Discussion of asymmetric districts

November 14, 2022

I. No Dominant District

Assume that there are three district and the number of votes in these districts are $N_1 = 1$, $N_2 = N_3 = \alpha > 1/2$. Contional on the winning districts, a candidate's probability of winning the election is given by the following table:

No. of winning districts	Electoral College	Popular Vote
$1+2+3$	1	1
$1+2$ *or* 3	1	p_1
$2+3$	1	p_2
2 *or* 3	0	$1-p_1$
1	0	$1-p_2$

In the table, p_1 and p_2 are given by the following equations:

$$p_1 = \int_{z_1+\alpha z_2+\alpha z_3 \geq \frac{1+2\alpha}{2}} dF(z_1)dF(z_2)dG(z_3) \tag{1}$$

$$p_2 = \int_{z_1+\alpha z_2+\alpha z_3 \geq \frac{1+2\alpha}{2}} dG(z_1)dF(z_2)dF(z_3) \tag{2}$$

Since distribution F is first order stochastic dominates G, we should have $p_1 > p_2 > 1-p_1$.

Proposition 1. *Given opponent's strategy (c, d, d), any strategy (a, b, c) is dominated by some strategy (a, k, k).*

Proof. Let (c, d, d) be candidate $-i$'s strategy. Given any arbitrary strategy (a, b, e) of candidate i, his payoff is

$$\pi_i = \frac{abe + p_1(abd + aed) + p_2 cbe + (1-p_1)(cde + cdb) + (1-p_2)ad^2}{(a+c)(b+d)(e+d)} - c_i(a+b+e) \quad (3)$$

Differentiate π_i with respect to b and e respectively, we have

$$\frac{\partial \pi_i}{\partial b} = \frac{d}{b+d} \cdot \frac{(1-p_1)(ae+cd) + (p_1+p_2-1)(ad+ce)}{(a+c)(b+d)(e+d)} - c_i \quad (4)$$

$$\frac{\partial \pi_i}{\partial e} = \frac{d}{e+d} \cdot \frac{(1-p_1)(ab+cd) + (p_1+p_2-1)(ad+bc)}{(a+c)(b+d)(e+d)} - c_i \quad (5)$$

Take the difference of the above two partial derivatives, we have

$$\begin{aligned}
\frac{\partial \pi_H}{\partial b} - \frac{\partial \pi_H}{\partial e} =& \frac{d}{(a+c)(b+d)(e+d)} \cdot \left[\frac{(1-p_1)(ae+cd) + (p_1+p_2-1)(ad+ce)}{b+d} \right. \\
& \left. - \frac{(1-p_1)(ab+cd) + (p_1+p_2-1)(ad+bc)}{e+d} \right] \\
=& \frac{d}{(a+c)(b+d)(e+d)} \cdot \left[\frac{(1-p_1)(ae^2 + ced + aed + cd^2 - ab^2 - cbd - abd - cd^2)}{(b+d)(e+d)} \right. \\
& \left. + \frac{(p_1+p_2-1)(ade + ce^2 + ad^2 + ced - adb - b^2c - ad^2 - bcd)}{(b+d)(e+d)} \right] \\
=& \frac{d}{(a+c)(b+d)(e+d)} \cdot \left\{ \frac{(1-p_1)\left[a(e^2-b^2) + cd(e-b) + ad(e-b)\right]}{(b+d)(e+d)} \right. \\
& \left. + \frac{(p_1+p_2-1)\left[ad(e-b) + c(e^2-b^2) + cd(e-b)\right]}{(b+d)(e+d)} \right\} \\
=& \frac{d(e-b)}{(a+c)(b+d)(e+d)} \cdot \left\{ \frac{(1-p_1)\left[a(e+b) + cd + ad\right]}{(b+d)(e+d)} + \frac{(p_1+p_2-1)\left[ad + c(e+b) + cd\right]}{(b+d)(e+d)} \right\} \\
=& \frac{d(e-b)}{(a+c)(b+d)(e+d)} \cdot \frac{p_2(c+a)d + (e+b)\left[(p_1+p_2-1)c + (1-p_1)a\right]}{(b+d)(e+d)}.
\end{aligned} \quad (6)$$

Since $1 > p_1$, $p_1 > 1 - p_2$, and $p_2 > 0$, we should have

$$(b-e)\left(\frac{\partial \pi_H}{\partial b} - \frac{\partial \pi_H}{\partial e}\right) < 0.$$

Clearly, the Schur Ostrowski criterion is satisfied. Thus, for any strategy (a,b,e), there exists some strategy (a,k,k) that generates a higher payoff.

$\square$

In the following, we proceed to look for an equilibrium that candidates adopt the same act in the two districts whose number of votes is α.

Given the strategies of H and L, (a, b, b) and (c, d, d), the expected payoffs are

$$\pi_H = \frac{ab^2 + 2p_1abd + p_2cb^2 + 2(1-p_1)bcd + (1-p_2)ad^2}{(a+c)(b+d)^2} - c_H(a+2b) \tag{7}$$

$$\pi_L = \frac{cd^2 + 2p_1cdb + p_2ad^2 + 2(1-p_1)dab + (1-p_2)cb^2}{(a+c)(b+d)^2} - c_L(c+2d) \tag{8}$$

The first order conditions are

$$\frac{\partial \pi_H}{\partial a} = c \cdot \frac{(1-p_2)b^2 + 2(2p_1-1)bd + (1-p_2)d^2}{(a+c)^2(b+d)^2} - c_H = 0 \tag{9}$$

$$\frac{\partial \pi_H}{\partial b} = 2d \cdot \frac{(1-p_1)(ab+cd) + (p_1+p_2-1)(cb+ad)}{(a+c)(b+d)^3} - 2c_H = 0 \tag{10}$$

$$\frac{\partial \pi_L}{\partial c} = a \cdot \frac{(1-p_2)d^2 + 2(2p_1-1)db + (1-p_2)b^2}{(a+c)^2(b+d)^2} - c_L = 0 \tag{11}$$

$$\frac{\partial \pi_L}{\partial d} = 2b \cdot \frac{(1-p_1)(cd+ab) + (p_1+p_2-1)(ad+cb)}{(a+c)(b+d)^3} - 2c_L = 0 \tag{12}$$

