Experiment 211 - Coupled Pendulums

Felix Fleischle

17.1.2022

Introduction

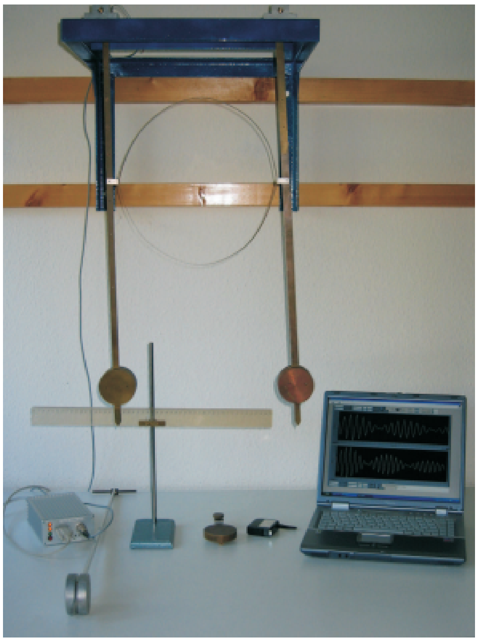


Figure 1: Experimental setup

In this experiment, we are going to determine the frequency of symmetric and asymmetric natural oscillations of two identical coupled brass pendulums, for three different coupling strenghts, as well as the frequencies of the pendulum and beat for beat oscillation for those coupling strenghts.

Fundamentals

The motion of the free pendulum can be described by a differential equation:

$$J\ddot{\varphi} = -D\varphi = -mgL\varphi \tag{1}$$

with the moment of inertia J, the mass m, the gravitational acceleration g, and the pendulum length L. The solution is a harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{g}{L}} \tag{2}$$

The coupling spring exerts an additional moment of torque on the pendulums, depending on the angular deviations φ_1 and φ_2 of the pendulums.

$$M_1 = D'(\varphi_2 - \varphi_1) \tag{3}$$

$$M_2 = D'(\varphi_1 - \varphi_2) \tag{4}$$

If we add those moments of torque to the equations of motion, we get

$$J\ddot{\varphi}_1 = -D\varphi_1 + D'(\varphi_2 - \varphi_1) \tag{5}$$

$$J\ddot{\varphi}_2 = -D\varphi_2 + D'(\varphi_1 - \varphi_2) \tag{6}$$

By substituting $u = \varphi_1 + \varphi_2$ and $v = \varphi_1 - \varphi_2$ we get two independent differential equations:

$$J\ddot{u} + Du = 0 \tag{7}$$

$$J\ddot{v} + (D + 2D')v = 0 \tag{8}$$

with the solutions

$$u(t) = A_1 \cos\omega_1 t + B_1 \sin\omega_1 t \tag{9}$$

$$v(t) = A_1 \cos\omega_2 t + B_1 \sin\omega_2 t \tag{10}$$

and with substituting $\varphi_1 = \frac{1}{2}(u + v)$ and $\varphi_2 = \frac{1}{2}(u - v)$

$$\varphi_1(t) = \frac{1}{2}(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t + A_2 \cos\omega_2 t + B_2 \sin\omega_2 t) \tag{11}$$

$$\varphi_2(t) = \frac{1}{2}(A_1 \cos\omega_1 t - B_1 \sin\omega_1 t - A_2 \cos\omega_2 t + B_2 \sin\omega_2 t) \tag{12}$$

with

$$\omega_1 = \sqrt{\frac{D}{J}} \tag{13}$$

$$\omega_2 = \sqrt{\frac{D + 2D'}{J}} \tag{14}$$

For different starting conditions, we get different coefficients and different motions for the two pendulums:

Symmetric Oscillation

For $\varphi_1(0) = \varphi_2(0) = \varphi_0$ and $\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0$ we get $A_1 = 2\varphi_0$ and $A_2 = B_1 = B_2 = 0$ and thus

$$\varphi_1(t) = \varphi_2(t) = \varphi_0 \cos \omega_1 t \tag{15}$$

Both pendulums oscillate in phase, as if they were not coupled at all, since ω_1 does not depend on the coupling D'. This can be easily observed, since the coupling spring will not get compressed or streched at all.

Asymmetric Oscillation

For $\varphi_1(0) = -\varphi_2(0) = \varphi_0$ and $\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0$ we get $A_2 = 2\varphi_0$ and $A_1 = B_1 = B_2 = 0$ and thus

$$\varphi_1(t) = -\varphi_2(t) = \varphi_0 \cos \omega_2 t \tag{16}$$

Both pendulums oscillate out of phase, with frequency ω_2, which is dependent on the momentum of the coupling D'.

Beat Oscillation

For $\varphi_1(0) = 0$, $\varphi_2(0) = \varphi_0$ and $\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0$ we get $A_1 = -A_2 = \varphi_0$ and $B_1 = B_2 = 0$ and thus

$$\varphi_1(t) = \varphi_0 \sin \frac{\omega_2 - \omega_1}{2} t \sin \frac{\omega_2 + \omega_1}{2} t \tag{17}$$

$$\varphi_2(t) = \varphi_0 \cos \frac{\omega_2 - \omega_1}{2} t \cos \frac{\omega_2 + \omega_1}{2} t \tag{18}$$

Here the pendulum that was deflected at the start gradually transfers its energy to the resting pendulum until it stands still itsself. The process then reverses. The oscillations may be described by two frequencies:

$$\omega_I = \frac{1}{2}(\omega_2 + \omega_1) \tag{19}$$

with which each individual pendulum oscillates, and the beat frequency

$$\omega_{II} = \frac{1}{2}(\omega_2 - \omega_1) \tag{20}$$

with which the energy oscillates between the pendulums.

Coupling Strength

The coupling strength κ can be quantified through

$$\kappa = \frac{D'}{D + D'} = \frac{T_1^2 - T_2^2}{T_1^2 + T_2^2} \tag{21}$$

Measuring Technique

In this experiment, we are going to measure the angular deviation magnetically, using the Hall-Effect. When the pendulum oscillates, the hall sensor moves relative to the bar magnets, creating a voltage proportional to the angular deviation. The voltage is then digitalised by an Analog-Digital converter and read by the PC.

Experimental Procedure

First, we are going to measure the oscillation frequency for each of the brass pendulums independently, by letting the pendulums swing, and then using Python to determine the frequency of the pendulums. Additionally we are going to save an image of both the angular deviation over time, and the frequency spectrum with gauß-fit.

Next we are going to measure the frequency for symmetric and asymmetric oscillation, for three different coupling strenghts. The different coupling strenghts are achieved through attaching the spring at different lenghts from the pendulum axis. We measure the different distances of the three settings.

After that we are going to measure the frequency of the beat oscillation for the three different coupling strenghts.

In the end we are going to observe another coupled oscillation, with two resonant circuits and an oscilloscope. The coupling occurs inductively through the two coils. We are going to vary the coupling strength by changing the coil seperation and observe the effects onto the oscillations.

COUPLED PENDULUMS (211)

17.01.2022 (14.00 - 17.00)

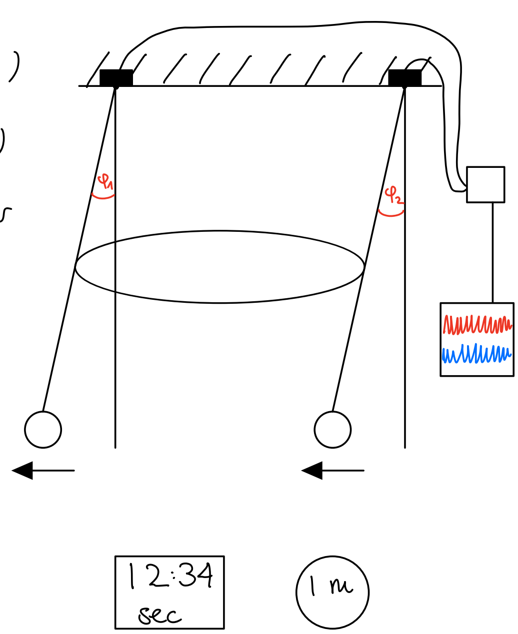
Tobias Heintz, Tutor: Vasu Dipakkumar Pipwala
Felix Fleischle

TASKS

- With three couplings of diffrent strengths, the frequencies of the symmetrical and antisymmetrical natural oscillation of two coupled brass pendulums of the same type are to be determined.
- In addition for the above couplings beatings are to be generated and determine the frequency of the pendulum and the beat.

