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1 Linear SISO plant

Control problem

Given plant G and reference model M .

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$

$$\text{IC } x_p(0) \in \mathbb{R}$$

a_p - pole of plant
 k_p - input gain of plant

$$M : \dot{x}_m(t) = a_m x_m(t) + k_m r(t), \quad (2)$$

$$\text{IC } x_m(0) \in \mathbb{R}$$

a_m - pole of reference model
 k_m - input gain of reference model
 $r(t)$ - reference signal

The **reference model parameters** are set by the user and are therefore known.

Task find a control $u(t)$ such that $x_p(t) \rightarrow x_m(t)$ for $t \rightarrow \infty$.

$$G_{\text{des}} : \dot{x}_p(t) = a_m x_p(t) + k_m r(t) \quad (3)$$

$$\text{IC } x_p(0) \in \mathbb{R}$$

Solutions for $u(t)$ using

- Model reference control (MRC)
- Model reference adaptive control (MRAC)

1.1 Model reference control (MRC)

The **plant parameters** are assumed to be known.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$

$$\text{IC } x_p(0) \in \mathbb{R}$$

Pick $u(t)$ such that the dynamical behaviour of the closed loop is equal to that of the model. This is done by comparing (1) to (3).

$$\begin{aligned} u^*(t) &= \frac{1}{k_p} (-a_p x_p(t) + a_m x_p(t) + k_m r(t)) \\ &= \underbrace{\frac{a_m - a_p}{k_p}}_{a^*} x_p(t) + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t) \\ &= [a^* \quad k^*] \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix} \\ u^*(t) &= \theta^{*T} \phi(t) \end{aligned}$$

Using this input¹, now the dynamics of the plant G matches the dynamics of the model M , as in equation (3).

However, even though now the dynamics are the same, the initial conditions are not necessarily the same. Would this input also work for $x_m(0) \neq x_p(0)$? I.e., does this guarantee that $x_p(t) \rightarrow x_m(t)$ for $t \rightarrow \infty$?

Dependence on the initial conditions To check this, we examine the error dynamics and see if the error asymptotically goes to zero.

$$e(t) = x_p(t) - x_m(t)$$

$$\dot{e}(t) = \dot{x}_p(t) - \dot{x}_m(t)$$

Using eqns. (2) and (3):

$$\dot{e}(t) = a_m e(t) \quad (4)$$

If $a_m < 0$, the error dynamics are stable. That is, $e(t) \rightarrow 0$ for any ICs.

Conclusion

- Model reference control (MRC) works with the error dynamics of the reference model
- **We need to know all plant parameters very well**
 \Rightarrow Problem: **uncertainty in parameters**

1.2 Model reference adaptive control (MRAC)

The **plant parameters** are unknown. We assume $k_p > 0$.

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t), \quad (1)$$

$$\text{IC } x_p(0) \in \mathbb{R}$$

Control law We search for (learn) the value of θ and k , which are therefore functions of time.

$$\begin{aligned} u(t) &= [a(t) \quad k(t)] \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix} \\ &= \theta^T(t) \phi(t) \end{aligned} \quad (5)$$

Adaptive law Adapt the control parameters in the following fashion.

$$\begin{bmatrix} \dot{a}(t) \\ \dot{k}(t) \end{bmatrix} = -\text{sgn}(k_p) e(t) \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}$$

$$\Rightarrow \dot{\theta} = -\text{sgn}(k_p) e(t) \Gamma \Phi(t) \quad (6)$$

The equations in (6) are nonlinear ODEs.

¹The starred variables with * superscripts represent the ideal values of the control parameters.

Questions

- Is the closed loop stable?
- Does, with this, $e(t) \rightarrow 0$?
- Are the parameters $\theta(t)$ finite?
- Are the parameters $\theta(t)$ constant for $t \rightarrow \infty$?
- Do the parameters $\theta(t)$ approach their 'ideal' values θ^* for $t \rightarrow \infty$?

2 Nonlinear SISO plant

The linear SISO plant in the previous chapter is generalised to a nonlinear SISO plant.

Control problem

Given plant G and reference model M .

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t) + \alpha_p f(z) \quad (7)$$

$$M : \dot{x}_m(t) = a_m x_m(t) + k_m r(t) \quad (2)$$

- a_p, k_p, α_p are unknown but constant
- $f(z)$ is a nonlinear (external) function
- $\text{sgn}(k_p), f(\cdot)$ are known (z is a known signal)
- $\alpha_m f(z)$ is not necessary

Goal $x_p(t) \rightarrow x_m(t)$ for $t \rightarrow \infty$.

2.1 Control structures

Ideal control structure based on MRC

$$\begin{aligned} u^*(t) &= \frac{1}{k_p} (-a_p x_p(t) + a_m x_p(t) + k_m r(t) - \alpha_p f(z)) \\ &= \underbrace{\frac{(a_m - a_p)}{k_p}}_{a^*} x_p + \underbrace{\frac{k_m}{k_p}}_{k^*} r(t) + \underbrace{\frac{-\alpha_p}{k_p}}_{\alpha^*} f(z) \\ &= [a^* \quad k^* \quad \alpha^*] \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\ u^*(t) &= \theta^{*T} \phi(t) \end{aligned} \quad (8)$$

Control law using MRAC

$$\begin{aligned} u(t) &= [a(t) \quad k(t) \quad \alpha(t)] \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\ u(t) &= \theta^T(t) \phi(t) \end{aligned} \quad (9)$$

$a(t), k(t), \alpha(t)$ unknown.

