Contents

1 Linear SISO plant

Control problem

Given plant G and reference model M.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

a_p - pole of plant
k_p - input gain of plant

$$M : \dot{x}_m(t) = a_m x_m(t) + k_m r(t), \quad (2)$$
$$\text{IC } x_m(0) \in \mathbb{R}$$

a_m - pole of reference model
k_m - input gain of reference model
$r(t)$ - reference signal

The reference model parameters are set by the user and are therefore known.

Task find a control $u(t)$ such that $x_p(t) \to x_m(t)$ for $t \to \infty$.

$$G_{des} : \dot{x}_p(t) = a_m x_p(t) + k_m r(t) \quad (3)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Solutions for $u(t)$ using

- Model reference control (MRC)
- Model reference adaptive control (MRAC)

1.1 Model reference control (MRC)

The plant parameters are assumed to be known.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Pick $u(t)$ such that the dynamical behaviour of the closed loop is equal to that of the model. This is done by comparing (1) to (3).

$$\begin{aligned} u^*(t) &= \frac{1}{k_p}\left(-a_p x_p(t) + a_m x_p(t) + k_m r(t)\right) \\ &= \underbrace{\frac{a_m - a_p}{k_p}}_{a^*} x_p(t) + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t) \\ &= \begin{bmatrix} a^* & k^* \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix} \end{aligned}$$
$$u^*(t) = \theta^{*T} \phi(t)$$

Using this input[1], now the dynamics of the plant G matches the dynamics of the model M, as in equation (3).

However, even though now the dynamics are the same, the initial conditions are not necessarily the same. Would this input also work for $x_m(0) \neq x_p(0)$? I.e., does this guarantee that $x_p(t) \to x_m(t)$ for $t \to \infty$?

Dependence on the initial conditions To check this, we examine the error dynamics and see if the error asymptotically goes to zero.

$$e(t) = x_p(t) - x_m(t)$$
$$\dot{e}(t) = \dot{x}_p(t) - \dot{x}_m(t)$$

Using eqns. (2) and (3):

$$\dot{e}(t) = a_m e(t) \quad (4)$$

If $a_m < 0$, the error dynamics are stable. That is, $e(t) \to 0$ for any ICs.

Conclusion

- Model reference control (MRC) works with the error dynamics of the reference model
- We need to know all plant parameters very well
  $\Rightarrow$ Problem: uncertainty in parameters

1.2 Model reference adaptive control (MRAC)

The plant parameters are unknown. We assume $k_p > 0$.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Control law We search for (learn) the value of θ and k, which are therefore functions of time.

$$\begin{aligned} u(t) &= \begin{bmatrix} a(t) & k(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix} \\ &= \theta^T(t) \phi(t) \end{aligned} \quad (5)$$

Adaptive law Adapt the control parameters in the following fashion.

$$\begin{bmatrix} \dot{a}(t) \\ \dot{k}(t) \end{bmatrix} = -\operatorname{sgn}(k_p)\, e(t) \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}$$
$$\Rightarrow \dot{\theta} = -\operatorname{sgn}(k_p)\, e(t) \Gamma \phi(t) \quad (6)$$

The equations in (6) are nonlinear ODEs.

[1] The starred variables with $*$ superscripts represent the ideal values of the control parameters.

Questions

- Is the closed loop stable?
- Does, with this, $e(t) \to 0$?
- Are the parameters $\theta(t)$ finite?
- Are the parameters $\theta(t)$ constant for $t \to \infty$?
- Do the parameters $\theta(t)$ approach their 'ideal' values θ^* for $t \to \infty$?

2 Nonlinear SISO plant

The linear SISO plant in the previous chapter is generalised to a nonlinear SISO plant.

Control problem

Given plant G and reference model M.

$$
\begin{aligned}
G:\ \dot{x}_p(t) &= a_p x_p(t) + k_p u(t) \\
&\quad + \alpha_p f(z) && (7) \\
M:\dot{x}_m(t) &= a_m x_m(t) + k_m r(t) && (2)
\end{aligned}
$$

- a_p, k_p, α_p are unknown but constant
- $f(z)$ is a nonlinear (external) function
- $\operatorname{sgn}(k_p), f(.)$ are known (z is a known signal)
- $\alpha_m f(z)$ is not necessary

Goal $x_p(t) \to x_m(t)$ for $t \to \infty$.

2.1 Control structures

Ideal control structure based on MRC

$$
\begin{aligned}
u^*(t) &= \frac{1}{k_p}\left(-a_p x_p(t) + a_m x_p(t)\right. \\
&\qquad \left. + k_m r(t) - \alpha_p f(z)\right) \\
&= \underbrace{\frac{(a_m - a_p)}{k_p}}_{a^*} x_p + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t) + \underbrace{\frac{-\alpha_p}{k_p}}_{\alpha^*} f(z) \\
&= \begin{bmatrix} a^* & k^* & \alpha^* \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix}
\end{aligned}
$$

$$
u^*(t) = \theta^{*T} \phi(t) \tag{8}
$$

Control law using MRAC

$$
u(t) = \begin{bmatrix} a(t) & k(t) & \alpha(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix}
$$

$$
u(t) = \theta^T(t)\phi(t) \tag{9}
$$

$a(t),\ k(t),\ \alpha(t)$ unknown.