INSTRUMENTAL SETUP

- 2 brass pendulums ($\rho = 7.5$ g/cm³)
- coupling spring (phosphorus bronze band)
- Magnetic angle - measurement recorder
- Analog - Digital converter
- PC with printer



1. First, the frequency is measured for both brass pendulums (without coupling spring):

Table 1: Frequences of the decoupled pendulums

#	μ [Hz]	σ [Hz]	FWHW [Hz]
1	0.618	0.005	0.011

2. For the three couplings of diffrent strengths, the symmetrical and the asymmetrical natural oscillation are excited and the frequences are determined.

Table 2: Positions of the mounting holes

#	l [cm]
1	$9.15 + 10 \pm 0.05$
2	$19.20 + 10 \pm 0.05$
3	$29.35 + 10 \pm 0.05$

Table 3a: Frequences of the symmetrical natural oscillation

#	μ [Hz]	σ [Hz]	FWHW [Hz]
1	0.620	0.015	0.035
2	0.621	0.005	0.013
3	0.621	0.007	0.017

Table 3b: Frequences of the asymmetrical natural oscillation

#	μ [Hz]	σ [Hz]	FWHW [Hz]
1	0.65	0.007	0.016
2	0.687	0.009	0.021
3	0.741	0.009	0.022

3. For equal couplings, beats are generated by releasing one pendulum at maximum deflection while holding the other pendulum at rest. Oscillation and beat frequences are measured.

Table 4: Oscillation and beat frequency

#	μ_1 [Hz]	σ_1 [Hz]	FWHW$_1$ [Hz]	μ_2 [Hz]	σ_2 [Hz]	FWHW$_2$ [Hz]
1	0.621	0.006	0.014	0.648	0.007	0.016
2	0.622	0.006	0.014	0.685	0.007	0.016
3	0.621	0.006	0.015	0.735	0.007	0.016

4. The oscillations of the two coupled electric oscillating circuits are observed for diffrent couplings.

The frequency decreases if the distance between the coils is increased

Analysis

The calculations were done in Python, the code can be found at the end of the document.
First, we are going to calculate the different angular frequencies for the different frequencies that we measured during the experiment using $\omega = 2\pi f$, and compare the results:

Coupling spring position	ω [1/s]	ω_1 [1/s]	ω_2 [1/s]	Beat ω_1 [1/s]	Beat ω_2 [1/s]
top		$3,90 \pm 0,09$	$4,08 \pm 0,04$	$3,90 \pm 0,04$	$4,07 \pm 0,04$
middle	$3,88 \pm 0,03$	$3,90 \pm 0,03$	$4,32 \pm 0,06$	$3,91 \pm 0,04$	$4,30 \pm 0,04$
bottom		$3,90 \pm 0,04$	$4,66 \pm 0,06$	$3,90 \pm 0,04$	$4,62 \pm 0,04$

Table 1: Comparison of the angular frequencies

where the independent angular frequency ω was determined in figure 2, the symmetric oscillation angular frequencies ω_1 were measured in figures 4, 6 and 8, the asymmetrical oscillation angular frequencies were measured in figures 10, 12 and 14, and the beat oscillation frequencies ω_1 and ω_2 were measured in figures 16, 18 and 20.
We can see that $\omega_1 = \sqrt{\frac{D}{J}}$ does not depend on the coupling strength as expected, in both the symmetric and the beat oscillation, whereas $\omega_2 = \sqrt{\frac{D+2D'}{J}}$ does indeed depend on the position of the coupling spring.