2.2 Error dynamics

In adaptive control, the current estimated parameters are varying. We therefore have the following error in parameters as deviations from the unknown but ideal and constant real parameters:

$$\left. \begin{aligned} \tilde{a}(t) &= a(t) - a^* \\ \tilde{k}(t) &= k(t) - k^* \\ \tilde{\alpha}(t) &= \alpha(t) - \alpha^* \end{aligned} \right\} \tilde{\theta}(t) = \theta(t) - \theta^*$$

$$\begin{aligned} \dot{e}(t) &= \dot{x}_p(t) - \dot{x}_m(t) \\ &= a_p x_p(t) + k_p (a(t)x_p(t) + k(t)r(t) + \alpha(t)f(z)) + \alpha_p f(z) \\ &\quad - (a_m x_m(t) + k_m r(t)) \\ &= a_p x_p(t) - a_m x_m(t) + k_p a(t)x_p(t) + k_p \underbrace{\left(k(t) - \frac{k_m}{k_p}\right)}_{\tilde{k}(t)} r(t) \\ &\quad + k_p \underbrace{\left(\alpha(t) - \frac{\alpha_p}{k_p}\right)}_{\tilde{\alpha}(t)} f(z) \\ &= \underbrace{(a_m - k_p a^*)}_{a_p} x_p(t) - a_m x_m(t) + k_p a(t)x_p(t) \\ &\quad + k_p \tilde{k}(t)r(t) + k_p \tilde{\alpha}(t)f(z) \\ &= a_m e(t) + k_p \tilde{a}(t)x_p(t) + k_p \tilde{k}(t)r(t) + k_p \tilde{\alpha}(t)f(z) \end{aligned}$$

$$\begin{aligned} \dot{e}(t) &= a_m e(t) + k_p [\tilde{a}(t) \quad \tilde{k}(t) \quad \tilde{\alpha}(t)] \begin{bmatrix} x_p(t) \\ r(t) \\ f(z) \end{bmatrix} \\ \dot{e}(t) &= a_m e(t) + \frac{1}{k^*} k_m \tilde{\theta}^T(t) \phi(t) \end{aligned} \quad (10)$$

The error dynamics can be rewritten using an operator $M(s) = \frac{k_m}{s - a_m}$, which is non other than the transfer function of the reference model!

$$\begin{aligned} \dot{e}(t) &= a_m e(t) + \frac{1}{k^*} k_m \theta^T(t) \phi(t) \\ (s - a_m) e(t) &= \frac{1}{k^*} k_m \theta^T(t) \phi(t) \\ e(t) &= \frac{1}{k^*} M(s) \theta^T(t) \phi(t) \end{aligned} \quad (10)$$

All unknown parameters appear linearly (affine, 'linear in the parameters'). The error dynamics (10) is a nonlinear differential equation. When is it stable? \rightarrow Lyapunov.

2.3 Lyapunov-like function

New interpretation of Lyapunov Nothing to do with energy. V affects the scaling of the distance of x from the origin in the phase portrait.

$$\|x\|_V^2 = x^T V x, \quad V \succ 0$$

All Lyapunov says is: how far is x from the origin? We want to find some type of measure for that.

Lyapunov function (Lyapunov-like)

We want the output error $e(t)$ as well as the parameter error $\tilde{\theta}(t)$ to go to zero. $\Gamma \succ 0$ symmetrical, positive definite.

$$V(e, \tilde{\theta}) = \frac{1}{2}e^2 + \frac{1}{2}|k_p| \left(\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \right) \quad (11)$$

$$\dot{V} = e\dot{e} + \frac{1}{2}|k_p| \underbrace{\left(2\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \right)}_2$$

Substitute \dot{e} using equation (10).

$$\begin{aligned} \dot{V} &= a_m e^2 + e k_p \tilde{\theta}^T \Phi + \frac{1}{2}|k_p| \left(2\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \right) \\ &= a_m e^2 + |k_p| \tilde{\theta}^T \underbrace{\left(\text{sgn}(k_p) e \Phi + \Gamma^{-1} \dot{\tilde{\theta}} \right)}_{\stackrel{!}{=} 0} \end{aligned}$$

The second term is set to 0, because we want $V \preceq 0$ and we don't know all the signs of the terms. This will define the **adaptive law**.

$$\begin{aligned} \dot{\tilde{\theta}}(t) &= -\text{sgn}(k_p) \Gamma \Phi(t) e(t) \\ \dot{\theta}(t) &= -\text{sgn}(k_p) \Gamma \Phi(t) e(t) \end{aligned}$$

With the adaptive law, we obtain for \dot{V} :

$$\dot{V} = a_m e^2 \preceq 0 \quad (12)$$

Remark $e(t)$ does not have to be 0 – why?

