1 Linear SISO plant

Control problem

Given plant G and reference model M.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \qquad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

a_p - pole of plant
k_p - input gain of plant

$$M : \dot{x}_m(t) = a_m x_m(t) + k_m r(t), \qquad (2)$$
$$\text{IC } x_m(0) \in \mathbb{R}$$

a_m - pole of reference model
k_m - input gain of reference model
$r(t)$ - reference signal

The **reference model parameters** are set by the user and are therefore known.

Task find a control $u(t)$ such that $x_p(t) \to x_m(t)$ for $t \to \infty$.

$$G_{des} : \dot{x}_p(t) = a_m x_p(t) + k_m r(t) \qquad (3)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Solutions for $u(t)$ using

- Model reference control (MRC)
- Model reference adaptive control (MRAC)

1.1 Model reference control (MRC)

The **plant parameters** are assumed to be known.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \qquad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Pick $u(t)$ such that the dynamical behaviour of the closed loop is equal to that of the model. This is done by comparing (1) to (3).

$$u^*(t) = \frac{1}{k_p}\left(-a_p x_p(t) + a_m x_p(t) + k_m r(t)\right)$$
$$= \underbrace{\frac{a_m - a_p}{k_p}}_{a^*} x_p(t) + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t)$$
$$= \begin{bmatrix} a^* & k^* \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}$$
$$u^*(t) = \theta^{*T} \phi(t)$$

Using this input[1], now the dynamics of the plant G matches the dynamics of the model M, as in equation (3).

However, even though now the dynamics are the same, the initial conditions are not necessarily the same. Would this input also work for $x_m(0) \neq x_p(0)$? I.e., does this guarantee that $x_p(t) \to x_m(t)$ for $t \to \infty$?

Dependence on the initial conditions To check this, we examine the error dynamics and see if the error asymptotically goes to zero.

$$e(t) = x_p(t) - x_m(t)$$
$$\dot{e}(t) = \dot{x}_p(t) - \dot{x}_m(t)$$

Using eqns. (2) and (3):

$$\dot{e}(t) = a_m e(t) \qquad (4)$$

If $a_m < 0$, the error dynamics are stable. That is, $e(t) \to 0$ for any ICs.

Conclusion

- Model reference control (MRC) works with the error dynamics of the reference model
- **We need to know all plant parameters very well**
  $\Rightarrow$ Problem: **uncertainty in parameters**

1.2 Model reference adaptive control (MRAC)

The **plant parameters** are unknown. We assume $k_p > 0$.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \qquad (1)$$
$$\text{IC } x_p(0) \in \mathbb{R}$$

Control law We search for (learn) the value of θ and k, which are therefore functions of time.

$$u(t) = \begin{bmatrix} a(t) & k(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}$$
$$= \theta^T(t)\phi(t) \qquad (5)$$

Adaptive law Adapt the control parameters in the following fashion.

$$\begin{bmatrix} \dot{a}(t) \\ \dot{k}(t) \end{bmatrix} = -\operatorname{sgn}(k_p)\, e(t) \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}$$
$$\Rightarrow \dot{\theta} = -\operatorname{sgn}(k_p)\, e(t) \Gamma \phi(t) \qquad (6)$$

The equations in (6) are nonlinear ODEs.

[1] The starred variables with $*$ superscripts represent the ideal values of the control parameters.

Questions

- Is the closed loop stable?
- Does, with this, $e(t) \to 0$?
- Are the parameters $\theta(t)$ finite?
- Are the parameters $\theta(t)$ constant for $t \to \infty$?
- Do the parameters $\theta(t)$ approach their 'ideal' values θ^* for $t \to \infty$?

2 Nonlinear SISO plant

The linear SISO plant in the previous chapter is linearised to a nonlinear SISO plant.

Control problem

Given plant G and reference model M.

$$G: \dot{x}_p(t) = a_p x_p(t) + k_p u(t) + \alpha_p f(z) \qquad (7)$$

$$M: \dot{x}_m(t) = a_m x_m(t) + k_m r(t) \qquad (2)$$

- a_p, k_p, α_p are unknown but constant
- $f(z)$ is a nonlinear (external) function
- $\operatorname{sgn}(k_p)$, $f(.)$ are known (z is a known signal)
- $\alpha_m f(z)$ is not necessary

Goal $x_p(t) \to x_m(t)$ for $t \to \infty$.