2.2 Error dynamics

In adaptive control, the current estimated parameters are varying. We therefore have the following error in parameters as deviations from the unknown but ideal and constant real parameters:

$$
\left.\begin{aligned}
\tilde{a}(t) &= a(t) - a^* \\
\tilde{k}(t) &= k(t) - k^* \\
\tilde{\alpha}(t) &= \alpha(t) - \alpha^*
\end{aligned}\right\} \tilde{\theta}(t) = \theta(t) - \theta^*
$$

$$
\begin{aligned}
\dot{e}(t) &= \dot{x}_p(t) - \dot{x}_m(t) \\
&= a_p x_p(t) \\
&\quad + k_p\left(a(t)x_p(t) + k(t)r(t) + \alpha(t)f(z)\right) \\
&\quad + \alpha_p f(z) \\
&\quad - (a_m x_m(t) + k_m r(t)) \\
&= a_p x_p(t) - a_m x_m(t) \\
&\quad + k_p a(t) x_p(t) \\
&\quad + k_p \underbrace{\left(k(t) - \frac{k_m}{k_p}\right)}_{\tilde{k}(t)} r(t) \\
&\quad + k_p \underbrace{\left(\alpha(t) - \frac{\alpha_p}{k_p}\right)}_{\tilde{\alpha}(t)} f(z) \\
&= \underbrace{(a_m - k_p a^*)}_{a_p} x_p(t) - a_m x_m(t) \\
&\quad + k_p a(t) x_p(t) \\
&\quad + k_p \tilde{k}(t) r(t) + k_p \tilde{\alpha}(t) f(z) \\
&= a_m e(t) + k_p \tilde{a}(t) x_p(t) \\
&\quad + k_p \tilde{k}(t) r(t) + k_p \tilde{\alpha}(t) f(z)
\end{aligned}
$$

$$
\dot{e}(t) = a_m e(t) + k_p \begin{bmatrix} \tilde{a}(t) & \tilde{k}(t) & \tilde{\alpha}(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix}
$$

$$
\dot{e}(t) = a_m e(t) + \frac{1}{k^*} k_m \tilde{\theta}^T(t) \phi(t) \tag{10}
$$

The error dynamics can be rewritten using an operator $M(s) = \frac{k_m}{s - a_m}$, which is non other than the transfer function of the reference model!

$$
\begin{aligned}
\dot{e}(t) &= a_m e(t) + \frac{1}{k^*} k_m \theta^T(t) \phi(t) \\
(s - a_m)\, e(t) &= \frac{1}{k^*} k_m \theta^T(t) \phi(t) \\
e(t) &= \frac{1}{k^*} M(s) \theta^T(t) \phi(t) && (10)
\end{aligned}
$$

All unknown parameters appear linearly (affine, 'linear in the parameters'). The error dynamics (10) is a nonlinear differential equation. When is it stable? → Lyapunov.

2.3 Lyapunov-like function

New interpretation of Lyapunov Nothing to do with energy. $\boldsymbol{V}$ affects the scaling of the distance of $\boldsymbol{x}$ from the origin in the phase portrait.

$$
\|\boldsymbol{x}\|_{\boldsymbol{V}}^2 = \boldsymbol{x}^T \boldsymbol{V} \boldsymbol{x}, \qquad \boldsymbol{V} \succ 0
$$

> **All Lyapunov says is**: how far is x from the origin? We want to find some type of measure for that.

Lyapunov function (Lyapunov-like)
We want the output error $e(t)$ as well as the parameter error $\tilde{\theta}(t)$ to go to zero. $\Gamma \succ 0$ symmetrical, positive definite.

$$V(e,\tilde{\theta}) = \frac{1}{2}e^2 + \frac{1}{2}|k_p|\left(\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}\right) \tag{11}$$
$$\dot{V} = e\dot{e} + \frac{1}{2}|k_p|\underbrace{\left(2\tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}}\right)}_{2}$$

Substitute $\dot{e}$ using equation (10).

$$\begin{aligned}\dot{V} &= a_m e^2 + e k_p \tilde{\theta}^T \phi + \frac{1}{2}|k_p|\left(2\theta^T \tilde{\Gamma}^{-1}\dot{\tilde{\theta}}\right)\\ &= a_m e^2 + |k_p|\tilde{\theta}^T \underbrace{\left(\operatorname{sgn}(k_p)\, e\phi + \Gamma^{-1}\dot{\tilde{\theta}}\right)}_{\overset{!}{=}0}\end{aligned}$$

The second term is set to 0, because we want $V \preceq 0$ and we don't know all the signs of the terms. This will define the **adaptive law**.

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -\operatorname{sgn}(k_p)\,\Gamma\phi(t)e(t)\\ \dot{\theta}(t) &= -\operatorname{sgn}(k_p)\,\Gamma\phi(t)e(t)\end{aligned}$$

With the adaptive law, we obtain for $\dot{V}$:

$$\dot{V} = a_m e^2 \preceq 0 \tag{12}$$

Remark $e(t)$ does not have to be 0 – why?

