Electoral College versus Popular Vote*

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Abstract

We compare the Electoral College and the popular vote in a stylized two-candidate, three-district election game. Under the Electoral College, the winner of a district acquires all votes of the electorate; under the popular vote, each candidate acquires the votes s/he actually wins in each district. Compared with the popular vote, the Electoral College entails a higher winning chance for the stronger candidate, a lower election inversion rate, and a higher total campaign expenditure with relatively symmetric candidates. Our study contributes further insights and findings to the heated debate on the reform of the US presidential election system.

Keywords: Electoral College; Inversion Rate; Multi-battle Contests; Political Campaigns; Popular Vote; Presidential Elections.

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Over the last two centuries, intense debate has persisted regarding whether the US should abolish the Electoral College and switch to the popular vote for presidential

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elections. The most commonly argued merits of the Electoral College concern the power balance between Congress and state legislatures, checks and balances and the reduced dominance of big states in presidential politics. However, there are also well-documented drawbacks. In particular, the phenomenon of "election inversion" looms significantly larger recently: discrepancies in the candidate elected based on the popular vote versus the Electoral College happen much more frequently since the 2000 election. In two of the last six presidential elections, the candidates who won the presidency under the Electoral College lost the nationwide popular vote.[1] These discrepancies, which undermine public confidence in the American electoral system and democracy, have triggered continuing controversy.

In this paper, we compare of the two election schemes from the perspective of game theoretical analysis. We model the presidential election as a simultaneous multi-battle contest between two opposing candidates who spend resources and/or exert irreversible costly effort in each state during their campaigns. We focus on analyzing candidates' winning chances, campaign expenditures, and inversion rates under different schemes, which, to our best knowledge, have not been studied in the literature so far. These issues are of great interest to politicians, the public, researchers, and electoral system designers.

In this study, we adopt a highly stylized model with three symmetric electoral districts, each endowed with a mass of one vote in total by normalization. There are two competing candidates who are asymmetric in terms of their popularity or efficiency in spending campaign resources to win voters' support. We use the competitors' marginal effort costs to integrate all of the factors that jointly determine their strengths in the competition, and assume that a candidate's marginal effort cost remains the same across districts. Both competitors simultaneously decide on a 3-dimensional effort profile, with each dimension targeting a different district.[2] In each district, the two candidates' corresponding dimensional efforts jointly determine the winner—i.e., the candidate who wins more actual votes in the district—randomly through a lottery contest. Specifically, the winner's number of actual votes in a district follows a general distribution, and the loser wins the rest. Under the Electoral College, the winner of a district acquires all votes of the district and the loser acquires no votes. Under the popular vote, both candidates acquire the votes

[1] In 2000, the popular vote winner, Al Gore, did not become president because George W. Bush had 4 more electoral votes. In 2016, Donald Trump had 77 more electoral votes than Hillary Clinton, even though Clinton had more popular votes.

[2] We assume that under Popular Vote, the candidates still need to battle in each district by exerting district-specific effort as they do under Electoral College.

they actually win in each district.

Under both schemes, the grand winner of the election is determined by the total numbers of the candidates' acquired votes across all districts, and the one with more acquired votes becomes the president-elect. With symmetric districts, the ultimate winner under the Electoral College only needs to win more districts than his/her opponent due to the scheme's winner-take-all practice. Under the popular vote, s/he has to collect more total votes from all districts than his/her opponent. Different election winners may be produced under the two schemes when a candidate wins exactly two districts.[3] The difference in how candidates' acquired votes are counted across the two schemes leads to the diverging equilibrium campaign strategies.

To characterize the equilibrium of the election game under each scheme, we first construct a general election game, of which the election games under the two schemes are simply two special cases, and then use the symmetry between districts to construct a restricted game in which each candidate is restricted to exert effort in the same way across districts, no matter whether s/he adopts a pure or mixed strategy. We find that the unique equilibrium of the restricted game is an equilibrium of the general election game, which is also unique. This approach greatly facilitates characterization of the equilibrium of the two election games under consideration. Specifically, we find that under both schemes, the stronger candidate always adopts a uniform pure strategy at the equilibrium, while the weaker candidate adopts a uniform pure strategy if and only if they are sufficiently symmetric.[4] When they are sufficiently asymmetric, the stronger candidate still adopts a uniform pure strategy and the weaker candidate adopts a uniform mixed strategy, in which s/he randomizes between being inactive and exerting a fixed positive level of effort.[5] Moreover, such a mixed strategy equilibrium is more likely to prevail under the Electoral College than under the popular vote, which can be explained by the following observation. Compared with the popular vote, the Electoral College scheme as a winner-selection mechanism has a higher discriminatory power in the language of the contest literature. In other words, when candidates exert different (uniform) effort, the one exerting higher effort

[3]Given the three districts' voting outcomes, even when a candidate loses (resp. wins) two districts and thus loses (resp. wins) the election under the Electoral College, s/he can still win (resp. lose) under the popular vote. For example, when candidates' vote shares in three districts are (49%, 51%), (48%, 52%), and (60%,40%), respectively, in the three districts, the first candidate would lose under the Electoral College but would win under the popular vote.

[4]Candidates' efforts differ across the two schemes.

[5]The level of positive effort differs across the two schemes.

wins with a higher chance under the Electoral College than under the popular vote.[6] Under either scheme, the stronger candidate is the one who exerts higher effort, and therefore his/her advantage over the weaker candidate is strengthened under the Electoral College, which discourages the weaker candidate from exerting effort.

Our equilibrium analysis means that in general, the candidates' winning chances, total campaign expenditure, and inversion rates must differ across the two schemes. Our main findings are as follows. First, the Electoral College always elects the stronger candidate with a higher probability than the popular vote. Recall that the Electoral College scheme is more discriminatory than the popular vote scheme. The advantage of the stronger candidate, who is more efficient in exerting effort, over the weaker candidate is thus strengthened under the Electoral College, which leads to a (weakly) higher election winning chance for the stronger candidate in each district under the Electoral College than under the popular vote. We call this a "strategic effect." Moreover, there is a "rounding effect," which further strengthens the advantage of the stronger player: Even if the candidates win each district with the same chances across the two schemes, the stronger player would win the whole election with a higher chance under the Electoral College than under the popular vote. The stronger candidate clearly has a higher chance of winning two districts than winning only one; and conditional on winning two districts, due to the rounding of votes under the Electoral College, the stronger candidate wins the whole election with a higher chance under the Electoral College than under the popular vote.

Second, comparison of total campaign expenditures under the two schemes depends on the degree of asymmetry across the candidates.[7] When they are sufficiently symmetric (resp. asymmetric), the Electoral College generates higher (resp. lower) total expenditure than the popular vote. This is because the Electoral College is more discriminatory than the popular vote. When candidates are sufficiently asymmetric, i.e., the competition is very unbalanced, a more discriminatory contest would dampen candidates' incentives to exert high effort. As a result, total effort cost or campaign expenditure is lower under the Electoral College than under the popular vote. When the competition is quite balanced, a more discriminatory contest would better motivate candidates to exert high effort. Thus,

[6]The reason is that the candidate exerting higher effort has a higher chance of winning two districts than winning only one district; and conditional on winning two districts, due to the rounding of votes under the Electoral College, the candidate wins the whole election with a higher chance under the Electoral College than under the popular vote.

[7]Total effort costs in our model can alternatively be interpreted as total campaign expenditures.

total effort cost or campaign expenditure is higher under the Electoral College than under the popular vote.

Third, the Electoral College generates a (weakly) lower inversion rate than the popular vote.[8] Under either election scheme, election inversion can happen if and only if no candidate wins all districts and the inversion rate is proportional to the probability of this event. Therefore, to compare election inversion rates across the two schemes, we should inspect the probabilities that no candidate wins all districts under the two schemes. Since the Electoral College is more discriminatory than the popular vote, the stronger candidate's advantage over the weaker candidate is strengthened under the Electoral College. That is, the actual asymmetry between candidates is enlarged under the Electoral College. As candidates become more asymmetric, it is more likely that a candidate will win all districts. We thus observe a (weakly) lower election inversion rate under the Electoral College.

Our paper primarily belongs to the well-established literature that studies political competitions using multi-battle contest models.[9] Many papers in this literature employ variants of the seminal Myerson (1993) contest model of campaign promises to study issues related to political and public economy.[10] Myerson (1993) studies election candidates' incentives to cultivate favored minorities in a setting in which two candidates compete for a continuum of voters by making binding promises to each of them. Each voter votes for the one who makes the higher offer to him/her, and the candidate who collects the most votes wins.[11] Lizzeri and Persico (2001) investigate how a prize allocation rule (a winner-take-all system versus a proportional system) would affect the provision of public goods in a generalized model of Myerson (1993). They further conduct a comparison of the Electoral College and the popular vote in terms of their impacts on public good provision by introducing a continuum of measure 1 of districts, with each district endowed with

[8]Under the popular vote, we define election inversion as a situation in which a candidate wins more than half of the total votes across all districts, but loses more than half of the districts. If the US switches to the popular vote, this situation could become relevant. This type of election inversion could be a concern in a country like the US, in which the preferences of its states are well respected by convention and the Constitution.

[9]Due to space limits, we cannot provide an exhaustive review of relevant studies. Please refer to Boyer et al. (2017) for an excellent review of this literature.

[10]This family of models is closely related to models of redistributive politics, such as those of Cox and McCubbins (1986) and Lindbeck and Weibull (1987).

[11]Ueda (1998) and Lizzeri (1999) endogenize campaign spending in Myerson (1993). Boyer et al. (2017) further accommodate loyal voters and a vote-maximization objective while assuming an endogenous campaign budget.

a continuum of voters.[12] Laslier and Picard (2002) study a discrete version (with finite voters) of the Myerson (1993) model. Roberson (2008) introduces jurisdiction-specific local projects to the Laslier and Picard (2002) model and studies pork-barrel politics, targetable policies, and federalism. Dekel et al. (2008) study vote buying in general elections in a sequential bidding game with finite voters.

The vote-generating technique in our election model is in line with many studies that adopt reduced-form models to determine the numbers of votes won by each candidate, without specifically modeling each voter's decision. Snyder (1989) proposes an election model with multiple legislative districts, each endowed with one seat in the legislature, and two parties compete in all districts. Kovenock and Roberson (2009b) extend Snyder (1989) to two periods and investigate the optimality of a 50-state strategy adopted by the Democratic National Committee. Klumpp and Polborn (2006) further accommodate a fully sequential variant of Snyder (1989) and compare a sequential organization and simultaneous elections. They illustrate the New Hampshire effect in US primaries, and find that sequential elections lead to lower campaign expenditures. Stromberg (2008) develops and estimates a multi-district probabilistic-voting model of US presidential elections with two symmetric candidates to study the impact of a hypothetical electoral reform of switching to a nationwide popular vote on candidates' equilibrium allocation of campaign time. Fu et al. (2015) propose a team contest model with pairwise battles fought by different team members. In each district, votes are shared based on a homogeneous-of-degree-zero contest success function. Their model can be applied to study general elections in most democracies, such as House of Representatives and Senate elections in the US. Differentiating from these studies, we develop a convenient multi-district model of presidential elections with two asymmetric candidates, which enables us to analytically compare the Electoral College and the popular vote in terms of winning chances, campaign expenditures, and inversion rates.

Technically, our paper contributes to equilibrium characterization in contests with multiple but finite battles, asymmetric candidates, and endogenous budgets.[13] Under the Electoral College, our model reduces to that of Klumpp and Polborn (2006). However, under the popular vote, our model exhibits additional synergies/interactions across battles, since even votes won in a losing district can contribute positively to a final win. As

[12] Kovenock and Roberson (2009a) and Crutzen and Sahuguet (2009) study the impacts of inefficiencies in collecting and reallocating resources using the Lizzeri and Persico (2001) model. Bierbrauer and Boyer (2016) extend Lizzeri and Persico (2001) to an environment with incomplete information.

[13] Please refer to Kovenock and Roberson (2012) for a comprehensive survey of multi-battle conflicts.

in Klumpp and Polborn (2006), we show that candidates adopt uniform strategies at equilibrium under both schemes. We further explicitly pin down the equilibria for a full range of asymmetry between the candidates. The equilibrium is in pure strategy when the candidates are relatively symmetric, and in mixed strategy with a single positive level of effort when they are more asymmetric.[14] To our best knowledge, this is the first time in the literature that these equilibria have been analytically pinned down in contests with finite battles and sufficiently asymmetric candidates.

The paper proceeds as follows. In Section I, we present the model setup. In Section II, we characterize the equilibria of election games under the two schemes. Section III conducts the comparison between the two election schemes. We conclude in Section IV. Omitted proofs are provided in the Appendix.

