1 Linear SISO plant

Control problem

Given plant G and reference model M.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

a_p - pole of plant
k_p - input gain of plant

$$M : \dot{x}_m(t) = a_m x_m(t) + k_m r(t), \quad (2)$$
$$\text{IC } x_m(0) \in \mathbb{R}$$

a_m - pole of reference model
k_m - input gain of reference model
$r(t)$ - reference signal

The reference model parameters are set by the user and are therefore known.

Task find a control $u(t)$ such that $x_p(t) \to x_m(t)$ for $t \to \infty$.

$$G_{des} : \dot{x}_p(t) = a_m x_p(t) + k_m r(t) \quad (3)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Solutions for $u(t)$ using
- Model reference control (MRC)
- Model reference adaptive control (MRAC)

1.1 Model reference control (MRC)

The plant parameters are assumed to be known.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Pick $u(t)$ such that the dynamical behaviour of the closed loop is equal to that of the model. This is done by comparing (1) to (3).

$$\begin{aligned} u^*(t) &= \frac{1}{k_p}\left(-a_p x_p(t) + a_m x_p(t) + k_m r(t)\right) \\ &= \underbrace{\frac{a_m - a_p}{k_p}}_{a^*} x_p(t) + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t) \\ &= \begin{bmatrix} a^* & k^* \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix} \\ u^*(t) &= \theta^{*T} \boldsymbol{\phi}(t) \end{aligned}$$

Using this input[1], now the dynamics of the plant G matches the dynamics of the model M, as in equation (3).

However, even though now the dynamics are the same, the initial conditions are not necessarily the same. Would this input also work for $x_m(0) \neq x_p(0)$? I.e., does this guarantee that $x_p(t) \to x_m(t)$ for $t \to \infty$?

Dependence on the initial conditions To check this, we examine the error dynamics and see if the error asymptotically goes to zero.

$$e(t) = x_p(t) - x_m(t)$$
$$\dot{e}(t) = \dot{x}_p(t) - \dot{x}_m(t)$$

Using eqns. (2) and (3):

$$\dot{e}(t) = a_m e(t) \quad (4)$$

If $a_m < 0$, the error dynamics are stable. That is, $e(t) \to 0$ for any ICs.

Conclusion
- Model reference control (MRC) works with the error dynamics of the reference model
- We need to know all plant parameters very well

  $\Rightarrow$ Problem: uncertainty in parameters

1.2 Model reference adaptive control (MRAC)

The plant parameters are unknown. We assume $k_p > 0$.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Control law We search for (learn) the value of θ and k, which are therefore functions of time.

$$\begin{aligned} u(t) &= \begin{bmatrix} a(t) & k(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix} \\ &= \theta^T(t) \boldsymbol{\phi}(t) \end{aligned} \quad (5)$$

Adaptive law Adapt the control parameters in the following fashion.

$$\begin{bmatrix} \dot{a}(t) \\ \dot{k}(t) \end{bmatrix} = -\operatorname{sgn}(k_p)\, e(t) \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}$$
$$\Rightarrow \dot{\theta} = -\operatorname{sgn}(k_p)\, e(t) \boldsymbol{\Gamma} \boldsymbol{\phi}(t) \quad (6)$$

The equations in (6) are nonlinear ODEs.

[1] The starred variables with $*$ superscripts represent the ideal values of the control parameters.

Questions

- Is the closed loop stable?
- Does, with this, $e(t) \to 0$?
- Are the parameters $\theta(t)$ finite?
- Are the parameters $\theta(t)$ constant for $t \to \infty$?
- Do the parameters $\theta(t)$ approach their 'ideal' values θ^* for $t \to \infty$?

2 Nonlinear SISO plant

The linear SISO plant in the previous chapter is generalised to a nonlinear SISO plant.