2.1 Control structures

Ideal control structure based on MRC

$$\begin{aligned} u^*(t) &= \frac{1}{k_p}\left(-a_p x_p(t) + a_m x_p(t) + k_m r(t) - \alpha_p f(z)\right) \\ &= \underbrace{\frac{(a_m - a_p)}{k_p}}_{a^*} x_p + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t) + \underbrace{\frac{-\alpha_p}{k_p}}_{\alpha^*} f(z) \\ &= \begin{bmatrix} a^* & k^* & \alpha^* \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \end{aligned}$$

$$u^*(t) = \theta^{*T} \phi(t) \qquad (8)$$

Control law using MRAC

$$u(t) = \begin{bmatrix} a(t) & k(t) & \alpha(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix}$$

$$u(t) = \theta^T(t) \phi(t) \qquad (9)$$

$a(t)$, $k(t)$, $\alpha(t)$ unknown.

2.2 Error dynamics

In adaptive control, the current estimated parameters are varying. We therefore have the following error in parameters as deviations from the unknown but ideal and constant real parameters:

$$\left.\begin{aligned} \tilde{a}(t) &= a(t) - a^* \\ \tilde{k}(t) &= k(t) - k^* \\ \tilde{\alpha}(t) &= \alpha(t) - \alpha^* \end{aligned}\right\} \tilde{\theta}(t) = \theta(t) - \theta^*$$

$$\begin{aligned} \dot{e}(t) &= \dot{x}_p(t) - \dot{x}_m(t) \\ &= a_p x_p(t) \\ &\quad + k_p\left(a(t)x_p(t) + k(t)r(t) + \alpha(t)f(z)\right) \\ &\quad + \alpha_p f(z) \\ &\quad - \left(a_m x_m(t) + k_m r(t)\right) \\ &= a_p x_p(t) - a_m x_m(t) \\ &\quad + k_p a(t) x_p(t) \\ &\quad + k_p \underbrace{\left(k(t) - \frac{k_m}{k_p}\right)}_{\tilde{k}(t)} r(t) \\ &\quad + k_p \underbrace{\left(\alpha(t) - \frac{\alpha_p}{k_p}\right)}_{\tilde{\alpha}(t)} f(z) \\ &= \underbrace{(a_m - k_p a^*)}_{a_p} x_p(t) - a_m x_m(t) \\ &\quad + k_p a(t) x_p(t) \\ &\quad + k_p \tilde{k}(t) r(t) + k_p \tilde{\alpha}(t) f(z) \\ &= a_m e(t) + k_p \tilde{a}(t) x_p(t) \\ &\quad + k_p \tilde{k}(t) r(t) + k_p \tilde{\alpha}(t) f(z) \end{aligned}$$

$$\dot{e}(t) = a_m e(t) + k_p \begin{bmatrix} \tilde{a}(t) & \tilde{k}(t) & \tilde{\alpha}(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix}$$

$$\dot{e}(t) = a_m e(t) + \frac{1}{k^*} k_m \theta^T(t) \phi(t) \qquad (10)$$

The error dynamics can be rewritten using an operator $M(s) = \dfrac{k_m}{s - a_m}$.

$$\begin{aligned} \dot{e}(t) &= a_m e(t) + \frac{1}{k^*} k_m \theta^T(t) \phi(t) \\ (s - a_m)\, e(t) &= \frac{1}{k^*} k_m \theta^T(t) \phi(t) \\ e(t) &= \frac{1}{k^*} M(s) \theta^T(t) \phi(t) \end{aligned} \qquad (10)$$

All unknown parameters appear linearly (affine, 'linear in the parameters'). The error dynamics (10) is a nonlinear differential equation. When is it stable? → Lyapunov.

3 Lyapunov-like functions

New interpretation of Lyapunov Nothing to do with energy. $\mathbf{V}$ affects the scaling of the distance of $\mathbf{x}$ from the origin in the phase portrait.

$$\|\mathbf{x}\|_{\mathbf{V}}^2 = \mathbf{x}^T \mathbf{V} \mathbf{x}, \qquad \mathbf{V} \succ 0$$

> **All Lyapunov says is**: how far is x from the origin? We want to find some type of measure for that.