I. The Model

Two candidates, H and L, compete in an election with three districts. In each district, candidates exert costly campaign efforts to influence voters as specified by a campaign technology. Votes at district level are then aggregated according to the requirements of the chosen election scheme. Whoever acquires more aggregated votes wins the grand election. In the following, we will introduce the campaign technology and election schemes in detail.

Campaign technology. Assume that the three districts are symmetric and normalize the number of votes in each district to 1. For any district $k \in \{1, 2, 3\}$, let $s_k \in [0, 1]$ denote candidate H's votes and $1 - s_k$ denote L's votes. Given candidates' campaign efforts $x_{k,H}$ and $x_{k,L}$ in the district, the campaign technology generates s_k in the following way:

(i) with probability $x_{k,H}/(x_{k,H} + x_{k,L})$, s_k is randomly drawn from distribution $F(s)$ on $[1/2, 1]$;

(ii) with probability $x_{k,L}/(x_{k,H} + x_{k,L})$, s_k is randomly drawn from distribution $G(s)$ on $[0, 1/2]$.

We assume that the campaign technology is unbiased, i.e., when both candidates exert the same amount of effort in district k, H's votes, s_k, has the same probability distribution as

[14] In this sense, our equilibrium characterizations share some features with those in single-battle Tullock contests with asymmetric candidates, which have been studied by Nti (1999); Alcalde and Dahm (2010); and Wang (2010), among others.

that of L's votes, $1 - s_k$. It is easy to see that this unbiasedness condition implies that F and G satisfy $G(s) = 1 - F(1 - s), \forall s \in [0, 1/2]$.

The above campaign technology is probabilistic in that, given campaign efforts, candidates' votes are not deterministic, but rather exhibit some randomness. The randomness captures unexpected incidents that may occur during the election; for example, unpredictable and vital news about candidates released by a third-party source, eligible voters' willingness to cast ballots, and changes in economic conditions. In spite of the randomness, a candidate who exerts more effort is awarded higher odds of wining more votes (recall that the supports of F and G only overlap at $1/2$). If a candidate wins more votes in a district, then s/he wins the district. Although this campaign technology is not well defined when both candidates exert zero effort in the same district, we show in Section II that this will not be a concern, since candidates never exert zero effort at the same time in equilibrium.

Election schemes. Two election schemes are considered. The first one is the Electoral College (EC hereafter), which features a winner-take-all practice at district level. That is, a candidate who wins the majority votes in a district (district winner) acquires all the votes of that district. The second one is the popular vote (PV), in which there is no such winner-take-all practice and a candidate only acquires the votes s/he wins regardless of whether s/he is the district winner. Essentially, the winner-take-all practice is a rounding procedure for aggregating candidates' acquired votes, which rounds any district vote above $1/2$ up to 1 and below $1/2$ down to 0.

In sum, the campaign technology, together with candidates' campaign efforts, determines candidates' stochastic votes at district level, and the prevailing election scheme determines how votes are aggregated. Since an election winner needs to acquire more aggregated votes, the winning rules under the two schemes can effectively be interpreted as follows:

(1) under the Electoral College, H wins the grand election if and only if[15]

$$\mathbb{1}_{\{s_1 \geq 1/2\}} + \mathbb{1}_{\{s_2 \geq 1/2\}} + \mathbb{1}_{\{s_3 \geq 1/2\}} \geq 3/2; \tag{1}$$

(2) under the popular vote, H wins the grand election if and only if

$$s_1 + s_2 + s_3 \geq 3/2. \tag{2}$$

[15]Here $\mathbb{1}_{\{A\}}$ is the indicator function, which equals 1 (0) if event A is true (false).

A pure strategy of candidate $i \in \{H, L\}$ is denoted by $\mathbf{x}_i = (x_{1,i}, x_{2,i}, x_{3,i})$, in which $x_{k,i}$ ($k \in \{1, 2, 3\}$) is the campaign effort in district k. Let $\mathbf{X}_i = \mathbb{R}^3_+$ denote the set of all possible pure strategies for i. With the value of winning the grand election being common to both candidates and normalized to 1, i's payoff from competing in the election with strategy $\mathbf{x}_i$ is given by

$$\begin{cases} 1 - c_i(x_{1,i} + x_{2,i} + x_{3,i}), & \text{if } i \text{ wins the election,} \\ -c_i(x_{1,i} + x_{2,i} + x_{3,i}), & \text{if otherwise,} \end{cases} \tag{3}$$

in which c_i denotes i's marginal effort cost. Specifically, c_i measures candidate i's efficiency in exerting effort; for example, his/her ability in raising campaign funds, managing campaign expenditures, and attracting votes. We assume that candidate H is stronger than L in the sense that s/he has a lower marginal effort cost, i.e., $0 < c_H \leq c_L$.

Both candidates are risk neutral and maximize expected payoffs. Under each election scheme, we solve the Nash equilibrium of the simultaneous-move election game.[16]

Discussion:

(1) A closely related type of game is the Colonel Blotto game, in which players compete by simultaneously distributing limited resources over several battlefields; see Konrad (2011). The key difference between the election games considered in this paper and the Blotto game is in the interdependence among battlefields. More specifically, a player's payoff function is additively separable of the rewards from battlefields in a Blotto game, while in our paper it is nonseparable. This means that we have interdependent districts.

(2) The campaign technology we adopt is similar to that in Snyder (1989) and Klumpp and Polborn (2006)), which links candidates' efforts to their district-winning probabilities. However, unlike these two papers in which only the winning of a district is relevant, in this paper a candidate's exact number of votes in each district matters under the popular vote scheme. Given this, we modified the campaign technology in Snyder (1989) and Klumpp and Polborn (2006) by further specifying how candidates' votes are generated.

[16]The competitions in the three districts are organized simultaneously, in the sense that candidates exert effort in a district without observing the outcomes in others. Therefore, even though the timing of competitions may differ across districts, candidates' effort choices can be viewed as being made simultaneously.

II. Equilibrium Analysis

This section characterizes the equilibria of election games under the Electoral College and the popular vote, respectively. We first construct a general election game, of which the election games under consideration are simply two special cases. Using the symmetry among districts (i.e., identical number of votes, unbiased campaign technology, and simultaneous competitions in all three districts), we construct a restricted game which requires that candidates exert effort in the same way across districts. We show that the unique equilibrium of this restricted game is also an equilibrium of the original general election game, and also establish its uniqueness. Then the equilibria of the election games under our two schemes, as special cases, follow naturally.

A. A General Election Game

Table 1 compares candidate H's chance of being elected contingent on his/her winning a given number of districts under the two schemes.[17] Here,

$$q^{EC} \triangleq 1, \quad \textit{and} \quad q^{PV} \triangleq \int_{z_1+z_2+z_3 \geq 3/2} dF(z_1)dF(z_2)dG(z_3) \in (1/2, 1). \tag{4}$$

No. of winning districts	Electoral College	Popular Vote
3	1	1
2	q^{EC}	q^{PV}
1	$1 - q^{EC}$	$1 - q^{PV}$
0	0	0

Table 1: Candidate H's conditional chance of being elected

As shown by rows 1 and 4 of Table 1, a candidate will be elected for sure if s/he wins all three districts, regardless of which election scheme is adopted. However, when neither candidate wins all three districts, the discrepancy between the two schemes arises, because the conditional chance of being elected depends on the prevailing election scheme (see Table 1, rows 2 and 3). For example, suppose H wins two of the three districts: (i) under EC, H will be elected with certainty, i.e., $q^{EC} = 1$. This is due to the winner-take-all

[17]For candidate L, we have the same table.

practice under EC, which rounds any district votes above 1/2 up to 1 and below 1/2 down to 0. Note that q^{EC} always equals one, and does not depend on distributions F and G. (ii) under PV, H is more likely to win the election compared with L, but does not win for sure. The exact conditional chance in this case, q^{PV} in equation (4), is an integration involving F and G.[18]

The case in which H wins only one of the three districts is exactly the complement to the case in which H wins two: (i) under EC, H loses for sure; (ii) under PV, H's conditional chance of being elected equals another integration, $\int_{z_1+z_2+z_3\geq 3/2} dF(z_1)dG(z_2)dG(z_3)$, which can be shown to be equal to $1-q^{PV}$ using the unbiasedness condition of the campaign technology.[19]

Since (i) L's winning districts are precisely H's losing districts and (ii) both candidates' chances of being elected add up to one, Table 1 also gives the corresponding conditional election winning chances for candidate L.

As seen in Table 1, q^{EC} and q^{PV} are, respectively, the summary statistics of the election game under EC and PV. To avoid repetition in solving the equilibria of our two election games separately, we construct a general election game $\mathcal{G}(q)$ parameterized by $q \in [1/2, 1]$, of which the two games under EC and PV are just two special cases. Here, q stands for a candidate's chance of winning the whole election conditional on his/her winning two districts. Specifically, in game $\mathcal{G}(q)$, each candidate $i \in \{H, L\}$ chooses $\mathbf{x}_i = (x_{1,i}, x_{2,i}, x_{3,i}) \in \mathbf{X}_i = \mathbb{R}^3_+$. Given any profile $(\mathbf{x}_H, \mathbf{x}_L) \in \mathbf{X}_H \times \mathbf{X}_L$, candidate i's payoff in $\mathcal{G}(q)$ equals

$$\pi_i(\mathbf{x}_H, \mathbf{x}_L; q) = p_i(\mathbf{x}_H, \mathbf{x}_L; q) - c_i \sum_{k=1}^{3} x_{k,i}, \tag{5}$$

[18]Suppose $F(z) = 2z-1, z \in [1/2, 1]$ and $G(w) = 2w, w \in [0, 1/2]$, we have $q^{PV} = 5/6$.

[19]We can show the following identity:

$$\int_{z_1+z_2+z_3\geq 3/2} dF(z_1)dF(z_2)dG(z_3) + \int_{z_1+z_2+z_3\geq 3/2} dF(z_1)dG(z_2)dG(z_3) = 1,$$

by changing variables of integrations and using $G(z) = 1 - F(1-z)$. Also, we have

$$\int_{z_1+z_2+z_3\geq 3/2} dF(z_1)dF(z_2)dF(z_3) = 1, \quad \int_{z_1+z_2+z_3\geq 3/2} dG(z_1)dG(z_2)dG(z_3) = 0.$$

where

$$p_i(\mathbf{x}_H, \mathbf{x}_L; q) = \frac{\prod_{k=1}^{3} x_{k,i} + q \sum_{k'=1}^{3} (x_{k',-i} \prod_{k \neq k'} x_{k,i}) + (1-q) \sum_{k'=1}^{3} (x_{k',i} \prod_{k \neq k'} x_{k,-i})}{(x_{1,H} + x_{1,L})(x_{2,H} + x_{2,L})(x_{3,H} + x_{3,L})}. \tag{6}$$

Here, $-i$ denotes the opponent of candidate $i \in \{H, L\}$.

The next lemma shows the relationship between $\mathcal{G}(q)$ and the election games described in Section I.

Lemma 1. $\mathcal{G}\big(q^{EC}\big) = \mathcal{G}(1)$ *is the election game under the Electoral College;* $\mathcal{G}\big(q^{PV}\big)$ *is the election game under the popular vote.*

By Lemma 1, to find the equilibria of election games under the two schemes, we only need to identify the equilibrium of the general election game for a generic q. Furthermore, the comparison between EC and PV boils down to the comparative statics with respect to q under $\mathcal{G}(q)$.[20]

B. Equilibrirum Characterization

In this subsection, we fully characterize the equilibrium of $\mathcal{G}(q)$. To facilitate the analysis, we introduce the following two concepts. If candidate i sets $x_{k,i} = x$, $\forall k \in \{1, 2, 3\}$, where $x \in \mathbb{R}_+$ is drawn from some cumulative distribution function Λ_i, then i is playing a *uniform strategy*. An equilibrium is *uniform* if both candidates adopt uniform strategies.[21] Note that a uniform equilibrium does not imply that both candidates adopt the same strategy in equilibrium. Instead, it requires that the candidates exert efforts (either deterministically or stochastically) in the same way across districts.

We characterize the equilibrium in three steps. First, we introduce a restricted game of $\mathcal{G}(q)$ which requires that candidates adopt uniform strategies. We denote it by $\tilde{\mathcal{G}}(q)$. Second, we characterize the unique equilibrium of $\tilde{\mathcal{G}}(q)$, which is further shown to be an equilibrium of the original game $\mathcal{G}(q)$. Third, we establish the uniqueness of the equilibrium of $\mathcal{G}(q)$.

Step 1. Solving the restricted game $\tilde{\mathcal{G}}(q)$.

[20] Lemma 1 can also be employed to analyze the effects of campaign technologies—for instance, the dispersions of F and G—on the equilibrium objectives, which are beyond of the scope of this paper.

[21] The definitions of uniform strategy and uniform equilibrium follow from Klumpp and Polborn (2006).

