1 Model reference adaptive control

Given plant G and reference model M

$$
\begin{aligned}
G:\ \dot{x}_p(t) &= a_p x_p(t) + k_p u(t), && (1)\\
&\text{IC } x_p(0) \in \mathbb{R}\\
M:\dot{x}_m(t) &= a_m x_m(t) + k_m r(t), && (2)\\
&\text{IC } x_m(0) \in \mathbb{R}
\end{aligned}
$$

a_p - pole of plant
k_p - input gain of plant
a_m - pole of reference model
$r(t)$ - reference signal

The plant parameters are assumed to be known. The reference model parameters are set by the user and are therefore known.

Task find a control $u(t)$ such that $x_p(t) \to x_m(t)$ for $t \to \infty$.

$$
\begin{aligned}
G_{des}:\dot{x}_p(t) &= a_m x_p(t) + k_p r(t) && (3)\\
&\text{IC } x_p(0) \in \mathbb{R}
\end{aligned}
$$

1.1 Solve with model reference control (MRC)

Pick $u(t)$ s.t. dynamical behaviour of closed loop is equal to that of the model.

$$
\begin{aligned}
u^*(t) &= \frac{1}{k_p}\left(-a_p x_p(t) + a_m x_p(t) + k_m r(t)\right)\\
&= \underbrace{\frac{a_m - a_p}{k_p}}_{a^*} x_p(t) + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t)\\
&= \begin{bmatrix} a^* & k^* \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}\\
u^*(t) &= \theta^{*T}(t)\boldsymbol{\phi}(t) && (4)
\end{aligned}
$$

Using this input, now the dynamics of the plant G matches the dynamics of the model M (Equation (**??**)). The starred variables with $*$ superscripts represent the correct values of the known parameters.

However, even though now the dynamics are the same, the initial conditions are not necessarily the same. Would this input also work for $x_m(0) \neq x_p(0)$? I.e., does this guarantee that $x_p(t) \to x_m(t)$ for $t \to \infty$?

To check this, we check the error dynamics and see if the error asymptotically goes to zero.

$$
\begin{aligned}
e(t) &= x_p(t) - x_m(t)\\
\dot{e}(t) &= \dot{x}_p(t) - \dot{x}_m(t)
\end{aligned}
$$

Using eqns. (2) and (3):

$$\dot{e}(t) = a_m e(t) \tag{5}$$

If $a_m < 0$, the error dynamics are stable, that is, $e(t) \to 0$ for any ICs.

Conclusion
- Model reference control (MRC) works with the error dynamics of the reference model
- We need to know all plant parameters very well
  $\Rightarrow$ Problem: uncertainty in parameters

1.2 Solve with model reference adaptive control

Now a_p, k_p unknown, $k_p > 0$

Control law We search for (learn) the value of θ and k, which are therefore functions of time.

$$
\begin{aligned}
u(t) &= \begin{bmatrix} a(t) & k(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}\\
&= \theta^T(t)\boldsymbol{\phi}(t)
\end{aligned}
$$

Adaptive law Adapt the control parameters in the following fashion.

$$
\begin{aligned}
&\begin{bmatrix} \dot{a}(t) \\ \dot{k}(t) \end{bmatrix} = -\operatorname{sign} k_p e(t) \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}\\
&\Rightarrow\ \dot{\theta} = -\operatorname{sign} k_p e(t) \Gamma \boldsymbol{\phi}(t) && (6)
\end{aligned}
$$

The equations in (6) are nonlinear ODEs.

Questions
- Is the closed loop stable?
- Does with this, $e(t) \to 0$?
- Are the parameters $\theta(t), k(t)$ finite?
- Are the parameters $\theta(t), k(t)$ constant for $t \to \infty$?
- Do the parameters $\theta(t), k(t)$ approach their 'correct' values θ^*, k^* for $t \to \infty$?