Lyapunov function (Lyapunov-like)
We want the output error $e(t)$ as well as the parameter error $\tilde{\theta}(t)$ to go to zero. $\Gamma \succ 0$ symmetrical, positive definite.

$$V(e,\tilde{\theta}) = \frac{1}{2}e^2 + \frac{1}{2}|k_p|\left(\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}\right) \tag{11}$$

$$\dot{V} = e\dot{e} + \frac{1}{2}|k_p|\underbrace{\left(2\tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}}\right)}_{2}$$

Substitute $\dot{e}$ using equation (10).

$$\begin{aligned}\dot{V} &= a_m e^2 + ek_p\tilde{\theta}^T\phi + \frac{1}{2}|k_p|\left(2\theta^T\tilde{\Gamma}^{-1}\dot{\tilde{\theta}}\right)\\ &= a_m e^2 + |k_p|\tilde{\theta}^T\underbrace{\left(\operatorname{sgn}(k_p)\,e\phi + \Gamma^{-1}\dot{\tilde{\theta}}\right)}_{\overset{!}{=}0}\end{aligned}$$

The second term is set to 0, because we want $V \preceq 0$ and we don't know all the signs of the terms. This will define the **adaptive law**.

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -\Gamma \operatorname{sgn} k_p \phi(t)e(t)\\ \dot{\theta}(t) &= -\Gamma \operatorname{sgn} k_p \phi(t)e(t)\end{aligned}$$

With the adaptive law, we obtain for $\dot{V}$:

$$\dot{V} = a_m e^2 \preceq 0 \tag{12}$$

Remark $e(t)$ does not have to be 0 – why?

- We do not need $\dot{V}$ to approach zero.
- $\dot{V} = 0$ does not imply that V has a limit as $t \to \infty$. (although this is known, see footnote below[3]).
- In other words, $\dot{V} = 0$ does not imply that the errors go to zero.

> If the derivative of a function $\to 0$, that does not imply that the function has a limit, and vice versa: if a function has a limit, that doesn't mean its derivative $\to 0$.
>
> $$\lim_{t\to\infty}\dot{f}(t) = 0 \not\Leftrightarrow \lim_{t\to\infty} f(t) = k$$

[2] Possible due to Γ symmetrical.
[3] A function V that is bounded from below $V \succeq 0$ and **non-increasing** $\dot{V} \preceq 0$ has a limit as $t \to \infty$

Counterexamples:

$$\nexists \lim_{t\to\infty} f(t),\quad \begin{aligned} f(t) &= \sin(\ln t)\\ \dot{f}(t) &= \frac{\cos(\ln t)}{t} \to 0\end{aligned}$$

$$\begin{aligned} f(t) &= e^{-t}\sin(e^{2t})\\ \lim_{t\to\infty} f(t) &= 0\\ \dot{f}(t) &= -e^{-t}\sin(e^{2t}) + e^{t}\sin(e^{2t})\\ &\to \text{ explodes!}\end{aligned}$$

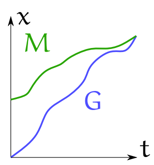
Are the error dynamics stable?

- Measure (some of) the states
- Apply $V = f(e,\tilde{\theta})$
- $V \to \infty$? Or $V \downarrow$?
  $\Rightarrow$ analyse time derivative $\dot{V}$
- If we show $\dot{V} \to 0$, then $e \to 0$.

Extensions to Lyapunov There are two well-known extensions to Lyapunov to prove asymptotic stability, even if $\dot{V} \preceq 0$.

(i) LaSalle's invariance principle (**only for autonomous systems**)
(ii) Barbalat's lemma (**OK for non-autonomous systems**)

Our system's error dynamics are non-autonomous, $\dot{e} = f(t,\dots)$, due to following another system.



