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Mehrotra-type predictor–corrector algorithms for sufficient linear complementarity problem ☆

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ABSTRACT

Two Mehrotra-type predictor–corrector algorithms are proposed for solving sufficient linear complementarity problems. Both algorithms are independent on the handicap χ of the problems. The first version of the Mehrotra-type algorithm is a generalization of the safeguard based Mehrotra-type algorithm for linear programming, that was proposed by Salahi et al. [M. Salahi, J. Peng, T. Terlaky, On Mehrotra-type predictor–corrector algorithms, SIAM J. Optim. 18 (2007) 1377–1397]. We also present a new variant of Mehrotra-type predictor–corrector algorithm using a new adaptive updating strategy of the centering parameter. We show that both algorithms enjoy $\mathcal{O}(n(1+\chi)\log((x^0)^T s^0/\epsilon))$ iteration complexity. Some numerical results are reported as well.

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1. Introduction

Since Karmarkar's landmark paper [8], interior point methods (IPMs) have become one of the most active research areas. Many strong theoretical results are obtained by using primal–dual IPMs, and IPMs have been successfully implemented in software packages for solving linear programming (LP), quadratic programming (QP), semidefinite programming (SDP), and many other problems [3,10,13,17,23]. For more detailed account of the primal–dual IPMs and their implementation the reader can see the monograph of Wright [22]. Mizuno–Todd–Ye (MTY) method was proposed in [15] and it has the best iteration complexity $\mathcal{O}(\sqrt{n}L)$ and quadratic convergence [24]. MTY method was generalized to linear complementarity problem (LCP) in [6] and the resulting algorithm was proved to have $\mathcal{O}(\sqrt{n}L)$ iteration complexity under general conditions, and superlinear convergence under the assumption that the LCP has a strictly complementary solution and the iteration sequence converges. We mention that the former assumption is not restrictive [16] and the latter assumption always holds [1]. In 1992 Mehrotra [12] proposed a primal–dual infeasible-interior-point algorithm for LPs. By solving linear systems with the same coefficient matrix, it obtains three directions: predictor, corrector and centering directions. The new iterate is chosen by moving along the combination of these directions, from the current iterate. Unlike the MTY predictor–corrector algorithm, Mehrotra's algorithm does not require an extra matrix factorization for the corrector step. Mehrotra's algorithmic framework is widely regarded as the most efficient in practice and has been implemented in the highly successful interior-point code OB1 [11]. The practical importance and lack of theoretical analysis of Mehrotra's algorithm motivated many experts to study Mehrotra-type algorithm [19,20,27,28]. Zhang and Zhang [27] studied the theoretical convergence properties of Mehrotra's algorithmic framework on a horizontal linear complementarity problem (HLCP) and established

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polynomial complexity bounds for two variants of the Mehrotra-type algorithms. By a numerical example, Salahi et al. [20] showed that Mehrotra's algorithm might make very small steps in order to keep the iterate in a certain neighborhood of the central path, which implies that the algorithm needs to take many iterations to convergence. Motivated by this observation they incorporated a safeguard in their algorithmic scheme that gives a lower bound for the step size at each iteration. They obtained the iteration complexity $\mathcal{O}(nL)$. By employing a new primal–dual corrector algorithm, Cartis [2] investigated the impact that corrector directions may have on the convergence behaviour of predictor–corrector methods. She presented examples that showed that the Mehrotra's algorithm may fail to converge to a solution of the LP, in both exact and finite arithmetic, regardless of the choice of stepsize that is employed. The cause of this bad behaviour is that the correctors exert too much influence on the direction in which the iterates move. By postponing the choice of the barrier parameter, Salahi and Terlaky [19] proposed a Mehrotra-type predictor–corrector method for LPs, which does not require any safeguard, and obtained the iteration complexity $\mathcal{O}(nL)$.

LCP is a fundamental problem in mathematical programming, and many different optimization problems such as LPs and QPs can be reduced to LCPs. Also many equilibrium problems from game theory or simulation of mechanical systems with contacts can be formulated as LCPs. By introducing the class of $P_*(\kappa)$-matrices, Kojima et al. [9] provided a unified framework for studying IPMs. Miao [14] extended the MTY predictor–corrector method for $P_*(\kappa)$-LCPs. The algorithm uses the l_2 neighborhood of the central path and has $\mathcal{O}((1+\kappa)\sqrt{n}L)$ iteration complexity, and is quadratically convergent for nondegenerate problems. However, the constant κ is explicitly used in the construction of the algorithm. It is well known that this constant is very difficult to estimate for many $P_*(\kappa)$-LCPs, which means that the algorithm application has heavy limitation. Gurtuna et al. [5] presented a corrector-predictor method for solving $P_*(\kappa)$-LCPs for which a sufficiently centered feasible starting point is available. The method does not depend on the handicap κ of the problem and has $\mathcal{O}((1+\kappa)\sqrt{n}L)$ iteration complexity and is quadratically convergent for nondegenerate problems. Zhang and Lü [26] proposed a second-order Mehrotra's predictor–corrector algorithm for $P_*(\kappa)$-LCPs, which works in the 'wide' neighborhood and has $\mathcal{O}((4\kappa+3)\sqrt{14\kappa+5}nL)$ iteration complexity. For more applications and fundamental theoretical properties of LCPs, we refer to Cottle et al. [4] and Kojima et al. [9].

Motivated by the works mentioned above, we propose two Mehrotra-type predictor–corrector algorithms for sufficient HLCP in the $\mathcal{N}_\infty^-(\gamma)$ neighborhood of the central path, which do not depend on the handicap of the problem. The first one is an extension of Salahi et al. in [20]. The second one uses a new adaptive updating technique of the centering parameter. We show that both algorithms enjoy $\mathcal{O}(n(1+\chi)\log((x^0)^T s^0/\epsilon))$ iteration complexity, where χ is the handicap of the problem.

Throughout the paper $\|x\|$ denotes the 2-norm of vector x. We use $\mathcal{R}^n_+(\mathcal{R}^n_{++})$ to denote the nonnegative (positive) orthant in $\mathcal{R}^n$ and denote by $\mathcal{I}$ the index set $\{1,2,\ldots,n\}$. For any two vectors x and s, xs denotes the componentwise product of the two vectors. We denote by e the vector with all components equal to one and by I the identity matrix.

The paper is organized as follows. In Section 2, we describe the sufficient HLCPs. The Mehrotra-type predictor–corrector algorithm and convergence analysis are showed in Section 3. In Section 4, we propose a new algorithm for HLCPs and establish the complexity. In Section 5, some illustrative numerical results are reported. The last section contains concluding remarks.

2. The sufficient horizontal linear complementarity problem

Given two matrices $Q, R \in \mathcal{R}^{n\times n}$, and a vector $b \in \mathcal{R}^n$, the HLCP is to find a vector pair $(x,s) \in \mathcal{R}^n_+ \times \mathcal{R}^n_+$ such that

$$\begin{aligned} &xs = 0,\\ &Qx + Rs = b,\\ &(x,s) \geqslant 0. \end{aligned} \tag{1}$$

This problem is sufficiently general to include the standard LCP, LP and QP problem [27].

The monotone LCP is the special case where $R = -I$ and Q is positive semidefinite. Let $\kappa \geqslant 0$ be a given constant. We say that (1) is a $P_*(\kappa)$-HLCP if $Qu + Rv = 0$ implies

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+} u_i v_i + \sum_{i\in\mathcal{I}_-} u_i v_i \geqslant 0 \tag{2}$$

for any $u, v \in \mathcal{R}^n$, where $\mathcal{I}_+ = \{i\colon\ u_i v_i > 0\}$ and $\mathcal{I}_- = \{i\colon\ u_i v_i \leqslant 0\}$. If the inequality (2) is satisfied we refer to (Q,R) as a $P_*(\kappa)$-pair and we write $(Q,R) \in P_*(\kappa)$.

