3 Complex numbers

3.1 Definitions

Complex numbers are used extensively in quantum mechanics.[1] A generic complex number z may be written as

$$z = x + iy, \tag{1}$$

where $x, y \in \mathbb{R}$ are real numbers and i is the **imaginary unit**. The square of the imaginary unit is

$$i^2 = -1. \tag{2}$$

The components x and y of a complex number are called the **real** and **imaginary** parts of z. We also define two functions, Re and Im, which give the real and imaginary parts of z:

$$\mathrm{Re}(z) = x \tag{3}$$

$$\mathrm{Im}(z) = y. \tag{4}$$

The complex conjugate of a complex number z is written as z^* and is given by

$$z^* = (x + iy)^* = x - iy. \tag{5}$$

The modulus square of a complex number z is written as $|z|^2$ and is defined as

$$|z|^2 = z^* z = z z^* = (x - iy)(x + iy) = x^2 + y^2. \tag{6}$$

Note that $|z|^2$ is a real nonnegative number ($|z|^2 \geq 0$) and its square root measures the magnitude of z:

$$|z| = \sqrt{x^2 + y^2} \tag{7}$$

In contrast, the square of z is a complex number with both real and imaginary parts.

$$z^2 = (x + iy)(x + iy) = x^2 - y^2 + 2ixy. \tag{8}$$

[1] Complex numbers were originally introduced to deal with algebraic equations like $x^2 + 1 = 0$, which has no solution in the set of real numbers. If we define $i = \sqrt{-1}$, then we see that i solves this equation as it can be easily verified: $i^2 + 1 = (\sqrt{-1})^2 + 1 = -1 + 1 = 0$.

3.2 Polar representation of complex numbers

Complex numbers can be written also in an equivalent form called the **polar representation**. In the polar representation we express a complex number as a the product of a non-negative real number r and an angle θ

$$z = re^{i\theta}. \tag{9}$$

To relate the Cartesian (x, y) and polar (r, θ) representations of a complex number we use Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{10}$$

This is an important identity and should be memorized. Plugging in Euler's formula in the polar representation of z we get

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta) = \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y} \tag{11}$$

This shows that the Cartesian components are related to the polar ones via

$$x = r\cos\theta \tag{12}$$

$$y = r\sin\theta. \tag{13}$$

From these two equations it is easy to show that the radius is given by

$$r = \sqrt{x^2 + y^2}, \tag{14}$$

and the angle θ by[2]

$$\theta = \arctan \frac{y}{x}. \tag{15}$$

[2] Here, the right function is not really $\arctan(y/x)$ rather the function $\mathrm{atan2}(y, x)$, which is defined for angles $\theta \in [-\pi, \pi]$. See, e.g. `https://en.wikipedia.org/wiki/Atan2`.

Let us compute the modulus square of a complex number in polar coordinates

$$|z|^2 = zz^* = re^{i\theta}\left(re^{i\theta}\right)^* = re^{i\theta}re^{-i\theta} = r^2 \underbrace{e^{i\theta}e^{-i\theta}}_{e^{i\theta-i\theta}=e^0=1} = r^2. \tag{16}$$

This shows that the modulus square $|z|^2$ depends only on r. Note that in this derivation we used the fact that when we take the complex conjugate of $e^{i\theta}$ we change the sign of i in the exponent, and this leads to a cancellation of the exponential factors.

3.3 Differentiating complex functions

In quantum mechanics we occasionally have to take derivatives with respect to real variables (e.g. x) of a complex function $f(x)$. An example is the function $f(x) = x + ix^2$, which has both a real and an imaginary part, or a function like $f(x) = e^{ix}$, which at first glance may not be obvious how to split into real and imaginary parts. In both cases, the common rules of differentiation apply. Let us consider the function $f(x) = x^2 + ix^3$. When taking its derivative with respect to x we can separate the real and imaginary parts

$$\frac{df(x)}{dx} = \frac{d(x^2 + ix^3)}{dx} = \frac{dx^2}{dx} + i\frac{dx^3}{dx} = 2x + i3x^2. \tag{17}$$

In the case of e^{-ix} the derivative is instead obtained using the chain rule

$$\frac{de^{-ix}}{dx} = -ie^{ix}. \tag{18}$$

The process of differentiation can be generalized to higher derivatives. For example, the second derivative of e^{ix} is

$$\frac{d^2e^{-ix}}{dx^2} = -i\frac{de^{-ix}}{dx} = i^2e^{ix} = -e^{ix}. \tag{19}$$