> If the derivative of a function $\rightarrow 0$, that does not imply that the function has a limit, and vice versa: if a function has a limit, that doesn't mean its derivative $\rightarrow 0$.
>
> $$\lim_{t\to\infty} \dot{f}(t) = 0 \not\Leftrightarrow \lim_{t\to\infty} f(t) = k$$

Counterexamples:

$$f(t) = \sin(\ln t)$$
$$\nexists \lim_{t\to\infty} f(t), \quad \dot{f}(t) = \frac{\cos(\ln t)}{t} \rightarrow 0$$

$$f(t) = e^{-t}\sin(e^{2t})$$
$$\lim_{t\to\infty} f(t) = 0$$
$$\dot{f}(t) = -e^{-t}\sin(e^{2t}) + e^{t}\sin(e^{2t})$$
$$\rightarrow \text{ explodes!}$$

[2]Possible due to Γ symmetrical.

Are the error dynamics stable?
- Measure (some of) the states
- Apply $V = f(e, \tilde{\theta})$
- $V \rightarrow \infty$? Or $V \downarrow$?
  $\Rightarrow$ analyse time derivative $\dot{V}$
- If we show $\dot{V} \rightarrow 0$, then $e \rightarrow 0$.

Extensions to Lyapunov There are two well-known extensions to Lyapunov to prove asymptotic stability, even if $\dot{V} \preceq 0$.
(i) LaSalle's invariance principle (only for autonomous systems)
(ii) Barbalat's lemma (OK for non-autonomous systems)

Our system's error dynamics are non-autonomous, $\dot{e} = f(t, \ldots)$, due to following another system (Figure 1).

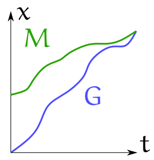

Figure 1: Non-autonomous dynamics

2.4 Closed loop stability analysis

Using Barbalat's Lemma (Variant A)[3] on the function V, we need to fulfill the following conditions:
(i) V is differentiable
Yes, $\exists\, \dot{V} = a_m e^2(t)$
(ii) V has a finite limit as $t \rightarrow \infty$
Yes, as $V \succeq 0$ and $\dot{V} \preceq 0$
(iii) $\dot{V}$ is uniformly continuous $\Leftarrow \exists\, \ddot{V}$ (sufficient condition[4])
Is $\ddot{V} = 2a_m e\dot{e}$ bounded?

Boundedness of $e(t)$
- As V is bounded from below and non-increasing, V has a limit as $t \rightarrow \infty$.
- Tracking error $e(t)$ and parameter errors $\tilde{\theta}(t)$ are bounded.
- As $\tilde{\theta}(t)$ bounded and θ^* constant, $\theta(t)$ is bounded.

Boundedness of $\dot{e}(t)$
- Assume $r(t)$ bounded[5], then, from the reference model equation (2):
  $x_m(t), \dot{x}_m(t)$ bounded ($\because$ M is stable)
- $x_p(t) = \underset{\text{bd.}}{e(t)} + \underset{\text{bd.}}{x_m(t)}$
  $\Rightarrow x_p(t)$ bounded.
- $u(t) = \theta^T(t)\phi(t)$ bounded if $\phi(t)$ bounded.
  $$\phi(t) = \begin{bmatrix} \underset{\text{bd.}}{r(t)} & \underset{\text{bd.}}{x_p(t)} & \underset{\text{bd.}}{f(z)} \end{bmatrix}^T$$

[3]Chapter 2.7 on page 7
[4]Chapter 2.6 on page 7
[5]Reasonable assumption, because why would we want to use an unbounded input?

(new requirement: $f(z)$ needs to be bounded).

- $\dot{x}_p(t) = a_p \underset{\text{bd.}}{x_p(t)} + k_p \underset{\text{bd.}}{u(t)}$
  $\Rightarrow \dot{x}_p(t)$ bounded
- $\dot{e}(t) = \underset{\text{bd.}}{\dot{x}_p(t)} - \underset{\text{bd.}}{\dot{x}_m(t)}$ is bounded.

All the conditions of Barbalat's lemma thus fulfilled, we can conclude that the derivative of V approaches zero for $t \to \infty$.

$$\lim_{t\to\infty} \dot{V} = 0$$
$$\Rightarrow \lim_{t\to\infty} e(t) = 0$$

2.5 Nonlinearities

THEOREM: Nonlinear SISO plant

Given plant G and reference model M.

$$G: \ \dot{x}_p(t) = a_p x_p(t) + k_p u(t) + \alpha_p f(z) \qquad (7)$$
$$M: \dot{x}_m(t) = a_m x_m(t) + k_m r(t) \qquad (2)$$

The input

$$u(t) = \theta^T(t)\boldsymbol{\phi}(t) \qquad (9)$$
$$\text{with } \dot{\theta} = -\operatorname{sgn}(k_p)\,\boldsymbol{\Gamma}\boldsymbol{\phi}e \qquad (6)$$
$$\text{and } \boldsymbol{\phi}(t) = \begin{bmatrix} r(t) & x_p(t) & f(z) \end{bmatrix}^T$$

renders the origin asymptotically stable and guarantees $x_p(t) \to x_m(t)$ as $t \to \infty$.