In case $R = -I$, $(Q,-I)$ is a $P_*(\kappa)$-pair if and only if Q is a $P_*(\kappa)$-matrix in the sense that:

$$(1+4\kappa)\sum_{i\in\hat{\mathcal{I}}_+} x_i(Qx)_i + \sum_{i\in\hat{\mathcal{I}}_-} x_i(Qx)_i \geqslant 0, \quad \forall x \in \mathcal{R}^n,$$

where $\hat{\mathcal{I}}_+ = \{i\colon\ x_i(Qx)_i > 0\}$, $\hat{\mathcal{I}}_- = \{i\colon\ x_i(Qx)_i \leqslant 0\}$. If (Q,R) belongs to the class

$$P_* = \bigcup_{\kappa\geqslant 0} P_*(\kappa),$$

we say that (Q,R) is a P_*-pair and (1) is a P_*-HLCP.

It is well known that a matrix Q is called column sufficient if

$$x_i(Qx)_i \leqslant 0, \quad i=1,2,\ldots,n \quad \Rightarrow \quad x_i(Qx)_i = 0, \quad i=1,2,\ldots,n$$

for any $x \in \mathcal{R}^n$ and row sufficient if Q^T is column sufficient. A matrix that is both row sufficient and column sufficient is called a sufficient matrix. Väliaho's result [21] states that a matrix is sufficient if and only if it is a $P_*(\kappa)$-matrix for some $\kappa \geqslant 0$. Then, a P_*-HLCP will be called a sufficient HLCP and a P_*-pair will be called a sufficient pair. The handicap of a sufficient pair (Q, R) is defined as

$$\chi(Q,R) := \min\{\kappa:\ \kappa \geqslant 0, (Q,R) \in P_*(\kappa)\}. \tag{3}$$

We will use the notation: $\chi := \chi(Q,R)$.

We denote the feasible set of (1) by

$$\mathcal{F} = \{(x,s) \in \mathcal{R}^n_+ \times \mathcal{R}^n_+:\ Qx + Rs = b\}$$

and its solution set by

$$\mathcal{F}^* = \{(x^*,s^*) \in \mathcal{R}^n_+ \times \mathcal{R}^n_+:\ (x^*)^T s^* = 0,\ Qx^* + Rs^* = b\}.$$

For any $\epsilon > 0$, we define the set of *ϵ-approximate solution* of (1) as

$$\mathcal{F}^*_\epsilon = \{(x^*,s^*) \in \mathcal{R}^n_+ \times \mathcal{R}^n_+:\ (x^*)^T s^* \leqslant \epsilon,\ Qx^* + Rs^* = b\}.$$

The relative interior of $\mathcal{F}$

$$\mathcal{F}^0 = \{(x,s) \in \mathcal{R}^n_{++} \times \mathcal{R}^n_{++}:\ Qx + Rs = b\}$$

will be called the set of strictly feasible points. It is known [9] that if $\mathcal{F}^0$ is nonempty and (Q,R) is $P_*(\kappa)$-pair, then the nonlinear system

$$\begin{aligned} &xs = \mu e, \\ &Qx + Rs = b \end{aligned}$$

has a unique solution for any $\mu > 0$. The set of all such solutions defines the central path $\mathcal{C}$ of the $P_*(\kappa)$-HLCP, i.e.,

$$\mathcal{C} = \{z \in \mathcal{R}^n_{++} \times \mathcal{R}^n_{++}:\ xs = \mu e,\ Qx + Rs = b,\ \mu > 0\}.$$

The accumulation point of the central path exists and every accumulation point is a solution of the $P_*(\kappa)$-HLCP [9]. In this paper we assume that $\mathcal{F}^0 \neq \emptyset$. Our algorithm will restrict the iterates to the following neighborhood, called the minus-infinity neighborhood, of the central path:

$$\mathcal{N}^-_\infty(\gamma) = \{(x,s) \in \mathcal{F}^0:\ x_i s_i \geqslant \gamma\mu_g,\ \forall i = 1,2,\ldots,n\},$$

where $\mu_g = \frac{x^T s}{n}$ and $\gamma \in (0,1)$.

3. The Mehrotra-type predictor–corrector algorithm and convergence analysis

3.1. The Mehrotra-type predictor–corrector algorithm

In what follows we describe briefly our version of Mehrotra-type method. In the predictor step it computes the affine scaling search direction $(\Delta x^a, \Delta s^a)$, which is the solution of the following linear system:

$$\begin{aligned} &Q\Delta x^a + R\Delta s^a = 0, \\ &s\Delta x^a + x\Delta s^a = -xs. \end{aligned} \tag{4}$$

Then the maximum feasible step size in this direction is computed, i.e., the largest $\alpha_a \in [0,1]$ such that

$$(x(\alpha_a), s(\alpha_a)) := (x + \alpha_a\Delta x^a, s + \alpha_a\Delta s^a) \geqslant 0. \tag{5}$$

However, the algorithm does not take such a step right away. In the remainder of the paper, we let

$$\mathcal{I}_+ = \{i:\ \Delta x^a_i \Delta s^a_i > 0\}, \qquad \mathcal{I}_- = \{i:\ \Delta x^a_i \Delta s^a_i \leqslant 0\}.$$

By $(Q,R) \in P_*(\kappa)$, we have

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a+\sum_{i\in\mathcal{I}_-}\Delta x_i^a\Delta s_i^a\geqslant 0. \tag{6}$$

If $\mathcal{I}_+=\emptyset$, from (6) one has $\Delta x_i^a\Delta s_i^a=0, i=1,2,\dots,n$. If $\Delta x_i^a=0$, by the second equation of (4), we have $s_i+\Delta s_i^a=0$. On the other hand, if $\Delta s_i^a=0$, we have $x_i+\Delta x_i^a=0$, which implies $\alpha_a=1$ and

$$\left(x_i+\alpha_a\Delta x_i^a\right)\left(s_i+\alpha_a\Delta s_i^a\right)=0,\quad i=1,2,\dots,n.$$

Then the algorithm terminates. Therefore, we assume $\mathcal{I}_+\neq\emptyset$ in the rest of the paper.

Let

$$k_a:=\max\left\{\frac{\sum_{i\in\mathcal{I}_-}|\Delta x_i^a\Delta s_i^a|}{\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a},1\right\}.$$

From (3), one has easily that

$$1\leqslant k_a\leqslant 1+4\chi. \tag{7}$$

Then a vector $\bar{\alpha}_a\in R^n$ is defined by k_a and α_a, whose ith component is

$$(\bar{\alpha}_a)_i=\begin{cases}\alpha_a, & i\in\mathcal{I}_+,\\ \frac{\alpha_a}{k_a}, & i\in\mathcal{I}_-.\end{cases} \tag{8}$$

Using the information from the predictor step our algorithm computes the corrector direction $(\Delta x,\Delta s)$ that is defined as follows:

$$\begin{aligned}&Q\,\Delta x+R\Delta s=0,\\ &s\Delta x+x\Delta s=\mu e-xs-\bar{\alpha}_a\left(\Delta x^a\Delta s^a\right),\end{aligned} \tag{9}$$

where μ is defined adaptively by [12]

$$\mu=\left(\frac{g_a}{g}\right)^2\frac{g_a}{n},$$

where $g_a=(x+\alpha_a\Delta x^a)^T(s+\alpha_a\Delta s^a),\ g=x^Ts$. By a simple calculation, one has

$$\mu=\tau\mu_g,$$

where

$$\tau:=\left(1-\alpha_a+\alpha_a^2\frac{(\Delta x^a)^T\Delta s^a}{x^Ts}\right)^3. \tag{10}$$

Let

$$\left(x(\alpha),s(\alpha)\right)=(x,s)+\alpha(\Delta x,\Delta s).$$

Then it computes the maximum corrector step α_c such that

$$\left(x(\alpha),s(\alpha)\right)\in\mathcal{N}_\infty^-(\gamma)\quad\text{for any }\alpha\in(0,\alpha_c]. \tag{11}$$

In the case of LPs, Mehrotra's algorithm may result in very small steps in order to keep the iterate in a certain neighborhood of the central path [20]. Therefore, Mehrotra's adaptive updating scheme of the centering parameter has to be combined with certain safeguard to get a warranted step size at each iteration. We describe the Mehrotra-type predictor–corrector algorithm in detail.

Algorithm 1.
Input: An accuracy parameter $\epsilon>0$, neighborhood parameters $\gamma\in(0,0.4]$ and initial point $(x^0,s^0)\in\mathcal{N}_\infty^-(\gamma)$. Set $k:=0$.