The restricted game $\tilde{\mathcal{G}}(q)$ is relatively simpler to analyze than $\mathcal{G}(q)$, because each candidate's strategy is effectively of one dimension in $\tilde{\mathcal{G}}(q)$ instead of being three-dimensional, as in the original game $\mathcal{G}(q)$. Specifically, in game $\tilde{\mathcal{G}}(q)$, given a uniform strategy profile $((x_H, x_H, x_H), (x_L, x_L, x_L))$, candidate i's payoff is

$$\tilde{\pi}_i\big((x_H, x_H, x_H), (x_L, x_L, x_L); q\big) = p_i\big((x_H, x_H, x_H), (x_L, x_L, x_L); q\big) - 3c_i x_i. \tag{7}$$

We first define a cutoff that facilitates the equilibrium characterization as follows. Lemma 2 below further illustrates its properties.

Definition 1. *For each $q \in [1/2, 1]$, define a cutoff for the marginal effort cost ratio $c = c_H/c_L$,*[22]

$$\kappa(q) \triangleq \begin{cases} 0, & \text{if } 1/2 \leq q < 3/4, \\ -(3q-1) + \sqrt{9q^2 + 6q - 8}, & \text{if } 3/4 \leq q \leq 1. \end{cases} \tag{8}$$

Lemma 2. *$\kappa(q)$ is continuous and weakly increasing in q. Specifically, $\kappa(1) = -2 + \sqrt{7} \approx 0.646$.*

The following proposition presents the equilibrium of $\tilde{\mathcal{G}}(q)$.

Proposition 1. *For game $\tilde{\mathcal{G}}(q)$,*

1. *when $c \geq \kappa(q)$, there exists a pure strategy equilibrium, in which candidate $i \in \{H, L\}$ adopts a uniform pure strategy $(\tilde{x}_i^*(q), \tilde{x}_i^*(q), \tilde{x}_i^*(q))$ with*

$$\tilde{x}_i^*(q) = \frac{c}{c_i} \cdot \frac{(1-q)(1+c^2) + 2(2q-1)c}{(1+c)^4}; \tag{9}$$

2. *when $c < \kappa(q)$, there is a mixed strategy equilibrium, in which H adopts uniform pure strategy $\big(\tilde{x}_H^*(q), \tilde{x}_H^*(q), \tilde{x}_H^*(q)\big)$ with*

$$\tilde{x}_H^*(q) = \frac{c}{c_H} \cdot \frac{(1-q)\big(1+\kappa(q)^2\big) + 2(2q-1)\kappa(q)}{\big(1+\kappa(q)\big)^4} \tag{10}$$

[22] $\kappa(q)$ actually is the cutoff at which L's expected equilibrium payoff is 0.

and L adopts the following uniform mixed strategy

$$\begin{cases} \big(\tilde{x}_L^*(q), \tilde{x}_L^*(q), \tilde{x}_L^*(q)\big) & \textit{with probability } c/\kappa(q), \\ (0,0,0) & \textit{with probability } 1 - c/\kappa(q), \end{cases} \tag{11}$$

where

$$\tilde{x}_L^*(q) = \frac{\kappa(q)}{c_L} \cdot \frac{(1-q)\big(1+\kappa(q)^2\big) + 2(2q-1)\kappa(q)}{\big(1+\kappa(q)\big)^4}. \tag{12}$$

Furthermore, the equilibrium of $\tilde{G}(q)$ is unique.

A key observation of $\tilde{G}(q)$ is that the winning probability function $p_i(\cdot)$ is homogeneous with degree zero, which implies that the equilibrium effort ratio, when both play pure strategies, equals the inverse of the marginal cost ratio (see Corollary 1 (i) below). Furthermore, using the effort ratio and homogeneity property of p_i, we can pin down the equilibrium winning probability and each candidate's equilibrium effort in game $\tilde{G}(q)$, since only relative effort matters for the evaluation of $p_i(\cdot)$ and its partial derivatives.

A challenge here is that the weaker candidate L might play a mixed strategy when the degree of asymmetry between two candidates is sufficiently large. Here we provide a sketch of the proof of Proposition 1 (the detailed proof is relegated to the Appendix). We first focus on the case with a pure strategy equilibrium. Since candidates' strategies are effectively one-dimensional, using first-order conditions, we can easily identify a candidate for a pure strategy equilibrium. We next identify the range of c so that this candidate's strategy profile is indeed an equilibrium of $\tilde{G}(q)$. For any c outside this range, we consider a particular form of mixed strategy equilibrium, in which only candidate L adopts a mixed strategy and explicitly construct and verify a mixed strategy equilibrium in this form for $\tilde{\mathcal{G}}(q)$.[23]

Step 2. Identifying an equilibrium of $\mathcal{G}(q)$.

Next, we establish a connection between the equilibrium of $\tilde{\mathcal{G}}(q)$ and that of $\mathcal{G}(q)$. Since candidates can choose any strategy—for instance, a non-uniform one—in the general election game $\mathcal{G}(q)$, an equilibrium of $\tilde{\mathcal{G}}(q)$ is not necessarily an equilibrium of $\mathcal{G}(q)$. However, we find that, given an opponent's uniform strategy, a candidate's expected payoff function is Schur-concave in his/her strategy, which implies that any nonuniform strategy

[23]This type of mixed strategy equilibrium bears some similarity to the mixed strategy equilibrium in single-battle two-player asymmetric Tullock contests; see Wang (2010). However, the payoff in (7) differs from the payoff in Tullock contests.

is weakly dominated by a uniform strategy.[24] Consequently, the equilibrium of $\tilde{\mathcal{G}}(q)$, which is uniform by definition, must also be an equilibrium of $\mathcal{G}(q)$.

Proposition 2. *The unique equilibrium of $\tilde{\mathcal{G}}(q)$ characterized in Proposition 1 is an equilibrium of $\mathcal{G}(q)$.*

Step 3. Establishing the uniqueness of the equilibrium of $\mathcal{G}(q)$.

By now we have constructed an equilibrium of $\mathcal{G}(q)$. We next establish the interchangeability of equilibria of $\mathcal{G}(q)$ by employing certain special feature of its payoffs and obtain the following uniqueness result.

Proposition 3. *The equilibrium of the general election game $\mathcal{G}(q)$ is unique.*

C. Properties of the Equilibrium

Propositions 1, 2, and 3 together fully characterize the unique equilibrium of the general election $\mathcal{G}(q)$. Let $\mathbf{x}_H^*(q)$ denote H's equilibrium strategy and $\mathbf{x}_L^*(q)$ denote L's equilibrium strategy. The equilibrium could be either pure or mixed, depending on the candidates' relative strength, i.e., $c = c_H/c_L$. For any c above $\kappa(q)$, i.e., candidates are sufficiently symmetric, both $\mathbf{x}_H^*(q)$ and $\mathbf{x}_L^*(q)$ are pure strategies, with $\mathbf{x}_H^*(q) = \big(\tilde{x}_H^*(q), \tilde{x}_H^*(q), \tilde{x}_H^*(q)\big)$ and $\mathbf{x}_L^*(q) = \big(\tilde{x}_L^*(q), \tilde{x}_L^*(q), \tilde{x}_L^*(q)\big)$. For any c below $\kappa(q)$, i.e., candidates are sufficiently asymmetric, the unique equilibrium becomes a mixed one, in which H's equilibrium strategy $\mathbf{x}_H^*(q)$ is pure with $\mathbf{x}_H^*(q) = \big(\tilde{x}_H^*(q), \tilde{x}_H^*(q), \tilde{x}_H^*(q)\big)$ and L's equilibrium strategy $\mathbf{x}_L^*(q)$ is mixed with $\mathbf{x}_L^*(q) = \big(c/\kappa(q), (\tilde{x}_L^*(q), \tilde{x}_L^*(q), \tilde{x}_L^*(q)); 1 - c/\kappa(q), (0,0,0)\big)$.

The following corollary, which comes from the unique equilibrium of $\mathcal{G}(q)$, collects useful properties of the equilibrium, which will be used in subsequent analysis.

Corollary 1. *In the unique equilibrium of election game $\mathcal{G}(q)$, the following holds:*

(i) ***Effort ratio****. If $c \geq \kappa(q)$, $\tilde{x}_L^*(q)/\tilde{x}_H^*(q) = c$; if $c < \kappa(q)$, $\tilde{x}_L^*(q)/\tilde{x}_H^*(q) = \kappa(q)$.*

(ii) ***Total effort costs****. Candidate H's total effort cost equals L's expected total effort cost.*

[24] See Appendix for the definition, criteria, and implications of Schur concavity.

(iii) ***Winning chance****. H's election winning probability is*[25]

$$WP_H(q) \triangleq p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) = \begin{cases} \frac{1+3qc+3(1-q)c^2}{(1+c)^3}, & \textit{if } c \geq \kappa(q); \\ 1 - c \cdot \frac{\kappa(q)^2+3q\kappa(q)+3(1-q)}{[1+\kappa(q)]^3}, & \textit{if } c < \kappa(q). \end{cases}$$

Furthermore, WP_H *is at least* $1/2$*.*

(iv) ***Expected payoffs.*** *H's expected payoff is*

$$\pi_H^*(q) \triangleq \pi_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) = \begin{cases} \frac{1+(6q-2)c+(9-12q)c^2}{(1+c)^4}, & \textit{if } c \geq \kappa(q); \\ 2p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) - 1, & \textit{if } c < \kappa(q), \end{cases}$$

which is strictly positive; L's expected payoff is

$$\pi_L^*(q) \triangleq \pi_L\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) = \max\left\{\frac{c^2 + (6q-2)c + (9-12q)}{(1+c)^4}, 0\right\},$$

which is strictly positive only when $c > \kappa(q)$*.*

III. Comparing EC and PV

In this section, we compare equilibrium outcomes of election games under the Electoral College and the popular vote in three dimensions: winning probability, total campaign expenditure, and election inversion rate. Note that the election games under EC and PV are simply two special cases of the general election game, and their equilibria and equilibrium properties follow naturally according to Lemma 1. Thus, it suffices to conduct comparative statics with respect to q in the game $\mathcal{G}(q)$ across the three dimensions.

Before proceeding, we first state a key observation that plays a critical role in driving all of our comparsion results in the following lemma.

Lemma 3. *As a winner-selection mechanism, EC is more discriminatory than PV in the following sense: Assume both candidates adopt uniform pure strategies. The candidate who exerts higher effort enjoys a higher probability of winning the election under EC than under PV.*

An immediate implication of the above lemma is that a mixed strategy equilibrium is more likely to arise under the Electoral College than under the popular vote. To see

[25] Candidate L's election winning probability is just $1 - WP_H$.

this, we first define two cutoffs,

$$\kappa_{EC} \triangleq \kappa\big(q^{EC}\big), \text{ and } \kappa_{PV} \triangleq \kappa\big(q^{PV}\big).$$

Then we have $\kappa_{PV} < \kappa_{EC}$, since $\kappa(q)$ increases with q and $q^{PV} < q^{EC} = 1$.[26] Under either scheme, a mixed strategy equilibrium arises if and only if the marginal effort cost ratio c is below the scheme-specific cutoff (i.e., κ_{EC} under EC and κ_{PV} under PV). Since in equilibrium the stronger candidate uniformly exerts more effort than the weaker candidate, EC, as a more discriminatory winner-selection mechanism, enlarges the stronger candidate's advantage over the weaker candidate and thus discourages the weaker candidate from exerting effort more than the popular vote does.[27]

A. Winning Probability

Recall that candidate H's equilibrium election winning probability is

$$WP_H(q) = p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big).$$

By Lemma 1, $WP_H\big(q^{EC}\big)$ and $WP_H\big(q^{PV}\big)$ are candidate H's election winning chances under EC and PV, respectively. Comparing these two probabilities yields the following:

Theorem 1. *The stronger candidate's winning chance under the Electoral College is always higher than that under the popular vote.*

Figure 1 depicts the comparison result presented in Theorem 1. The solid red curve, representing H's election winning probability under EC, lies above the dashed blue curve, representing that under PV.[28]

To see the intuition, we decompose the difference between $WP_H\big(q^{EC}\big)$ and $WP_H\big(q^{PV}\big)$

[26]When F and G are uniform distributions, $q^{PV} = 5/6$ and $q^{EC} = 1$. Then we have $\kappa(5/6) = (\sqrt{13} - 3)/2 \approx 0.303$ and $\kappa(1) = -2 + \sqrt{7} \approx 0.646$.

[27]To give another perspective, note that L's equilibrium payoff is always lower in EC than in PV (see Corollary 2 (i)). Thus, the range in which L obtains zero payoff (or, equivalently, L plays a mixed strategy), is larger under EC (see the two dashed curves in Figure 3).

[28]We set $F(z) = 2z - 1$, $z \in [1/2, 1]$ and $G(w) = 2w$, $w \in [0, 1/2]$ in this figure and subsequent ones.