Control problem

Given plant G and reference model M.

$$\begin{aligned} G:\ \dot{x}_p(t) &= a_p x_p(t) + k_p u(t) \\ &\quad + \alpha_p f(z) && (7) \\ M:\dot{x}_m(t) &= a_m x_m(t) + k_m r(t) && (2) \end{aligned}$$

- a_p, k_p, α_p are unknown but constant
- $f(z)$ is a nonlinear (external) function
- $\operatorname{sgn}(k_p)$, $f(.)$ are known (z is a known signal)
- $\alpha_m f(z)$ is not necessary

Goal $x_p(t) \to x_m(t)$ for $t \to \infty$.

2.1 Control structures

Ideal control structure based on MRC

$$\begin{aligned} u^*(t) &= \frac{1}{k_p}\left(-a_p x_p(t) + a_m x_p(t)\right. \\ &\qquad \left. + k_m r(t) - \alpha_p f(z)\right) \\ &= \underbrace{\frac{(a_m - a_p)}{k_p}}_{a^*} x_p + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t) + \underbrace{\frac{-\alpha_p}{k_p}}_{\alpha^*} f(z) \\ &= \begin{bmatrix} a^* & k^* & \alpha^* \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\ u^*(t) &= \theta^{*T} \phi(t) && (8) \end{aligned}$$

Control law using MRAC

$$\begin{aligned} u(t) &= \begin{bmatrix} a(t) & k(t) & \alpha(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\ u(t) &= \theta^T(t)\phi(t) && (9) \end{aligned}$$

$a(t)$, $k(t)$, $\alpha(t)$ unknown.

2.2 Error dynamics

In adaptive control, the current estimated parameters are varying. We therefore have the following error in parameters as deviations from the unknown but ideal and constant real parameters:

$$\left.\begin{aligned} \tilde{a}(t) &= a(t) - a^* \\ \tilde{k}(t) &= k(t) - k^* \\ \tilde{\alpha}(t) &= \alpha(t) - \alpha^* \end{aligned}\right\} \tilde{\theta}(t) = \theta(t) - \theta^*$$

$$\begin{aligned} \dot{e}(t) &= \dot{x}_p(t) - \dot{x}_m(t) \\ &= a_p x_p(t) \\ &\quad + k_p\left(a(t)x_p(t) + k(t)r(t) + \alpha(t)f(z)\right) \\ &\quad + \alpha_p f(z) \\ &\quad - \left(a_m x_m(t) + k_m r(t)\right) \\ &= a_p x_p(t) - a_m x_m(t) \\ &\quad + k_p a(t) x_p(t) \\ &\quad + k_p \underbrace{\left(k(t) - \frac{k_m}{k_p}\right)}_{\tilde{k}(t)} r(t) \\ &\quad + k_p \underbrace{\left(\alpha(t) - \frac{\alpha_p}{k_p}\right)}_{\tilde{\alpha}(t)} f(z) \\ &= \underbrace{(a_m - k_p a^*)}_{a_p} x_p(t) - a_m x_m(t) \\ &\quad + k_p a(t) x_p(t) \\ &\quad + k_p \tilde{k}(t) r(t) + k_p \tilde{\alpha}(t) f(z) \\ &= a_m e(t) + k_p \tilde{a}(t) x_p(t) \\ &\quad + k_p \tilde{k}(t) r(t) + k_p \tilde{\alpha}(t) f(z) \end{aligned}$$

$$\begin{aligned} \dot{e}(t) &= a_m e(t) + k_p \begin{bmatrix} \tilde{a}(t) & \tilde{k(t)} & \tilde{\alpha}(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\ \dot{e}(t) &= a_m e(t) + \frac{1}{k^*} k_m \tilde{\theta}^T(t) \phi(t) && (10) \end{aligned}$$

The error dynamics can be rewritten using an operator $M(s) = \dfrac{k_m}{s - a_m}$, which is non other than the transfer function of the reference model!

$$\begin{aligned} \dot{e}(t) &= a_m e(t) + \frac{1}{k^*} k_m \theta^T(t) \phi(t) \\ (s - a_m)\, e(t) &= \frac{1}{k^*} k_m \theta^T(t) \phi(t) \\ e(t) &= \frac{1}{k^*} M(s) \theta^T(t) \phi(t) && (10) \end{aligned}$$

All unknown parameters appear linearly (affine, 'linear in the parameters'). The error dynamics (10) is a nonlinear differential equation. When is it stable? → Lyapunov.