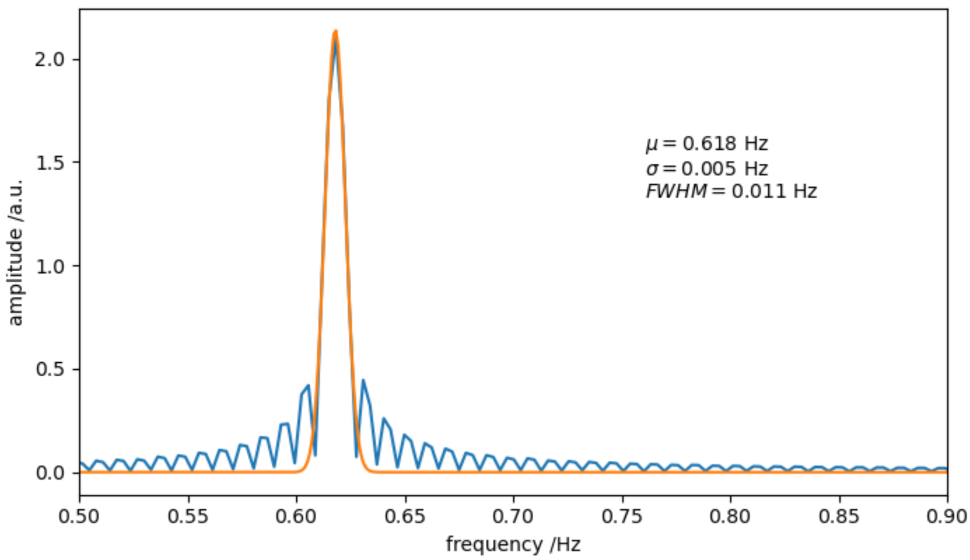


Figure 2: Frequency spectrum - independent oscillation

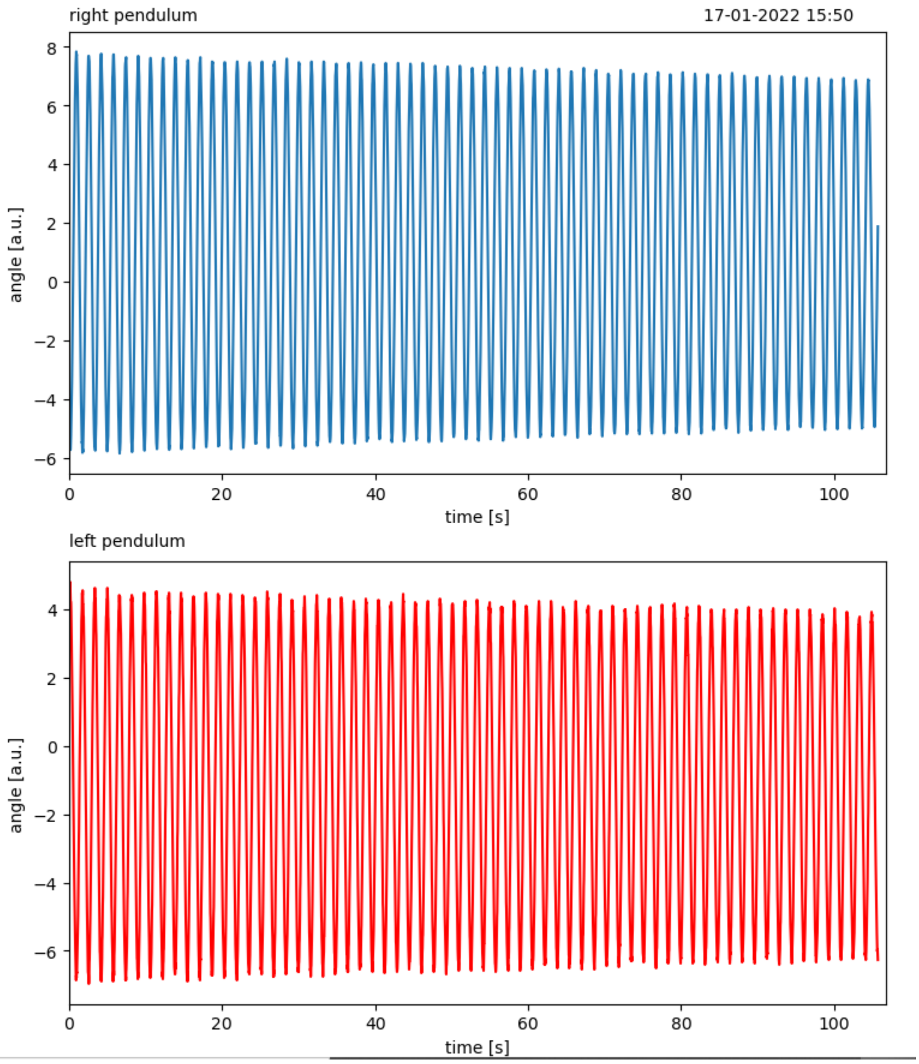


Figure 3: Angular deviation - independent oscillation

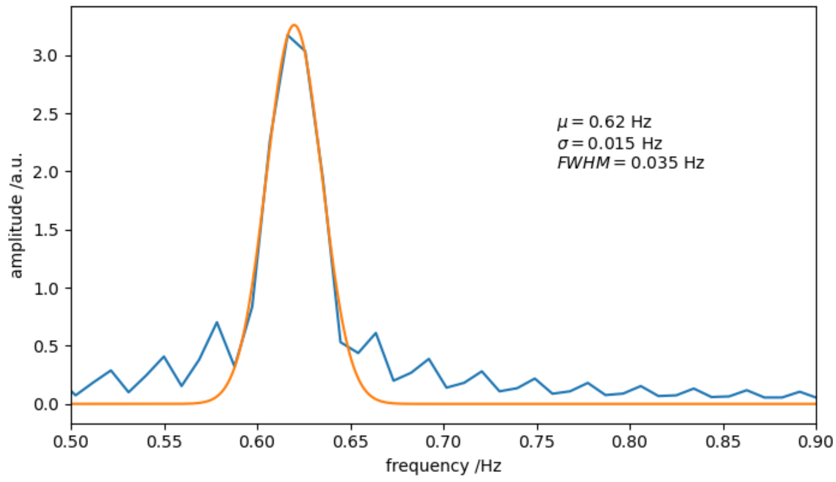


Figure 4: Frequency spectrum - symmetric oscillation - spring in top position

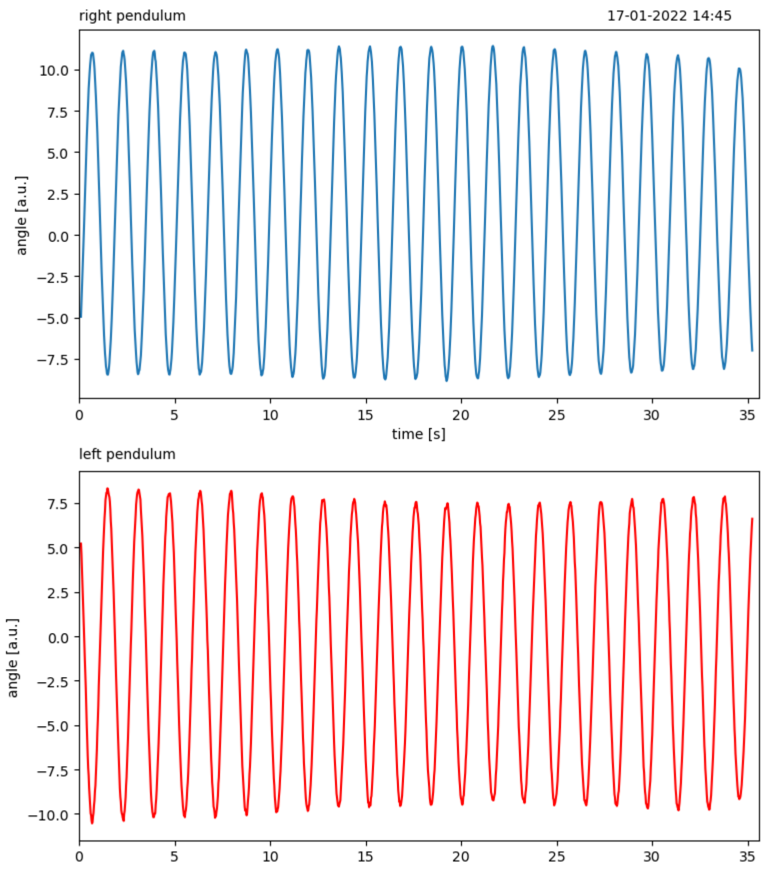


Figure 5: Angular deviation - symmetric oscillation - spring in top position

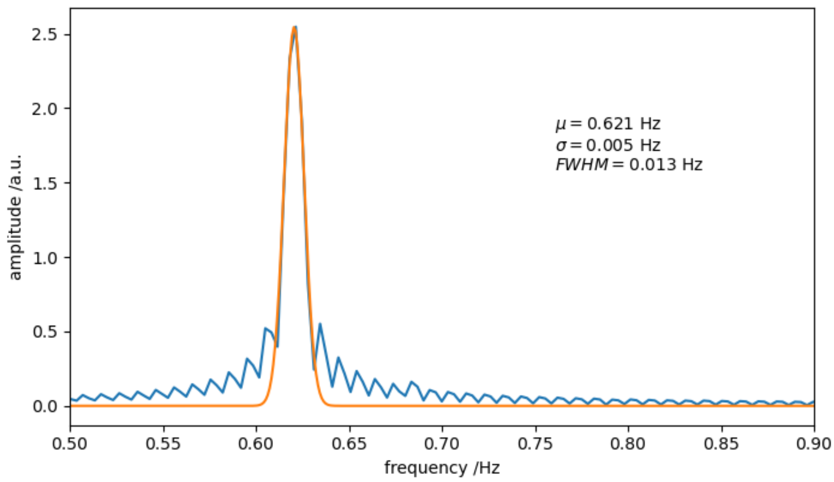


Figure 6: Frequency spectrum - symmetric oscillation - spring in middle position

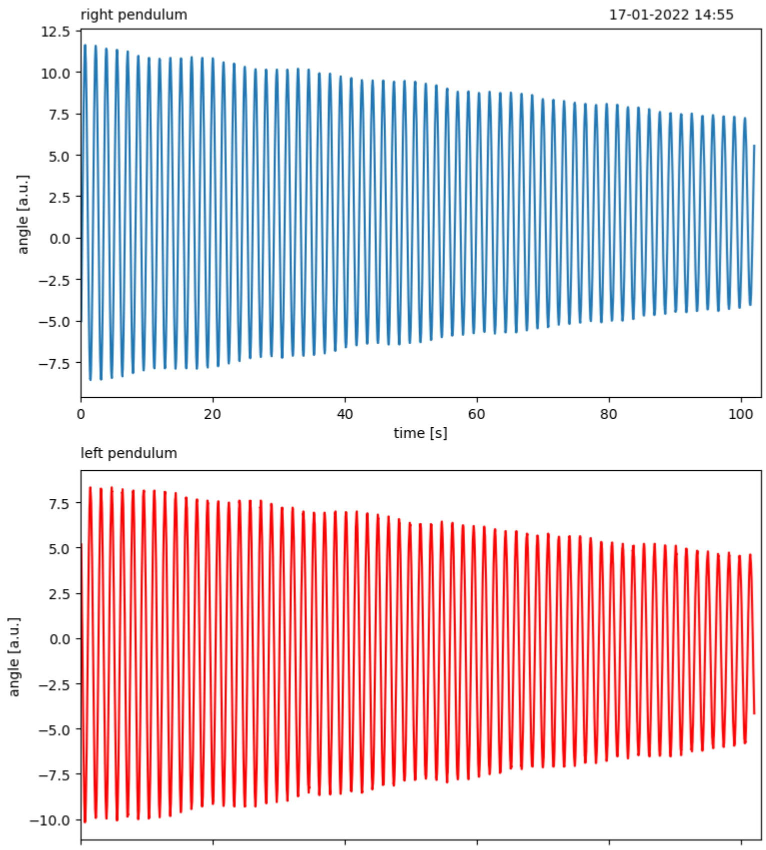


Figure 7: Angular deviation - symmetric oscillation - spring in middle position