Step 1. If $(x^k)^Ts^k\leqslant\epsilon$, then report $(x^k,s^k)\in\mathcal{F}_\epsilon^*$ and terminate; otherwise, set $x=x^k, s=s^k$;
Step 2. Solve (4) and computer the maximum step size α_a from (5);
Step 3. If $x(\alpha_a)^Ts(\alpha_a)<\epsilon$, then report $(x,s)=(x(\alpha_a),s(\alpha_a))\in\mathcal{F}_\epsilon^*$ and terminate. Otherwise compute $\bar{\alpha}_a\in\mathcal{R}^n$ from (8);
Step 4. If $\alpha_a\geqslant 0.1$, solve (9) with $\mu=\tau\mu_g$, where τ is computed by (10), and compute the maximum step size α_c from (11);

Step 5. If $\alpha_c \geqslant \frac{\gamma}{3n}$, do: if $(\Delta x)^T \Delta s \leqslant 0$, set $\check{\alpha} = \alpha_c$, otherwise compute $\bar{\alpha}_c = \frac{(1-\tau)x^T s}{2(\Delta x)^T \Delta s}$ and set $\check{\alpha} = \min(\alpha_c, \bar{\alpha}_c)$ and go to Step 7;
Step 6. If $\alpha_a < 0.1$ or $\alpha_c < \frac{\gamma}{3n}$, solve the system (9) with $\mu = \frac{\gamma}{1-\gamma}\mu_g$ and compute the maximum step α_c from (11). If $(\Delta x)^T \Delta s \leqslant 0$, set $\check{\alpha} = \alpha_c$, otherwise compute $\hat{\alpha}_c = \frac{(1-\varsigma)x^T s}{2(\Delta x)^T \Delta s}$, where $\varsigma = \frac{\gamma}{1-\gamma}$, and set $\check{\alpha} = \min(\alpha_c, \hat{\alpha}_c)$;
Step 7. Set $x^{k+1} = x^k + \check{\alpha}\Delta x$, $s^{k+1} = s^k + \check{\alpha}\Delta s$, $k := k+1$, and return Step 1.

Remark 1. (i) For monotone LCP, we have $k_a = 1$. Then, $(\bar{\alpha}_a)_i = \alpha_a$, $\forall i = 1, 2, \ldots, n$. Therefore, our algorithm is an extension of that in [20].
(ii) By (14), we have:

$$\mu_g(\alpha) = \frac{x(\alpha)^T s(\alpha)}{n} \leqslant \left(1 - (1-\sigma)\alpha + \frac{\Delta x^T \Delta s}{x^T s}\alpha^2\right)\mu_g, \quad \forall \sigma \in (0, 1).$$

To minimize complementarity gap as much as possible at each iteration, in Steps 5 and 6, we compute the critical point of the quadratic $f(\alpha) = 1 - (1-\sigma)\alpha + \frac{\Delta x^T \Delta s}{x^T s}\alpha^2$, $\Delta x^T \Delta s > 0$.

3.2. Polynomial complexity of Algorithm 1

In this section, we mainly characterize the polynomial complexities of Algorithm 1. We list some lemmas and theorems, which will play an important role in the complexity analysis.

Lemma 1. *(See [5].) If HLCP is $P_*(\kappa)$, then for any $(x, s) \in \mathcal{R}^n_+ \times \mathcal{R}^n_+$ and any $a \in \mathcal{R}^n$ the linear system*

$$Qu + Rv = 0,$$
$$su + xv = a$$

has a unique solution (u, v) for which the following estimates hold

$$\|uv\| \leqslant \left(2^{-\frac{3}{2}} + \chi\right)\|\tilde{a}\|^2,$$
$$u^T v \leqslant \frac{1}{4}\|\tilde{a}\|^2,$$

where $\tilde{a} = (xs)^{-1/2}a$.

Lemma 2. *Suppose that $(x, s) \in \mathcal{N}^-_\infty(\gamma)$ and let $(\Delta x^a, \Delta s^a)$ be the solution of (4), then*

1. $\Delta x^a_i \Delta s^a_i \leqslant \frac{x_i s_i}{4}$, $\forall i \in \mathcal{I}_+$,
2. $\frac{1}{k_a}\sum_{i\in\mathcal{I}_-} |\Delta x^a_i \Delta s^a_i| \leqslant \sum_{i\in\mathcal{I}_+} \Delta x^a_i \Delta s^a_i \leqslant \frac{x^T s}{4} = \frac{1}{4}n\mu_g$.

Proof. The first part follows from Lemma A.1 in [18], and the second from the definition of k_a. □

Corollary 1. *Suppose that $(x, s) \in \mathcal{N}^-_\infty(\gamma)$ and let $(\Delta x^a, \Delta s^a)$ be the solution of (4), then*

$$\sum_{i\in\mathcal{I}} (\bar{\alpha}_a)_i \Delta x^a_i \Delta s^a_i = \frac{\alpha_a}{k_a}\sum_{i\in\mathcal{I}_-} \Delta x^a_i \Delta s^a_i + \alpha_a \sum_{i\in\mathcal{I}_+} \Delta x^a_i \Delta s^a_i \geqslant 0.$$

The following lemma shows that there exists a guaranteed positive step size in the predictor step of the algorithm.

Lemma 3. *Suppose that $(x, s) \in \mathcal{N}^-_\infty(\gamma)$ and let $(\Delta x^a, \Delta s^a)$ be the solution of (4). Then the maximum feasible step size α_a defined by (5), satisfies*

$$\alpha_a \geqslant \frac{2\sqrt{\gamma^2 + (1+4\chi)\gamma n} - 2\gamma}{(1+4\chi)n}.$$

Proof. Case 1: $\mathcal{I}_- = \emptyset$. We have $i \in \mathcal{I}_+$, $i = 1, 2, \ldots, n$. From (4), we have $\Delta x^a_i < 0, \Delta s^a_i < 0$, and $\frac{\Delta x^a_i}{x_i} + \frac{\Delta s^a_i}{s_i} = -1$. So $\alpha_a = \min\{|\frac{x_i}{\Delta x^a_i}|, |\frac{s_i}{\Delta s^a_i}|\} \geqslant 1$. Then, we have $\alpha_a = 1$.
Case 2: $\mathcal{I}_- \neq \emptyset$. For $i \in \mathcal{I}_-$, since $(x, s) \in \mathcal{N}^-_\infty(\gamma)$, using (7) and Lemma 2, we have that

$$\begin{aligned}\big(x_i+\alpha\Delta x_i^a\big)\big(s_i+\alpha\Delta s_i^a\big)&=(1-\alpha)x_is_i+\alpha^2\Delta x_i^a\Delta s_i^a\geqslant\gamma\left(1-\alpha-\frac{k_an}{4\gamma}\alpha^2\right)\mu_g\\&\geqslant\gamma\left(1-\alpha-\frac{(1+4\chi)n}{4\gamma}\alpha^2\right)\mu_g.\end{aligned}\tag{12}$$

By $i\in\mathcal{I}_-$, we have $\triangle x_i^a\Delta s_i^a\leqslant 0$. Without loss of generality, we may assume that $\triangle x_i^a\geqslant 0$. Then $x_i+\alpha\triangle x_i^a>0$ for any $\alpha>0$. Hence $(x_i+\alpha\Delta x_i^a)(s_i+\alpha\Delta s_i^a)\geqslant 0$ implies $s_i+\alpha\Delta s_i^a\geqslant 0$. By (12), the inequality $(x_i+\alpha\Delta x_i^a)(s_i+\alpha\Delta s_i^a)\geqslant 0$ holds if

$$\gamma\left(1-\alpha-\frac{(1+4\chi)n}{4\gamma}\alpha^2\right)\mu_g\geqslant 0,$$

which is true if

$$\alpha\in\left[\frac{-2\sqrt{\gamma^2+(1+4\chi)\gamma n}-2\gamma}{(1+4\chi)n},\frac{2\sqrt{\gamma^2+(1+4\chi)\gamma n}-2\gamma}{(1+4\chi)n}\right].$$

Therefore we have $\alpha_a\geqslant\frac{2\sqrt{\gamma^2+(1+4\chi)n\gamma}-2\gamma}{(1+4\chi)n}$. $\square$

From (5) we have the following lemma.

