homework1

March 10, 2025

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
import math as math
```

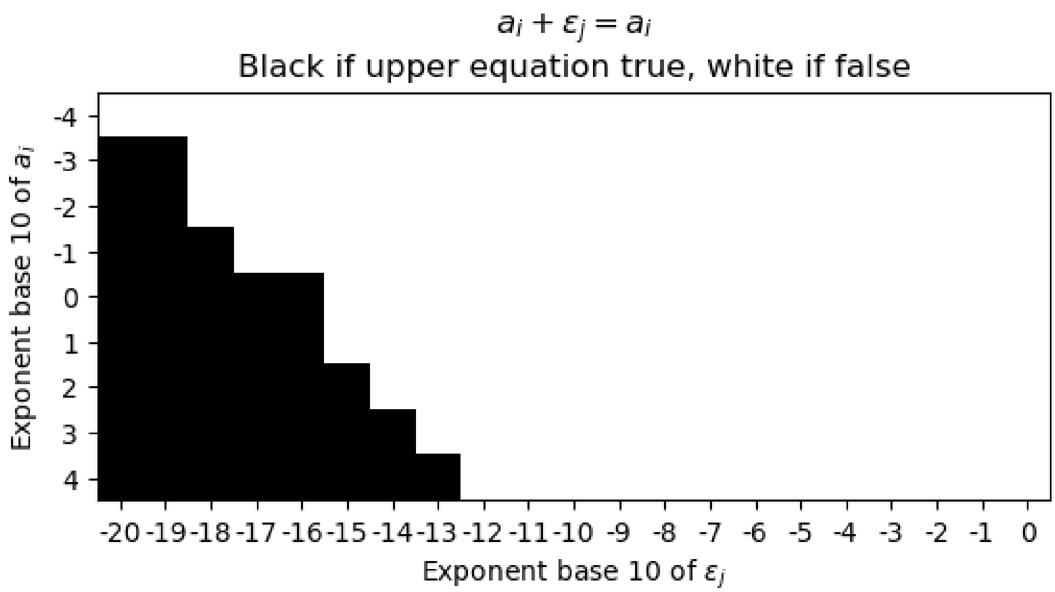
1 Exercise 3

[2]:
```
I = np.arange(-4,5)
J = np.arange(-20,1)

a_i = 10.**I
eps_j = 10.**J

# like an outer product of vectors, only its a sum (row vector + column vector)
B = a_i[:, np.newaxis] + eps_j[np.newaxis, :]
# a_i column copied 21 times
A_i = np.tile(a_i , (len(J),1)).T

boolean_Matrix = B == A_i
```

[3]:
```
max_indices = np.array([
    np.max(np.where(row)[0]) if row.any() else None
    for row in boolean_Matrix
])
print([index.item()-20 if index!=None else None for index in max_indices ])
```

```
[None, -19, -19, -18, -16, -16, -15, -14, -13]
```

[4]:
```
plt.imshow(boolean_Matrix, cmap='binary')
# Set custom tick positions and labels for both axes
plt.xticks(np.arange(21),np.arange(-20, 1))
plt.xlabel('Exponent base 10 of $\\varepsilon_j$')
plt.yticks(np.arange(9),np.arange(-4, 5))
plt.ylabel('Exponent base 10 of $a_i$')
plt.title('$a_i + \\varepsilon_j = a_i$\nBlack if upper equation true, white if␣
 ↪false')
plt.show()
```



We see that the error for adding two floats is relative to the exponent. According to the plot the relative precision for $i = 0$ i.e. for $a_0 = 1$ is in the magnitude of $\varepsilon_0 = 10^{-16}$

[7]:
```
error = math.nextafter(1.,2) - 1
print(f"real error: {error:.2e}")
```