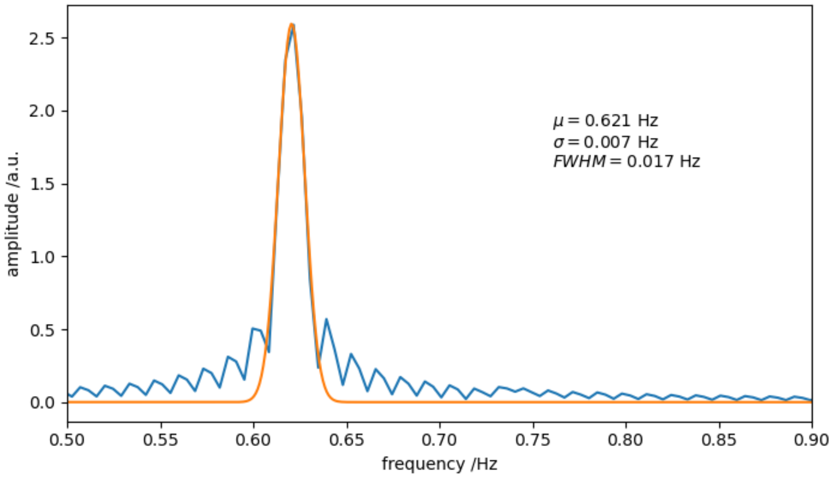


Figure 8: Frequency spectrum - symmetric oscillation - spring in bottom position

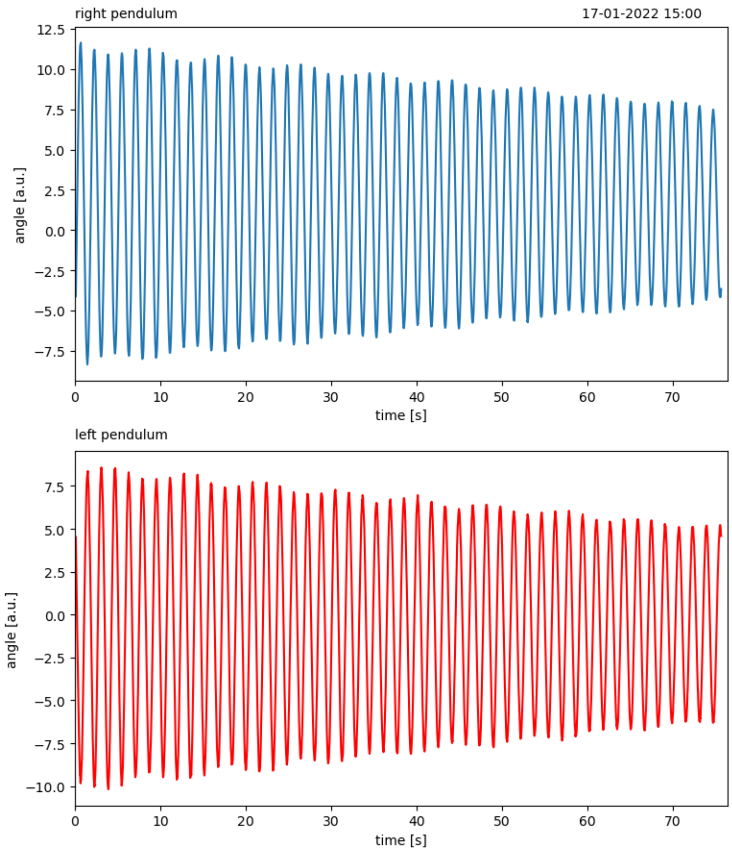


Figure 9: Angular deviation - symmetric oscillation - spring in bottom position

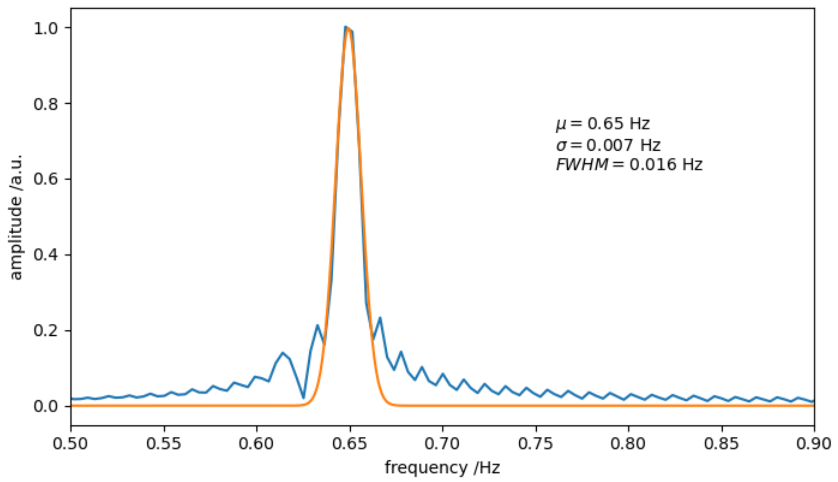


Figure 10: Frequency spectrum - asymmetric oscillation - spring in top position

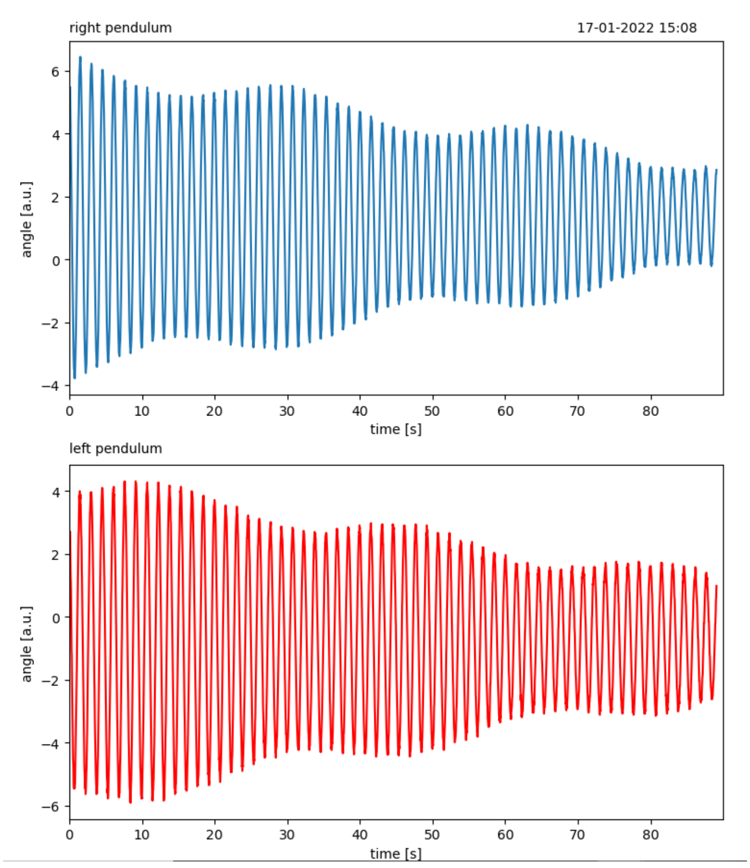


Figure 11: Angular deviation - asymmetric oscillation - spring in top position

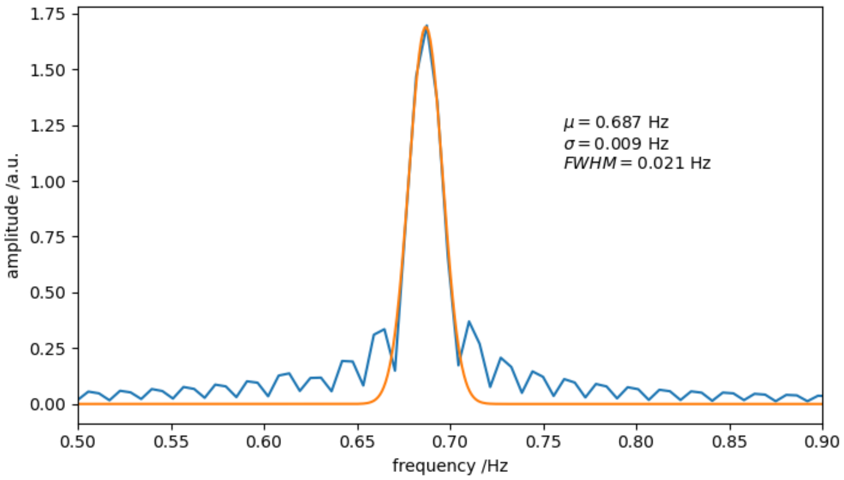


Figure 12: Frequency spectrum - asymmetric oscillation - spring in middle position