Lemma 4. *Suppose that $(x,s)\in\mathcal{N}_\infty^-(\gamma)$ and let $(\Delta x^a,\Delta s^a)$ be the solution of (4), then*

$$\alpha_a^2\big|\Delta x_i^a\Delta s_i^a\big|\leqslant(1-\alpha_a)x_is_i,\quad\forall i\in\mathcal{I}_-.$$

In what follows we establish the estimates of the upper bounds on $\|\Delta x\Delta s\|$ and $(\Delta x)^T\Delta s$, which will be used in the next theorem.

Lemma 5. *Suppose that $(x,s)\in\mathcal{N}_\infty^-(\gamma)$ and let $(\Delta x,\Delta s)$ be the solution of (9). Then we have*

$$\|\Delta x\Delta s\|\leqslant\big(2^{\frac{-3}{2}}+\chi\big)\left(\frac{1}{\gamma}\left(\frac{\mu}{\mu_g}\right)^2-\left(2-\frac{\alpha_a}{2\gamma}\right)\frac{\mu}{\mu_g}+\frac{20+4\alpha_a+\alpha_a^2}{16}\right)n\mu_g,$$

$$(\Delta x)^T\Delta s\leqslant\frac{1}{4}\left(\frac{1}{\gamma}\left(\frac{\mu}{\mu_g}\right)^2-\left(2-\frac{\alpha_a}{2\gamma}\right)\frac{\mu}{\mu_g}+\frac{20+4\alpha_a+\alpha_a^2}{16}\right)n\mu_g.$$

Proof. By Lemma 1, we have that

$$\begin{aligned}\frac{\|\Delta x\Delta s\|}{(2^{-\frac{3}{2}}+\chi)}&\leqslant\big\|(xs)^{-\frac{1}{2}}\big(\mu e-xs-\bar{\alpha}_a\big(\Delta x^a\Delta s^a\big)\big)\big\|^2\\&=\sum_{i\in\mathcal{I}}\left(\frac{\mu}{\sqrt{x_is_i}}-\sqrt{x_is_i}-(\bar{\alpha}_a)_i\frac{\Delta x_i^a\Delta s_i^a}{\sqrt{x_is_i}}\right)^2\\&\leqslant\sum_{i\in\mathcal{I}}\frac{\mu^2}{x_is_i}+n\mu_g+\alpha_a^2\sum_{i\in\mathcal{I}_+}\frac{(\Delta x_i^a\Delta s_i^a)^2}{x_is_i}+\frac{\alpha_a^2}{k_a^2}\sum_{i\in\mathcal{I}_-}\frac{(\Delta x_i^a\Delta s_i^a)^2}{x_is_i}\\&\quad-2n\mu-\frac{2\mu\alpha_a}{k_a}\sum_{i\in\mathcal{I}_-}\frac{\Delta x_i^a\Delta s_i^a}{x_is_i}+2\alpha_a\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a\\&\leqslant\sum_{i\in\mathcal{I}}\frac{\mu^2}{x_is_i}+n\mu_g+\frac{\alpha_a^2}{16}\sum_{i\in\mathcal{I}_+}x_is_i+\sum_{i\in\mathcal{I}_-}\frac{\alpha_a^2|\Delta x_i^a\Delta s_i^a|}{x_is_i}\frac{|\Delta x_i^a\Delta s_i^a|}{k_a^2}\\&\quad-2n\mu+\frac{2\mu\alpha_a}{k_a}\sum_{i\in\mathcal{I}_-}\frac{|\Delta x_i^a\Delta s_i^a|}{x_is_i}+2\alpha_a\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a\\&\leqslant\frac{n\mu^2}{\gamma\mu_g}+n\mu_g+\frac{\alpha_a^2}{16}n\mu_g+(1-\alpha_a)\sum_{i\in\mathcal{I}_-}\frac{|\Delta x_i^a\Delta s_i^a|}{k_a^2}\\&\quad-2n\mu+\frac{2\mu\alpha_a}{\gamma\mu_g}\frac{1}{k_a}\sum_{i\in\mathcal{I}_-}\big|\Delta x_i^a\Delta s_i^a\big|+2\alpha_a\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a\end{aligned}$$

$$\leqslant \left[\frac{1}{\gamma}\left(\frac{\mu}{\mu_g}\right)^2 + 1 + \frac{\alpha_a^2}{16} + \frac{1-\alpha_a}{4} - 2\frac{\mu}{\mu_g} + \frac{\alpha_a \mu}{2\gamma \mu_g} + \frac{\alpha_a}{2}\right] n\mu_g$$
$$= \left[\frac{1}{\gamma}\left(\frac{\mu}{\mu_g}\right)^2 - \left(2 - \frac{\alpha_a}{2\gamma}\right)\frac{\mu}{\mu_g} + \frac{20 + 4\alpha_a + \alpha_a^2}{16}\right] n\mu_g,$$

where the second inequality follows from the definition of $\mathcal{I}_+$ and $\mathcal{I}_-$. The next three inequalities follow from Lemmas 2, 4 and $(x, s) \in \mathcal{N}_\infty^-(\gamma)$. The argument for the second inequality is similar. □

The following theorem gives the explicit upper bounds on $\|\Delta x \Delta s\|$ and $(\Delta x)^T \Delta s$ for a specific μ.

Theorem 1. *Suppose that $(x, s) \in \mathcal{N}_\infty^-(\gamma)$ and let $(\Delta x, \Delta s)$ be the solution of (9) with $\mu = \frac{\gamma}{1-\gamma}\mu_g$, where $\gamma \in (0, 0.4]$, then*

$$\|\Delta x \Delta s\| \leqslant \frac{73}{16}\left(2^{-\frac{3}{2}} + \chi\right) n\mu_g,$$
$$(\Delta x)^T \Delta s \leqslant \frac{73}{64} n\mu_g.$$

Proof. By Lemma 5, we have

$$\|\Delta x \Delta s\| \leqslant \left(2^{-\frac{3}{2}} + \chi\right)\left(\frac{1}{\gamma}\left(\frac{\gamma}{1-\gamma}\right)^2 - \left(2 - \frac{\alpha_a}{2\gamma}\right)\frac{\gamma}{1-\gamma} + \frac{20 + 4\alpha_a + \alpha_a^2}{16}\right) n\mu_g$$
$$\leqslant \left(2^{-\frac{3}{2}} + \chi\right)\left(\frac{1}{1-\gamma} + \frac{1}{2(1-\gamma)} + \frac{25}{16}\right) n\mu_g$$
$$\leqslant \left(2^{-\frac{3}{2}} + \chi\right)\left(2 + 1 + \frac{25}{16}\right) n\mu_g$$
$$\leqslant \frac{73}{16}\left(2^{-\frac{3}{2}} + \chi\right) n\mu_g.$$

The argument for the second part is similar. □

The following theorem estimates the lower bound for the maximum step size α_c for a specific choice of μ.

Theorem 2. *Suppose that $(x, s) \in \mathcal{N}_\infty^-(\gamma)$, where $\gamma \in (0, 0.4]$. Let $(\Delta x, \Delta s)$ be the solution of (9) with $\mu = \gamma\mu_g/(1-\gamma)$. Then the maximum step size α_c, such that $(x(\alpha), s(\alpha)) \in \mathcal{N}_\infty^-(\gamma)$ for any $\alpha \in (0, \alpha_c]$ satisfies*

$$\alpha_c \geqslant \frac{\gamma}{(3 + 7\chi)n}.$$

Proof. The goal is to find the maximum α for which $x_i(\alpha)s_i(\alpha) \geqslant \gamma\mu_g(\alpha)$ holds, where $\mu_g(\alpha) = x(\alpha)^T s(\alpha)/n$. If the previous inequality is true for $\alpha > \frac{1}{1+t}$, where

$$t := \max_{i \in \mathcal{I}_+}\left\{\frac{\Delta x_i^a \Delta s_i^a}{x_i s_i}\right\} \tag{13}$$

and $t \leqslant \frac{1}{4}$ by Lemma 2, then we obtain the result. For $\alpha \leqslant \frac{1}{1+t}$, we have

$$x_i(\alpha)s_i(\alpha) = x_i s_i + \alpha(x_i \Delta s_i + s_i \Delta x_i) + \alpha^2 \Delta x_i \Delta s_i$$
$$= (1-\alpha)x_i s_i + \alpha\left(\mu - (\bar{\alpha}_a)_i \Delta x_i^a \Delta s_i^a\right) + \alpha^2 \Delta x_i \Delta s_i$$
$$\geqslant (1-\alpha)x_i s_i + \alpha\mu - \alpha(\bar{\alpha}_a)_i t x_i s_i + \alpha^2 \Delta x_i \Delta s_i$$
$$\geqslant (1-\alpha)x_i s_i + \alpha\mu - \alpha\alpha_a t x_i s_i + \alpha^2 \Delta x_i \Delta s_i$$
$$= \left(1 - \alpha(1 + \alpha_a t)\right)x_i s_i + \alpha\mu + \alpha^2 \Delta x_i \Delta s_i$$
$$\geqslant \gamma\left(1 - \alpha(1 + \alpha_a t)\alpha\right)\mu_g + \alpha\mu - \alpha^2\left(\frac{73}{16}\left(2^{-\frac{3}{2}} + \chi\right) n\mu_g\right),$$

where the first inequality follows from (13), and the last from Theorem 1.