Combining these equations, we have $c/a = c_H/c_L = k$ and $d/b = c_H/c_L = k$. Plug the ratios back into the FOCs, we have

$$a^* = \frac{k}{c_H} \cdot \frac{(1-p_2)(1+k^2) + 2(2p_1-1)k}{(1+k)^4} \tag{13}$$

$$b^* = \frac{k}{c_H} \cdot \frac{(1-p_1)(1+k^2) + 2(p_1+p_2-1)k}{(1+k)^4} \tag{14}$$

$$c^* = \frac{k}{c_L} \cdot \frac{(1-p_2)(1+k^2) + 2(2p_1-1)k}{(1+k)^4} \tag{15}$$

$$d^* = \frac{k}{c_L} \cdot \frac{(1-p_1)(1+k^2) + 2(p_1+p_2-1)k}{(1+k)^4} \tag{16}$$

A. Second order condition

Now we check the second order conditions to see whether the solution to FOCs is the payoff maximizer or not . Withou loss of generality, we consider the second order condition for candidate H's payoff functions:

$$
\begin{aligned}
\frac{\partial^2 \pi_H}{\partial a^2}|_{a=a^*,b=b^*} =& -2c^* \cdot \frac{(1-p_2)b^2 + 2(2p_1-1)bd^* + (1-p_2)(d^*)^2}{(a+c^*)^3(b+d^*)^2}|_{a=a^*,b=b^*} \\
=& -2k \cdot \frac{(1-p_2) + 2(2p_1-1)k + (1-p_2)k^2}{(a^*)^2(1+k)^5} \\
=& \frac{-2c_H}{a^*(1+k)} < 0
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\frac{\partial^2 \pi_H}{\partial b^2}|_{a=a^*,b=b^*} =& \frac{2d^*}{(a^*+c^*)} \cdot \left[\frac{(1-p_1)a^* + (p_1+p_2-1)c^*}{(b^*+d^*)^3} \right. \\
& \left. -3\frac{(1-p_1)(a^*b^*+c^*d^*) + (p_1+p_2-1)(c^*b^*+a^*d^*)}{(b^*+d^*)^4} \right] \\
=& \frac{2d^*}{(a^*+c^*)} \cdot \left\{ \frac{[(1-p_1)a^* + (p_1+p_2-1)c^*]b^* + [(1-p_1)a^* + (p_1+p_2-1)c^*]d^*}{(b^*+d^*)^4} \right. \\
& \left. -3\frac{[(1-p_1)a^* + (p_1+p_2-1)c^*]b^* + [(1-p_1)c^* + (p_1+p_2-1)a^*]d^*}{(b^*+d^*)^4} \right\} \\
=& \frac{2d^*}{(a^*+c^*)} \cdot \left\{ \frac{[(1-p_1)a^* + (p_1+p_2-1)c^*]d^*}{(b^*+d^*)^4} \right. \\
& \left. -\frac{2[(1-p_1)a^* + (p_1+p_2-1)c^*]b^* + 3[(1-p_1)c^* + (p_1+p_2-1)a^*]d^*}{(b^*+d^*)^4} \right\} \\
=& \frac{2k}{(1+k)} \cdot \left\{ \frac{(1-p_1)k + (p_1+p_2-1)k^2}{(b^*)^2(1+k)^4} \right. \\
& \left. -\frac{2(1-p_1) + 2k(p_1+p_2-1) + 3k^2(1-p_1) + 3k(p_1+p_2-1)}{(b^*)^2(1+k)^4} \right\} \\
=& \frac{2k}{(1+k)} \cdot \left\{ \frac{(1-p_1)k + (p_1+p_2-1)k^2 - k^2(1-p_1) - k(p_1+p_2-1)}{(b^*)^2(1+k)^4} \right. \\
& \left. -\frac{2(1-p_1) + 4k(p_1+p_2-1) + 2k^2(1-p_1)}{(b^*)^2(1+k)^4} \right\} \\
=& \frac{2k}{(1+k)} \cdot \left\{ \frac{(2-2p_1-p_2)k(1-k)}{(b^*)^2(1+k)^4} - \frac{2[(1-p_1)(1+k^2) + 2k(p_1+p_2-1)]}{(b^*)^2(1+k)^4} \right\}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\frac{\partial^2\pi_H}{\partial a\partial b}|_{a=a^*,b=b^*} =& \frac{2c^*d^*}{(a^*+c^*)^2}\cdot\frac{(2-2p_1-p_2)(b^*-d^*)}{(b^*+d^*)^3} \\
=& \frac{2(2-2p_1-p_2)k^2(1-k)}{a^*b^*(1+k)^5}
\end{aligned}
\tag{19}
$$

Let

$$
D=\frac{\partial^2\pi_H}{\partial a^2}|_{a=a^*,b=b^*}\times\frac{\partial^2\pi_H}{\partial b^2}|_{a=a^*,b=b^*}-\left(\frac{\partial^2\pi_H}{\partial b^2}|_{a=a^*,b=b^*}\right)^2
$$

Then $D>0$ is equivalent to

$$
\begin{aligned}
&\frac{-2c_H}{a^*(1+k)}\frac{2k}{(1+k)}\cdot\left\{\frac{(2-2p_1-p_2)k(1-k)}{(b^*)^2(1+k)^4}-\frac{2[(1-p_1)(1+k^2)+2k(p_1+p_2-1)]}{(b^*)^2(1+k)^4}\right\} \\
>&\frac{4(2-2p_1-p_2)^2k^4(1-k)^2}{(a^*b^*)^2(1+k)^{10}},
\end{aligned}
$$

which is

$$
\begin{aligned}
&-c_Hk\cdot\left\{\frac{(2-2p_1-p_2)k(1-k)}{b^*(1+k)^4}-\frac{2c_H}{k}\right\}>\frac{(2-2p_1-p_2)^2k^4(1-k)^2}{a^*b^*(1+k)^8} \\
&-c_Hk\cdot\frac{(2-2p_1-p_2)k(1-k)}{b^*(1+k)^4}+2c_H^2>\frac{(2-2p_1-p_2)^2k^4(1-k)^2}{a^*b^*(1+k)^8}
\end{aligned}
$$