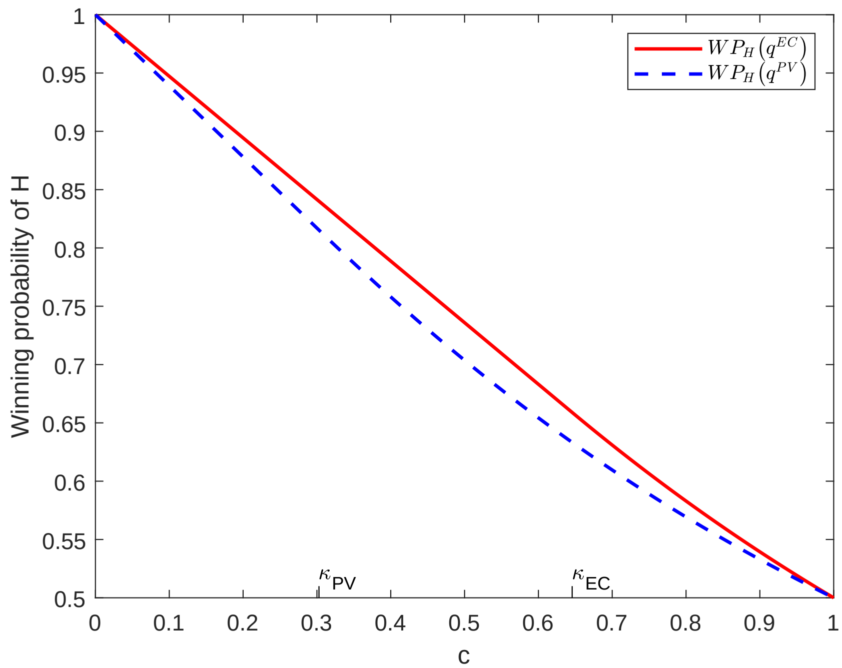

Figure 1: The stronger candidate's winning probability

into two components:

$$
\begin{aligned}
&WP_H\big(q^{EC}\big) - WP_H\big(q^{PV}\big)\\
=&\underbrace{p_H\big(\mathbf{x}_H^*\big(q^{EC}\big), \mathbf{x}_L^*\big(q^{EC}\big); q^{EC}\big) - p_H\big(\mathbf{x}_H^*\big(q^{EC}\big), \mathbf{x}_L^*\big(q^{EC}\big); q^{PV}\big)}_{\text{``rounding effect''}}\\
&+\underbrace{p_H\big(\mathbf{x}_H^*\big(q^{EC}\big), \mathbf{x}_L^*\big(q^{EC}\big); q^{PV}\big) - p_H\big(\mathbf{x}_H^*\big(q^{PV}\big), \mathbf{x}_L^*\big(q^{PV}\big); q^{PV}\big)}_{\text{``strategic effect''}},
\end{aligned}
$$

where the wining probability $p_H(\mathbf{x}_H, \mathbf{x}_L; q)$, is defined in (6).[29]

The first component, the rounding effect, refers to the direct effect caused by the procedure of rounding any votes above 1/2 up to 1 and below 1/2 down to 0 when the distribution of candidates' votes in each district is given by $\big(\mathbf{x}_H^*\big(q^{EC}\big), \mathbf{x}_L^*\big(q^{EC}\big)\big)$ under both schemes. The second component, the strategic effect, refers to the indirect effect caused by the difference in equilibrium strategies when the method for aggregating votes (or the election scheme) is fixed, as in PV.

Since in equilibrium the stronger candidate H exerts a higher effort than L in each district, H is more likely to draw votes greater than 1/2 in each district and enjoys the

[29]Here we extend the domain of p_H to allow for possibly mixed strategies.

benefit of the rounding procedure more often. Thus, the rounding effect on H's winning probability must always be positive.

The strategic effect captures the impact of the difference in candidates' equilibrium strategies. Note that different equilibrium strategies do not necessarily lead to different election winning probabilities. What really matters for the election winning probability is the equilibrium effort ratio as well as the probabilities of candidates' remaining active. When $\kappa_{EC} \leq c$, both EC and PV induce a pure strategy equilibrium in which the two candidates are always active. The equilibrium effort ratios under the two schemes are the same even though the equilibrium strategies are different (see the first part of Corollary 1 (i)). Therefore, the strategic effect vanishes in this case. However, as candidates become more and more asymmetric, i.e., $c < \kappa_{EC}$, EC strengthens H's advantage over L because H benefits more from the rounding procedure and thus discourages L from exerting effort. Specifically, EC always induces a mixed strategy equilibrium in which L stays inactive with a positive probability, while PV induces a pure strategy equilibrium if $c \geq \kappa_{PV}$ and a mixed strategy equilibrium if $c < \kappa_{PV}$. Regardless of whether the equilibrium induced under PV is pure or mixed, the probability of L staying inactive is greater under EC than under PV. Although the ratio of L's equilibrium effort over H's in each district is higher under EC than under PV (see Corollary 1 (i); under EC, the ratio is κ_{EC}; under PV, the ratio is c if $c \geq \kappa_{PV}$ and κ_{PV} if $c < \kappa_{PV}$), the effect of a lower probability of L staying active dominates the effect of a relatively higher effort ratio of L over H, which finally leads to a positive strategic effect on H's election winning probability.

B. Total Campaign Expenditure

The total effort cost of both candidates can be interpreted as the total campaign expenditure in the election. By Propositions 1 and 2, the total campaign expenditure under the equilibrium of election game $\mathcal{G}(q)$ equals

$$TCE(q) = \begin{cases} 6c \cdot \frac{(1-q)(1+c^2)+2(2q-1)c}{(1+c)^4}, & \text{if } c \geq \kappa(q); \\ 6c \cdot \frac{(1-q)\left[1+\kappa(q)^2\right]+2(2q-1)\kappa(q)}{[1+\kappa(q)]^4}, & \text{if } c < \kappa(q). \end{cases} \tag{13}$$

By Lemma 1, the total campaign expenditure is $TCE\left(q^{EC}\right)$ under the Electoral College and $TCE\left(q^{PV}\right)$ under the popular vote. Comparing them yields the following result.

Theorem 2. *There exists a cutoff $\hat{c} \in (\kappa_{PV}, \kappa_{EC})$ such that the total campaign expenditure is higher under EC when $c > \hat{c}$ and higher under PV when $c < \hat{c}$.*

Figure 2 depicts the comparison of total campaign expenditures under the two schemes. The solid red curve describes the total campaign expenditure under EC and the dashed blue curve corresponds to the expenditure under PV. The two curves start from the same point and then cross only once at $\hat{c}$ between κ_{PV} and κ_{EC}. For each scheme, the total campaign expenditure curve is linear in c when c is smaller than its scheme-specific cutoff,[30] with a steeper slope under PV.

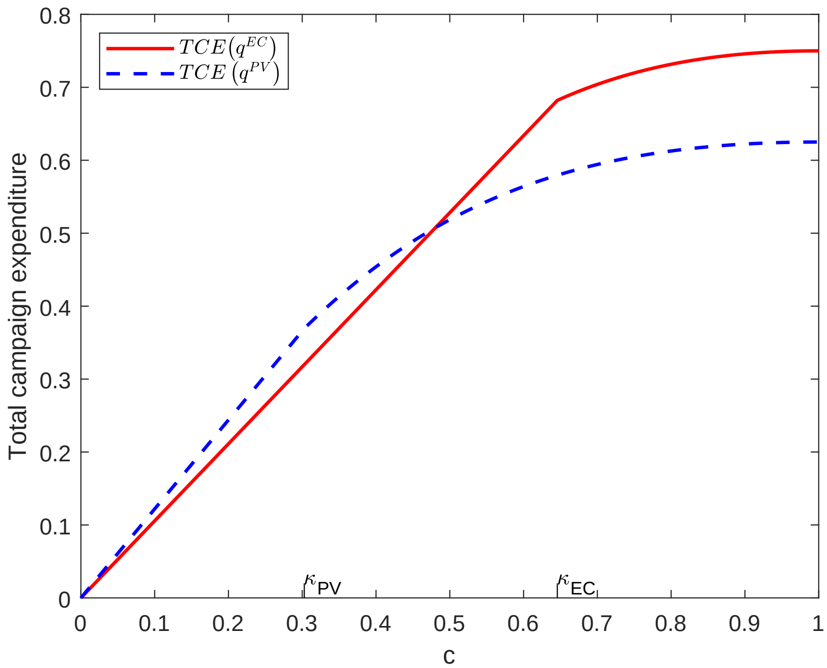

Figure 2: Comparison of total campaign expenditure

The intuition behind the comparison result is as follows. Recall that the Electoral College is more discriminatory than the popular vote. When candidates are relatively asymmetric, i.e., $c \leq \hat{c}$, a more discriminatory scheme would dampen candidates' incentives to exert high effort. As a result, the total campaign expenditure must be lower under the Electoral College than under the popular vote. When candidates are relatively symmetric, i.e., $c \geq \hat{c}$, a more discriminatory scheme would provide better incentives for candidates to exert high effort. Thus, the total campaign expenditure must be higher under the Electoral College than under the popular vote.

Having studied the winning probabilities and campaign expenditures, we now combine Theorems 1 and 2 to obtain the following payoff comparison results.

[30]Recall that the scheme-specific cutoff is κ_{EC} under EC and κ_{PV} under PV.

Corollary 2.

(i) $\pi_L^*(q^{PV}) \geq \pi_L^*(q^{EC})$ *for any* $c \in (0, 1]$*, with strict inequality when* $c > \kappa_{PV}$*.*

(ii) $\pi_H^*(q^{PV}) > \pi_H(q^{EC})$ *when* $c \geq \kappa_{EC}$*, while* $\pi_H^*(q^{EC}) > \pi_H(q^{PV})$ *when* $c \leq \hat{c}$ *(here* $\hat{c}$ *is the cutoff given in Theorem 2).*

Proof. By Corollary 1 (iv), L's expected payoff is

$$\pi_L^*\left(q^{EC}\right) = \max\left\{\frac{c^2 + (6q^{EC} - 2)c + (9 - 12q^{EC})}{(1+c)^4}, 0\right\}$$

under EC and

$$\pi_L^*\left(q^{PV}\right) = \max\left\{\frac{c^2 + (6q^{PV} - 2)c + (9 - 12q^{PV})}{(1+c)^4}, 0\right\}$$

under PV. Since $d\left[c^2 + (6q - 2)c + (9 - 12q)\right]/dq = 6(c - 2) < 0$ and $q^{EC} > q^{PV}$, we have $\pi_L^*\left(q^{PV}\right) \geq \pi_L^*\left(q^{EC}\right)$.

To compare the payoffs for candidate H under the two schemes, we first present two useful identities in election game $\mathcal{G}(q)$:

$$\begin{cases} \pi_H^*(q) + \pi_L^*(q) = 1 - TCE(q); \\ \pi_H^*(q) - \pi_L^*(q) = 2WP_H(q) - 1. \end{cases} \tag{14}$$

The first equality comes from the fact that each candidate's payoff equals his/her expected winning probability minus his expected cost. Since the winning probabilities of H and L add up to one, the sum of their equilibrium payoffs equals one minus the total campaign expenditure.[31] Furthermore, both candidates have the same expected cost by Corollary 1 (ii); the difference between candidates' equilibrium payoffs is exactly the difference between their winning probabilities, which is precisely the second identity. Combining these two identities yields the third one: $\pi_H^*(q) = WP_H(q) - TCE(q)/2$.

For any $c \leq \hat{c}$, since $WP_H\left(q^{EC}\right) > WP_H\left(q^{PV}\right)$ by Theorem 1 and $TCE\left(q^{PV}\right) \geq TCE(q^{EC})$ by Theorem 2, we have $\pi_H^*\left(q^{EC}\right) > \pi_H^*\left(q^{PV}\right)$. For any $c \geq \kappa_{EC}$, both schemes induce a pure strategy equilibrium, in which H's expected payoff, by Corollary 1 (iv), equals $\left[1 + (6q-2)c + (9-12q)c^2\right]/(1+c)^4$, $q \in \left\{q^{EC}, q^{PV}\right\}$. Clearly, the payoff expression is decreasing in c when $c \geq 1/2$. Since $\kappa_{EC} = \kappa(1) \approx 0.646 > 1/2$, we thus have $\pi_H^*\left(q^{PV}\right) \geq \pi_H^*\left(q^{EC}\right)$ in this range of c. ■

[31]The value of winning the election is normalized as 1 for both candidates.