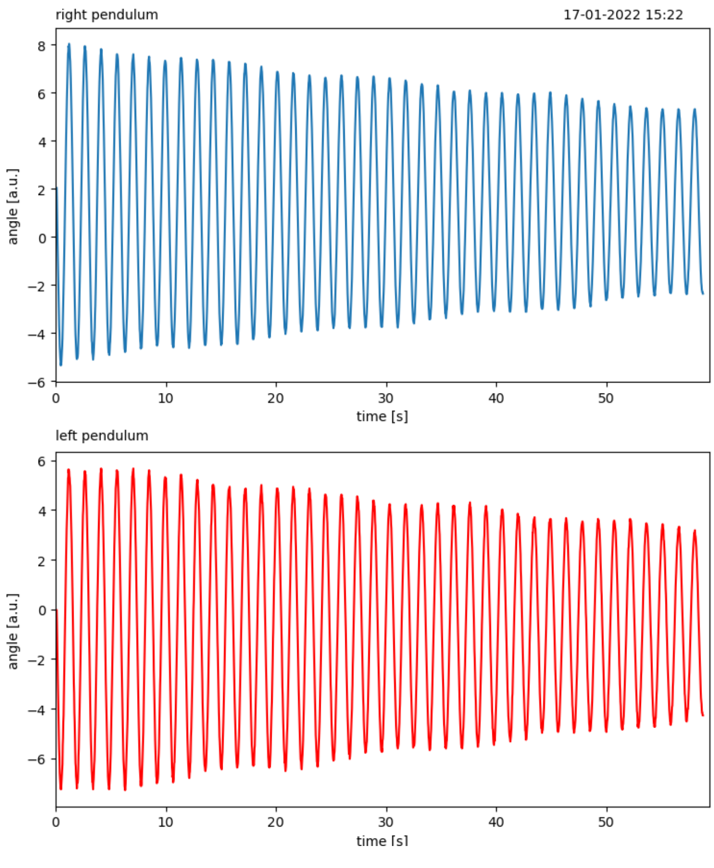


Figure 13: Angular deviation - asymmetric oscillation - spring in middle position

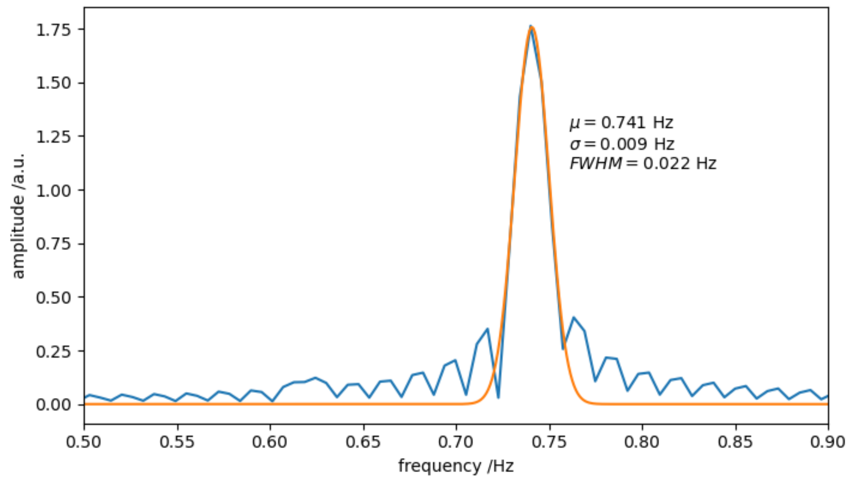


Figure 14: Frequency spectrum - asymmetric oscillation - spring in bottom position

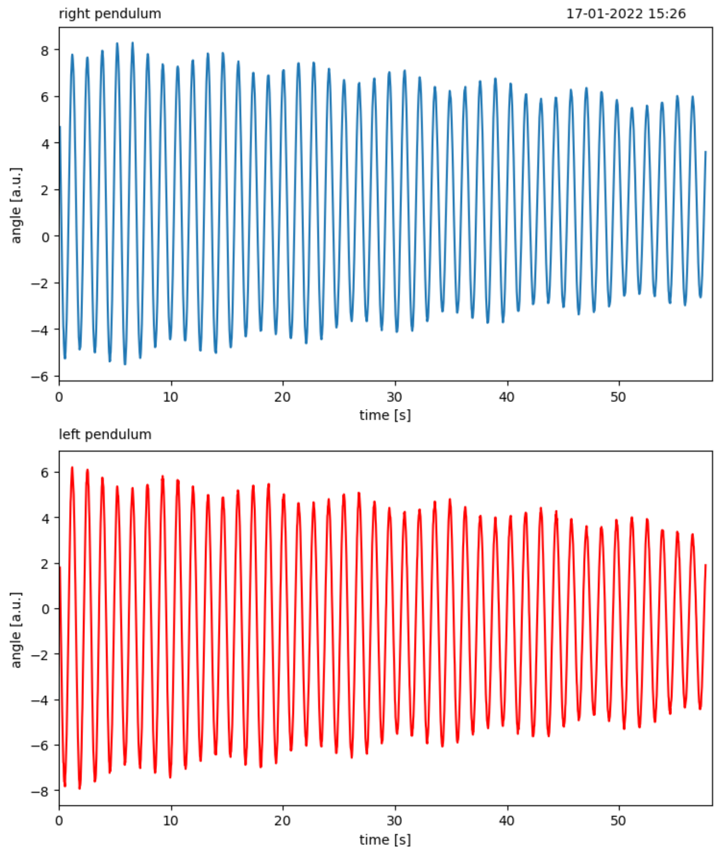


Figure 15: Angular deviation - asymmetric oscillation - spring in bottom position