Let $A=(1-p_2)(1+k^2)+2(2p_1-1)k$ and $B=(1-p_1)(1+k^2)+2(p_1+p_2-1)k$. Then $c_H=\frac{Bk}{b^*(1+k)^4}=\frac{Ak}{a^*(1+k)^4}$ and

$$
\begin{aligned}
&-k\cdot\frac{(2-2p_1-p_2)k(1-k)}{b^*(1+k)^4}+2c_H>\frac{(2-2p_1-p_2)^2k^4(1-k)^2}{a^*c_Hb^*(1+k)^8} \\
&-k\cdot\frac{(2-2p_1-p_2)k(1-k)}{b^*(1+k)^4}+2c_H>\frac{(2-2p_1-p_2)^2k^4(1-k)^2}{Akb^*(1+k)^4} \\
&-k\cdot\frac{(2-2p_1-p_2)k(1-k)}{Bk}c_H+2c_H>\frac{(2-2p_1-p_2)^2k^2(1-k)^2c_H}{AB} \\
&2>\frac{(2-2p_1-p_2)^2k^2(1-k)^2}{AB}+\frac{(2-2p_1-p_2)k(1-k)}{B}
\end{aligned}
$$

$A-B=(p_1-p_2)(1+k)^2>0$. We should have $A\geq B$, which implies that

$$
\frac{(2-2p_1-p_2)^2k^2(1-k)^2}{AB}+\frac{(2-2p_1-p_2)k(1-k)}{B}\leq\frac{(2-2p_1-p_2)^2k^2(1-k)^2}{B^2}+\frac{(2-2p_1-p_2)k(1-k)}{B}
$$

If $\left(\frac{(2-2p_1-p_2)k(1-k)}{B}+1/2\right)^2 < 9/4$, then $D>0$. If we can prove that

$$-2 < \frac{(2-2p_1-p_2)k(1-k)}{B} < 1, \tag{20}$$

then we could have $D > 0$.

Suppose that both F and G are uniform distributions. Then $p_1 = 1-\alpha/6$, $p_2 = 2-\frac{\alpha}{3}-\frac{1}{\alpha}+\frac{1}{6\alpha^2}$.

Then we have

$$2-2p_1-p_2 = \frac{2\alpha}{3}+\frac{1}{\alpha}-\frac{1}{6\alpha^2}-2.$$

Let $g(\alpha)=\frac{2\alpha}{3}+\frac{1}{\alpha}-\frac{1}{6\alpha^2}-2$. The first order condition of g is

$$g'(\alpha)=\frac{2}{3}-\frac{1}{a^2}+\frac{1}{3\alpha^3}=\frac{2}{3}+\frac{1-3\alpha}{3\alpha^3}$$

The second order condition is

$$g''(\alpha)=\frac{2}{\alpha^3}-\frac{1}{\alpha^4}=\frac{1}{\alpha^3}(2-\frac{1}{\alpha})$$

Since $\alpha>1/2$, we should have $g''(\alpha)>0$, which implies that g' is increasing when $\alpha>1/2$. Note that $g'(1)=0$, we should have $g'(\alpha)\leq 0$ when $1\geq\alpha>1/2$. Thus, g must be decreasing in α if $\alpha>1/2$. Note that $g(1/2)=-1/3$ and $g(1)=-1/2$. Using the decreasing property, we should have $g(\alpha)<0$ always holds.

Obviously, the right-hand side of (20) always hold. We consider the right-hand side. Since $B>1-p_1$

$$\frac{(2-2p_1-p_2)k(1-k)}{B}\geq 6k(1-k)\cdot\frac{g(\alpha)}{\alpha}\geq\frac{3}{2}\cdot\frac{g(\alpha)}{\alpha}$$

Let

$$h(\alpha)=\frac{g(\alpha)}{\alpha}=\frac{2}{3}+\frac{1}{\alpha^2}-\frac{1}{6\alpha^3}-\frac{2}{\alpha}$$

The foc is

$$h'(\alpha)=-\frac{2}{\alpha^3}+\frac{1}{2\alpha^4}+\frac{2}{\alpha^2}=\frac{1}{2\alpha^4}(-4\alpha+1+4\alpha^2)\geq 0$$

Clearly, $h(\alpha)$ must be increasing in α. $h(1)=-1/2$ and $h(1/2)=-2/3$, which implies that

$$\frac{(2-2p_1-p_2)k(1-k)}{B}\geq -1$$

Thus, the left-hand side of (20) holds as well.

$$
\begin{aligned}
p_1 &= \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1-z_3} \int_{\frac{1+2\alpha}{2}-\alpha z_2-\alpha z_3}^{1} 8dz_1dz_2dz_3 + \int_0^{\frac{1}{2}} \int_{1-z_3}^{1} 4dz_2dz_3 \\
&= 8\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1-z_3} [\alpha(z_2+z_3-1)+\frac{1}{2}]dz_2dz_3 + 4\int_0^{\frac{1}{2}} z_3dz_3 \\
&= 8\int_0^{\frac{1}{2}} \{\frac{\alpha}{2}(\frac{3}{2}-z_3)(\frac{1}{2}-z_3)+[\alpha(z_3-1)+\frac{1}{2}](\frac{1}{2}-z_3)\}dz_3 + \frac{1}{2} \\
&= 8\int_0^{\frac{1}{2}} [\frac{\alpha}{2}(z_3^2-2z_3+\frac{3}{4})+\alpha(-z_3^2+\frac{3}{2}z_3-\frac{1}{2})+\frac{1}{2}(\frac{1}{2}-z_3)]dz_3 + \frac{1}{2} \\
&= 8\int_0^{\frac{1}{2}} [\frac{\alpha}{2}(-z_3^2+z_3-\frac{1}{4})+\frac{1}{2}(\frac{1}{2}-z_3)]dz_3 + \frac{1}{2} \\
&= 8[\frac{\alpha}{2}(-\frac{1}{3}\cdot\frac{1}{8}+\frac{1}{8}-\frac{1}{8})+\frac{1}{16}]+\frac{1}{2} \\
&= 8(-\frac{\alpha}{48}+\frac{1}{16})+\frac{1}{2} \\
&= 1-\frac{\alpha}{6}
\end{aligned}
$$