If the derivative of a function $\rightarrow 0$, that **does not** imply that the function has a limit, and vice versa: if a function has a limit, that doesn't mean its derivative $\rightarrow 0$.

$$\lim_{t \rightarrow \infty} \dot{f}(t) = 0 \not\Leftarrow \lim_{t \rightarrow \infty} f(t) = k$$

Counterexamples:

$$\begin{aligned} f(t) &= \sin(\ln t) \\ \nexists \lim_{t \rightarrow \infty} f(t), \quad \dot{f}(t) &= \frac{\cos(\ln t)}{t} \rightarrow 0 \\ f(t) &= e^{-t} \sin(e^{2t}) \\ \lim_{t \rightarrow \infty} f(t) &= 0 \\ \dot{f}(t) &= -e^{-t} \sin(e^{2t}) + e^t \sin(e^{2t}) \\ &\rightarrow \text{explodes!} \end{aligned}$$

Are the error dynamics stable?

- Measure (some of) the states
- Apply $V = f(e, \tilde{\theta})$
- $V \rightarrow \infty$? Or $V \downarrow$?
 \Rightarrow analyse time derivative \dot{V}
- If we show $\dot{V} \rightarrow 0$, then $e \rightarrow 0$.

²Possible due to Γ symmetrical.

Extensions to Lyapunov There are two well-known extensions to Lyapunov to prove asymptotic stability, even if $\dot{V} \preceq 0$.

- LaSalle's invariance principle (**only for autonomous systems**)
- Barbalat's lemma (**OK for non-autonomous systems**)

Our system's error dynamics are non-autonomous, $\dot{e} = f(t, \dots)$, due to following another system (Figure 1).

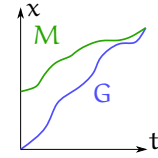


Figure 1: Non-autonomous dynamics

2.4 Closed loop stability analysis

Using Barbalat's Lemma (Variant A)³ on the function V , we need to fulfill the following conditions:

- V is differentiable
Yes, $\exists \dot{V} = a_m e^2(t)$
- V has a finite limit as $t \rightarrow \infty$
Yes, as $V \succeq 0$ and $\dot{V} \preceq 0$
- \dot{V} is uniformly continuous $\Leftarrow \exists \ddot{V}$ (sufficient condition⁴)
Is $\ddot{V} = 2a_m e\dot{e}$ bounded?

Boundedness of $e(t)$

- As V is bounded from below and non-increasing, V has a limit as $t \rightarrow \infty$.
- Tracking error $e(t)$ and parameter errors $\tilde{\theta}(t)$ are bounded.
- As $\tilde{\theta}(t)$ bounded and θ^* constant, $\theta(t)$ is bounded.

Boundedness of $\dot{e}(t)$

- Assume $r(t)$ bounded⁵, then, from the reference model equation (2):
 $x_m(t), \dot{x}_m(t)$ bounded ($\because M$ is stable)
- $x_p(t) = e(t) + x_m(t)$
 $\Rightarrow x_p(t)$ **bd.**
- $u(t) = \theta^T(t) \Phi(t)$ bounded if $\Phi(t)$ bounded.
 $\Phi(t) = \begin{bmatrix} r(t) & x_p(t) & f(z) \end{bmatrix}^T$
 $\begin{bmatrix} \text{bd.} & \text{bd.} & \text{bd.} \end{bmatrix}$
(new requirement: $f(z)$ needs to be bounded).
- $\dot{x}_p(t) = a_p x_p(t) + k_p u(t)$
 $\Rightarrow \dot{x}_p(t)$ **bd.**
- $\dot{e}(t) = \dot{x}_p(t) - \dot{x}_m(t)$ is bounded.
 $\begin{bmatrix} \text{bd.} & \text{bd.} \end{bmatrix}$

³Chapter 2.7 on page 7

⁴Chapter 2.6 on page 7

⁵Reasonable assumption, because why would we want to use an unbounded input?

All the conditions of Barbalat's lemma thus fulfilled, we can conclude that the derivative of V approaches zero for $t \rightarrow \infty$.

$$\lim_{t \rightarrow \infty} \dot{V} = 0 \\ \Rightarrow \lim_{t \rightarrow \infty} e(t) = 0$$

2.5 Nonlinearities

THEOREM: Nonlinear SISO plant

Given plant G and reference model M .

$$G : \dot{x}_p(t) = a_p x_p(t) + k_p u(t) + \alpha_p f(z) \quad (7)$$

$$M : \dot{x}_m(t) = a_m x_m(t) + k_m r(t) \quad (2)$$

The input

$$u(t) = \theta^T(t) \phi(t) \quad (9)$$

$$\text{with } \dot{\theta} = -\text{sgn}(k_p) \Gamma \phi e \quad (6)$$

$$\text{and } \phi(t) = [r(t) \quad x_p(t) \quad f(z)]^T$$

renders the origin asymptotically stable and guarantees $x_p(t) \rightarrow x_m(t)$ as $t \rightarrow \infty$.

