f_exact = (A @ x_exact.T).T

x_algo = np.array([tridiagonal_lu_solve(l, u, b, f) for f in f_exact])

errors = np.linalg.norm(x_exact - x_algo, axis=1)/np.linalg.norm(x_exact,␣
↪axis=1)

plt.scatter(range(100), errors)
```

[210]: <matplotlib.collections.PathCollection at 0x12101d400>



We see that our algorithm is exact, because the errors are in the magnitude of machine precision