$$
\begin{aligned}
p_2 &= \int_{\frac{1}{2\alpha}}^{1} \int_{\frac{1}{2}}^{\frac{1}{2\alpha}+1-z_2} \int_{\frac{1+2\alpha}{2}-\alpha(z_2+z_3)}^{\frac{1}{2}} 8dz_1dz_3dz_2 + \int_{\frac{1}{2}}^{\frac{1}{2\alpha}} \int_{\frac{1}{2}}^{1} \int_{\frac{1+2\alpha}{2}-\alpha(z_2+z_3)}^{\frac{1}{2}} 8dz_1dz_3dz_2 + \int_{\frac{1}{2\alpha}}^{1} \int_{\frac{1}{2\alpha}+1-z_2}^{1} 4dz_3dz_2 \\
&= 4\alpha\int_{\frac{1}{2\alpha}}^{1} (-z_2^2+z_2+(\frac{1}{4\alpha^2}-\frac{1}{4}))dz_2 + 8\int 2\alpha \int_{\frac{1}{2}}^{1} \alpha(z_2+z_3-1)dz_3dz_2 + \int_{\frac{1}{2\alpha}}^{1} 4(z_2-\frac{1}{2\alpha})dz_2 \\
&= (\frac{1}{2}+\frac{1}{2\alpha}-\frac{1}{3\alpha^2}-\frac{\alpha}{3}) + 8\int_{\frac{1}{2}}^{\frac{1}{2\alpha}} [\frac{3\alpha}{8}+\frac{\alpha}{2}(z_2-1)]dz_2 + 2(1-\frac{1}{2\alpha})^2 \\
&= (\frac{1}{2}+\frac{1}{2\alpha}-\frac{1}{3\alpha^2}-\frac{\alpha}{3}) + (\frac{1}{2\alpha}-\frac{1}{2}) + 2(1-\frac{1}{2\alpha})^2 \\
&= 2-\frac{\alpha}{3}-\frac{1}{\alpha}+\frac{1}{6\alpha^2}
\end{aligned}
$$

Given strategy profile (a^*, b^*, b^*) and (c^*, d^*, d^*), the two candidates' expected payoffs are

$$\pi_H^* = \frac{1 + 2p_1 k + p_2 k + 2(1-p_1)k^2 + (1-p_2)k^2}{(1+k)^3} - c_H(a+2b) \tag{21}$$

$$\pi_L^* = \frac{k^3 + (2p_1 + p_2)k^2 + 2(1-p_1)k + (1-p_2)k}{(1+k)^3} - c_L(c+2d) \tag{22}$$

Take the difference of π_H^* and π_L^*, we have

$$\pi_H^* - \pi_L^* = \frac{(1-k)\left[1 + k^2 + 2(2p_1 + p_2 - 1)k\right]}{(1+k)^3} > 0,$$

since $2p_1 - 1 > 0$.

In the following, we identify the condition for π_L^* to be nonnegative.

$$\begin{aligned}
\pi_L^* =& \frac{k}{(1+k)^4} \cdot \Big\{ \left[k^2 + (2p_1 + p_2)k + 2(1-p_1) + (1-p_2)\right](1+k) \\
& -(1-p_2)(1+k^2) - 2(2p_1-1)k - 2(1-p_1)(1+k^2) - 4(p_1+p_2-1)k \Big\} \\
=& \frac{k}{(1+k)^4} \cdot \Big\{ k^2 + (2p_1+p_2)k + 2(1-p_1) + (1-p_2) + k^3 + (2p_1+p_2)k^2 + 2(1-p_1)k + (1-p_2)k \\
& -(1-p_2)(1+k^2) - 2(2p_1-1)k - 2(1-p_1)(1+k^2) - 4(p_1+p_2-1)k \Big\} \\
=& \frac{k^2}{(1+k)^4} \cdot \Big\{ k^2 + (4p_1 + 2p_2 - 2)k + (9 - 8p_1 - 4p_2) \Big\}.
\end{aligned}$$

$\pi_L^* \geq 0$ if and only if

$$k^2 + (4p_1 + 2p_2 - 2)k + (9 - 8p_1 - 4p_2) \geq 0.$$

For the left-hand side of the above inequality, we have

$$\begin{aligned}
\Delta =& 4\left[(2p_1 + p_2 - 1)^2 - (9 - 8p_1 - 4p_2)\right] \\
=& 4\left[(2p_1 + p_2)^2 - 2(2p_1 + p_2) + 1 - 9 + 4(2p_1 + p_2)\right] \\
=& 4\left[(2p_1 + p_2 + 1)^2 - 9\right] > 0.
\end{aligned}$$

Thus, with $k > 0$, the inequality hold if and only if

$$k > -(2p_1 + p_2 - 1) + \sqrt{(2p_1 + p_2 + 1)^2 - 9} = k_0.$$

Previously, we have shown that $-1/2 \leq 2-(2p_1+p_2) \leq -1/3$, which implies that $(2p_1+p_2) \in [7/3, 5/2]$ and $9-4(2p_1+p_2) < 0$. Thus, $1 > k_0 > 0$.

Specifically, it is easy to check that k_0 is increasing in $2p_1+p_2$. That is, $k_0^{EC} > k_0^{PV}$.

B. Mixed strategy equilibrium

When $k < k_0 = -(2p_1+p_2-1)+\sqrt{(2p_1+p_2+1)^2-9}$, a mixed strategy equilibrium exists. Assume that the mixed strategy equilibrium takes the form that H adopts a nonzero pure strategy (a,b,b) and L mixes between inactive and a nonzero pure strategy (c,d,d). Let p^* be the probability of being active and adopting the pure strategy.

$$\begin{aligned}\widetilde{\pi}_H =&(1-p^*)+p^*p_H - c_H(a+2b)\\ =&1-p^*+p^*\left[p_H-\frac{c_H}{p^*}(a+2b)\right]\end{aligned} \tag{23}$$

$$\widetilde{\pi}_L = p^*(1-p_H-c_L(c+2d)), \tag{24}$$

in which p_H is defined as

$$p_H = \frac{abe+p_1(abd+aed)+p_2cbe+(1-p_1)(cde+cdb)+(1-p_2)ad^2}{(a+c)(b+d)(e+d)}.$$

As long as L is active, s/he is effectively competing with a candidate with marginal effort cost c_H/p^*. We thus have $c^*/a^* = d^*/b^* = k/p^*$.

Since L mixes between active and inactive, his expected payoff from the mixed strategy should be 0, which implies that $k/p^* = k_0$. Thus, we have $p^* = k/k_0$.