We can add, arbitrarily, many 'nonlinearities' $f_i(z_j)$ with unknown gains α_i (Figure 2). The function $z(t)$ z is a placeholder. $z(t)$ can be an external or an internal signal. The nonlinearity functions need not be continuous. The only requirement:

$$f_i(z_j) \in \mathcal{L}_\infty$$

Nonlinearities are bounded at all times[a].

[a] For the boundedness of $u(t)$ and therefore $\dot{e}(t)$

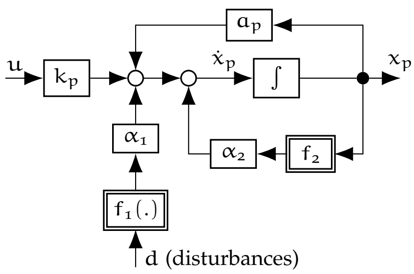

Figure 2: Plant G with nonlinearities

Questions

- Can we do $f(u)$?
  Possible, but solving for $u = \cdots f(u) \cdots$ is difficult.
- Can $f(.)$ be a differential operator (filter)?
  Yes (Figure 3). If the filter is linear, then a solution definitely exists.

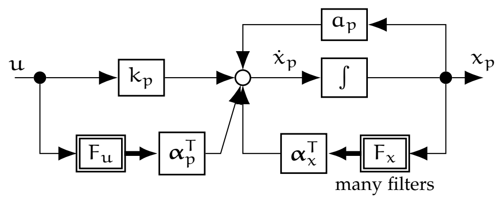

Figure 3: Plant G with nonlinearities as filters

The filters F_u and F_x are stable dynamical systems. If they are not stable, there is a higher chance of getting an infinite output $\to$ conflicts with bounded output requirement.

If no such filters are present in the plant G, then the plant has an order of 1.

Let's say we have an offset in the plant input of unknown magnitude, and the plant has otherwise known parameters.

$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t) + \alpha_p \cdot 1$$
$$u(t) = a^* x_p(t) + k^* r(t) - \frac{\alpha(t)}{k_p}$$

Closed loop becomes

$$\dot{x}_p(t) = a_m x_p(t) + k_m r(t) + \tilde{\alpha}(t)$$

Error dynamics

$$\dot{e} = a_m e(t) + \tilde{\alpha}(t)$$

Lyapunov-like function

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{\alpha}^2$$
$$\dot{V} = e\dot{e} + \tilde{\alpha}\dot{\tilde{\alpha}} + \cdots$$
$$= a_m e^2 + \tilde{\alpha}\underbrace{\left(e + \dot{\tilde{\alpha}}\right)}_{\text{to set to 0}}$$

Setting the second term to zero ensures the negative semidefiniteness of $\dot{V}$.

$$\dot{\tilde{\alpha}} = -e \qquad (13)$$

Equation (13) above implies $u = \cdots + \int e\,dt + \cdots$, i.e., that the controller contains an I-part. As a result of this, there are no steady state errors caused by model uncertainties. The controller eliminates offset at input of plant.

Equation (13) is a pure integrator acting on control error. This is a linear controller! We have learned: integrators 'learn' input offsets of the plant and correct them.

Adaptive controllers can be interpreted as nonlinear PI controllers.

2.6 Uniformly continuous functions

Def: Uniformly continuous function

A function $f(t) : \mathbb{R} \to \mathbb{R}$ is uniformly contінous, if
$\forall \varepsilon > 0: \quad \exists\, \delta = \delta(\varepsilon) > 0,$

$$\forall |t_2 - t_1| \leqslant \delta$$
$$\Rightarrow |f(t_2) - f(t_1)| \leqslant \varepsilon$$

Sufficient condition for uniformly continuous functions: If the derivative $\dot{f}(t)$ exists (i.e. bounded), $\Rightarrow f(t)$ is uniformly continuous.

2.7 Barbalat's lemma

Lemma: Barbalat Variant A

If $f(t) : \mathbb{R} \to \mathbb{R}$
(i) is a differentiable function, $\dot{f} \in \mathcal{L}_\infty$
(ii) has a finite limit as $t \to \infty$, $f \in \mathcal{L}_\infty$
(iii) $\dot{f}(t)$ is uniformly continuous, $\ddot{f} \in \mathcal{L}_\infty$

$$\Rightarrow \lim_{t\to\infty} \dot{f}(t) = 0$$

Lemma: Barbalat Variant B

If
(i) $f(t) : \mathbb{R} \to \mathbb{R}$ is uniformly continous $\forall t$
(ii) $\exists \lim_{t\to\infty} \int_0^t f(\tau)d\tau$

$$\Rightarrow \lim_{t\to\infty} f(t) = 0$$

Lemma: Barbalat Variant C

If
(i) $f \in \mathcal{L}_\infty$
(ii) $\dot{f} \in \mathcal{L}_\infty$
(iii) $f \in \mathcal{L}_2$,

$$\Rightarrow |f(t)| \to 0 \text{ as } t \to \infty$$

2.8 Signal norms and functional spaces

Idea quantify magnitude of a signal $x(t)$ – "How big is a signal?"