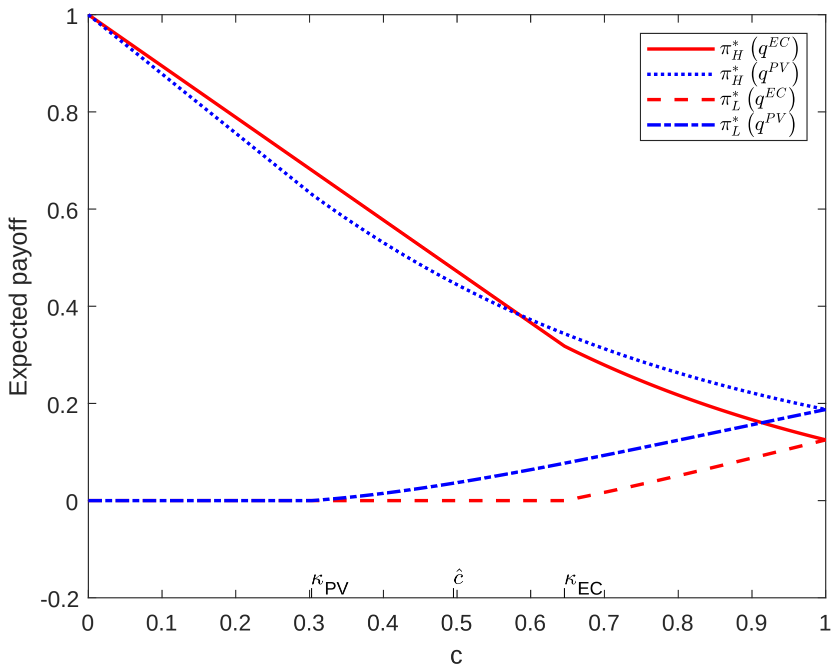

Figure 3: Comparison of equilibrium payoffs

Figure 3 depicts the comparison of candidates' payoffs under the two election schemes. The solid red curve and dashed red curve describe the payoffs for H and L under EC, respectively. The dotted blue curve and dash-dot blue curve correspond to the payoffs for H and L under PV, respectively. We observe that (i) L's payoff is weakly higher under PV than under EC, and (ii) H's payoff is higher under EC when H and L are relatively asymmetric ($c < \hat{c}$) and higher under PV when H and L are relatively symmetric ($c > \kappa_{EC}$).

C. Election Inversion Rate

Since votes are aggregated in different ways under EC and PV, they can can generate different election winners; This phenomenon is known as election inversion. Formally, define

$$E_1 = \left\{\mathbb{1}_{\{s_1 \geq 1/2\}} + \mathbb{1}_{\{s_2 \geq 1/2\}} + \mathbb{1}_{\{s_3 \geq 1/2\}} \geq 3/2\right\}, \text{ and } E_2 = \left\{s_1 + s_2 + s_3 \geq 3/2\right\},$$

which denote the events that H wins the election under EC and PV, respectively. Clearly, election inversion occurs only in the event $E_1 \Delta E_2$.[32] We define the election inversion rate under any strategy profile $(\mathbf{x}_H, \mathbf{x}_L)$ as $Pr_{(\mathbf{x}_H, \mathbf{x}_L)}(E_1 \Delta E_2)$, with the understanding that the probability distributions of s_1, s_2, s_3 are generated by the campaign technologies under the profile $(\mathbf{x}_H, \mathbf{x}_L)$.

Lemma 4. *The equilibrium election inversion rate in election game $\mathcal{G}(q)$ is*

$$EIR(q) = Pr_{\left(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q)\right)}(E_1 \Delta E_2) = \begin{cases} \left(1 - q^{PV}\right) \frac{3c}{(1+c)^2} & \textit{if } c \geq \kappa(q); \\ \left(1 - q^{PV}\right) \frac{c}{\kappa(q)} \cdot \frac{3\kappa(q)}{(1+\kappa(q))^2} & \textit{if } c < \kappa(q). \end{cases} \tag{15}$$

$EIR(q)$ strictly increases in c as both expressions in (15) strictly increase in c.

Given the votes (before aggregation) won by the candidates in the districts, as is shown in Table 1, EC and PV might generate different election winners only in the event that no candidate wins all three districts. When $c \geq \kappa(q)$, so that both candidates play pure strategies, the chance that H wins exactly two districts is $3\left(\frac{1}{1+c}\right)^2\left(\frac{c}{1+c}\right)$, since the effort ratio equals $1/c$ by Corollary 1 (i). Conditional on winning two districts, election inversion happens with probability $q^{EC} - q^{PV} = 1 - q^{PV}$. Furthermore, H wins only one district with probability $3\left(\frac{1}{1+c}\right)\left(\frac{c}{1+c}\right)^2$, in which election inversion also occurs with conditional probability $1 - q^{PV}$. Taking the sum yields the first expression in (15).

When $c < \kappa(q)$, election inversion occurs only when L is active, which happens with probability $\frac{c}{\kappa(q)}$.[33] Conditional on L being active, the effort ratio is $1/\kappa(q)$ by Corollary 1 (i). Combining these observations yields the second expression in (15).

To facilitate the comparison, we can further simplify (15) as

$$EIR(q) = (1 - q^{PV})3c \cdot \frac{1}{(1 + \max\{c, \kappa(q)\})^2}. \tag{16}$$

According to Lemma 1, the election inversion rate is $EIR\left(q^{EC}\right)$ under EC and $EIR\left(q^{PV}\right)$ under PV. Since $\kappa_{EC} > \kappa_{PV}$, it follows that $\max\{c, \kappa_{EC}\} \geq \max\{c, \kappa_{PV}\}$, and equality holds when $c \geq \kappa_{EC}$. Thus, we have $EIR\left(q^{EC}\right) \leq EIR\left(q^{PV}\right)$, which yields the following theorem.

[32] For instance, when $(s_1, s_2, s_3) = (0.4, 0.4, 0.9)$, the election winner is $H(L)$ under PV (EC). On the other hand, when $(s_1, s_2, s_3) = (0.6, 0.6, 0.2)$, the winner is $L(H)$ under PV (EC).

[33] Obviously, when L is inactive, H wins for sure under both schemes.

Theorem 3. *EC has a weakly lower inversion rate than PV, i.e.,*

$$EIR\left(q^{EC}\right) \leq EIR\left(q^{PV}\right).$$

Specifically, equality holds if and only if $\kappa_{EC} \leq c$.

Election inversion happens only when no candidate wins all three districts, and its rate is proportional to the probability of that event. From Table 1 (specifically, Rows 2 and 3), we can infer that the proportion is $q^{EC} - q^{PV}$ under either scheme, and thus the comparison of inversion rates degenerates to the comparison of the probability that no candidate wins all districts. When $c \geq \kappa_{EC}$, both EC and PV induce a pure strategy equilibrium, and the equilibrium effort ratios are the same across the two schemes (see Corollary 1 (i) and also the discussion after Proposition 1). In this case, the probability that no candidate wins all districts under EC equals that under PV, which implies the same election inversion rates across the two schemes. When $c < \kappa_{EC}$, EC always induces a mixed strategy equilibrium, and PV, depending on whether $c \geq \kappa_{PV}$ or not, either induces a pure strategy equilibrium or a mixed strategy equilibrium. EC, as a more discriminatory scheme than PV, enlarges the asymmetry between candidates and thus generates a lower probability of the event that no candidate wins all districts. A lower election inversion rate under EC thus follows.

Figure 4 depicts the comparison of election inversion rates under EC and PV. We observe that the solid red curve for EC is uniformly below the dashed blue curve for PV, and the two curves coincide when $c \geq \kappa_{EC}$.

IV. Conclusion

In this paper, we contribute to the heated debate regarding two election schemes in American politics and democracy: the Electoral College (EC) versus the popular vote (PV).

We propose a stylized model of an election game with two asymmetric candidates and multiple symmetric districts. In each district, the candidates compete for votes through a general stochastic vote-generating technology by exerting costly campaigning effort. Under different election schemes, the candidate's votes at district level are aggregated across districts in different ways. Under EC, the candidate who wins the majority of votes in a district acquires all the votes of that district. Under PV, each candidate acquires

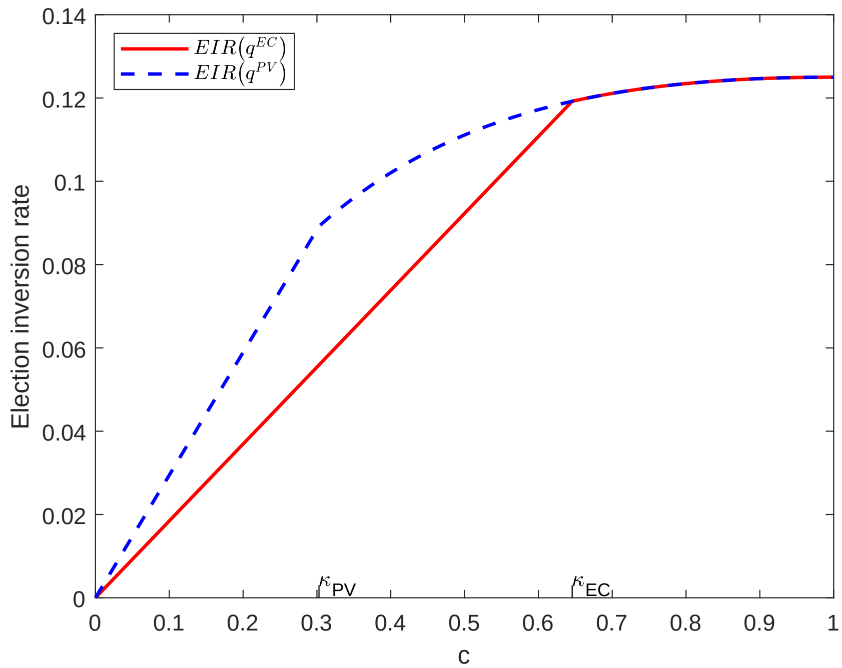

Figure 4: Comparison of election inversion rate

the votes s/he actually wins. Under both schemes, the candidate with more aggregate acquired votes wins the election. We find that EV as a winner-selection mechanism is more discriminatory than PV, in the sense that a candidate who exerts more effort enjoys a relatively higher winning probability. This key observation drives our comparison results across the two schemes.

Our stylized model allows us to explicitly pin down the unique equilibrium under each scheme. Under either scheme, when the candidates become sufficiently asymmetric, the weaker candidate would play a mixed strategy in every district at equilibrium and stays inactive with a positive probability. Since a more discriminatory winner selection mechanism tends to discourage the weaker player from exerting high effort, it is more likely for the weaker candidate to stay inactive under EC.

The equilibrium analysis allows us to compare the two election schemes in several respects, which are important for policymakers and the general public. First, the stronger candidate wins with higher chance under EC than under PV. This finding is intuitive, given that EC is more discriminatory and thus gives the stronger candidate an advantage, and EC has the additional advantage of a rounding effect. Second, when candidates are sufficiently symmetric (resp. asymmetric), EC generates higher (resp. lower) total expenditure than PV. Note that EC is more discriminatory than PV. When candidates are sufficiently asymmetric, and thus the competition is very unbalanced, EC would dampen candidates' incentives to exert high effort, which leads to a lower total campaign expenditure. Third, EC generates a (weakly) lower inversion rate than PV. Under either scheme, election inversion can happen if and only if no candidate wins all districts. Since EC is more discriminatory than PV, the stronger candidate's advantage over the weaker candidate is strengthened under EC. As candidates get more asymmetric, it is more likely that a candidate will win all districts under EC than under PV, which entails a (weakly) lower election inversion rate under EC.

However, our equilibrium analysis and the comparison results largely depend on the assumption of identical districts and a small number of districts. When districts are asymmetric (for example, asymmetric campaign technology or asymmetric number of total votes across districts), our procedure of characterizing the equilibrium no longer applies. A similar issue arises when the number of districts increases. Extending the analysis to more general environments and investigating the robustness of our findings are clearly important, though they are demanding technically. We leave these to future work.

Our study is a first step in comparing EC and PV from the perspectives of winning chance, campaign expenditure, and inversion rate. The analysis, though relies on our stylized model, yields meaningful insights that are informative for policymakers and the general public when they weigh the possible advantages and disadvantages of different election schemes.

Broadly speaking, our work contributes to the literature on optimal design of multi-battle contests including sports, R&D races, political competitions, etc. The analysis and findings are useful when comparing different rules of allocating a single prize in environments where players compete in multiple battlefields.

APPENDIX

This appendix contains some mathematics preliminaries (Section A) and omitted proofs of results in the main text (Section B).

A. Mathematics preliminaries

Definition 2. *x is **majorized** by y, denoted by $x \prec y$, if the rearrangement of the components of x and y such that $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$, $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ satisfies $\Sigma_{i=1}^{k} x_{[i]} \leq \Sigma_{i=1}^{k} y_{[i]}$ for $1 \leq k \leq n-1$ and $\Sigma_{i=1}^{n} x_{[i]} = \Sigma_{i=1}^{n} y_{[i]}$.*

Definition 3. *The function f is called Schur-convex if $x \prec y$ implies $f(x) \leq f(y)$. Any such function is called Schur-concave if $-f$ is Schur-convex.*

Schur-Ostrowski criterion: [Marshall et al. (1979)] If $f : \mathbb{R}^d \to \mathbb{R}$ is a symmetric function and all first partial derivatives exist, then f is Schur-concave if and only if

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \leq 0, \text{ for all } x \in \mathbb{R}^d, \tag{17}$$

holds for all $i, j = 1, \ldots, d$.