2.3 Lyapunov-like function

New interpretation of Lyapunov Nothing to do with energy. $\boldsymbol{V}$ affects the scaling of the distance of $\boldsymbol{x}$ from the origin in the phase portrait.

$$\|\boldsymbol{x}\|_{\boldsymbol{V}}^2 = \boldsymbol{x}^T \boldsymbol{V} \boldsymbol{x}, \qquad \boldsymbol{V} \succ 0$$

> **All Lyapunov says is**: how far is x from the origin? We want to find some type of measure for that.

Lyapunov function (Lyapunov-like)
We want the output error $e(t)$ as well as the parameter error $\tilde{\theta}(t)$ to go to zero. $\Gamma \succ 0$ symmetrical, positive definite.

$$V(e,\tilde{\theta}) = \frac{1}{2}e^2 + \frac{1}{2}|k_p|\left(\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}\right) \tag{11}$$

$$\dot{V} = e\dot{e} + \frac{1}{2}|k_p|\underbrace{\left(2\tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}}\right)}_{2}$$

Substitute $\dot{e}$ using equation (10).

$$\begin{aligned}\dot{V} &= a_m e^2 + ek_p\tilde{\theta}^T\phi + \frac{1}{2}|k_p|\left(2\theta^T\tilde{\Gamma}^{-1}\dot{\tilde{\theta}}\right)\\ &= a_m e^2 + |k_p|\tilde{\theta}^T\underbrace{\left(\operatorname{sgn}(k_p)\,e\phi + \Gamma^{-1}\dot{\tilde{\theta}}\right)}_{\overset{!}{=}0}\end{aligned}$$

The second term is set to 0, because we want $V \preceq 0$ and we don't know all the signs of the terms. This will define the **adaptive law**.

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -\operatorname{sgn}(k_p)\,\Gamma\phi(t)e(t)\\ \dot{\theta}(t) &= -\operatorname{sgn}(k_p)\,\Gamma\phi(t)e(t)\end{aligned}$$

With the adaptive law, we obtain for $\dot{V}$:

$$\dot{V} = a_m e^2 \preceq 0 \tag{12}$$

Remark $e(t)$ does not have to be 0 – why?

> If the derivative of a function $\to 0$, that does not imply that the function has a limit, and vice versa: if a function has a limit, that doesn't mean its derivative $\to 0$.
>
> $$\lim_{t\to\infty}\dot{f}(t) = 0 \not\Leftrightarrow \lim_{t\to\infty} f(t) = k$$

Counterexamples:

$$f(t) = \sin(\ln t)$$

$$\nexists \lim_{t\to\infty} f(t), \quad \dot{f}(t) = \frac{\cos(\ln t)}{t} \to 0$$

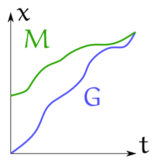
$$\begin{aligned}f(t) &= e^{-t}\sin(e^{2t})\\ \lim_{t\to\infty} f(t) &= 0\\ \dot{f}(t) &= -e^{-t}\sin(e^{2t}) + e^{t}\sin(e^{2t})\\ &\to \text{explodes!}\end{aligned}$$

[2] Possible due to Γ symmetrical.

Are the error dynamics stable?
- Measure (some of) the states
- Apply $V = f(e,\tilde{\theta})$
- $V \to \infty$? Or $V \downarrow$?
  $\Rightarrow$ analyse time derivative $\dot{V}$
- If we show $\dot{V} \to 0$, then $e \to 0$.

Extensions to Lyapunov There are two well-known extensions to Lyapunov to prove asymptotic stability, even if $\dot{V} \preceq 0$.
(i) LaSalle's invariance principle (only for autonomous systems)
(ii) Barbalat's lemma (OK for non-autonomous systems)

Our system's error dynamics are non-autonomous, $\dot{e} = f(t,\dots)$, due to following another system.