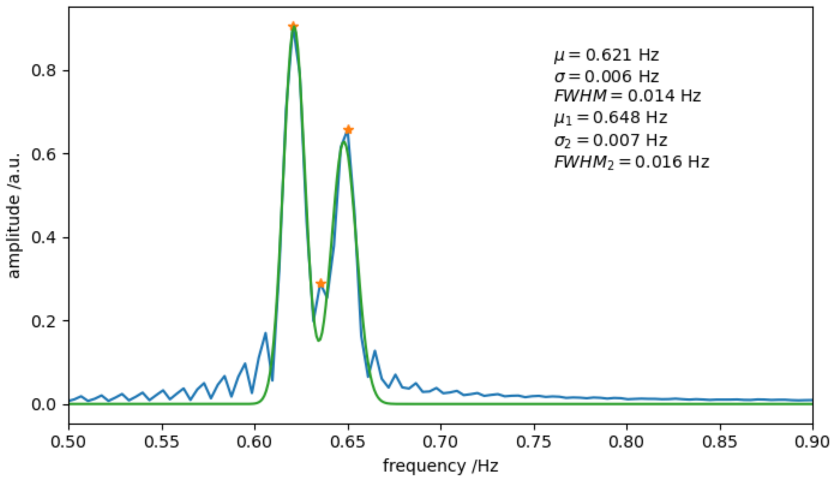


Figure 16: Frequency spectrum - beat oscillation - spring in top position

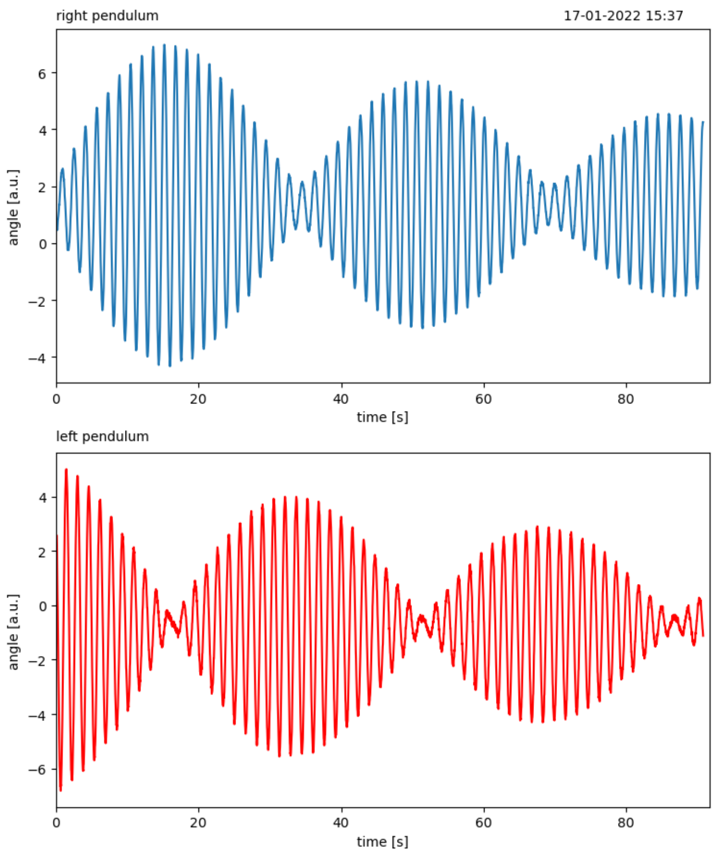


Figure 17: Angular deviation - beat oscillation - spring in top position

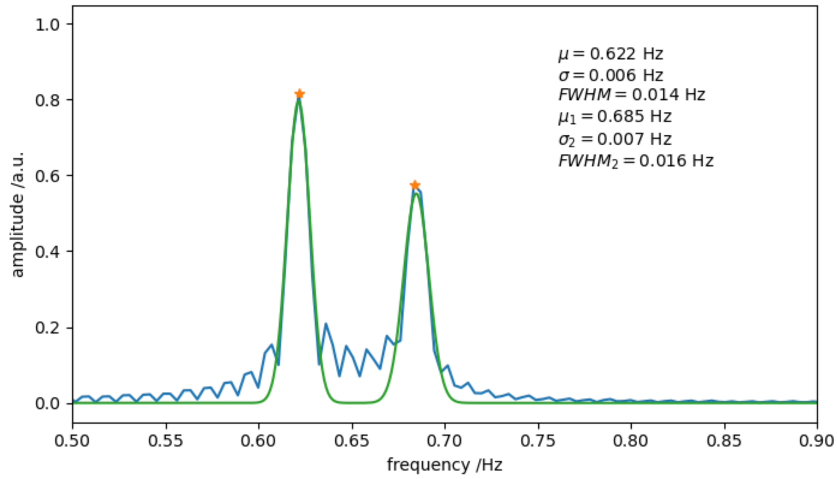


Figure 18: Frequency spectrum - beat oscillation - spring in middle position

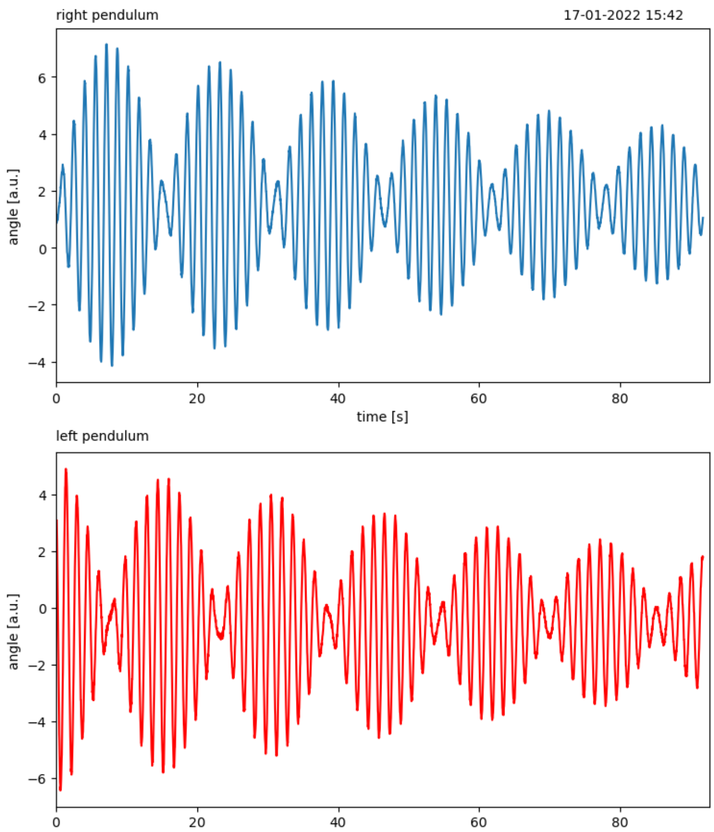


Figure 19: Angular deviation - beat oscillation - spring in middle position

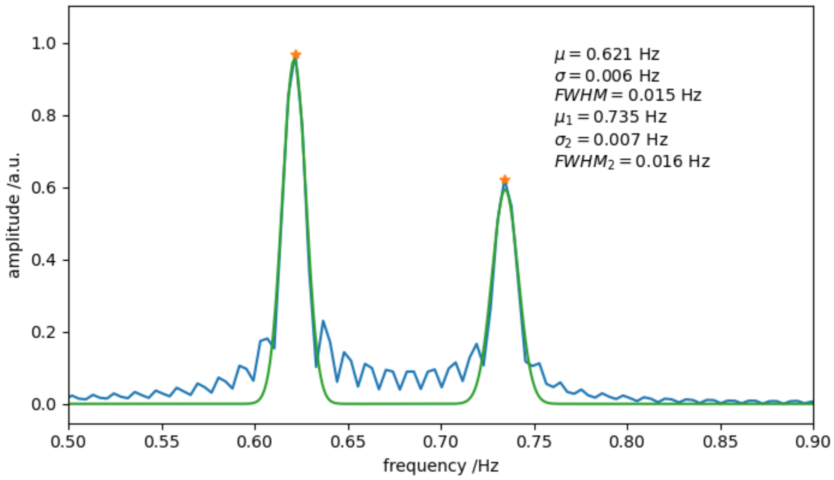


Figure 20: Frequency spectrum - beat oscillation - spring in bottom position

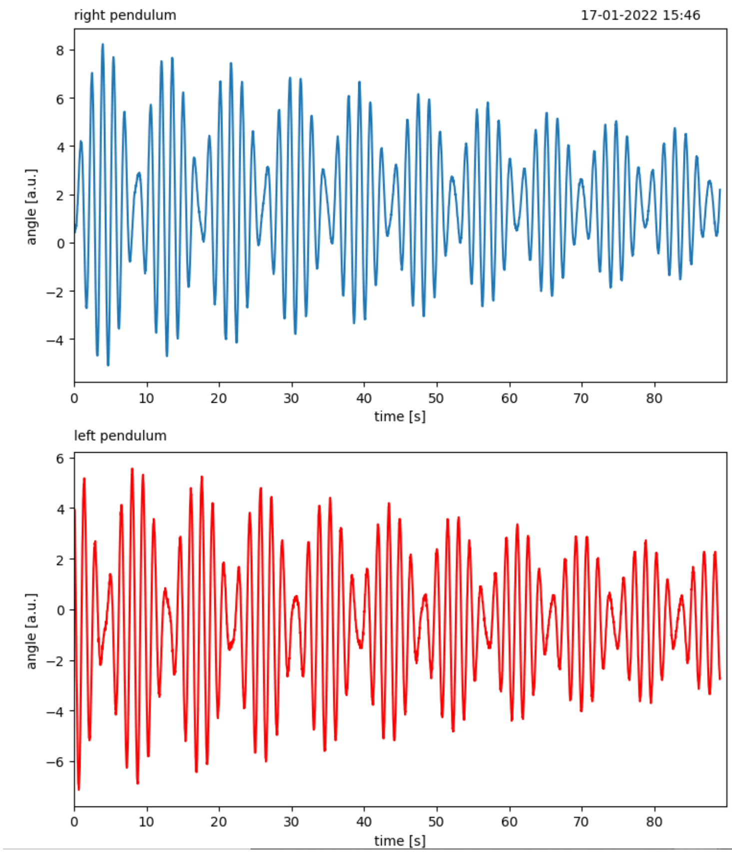


Figure 21: Angular deviation - beat oscillation - spring in bottom position

Calculating the beat frequencies ω_I and ω_{II}

From the symmetric and asymmetric frequencies ω_1 and ω_2 that we measured during the beat oscillation, we can calculate the mixed frequency $\omega_I = \frac{1}{2}(\omega_2 + \omega_1)$ that describes the oscillation of the individual pendulums, as well as $\omega_{II} = \frac{1}{2}(\omega_2 - \omega_1)$ that describes the energy oscillating between the two pendulums. Additionally, we can determine experimental values for those frequencies from figures 17, 19 and 21, and compare those values. Our results are