1.3 Generalisation

$$
\begin{aligned}
G:\ \dot{x}_p(t) &= a_p x_p(t) + k_p u(t) + \alpha_p f(z) && (7)\\
M:\dot{x}_m(t) &= a_m x_m(t) + k_m r(t) + \cancel{\alpha_m f(z)} && (2)
\end{aligned}
$$

$f(z)$ - Nonlinear function, (external)

- a_p, k_p, α_p are unknown but constant
- $\operatorname{sign}(k_p)$, $f(.)$ are known (z is a known signal)
- $\alpha_m f(z)$ is not necessary

We define the following control structure

$$
\begin{aligned}
u^*(t) &= \frac{1}{k_p}\left(-a_p x_p(t) + a_m x_p(t)\right. \\
&\qquad \left. + k_m r(t) - \alpha_p f(z)\right) \\
&= \underbrace{\frac{(a_m - a_p)}{k_p}}_{a^*} x_p + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t) + \underbrace{\frac{-\alpha_p}{k_p}}_{\alpha^*} f(z) \\
&= \begin{bmatrix} a^* & k^* & \alpha^* \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\
u^*(t) &= \theta^{*T}(t)\boldsymbol{\phi}(t) \qquad (8)
\end{aligned}
$$

Choose

$$
\begin{aligned}
u(t) &= a(t)x_p(t) + k(t)r(t) + \alpha(t)f(z) \\
&= \begin{bmatrix} a(t) & k(t) & \alpha(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\
u(t) &= \theta^T(t)\boldsymbol{\phi}(t) \qquad (9)
\end{aligned}
$$

$a(t)$, $k(t)$, $\alpha(t)$ unknown.

Goal $x_p(t) \to x_m(t)$ for $t \to \infty$.

Error dynamics In AC, the current estimated parameters are varying. We therefore have the following error in parameters as deviations from the unknown but exact and constant real parameters:

$$
\left.\begin{aligned}
\tilde{a}(t) &= a(t) - a^* \\
\tilde{k}(t) &= k(t) - k^* \\
\tilde{\alpha}(t) &= \alpha(t) - \alpha^*
\end{aligned}\right\} \tilde{\theta}(t) = \theta(t) - \theta^*
$$

$$
\begin{aligned}
\dot{e}(t) &= a_p x_p(t) \\
&\quad + k_p\left(a(t)x_p(t) + k(t)r(t) + \alpha(t)f(z)\right) \\
&\quad + \alpha_p f(z) \\
&\quad - \left(a_m x_m(t) + k_m r(t)\right) \\
&= a_p x_p(t) - a_m x_m(t) \\
&\quad + k_p a(t) x_p(t) \\
&\quad + k_p \underbrace{\left(k(t) - \frac{k_m}{k_p}\right)}_{\tilde{k}(t)} r(t) \\
&\quad + k_p \underbrace{\left(\alpha(t) - \frac{\alpha_p}{k_p}\right)}_{\tilde{\alpha}(t)} f(z) \\
&= \underbrace{(a_m - k_p a^*)}_{a_p} x_p(t) - a_m x_m(t) + k_p a(t) x_p(t) \\
&\quad + k_p \tilde{k}(t) r(t) + k_p \tilde{\alpha}(t) f(z) \\
\dot{e}(t) &= a_m e(t) + k_p \tilde{a}(t) x_p(t) + k_p \tilde{k}(t) r(t) + k_p \tilde{\alpha}(t) f(z)
\end{aligned}
$$

var - known/measured
var - unknown

Note All unknown parameters that appear linearly (affine, "linear in the parameters") can be collected in a vector $\theta(t)$. Same with the measurable functions of time $\phi(t)$ (regressor).

$$
\begin{aligned}
\dot{e}(t) &= a_m e(t) + k_p \begin{bmatrix} \tilde{a}(t) & \tilde{k}(t) & \tilde{\alpha}(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\
\dot{e}(t) &= a_m e(t) + \frac{1}{k^*} k_m \theta^T(t) \boldsymbol{\phi}(t) \qquad (10) \\
e(t) &= \frac{1}{k^*} M(s) \theta^T(t) \boldsymbol{\phi}(t) \\
&\text{Operator: } M(s) = \frac{k_m}{s - a_m}
\end{aligned}
$$

The above is a nonlinear differential equation. When is it stable? $\to$ Lyapunov

1.4 Lyapunov-like functions

New interpretation of Lyapunov Nothing to do with energy. $\mathbf{V}$ affects the scaling of the distance of $\mathbf{x}$ from the origin in the phase portrait: $\|\mathbf{x}\|_{\mathbf{V}}^2 = \mathbf{x}^T \mathbf{V} \mathbf{x}$, $\mathbf{V} \succ 0$.
$\Rightarrow$ all Lyapunov says: how far is $\mathbf{x}$ from the origin? We want to find some type of measure for that.