2.4 Closed loop stability analysis

Using Barbalat's Lemma (Variant A)[3] on the function V, we need to fulfill the following conditions:
(i) V is differentiable
Yes, $\exists\ \dot{V} = a_m e^2(t)$
(ii) V has a finite limit as $t \to \infty$
Yes, as $V \succeq 0$ and $\dot{V} \preceq 0$
(iii) $\dot{V}$ is uniformly continuous $\Leftarrow \exists\ \ddot{V}$ (sufficient condition[4])
Is $\ddot{V} = 2a_m e\dot{e}$ bounded?

Boundedness of $e(t)$
- As V is bounded from below and non-increasing, V has a limit as $t \to \infty$.
- Tracking error $e(t)$ and parameter errors $\tilde{\theta}(t)$ are bounded.
- As $\tilde{\theta}(t)$ bounded and θ^* constant, $\theta(t)$ is bounded.

Boundedness of $\dot{e}(t)$
- Assume $r(t)$ bounded[5], then, from the reference model equation (2):
  $x_m(t), \dot{x}_m(t)$ bounded ($\because$ M is stable)
- $x_p(t) = \underset{\text{bd.}}{e(t)} + \underset{\text{bd.}}{x_m(t)}$
  $\Rightarrow x_p(t)$ bounded.
- $u(t) = \theta^T(t)\phi(t)$ bounded if $\phi(t)$ bounded.
  $\phi(t) = \begin{bmatrix}\underset{\text{bd.}}{r(t)} & \underset{\text{bd.}}{x_p(t)} & \underset{\text{bd.}}{f(z)}\end{bmatrix}^T$

[3] Chapter 4.2 on page 6
[4] Chapter 4.1 on page 5
[5] Reasonable assumption, because why would we want to use an unbounded input?

(new requirement: $f(z)$ needs to be bounded).

- $\dot{x}_p(t) = \underbrace{a_p x_p(t)}_{\text{bd.}} + \underbrace{k_p u(t)}_{\text{bd.}}$
  $\Rightarrow \dot{x}_p(t)$ bounded
- $\dot{e}(t) = \underbrace{\dot{x}_p(t)}_{\text{bd.}} - \underbrace{\dot{x}_m(t)}_{\text{bd.}}$ is bounded.

All the conditions of Barbalat's lemma thus fulfilled, we can conclude that the derivative of V approaches zero for $t \to \infty$.

$$\lim_{t\to\infty} \dot{V} = 0$$
$$\Rightarrow \lim_{t\to\infty} e(t) = 0$$

2.5 Nonlinearities

Theorem: Nonlinear SISO plant

Given plant G and reference model M.

$$G: \ \dot{x}_p(t) = a_p x_p(t) + k_p u(t) + \alpha_p f(z) \qquad (7)$$
$$M: \dot{x}_m(t) = a_m x_m(t) + k_m r(t) \qquad (2)$$

The input

$$u(t) = \theta^T(t)\phi(t) \qquad (9)$$
$$\text{with } \dot{\theta} = -\operatorname{sgn}(k_p)\,\Gamma\phi e \qquad (6)$$
$$\text{and } \phi(t) = \begin{bmatrix} r(t) & x_p(t) & f(z) \end{bmatrix}^T$$

renders the origin asymptotically stable and guarantees $x_p(t) \to x_m(t)$ as $t \to \infty$.

We can add, arbitrarily, many 'nonlinearities' $f_i(z_j)$ with unknown gains α_i. The nonlinearity functions need not be continuous. The only requirement:

$$f_i(z_j) \in \mathcal{L}_\infty$$

Nonlinearities are bounded at all times[a].

[a]For the boundedness of $u(t)$ and therefore $\dot{e}(t)$

The function $z(t)$ z is a placeholder. $z(t)$ can be an external or an internal signal.