Def: Signal norm

Given

$$x(t) : \mathbb{R}^+ \to \mathbb{R}^n, \qquad \mathbb{R}^+ = [0, \infty)$$

p-Norm

$$\|x_p\| = \left(\int_0^\infty |x(t)|^p \, dt\right)^{1/p} \qquad p \in (0, \infty)$$

Distance vector 2-norm of $\mathbf{x}(t)$, i.e. $|\mathbf{x}|$.

Max. value "When power ∞, only the greatest value survives"

$$\|x\|_\infty = \sup_{t\in\mathbb{R}^+} |x(t)|$$
$$\hat{=} \text{ highest value of } x(t).$$

Def: Functional space

$$\mathcal{L}_p = \{x(t) \in \mathbb{R}^n : \underbrace{\|x\|_p < \infty}_{\text{exists}}\}$$

$x(t) \in \mathcal{L}_p$
- x is bounded
- "x's highest value exists and is not infinity."

Ex. 2.1 Functional space

Show that $e \in \mathcal{L}_\infty$
e is in V and V is bounded, $\therefore e \in \mathcal{L}_\infty$.

Ex. 2.2 Functional space

Show that $e \in \mathcal{L}_2$

$$\int_0^\infty \dot{V} dt = V(\infty) - V(0), \qquad \text{is bounded.}$$
$$a_m \int_0^\infty e^2 dt \in \mathcal{L}_\infty$$
$$\|e\|^2 \in \mathcal{L}_\infty$$
$$\Rightarrow e \in \mathcal{L}_2$$

3 Positive real functions

Def: Positive real function I

A rational function $H(s) : \mathbb{Z} \to \mathbb{Z}$, $s = \sigma + j\omega$ is positive real (PR), if

(i) $H(s)$ is real for real s

(ii) $\Re\{H(s)\} \geqslant 0$ for $\Re\{s\} > 0$

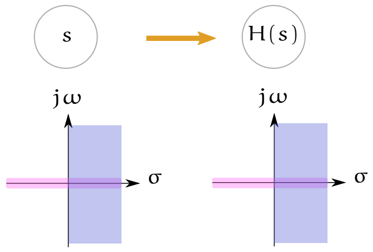

Figure 4: Positive real mapping $s \to H(s)$

Def: Positive real function II

A rational function $H(s)$ is positive real, if

(i) $H(s)$ is analytic in $\Re\{s\} > 0$
- $H(s)$ has no poles in RHP ($\Re\{s\} > 0$)
- $H(s)$ is stable

(ii) $\Re\{H(j\omega)\} \geqslant 0, \quad \forall\omega \in [0, \infty]$
- Nyquist of $H(s)$ is in the RHP
- phase $\angle H(j\omega) \in [-90°, +90°]$
- rel. degree of $H(s)$ is 0 or 1

(iii) any pure imaginary pole $j\omega$ of $H(s)$ is a simple pole, and the residue

$$\lim_{s \to j\omega} (s - j\omega)\, H(s)$$

is positive semidefinite.

Alternatively: $H(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^2 \Re\{H(j\omega)\} \geqslant 0$$

Def: Strictly positive real functions

$H(s)$ is strictly positive real (SPR) if $H(s - \varepsilon)$ is PR for some $\varepsilon > 0$.

Note relative degree of a system corresponds to its response delay.

Lemma: SPR lemma

$H(s)$ is SPR if

(i) $H(s)$ is Hurwitz

all poles on LHP, none are purely imaginary

(ii) $\Re\{H(j\omega)\} > 0, \quad \forall\omega \in \mathbb{R}$
- Nyquist of $H(s)$ is in the RHP and not on the imaginary axis.
- phase $\angle H(j\omega) \in (-90°, +90°)$
- rel. degree of $H(s) \in \{0, 1\}$

phase $\in (-90°, 90°)$

rel. degree $\in \{0, 1\}$

(iii) $H(\infty) > 0$ (positive gain for proper H) or

$$\lim_{\omega \to \infty} \omega^2 \Re\{H(j\omega)\} > 0$$

positive gain for relative degree 1

Discussion If $H(s)$ is SPR, then so is the inverse $H^{-1}(s)$ (stable poles, stable zeroes).

SPR $\Rightarrow$ $H(s)$ is stable, minimal-phase.
I.e., only stable zeroes, because zeroes are in LHP.
- phase $\angle H(j\omega) \in (-90°, +90°)$
- rel. degree of $H(s) \leqslant 1 (-1, 0, 1)$
- positive gain $\forall\ \omega$

Ex. 3.3 PR

$G(s) = \frac{1}{s}$ has a single pole $s = 0$, with a residue of 1.

$$\Re\{G(j\omega)\} = \Re\left\{\frac{1}{j\omega}\right\} = 0 \qquad \forall\omega \neq 0$$

Hence, $G(s)$ is PR but not SPR, as $\frac{1}{s-\varepsilon}$ has a pole in $\Re\{s\} \geqslant 0$ for any $\varepsilon > 0$.