We can add, arbitrarily, many 'nonlinearities' $f_i(z_j)$ with unknown gains α_i (Figure 2). The function $z(t)$ z is a placeholder. $z(t)$ can be an external or an internal signal. The nonlinearity functions **need not be continuous**. The only requirement:

$$f_i(z_j) \in \mathcal{L}_\infty$$

Nonlinearities are bounded at all times^a.

^aFor the boundedness of $u(t)$ and therefore $\dot{e}(t)$

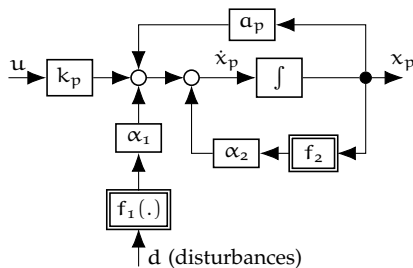


Figure 2: Plant G with nonlinearities

Questions

- Can we do $f(u)$?
Possible, but **solving for $u = \dots f(u) \dots$ is difficult**.

- Can $f(\cdot)$ be a differential operator (filter)?

Yes (Figure 3). If the filter is linear, then a solution definitely exists.

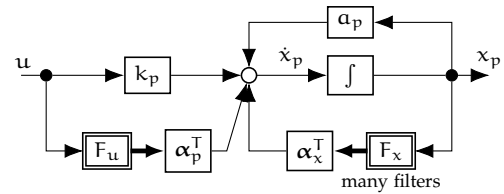


Figure 3: Plant G with nonlinearities as filters

The filters F_u and F_x are stable dynamical systems. **If they are not stable, there is a higher chance of getting an infinite output \rightarrow conflicts with bounded output requirement.**

If no such filters are present in the plant G , then the plant has an order of 1.

Let's say we have an **offset in the plant input** of unknown magnitude, and the plant has otherwise **known parameters**.

$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t) + \alpha_p \cdot 1 \\ u(t) = a^* x_p(t) + k^* r(t) - \frac{\alpha(t)}{k_p}$$

Closed loop becomes

$$\dot{x}_p(t) = a_m x_p(t) + k_m r(t) + \tilde{\alpha}(t)$$

Error dynamics

$$\dot{e} = a_m e(t) + \tilde{\alpha}(t)$$

Lyapunov-like function

$$V = \frac{1}{2} e^2 + \frac{1}{2} \tilde{\alpha}^2 \\ \dot{V} = e \dot{e} + \tilde{\alpha} \dot{\tilde{\alpha}} + \dots \\ = a_m e^2 + \tilde{\alpha} \underbrace{(e + \dot{\tilde{\alpha}})}_{\text{to set to 0}}$$

Setting the second term to zero ensures the negative semidefiniteness of \dot{V} .

$$\dot{\tilde{\alpha}} = -e \quad (13)$$

Equation (13) above implies $u = \dots + \int e dt + \dots$, i.e., that the controller contains an I-part. As a result of this, there are no steady state errors caused by model uncertainties. The controller eliminates offset at input of plant.

Equation (13) is a pure integrator acting on control error. This is a linear controller! We have learned: integrators 'learn' input offsets of the plant and correct them.

Adaptive controllers can be interpreted as nonlinear PI controllers.

2.6 Uniformly continuous functions

DEF: Uniformly continuous function

A function $f(t) : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous, if

$$\forall \varepsilon > 0 : \exists \delta = \delta(\varepsilon) > 0,$$

$$\begin{aligned} \forall |t_2 - t_1| \leq \delta \\ \Rightarrow |f(t_2) - f(t_1)| \leq \varepsilon \end{aligned}$$

Sufficient condition for uniformly continuous functions: If the derivative $\dot{f}(t)$ exists (i.e. bounded), $\Rightarrow f(t)$ is uniformly continuous.

2.7 Barbalat's lemma

LEMMA: Barbalat Variant A

If $f(t) : \mathbb{R} \rightarrow \mathbb{R}$

- (i) is a differentiable function, $\dot{f} \in \mathcal{L}_\infty$
- (ii) has a finite limit as $t \rightarrow \infty$, $f \in \mathcal{L}_\infty$
- (iii) $\dot{f}(t)$ is uniformly continuous, $\ddot{f} \in \mathcal{L}_\infty$

$$\Rightarrow \lim_{t \rightarrow \infty} \dot{f}(t) = 0$$

LEMMA: Barbalat Variant B

If

- (i) $f(t) : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous $\forall t$
- (ii) $\exists \lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau$

$$\Rightarrow \lim_{t \rightarrow \infty} f(t) = 0$$

LEMMA: Barbalat Variant C

If

- (i) $f \in \mathcal{L}_\infty$
- (ii) $\dot{f} \in \mathcal{L}_\infty$
- (iii) $f \in \mathcal{L}_2$,

$$\Rightarrow |f(t)| \rightarrow 0 \text{ as } t \rightarrow \infty$$

2.8 Signal norms and functional spaces

Idea quantify magnitude of a signal $x(t)$ – “How big is a signal?”