> **Def: Uniformly continuous function**
>
> A function $f(t) : \mathbb{R} \to \mathbb{R}$ is uniformly contínous, if
> $\forall\varepsilon > 0 :\quad \exists\,\delta = \delta(\varepsilon) > 0,$
>
> $$\forall|t_2 - t_1| \leqslant \delta$$
> $$\Rightarrow |f(t_2) - f(t_1)| \leqslant \varepsilon$$

> **Sufficient condition for uniformly continuous functions**: If the derivative $\dot{f}(t)$ exists (i.e. bounded), $\Rightarrow f(t)$ is uniformly constant.

4 Barbalat's lemma

Lemma: Barbalat Variant A

If $f(t) : \mathbb{R} \to \mathbb{R}$
(i) is a differentiable function
(ii) has a finite limit as $t \to \infty$
(iii) $\dot{f}(t)$ is uniformly constant
$\Rightarrow \lim_{t\to\infty} \dot{f}(t) = 0$

Lemma: Barbalat Variant B

If
(i) $f(t) : \mathbb{R} \to \mathbb{R}$ is uniformly continous $\forall t$
(ii) $\exists \lim_{t\to\infty} \int_0^t f(\tau) d\tau$
$\Rightarrow \lim_{t\to\infty} f(t) = 0$

Lemma: Barbalat Variant C

If $f, \dot{f} \in \mathcal{L}_\infty$ and $f \in \mathcal{L}_2$, then $|f(t)| \to 0$ as $t \to \infty$.

Closed loop stability analysis By definition, $V \succeq 0$ and $\dot{V} \preceq 0$.

Boundedness of $e(t)$
- As V is bounded from below and non-increasing, V has a limit as $t \to \infty$.
- Tracking error $e(t)$ and parameter errors $\tilde{\theta}(t)$ are bounded.
- As $\tilde{\theta}(t)$ bounded and θ^* constant, $\theta(t)$ is bounded.

Boundedness of $\dot{e}(t)$
- Assume $r(t)$ bounded, then $x_m(t), \dot{x}_m(t)$ bounded ($\because$ M is stable)
- $x_p(t) = \underset{\text{bd.}}{e(t)} + \underset{\text{bd.}}{x_m(t)}$
  $\Rightarrow x_p(t)$ bounded.
- $u(t)$ bounded if $\phi(t)$ bounded.
  $\phi(t) = \begin{bmatrix} \underset{\text{bd.}}{r(t)} & \underset{\text{bd.}}{x_p(t)} & \underset{\text{bd.}}{f(z)} \end{bmatrix}$
  (new requirement: $f(z)$ needs to be bounded).
- $\dot{x}_p(t) = a_p \underset{\text{bd.}}{x_p(t)} + k_p \underset{\text{bd.}}{u(t)}$
  $\Rightarrow \dot{x}_p(t)$ bounded

- $\dot{e}(t) = \underset{\text{bd.}}{\dot{x}_p(t)} - \underset{\text{bd.}}{\dot{x}_m(t)}$ is bounded.

With $\ddot{V} = 2a_m \underset{\text{bd.}}{e} \underset{\text{bd.}}{\dot{e}}$, $\ddot{V}$ bounded $\to \dot{V}$ uniformly constant.

Using Barbalat's lemma Variant A,
- V is differentiable
- V has a finite limit as $t \to \infty$
  (bounded from below and non-increasing)
- $\dot{V}$ is uniformly constant

$$\Rightarrow \lim_{t\to\infty} \dot{V} = 0$$
$$\Rightarrow \lim_{t\to\infty} e(t) = 0$$

Signal norm, $\mathcal{L}_p$ space "how big is a signal?"
- **Idea**: quantify magnitude of a signal $x(t)$
  $x(t) : \mathbb{R}^+ \to \mathbb{R}^n, \quad \mathbb{R}^+ = [0, \infty)$
- p-Norm

$$\|x_p\| = \left(\int_0^\infty |x(t)|^p \, dt \right)^{1/p} \qquad p \in (0, \infty)$$

- If $\mathbf{x}(t)$ vector, $|.|$ is the vector 2-norm, 'distance'.
- $\|x\|_\infty = \sup_{t\in\mathbb{R}^+} |x(t)| \triangleq$ highest value of $\mathbf{x}$
  "When power ∞, only the greatest value survives"