Ex. 3.4 PR

$G(s) = \dfrac{1}{s+a}$, $a > 0$ is Hurwitz.

$$\Re\{G(j\omega)\} = \frac{a}{\omega^2 + a^2} > 0 \qquad \forall\omega \in [0, \infty]$$

$$\lim_{\omega \to \infty} \omega^2 \Re\{G(j\omega)\} = \lim_{\omega \to \infty} \frac{\omega^2 a}{\omega^2 + a^2} = a, \qquad a > 0$$

Ex. 3.5 PR

$$
\begin{aligned}
G(s) &= \frac{1}{s^2+s+1} \\
\Re\{G(j\omega)\} &= \frac{1-\omega^2}{(1-\omega^2)^2+\omega^2} \\
&\not> 0 \qquad \forall\omega \\
&\Rightarrow \text{G is not PR.}
\end{aligned}
$$

4 Kalman-Yakobovich lemma (KY)

"Maier version; how to design a controller given SPR"

Lemma: Kalman-Yakobovich Lemma

Given

- a scalar $\gamma \geqslant 0$
- vectors $\mathbf{b}$ and $\mathbf{c}$,
- an asymptotically stable matrix $\mathbf{A}$[a], and
- a positive definite matrix $\mathbf{L} \succ 0$,

$$
\begin{aligned}
&\text{If } H(s) \triangleq \frac{1}{2}\gamma + \mathbf{c}^T(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} \\
&\Rightarrow H(s) \text{ is SPR}
\end{aligned}
$$

Then, there exist

- a scalar $\varepsilon > 0$
- a vector $\mathbf{q}$, and
- a symmetric positive definite matrix $\mathbf{P}$,

s.t.

$$
\begin{aligned}
\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} &= -\mathbf{q}\mathbf{q}^T - \varepsilon\mathbf{L} &(14)\\
\mathbf{P}\mathbf{b} - \mathbf{c} &= \sqrt{\gamma}\mathbf{q} &(15)
\end{aligned}
$$

[a] positive eigenvalues

Using it We only need $\gamma = 0$ in all cases (in this course). Hence we can say: if $H(s)$ is SPR $\Rightarrow \exists\ \mathbf{P} = \mathbf{P}^T > 0$,
s.t.

$$
\begin{aligned}
\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} &= -\mathbf{Q} \\
\mathbf{P}\mathbf{b} &= \mathbf{c} \qquad \text{(s. note)[6]}
\end{aligned}
$$

where $\mathbf{Q} = \mathbf{Q}^T > 0$

[6] 'boundary cond., means SPR'

Lemma: Adaptive laws based on Lyapunov

(For rel. degree 1 plants)
Consider the dynamical system below[abc].

$$
\begin{aligned}
\dot{x}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}\theta^T(t)\phi(t) &(16)\\
y(t) &= \mathbf{c}^T\mathbf{x}(t) \\
z_1(t) &= ky(t)
\end{aligned}
$$

where

- $(\mathbf{A}, \mathbf{b})$ is stabilisable
- $(\mathbf{c}^T, \mathbf{A})$ is detectable
- $\mathbf{c}^T(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} \triangleq H(s)$ is SPR

Let $\theta(t)$ be a vector of adjustable parameters.
Let $\phi(t)$ and $z_1(t)$ be time-varying functions that can be measured.

Then, if $\theta(t)$ is adjusted as

$$
\dot{\theta}(t) = -\operatorname{sgn} k z_1(t)\phi(t) \qquad (17)
$$

$\Rightarrow$ the equilibrium state $(x = 0, \theta = 0)$ is uniformly stable[d] in the large[e].

[a] This refers to the error dynamics, not the plant!
[b] z_1 allows change of symbol with respect to the output $y(t)$
[c] $x \in \mathbb{R}^n, \quad y, z \in \mathbb{R}^1, \quad \phi, \theta \in \mathbb{R}^k$
[d] uniformly stable: not dependent on time nor on IC
[e] in the large: IC don't matter anywhere in $\mathbb{R}^n$

Proof Since $H(s)$ is SPR, it follows from the KY-lemma that $\exists\ \mathbf{P} = \mathbf{P}^T > 0$, such that

$$
\begin{aligned}
\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} &= -\mathbf{Q}, \quad \mathbf{Q} = \mathbf{Q}^T > 0 &(14)\\
\mathbf{P}\mathbf{b} &= \mathbf{c} &(15)
\end{aligned}
$$

Let V be a positive definite function

$$
\begin{aligned}
V &= \mathbf{x}^T\mathbf{P}\mathbf{x} + \frac{1}{|k|}\theta^T\theta \\
\dot{V} &= \mathbf{x}^T\left(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P}\right)\mathbf{x} \\
&\quad + 2\mathbf{x}^T\mathbf{P}\mathbf{b}\theta^T\phi - 2\theta^T y\phi \\
&= -\mathbf{x}^T\mathbf{Q}\mathbf{x} \leqslant 0
\end{aligned}
$$

Therefore, the origin of the system (16) together with the adaptive law (17) is stable.