```
real error: 2.22e-16
```

2 Exercise 4

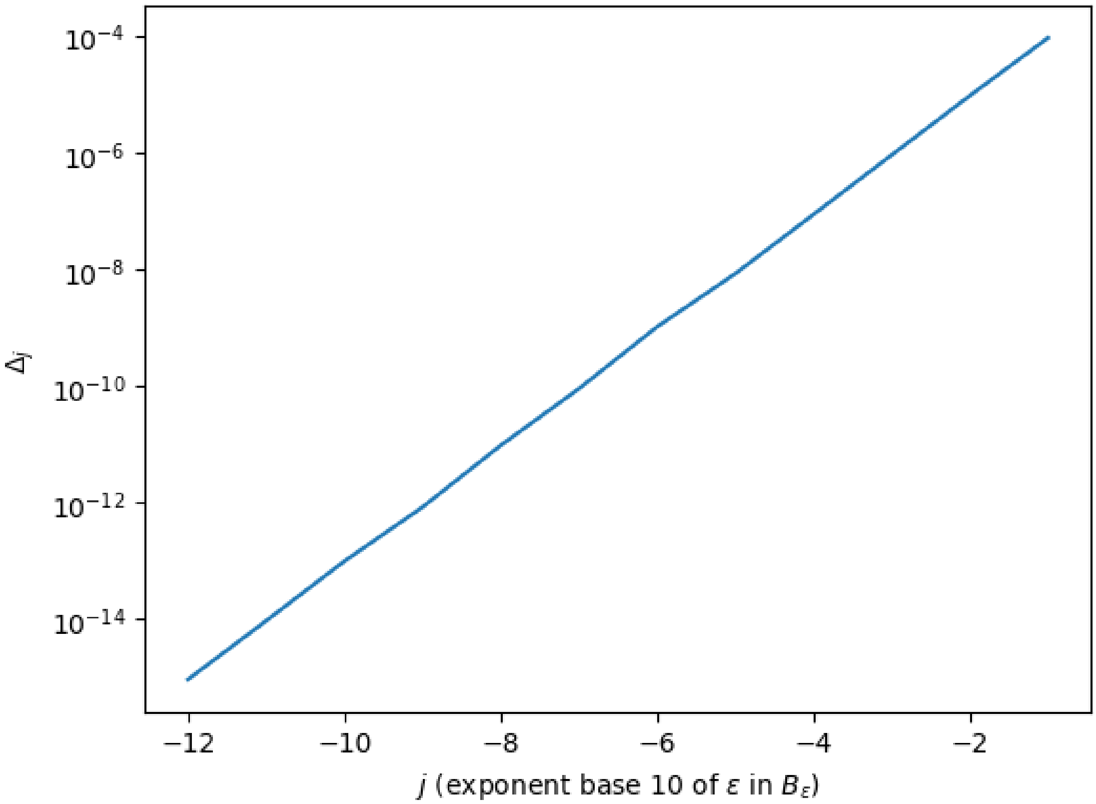
[8]:
```
A = np.array([[np.sqrt(2)/2, -np.sqrt(2)/2],[np.sqrt(2)/2,np.sqrt(2)/2]])
u = np.random.random(size=(100,2))
f = np.array([A @ u_i for u_i in u])
u_hat = np.array([np.linalg.solve(A,f_i) for f_i in f])
Delta = 1/u.shape[0] * np.sum(np.linalg.norm(u_hat-u, axis=1)/np.linalg.norm(u,␣
↪axis=1))
print(f"Delta = {Delta:.2e}")
```

```
Delta = 7.84e-17
```

[9]:
```
B = lambda eps: np.array([[1, 1+eps],[1,1]])
Delta_B = np.empty(12)
u = np.random.random(size=(100,2))

for j in reversed(range(1,13)):
    B_eps = B(10**(-j))
    f = np.array([B_eps @ u_i for u_i in u])
```

```
    u_hat = np.array([np.linalg.solve(B_eps,f_i) for f_i in f])
    Delta_B[j-1]=(1/u.shape[0] * np.sum(np.linalg.norm(u_hat-u, axis=1)/np.
 ↪linalg.norm(u, axis=1)))
```

[10]:
```
plt.plot(np.arange(-12,0), Delta_B)
plt.yscale('log')
plt.xlabel('$j$ (exponent base 10 of $\\varepsilon$ in $B_\\varepsilon$)')
plt.ylabel('$\\Delta_j$')
plt.show()
```



We see that the solution error Δ_j increases rapidly as ε_j increases. This may be due to the fact that for very small ε the numbers 1 and $1+\varepsilon$ are indistinguishable, and then we have a singular matrix, and the built-in method uses some exact algorithm to solve the problems. But when ε_j is large, this does not happen and we have a nearly singular matrix, which is usually very hard to solve with some numerical algorithms, which Python probably uses in this case.