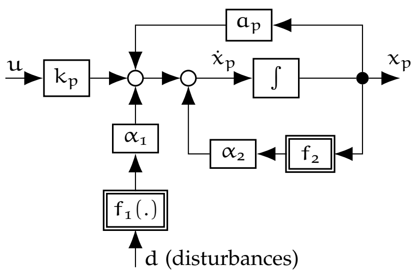


Figure 1: Plant G with nonlinearities

Questions

- Can we do $f(u)$?
  Possible, but solving for $u = \cdots f(u) \cdots$ is difficult.
- Can $f(.)$ be a differential operator (filter)?
  Yes. If the filter is linear, then a solution definitely exists.

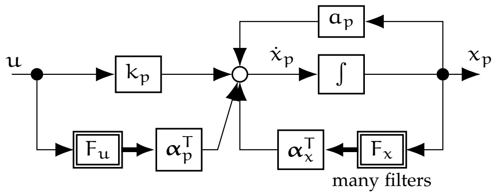


Figure 2: Plant G with nonlinearities as filters

The filters F_u and F_x are stable dynamical systems. If they are not stable, there is a higher chance of getting an infinite output $\to$ conflicts with bounded output requirement.

If no such filters are present in the plant G, then the plant has an order of 1.

Let's say we have an offset in the plant input of unknown magnitude, and the plant has otherwise known parameters.

$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t) + \alpha_p \cdot 1$$
$$u(t) = a^* x_p(t) + k^* r(t) - \frac{\alpha(t)}{k_p}$$

Closed loop becomes

$$\dot{x}_p(t) = a_m x_p(t) + k_m r(t) + \tilde{\alpha}(t)$$

Error dynamics

$$\dot{e} = a_m e(t) + \tilde{\alpha}(t)$$

Lyapunov-like function

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{\alpha}^2$$
$$\dot{V} = e\dot{e} + \tilde{\alpha}\dot{\tilde{\alpha}} + \cdots$$
$$= a_m e^2 + \tilde{\alpha}\underbrace{\left(e + \dot{\tilde{\alpha}}\right)}_{\text{to set to 0}}$$

Setting the second term to zero ensures the negative semidefiniteness of $\dot{V}$.

$$\dot{\tilde{\alpha}} = -e \qquad (13)$$

Equation (13) above implies $u = \cdots + \int e dt + \cdots$, i.e., that the controller contains an I-part. As a result of this, there are no steady state errors caused by model uncertainties. The controller eliminates offset at input of plant.

Equation (13) is a pure integrator acting on control error. This is a linear controller! We have learned: integrators 'learn' input offsets of the plant and correct them.

Adaptive controllers can be interpreted as nonlinear PI controllers.

3 Positive real functions

Def: Positive real function 1

A rational function $H(s) : \mathbb{Z} \to \mathbb{Z}$, $s = \sigma + j\omega$ is positive real (PR), if

(i) $H(s)$ is real for real s

(ii) $\Re\{H(s)\} \geqslant 0$ for $\Re\{s\} > 0$

Def: Positive real function 2

A rational function $H(s)$ is positive real, if

(i) $H(s)$ is analytic in $\Re\{s\} > 0$
- $H(s)$ has no poles in RHP ($\Re\{s\} > 0$)
- $H(s)$ is stable

(ii) $\Re\{H(j\omega)\} \geqslant 0, \quad \forall\omega \in [0, \infty]$
- Nyquist of $H(s)$ is in the RHP
- phase $\angle H(j\omega) \in [-90^\circ, +90^\circ]$
- rel. degree of $H(s)$ is 0 or 1

(iii) any pure imaginary pole $j\omega$ of $H(s)$ is a simple pole, and the residue

$$\lim_{s \to j\omega} (s - j\omega)\, H(s)$$

is positive semidefinite.

Alternatively: $H(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^2 \Re\{H(j\omega)\} \geqslant 0$$

Def: Strictly positive real functions

$H(s)$ is strictly positive real (SPR) if $H(s - \varepsilon)$ is PR for some $\varepsilon > 0$.