DEF: Signal norm

Given

$$x(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^n, \quad \mathbb{R}^+ = [0, \infty)$$

p-Norm

$$\|x_p\| = \left(\int_0^\infty |x(t)|^p dt \right)^{1/p} \quad p \in (0, \infty)$$

Distance vector 2-norm of $x(t)$, i.e. $|x|$.

Max. value “When power ∞ , only the greatest value survives”

$$\begin{aligned} \|x\|_\infty &= \sup_{t \in \mathbb{R}^+} |x(t)| \\ &\triangleq \text{highest value of } x(t). \end{aligned}$$

DEF: Functional space

$$\mathcal{L}_p = \{x(t) \in \mathbb{R}^n : \underbrace{\|x\|_p < \infty}_{\text{exists}}\}$$

$$x(t) \in \mathcal{L}_p$$

- x is bounded
- “ x ’s highest value exists and is not infinity.”

Ex. 2.1 Functional space

Show that $e \in \mathcal{L}_\infty$

e is in V and V is bounded, $\therefore e \in \mathcal{L}_\infty$.

Ex. 2.2 Functional space

Show that $e \in \mathcal{L}_2$

$$\begin{aligned} \int_0^\infty \dot{V} dt &= V(\infty) - V(0), \quad \text{is bounded.} \\ a_m \int_0^\infty e^2 dt &\in \mathcal{L}_\infty \\ \|e\|^2 &\in \mathcal{L}_\infty \\ \Rightarrow e &\in \mathcal{L}_2 \end{aligned}$$

3 Positive real functions

DEF: Positive real function I

A rational function $H(s) : \mathbb{Z} \rightarrow \mathbb{Z}$, $s = \sigma + j\omega$ is positive real (PR), if

- (i) $H(s)$ is real for real s
- (ii) $\Re\{H(s)\} \geq 0$ for $\Re\{s\} > 0$

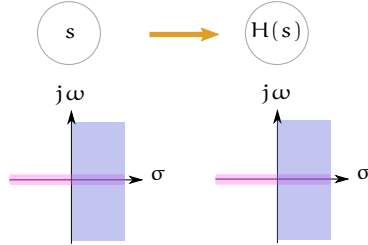


Figure 4: Positive real mapping $s \rightarrow H(s)$

DEF: Positive real function II

A rational function $H(s)$ is positive real, if

- (i) $H(s)$ is analytic in $\Re\{s\} > 0$
 - $H(s)$ has no poles in RHP ($\Re\{s\} > 0$)
 - $H(s)$ is stable
- (ii) $\Re\{H(j\omega)\} \geq 0$, $\forall \omega \in [0, \infty]$
 - Nyquist of $H(s)$ is in the RHP
 - phase $\angle H(j\omega) \in [-90^\circ, +90^\circ]$
 - rel. degree of $H(s)$ is 0 or 1
- (iii) any pure imaginary pole $j\omega$ of $H(s)$ is a simple pole, and the residue

$$\lim_{s \rightarrow j\omega} (s - j\omega) H(s)$$

is positive semidefinite.

Alternatively: $H(\infty) > 0$ or

$$\lim_{\omega \rightarrow \infty} \omega^2 \Re\{H(j\omega)\} \geq 0$$

DEF: Strictly positive real functions

$H(s)$ is strictly positive real (SPR) if $H(s - \epsilon)$ is PR for some $\epsilon > 0$.

Note relative degree of a system corresponds to its response delay.

LEMMA: SPR lemma

$H(s)$ is SPR if

- (i) $H(s)$ is Hurwitz
 - all poles on LHP, none are purely imaginary
- (ii) $\Re\{H(j\omega)\} > 0$, $\forall \omega \in \mathbb{R}$
 - Nyquist of $H(s)$ is in the RHP and **not on the imaginary axis**.
 - phase $\angle H(j\omega) \in (-90^\circ, +90^\circ)$
 - rel. degree of $H(s) \in \{0, 1\}$
- (iii) $H(\infty) > 0$ (positive gain for proper H) or

$$\lim_{\omega \rightarrow \infty} \omega^2 \Re\{H(j\omega)\} > 0$$

positive gain for relative degree 1

Discussion If $H(s)$ is SPR, then so is the inverse $H^{-1}(s)$ (stable poles, stable zeroes).

SPR $\Rightarrow H(s)$ is stable, minimal-phase.

I.e., only stable zeroes, because zeroes are in LHP.

- phase $\angle H(j\omega) \in (-90^\circ, +90^\circ)$
- rel. degree of $H(s) \leq 1(-1, 0, 1)$
- positive gain $\forall \omega$

Ex. 3.3 PR

$G(s) = \frac{1}{s}$ has a single pole $s = 0$, with a residue of 1.

$$\Re\{G(j\omega)\} = \Re\left\{\frac{1}{j\omega}\right\} = 0 \quad \forall \omega \neq 0$$

Hence, $G(s)$ is PR but not SPR, as $\frac{1}{s-\epsilon}$ has a pole in $\Re\{s\} \geq 0$ for any $\epsilon > 0$.