Lemma 5. *Fix $y > 0$ and $1 \geq q \geq 1/2$. The following function $f : \mathbb{R}_+^3 \to \mathbb{R}$:*

$$f(x_1, x_2, x_3) = \frac{x_1x_2x_3 + q(x_1x_2 + x_1x_3 + x_2x_3)y + (1-q)(x_1 + x_2 + x_3)y^2}{(x_1 + y)(x_2 + y)(x_3 + y)} - c\sum_{i=1}^{3} x_i,$$

is Schur-concave.

Proof. Clearly f is a symmetric function. Fix $t \in \mathcal{I} = \{1, 2, 3\}$. Differentiating f with respect to x_t yields

$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_t} = y\frac{(1-q)\prod_{i\in\mathcal{I}\setminus\{t\}} x_i + (2q-1)y\sum_{i\in\mathcal{I}\setminus\{t\}} x_i + (1-q)y^2}{(x_t + y)^2\prod_{i\in\mathcal{I}\setminus\{t\}}(x_i + y)} - c. \tag{18}$$

Define

$$A_{\mathcal{I}\setminus\{t\}} = (1-q)\prod_{i\in\mathcal{I}\setminus\{t\}} x_i + (2q-1)y\sum_{i\in\mathcal{I}\setminus\{t\}} x_i + (1-q)y^2,$$

which is nonnegative since $(2q-1) \geq 0, (1-q) \geq 0$, $y > 0$ and $x_i \geq 0, \forall i$. Then we have

$$\frac{\partial f}{\partial x_t} = \frac{y}{\prod_{i \in \mathcal{I}}(x_i + y)} \frac{A_{\mathcal{I}\setminus\{t\}}}{x_t + y} - c. \tag{19}$$

Similarly, we have

$$\frac{\partial f}{\partial x_s} = \frac{y}{\prod_{i \in \mathcal{I}}(x_i + y)} \frac{A_{\mathcal{I}\setminus\{s\}}}{x_s + y} - c.$$

Without loss of generality, assume that $x_t \geq x_s$. Then we have $A_{\mathcal{I}\setminus\{t\}} \leq A_{\mathcal{I}\setminus\{s\}}$ and $1/(x_t + y) \leq 1/(x_s + y)$; thus,

$$\frac{\partial f}{\partial x_t} - \frac{\partial f}{\partial x_s} = \frac{y}{\prod_{i \in \mathcal{I}}(x_i + y)} \cdot \left\{ \frac{A_{\mathcal{I}\setminus\{t\}}}{x_t + y} - \frac{A_{\mathcal{I}\setminus\{s\}}}{x_s + y} \right\} \leq 0.$$

In other words,

$$(x_t - x_s) \left(\frac{\partial f}{\partial x_t} - \frac{\partial f}{\partial x_s} \right) \leq 0.$$

So f is Schur-concave by the Schur-Ostrowski criterion. ■

Note that for any vector $(x_1, x_2, x_3) \in \mathbb{R}^3$, we have

$$(x_1, x_2, x_3) \prec \left(\frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + x_3}{3} \right), \tag{20}$$

which leads to the following useful remark.

Remark 1. *For any Schur-concave f,*

$$f(x_1, x_2, x_3) \leq f\left(\frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + x_3}{3} \right), \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3.$$

B. Omitted Proofs

PROOF OF LEMMA 1:

Without loss of generality, we focus on candidate H since the case for candidate L is similar. First note that $\frac{x_{1,H}x_{2,H}x_{3,H}}{(x_{1,H}+x_{1,L})(x_{2,H}+x_{2,L})(x_{3,H}+x_{3,L})}$, $\frac{x_{1,L}x_{2,H}x_{3,H}+x_{1,H}x_{2,L}x_{3,H}+x_{1,H}x_{2,H}x_{3,L}}{(x_{1,H}+x_{1,L})(x_{2,H}+x_{2,L})(x_{3,H}+x_{3,L})}$, and $\frac{x_{1,L}x_{2,L}x_{3,H}+x_{1,L}x_{2,H}x_{3,L}+x_{1,H}x_{2,L}x_{3,L}}{(x_{1,H}+x_{1,L})(x_{2,H}+x_{2,L})(x_{3,H}+x_{3,L})}$ are, respectively, the probabilities that H wins exactly 3, 2, and 1 out of three districts under the strategy profile $(\mathbf{x}_H, \mathbf{x}_L)$, respectively. Under either EC or PV, H's odds of winning the election is the sum of his probabilities of winning 3, 2, and 1 districts, adjusted by the corresponding conditional chances as shown in Table 1, respectively. Thus, H's payoff in the election game, as described in Section I, is precisely equal to that in $\mathcal{G}\big(q^{PV}\big)$ under PV and equal to that in $\mathcal{G}(1)$ under EC. ■

PROOF OF LEMMA 2:

Recall that when $q \in [3/4, 1]$, the cutoff is

$$\kappa(q) = -(3q-1) + \sqrt{9q^2 + 6q - 8}. \tag{21}$$

First, we have

$$\kappa\left(\frac{3}{4}\right) = -\left(\frac{9}{4} - 1\right) + \sqrt{\frac{81}{16} + \frac{18}{4} - 8} = -\frac{5}{4} + \sqrt{\frac{25}{16}} = 0. \tag{22}$$

By definition, $\kappa(q) = 0$ when $q \in [1/2, 3/4)$. Thus, $\kappa(q)$ is continuous in q.

When $q \in [3/4, 1]$, the first-order derivative of $\kappa(q)$ is

$$\kappa'(q) = 3\left[\frac{3q+1}{\sqrt{9q^2+6q-8}} - 1\right] = 3\left[\frac{\sqrt{9q^2+6q+1}}{\sqrt{9q^2+6q-8}} - 1\right] > 0. \tag{23}$$

Thus, $\kappa(q)$ increases in q when $q \in [3/4, 1]$. Overall, $\kappa(q)$ is weakly increasing in q.

When $q = 1$, we have $\kappa(1) = -(3-1) + \sqrt{9+6-8} = -2 + \sqrt{7} \approx 0.646$. ■

PROOF OF PROPOSITION 1:

In this proof, we first identify a candidate for the pure strategy equilibrium for $\tilde{\mathcal{G}}(q)$, identify a condition for it to be an equilibrium, and establish its uniqueness under the condition. Then, we find a special form of mixed equilibrium when the condition for a pure strategy equilibrium does not hold and prove its uniqueness.

Step 1: Identify a uniform pure-strategy equilibrium and the required condition. Given a uniform strategy profile $\big((x_H, x_H, x_H), (x_L, x_L, x_L)\big)$, the payoff for candidate $i \in \{H, L\}$ is

$$\tilde{\pi}_i\big((x_H, x_H, x_H), (x_L, x_L, x_L); q\big) = \frac{x_i^3 + 3q x_i^2 x_{-i} + 3(1-q) x_i x_{-i}^2}{(x_i + x_{-i})^3} - 3c_i x_i. \tag{24}$$

The first-order condition implies that

$$\frac{d\tilde{\pi}_i}{dx_i} = 3\frac{(1-q)x_i^2 x_{-i} + 2(2q-1)x_i x_{-i}^2 + (1-q)x_{-i}^3}{(x_i + x_{-i})^4} - 3c_i = 0. \tag{25}$$

For candidate H, we have

$$\frac{(1-q)x_H^2 x_L + 2(2q-1)x_H x_L^2 + (1-q)x_L^3}{(x_H+x_L)^4} = c_H. \tag{26}$$

For candidate L, we have

$$\frac{(1-q)x_L^2 x_H + 2(2q-1)x_L x_H^2 + (1-q)x_H^3}{(x_H+x_L)^4} = c_L. \tag{27}$$

Combining equations (26) and (27), we obtain $\tilde{x}_L^*/\tilde{x}_H^* = c_H/c_L = c$ at optimum. Substituting the relation back into equations (26) and (27), we have

$$\tilde{x}_i^* = \frac{c}{c_i} \cdot \frac{(1-q)(1+c^2) + 2(2q-1)c}{(1+c)^4}, \quad \forall i \in \{H, L\}. \tag{28}$$

Since $\tilde{x}_H^* c_H = \tilde{x}_L^* c_L$ and $c_H < c_L$, we should have $\tilde{x}_H^* \geq \tilde{x}_L^*$, which implies that H has a higher winning probability than L and thus has a greater payoff than L. To save space, the details are omitted.

Under $\big((\tilde{x}_H^*, \tilde{x}_H^*, \tilde{x}_H^*), (\tilde{x}_L^*, \tilde{x}_L^*, \tilde{x}_L^*)\big)$, L's payoff is

$$\tilde{\pi}_L^* = \frac{c^3 + 3qc^2 + 3(1-q)c}{(1+c)^3} - 3c\frac{(1-q)(1+c^2) + 2(2q-1)c}{(1+c)^4}. \tag{29}$$

$\pi_L^* \geq 0$ implies that $\big[c^2 + 3qc + 3(1-q)\big](1+c) \geq 3\big[(1-q)(1+c^2) + 2(2q-1)c\big]$. Rearranging it, we obtain

$$c^2 + (6q-2)c + (9-12q) \geq 0. \tag{30}$$

Depending on the sign of $9-12q$, we have the following scenarios:

(1) if $1/2 \leq q \leq 3/4$, inequality (30) always holds, which implies that for all c, $\pi_L^* \geq 0$;

(2) if $q \geq 3/4$, we have $\pi_L^* \geq 0$ if and only if $c \geq -(3q-1) + \sqrt{9q^2 + 6q - 8}$.

That is, $\tilde{\pi}_L^* \geq 0$ if and only if $c \geq \kappa(q)$. By Lemma 2, $\kappa(q) < 1$ for any $q \in [1/2, 1]$.

According to Lemma 6, $\tilde{x}_i^*$, as a solution to $\frac{d\tilde{\pi}_i}{dx_i} = 0$, is the unique best response to $\tilde{x}_{-i}^* > 0$ when $c \geq \kappa(q)$. Thereafter, $\big((\tilde{x}_H^*, \tilde{x}_H^*, \tilde{x}_H^*), (\tilde{x}_L^*, \tilde{x}_L^*, \tilde{x}_L^*)\big)$ is an equilibrium of election game $\tilde{\mathcal{G}}(q)$.

Moreover, by Remark 3 of Lemma 7, if $\tilde{\mathcal{G}}$ has multiple equilibria, those equilibria must be interchangeable, i.e., some other strategy $\hat{x}_i$ and $\tilde{x}_{-i}^*$ also consist of an equilibrium. But by the uniqueness of best response, that is impossible. Therefore, $\big((\tilde{x}_H^*, \tilde{x}_H^*, \tilde{x}_H^*), (\tilde{x}_L^*, \tilde{x}_L^*, \tilde{x}_L^*)\big)$ must be the unique equilibrium of $\tilde{\mathcal{G}}(q)$.

Step 2: Identify a mixed strategy equilibrium when $c \leq \kappa(q)$. We consider a special form of mixed strategy equilibrium: (i) H adopts a pure uniform strategy $\sigma_H^* = (\tilde{x}_H^*, \tilde{x}_H^*, \tilde{x}_H^*)$; (ii) L adopts a mixed uniform strategy σ_L^*, in which $(\tilde{x}_L^*, \tilde{x}_L^*, \tilde{x}_L^*)$ is played with probability p^* and $(0,0,0)$ is played with probability $1-p^*$. Then candidates' payoffs are

$$\tilde{\pi}_H(\sigma_H^*, \sigma_L^*; q) = 1 - p^* + p^* \cdot p_H - 3c_H \tilde{x}_H^*, \tag{31}$$

$$\tilde{\pi}_L(\sigma_H^*, \sigma_L^*; q) = p^*\big(1 - p_H - 3c_L \tilde{x}_L^*\big), \tag{32}$$

in which p_H is H's winning probability under effort profile $\big((\tilde{x}_H^*, \tilde{x}_H^*, \tilde{x}_H^*), (\tilde{x}_L^*, \tilde{x}_L^*, \tilde{x}_L^*)\big)$.

Note that equation (31) can be rewritten as

$$\tilde{\pi}_H(\sigma_H^*, \sigma_L^*; q) = 1 - p^* + p^* \left(p_H - 3\frac{c_H}{p^*}\tilde{x}_H^*\right). \tag{33}$$

Thus, as long as L is active, s/he is effectively competing with a candidate with marginal effort cost c_H/p. By the result in Step 1, we should have $\tilde{x}_L^*/\tilde{x}_H^* = (c_H/p^*)/c_L = c/p^*$.