Since

$$\begin{aligned} x(\alpha)^T s(\alpha) &= x^T s + \alpha\left(x^T \Delta s + s^T \Delta x\right) + \alpha^2 (\Delta x)^T \Delta s \\ &= (1-\alpha)x^T s + \alpha\left(n\mu - \sum_{i \in I} (\bar{\alpha}_a)_i \Delta x_i^a \Delta s_i^a\right) + \alpha^2 (\Delta x)^T \Delta s \\ &\leqslant (1-\alpha)x^T s + \alpha n\mu + \alpha^2 (\Delta x)^T \Delta s, \end{aligned}$$

where the inequality follows from Corollary 1. Therefore, by Theorem 1 we have

$$\begin{aligned} \mu_g(\alpha) &= x(\alpha)^T s(\alpha)/n \\ &\leqslant \frac{1}{n}\left((1-\alpha)x^T s + \alpha n\mu + \alpha^2 (\Delta x)^T \Delta s\right) \\ &= \left(1 - \left(1 - \frac{\mu}{\mu_g}\right)\alpha + \frac{(\Delta x)^T \Delta s}{x^T s}\alpha^2\right)\mu_g \\ &\leqslant \left(1 - \left(1 - \frac{\mu}{\mu_g}\right)\alpha + \frac{73}{64}\alpha^2\right)\mu_g. \end{aligned} \tag{14}$$

Hence $(x(\alpha), s(\alpha)) \in \mathcal{N}_\infty^-(\gamma)$ holds if

$$\gamma\left(1 - (1+\alpha_a t)\alpha\right)\mu_g + \alpha\mu - \alpha^2\left(\frac{73}{16}\left(2^{-\frac{3}{2}} + \chi\right)n\mu_g\right) \geqslant \gamma\left(1 - \left(1 - \frac{\mu}{\mu_g}\right)\alpha + \frac{73}{64}\alpha^2\right)\mu_g,$$

which holds if

$$\alpha \leqslant \frac{(1-\gamma)\frac{\mu}{\mu_g} - \gamma\alpha_a t}{\frac{73}{64}\left(\gamma + 4\left(2^{-\frac{3}{2}} + \chi\right)n\right)}.$$

Since

$$\begin{aligned} \frac{(1-\gamma)\frac{\mu}{\mu_g} - \gamma\alpha_a t}{\frac{73}{64}\left(\gamma + 4\left(2^{-\frac{3}{2}} + \chi\right)n\right)} &= \frac{(1-\alpha_a t)\gamma}{\frac{73}{64}\left(\gamma + 4\left(2^{-\frac{3}{2}} + \chi\right)n\right)} \geqslant \frac{3\gamma}{\frac{73}{16}\left(\gamma + 4\left(2^{-\frac{3}{2}} + \chi\right)n\right)} \\ &\geqslant \frac{\gamma}{\frac{73}{96} + \frac{73}{12}\left(2^{-\frac{3}{2}} + \chi\right)n} \geqslant \frac{\gamma}{(3+7\chi)n}, \end{aligned}$$

one has

$$\alpha_c \geqslant \frac{\gamma}{(3+7\chi)n}.$$

That completes the proof. □

To obtain lower bound for the step size $\check{\alpha}$, which is defined in Algorithm 1, we need the following two technical lemmas.

Lemma 6. *If $\alpha_a \geqslant 0.1$, $\Delta x^T \Delta s > 0$ and $n \geqslant 2$, then*

$$\bar{\alpha}_c = \frac{(1-\tau)x^T s}{2\Delta x^T \Delta s} \geqslant (1-\tau)\gamma \geqslant \frac{\gamma}{(3+7\chi)n}.$$

Proof. By Lemma 2, we have

$$\begin{aligned} \left(\Delta x^a\right)^T \Delta s^a &= \sum_{i \in \mathcal{I}_+} \Delta x_i^a \Delta s_i^a + \sum_{i \in \mathcal{I}_-} \Delta x_i^a \Delta s_i^a \\ &\leqslant \frac{1}{4}\sum_{i \in \mathcal{I}_+} x_i s_i \leqslant \frac{1}{4} x^T s. \end{aligned}$$

Therefore,

$$\tau = \left(1 - \alpha_a + \alpha_a^2 \frac{(\Delta x^a)^T \Delta s^a}{x^T s}\right)^3$$
$$\leqslant \left(1 - \alpha_a + \frac{1}{4}\alpha_a^2\right)^3 \leqslant \left(1 - \frac{1}{2}\alpha_a\right)^6 \leqslant (0.95)^6.$$

So, we have

$$1 - \tau \geqslant 1 - (0.95)^6 \geqslant \frac{1}{6}. \tag{15}$$

Then for any $n \geqslant 2$, we have $(1-\tau)\gamma \geqslant \frac{\gamma}{(3+7\chi)n}$. The first is true only if

$$\frac{\Delta x^T \Delta s}{x^T s} \leqslant \frac{1}{2\gamma}.$$

By Lemma 5 and $\tau \leqslant 0.95^6$, we have that

$$\frac{\Delta x^T \Delta s}{x^T s} \leqslant \frac{1}{4\gamma}\left(\tau^2 - \left(2\gamma - \frac{\alpha_a}{2}\right)\tau + \frac{\gamma(\alpha_a^2 + 4\alpha_a + 20)}{16}\right)$$
$$\leqslant \frac{1}{4\gamma}\left(\tau^2 + \frac{1}{2}\tau + \frac{25}{32}\right) \leqslant \frac{1}{2\gamma}, \tag{16}$$

that completes the proof. □

For $\alpha_a < 0.1$ or $\alpha_c < \frac{\gamma}{3n}$, Algorithm 1 solves the system (9) with $\mu = \frac{\gamma}{1-\gamma}\mu_g$. We have the following lemma.

Lemma 7. *If $\alpha_a < 0.1$ or $\alpha_c < \frac{\gamma}{3n}$, $\Delta x^T \Delta s > 0$ and $n \geqslant 2$, then*

$$\hat{\alpha}_c = \frac{(1-\varsigma)x^T s}{2(\Delta x)^T \Delta s} \geqslant \gamma(1-\varsigma) \geqslant \frac{\gamma}{(3+7\chi)n},$$

where $\varsigma = \frac{\gamma}{1-\gamma}$.

Proof. Since $\gamma \in (0, 0.4]$, we have

$$1 - \varsigma = 1 - \frac{\gamma}{1-\gamma} \in \left[\frac{1}{3}, 1\right).$$

By Theorem 1, we have $\frac{x^T s}{2\Delta x^T \Delta s} \geqslant 32/73 > 0.4 \geqslant \gamma$. Hence we have that

$$\frac{(1-\varsigma)x^T s}{2(\Delta x)^T \Delta s} \geqslant \gamma(1-\varsigma) \geqslant \frac{\gamma}{3} \geqslant \frac{\gamma}{(3+7\chi)n},$$

that completes the proof. □

The following corollary gives the lower bound for step size $\check{\alpha}$.

Corollary 2. *If $\gamma \in (0, 0.4]$ and $n \geqslant 2$, we have*

$$\check{\alpha} \geqslant \frac{\gamma}{(3+7\chi)n}.$$

Proof. By Steps 5 and 6 of Algorithm 1 and Theorem 2, for $\Delta x \Delta s \leqslant 0$, we have

$$\check{\alpha} = \alpha_c \geqslant \frac{\gamma}{(3+7\chi)n}.$$

For $\Delta x \Delta s > 0$, by Lemmas 6 and 7, we also have

$$\check{\alpha} \geqslant \frac{\gamma}{(3+7\chi)n}. \qquad \square$$

In what follows we estimate the complementarity gap of the sequence produced by Algorithm 1, which is crucial for iteration complexity.

