homework4

March 28, 2025

[69]:
```python
import numpy as np
import matplotlib.pyplot as plt
```

Element-based formula of Jacobi method

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n$$

Element-based formula of Gauß Seidel

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n$$

We can also express the two sums as $ *{j i}* a{ij} x_j $ with x is part old vector, part new vector. We only need one storage vector. I dont implement this because then i have no stopping condition.

[70]:
```python
def jacobi_iteration(A, b, x_0, x_sol, max_iterations = 1000, tol = 1e-10):
    n = A.shape[0]
    x = x_0
    errors = []
    for iter_count in range(max_iterations):
        x_new = np.zeros(n)
        for i in range(n):
            sigma = A[i, :i] @ x[:i] + A[i, i+1:] @ x[i+1:]
            x_new[i] = (1 / A[i, i]) * (b[i] - sigma)

        errors.append(np.linalg.norm(x_sol - x_new))
        # in real life we wouldnt have x_sol, so we check against the previous x
        if np.allclose(x, x_new, atol=tol, rtol=0):
            print(f"Converged after {iter_count} iterations.")
            return x_new, errors

        x = x_new

    print("Maximum iterations reached without convergence.")
    return x
```

[71]:
```python
def gauss_seidel_iteration(A, b, x_0, x_sol, max_iterations = 1000, tol =␣
↪1e-10):
    n = A.shape[0]
    x = x_0
    errors = []
    for iter_count in range(max_iterations):
        x_new = np.zeros(n)
        for i in range(n):
            # Note that the x_new here is the only difference to␣
↪jacobi_iteration
            sigma = A[i, :i] @ x_new[:i] + A[i, i+1:] @ x[i+1:]
            x_new[i] = (1 / A[i, i]) * (b[i] - sigma)

        errors.append(np.linalg.norm(x_sol - x_new))
        # in real life we wouldnt have x_sol, so we check against the previous x
        if np.allclose(x, x_new, atol=tol, rtol=0):
            print(f"Converged after {iter_count} iterations.")
            return x_new, errors

        x = x_new

    print("Maximum iterations reached without convergence.")
    return x
```

[72]:
```python
def generate_matrix(n, spectral_gap=5):
    A = np.random.rand(n, n)
    rowsums = A.sum(axis=1)
    A = A + spectral_gap * np.diag(rowsums)
    return A
```

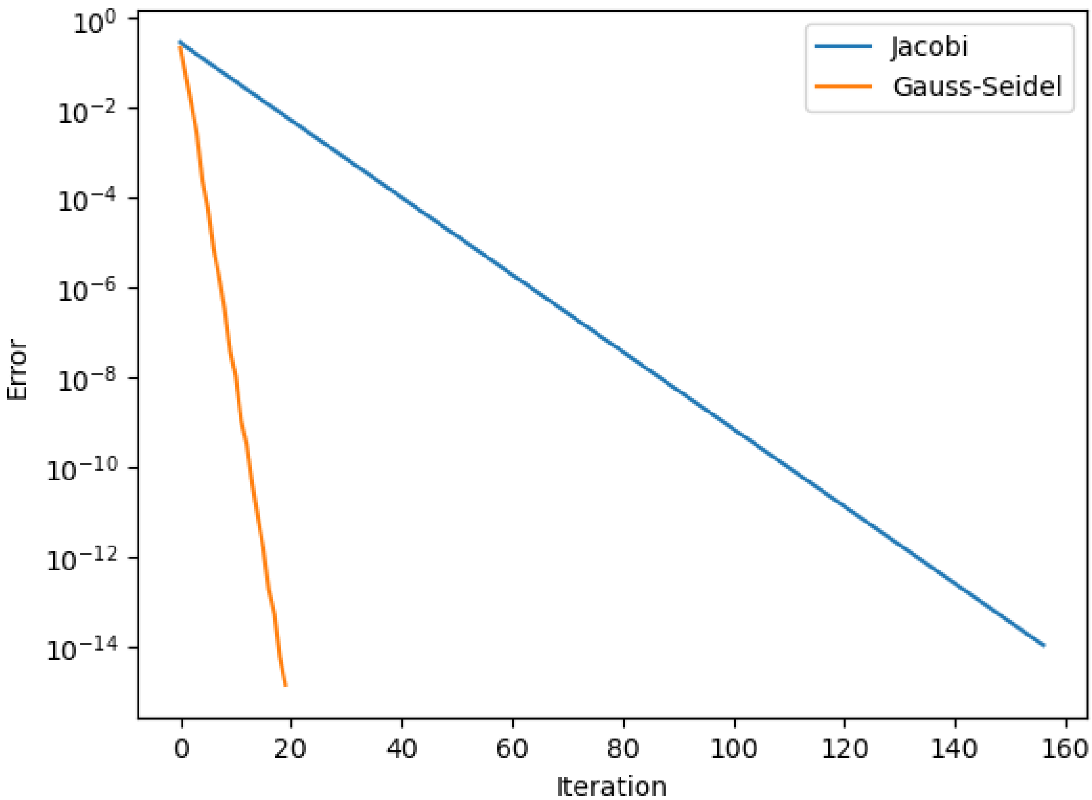
[73]:
```python
n = 10
A = generate_matrix(n, spectral_gap=1)
x_sol = np.random.rand(n)
b = A @ x_sol
x_0 = np.random.rand(n)
cond = np.linalg.cond(A)
print(f"Condition number of A: {cond}")
x_jacobi, errors_jacobi = jacobi_iteration(A, b, x_0, x_sol, tol=1e-14)
x_gauss_seidel, errors_gauss_seidel = gauss_seidel_iteration(A, b, x_0, x_sol,␣
↪tol=1e-14)
```

```
Condition number of A: 3.647703324781633
Converged after 156 iterations.
Converged after 19 iterations.
```

[74]:
```python
plt.plot(errors_jacobi, label='Jacobi')
plt.plot(errors_gauss_seidel, label='Gauss-Seidel')
```

```python
plt.yscale('log')
plt.xlabel('Iteration')
plt.ylabel('Error')
plt.legend()
```

[74]: <matplotlib.legend.Legend at 0x120e48410>



Note that Gauss-Seidel converges much faster, especially for worse conditioned matrices