Lyapunov function (Lyapunov-like)
$\boldsymbol{\Gamma} \succ 0$ symmetrical, positive definite.

$$
\begin{aligned}
V(e, \tilde{\theta}) &= \frac{1}{2} e^2 + \frac{1}{2} |k_p| \left(\tilde{\theta}^T \boldsymbol{\Gamma}^{-1} \tilde{\theta}\right) \qquad (11) \\
\dot{V} &= e\dot{e} + \frac{1}{2} |k_p| \left(2\tilde{\theta}^T \boldsymbol{\Gamma}^{-1} \dot{\tilde{\theta}}\right)
\end{aligned}
$$

substitute $\dot{e}$ using equation (10)

$$
\begin{aligned}
&= a_m e^2 + e k_p \tilde{\theta}^T \boldsymbol{\phi} + \frac{1}{2} |k_p| \left(2\theta^T \tilde{\Gamma}^{-1} \dot{\tilde{\theta}}\right) \\
&= a_m e^2 + |k_p| \tilde{\theta}^T \underbrace{\left(\operatorname{sign} k_p e \phi + \boldsymbol{\Gamma}^{-1} \dot{\tilde{\theta}}\right)}_{\overset{!}{=} 0}
\end{aligned}
$$

set second term = 0, $\because$ we want $V \preceq 0$ and we don't know all the signs of the terms

$$
\dot{V} = a_m e^2 \preceq 0 \qquad (12)
$$

Adaptive law

$$
\begin{aligned}
\dot{\tilde{\theta}}(t) &= -\boldsymbol{\Gamma} \operatorname{sign} k_p \boldsymbol{\phi}(t) e(t) \\
\dot{\theta}(t) &= -\boldsymbol{\Gamma} \operatorname{sign} k_p \boldsymbol{\phi}(t) e(t)
\end{aligned}
$$

Remark $e(t)$ does not have to be 0 – why?
A function that is bounded from below and non-increasing (V) has a limit as $t \to \infty$.

If the derivative of a function $\to 0$, that does not imply that the function has a limit, and vice versa: if a function has a limit, that doesn't mean its derivative $\to 0$.

$$\lim_{t\to\infty} \dot{f}(t) = 0 \not\Leftrightarrow \lim_{t\to\infty} f(t) = k$$

Counterexamples:

- $f(t) = \sin(\ln t)$
  $\nexists \lim_{t\to\infty} f(t)$
  $\dot{f}(t) = \frac{\cos(\ln t)}{t} \to 0$
- $f(t) = e^{-t}\sin(e^{2t})$
  $\lim_{t\to\infty} f(t) = 0$
  $\dot{f}(t) = -e^{-t}\sin(e^{2t}) + e^{t}\sin(e^{2t}) \to$ explodes!

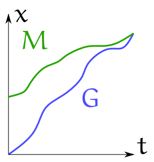
Are the error dynamics stable?

- Measure (some of) the states
- using $V = f(e, \tilde{\theta})$
- $V \to \infty$? Or $V \downarrow$?
  $\Rightarrow$ analyse time derivative $\dot{V}$
- If we show $\dot{V} \to 0$, then $e \to 0$.

Extensions to Lyapunov There are two well-known extensions to Lyapunov to prove asymptotic stability, even if $\dot{V} \preceq 0$.

(i) LaSalle's invariance principle (only for autonomous systems)
(ii) Barbalat's lemma (OK for non-autonomous systems)

Our system's error dynamics are non-autonomous, $\dot{e} = f(t, \dots)$, $\because$ following another system.