Using the same way as in Section 1, we have

$$\widetilde{a} = \frac{k}{c_H}\cdot\frac{(1-p_2)(1+k_0^2)+2(2p_1-1)k_0}{(1+k_0)^4} \tag{25}$$

$$\widetilde{b} = \frac{k}{c_H}\cdot\frac{(1-p_1)(1+k_0^2)+2(p_1+p_2-1)k_0}{(1+k_0)^4} \tag{26}$$

$$\widetilde{c} = \frac{k_0}{c_L}\cdot\frac{(1-p_2)(1+k_0^2)+2(2p_1-1)k_0}{(1+k_0)^4} \tag{27}$$

$$\widetilde{d} = \frac{k_0}{c_L}\cdot\frac{(1-p_1)(1+k_0^2)+2(p_1+p_2-1)k_0}{(1+k_0)^4}. \tag{28}$$

$$\tag{29}$$

C. Summary of Equilibrium

In the unqiue equilibrium of our election game, the following holds:

1. Winning chance of candidate H is

$$WP_H = \begin{cases} \frac{1+2p_1k+p_2k+2(1-p_1)k^2+(1-p_2)k^2}{(1+k)^3} & \text{if } k \geq k_0 \\ 1 - k \cdot \frac{k_0^2+2p_1k_0+p_2k_0+2(1-p_1)+(1-p_2)}{(1+k_0)^3} & \text{if } k \leq k_0 \end{cases} \tag{30}$$

2. Expected payoffs are

$$\pi_H^* = \begin{cases} \frac{1+2p_1k+p_2k+2(1-p_1)k^2+(1-p_2)k^2}{(1+k)^3} & \text{if } k \geq k_0 \\ 1 - k \cdot \frac{k_0^2+2p_1k_0+p_2k_0+2(1-p_1)+(1-p_2)}{(1+k_0)^3} & \text{if } k \leq k_0 \end{cases} \tag{31}$$

D. Comparison When There Is No Dominant District

When $k > k_0^i$, the election game under scheme $i \in \{EC, PV\}$ has a pure strategy equilibrium, (a^*, b^*, b^*) and (c^*, d^*, d^*). When $k < k_0^i$, the election game under scheme $i \in \{EC, PV\}$ has a mixed strategy equilibrium. The uniqueness is ensured by the interchangeability of the election game' equilibria.

D.1. Comparison of Winning probability

In the pure strategy equilibrium, candidate H's election winning probability is

$$p_H = \frac{1 + 2p_1k + p_2k + 2(1-p_1)k^2 + (1-p_2)k^2}{(1+k)^3}$$

It is easy to show that p_H is increasing in p_1 and p_2. When $k > k_0^{EC}$, both schemes induce a pure strategy equilibrium. In this case, the winning probability should be higher under EC.

When $k_0^{EC} \geq k > k_0^{PV}$, EC induces a mixed strategy equilibrium while PV induces a pure strategy equilibrium, the difference between the winning probabilities under the two schemes are

$$
\begin{aligned}
&WP_H^{EC} - WP_H^{PV} \\
=&1 - k \cdot \frac{(k_0^{EC})^2 + 2p_1^{EC} k_0^{EC} + p_2^{EC} k_0^{EC} + 2(1-p_1^{EC}) + (1-p_2^{EC})}{(1+k_0^{EC})^3} \\
&- \left[1 - k \cdot \frac{(k_0^{EC})^2 + 2p_1^{PV} k_0^{EC} + p_2^{PV} k_0^{EC} + 2(1-p_1^{PV}) + (1-p_2^{PV})}{(1+k_0^{EC})^3}\right] \\
&+ 1 - k \cdot \frac{(k_0^{EC})^2 + 2p_1^{PV} k_0^{EC} + p_2^{PV} k_0^{EC} + 2(1-p_1^{PV}) + (1-p_2^{PV})}{(1+k_0^{EC})^3} \\
&- \frac{1 + 2p_1^{PV} k + p_2^{PV} k + 2(1-p_1^{PV})k^2 + (1-p_2^{PV})k^2}{(1+k)^3}
\end{aligned} \tag{32}
$$

Note that $\frac{(k_0^{EC})^2 + 2p_1 k_0^{EC} + p_2 k_0^{EC} + 2(1-p_1) + (1-p_2)}{(1+k_0^{EC})^3}$ is decreasing in p_1 and p_2. So the first two rows after the equality, which measures rounding effect, must be positive. The last two rows, which measures strategic effect, can be rewritten as k times

$$
\frac{k^2 + 2p_1^{PV} k + p_2^{PV} k + 2(1-p_1^{PV}) + 1 - p_2^{PV}}{(1+k)^3} - \frac{(k_0^{EC})^2 + 2p_1^{PV} k_0^{EC} + p_2^{PV} k_0^{EC} + 2(1-p_1^{PV}) + 1 - p_2^{PV}}{(1+k_0^{EC})^3} \tag{33}
$$

Consider the derivative,

$$
\begin{aligned}
&\frac{d}{dk}\left(\frac{k^2 + 2p_1^{PV} k + p_2^{PV} k + 2(1-p_1^{PV}) + 1 - p_2^{PV}}{(1+k)^3}\right) \\
=&\frac{2k + 2p_1^{PV} + p_2^{PV} + 2k^2 + 2p_1^{PV} k + p_2^{PV} k - 3k^2 - 6p_1^{PV} k - 3p_2^{PV} k - 6(1-p_1^{PV}) - 3(1-p_2^{PV})}{(1+k)^4} \\
=&-\frac{k^2 + 2\left(2p_1^{PV} + p_2^{PV} - 1\right)k + (9 - 8p_1^{PV} - 4p_2^{PV})}{(1+k)^4}
\end{aligned} \tag{34}
$$

Since $k > k_0^{PV}$, the partial derivative in (34) must be negative, which implies that (33) must be positive since $k \leq k_0^{EC}$. That is, the strategic effect is postive as well.