Ex. 3.4 PR

$G(s) = \frac{1}{s+a}$, $a > 0$ is Hurwitz.

$$\Re\{G(j\omega)\} = \frac{a}{\omega^2 + a^2} > 0 \quad \forall \omega \in [0, \infty]$$

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \omega^2 \Re\{G(j\omega)\} &= \lim_{\omega \rightarrow \infty} \frac{\omega^2 a}{\omega^2 + a^2} \\ &= a, \quad a > 0 \end{aligned}$$

Ex. 3.5 PR

$$G(s) = \frac{1}{s^2 + s + 1}$$

$$\Re\{G(j\omega)\} = \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2}$$

$$\not\geq 0 \quad \forall \omega$$

$\Rightarrow G$ is not PR.

4 Kalman-Yakobovich lemma (KY)

“Maier version; how to design a controller given SPR”

LEMMA: Kalman-Yakobovich Lemma

Given

- a scalar $\gamma \geq 0$
- vectors \mathbf{b} and \mathbf{c} ,
- an asymptotically stable matrix \mathbf{A}^a , and
- a positive definite matrix $\mathbf{L} \succ 0$,

$$\text{If } H(s) \triangleq \frac{1}{2}\gamma + \mathbf{c}^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} \\ \Rightarrow H(s) \text{ is SPR}$$

Then, there exist

- a scalar $\varepsilon > 0$
- a vector \mathbf{q} , and
- a symmetric positive definite matrix \mathbf{P} ,

s.t.

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} = -\mathbf{q} \mathbf{q}^T - \varepsilon \mathbf{L} \quad (14)$$

$$\mathbf{P} \mathbf{b} - \mathbf{c} = \sqrt{\gamma} \mathbf{q} \quad (15)$$

^apositive eigenvalues

Using it We only need $\gamma = 0$ in all cases (in this course). Hence we can say: if $H(s)$ is SPR $\Rightarrow \exists \mathbf{P} = \mathbf{P}^T > 0$, s.t.

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} = -\mathbf{Q} \\ \mathbf{P} \mathbf{b} = \mathbf{c} \quad (\text{s. note})^6$$

where $\mathbf{Q} = \mathbf{Q}^T > 0$

LEMMA: Adaptive laws based on Lyapunov

(For rel. degree 1 plants)

Consider the dynamical system below^{abc}.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\theta^T(t)\boldsymbol{\phi}(t) \quad (16)$$

$$y(t) = \mathbf{c}^T \mathbf{x}(t)$$

$$z_1(t) = ky(t)$$

where

- (\mathbf{A}, \mathbf{b}) is stabilisable
- $(\mathbf{c}^T, \mathbf{A})$ is detectable
- $\mathbf{c}^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} \triangleq H(s)$ is SPR

Let $\theta(t)$ be a vector of adjustable parameters.

Let $\boldsymbol{\phi}(t)$ and $z_1(t)$ be time-varying functions that can be measured.

Then, if $\theta(t)$ is adjusted as

$$\dot{\theta}(t) = -\text{sgn } kz_1(t)\boldsymbol{\phi}(t) \quad (17)$$

\Rightarrow the **equilibrium state** ($\mathbf{x} = 0, \theta = 0$) is **uniformly stable^d in the large^e**.

^aThis refers to the error dynamics, **not the plant!**

^b z_1 allows change of symbol with respect to the output $y(t)$

^c $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y}, \mathbf{z} \in \mathbb{R}^1$, $\boldsymbol{\phi}, \theta \in \mathbb{R}^k$

^duniformly stable: not dependent on time nor on IC

^ein the large: IC don't matter anywhere in \mathbb{R}^n

Proof Since $H(s)$ is SPR, it follows from the KY-lemma that $\exists \mathbf{P} = \mathbf{P}^T > 0$, such that

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} = -\mathbf{Q}, \quad \mathbf{Q} = \mathbf{Q}^T > 0 \quad (14)$$

$$\mathbf{P} \mathbf{b} = \mathbf{c} \quad (15)$$

Let V be a positive definite function

$$V = \mathbf{x}^T \mathbf{P} \mathbf{x} + \frac{1}{|k|} \theta^T \theta \\ \dot{V} = \mathbf{x}^T (\mathbf{P} \mathbf{A} + \mathbf{A}^T \mathbf{P}) \mathbf{x} \\ + 2\mathbf{x}^T \mathbf{P} \mathbf{b} \theta^T \boldsymbol{\phi} - 2\theta^T y \boldsymbol{\phi} \\ = -\mathbf{x}^T \mathbf{Q} \mathbf{x} \leq 0$$

Therefore, the origin of the system (16) together with the adaptive law (17) is stable.

Discussion We now have a simple tool for finding adaptive laws with error dynamics below (10), where $M(s)$ is SPR, stabilisable and detectable, and the adaptive laws are defined as in (18)

$$\mathbf{e}(t) = \frac{1}{k^*} M(s) [\tilde{\theta}^T \boldsymbol{\phi}] \quad (10)$$

$$\dot{\theta}(t) = -\text{sgn } \varepsilon \Gamma \boldsymbol{\phi} \quad (18)$$

⁶boundary cond., means SPR'

If $M(s)$ has a relative degree of 1, it is obvious that $e(t) \rightarrow 0$ for $t \rightarrow \infty$ ⁷.