Let

$$f(\alpha) := 1 - (1-\sigma)\alpha + \alpha^2 \frac{(\Delta x)^T \Delta s}{x^T s}, \quad 0 \leqslant \sigma < 1.$$

If $(\Delta x)^T \Delta s \leqslant 0$, then $f(\alpha)$ is monotonically decreasing in $[0, 1]$. If $(\Delta x)^T \Delta s > 0$, $f(\alpha)$ is monotonically decreasing in $[0, \beta(\sigma)]$, where

$$\beta(\sigma) := \frac{(1-\sigma)x^T s}{2(\Delta x)^T \Delta s}.$$

Lemma 8. *If $\alpha_a \geqslant 0.1$ and $\alpha_c \geqslant \frac{\gamma}{3n}$, one has*

$$\mu_g(\alpha) \leqslant \left(1 - \frac{\gamma}{(36 + 84\chi)n}\right)\mu_g.$$

Proof. Case 1: $(\Delta x)^T \Delta s \leqslant 0$. By (14) and (15), we obtain

$$\begin{aligned}
\mu_g(\alpha) &\leqslant \left[1 - (1-\tau)\alpha_c\right]\mu_g \\
&\leqslant \left(1 - \frac{(1-\tau)\gamma}{(3+7\chi)n}\right)\mu_g \\
&\leqslant \left(1 - \frac{\gamma}{(18+42\chi)n}\right)\mu_g.
\end{aligned}$$

Case 2: $(\Delta x)^T \Delta s > 0$. By Step 5, we have that $f(\alpha)$ is monotonically decreasing in $[0, \beta(\tau)]$ and $\check{\alpha} \leqslant \beta(\tau)$. Therefore, using Corollary 2, (14) and (16), we have

$$\begin{aligned}
\mu_g(\alpha) &\leqslant \left(1 - (1-\tau)\frac{\gamma}{(3+7\chi)n} + \frac{(\Delta x)^T \Delta s}{x^T s}\left(\frac{\gamma}{(3+7\chi)n}\right)^2\right)\mu_g \\
&\leqslant \left(1 - (1-\tau)\frac{\gamma}{(3+7\chi)n} + \frac{1}{2\gamma}\left(\frac{\gamma}{(3+7\chi)n}\right)^2\right)\mu_g \\
&\leqslant \left(1 - (1-\tau)\frac{\gamma}{(3+7\chi)n} + \frac{1}{2}(1-\tau)\left(\frac{\gamma}{(3+7\chi)n}\right)\right)\mu_g \\
&\leqslant \left(1 - \frac{1}{12}\frac{\gamma}{(3+7\chi)n}\right)\mu_g \\
&= \left(1 - \frac{\gamma}{(36+84\chi)n}\right)\mu_g,
\end{aligned}$$

where the third inequality follows from Lemma 6, and the fourth from (15). □

Lemma 9. *If $\alpha_a < 0.1$ or $\alpha_c < \frac{\gamma}{3n}$, one has*

$$\mu_g(\alpha) \leqslant \left(1 - \frac{\gamma}{(18 + 42\chi)n}\right)\mu_g.$$

Proof. By Corollary 2 and the proof of Lemma 8, if $(\Delta x)^T \Delta s \leqslant 0$, we have

$$\begin{aligned}
\mu_g(\alpha) &\leqslant \left(1 - \frac{(1-\varsigma)\gamma}{(3+7\chi)n}\right)\mu_g \\
&\leqslant \left(1 - \frac{\gamma}{(9+21\chi)n}\right)\mu_g.
\end{aligned}$$

If $(\Delta x)^T \Delta s > 0$, we have

$$\begin{aligned}
\mu_g(\alpha) &\leqslant \left(1 - (1-\varsigma)\frac{\gamma}{(3+7\chi)n} + \frac{73}{64}(1-\varsigma)\left(\frac{\gamma}{(3+7\chi)n}\right)\right)\mu_g \\
&\leqslant \left(1 - \frac{73}{128}(1-\varsigma)\frac{\gamma}{(3+7\chi)n}\right)\mu_g \\
&\leqslant \left(1 - \frac{\gamma}{(18+42\chi)n}\right)\mu_g.
\end{aligned}$$

We complete the proof. □

Our main result in this section is the following theorem which describes the iteration complexity for Algorithm 1.

Theorem 3. *Algorithm* 1 *stops after at most*

$$\mathcal{O}\left((1+\chi)n\log\frac{(x^0)^T s^0}{\epsilon}\right)$$

iterations with a solution for which $x^T s \leqslant \epsilon$.

Proof. By Lemmas 8 and 9, we have

$$\mu_g(\alpha) \leqslant \left(1-\frac{\gamma}{(36+84\chi)n}\right)\mu_g,$$

which completes the proof conforming to Theorem 3.2 of [22]. □

4. A new Mehrotra-type predictor–corrector algorithm and convergence analysis

4.1. A new Mehrotra-type predictor–corrector algorithm

In this section, a new adaptive technique to compute the centering parameter is presented. Salahi et al. [20] pointed out that a too small centering parameter might force the algorithm to make a very small step in order to keep the iterate in a certain neighborhood of the central path. This further implies that the algorithm needs to perform a large number of iterations to get a sufficiently accurate approximate solution. On the other hand, centering parameters approaching 1 yields a slow reduction of the duality gap since in practice the algorithm takes pure centering steps. Motivated by these observations we introduce the following new rule to update the centering parameter ρ:

$$\rho = \min\left\{\max\left\{\rho_0, \left(1-\alpha_a+\alpha_a^2\frac{(\Delta x^a)^T\Delta s^a}{x^T s}\right)^3\right\}, \rho_1\right\}, \tag{17}$$

where $0 < \rho_0 < \rho_1 \leqslant 0.9$.

Algorithm 2.
Input: $\epsilon > 0$, $0 < \rho_0 < \rho_1 \leqslant 0.9$ and $\gamma = \rho_0/(1+\rho_0)$ and initial point $(x^0, s^0) \in \mathcal{N}_\infty^-(\gamma)$. Set $k := 0$.

Step 1. If $(x^k)^T s^k \leqslant \epsilon$, then report $(x^k, s^k) \in \mathcal{F}_\epsilon^*$ and terminate; otherwise, set $x = x^k$, $s = s^k$;
Step 2. Solve (4) and compute the maximum step size α_a from (5);
Step 3. If $x(\alpha_a)^T s(\alpha_a) < \epsilon$, then report $(x(\alpha_a), s(\alpha_a)) \in \mathcal{F}_\epsilon^*$ and terminate. Otherwise compute $\bar{\alpha}_a$ from (8) and ρ from (17);
Step 4. Solve (9) with $\mu = \rho\mu_g$ and compute the maximum step size α_c from (11);
Step 5. If $(\Delta x)^T\Delta s \leqslant 0$, set $\check{\alpha} = \alpha_c$. Otherwise, compute $\bar{\alpha}_c = \frac{(1-\rho)x^T s}{2(\Delta x)^T\Delta s}$, and set $\check{\alpha} = \min\{\alpha_c, \bar{\alpha}_c\}$;
Step 6. Set $x^{k+1} = x^k + \check{\alpha}\Delta x$, $s^{k+1} = s^k + \check{\alpha}\Delta s$, $k := k+1$, and return Step 1.

4.2. Polynomial complexity of Algorithm 2

In this section, we establish our main complexity results for Algorithm 2. The following theorem gives the upper bounds on $\|\Delta x\Delta s\|$ and $(\Delta x)^T\Delta s$ for a specific μ.