When $k \leq k_0^{PV}$, both schemes induce a mixed strategy equilibrium. The difference between the winning probabilities under the two schemes are

$$
WP_H^{EC} - WP_H^{PV} = \text{rounding effect} + \text{strategic effect}. \tag{35}
$$

The rounding effect is exactly the same as that in the case above, which is shown to be positive. The strategic effect is

$$
\begin{aligned}
&1-k\cdot\frac{(k_0^{EC})^2+2p_1^{PV}k_0^{EC}+p_2^{PV}k_0^{EC}+2(1-p_1^{PV})+1-p_2^{PV}}{(1+k_0^{EC})^3}\\
&-\left[1-k\cdot\frac{(k_0^{PV})^2+2p_1^{PV}k_0^{PV}+p_2^{PV}k_0^{PV}+2(1-p_1^{PV})+1-p_2^{PV}}{(1+k_0^{PV})^3}\right],
\end{aligned}
\tag{36}
$$

which can be rewritten as k times

$$
\begin{aligned}
&-\frac{(k_0^{EC})^2+2p_1^{PV}k_0^{EC}+p_2^{PV}k_0^{EC}+2(1-p_1^{PV})+1-p_2^{PV}}{(1+k_0^{EC})^3}\\
&\frac{(k_0^{PV})^2+2p_1^{PV}k_0^{PV}+p_2^{PV}k_0^{PV}+2(1-p_1^{PV})+1-p_2^{PV}}{(1+k_0^{PV})^3}
\end{aligned}
\tag{37}
$$

Similarly, the negative sign of (34) implies that the stategic effect should be positive as well.

D.2. Comparison of Total Expenditure

Compare total expenditures under EC and PV for $k > k_0^{EC}$, $k_0^{EC} \geq k > k_0^{PV}$, and $k \leq k_0^{PV}$ separately. Before proceeding to conduct the comparison, we first define

$$
\begin{aligned}
h(k;p_1,p_2) =& \frac{(1-p_2)(1+k^2)+2(2p_1-1)k}{(1+k)^4} + 2\frac{(1-p_1)(1+k^2)+2(p_1+p_2-1)k}{(1+k)^4}\\
=& \frac{(3-p_2-2p_1)(1+k^2)+(8p_1+4p_2-6)k}{(1+k)^4}.
\end{aligned}
\tag{38}
$$

The total expenditure under a scheme with p_1, p_2, and p_0 is

$$
TCE(p_1,p_2) = 2k\cdot h(\max\{k,k_0\};p_1,p_2).
$$

When $k > k_0^{EC}$, each scheme induces a pure stratege equilibrium. The total campaign expenditure under EC and PV are $2k\cdot h(k;p_1^{EC},p_2^{EC})$ and $2k\cdot h(k;p_1^{PV},p_2^{PV})$. The sign of partial derivative of $h(k;p_1,p_2)$ with respect to p_1 (or p_2) depends on the sign of $-(1+k^2)+4k$, which is positive if and only if $2-\sqrt{3} < k < 2+\sqrt{3}$. Since $k > k_0^{EC} = \sqrt{7}-2 > 2-\sqrt{3}$, the condition required for $\partial h/\partial p_1$ (or $\partial h/\partial p_2$) to be positive is satisfied. Since $p_1^{EC} > p_1^{PV}$ and $p_2^{EC} > p_2^{PV}$, the total expenditure must be higher under EC than under PV.

When $k \leq k_0^{PV}$, each scheme induces a mixed strategy equilibrium in which L mixes between inactive and active. Thus, L's expected total effort cost (or total campaign expenditure) should equal his expected winning probability, which has been shown to be higher under PV. Since the total effort cost is the same across the two candidates, the total campaign expenditure must be higher under PV than under EC in this case.

When k is between k_0^{PV} and k_0^{EC}, EC induces a mixed strategy equilibrium while PV induces a pure strategy equilibrium. The total expenditure is $2k \cdot h(k_0^{EC}; p_1^{EC}, p_2^{EC})$ under EC and $2k \cdot h(k; p_1^{PV}, p_2^{PV})$ under PV. The first order partial derivative of $h(k; p_1^{PV}, p_2^{PV})$ with respect to k is

$$\begin{aligned}\frac{\partial h(k; p_1^{PV}, p_2^{PV})}{\partial k} =& \frac{(3 - p_2^{PV} - 2p_1^{PV})2k + 8p_1^{PV} + 4p_2^{PV} - 6 + (3 - p_2^{PV} - 2p_1^{PV})2k^2 + (8p_1^{PV} + 4p_2^{PV} - 6)k}{(1+k)^5} \\ & - \frac{4(3 - p_2^{PV} - 2p_1^{PV})(1+k^2) + 4(8p_1^{PV} + 4p_2^{PV} - 6)k}{(1+k)^5} \\ =& 2 \cdot \frac{(2p_1^{PV} + p_2^{PV} - 3)k^2 - (14p_1^{PV} + 7p_2^{PV} - 12)k + 8p_1^{PV} + 4p_2^{PV} - 9}{(1+k)^5}\end{aligned} \tag{39}$$

Let $T = 2p_1^{PV} + p_2^{PV}$. Since now k induces a pure strategy under PV, we should have $k^2 + (2T - 2)k > 4T - 9$. Then we have the following result about the numerator of (39)

$$\begin{aligned}(T-3)k^2 - (7T-12)k + 4T - 9 <& (T-3)k^2 - (7T-12)k + k^2 + (2T-2)k \\ =& (T-2)k^2 - (5T-10)k = (T-2)k(k-5)\end{aligned} \tag{40}$$

Previously, we have shown that, with F and G being uniform distribution, $T - 2 = 2p_1 + p_2 - 2 > 0$. It implies that $(T-2)k(k-5) < 0$ since $k \in (0,1)$. That is,

$$\frac{\partial h(k; p_1^{PV}, p_2^{PV})}{\partial k} < 0.$$

Note that (i) $h(k_0^{EC}, p_1^{EC}, p_2^{EC}) > h(k_0^{EC}, p_1^{PV}, p_2^{PV})$, (ii) $h(k_0^{PV}, p_1^{EC}, p_2^{EC}) = h(k_0^{EC}, p_1^{EC}, p_2^{EC}) < h(k_0^{PV}, p_1^{PV}, p_2^{PV})$, and (iii) $h(k; p_1^{PV}, p_2^{PV})$ is decreasing in k on $[k_0^{EC}, k_0^{PV}]$. $h(k; p_1^{PV}, p_2^{PV})$ takes value $h(k_0^{EC}), p_1^{EC}, p_2^{EC}$ only once on interval $[k_0^{EC}, k_0^{PV}]$.