⁷Why?

5 Performance considerations

Performance criteria:

Performance	Noise
Disturbances	Robustness

- Increasing γ , we are unhappy with the **oscillations of our parameters θ** and therefore with the oscillations of $u(t)$.
- We have no clue what the adaptive closed loop will do between $t = 0$ and $t = \infty$ other than boundedness

5.1 Adaptation with a closed loop reference model

Now Deal with transient response

Idea Adaptation changes with signals

$$\dot{\theta} = -\text{sgn}(\varepsilon) \varepsilon \gamma \phi$$

where the value of ε and γ are changeable.
 \Rightarrow we can alter the transient with γ (leads to oscillations), **or we can change $\varepsilon(t)$** .

So far Open loop reference model (ORM)

$$\dot{x}_m^o(t) = a_m x_m^o(t) + k_m r(t) \quad (2)$$

Now Closed loop reference model (CRM)

$$\dot{x}_m^c(t) = a_m x_m^c(t) + k_m r(t) - l e^c(t) \quad (19)$$

ORM = CRM if $l = 0$.

"CRM is observer-like; M helps G by moving towards G and retreating to original position."
Through the movement, the reference model now has a different behaviour ($M \rightarrow M'$) and the plant P is trying to follow M' .

- γ - learning effect
 - decreasing γ helps P follow M' ,
 - but the learning becomes slower
- l - movement to P
 - increasing l helps P follow M'

5.2 Stability proof

$$\dot{x}_m^c(t) = a_m x_m^c(t) + k_m r(t) - l e^c(t) \quad (19)$$

$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t) \quad (1)$$

Input is

$$u = \begin{bmatrix} a(t) & k(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix} = \theta^T(t) \phi(t)$$

$$\dot{x}_p(t) = a_m x_p(t) + k_m r(t) + k_p \theta^T(t) \phi(t)$$

Tracking error

$$\begin{aligned} \dot{e}^c(t) &= \dot{x}_p(t) - \dot{x}_m^c(t) \\ &= (a_m + l) e^c(t) + k_p \theta^T(t) \phi(t) \end{aligned}$$

Lyapunov-like function

$$\begin{aligned} V(e^c, \tilde{\theta}) &= \frac{1}{2} (e^c)^2 + \frac{1}{2} \Gamma^{-1} |k_p| \tilde{\theta}^T(t) \tilde{\theta}(t) \\ \dot{V} &= e^c \dot{e}^c + \Gamma^{-1} |k_p| \tilde{\theta}^T \dot{\tilde{\theta}} = \dots \\ &= (a_m + l) (e^c)^2 \\ &\quad + \underbrace{e^c k_p \tilde{\theta} \phi + \Gamma^{-1} |k_p| \tilde{\theta}^T \dot{\tilde{\theta}}}_{\stackrel{!}{=} 0} \\ \dot{V} &= (a_m + l) (e^c)^2 \leq 0, \quad l < 0 \end{aligned}$$

Adaptive law

$$\dot{\theta} = -\Gamma \text{sgn } k_p e^c \phi$$

Proof as before. $e^c(t) \rightarrow 0$ for $t \rightarrow \infty$ ⁸.

Questions

- How do we show increased performance? (using $\|e^c(t)\|_{\mathcal{L}_2}$ as a performance criterion)
- How do we show that the oscillations decrease?

5.3 Analysing transient performance

Check the performance criterion \mathcal{L}_2 -norm of e^c

$$\begin{aligned} \int_0^\infty \dot{V}(e^c, \theta) d\tau &= V(\infty) - V(0) \\ -|a_m + l| \int_0^\infty e^{c^2} d\tau &= V(\infty) - V(0) \\ V(0) &= \underbrace{V(\infty)}_{\geq 0} + |a_m + l| \cdot \|e^c\|_2^2 \\ V(0) &\geq |a_m + l| \cdot \|e^c\|_2^2 \\ \|e^c\|_2 &\leq \sqrt{\frac{V(0)}{|a_m + l|}} \end{aligned}$$

$$\|e^c\|_2^2 \leq \frac{1}{2} \frac{(e^c(0))^2 + \frac{|k_p|}{\gamma} \theta^T(0) \theta(0)}{|a_m + l|} \quad (20)$$

⁸We assume here that $e^c(t) \rightarrow 0$ follows from $e^o(t) \rightarrow 0$. In actuality, though, $e^o(t)$ can't be proven for special functions. However, these cases are usually not relevant to engineering/industry. Therefore, **strictly speaking**, we can't actually assume that $e^c(t) \rightarrow 0$

Discussion

- Increasing γ reduces $\|e^c\|_{\mathcal{L}_2}$ depending on the parameter errors $\tilde{\theta}$
- Increasing the value of l reduces $\|e^c\|_{\mathcal{L}_2}$ also from $e^c(o)$