Theorem 4. *Let* $\mu = \rho\mu_g$, *then*

$$\|\Delta x\Delta s\| \leqslant \frac{3}{2}\left(2^{-\frac{3}{2}}+\chi\right)\frac{n\mu_g}{\gamma},$$

$$(\Delta x)^T\Delta s \leqslant \frac{3n\mu_g}{8\gamma}.$$

Proof. Let

$$h(t,\gamma) := t^2 + \frac{1}{2}t - 2t\gamma + \frac{25}{16}\gamma, \quad 0 \leqslant t \leqslant 1,\ 0 \leqslant \gamma \leqslant \frac{1}{2}.$$

One can easily show that

$$\max_{(t,\gamma)\in[0,1]\times[0,\frac{1}{2}]} h(t,\gamma) = h(1,0) \leqslant \frac{3}{2}.$$

Then, by Lemma 5 we have

$$\begin{aligned}
\|\Delta x\Delta s\| &\leqslant \left(2^{-\frac{3}{2}}+\chi\right)\left(\frac{1}{\gamma}\rho^2-\left(2-\frac{\alpha_a}{2\gamma}\right)\rho+\frac{20+4\alpha_a+\alpha_a^2}{16}\right)n\mu_g \\
&\leqslant \left(2^{-\frac{3}{2}}+\chi\right)\left(\rho^2+\frac{1}{2}\rho-2\rho\gamma+\frac{25\gamma}{16}\right)\frac{n\mu_g}{\gamma} \\
&\leqslant \frac{3}{2}\left(2^{-\frac{3}{2}}+\chi\right)\frac{n\mu_g}{\gamma},
\end{aligned}$$

and

$$\begin{aligned}
(\Delta x)^T\Delta s &\leqslant \frac{1}{4}\left(\frac{1}{\gamma}\rho^2-\left(2-\frac{\alpha_a}{2\gamma}\right)\rho+\frac{20+4\alpha_a+\alpha_a^2}{16}\right)\frac{n\mu_g}{\gamma} \\
&\leqslant \frac{1}{4}\left(\rho^2+\frac{1}{2}\rho-2\rho\gamma+\frac{25\gamma}{16}\right)\frac{n\mu_g}{\gamma} \\
&\leqslant \frac{3n\mu_g}{8\gamma},
\end{aligned}$$

which complete the proof. □

The following theorem gives the lower bound for the maximum step size α_c defined in Step 4 of Algorithm 2.

Theorem 5. *Suppose that the current iterate $(x,s)\in\mathcal{N}_\infty^-(\gamma)$, where $\gamma\in(0,\frac{1}{2})$. Let $(\Delta x,\Delta s)$ be the solution of (9) with $\mu=\rho\mu_g$. Then the maximum step size α_c, such that $(x(\alpha_c),s(\alpha_c))\in\mathcal{N}_\infty^-(\gamma)$, satisfies*

$$\alpha_c \geqslant \frac{\gamma^2}{2(1+\chi)n}.$$

Proof. The goal is to find the maximum α for which $x_i(\alpha)s_i(\alpha)\geqslant\gamma\mu_g(\alpha)$. If the previous inequality is true for $\alpha>\frac{1}{1+t}$, where t is defined by (13), then we get the result. Otherwise, similar to the proof of Theorem 2, we have

$$x_i(\alpha)s_i(\alpha) \geqslant \gamma\left(1-(1+\alpha_a t)\alpha\right)\mu_g+\alpha\mu-\alpha^2\left(\frac{3}{2}\left(2^{-\frac{3}{2}}+\chi\right)\frac{n\mu_g}{\gamma}\right).$$

By (14) we have

$$\begin{aligned}
\mu_g(\alpha) &\leqslant \frac{1}{n}\left((1-\alpha)x^Ts+\alpha n\mu+\alpha^2(\Delta x)^T\Delta s\right) \\
&= \left(1-\left(1-\frac{\mu}{\mu_g}\right)\alpha+\frac{(\Delta x)^T\Delta s}{x^Ts}\alpha^2\right)\mu_g \\
&\leqslant \left(1-\left(1-\frac{\mu}{\mu_g}\right)\alpha+\frac{3}{8\gamma}\alpha^2\right)\mu_g, \qquad (18)
\end{aligned}$$

where the last inequality follows from Theorem 4. Our aim is to ensure that $x_i(\alpha)s_i(\alpha)\geqslant\gamma\mu_g(\alpha)$. For this it suffices to require that

$$\gamma\left(1-\alpha(1+\alpha_a t)\right)+\alpha\frac{\mu}{\mu_g}-\frac{3}{2}\left(2^{-\frac{3}{2}}+\chi\right)\frac{n}{\gamma}\alpha^2 \geqslant \gamma\left((1-\alpha)+\alpha\frac{\mu}{\mu_g}+\frac{3}{8\gamma}\alpha^2\right).$$

One can easily verify that this inequality holds when

$$\alpha \leqslant \frac{(1-\gamma)\frac{\mu}{\mu_g}-\gamma\alpha_a t}{\frac{3}{8}+\frac{3n}{2\gamma}\left(2^{-\frac{3}{2}}+\chi\right)}.$$

Since

$$\begin{aligned}
\frac{(1-\gamma)\frac{\mu}{\mu_g}-\gamma\alpha_a t}{\frac{3}{8}+\frac{3n}{2\gamma}(2^{-\frac{3}{2}}+\chi)} &\geqslant \frac{\gamma(1-\gamma)\rho-\frac{1}{4}\gamma^2}{\frac{3}{8}\gamma+\frac{3n}{2}(2^{-\frac{3}{2}}+\chi)} \geqslant \frac{\gamma(1-\gamma)\rho_0-\frac{1}{4}\gamma^2}{\frac{3}{8}\gamma+\frac{3n}{2}(2^{-\frac{3}{2}}+\chi)} \\
&= \frac{\frac{3}{4}\gamma^2}{\frac{3}{8}\gamma+\frac{3n}{2}(2^{-\frac{3}{2}}+\chi)} = \frac{\gamma^2}{\frac{\gamma}{2}+2(2^{-\frac{3}{2}}+\chi)n} \\
&\geqslant \frac{\gamma^2}{2(1+\chi)n},
\end{aligned}$$

we have $\alpha_c \geqslant \frac{\gamma^2}{2(1+\chi)n}$, which completes the proof. □

Lemma 10. *For any $n \geqslant 2$ and $(\Delta x)^T \Delta s > 0$, we have*

$$\frac{\gamma^2}{2(1+\chi)n} \leqslant \frac{4}{3}\gamma(1-\rho) \leqslant \bar{\alpha}_c,$$

where $\bar{\alpha}_c$ is defined in Step 5 of Algorithm 2.

Proof. The first part is true if

$$\frac{\gamma}{2n} \leqslant \frac{4}{3}(1-\rho). \tag{19}$$

Since $n \geqslant 2$, $\gamma < \frac{1}{2}$, one has $\frac{\gamma}{2n} < \frac{1}{8}$. Then, (19) is true if

$$\rho \leqslant \frac{29}{32} \leqslant 0.9063.$$

By $\rho \leqslant 0.9$ the first part holds. By Theorem 4, we have

$$\bar{\alpha}_c = \frac{(1-\rho)x^T s}{2(\Delta x)^T \Delta s} \geqslant \frac{1-\rho}{2}\frac{8\gamma}{3} = \frac{4}{3}(1-\rho)\gamma,$$

that completes the proof. □

Lemma 11. *Suppose that $(x, s) \in \mathcal{N}_\infty^-(\gamma)$, where $\gamma \in (0, \frac{1}{2})$. Then*

$$\mu_g(\alpha) \leqslant \left(1-\frac{\gamma^2}{40(1+\chi)n}\right)\mu_g.$$

Proof. If $(\Delta x)^T \Delta s \leqslant 0$, from (18) and Theorem 5, one has

$$\begin{aligned}
\mu_g(\alpha) &\leqslant \bigl(1-(1-\rho)\alpha_c\bigr)\mu_g \\
&\leqslant \left(1-(1-\rho)\frac{\gamma^2}{2(1+\chi)n}\right)\mu_g \\
&\leqslant \left(1-\frac{\gamma^2}{20(1+\chi)n}\right)\mu_g.
\end{aligned}$$

If $(\Delta x)^T \Delta s > 0$, one has $\check{\alpha} = \min\{\alpha_c, \bar{\alpha}_c\}$. If $\alpha_c \leqslant \bar{\alpha}_c$, from (18), one has

$$\begin{aligned}
\mu_g(\alpha) &\leqslant \left(1-(1-\rho)\frac{\gamma^2}{2(1+\chi)n}+\frac{3}{8\gamma}\left(\frac{\gamma^2}{2(1+\chi)n}\right)^2\right)\mu_g \\
&\leqslant \left(1-\frac{1}{2}(1-\rho)\frac{\gamma^2}{2(1+\chi)n}\right)\mu_g \\
&\leqslant \left(1-\frac{\gamma^2}{40(1+\chi)n}\right)\mu_g,
\end{aligned}$$

where the first inequality follows from the monotone decreasing property of $f(\alpha)$, the second from Theorem 4 and the last from Lemma 10.