Since L mixes between $(\tilde{x}_L^*, \tilde{x}_L^*, \tilde{x}_L^*)$ and $(0,0,0)$, L's payoff from playing $(\tilde{x}_L^*, \tilde{x}_L^*, \tilde{x}_L^*)$ should be the same as that from playing $(0,0,0)$, which is 0. Thus, we should have $c/p^* = \kappa(q)$, which implies that $p^* = c/\kappa(q)$ and

$$\tilde{x}_H^* = \frac{c}{c_H} \cdot \frac{(1-q)\left[1+\kappa(q)^2\right] + 2(2q-1)\kappa(q)}{\left[1+\kappa(q)\right]^4}, \tag{34}$$

$$\tilde{x}_L^* = \frac{\kappa(q)}{c_L} \cdot \frac{(1-q)\left[1+\kappa(q)^2\right] + 2(2q-1)\kappa(q)}{\left[1+\kappa(q)\right]^4}. \tag{35}$$

Clearly, $\tilde{x}_H^*$ is the unique best response to $(p^*, \tilde{x}_L^*)$, and both $\tilde{x}_L^*$ and 0 are the only two best responses to $\tilde{x}_H^*$. Given $\tilde{x}_H^*$, L would mixes between $\tilde{x}_L^*$ and 0, and the probability of choosing $\tilde{x}_L^*$ is uniquely pinned down by the equation $\tilde{x}_L^*/\tilde{x}_H^* = c/p^*$, which gives $p^* = c/\kappa(q)$. Therefore, by the unqiueness of mutual best responses and the interchangeability of the equilibria of $\tilde{\mathcal{G}}(q)$, (σ_H^*, σ_L^*) must be the unique equilibrium of $\tilde{\mathcal{G}}(q)$ for $c \leq \kappa(q)$. ■

Lemma 6. *Given any $x_{-i} \geq 0$, define function*

$$\hat{\pi}_i(x_i) = \frac{x_i^3 + 3q x_i^2 x_{-i} + 3(1-q) x_i x_{-i}^2}{(x_i + x_{-i})^3} - 3c_i x_i, \ \ 1/2 \leq q \leq 1.$$

If x_i^{**} is a solution to its FOC and $\hat{\pi}_i(x_i^{**}, x_i^{**}, x_i^{**}) > 0$, then x_i^{**} is its unique maximizer.

Proof. The first-order derivative of $\hat{\pi}_i$ is

$$\frac{d\hat{\pi}_i}{dx_i} = 3\left\{\frac{(1-q)x_i^2 x_{-i} + 2(2q-1)x_i x_{-i}^2 + (1-q)x_{-i}^3}{(x_i + x_{-i})^4} - c_i\right\}. \tag{36}$$

The second derivative is

$$\frac{d^2\hat{\pi}_i}{dx_i^2} = \frac{6x_{-i}}{(x_i + x_{-i})^5} \cdot \left\{-(1-q)x_i^2 + (4-7q)x_i x_{-i} + (4q-3)x_{-i}^2\right\}. \tag{37}$$

The sign of the second-order derivative depends on the part inside the bracket. Thus, we consider equation

$$(1-q)x_i^2 - (4-7q)x_i x_j - (4q-3)x_j^2 = 0, \tag{38}$$

for which the discriminant of the equation $\Delta = (11q-2)(3q-2)$. Depending on the value of q, there are the following two cases:

Case 1: $q > 3/4$. In this case, $\Delta > 0$, and it implies that there are two roots to equation (38). Let x^1 and x^2 be the two roots, and we have $x^1 \cdot x^2 = (3-4q)/(1-q) < 0$, that is, one root is positive and the other is negative. Therefore, $\frac{d\hat{\pi}_i}{dx_i}$ is concave in x_i. Depending on whether $\frac{d\hat{\pi}_i}{dx_i}|_{x_i=0} \geq 0$ or not, we have the following two scenarios: (i) $\hat{\pi}_i(x_i)$ is concave; (ii) $\hat{\pi}_i(x_i)$ first decreases, then increases, and lastly decreases.

Case 2: $q \leq 3/4$. In this case, either $\Delta \geq 0$ and both roots are negative or $\Delta < 0$. Under either scenario, we have $\frac{d^2\hat{\pi}_i}{dx_i^2} \leq 0$ if $x_i \geq 0$, which implies that $\frac{d\hat{\pi}_i}{dx_i}$ is decreasing in x_i on the range. Depending on whether $\frac{d\hat{\pi}_i}{dx_i}|_{x_i=0} \geq 0$ or not, $\frac{d\hat{\pi}_i}{dx_i} = 0$ either has a single solution or no solultion.

Under either case, as long as x_i^{**} is a solution to $\frac{d\hat{\pi}_i}{dx_i} = 0$ and $\hat{\pi}_i(x_i^{**}, x_i^{**}, x_i^{**}) > 0$, x_i^{**} must be the unique maximizer to $\hat{\pi}_i$. ■

Definition 4. *Given a two-player game, two equilibria—say, (μ_1^*, μ_2^*) and (μ_1^{**}, μ_2^{**})— are interchangeable if (μ_1^*, μ_2^{**}) and (μ_1^{**}, μ_2^*) are equilibrium as well.*

Lemma 7. *Given a game with two players, $i \in \{1, 2\}$, denote players' strategies by μ_1 and μ_2, respectively. If the players' payoff functions satisfy*

1. *Separable: $U_i(\mu_1, \mu_2) = F_i(\mu_1, \mu_2) - c_i(\mu_i)$,*

2. *Fixed sum: $F_1(\mu_1, \mu_2) + F_2(\mu_1, \mu_2) = constant$,*

then the equilibria of the game are interchangeable.

Proof. Consider two equilibria, $\mu^* = (\mu_1^*, \mu_2^*)$ and $\mu^{**} = (\mu_1^{**}, \mu_2^{**})$. By definition of equilibrium, we have

$$F_1(\mu_1^{**}, \mu_2^*) - c_1(\mu_1^{**}) \leq F_1(\mu_1^*, \mu_2^*) - c_1(\mu_1^*),$$

which implies that $F_1(\mu_1^{**}, \mu_2^*) - F_1(\mu_1^*, \mu_2^*) \leq c_1(\mu_1^{**}) - c_1(\mu_1^*)$.

Since for any (μ_1, μ_2), $F_1(\mu_1, \mu_2) + F_2(\mu_1, \mu_2) = constant$, it follows that

$$F_2(\mu_1^*, \mu_2^*) - F_2(\mu_1^{**}, \mu_2^*) \leq c_1(\mu_1^{**}) - c_1(\mu_1^*). \tag{39}$$

Similarly, for player 2, the definition of equilibrium implies that

$$F_2(\mu_1^{**}, \mu_2^*) - F_2(\mu_1^{**}, \mu_2^{**}) \leq c_2(\mu_2^*) - c_2(\mu_2^{**}). \tag{40}$$

Adding inequalities (39) and (40), we obtain

$$F_2(\mu_1^*, \mu_2^*) - F_2(\mu_1^{**}, \mu_2^{**}) \leq \big(c_1(\mu_1^{**}) - c_1(\mu_1^*)\big) + \big(c_2(\mu_2^*) - c_2(\mu_2^{**})\big). \tag{41}$$

Switching the role of μ^* and μ^{**}, we have

$$F_2(\mu_1^{**}, \mu_2^{**}) - F_2(\mu_1^*, \mu_2^*) \leq \big(c_1(\mu_1^*) - c_1(\mu_1^{**})\big) + \big(c_2(\mu_2^{**}) - c_2(\mu_2^*)\big). \tag{42}$$

Inequalities (41) and (42) together imply

$$F_2(\mu_1^{**}, \mu_2^{**}) - F_2(\mu_1^*, \mu_2^*) = \big(c_1(\mu_1^*) - c_1(\mu_1^{**})\big) + \big(c_2(\mu_2^{**}) - c_2(\mu_2^*)\big). \tag{43}$$

Using (41), (39), and (40), we can show that all of the above inequalities hold with equality. Thus, we have

$$U_1(\mu_1^{**}, \mu_2^*) = U_1(\mu_1^*, \mu_2^*),$$

$$U_2(\mu_1^{**}, \mu_2^*) = U_2(\mu_1^{**}, \mu_2^{**}),$$

which implies that (μ_1^{**}, μ_2^*) are mutual best responses. Analogously, we have

$$U_1(\mu_1^*, \mu_2^{**}) = U_1(\mu_1^{**}, \mu_2^{**}),$$

$$U_2(\mu_1^*, \mu_2^{**}) = U_2(\mu_1^*, \mu_2^*).$$

That is, μ_1^* and μ_2^{**} are mutual best responses. Similarly, we could also show that μ_1^{**} and μ_2^* are mutual best responses. ■

Remark 2. *The proof of interchangeability is adapted from that of Ewerhart (2017).*

Remark 3. *In a two-player election game, payoff functions satisfy the conditions in Lemma 7, and thus the equilibria of the game are interchangeable.*

PROOF OF PROPOSITION 2:

In the unrestricted game $\mathcal{G}(q)$, we have shown that, given his/her opponent's uniform strategy, a candidate's payoff function is Schur-concave in Lemma 5. By Remark 1, if the opponent adopts a uniform strategy, a candidate does better by also adopting a uniform strategy as well. Since the uniform strategies of H and L in the equilibrium of $\tilde{\mathcal{G}}(q)$ are mutual best responses, they must be mutual best responses in $\mathcal{G}(q)$. Thus, the equilibrium of $\tilde{\mathcal{G}}(q)$ is also an equilibrium of $\mathcal{G}(q)$. ■

PROOF OF PROPOSITION 3:

According to Remark 3 of Lemma 7, the equilibria of game $\mathcal{G}(q)$ are interchangeable. By interchangeablity, if there is another equilibrium of $\mathcal{G}(q)$, say, (μ_H^*, μ_L^*), then for any $i \in \{H, L\}$, μ_i^* and $-i$'s equilibrium strategies defined by Propositions 1 and 2 must be equilibrium as well. Note that i and $-i$'s equilibrium strategies defined by Propositions 1 and 2 are mutually unique best responses, and thus such (μ_H^*, μ_L^*) does not exist. Therefore, the equilibrium of $\mathcal{G}(q)$ must be unique. ■

PROOF OF COROLLARY 1:

(i) The results follow directly from the equilirbium characterization in Propositions 1 and 2 .

(ii) Based on the result of (i), we have $3\tilde{x}_L^*(q) \cdot c_L = 3\tilde{x}_H^*(q) \cdot c_H$ if $c \geq \kappa(q)$. When $c \leq \kappa(q)$, with probability $c/\kappa(q)$, L exerts $\tilde{x}_L^*(q)$ in each district, and we thus have $\big(c/\kappa(q)\big) \cdot \tilde{x}_L^*(q) \cdot c_L = \tilde{x}_H^*(q) \cdot c_H$. That is, L's expected total effort cost equals H's total effort cost.

(iii) $p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big)$ is obtained by directly plugging the equilibrium strategies characterized by Propositions 1 and 2 into equation (6).

When $\kappa(q) \leq c \leq 1$, we have

$$p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) = \frac{1 + 3qc + 3(1-q)c^2}{(1+c)^3} = 1 - \frac{c^3 + 3qc^2 + 3(1-q)c}{(1+c)^3}. \tag{44}$$

Since $q \geq 1/2$, we have $\frac{1+3qc+3(1-q)c^2}{(1+c)^3} - \frac{c^3+3qc^2+3(1-q)c}{(1+c)^3} = \frac{1-c^3+3(2q-1)c(1-c)}{(1+c)^3} \geq 0$. It follows that $p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) = \frac{1+3qc+3(1-q)c^2}{(1+c)^3} \geq 1/2$ for any $c \geq \kappa(q)$. Equality holds only when $c = 1$.

When $c \leq \kappa(q)$, we have

$$\begin{aligned} p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) =& 1 - \frac{c}{\kappa(q)} + \frac{c}{\kappa(q)} \cdot \frac{1 + 3q\kappa(q) + 3(1-q)\kappa(q)^2}{[1+\kappa(q)]^3} \\ =& 1 - c \cdot \frac{\kappa(q)^2 + 3q\kappa(q) + 3(1-q)}{[1+\kappa(q)]^3}, \end{aligned} \tag{45}$$

which is decreasing in c. Thus we have, for any $c \leq \kappa(q)$, $p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) \geq p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big)|_{c=\kappa(q)} > 1/2$.

(iv) According to Propositions 1 and 2 and their proofs, we have $\pi_L^*(q) = \frac{c^2+(6q-2)c+(9-12q)}{(1+c)^4}$ when $c \geq \kappa(q)$, $\pi_L^*(q) = 0$ when $c < \kappa(q)$, and $\pi_L^*(q) \geq 0$ with strict inequality when $c > \kappa(q)$.