Uniformly continuous function A function $f(t) : \mathbb{R} \to \mathbb{R}$ is uniformly continous, if
$\forall \varepsilon > 0 : \exists \delta = \delta(\varepsilon) > 0$,

$$\forall |t_2 - t_1| \leqslant \delta$$
$$\Rightarrow |f(t_2) - f(t_1)| \leqslant \varepsilon$$

Sufficient condition: If $\exists \dot{f}(t)$ (i.e. bounded), $\Rightarrow$ $f(t)$ is uniformly constant.

1.5 Barbalat's lemma

Variant A
If $f(t) : \mathbb{R} \to \mathbb{R}$

(i) is a differentiable function
(ii) has a finite limit as $t \to \infty$
(iii) $\dot{f}(t)$ is uniformly constant

$\Rightarrow \lim_{t\to\infty} \dot{f}(t) = 0$

Variant B
If

(i) $f(t) : \mathbb{R} \to \mathbb{R}$ is uniformly continous $\forall t$
(ii) $\exists \lim_{t\to\infty} \int_]^t f(\tau)d\tau$

$\Rightarrow \lim_{t\to\infty} f(t) = 0$

Variant C
If $f, \dot{f} \in \mathcal{L}_\infty$ and $f \in \mathcal{L}_2$, then $|f(t)| \to 0$ as $t \to \infty$.

Closed loop stability analysis By definition, $V \succeq 0$ and $\dot{V} \preceq 0$.

Boundedness of $e(t)$

- As V is bounded from below and non-increasing, V has a limit as $t \to \infty$.
- Tracking error $e(t)$ and parameter errors $\tilde{\theta}(t)$ are bounded.
- As $\tilde{\theta}(t)$ bounded and θ^* constant, $\theta(t)$ is bounded.

Boundedness of $\dot{e}(t)$

- Assume $r(t)$ bounded, then $x_m(t), \dot{x}_m(t)$ bounded ($\because$ M is stable)
- $x_p(t) = \underset{\text{bd.}}{e(t)} + \underset{\text{bd.}}{x_m(t)}$
  $\Rightarrow x_p(t)$ bounded.
- $u(t)$ bounded if $\phi(t)$ bounded.
  $\phi(t) = \begin{bmatrix} \underset{\text{bd.}}{r(t)} & \underset{\text{bd.}}{x_p(t)} & \underset{\text{bd.}}{f(z)} \end{bmatrix}$
  (new requirement: $f(z)$ needs to be bounded).
- $\dot{x}_p(t) = a_p \underset{\text{bd.}}{x_p(t)} + k_p \underset{\text{bd.}}{u(t)}$
  $\Rightarrow \dot{x}_p(t)$ bounded
- $\dot{e}(t) = \underset{\text{bd.}}{\dot{x}_p(t)} - \underset{\text{bd.}}{\dot{x}_m(t)}$ is bounded.

With $\ddot{V} = 2a_m \underset{\text{bd.}}{e}\,\underset{\text{bd.}}{\dot{e}}$, $\ddot{V}$ bounded $\to \dot{V}$ uniformly constant.

Using Barbalat's lemma Variant A,

- V is differentiable
- V has a finite limit as $t \to \infty$
  (bounded from below and non-increasing)
- $\dot{V}$ is uniformly constant

$$\Rightarrow \lim_{t\to\infty} \dot{V} = 0$$
$$\Rightarrow \lim_{t\to\infty} e(t) = 0$$

Signal norm, $\mathcal{L}_p$ space "how big is a signal?"

- **Idea**: quantify magnitude of a signal $x(t)$
  $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$, $\mathbb{R}^+ = [0, \infty)$
- p-Norm

$$\|x_p\| = \left(\int_0^\infty |x(t)|^p \, dt\right)^{1/p} \qquad p \in (0, \infty)$$

- If $\mathbf{x}(t)$ vector, $|.|$ is the vector 2-norm, 'distance'.
- $\|x\|_\infty = \sup_{t\in\mathbb{R}^+} |x(t)| \triangleq$ highest value of $\mathbf{x}$
  "When power ∞, only the greatest value survives"