By Lemma 10, we have $\bar{\alpha}_c \geqslant \frac{\gamma^2}{2(1+\chi)n}$. Thus the case where $\alpha_c > \bar{\alpha}_c$, is proved in the same way. Hence we get the desired result. □

Table 1
Numerical results of Example 1.

n	Algorithm 1	Algorithm 2	Z-L algorithm
	iter (std)	iter (std)	iter (std)
100	13.0 (0.00)	13.0 (0.00)	13.0 (0.00)
200	14.0 (0.00)	14.0 (0.00)	13.0 (0.00)
600	14.0 (0.00)	14.0 (0.00)	14.0 (0.00)
1000	15.0 (0.00)	15.0 (0.00)	14.0 (0.00)

Table 2
Numerical results of Example 2.

n	Algorithm 1	Algorithm 2	Z-L algorithm
	iter (std)	iter (std)	iter (std)
100	7.9 (1.72)	7.6 (1.42)	8.3 (1.83)
200	7.9 (2.23)	7.7 (1.95)	8.7 (2.54)
600	9.2 (2.25)	8.8 (1.75)	9.5 (2.01)
1000	11.3 (2.98)	10.5 (2.17)	11.4 (2.91)

The following theorem gives an upper bound for the number of iterations in which Algorithm 2 stops with an ϵ-*approximate solution*.

Theorem 6. *Algorithm* 2 *stops after at most*

$$\mathcal{O}\left((1+\chi)n\log\frac{(x^0)^T s^0}{\epsilon}\right)$$

iterations with a solution for which $x^T s \leqslant \epsilon$.

Proof. From Lemma 11, one has

$$\mu_g(\alpha) \leqslant \left(1 - \frac{\gamma^2}{40(1+\chi)n}\right)\mu_g,$$

which completes the proof by Theorem 3.2 of [22]. □

5. Numerical results

We test our algorithms on some instances in this section. We write simple MATLAB codes for Algorithms 1, 2 and Zhang and Lü's algorithm (Z-L algorithm for short) in [26]. In our experiments, we choose the step-length parameter $\omega = 0.9$ in the predictor step, i.e., multiply α_a by a factor of 0.9. We take $\gamma = 0.001$ for the algorithms and (e, e) as the starting point. We stop the iteration of each algorithm if the duality gap

$$x^T s \leqslant 1.0e - 8.$$

5.1. Test 1: Solving monotone LCP

For each dimension n, the entry in the column 'iter' of the tables is the average number of iterations of 10 monotone LCPs with the same n, and the number in the bracket is the standard deviation (std) of these 10 runs.

Example 1. (See Example 5.5 in [7].) The (i, j)th entry of Q equal to 2 (respectively, 1 and 0) if $j > i$ (respectively, $j = i$ and $j < i$), and with $R = -I, b = Qe - e$. See Table 1.

Example 2. (See [25].) This is a randomly generated LCP test problem. By taking $Q = E - E^T$, $E_{i,j} \in [-5, 5]$, $R = -I$ and $b = Qe - e$, we obtain a monotone LCP. See Table 2.

Example 3. (See [25].) This is a randomly generated LCP test problem. By taking $Q = EE^T + E - E^T$, $E_{i,j} \in [-5, 5]$, $R = -I$ and $b = Qe - e$ we obtain a monotone LCP. See Table 3.

Example 4. (See [25].) This is a randomly generated LCP test problem. By taking $Q = EE^T$, $E_{i,j} = 5(i - j)/n$, $R = -I$ and $b = Qe - e$ we obtain a monotone LCP. See Table 4.

Table 3
Numerical results of Example 3.

n	Algorithm 1	Algorithm 2	Z-L algorithm
	iter (std)	iter (std)	iter (std)
100	5.6 (0.52)	5.6 (0.52)	6.2 (0.42)
200	5.9 (0.31)	5.9 (0.31)	6.2 (0.42)
600	6.0 (0.00)	6.0 (0.00)	6.9 (0.31)
1000	6.0 (0.00)	6.0 (0.00)	7.0 (0.00)

Table 4
Numerical results of Example 4.

n	Algorithm 1	Algorithm 2	Z-L algorithm
	iter (std)	iter (std)	iter (std)
100	3.0 (0.00)	3.0 (0.00)	6.0 (0.00)
200	3.0 (0.00)	3.0 (0.00)	6.0 (0.00)
600	3.0 (0.00)	3.0 (0.00)	6.0 (0.00)
1000	3.0 (0.00)	3.0 (0.00)	6.0 (0.00)

Table 5
Numerical results of Example 5.

n	Algorithm 1	Algorithm 2	Z-L algorithm
	iter (std)	iter (std)	iter (std)
100	5.0 (0.00)	5.0 (0.00)	6.0 (0.00)
200	5.0 (0.00)	5.0 (0.00)	6.0 (0.00)
600	5.0 (0.00)	5.0 (0.00)	6.0 (0.00)
1000	5.0 (0.00)	5.0 (0.00)	6.0 (0.00)

Example 5. (See [25].) This is a randomly generated LCP test problem. By taking $Q = EE^T + B - B^T + D$, $E_{i,j} \in [-5, 5]$, $B_{i,j} \in [0, 5]$, $D_{i,i} \in (0, 0.3)$, where D is a diagonal matrix. $R = -I$ and $b = Qe - e$ we obtain a monotone LCP. See Table 5.

Based on the numerical results we have listed in the tables, all three algorithms are reliable in terms of number of iterations and the standard deviation. Besides, Algorithm 2 is the fastest, and Algorithms 1 and 2 are faster than Z-L algorithm.

5.2. Test 2: Solving P_*-HLCP

In this section we solve the $P_*(\kappa)$-HLCP with Algorithms 1 and 2. First we list the following lemmas which can be found in [5].

Lemma 12. *The matrices*

$$Q_2 = \begin{pmatrix} 0 & 1+4\kappa_1 \\ -1 & 0 \end{pmatrix} \in \mathcal{R}^{2\times 2}, \qquad Q_3 = \begin{pmatrix} 0 & 1+4\kappa_2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{R}^{3\times 3}$$

are $P_*(\kappa)$, $\forall \kappa_1, \kappa_2 \geqslant 0$.

By using Q_2 and Q_3, we construct a $P_*(\kappa)$ matrix of any size as block diagonal matrices having the matrices Q_2 and Q_3 on the diagonal, i.e.:

$$Q_{23} = \begin{pmatrix} Q_2 & & & & \\ & Q_3 & & & \\ & & \ddots & & \\ & & & Q_2 & \\ & & & & Q_3 \end{pmatrix}.$$

By [5], we know that the matrix Q_{23} is a $P_*(\kappa)$ matrix. In this test, we take $\kappa_1, \kappa_2 \in \{0, 1, 100, 1000\}$. The numerical experiment of testing P_*-HLCPs are generated as follows. We take $Q = Q_{23}$, $R = -I$ where $Q, R \in \mathcal{R}^{300\times 300}$ and $b = Qe - e$ to obtain a P_*-HLCP. For this class of P_*-HLCPs, all of this three algorithms stopped after 13 iterations for $(k_1, k_2) \neq 0$, and 12 iterations for $k_1 = k_2 = 0$. This shows that the proposed algorithms are efficient and reliable.

6. Conclusions

We have discussed the polynomial complexity of Mehrotra-type predictor–corrector algorithm for solving sufficient LCPs in the $N_{\infty}^{-}(\gamma)$ neighborhood of the central path. This is an extension of the safeguard based Mehrotra-type algorithms proposed by Salahi et al. [20]. We also proposed a new adaptive updating scheme of centering parameter, which make the proof of the polynomial complexity of our algorithm for sufficient LCPs more simple. Both of the algorithms are independent on the handicap of the problems, and enjoy $\mathcal{O}(n(1+\chi)\log((x^0)^T s^0/\epsilon))$ iteration complexity. Some numerical results verified the efficiency and reliability of the algorithms.

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