1. Show that $m(a+bX) = a + b \times m(X)$.

$$m(X) = \frac{1}{N}\sum_{i=1}^{N} x_i \quad , \quad Y = a+bX$$

$$m(Y) = m(a+bX) = \frac{1}{N}\sum_{i=1}^{N}(a+bx_i)$$

$$m(a+bX) = \frac{1}{N}\sum_{i=1}^{N} a + \frac{1}{N}\sum_{i=1}^{N} bx_i$$

$$m(a+bX) = \frac{Na}{N} + b \cdot \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\boxed{m(a+bX) = a+b \cdot m(X)}$$

2. Show that $\text{cov}(X, a+bY) = b \times \text{cov}(X,Y)$

$$\text{cov}(X,Y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y)) \quad , \quad Z = a+bY$$

$$m(Z) = m(a+bY) = a+b \cdot m(Y)$$

$$\text{cov}(X,Z) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(z_i - m(Z))$$

$$z_i = a+by_i \quad ; \quad m(Z) = a+bm(Y)$$

$$z_i - m(Z) = (a+by_i) - (a+bm(Y)) = b(y_i - m(Y))$$

$$\text{cov}(X,Z) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(b(y_i - m(Y)))$$

$$\text{cov}(X,Z) = b \cdot \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y))$$

$$\text{cov}(X,Z) = b \cdot \text{cov}(X,Y)$$

$$\boxed{\text{cov}(X, a+bY) = b \cdot \text{cov}(X,Y)}$$

3. Show that $\mathrm{cov}(a+bX, a+bX) = b^2\mathrm{cov}(X,X)$, and in particular that $\mathrm{cov}(X,X) = s^2$.

$$\mathrm{cov}(X,Y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y))$$

$$m(Z) = m(a+bX) = a + bm(X)$$

$$\mathrm{cov}(Z,Z) = \frac{1}{N}\sum_{i=1}^{N}(z_i - m(Z))(z_i - m(Z))$$

$$z_i = a+bx_i \quad ; \quad m(Z) = a+bm(X)$$

$$z_i - m(Z) = (a+bx_i) - (a+bm(X)) = b(x_i - m(X))$$

$$\mathrm{cov}(Z,Z) = \frac{1}{N}\sum_{i=1}^{N}(b(x_i - m(X)))(b(x_i - m(X)))$$

$$\mathrm{cov}(Z,Z) = b^2 \cdot \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))^2$$

$$s^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))^2$$

$$\mathrm{cov}(X,X) = s^2$$

$$\therefore \boxed{\mathrm{cov}(a+bX, a+bX) = b^2\mathrm{cov}(X,X) = b^2 s^2}$$

4. Instead of the mean, consider the median. Consider transformations that are non-decreasing (if $x \geq x'$, then $g(x) \geq g(x')$), like $2 + 5 \times X$ or $\operatorname{arcsinh}(X)$. Is a non-decreasing transformation of the median the median of the transformed variable? Explain. Does your answer apply to any quantile? The IQR? The range?

For non-decreasing transformation $g(x)$, the median of the transformed data is the transformation of the median: $\text{median}(g(x)) = g(\text{median}(x))$

This is b/c non-decreasing transformations preserve order, so the middle value (median) never moves.

$g(x) = 2+5x \rightarrow \text{median}(2+5x) = 2 + 5 \cdot \text{median}(x)$
$g(x) = \text{arcsinh}(x) \rightarrow \text{median}(\text{arcsinh}(x)) = \text{arcsinh}(\text{median}(x))$

This also applies to IQR & range. That is, as long as $g(x)$ is non-decreasing.

5. Consider a non-decreasing transformation $g()$. Is is always true that $m(g(X)) = g(m(X))$?

given $m(x) = \frac{1}{N}\sum_{i=1}^{N} x_i$

we compare : $m(g(x)) = \frac{1}{N}\sum_{i=1}^{N} g(x_i)$ and $g(m(x)) = g\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$

only equal when $g(x)$ linear :

For $g(x) = a+bx \rightarrow m(a+bx) = a+bm(x)$ ✓

For $g(x) = x^2$ & $x = \{-2, 0, 2\}$
↳ $m(x) = (-2+0+2)/3 = 0 \rightarrow g(m(x)) = 0^2 = 0$
↳ $m(g(x)) = m(x^2) = ((-2)^2 + 0^2 + 2^2)/3) = (4+0+4)/3 = 8/3$ $> \neq$

$\therefore m(g(x)) \neq g(m(x))$

$\therefore$ The transformation of the mean is not necessarily the mean of the transformed variables.