Def: SPR lemma

$H(s)$ is SPR if

(i) $H(s)$ is Hurwitz
all poles on LHP, none are purely imaginary

(ii) $\Re\{H(j\omega)\} > 0, \quad \forall\omega \in \mathbb{R}$
phase $\in (-90^\circ, 90^\circ)$
rel. degree $\in \{0, 1\}$

(iii) $H(\infty) > 0$ (positive gain for proper H) or

$$\lim_{\omega \to \infty} \omega^2 \Re\{H(j\omega)\} > 0$$

positive gain for relative degree 1

Note relative degree of a system corresponds to its response delay.

Discussion If $H(s)$ is SPR, then so is the inverse $H^{-1}(s)$ (stable poles, stable zeroes).

SPR $\to$ $H(s)$ is stable, minimal-phase. I.e., only stable zerioes, because zeroes are in LHP.

$$\text{rel. degr.} \leqslant 1 \qquad (-1, 0, 1)$$
$$\angle H(j\omega) \in (-90^\circ, 90^\circ)$$
$$\text{with positive gain } \forall\omega$$

4 Appendix

4.1 Uniformly continuous functions

Def: Uniformly continuous function

A function $f(t) : \mathbb{R} \to \mathbb{R}$ is uniformly contінous, if
$\forall\varepsilon > 0 : \quad \exists\, \delta = \delta(\varepsilon) > 0,$

$$\forall |t_2 - t_1| \leqslant \delta$$
$$\Rightarrow |f(t_2) - f(t_1)| \leqslant \varepsilon$$

Sufficient condition for uniformly continuous functions: If the derivative $\dot{f}(t)$ exists (i.e. bounded), $\Rightarrow$ $f(t)$ is uniformly constant.

4.2 Barbalat's lemma

Lemma: Barbalat Variant A

If $f(t) : \mathbb{R} \to \mathbb{R}$

(i) is a differentiable function

(ii) has a finite limit as $t \to \infty$

(iii) $\dot{f}(t)$ is uniformly continuous

$\Rightarrow \lim_{t\to\infty} \dot{f}(t) = 0$

Lemma: Barbalat Variant B

If

(i) $f(t) : \mathbb{R} \to \mathbb{R}$ is uniformly continuous $\forall t$

(ii) $\exists \lim_{t\to\infty} \int_0^t f(\tau)d\tau$

$\Rightarrow \lim_{t\to\infty} f(t) = 0$

Lemma: Barbalat Variant C

If $f, \dot{f} \in \mathcal{L}_\infty$ and $f \in \mathcal{L}_2$, then $|f(t)| \to 0$ as $t \to \infty$.

Def: Signal norm

Idea quantify magnitude of a signal $x(t)$ – "How big is a signal?" – also called the $\mathcal{L}_p$ space.

Given

$$x(t) : \mathbb{R}^+ \to \mathbb{R}^n, \qquad \mathbb{R}^+ = [0, \infty)$$

p-Norm

$$\|x_p\| = \left(\int_0^\infty |x(t)|^p \, dt\right)^{1/p} \qquad p \in (0, \infty)$$

- If $\mathbf{x}(t)$ vector, $|.|$ is the vector 2-norm, '**distance**'.
- $\|x\|_\infty = \sup_{t\in\mathbb{R}^+} |x(t)| \triangleq$ highest value of $\mathbf{x}$
"When power ∞, only the greatest value survives"

Def: Functional space

$$\mathcal{L}_p = \{x(t) \in \mathbb{R}^n : \underbrace{\|x\|_p < \infty}_{\text{exists}}\}$$

Ex. 4.1 Functional space

- $x(t) \in \mathcal{L}_p$: x is bounded.
"x's highest value exists and is not infinity."
- Show that $e \in \mathcal{L}_\infty$
e is in V and V is bounded, $\therefore e \in \mathcal{L}_\infty$.
- Show that $e \in \mathcal{L}_2$

$$\int_0^\infty \dot{V} dt = V(\infty) - V(0), \qquad \text{is bounded.}$$

$$a_m \int_0^\infty e^2 dt \in \mathcal{L}_\infty$$

$$\Rightarrow e \in \mathcal{L}_2$$