Discussion We now have a simple tool for finding adaptive laws with error dynamics below (10), where $M(s)$ is SPR, stabilisable and detectable, and the adaptive laws are defined as in (18)

$$
\begin{aligned}
e(t) &= \frac{1}{k^*}M(s)\left[\tilde{\theta}^T\phi\right] &(10)\\
\dot{\theta}(t) &= -\operatorname{sgn}\varepsilon e\Gamma\phi &(18)
\end{aligned}
$$

If $M(s)$ has a relative degree of 1, it is obvious that $e(t) \to 0$ for $t \to \infty$[7].

5 Performance considerations

Performance criteria:

Performance	Noise
Disturbances	Robustness

- Increasing γ, we are unhappy with the oscillations of our parameters θ and therefore with the oscillations of $u(t)$.
- We have no clue what the adaptive closed loop will do between $t = 0$ and $t = \infty$ other than boundedness

5.1 Adaptation with a closed loop reference model

Now Deal with transient response

Idea Adaptation changes with signals

$$\dot{\theta} = -\operatorname{sgn}(\varepsilon)\, e\gamma\phi$$

where the value of e and γ are changeable.
$\Rightarrow$ we can alter the transient with γ (leads to oscillations), or we can change $e(t)$.

So far Open loop reference model (ORM)

$$\dot{x}_m^o(t) = a_m x_m^o(t) + k_m r(t) \tag{2}$$

Now Closed loop reference model (CRM)

$$\dot{x}_m^c(t) = a_m x_m^c(t) + k_m r(t) - l e^c(t) \tag{19}$$

ORM = CRM if $l = 0$.

"CRM is observer-like; M helps G by moving towards G and retreating to original position." Through the movement, the reference model now has a different behaviour ($M \to M'$!) and the plant P is trying to follow M'.

γ - learning effect
- decreasing γ helps P follow M',
- but the learning becomes slower

l - movement to P
- increasing l helps P follow M'

[7] Why?

5.2 Stability proof

$$\dot{x}_m^c(t) = a_m x_m^c(t) + k_m r(t) - l e^c(t) \tag{19}$$
$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t) \tag{1}$$

Input is

$$u = \begin{bmatrix} a(t) & k(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix} = \theta^T(t)\boldsymbol{\phi}(t)$$

$$\dot{x}_p(t) = a_m x_p(t) + k_m r(t) + k_p \theta^T(t)\boldsymbol{\phi}(t)$$

Tracking error

$$\begin{aligned} \dot{e}^c(t) &= \dot{x}_p(t) - \dot{x}_m^c(t) \\ &= (a_m + l)\, e^c(t) + k_p \theta^T(t)\boldsymbol{\phi}(t) \end{aligned}$$

Lyapunov-like function

$$\begin{aligned} V(e^c, \tilde{\theta}) &= \frac{1}{2}(e^c)^2 + \frac{1}{2}\Gamma^{-1}|k_p|\tilde{\theta}^T(t)\tilde{\theta}(t) \\ \dot{V} &= e^c \dot{e}^c + \Gamma^{-1}|k_p|\tilde{\theta}^T\dot{\tilde{\theta}} = \dots \\ &= (a_m + l)\,(e^c)^2 \\ &\quad + \underbrace{e^c k_p \tilde{\theta}\boldsymbol{\phi} + \Gamma^{-1}|k_p|\tilde{\theta}^T\dot{\tilde{\theta}}}_{\stackrel{!}{=}0} \\ \dot{V} &= (a_m + l)\,(e^c)^2 \leqslant 0, \qquad l < 0 \end{aligned}$$

Adaptive law

$$\dot{\theta} = -\Gamma \operatorname{sgn} k_p e^c \boldsymbol{\phi}$$

Proof as before. $e^c(t) \to 0$ for $t \to \infty$ [8].

Questions
- How do we show increased performance? (using $\|e^c(t)\|_{\mathcal{L}_2}$ as a performance criterion)
- How do we show that the oscillations decrease?

5.3 Analysing transient performance

Check the performance criterion $\mathcal{L}_2$-norm of e^c

$$\begin{aligned} \int_0^\infty \dot{V}(e^c, \theta)\, d\tau &= V(\infty) - V(0) \\ -|a_m + l| \int_0^\infty e^{c2}\, d\tau &= V(\infty) - V(0) \\ V(0) &= \underbrace{V(\infty)}_{\geqslant 0} + |a_m + l| \cdot \|e^c\|_2^2 \\ V(0) &\geqslant |a_m + l| \cdot \|e^c\|_2^2 \\ \|e^c\|_2 &\leqslant \sqrt{\frac{V(0)}{|a_m + l|}} \end{aligned}$$

[8] We assume here that $e^c(t) \to 0$ follows from $e^o(t) \to 0$. In actuality, though, $e^o(t)$ can't be proven for special functions. However, these cases are usually not relevant to engineering/industry. Therefore, strictly speaking, we can't actually assume that $e^c(t) \to 0$

$$\|e^c\|_2^2 \leqslant \frac{1}{2}\frac{(e^c(0))^2 + \frac{|k_p|}{\gamma}\theta^T(0)\theta(0)}{|a_m + l|} \qquad (20)$$

Discussion

- Increasing γ reduces $\|e^c\|_{\mathcal{L}_2}$ depending on the parameter errors $\tilde{\theta}$
- Increasing the value of l reduces $\|e^c\|_{\mathcal{L}_2}$ also from $e^c(0)$