For candidate H, the equilibrium characterized in Propositions 1 and 2 implies that

$$\begin{aligned} \pi_H^*(q) =& \frac{1 + 3qc + 3(1-q)c^2}{(1+c)^3} - 3c \cdot \frac{(1-q)(1+c^2) + 2(2q-1)c}{(1+c)^4} \\ =& \frac{1 + (6q-2)c + (9-12q)c^2}{(1+c)^4}, \end{aligned} \tag{46}$$

when $c \geq \kappa(q)$. When $c < \kappa(q)$, $\pi_L^*(q) = 0$ implies that L's total effort cost equals $1 - p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big)$. Using (ii), we have $\pi_H^*(q) = 2p_H\big(\mathbf{x}_H^*(q), \mathbf{x}_L^*(q); q\big) - 1$.

According to (iii), H's winning probability is strictly greater than L's when $c \in (0,1)$ and equal to L's when $c = 1$. Thus, $\pi_H^*(q) \geq \pi_L^*(q)$ with equality for $c = 1$. By the property of $\pi_L^*(q)$, we have $\pi_H^*(q) > 0$ for all $c \in (0,1]$. ■

PROOF OF LEMMA 3

Recall that the election games under EC and PV are just two special cases of $\mathcal{G}(q)$. It thus suffices to inspect how candidates' winning probabilities change with q in $\mathcal{G}(q)$.

Given strategy profile $(\mathbf{x}_i, \mathbf{x}_{-i})$ with $\mathbf{x}_i = (x_i, x_i, x_i)$ and $\mathbf{x}_{-i} = (x_{-i}, x_{-i}, x_{-i})$, candidate i's election winning probability in $\mathcal{G}(q)$ is

$$p_i(\mathbf{x}_i, \mathbf{x}_{-i}; q) = \frac{x_i^3 + 3qx_i^2 x_{-i} + 3(1-q)x_i x_{-i}^2}{(x_i + x_{-i})^3}, \tag{47}$$

and candidate $-i$'s election winning probability is $1 - p_i(\mathbf{x}_i, \mathbf{x}_{-i}; q)$.

Without loss of generality, we assume that $x_i > x_{-i}$, that is, candidate i is the one who exerts relatively more effort. Differentiating p_i with respect to q, we obtain

$$\frac{\partial p_i}{\partial q} = \frac{3x_i x_{-i}}{(x_i + x_{-i})^3}(x_i - x_{-i}) > 0. \tag{48}$$

Since $q^{EC} > q^{PV}$, we have $p_i\left(\mathbf{x}_i, \mathbf{x}_{-i}; q^{EC}\right) > p_i\left(\mathbf{x}_i, \mathbf{x}_{-i}; q^{PV}\right)$. ■

PROOF OF THEOREM 1:

In this proof, we divide the whole range of c into three intervals and compare H's winning probabilities under the two schemes on each interval separately.

When $c > \kappa_{EC}$, the difference in candidate H's winning probabilities under the two schemes is

$$\begin{aligned}
&WP_H\left(q^{EC}\right) - WP_H\left(q^{PV}\right)\\
=&\frac{1 + 3q^{EC}c + 3\left(1 - q^{EC}\right)c^2}{(1+c)^3} - \frac{1 + 3q^{PV}c + 3\left(1 - q^{PV}\right)c^2}{(1+c)^3}\\
=&\frac{3\left(q^{EC} - q^{PV}\right)(c - c^2)}{(1+c)^3} \geq 0.
\end{aligned} \tag{49}$$

When $\kappa_{PV} < c \leq \kappa_{EC}$, the difference in H's equilibrium winning probabilities under the two schemes is

$$\begin{aligned}
&WP_H\left(q^{EC}\right) - WP_H\left(q^{PV}\right)\\
=&\underbrace{1 - c \cdot \frac{\kappa_{EC}^2 + 3q^{EC}\kappa_{EC} + 3\left(1 - q^{EC}\right)}{(1+\kappa_{EC})^3} - \left[1 - c \cdot \frac{\kappa_{EC}^2 + 3q^{PV}\kappa_{EC} + 3\left(1 - q^{PV}\right)}{(1+\kappa_{EC})^3}\right]}_{\text{“rounding effect”}}\\
&+\underbrace{1 - c \cdot \frac{\kappa_{EC}^2 + 3q^{PV}\kappa_{EC} + 3\left(1 - q^{PV}\right)}{(1+\kappa_{EC})^3} - \frac{1 + 3q^{PV}c + 3\left(1 - q^{PV}\right)c^2}{(1+c)^3}}_{\text{“strategic effect”}}.
\end{aligned} \tag{50}$$

The rounding effect on H's winning probability is positive, since $q^{EC} > q^{PV}$ and $\kappa_{EC} < 1$.

The strategic effect can be rewritten as

$$\begin{aligned}
&\frac{c^3 + 3q^{PV}c^2 + 3\left(1 - q^{PV}\right)c}{(1+c)^3} - c \cdot \frac{\kappa_{EC}^2 + 3q^{PV}\kappa_{EC} + 3\left(1 - q^{PV}\right)}{(1+\kappa_{EC})^3}\\
=&c \cdot \left[\frac{c^2 + 3q^{PV}c + 3\left(1 - q^{PV}\right)}{(1+c)^3} - \frac{\kappa_{EC}^2 + 3q^{PV}\kappa_{EC} + 3\left(1 - q^{PV}\right)}{(1+\kappa_{EC})^3}\right].
\end{aligned} \tag{51}$$

Consider the derivative of the first term in the above bracket,

$$\frac{d}{dc}\left(\frac{c^2+3q^{PV}c+3(1-q^{PV})}{(1+c)^3}\right)=\frac{\left(2c+3q^{PV}\right)(1+c)-3\left[c^2+3q^{PV}c+3\left(1-q^{PV}\right)\right]}{(1+c)^4}$$
$$=-\frac{c^2+\left(6q^{PV}-2\right)c+\left(9-12q^{PV}\right)}{(1+c)^4}. \tag{52}$$

Since $c \geq \kappa_{PV}$, we always have $d\left(\frac{c^2+3q^{PV}c+3\left(1-q^{PV}\right)}{(1+c)^3}\right)/dc \leq 0$, which implies that $\frac{c^2+3q^{PV}c+3(1-q^{PV})}{(1+c)^3}$ is decreasing in c for $c \geq \kappa_{PV}$. Since $c \leq \kappa_{EC}$ and the second term in the above bracket can be obtained by plugging κ_{EC} into the first term, the strategic effect should be positive. Thus, we have $WP_H\left(q^{EC}\right)-WP_H\left(q^{PV}\right)>0$.

When $c \leq \kappa_{PV}$, the difference in H's winning probabilities under the two schemes is

$$WP_H\left(q^{EC}\right)-WP_H\left(q^{PV}\right)=\text{rounding effect}+\text{strategic effect}.$$

The rounding effect is the same as above. The strategic effect now becomes

$$\begin{aligned}&1-c\cdot\frac{\kappa_{EC}^2+3q^{PV}\kappa_{EC}+3\left(1-q^{PV}\right)}{(1+\kappa_{EC})^3}-\left[1-c\cdot\frac{\kappa_{PV}^2+3q^{PV}\kappa_{PV}+3\left(1-q^{PV}\right)}{(1+\kappa_{PV})^3}\right]\\ =&c\cdot\left[\frac{\kappa_{PV}^2+3q^{PV}\kappa_{PV}+3\left(1-q^{PV}\right)}{(1+\kappa_{PV})^3}-\frac{\kappa_{EC}^2+3q^{PV}\kappa_{EC}+3\left(1-q^{PV}\right)}{(1+\kappa_{EC})^3}\right].\end{aligned} \tag{53}$$

With the same argument as above, the strategic effect is positive, which, together with the positive rounding effect, implies $WP_H\left(q^{EC}\right)-WP_H\left(q^{PV}\right)>0$. ■

PROOF OF THEOREM 2:

In this proof, we divide the whole range of c into three intervals and conduct a comparison on each of them separately. Before proceeding to conduct the comparison, we first define

$$h(c,q)=\frac{(1-q)(1+c^2)+2(2q-1)c}{(1+c)^4}.$$

The total campaign expenditure in election game $\mathcal{G}(q)$ can be rewritten as $TCE(q)=6c\cdot h\big(\max\{c,\kappa(q)\},q\big)$. Thus, comparing $TCE\left(q^{EC}\right)$ and $TCE\left(q^{PV}\right)$ is actually comparing $h\left(\max\{c,\kappa_{EC}\},q^{EC}\right)$ and $h\left(\max\{c,\kappa_{PV}\},q^{PV}\right)$.

When $c \geq \kappa_{EC}$, we have $\max\{c,\kappa_{EC}\}=c$ and $\max\{c,\kappa_{PV}\}=c$. The comparison becomes comparing $h\left(c,q^{EC}\right)$ and $h\left(c,q^{PV}\right)$. The first derivative of $h(c,q)$ with respect to q is

$$\partial h/\partial q = \frac{-1-c^2+4c}{(1+c)^4}. \tag{54}$$

Then we have $\partial h/\partial q > 0$ if and only if $2-\sqrt{3} \leq c \leq 2+\sqrt{3}$. Since $[\kappa_{EC}, 1] \in \left[2-\sqrt{3}, 2+\sqrt{3}\right]$, h must be increasing in q in this scenario, which implies that $TCE\left(q^{EC}\right) > TCE\left(q^{PV}\right)$ as $q^{EC} > q^{PV}$.

When $0 < c \leq \kappa_{PV}$, each scheme induces a mixed strategy equilibrium in which L mixes between staying active and and inactive. Thus, the payoff of L is zero under either scheme, which implies that for any $q \in \left\{q^{EC}, q^{PV}\right\}$, $p^* \cdot 3c_L\tilde{x}_L^*(q) = 1 - WP_H(q)$. Since equilibrium candidates' effort ratio is $\hat{x}_L^*/\hat{x}_H^* = c/p^*$, H's effort cost is $3c_H\hat{x}_H^* = p^* \cdot 3c_L\hat{x}_L$. Thus, the total campaign expenditure under either scheme can be written as

$$TCE(q) = 2\left[1 - WP_H(q)\right], \quad \forall q \in \{q^{EC}, q^{PV}\}. \tag{55}$$

By the proof of Theorem 1 (the last scenario), we have $TCE\left(q^{PV}\right) > TCE\left(q^{EC}\right)$ in this scenario.

When $\kappa_{PV} \leq c \leq \kappa_{EC}$, we have $\max\{c, \kappa_{EC}\} = \kappa_{EC}$ and $\max\{c, \kappa_{PV}\} = c$. The comparison becomes comparing $h\left(\kappa_{EC}, q^{EC}\right)$ and $h\left(c, q^{PV}\right)$.

The first partial derivative of $h\left(c, q^{PV}\right)$ with respect to c is

$$\frac{dh\left(c, q^{PV}\right)}{dc} = 2 \cdot \frac{(4c - c^2 - 3) + q^{PV}(c^2 + 4 - 7c)}{(1+c)^5}.$$

Note that now c induces a pure strategy equilibrium under PV. According to inequality (30), we have $-c^2 + 2c - 9 \leq (6c - 12)q^{PV}$. Then we obtain that

$$\begin{aligned}(4c - c^2 - 3) + q^{PV}(c^2 + 4 - 7c) \leq & 2c + 6 + (6c - 12)q^{PV} + q^{PV}(c^2 + 4 - 7c) \\ = & 2c + 6 + q^{PV}(c^2 - 8 - c) \leq 2c + 6 + c^2 - 8 - c \leq 0,\end{aligned}$$

which indicates that $\partial h\left(c, q^{PV}\right)/\partial c \leq 0$.

Recall that when $c = \kappa_{EC}$, $TCE\left(q^{EC}\right) > TCE\left(q^{PV}\right)$, which implies $h\left(\kappa_{EC}, q^{EC}\right) > h\left(\kappa_{EC}, q^{PV}\right)$; when $c = \kappa_{PV}$, $TCE\left(q^{EC}\right) < TCE\left(q^{PV}\right)$, which implies $h\left(\kappa_{EC}, q^{EC}\right) > h\left(\kappa_{PV}, q^{PV}\right)$. By the monotonicity of $h\left(c, q^{PV}\right)$ in c, $h\left(c, q^{PV}\right)$ takes value $h\left(\kappa_{EC}, q^{EC}\right)$ only once on interval $(\kappa_{PV}, \kappa_{EC})$.

In summary, there exists a cutoff $\hat{c}$ such that $TCE\left(q^{EC}\right) \geq TCE\left(q^{PV}\right)$ when $c \geq \hat{c}$ and $TCE\left(q^{EC}\right) \leq TCE\left(q^{PV}\right)$ when $c \leq \hat{c}$. ■

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