In summary, there exists a cutoff $\hat{k}$ such that $TCE(p_1^{EC}, p_2^{EC}) \geq TCE(p_1^{PV}, p_2^{PV})$ when $k \geq \hat{k}$ and $TCE(p_1^{EC}, p_2^{EC}) \leq TCE(p_1^{PV}, p_2^{PV})$ when $k \leq \hat{k}$.

D.3. Comparison of Inversion Rate

EC and PV provide different election winner only in the event that no candidate wins all three districts. When $k > k_0$, both candidates play pure strategies, and the chance that H

wins exactly two districts is $\underbrace{\left(\frac{1}{1+k}\right)^2\left(\frac{k}{1+k}\right)}_{\text{wins district 1 and 2}}+\underbrace{\left(\frac{1}{1+k}\right)^2\left(\frac{k}{1+k}\right)}_{\text{wins district 1 and 3}}+\underbrace{\left(\frac{1}{1+k}\right)^2\left(\frac{k}{1+k}\right)}_{\text{wins district 2 and 3}}$.
The probability that inversion happens conditional on H winning two districts is

$$(1-p_1^{PV})\underbrace{\left(\frac{1}{1+k}\right)^2\left(\frac{k}{1+k}\right)}_{\text{wins district 1 and 2}}+(1-p_1^{PV})\underbrace{\left(\frac{1}{1+k}\right)^2\left(\frac{k}{1+k}\right)}_{\text{wins district 1 and 3}}+(1-p_2^{PV})\underbrace{\left(\frac{1}{1+k}\right)^2\left(\frac{k}{1+k}\right)}_{\text{wins district 2 and 3}}.$$

The probability that inversion happens conditional on H winning one districts is

$$(1-p_2^{PV})\underbrace{\left(\frac{k}{1+k}\right)^2\left(\frac{1}{1+k}\right)}_{\text{wins district 1}}+(1-p_1^{PV})\underbrace{\left(\frac{k}{1+k}\right)^2\left(\frac{1}{1+k}\right)}_{\text{wins district 2}}+(1-p_1^{PV})\underbrace{\left(\frac{k}{1+k}\right)^2\left(\frac{1}{1+k}\right)}_{\text{wins district 3}}.$$

Thus, the inversion rate in election game with p_1 and p_2 is

$$EIR(p_1,p_2)=\begin{cases}(3-2p_1^{PV}-p_2^{PV})\frac{k}{(1+k)^2} & \text{if } k\geq k_0\\ (3-2p_1^{PV}-p_2^{PV})\frac{k}{k_0}\cdot\frac{k_0}{(1+k_0)^2} & \text{if } k<k_0\end{cases} \tag{41}$$

which can be rewritten as

$$EIR(p_1,p_2)=(3-2p_1^{PV}-p_2^{PV})\frac{k}{(1+\max\{k,k_0\})^2}.$$

Since $k_0^{EC}>k_0^{PV}$, we should have

$$EIR(p_1^{EC},p_2^{EC})\leq EIR(p_1^{PV},p_2^{PV}).$$

II. Dominant District

Assume that there are three district and the number of votes in these districts are $N_1=1$, $N_2=N_3=\alpha<1/2$. Contional on the winning districts, a candidate's probability of winning the election is given by the following table:

No. of winning districts	Electoral College	Popular Vote
$1+2+3$	1	1
$1+2$ *or* 3	1	p_1
$2+3$	0	p_2
2 *or* 3	0	$1-p_1$
1	1	$1-p_2$

When $\alpha < \frac{1}{2}$, no matter how much effort is exerted in district with votes α, a candidate wins the election as long as he wins District 1 if the prevailing election scheme is EC. Therefore, both players would focus on the district with votes 1, and the contest game degenerates to the one with a single battlefield contest under EC.

Let $(a, 0, 0)$ be the strategy of H and $(b, 0, 0)$ be the strategy of $L/$

$$\pi_H = \frac{a}{a+b} - c_H a \tag{42}$$
$$\pi_L = \frac{b}{a+b} - c_L b \tag{43}$$
$$\tag{44}$$

FOC gives

$$\frac{\partial \pi_H}{\partial a} = \frac{b}{(a+b)^2} - c_H \tag{45}$$
$$\frac{\partial \pi_H}{\partial b} = \frac{a}{(a+b)^2} - c_L \tag{46}$$

Let $k = \frac{c_H}{c_L}$. Combining the two FOCs, we have

$$a = \frac{k}{c_H(1+k)^2} \tag{47}$$
$$b = \frac{k}{c_L(1+k)^2} \tag{48}$$

Proposition 2. *In an election with a district that is endowed with votes more than the sum of the other two districts, the equilibrium under EC is given as following:*

1. *candidate H adopts strategy $(\frac{k}{c_H(1+k)^2}, 0, 0)$;*
2. *candidate L adopts strategy $(\frac{k}{c_L(1+k)^2}, 0, 0)$.*

Under PV, candidates still have to compete in all three districts. With uniform F and G, we have $p_1 = 1 - \alpha/6$ and $p_2 = \alpha$. It is easy to check that the sufficient second order condition is satisfied. The equilibrium identified in the previous section is also the equilibrium in this case, which is summarized as follows.

Proposition 3. *In an election with a district that is endowed with votes more than the sum of the other two districts, the equilibrium under PV is given as following:*

1. *if $k > k_0^{PV}$, there is a pure strategy in which candidate H adopts strategy (a^*, b^*, b^*) and L adopts strategy (c^*, d^*, d^*)*
2. *if $k \leq k_0^{PV}$, there is a mixed strategy equilibrium in which H adopts a pure strategy $(\tilde{a}, \tilde{b}, \tilde{b})$ and L stays active with probability k/k_0^{PV} playing $(\tilde{c}, \tilde{d}, \tilde{d})$*

A. Comparison When There Is Dominant District

A.1. Comparison of Winning probability

Under EC, for any $k \in (0, 1)$, candidate H's winning probability is

$$WP_H(p_1^{EC}, p_2^{EC}) = \frac{1}{1+k}$$

Under PV, candidate H's winning probability is

$$WP_H(p_1^{PV}, p_2^{PV}) = \begin{cases} \frac{1+2p_1^{PV}k + p_2^{PV}k + 2(1-p_1^{PV})k^2 + (1-p_2^{PV})k^2}{(1+k)^3} & \text{if } k \geq k_0^{PV} \\ 1 - k \cdot \frac{(k_0^{PV})^2 + 2p_1^{PV}k_0^{PV} + p_2^{PV}k_0^{PV} + 2(1-p_1^{PV}) + (1-p_2^{PV})}{(1+k_0^{PV})^3} & \text{if } k \leq k_0^{PV} \end{cases} \tag{49}$$