5.4 Analysing the signal oscillations

$\mathcal{L}_2$-norm of $\dot{k}$

$$\begin{aligned}
\dot{k} &= -\gamma \operatorname{sgn} k_p e^c r(t) \\
\int_0^\infty |\dot{k}|^2 d\tau &= \gamma^2 \int_0^\infty (e^c)^2 r^2 d\tau \\
&\qquad (r(t) \leqslant \|r\|_{\mathcal{L}_\infty}) \\
&\leqslant \gamma^2 \|r\|_{\mathcal{L}_\infty}^2 \int_0^\infty (e^c)^2 d\tau \\
&\leqslant \gamma^2 \|r\|_{\mathcal{L}_\infty}^2 \|e^c\|_{\mathcal{L}_2}^2
\end{aligned}$$

$$\|\dot{k}\|_2 \leqslant \gamma \|r\|_{\mathcal{L}_\infty} \sqrt{\frac{V(0)}{|a_m + l|}}$$

Increasing γ or reducing l causes $\dot{k}$ to decrease in magnitude.

$\mathcal{L}_2$-norm of $\dot{\theta}$

$$\begin{aligned}
\dot{\theta} &= -\gamma \operatorname{sgn} k_p e^c x_p(t) \\
&= -\gamma \operatorname{sgn} k_p e^c \left(e^c + x_m(t)\right) \\
|\dot{\theta}|^2 &= \gamma^2 (e^c)^2 \left(e^c + x_m(t)\right)^2 \\
&\qquad (a+b)^2 \leqslant 2a^2 + 2b^2 \\
&\leqslant 2\gamma^2 (e^c)^2 \left[(e^c)^2 (e^c)^2 + x_m^2\right] \\
\int_0^\infty |\dot{\theta}|^2 d\tau &\leqslant 2\gamma^2 \left[\int_0^\infty (e^c)^2 (e^c)^2 d\tau + \int_0^\infty (e^c)^2 x_m^2 d\tau\right] \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty |\dot{\theta}|^2 d\tau \\
&\leqslant 2\gamma^2 \frac{V(0)}{|a_m + l|}\left[V(0)\left(2 + \frac{l^2}{|a_m| \cdot |a_m + l|}\right) \right. \\
&\qquad \left. + 2\|\dot{x}_m\|_{\mathcal{L}_\infty}^2\right]
\end{aligned}$$

Discussion

- l reduces contribution of the ORM $\|\dot{x}_m(t)\|_{\mathcal{L}_\infty}$ on $\|\dot{\theta}\|_{\mathcal{L}_2}$
- l has no clear effect on the contributions of $V(0)$.
- γ always increases the oscillations, s. $\|\dot{\theta}\|_{\mathcal{L}_2}$

6 Output feedback adaptive control

Let $M(s)$ be a linear time-invariant, asymptotically stable reference model, with I/O $\{r(.), y_m(.)\}$. r is uniform, bounded, piecewise continuous function of time. The plant G is defined such that

$$G(s) = k_p \frac{Z_p(s)}{R_p(s)}$$

- Z_p, R_p are monic[9] polynomials of orders $n-1$ and n
- $G(s)$ is controllable, observable $\Leftrightarrow$ Z_p, R_p are coprime[10] polynomials

We know

(i) sign of high-frequency gain k_p
(ii) upper bound n on the order of $G(s)$
(iii) relative degree n^* [11]
(iv) zeroes of $Z_p(s)$ lie in $\mathbb{C}^-$
$\Rightarrow$ min. phase[12], no inverse response

Minimal phase systems "In minimal phase systems, we can predict the phase φ given the magnitude $|G|$ $\Leftarrow$ only if G stable with no time delay".

e.g. a plane, or a forklift, are examples of non minimal phase systems (they have different inverse responses depending on where you measure, s. Figure 5)

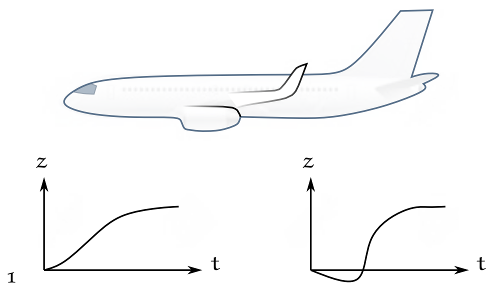

Figure 5: An ascending plane.

Solution in three parts

(i) k_p unknown
(ii) Z_p unknown

[9] monic: coefficient of the highest order is 1, e.g. $s^2 + 2s + 1$
[10] coprime: no cancellations, i.e. no common roots
[11] in this course, we only handle cases where $n^* = 1$
[12] there is no standard definition for minimal phase

(iii) R_p unknown

In each case

a) "Matching conditions"
   Show that $\exists$ parameters $\theta^* = \text{const.}$, such that the closed loop behaves exactly as the reference
b) Error dynamics Derive error model as

$$e = \frac{1}{k^*} M(s) \left[\tilde{\theta}^T \phi\right]$$

c) Use KY-lemma
d) Show that $e \to 0$ for $t \to \infty$ (Barbalat's lemma)

6.1 k_p unkown

Matching conditions