Coupling	ω_I [1/s]	ω_{II} [1/s]	$\omega_{I,exp}$ [1/s]	$\omega_{II,exp}$ [1/s]	σ-Deviation I	σ-Deviation II
top	$3,987 \pm 0,029$	$0,085 \pm 0,029$	$3,94 \pm 0,12$	$0,088290 \pm 0,00012$	$0,38$	$0,12$
middle	$4,106 \pm 0,029$	$0,198 \pm 0,029$	$3,97 \pm 0,13$	$0,2059 \pm 0,0007$	$1,04$	$0,28$
bottom	$4,260 \pm 0,029$	$0,358 \pm 0,029$	$4,45 \pm 0,16$	$0,3546 \pm 0,0020$	$1,17$	$0,12$

Table 2: Comparison of the mixed angular frequencies

As we can see, the deviations are not significant. As expected, the energy oscillates between the two pendulums with a way lower frequency than the oscillation of the pendulums themselves.

Calculating the coupling strengths

Now we will calculate the coupling strength κ with the approximated equation

$$\kappa = \frac{\omega_2^2 - \omega_1^2}{2\omega_1^2} = \frac{\omega_2^2}{2\omega_1^2} - \frac{1}{2} \tag{22}$$

with the error through propagation of uncertainty

$$\Delta\kappa = \sqrt{\left(\frac{\omega_2 \Delta\omega_2}{\omega_1^2}\right)^2 + \left(\frac{\omega_2^2 \Delta\omega_1}{\omega_1^3}\right)^2} \tag{23}$$

Our results are

Spring position	κ
top	$0,050 \pm 0,029$
middle	$0,112 \pm 0,019$
bottom	$0,212 \pm 0,024$

Table 3: Coupling strengths for different spring positions

As we can see, the coupling strength increases with the distance from the pendulum axis to the spring mount. In theory, it is proportional to l^2, which we are going to check in the next part of the experiment.

Comparing the ratios of length and coupling strength

We are going to calculate the ratios from bottom to top position, bottom to middle, and middle to top position of both the lengths that we measured for the distance from pendulum axis to spring mount, as well as the coupling strengths. In theory, those ratios should be roughly the same. Our results are

Spring positions	κ ratio	length^2 ratio	σ-deviation
middle/top	$2,26 \pm 0,14$	$2,325 \pm 0,015$	0,48
bottom/top	$4,28 \pm 0,27$	$4,222 \pm 0,025$	0,20
bottom/middle	$1,89 \pm 0,09$	$1,816 \pm 0,008$	0,90

Table 4: Comparing the ratios of length and coupling strength

As we can see, the values match well.

Discussion

Our goals in this experiment were to determine the frequency of symmetric and asymmetric natural oscillations of two identical coupled brass pendulums, for three different coupling strenghts, as well as the frequencies of the pendulum and beat for beat oscillation for those coupling strenghts. First, we compared our measured frequencies:

Coupling spring position	ω [1/s]	ω_1 [1/s]	ω_2 [1/s]	Beat ω_1 [1/s]	Beat ω_2 [1/s]
top		$3,90 \pm 0,09$	$4,08 \pm 0,04$	$3,90 \pm 0,04$	$4,07 \pm 0,04$
middle	$3,88 \pm 0,03$	$3,90 \pm 0,03$	$4,32 \pm 0,06$	$3,91 \pm 0,04$	$4,30 \pm 0,04$
bottom		$3,90 \pm 0,04$	$4,66 \pm 0,06$	$3,90 \pm 0,04$	$4,62 \pm 0,04$

Table 5: Comparison of the angular frequencies

As expected, the symmetric oscillation frequency does not depend on the coupling strength, as both pendulums swing as if they were independent. The asymmetric oscillation frequency does depend on the coupling strength.
Next, we calculated the mixed oscillation frequencies ω_I and ω_{II} from the symmetric and asymmetric oscillation frequencies of the beat oscillation. We also determined experimental values for those frequencies from the diagrams, and compared those values.

Coupling	ω_I [1/s]	ω_{II} [1/s]	$\omega_{I,exp}$ [1/s]	$\omega_{II,exp}$ [1/s]	σ-Deviation I	σ-Deviation II
top	$3,987 \pm 0,029$	$0,085 \pm 0,029$	$3,94 \pm 0,12$	$0,088290 \pm 0,00012$	$0,38$	$0,12$
middle	$4,106 \pm 0,029$	$0,198 \pm 0,029$	$3,97 \pm 0,13$	$0,2059 \pm 0,0007$	$1,04$	$0,28$
bottom	$4,260 \pm 0,029$	$0,358 \pm 0,029$	$4,45 \pm 0,16$	$0,3546 \pm 0,0020$	$1,17$	$0,12$

Table 6: Comparison of the mixed angular frequencies

The deviations were not significant. We could also observe that ω_{II}, which describes the energy oscillating between the pendulums, is way smaller than the individual oscillation frequency of the pendulums ω_I.
Then we calculated the coupling strength for the three different spring positions from the symmetric and asymmetric oscillation frequencies:

Spring position	κ
top	$0,050 \pm 0,029$
middle	$0,112 \pm 0,019$
bottom	$0,212 \pm 0,024$

Table 7: Coupling strengths for different spring positions

As we could see, the coupling strength depends on the length between pendulum axis and spring mount.
In the end we calculated the ratios of the coupling strengths for the different positions, as well as the ratios of the squared lengths, and compared these values:

Spring positions	κ ratio	length^2 ratio	σ-deviation
middle/top	$2,26 \pm 0,14$	$2,325 \pm 0,015$	0,48
bottom/top	$4,28 \pm 0,27$	$4,222 \pm 0,025$	0,20
bottom/middle	$1,89 \pm 0,09$	$1,816 \pm 0,008$	0,90

Table 8: Comparing the ratios of length and coupling strength

We could see that the coupling strength is indeed proportional to the squared length, as all deviations were not significant.

Experiment 211 - Coupled Pendulums

Felix Fleischle - 17.1.2022

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit
```

In [2]:
```
# Frequency of the independent pendulums
omega_ind = 2 * np.pi * 0.618 #Hz
omega_ind_err = 2 * np.pi * 0.005
print("Independent omega:", omega_ind, "+-", omega_ind_err, "[1/s]")

# Lenghts of the mounting holes
lengths = np.array([0.1915, 0.2920, 0.3935]) #m
lengths_err = np.array([0.0005, 0.0005, 0.0005])

# Symmetrical oscillation:
omega_sym = 2* np.pi * np.array([0.620, 0.621, 0.621])
omega_sym_err = 2* np.pi * np.array([0.015, 0.005, 0.007])
print("Omega symmetric:", omega_sym, "+-", omega_sym_err, "[1/s]")

# Asymmetrical oscillation:
omega_asym = 2* np.pi * np.array([0.650, 0.687, 0.741])
omega_asym_err = 2* np.pi * np.array([0.007, 0.009, 0.009])
print("Omega asymmetric:", omega_asym, "+-", omega_asym_err, "[1/s]")