- When $k \geq k_0^{PV}$,

$$\begin{aligned} WP_H(p_1^{PV}, p_2^{PV}) =& \frac{1 + 2k + k^2 + (2p_1^{PV} + p_2^{PV} - 2)k + 2(1-p_1^{PV})k^2 + (-p_2^{PV})k^2}{(1+k)^3} \\ =& \frac{1 + 2k + k^2 + (2p_1^{PV} + p_2^{PV} - 2)k(1-k)}{(1+k)^3} \end{aligned} \tag{50}$$

Since $2p_1^{PV} + p_2^{PV} - 2 = 2\alpha/3 > 0$, it follows that

$$WP_H(p_1^{PV}, p_2^{PV}) \geq \frac{1}{1+k} = WP_H(p_1^{EC}, p_2^{EC}).$$

- When $k < k_0^{PV}$,

$$\begin{aligned}
&\frac{(k_0^{PV})^2 + 2p_1^{PV}k_0^{PV} + p_2^{PV}k_0^{PV} + 2(1-p_1^{PV}) + (1-p_2^{PV})}{(1+k_0^{PV})^3} \\
=&\frac{(k_0^{PV})^2 + 2k_0^{PV} + 1 + (2p_1^{PV} + p_2^{PV} - 2)k_0^{PV} + 2(1-p_1^{PV}) - p_2^{PV}}{(1+k_0^{PV})^3} \\
=&\frac{(k_0^{PV}+1)^2 + (2p_1^{PV} + p_2^{PV} - 2)(k_0^{PV} - 1)}{(1+k_0^{PV})^3} \leq \frac{1}{1+k_0^{PV}} < \frac{1}{k+1}
\end{aligned} \tag{51}$$

Thus, we should have

$$WP_H(p_1^{PV}, p_2^{PV}) > 1 - k \cdot \frac{1}{1+k} = \frac{1}{1+k} = WP_H(p_1^{EC}, p_2^{EC}).$$

A.2. Comparison of Total Expenditure

Under EC, the total expenditure is

$$TCE(p_1^{EC}, p_2^{EC}) = 2k \cdot \frac{1}{(1+k)^2}.$$

Under PV, the total expenditure is

$$TCE(p_1^{PV}, p_2^{PV}) = 2k \cdot h(\max\{k, k_0^{PV}\}; p_1^{PV}, p_2^{PV}),$$

in which function h is defined as

$$\begin{aligned}
h(k; p_1, p_2) =&\frac{(1-p_2)(1+k^2) + 2(2p_1 - 1)k}{(1+k)^4} + 2\frac{(1-p_1)(1+k^2) + 2(p_1 + p_2 - 1)k}{(1+k)^4} \\
=&\frac{(3 - p_2 - 2p_1)(1+k^2) + (8p_1 + 4p_2 - 6)k}{(1+k)^4}.
\end{aligned} \tag{52}$$

Recall that we have denote $2p_1^{PV} + p_2^{PV}$ with T. Function h can be rewritten as

$$h(k; p_1^{PV}, p_2^{PV}) = \frac{(3-T)(1+k^2) + (8T-6)k}{(1+k)^4} \tag{53}$$

Previously, we have shown that $h(k; p_1^{PV}, p_2^{PV})$ is decreasing in k when $k \geq k_0^{PV}$. Thus, $h(\max\{k, k_0^{PV}\}; p_1^{PV}, p_2^{PV})$ stays at $h(k_0^{PV}; p_1^{PV}, p_2^{PV})$ till $k = k_0^{PV}$ and then decreases to $h(1; p_1^{PV}, p_2^{PV})$.

Consider the following difference:

$$\begin{aligned} &(3-T)\left(1+k^2\right)+(8T-6)k-(1+k)^2 \\ =&(2-T)\left(1+k^2\right)+8(T-1)k, \end{aligned} \tag{54}$$

which is increasing in k, since $(2-T)k+4(T-1)>0$ holds for any $k \leq 1$.

When $k=k_0^{PV}$, (54) equals

$$\begin{aligned} &(3-T)\left[1+(k_0^{PV})^2\right]+(8T-6)k_0^{PV}-(1+k_0^{PV})^2 \\ =&(2-T)\left[1+(k_0^{PV})^2\right]+8(T-1)k_0^{PV} \end{aligned} \tag{55}$$

By the definition of k_0^{PV}, we have $(k_0^{PV})^2+1+(2T-2)k_0^{PV}=4T-8$. The above expression can be rewritten as

$$\begin{aligned} &2(4T-8)-T\left[1+(k_0^{PV})^2\right]+4(T-1)k_0^{PV} \\ =&2(4T-8)-T(4T-8)-T(2T-2)k_0^{PV}+4(T-1)k_0^{PV} \\ =&-4(T-2)^2+2(T-1)k_0^{PV}(2-T)<0, \end{aligned} \tag{56}$$

since $T-2=2\alpha/3>0$.

When $k=1$, (54) equals

$$2(2-T)+8(T-1)=6T-4>0.$$

By the increasing property of (54), there exist a unique $\tilde{k} \in [k_0^{PV}, 1]$ such that $(2-T)\left(1+\tilde{k}^2\right)+8(T-1)\tilde{k}=0$. Therefore, we obtain that $TCE(p_1^{EC}, p_2^{EC}) > TCE(p_1^{PV}, p_2^{PV})$ when $k \leq \tilde{k}$ and $TCE(p_1^{EC}, p_2^{EC}) < TCE(p_1^{PV}, p_2^{PV})$ when $k \geq \tilde{k}$.

A.3. Comparison of Inversion Ratie

In the equilibrium of the election game under EC, candidates only exert effort in district 1. If the prevailing scheme is EC, then the candidate who wins district 1 wins the election. If the prevailing scheme is PV, the candidate who wins district 1 also winns the election. Clearly, election inversion does not happen at all.

In the equilibrium of the election game under PV, candidates exert effort in all three districts. Clearly, election inversion happens with a positive probaility.

Thus, the inversion rate is always lower under EC.