5.4 Analysing the signal oscillations

\mathcal{L}_2 -norm of \dot{k}

$$\begin{aligned}\dot{k} &= -\gamma \operatorname{sgn} k_p e^c r(t) \\ \int_0^\infty |\dot{k}|^2 d\tau &= \gamma^2 \int_0^\infty (e^c)^2 r^2 d\tau \\ (r(t) &\leq \|r\|_{\mathcal{L}_\infty}) \\ &\leq \gamma^2 \|r\|_{\mathcal{L}_\infty}^2 \int_0^\infty (e^c)^2 d\tau \\ &\leq \gamma^2 \|r\|_{\mathcal{L}_\infty}^2 \|e^c\|_{\mathcal{L}_2}^2\end{aligned}$$

$$\|\dot{k}\|_2 \leq \gamma \|r\|_{\mathcal{L}_\infty} \sqrt{\frac{V(o)}{|a_m + l|}}$$

Increasing γ or reducing l causes \dot{k} to decrease in magnitude.

\mathcal{L}_2 -norm of $\dot{\theta}$

$$\begin{aligned}\dot{\theta} &= -\gamma \operatorname{sgn} k_p e^c x_p(t) \\ &= -\gamma \operatorname{sgn} k_p e^c (e^c + x_m(t)) \\ |\dot{\theta}|^2 &= \gamma^2 (e^c)^2 (e^c + x_m(t))^2 \\ &\quad (a+b)^2 \leq 2a^2 + 2b^2 \\ &\leq 2\gamma^2 (e^c)^2 [(e^c)^2 (e^c)^2 + x_m^2] \\ \int_0^\infty |\dot{\theta}|^2 d\tau &\leq 2\gamma^2 \left[\int_0^\infty (e^c)^2 (e^c)^2 d\tau + \int_0^\infty (e^c)^2 x_m^2 d\tau \right] \\ &\vdots\end{aligned}$$

$$\begin{aligned}&\int_0^\infty |\dot{\theta}|^2 d\tau \\ &\leq 2\gamma^2 \frac{V(o)}{|a_m + l|} \left[V(o) \left(2 + \frac{l^2}{|a_m| \cdot |a_m + l|} \right) \right. \\ &\quad \left. + 2\|\dot{x}_m\|_{\mathcal{L}_\infty}^2 \right]\end{aligned}$$

Discussion

- l reduces contribution of the ORM $\|\dot{x}_m(t)\|_{\mathcal{L}_\infty}$ on $\|\dot{\theta}\|_{\mathcal{L}_2}$
- l has no clear effect on the contributions of $V(o)$.
- γ always increases the oscillations, s. $\|\dot{\theta}\|_{\mathcal{L}_2}$

6 Output feedback adaptive control

Let $M(s)$ be a linear time-invariant, asymptotically stable reference model, with I/O $\{r(\cdot), y_m(\cdot)\}$. r is uniform, bounded, piecewise continuous function of time. The plant G is defined such that

$$G(s) = k_p \frac{Z_p(s)}{R_p(s)}$$

- Z_p, R_p are monic⁹ polynomials of orders $n-1$ and n
- $G(s)$ is controllable, observable $\Leftrightarrow Z_p, R_p$ are coprime¹⁰ polynomials

We know

- (i) sign of high-frequency gain k_p
- (ii) upper bound n on the order of $G(s)$
- (iii) relative degree n^*^{11}
- (iv) zeroes of $Z_p(s)$ lie in \mathbb{C}^-
 \Rightarrow min. phase¹², no inverse response

Minimal phase systems “In minimal phase systems, we can predict the phase φ given the magnitude $|G| \Leftarrow$ only if G stable with no time delay”.

e.g. a plane, or a forklift, are examples of non minimal phase systems (they have different inverse responses depending on where you measure, s. Figure 5)

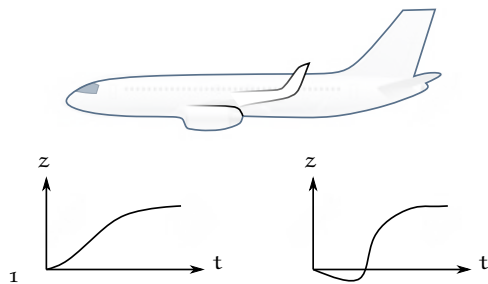


Figure 5: An ascending plane.

Solution in three parts

- (i) k_p unknown
- (ii) Z_p unknown
- (iii) R_p unknown

In each case

- a) “Matching conditions”

Show that \exists parameters $\theta^* = \text{const.}$, such that the closed loop behaves exactly as the reference

- b) Error dynamics Derive error model as

$$e = \frac{1}{k^*} M(s) [\theta^T \phi]$$

- c) Use KY-lemma
- d) Show that $e \rightarrow 0$ for $t \rightarrow \infty$ (Barbalat's lemma)

6.1 k_p unknown

Matching conditions

⁹monic: coefficient of the highest order is 1, e.g. $s^2 + 2s + 1$

¹⁰coprime: no cancellations, i.e. no common roots

¹¹in this course, we only handle cases where $n^* = 1$

¹²there is no standard definition for minimal phase