# Beat Oscillation:
omega_beat_1 = 2* np.pi * np.array([0.621, 0.622, 0.621])
omega_beat_1_err = 2* np.pi * np.array([0.006, 0.006, 0.006])
print("Omega 1 beat oscillation:", omega_beat_1, "+-", omega_beat_1_err, "[1/s]")

omega_beat_2 = 2* np.pi * np.array([0.648, 0.685, 0.735])
omega_beat_2_err = 2* np.pi * np.array([0.007, 0.007, 0.007])
print("Omega 2 beat oscillation:", omega_beat_2, "+-", omega_beat_2_err, "[1/s]")
```

```
Independent omega: 3.8830085198369844 +- 0.031415926535897934 [1/s]
Omega symmetric: [3.89557489 3.90185808 3.90185808] +- [0.09424778 0.03141593 0.0439823 ] [1/s]
Omega asymmetric: [4.08407045 4.31654831 4.65584031] +- [0.0439823  0.05654867 0.05654867] [1/s]
Omega 1 beat oscillation: [3.90185808 3.90814126 3.90185808] +- [0.03769911 0.03769911 0.03769911] [1/s]
Omega 2 beat oscillation: [4.07150408 4.30398194 4.6181412 ] +- [0.0439823 0.0439823 0.0439823] [1/s]
```

In [3]:
```
omega_beat_i = 1/2 * (omega_beat_1 + omega_beat_2)
omega_beat_i_err = np.sqrt((0.5 * omega_beat_1_err)**2 + (0.5 * omega_beat_2_err)**2)
omega_beat_ii = 1/2 * (omega_beat_2 - omega_beat_1)
omega_beat_ii_err = np.sqrt((0.5 * omega_beat_1_err)**2 + (0.5 * omega_beat_2_err)**2)

print("Omega_i:", omega_beat_i, "+-", omega_beat_i_err, "[Hz]")
print("Omega_ii:", omega_beat_ii, "+-", omega_beat_ii_err, "[Hz]")
```

```
Omega_i: [3.98668108 4.1060616  4.25999964] +- [0.02896405 0.02896405 0.02896405] [Hz]
Omega_ii: [0.084823   0.19792034 0.35814156] +- [0.02896405 0.02896405 0.02896405] [Hz]
```

In [4]:
```
# experimental Values for omega_i and omega_ii
T_i = np.array([1.5951, 1.5819, 1.4130])
T_i_err = np.array([0.05, 0.05, 0.05])
T_ii = np.array([71.1656, 30.5085, 17.7174])
T_ii_err = np.array([0.1, 0.1, 0.1])

omega_i_exp = 2 * np.pi / T_i
omega_i_exp_err = 2 * np.pi * T_i_err / (T_i)**2
print("Experimental omega_i:", omega_i_exp, "+-", omega_i_exp_err, "[1/s]")

omega_ii_exp = 2 * np.pi / T_ii
omega_ii_exp_err = 2 * np.pi * T_ii_err / (T_ii)**2
print("Experimental omega_ii:", omega_ii_exp, "+-", omega_ii_exp_err, "[1/s]")

# Deviations
sigma_omega_i = np.abs((omega_i_exp - omega_beat_i)/(np.sqrt(omega_i_exp_err**2 + omega_beat_i_err**2)))
print("Deviations omega_i:", sigma_omega_i)

sigma_omega_ii = np.abs((omega_ii_exp - omega_beat_ii)/(np.sqrt(omega_ii_exp_err**2 + omega_beat_ii_err**2)))
print("Deviations omega_ii:", sigma_omega_ii)
```

```
Experimental omega_i: [3.93905417 3.9719232  4.44669873] +- [0.12347358 0.1255428  0.15734957] [1/s]
Experimental omega_ii: [0.08828964 0.20594868 0.3546336 ] +- [0.00012406 0.00067505 0.00200161] [1/s]
Deviations omega_i: [0.37553177 1.04111867 1.16691942]
Deviations omega_ii: [0.11968656 0.27710774 0.12082625]
```

In [11]:
```
# Coupling strength
kappa = ((omega_asym)**2 / (2*omega_sym**2) )- 0.5
kappa_err = np.sqrt((omega_asym * omega_asym_err / (omega_sym)**2)**2 + (omega_asym**2 * omega_sym_err / (omega_sym)**3)**2)
print("Kappa:", kappa, "+-", kappa_err)
```

```
Kappa: [0.04955775 0.11192793 0.21190693] +- [0.02910694 0.01881909 0.02359326]
```

In [6]:
```python
# Ratios
kappa_r_mt = kappa[1]/kappa[0]
kappa_r_mt_err = np.sqrt((kappa_err[0]/kappa[0])**2 + (kappa_err[1]/kappa[1])**2) * kappa_r_mt
print("Ratio middle/top:", kappa_r_mt, "+-", kappa_r_mt_err)

kappa_r_bt = kappa[2]/kappa[0]
kappa_r_bt_err = np.sqrt((kappa_err[0]/kappa[0])**2 + (kappa_err[2]/kappa[2])**2) * kappa_r_bt
print("Ratio bottom/top:", kappa_r_bt, "+-", kappa_r_bt_err)

kappa_r_bm = kappa[2]/kappa[1]
kappa_r_bm_err = np.sqrt((kappa_err[1]/kappa[1])**2 + (kappa_err[2]/kappa[2])**2) * kappa_r_bm
print("Ratio bottom/middle:", kappa_r_bm, "+-", kappa_r_bm_err)
```

```
Ratio middle/top: 2.2585352984346216 +- 0.1383251140740547
Ratio bottom/top: 4.275959238904572 +- 0.2671546667624803
Ratio bottom/middle: 1.893244370308587 +- 0.08559717472243693
```

In [7]:
```python
# Ratio of lengths
lengths_sq = lengths**2
lengths_sq_err = 2 * lengths * lengths_err
print("Lengths squared", lengths_sq, "+-", lengths_err, "[m^2]")

length_r_mt = lengths_sq[1]/lengths_sq[0]
length_r_mt_err = np.sqrt((lengths_sq_err[0]/lengths_sq[0])**2 + (lengths_sq_err[1]/lengths_sq[1])**2) * length_r_mt
print("Length Ratio middle/top:", length_r_mt, "+-", length_r_mt_err)

length_r_bt = lengths_sq[2]/lengths_sq[0]
length_r_bt_err = np.sqrt((lengths_sq_err[0]/lengths_sq[0])**2 + (lengths_sq_err[2]/lengths_sq[2])**2) * length_r_bt
print("Length Ratio bottom/top:", length_r_bt, "+-", length_r_bt_err)

length_r_bm = lengths_sq[2]/lengths_sq[1]
length_r_bm_err = np.sqrt((lengths_sq_err[1]/lengths_sq[1])**2 + (lengths_sq_err[2]/lengths_sq[2])**2) * length_r_bm
print("Length Ratio bottom/mid:", length_r_bm, "+-", length_r_bm_err)
```

```
Lengths squared [0.03667225 0.085264   0.15484225] +- [0.0005 0.0005 0.0005] [m^2]
Length Ratio middle/top: 2.3250277798607937 +- 0.0145192082325063
Length Ratio bottom/top: 4.222327509220187 +- 0.024521059563612894
Length Ratio bottom/mid: 1.8160331441170954 +- 0.0077445807939464754
```

In [8]:
```python
# Deviations:
sigma_mt = np.abs((length_r_mt - kappa_r_mt)/(np.sqrt(length_r_mt_err**2 + kappa_r_mt_err**2)))
print("Deviation middle/top:", sigma_mt)

sigma_bt = np.abs((length_r_bt - kappa_r_bt)/(np.sqrt(length_r_bt_err**2 + kappa_r_bt_err**2)))
print("Deviation bottom/top:", sigma_bt)
```

```python
sigma_bm = np.abs((length_r_bm - kappa_r_bm)/(np.sqrt(length_r_bm_err**2 + kappa_r_bm_err**2)))
print("Deviation bottom/mid:", sigma_bm)
```

```
Deviation middle/top: 0.4780707392331711
Deviation bottom/top: 0.19991128358750593
Deviation bottom/mid: 0.8983605385427839
```

In [ ]:

```python

```

In [ ]:

```python

```