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Q1) First, we should find U.

$$\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow[\text{from row2}]{\text{subtract } 1 \times \text{row1}} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow[\text{from row3}]{\text{Subtract } 1/2 \times \text{row1}}$$

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[\text{from row3}]{\text{Subtract } 1/2 \times \text{row2}} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

$Ax = b$ $LUx = b$

$Ux = c$ $Lc = b$

Now, we should find c.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$c_1 = 2$

$2 + c_2 = 4$ $c_2 = 2$

$1 + 1 + c_3 = 6$ $c_3 = 4$

Finally, we can find the x.

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$4x_1 + 2x_2 = 2$ $x_1 = 1$

$2x_2 + 2x_3 = 2$ $x_2 = -1$

$2x_3 = 4$ $x_3 = 2$

Q2) Since the matrix is in proper form, no exchange is required.

$$\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 2 \\ 1/2 & 1 & 3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 2 \\ 1/2 & 1/2 & 2 \end{bmatrix}$$

we didn't change the rows so permutation matrix is:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\underset{P}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}} \cdot \underset{U}{\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$c_1 = 2$

$c_2 = 2$

$c_3 = 4$

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$x_1 = 1$

$x_2 = -1$

$x_3 = 2$

Q3) $x_a = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad x_1 + 2x_2 = 1$ $\quad 2x_1 + 4.01x_2 = 2$ $\quad \begin{bmatrix} 1 & 2 \\ 2 & 4.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4.01 \end{bmatrix} \xrightarrow{2} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 0.01 \end{bmatrix}}_{U} \qquad \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad c_1 = 1, \quad c_2 = 0$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0.01 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x_1 = 1, \quad x_2 = 0 \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

forward error $= \|x - x_a\|_\infty = \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\|_\infty =$

$$= \left\| \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\|_\infty = 2$$

backword error $= \|b - Ax_a\|_\infty = \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4.01 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\|_\infty =$

$$= \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2.01 \end{bmatrix} \right\|_\infty =$$

$$= \left\| \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \right\|_\infty = 0.01$$

Error magnification factor $= \dfrac{\|x - x_a\|_\infty \cdot \|b\|_\infty}{\|r\|_\infty \cdot \|x\|_\infty} = \dfrac{2 \cdot 2}{(0.01) \cdot 1}$

$$= \frac{4}{0.01} = 400$$

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Use Langrarge interpolation to find a polynomial that passes through the points
$(-1,0)$ $(2,1)$ $(3,1)$ $(5,2)$
$x_1\ y_1$ $x_2\ y_2$ $x_3\ y_3$ $x_4\ y_4$

$$P_3(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} \cdot y_1 + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \cdot y_2 + \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \cdot y_4$$

If we substitute the terms we obtain Langrange interpolation of the points. Let's substitute

$$= \frac{(x-2)(x-3)(x-5)}{(-3)(-4)(-6)} \cdot 0 + \frac{(x+1)(x-3)(x-5)}{(3)(-1)(-3)} \cdot 1 + \frac{(x+1)(x-2)(x-5)}{(4)(1)(-2)} \cdot 1$$

$$+ \frac{(x+1)(x-2)(x-3)}{6 \cdot 3 \cdot 2} \cdot 2$$

$$= (x+1)\left[\frac{x^2-8x+15}{\underset{(8)}{9}} - \frac{x^2-7x+10}{\underset{(9)}{8}} + \frac{x^2-5x+6}{\underset{(4)}{18}} \right]$$

$$= (x+1)\left[\frac{\cancel{8x^2} - \cancel{64x} + 120 - \cancel{9x^2} + 63x - 90 + \cancel{4x^2} - \cancel{20x} + 24}{72} \right]$$

$$= \frac{x^3 - 6x^2 + 11x + 18}{24}$$

Our polynomial $n-1=3$ degree polynomial

We can check that our polynomial satisfies all four points.

$P_3(-1) = 0$

$P_3(2) = 1$

$P_3(3) = 1$

$P_3(5) = 2$

Use Newton's divided differences to find the interpolating polynomial of the points in the previous exercise.

$$\underset{x_1\ y_1}{(-1, 0)} \quad \underset{x_2\ y_2}{(2,1)} \quad \underset{x_3\ y_3}{(3,1)} \quad \underset{x_4\ y_4}{(5\ 2)}$$

$$\begin{array}{r|lllll}
-1 & 0 & & & & \\
2 & 1 & \frac{1}{3} & \frac{-1/3}{4} \; \left(\frac{-1}{12}\right) & & \\
3 & 1 & 0 & & \frac{3/12}{6} = \frac{1}{24} & \\
5 & 2 & \frac{1}{2} & \frac{1/2}{3} \; \left(\frac{2}{12}\right) & &
\end{array}$$

$$P(x) = f[x_1] + f[x_1\ x_2](x - x_1) + f[x_1\ x_2\ x_3](x-x_1)(x-x_2)(x-x_3)$$

Substitute the terms, we yield

$$P(x) = 0 + \frac{1}{3}(x+1) - \frac{1}{12}(x+1)(x-2) + \frac{1}{24}(x+1)(x-2)(x-3)$$

$$= \frac{x^3 - 6x^2 + 11x + 18}{24}$$

At the end, we can observe that we obtain some polynomial as before with Langrange interpolation. Therefore we can say that we are on the correct way.

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Find the natural cubic spline through the points $(-1,1)$ $(1,1)$ $(2,4)$

$\delta_i = x_{i+1} - x_i$ $\qquad$ $\Delta_i = y_{i+1} - y_i$

$\delta_1 = x_2 - x_1$ $\qquad$ $\Delta_1 = y_2 - y_1$

$\delta_2 = x_3 - x_2$ $\qquad$ $\Delta_2 = y_3 - y_2$

$\delta_1 = 1-(-1) = 2$ $\qquad$ $\Delta_1 = 1-1 = 0$

$\delta_2 = 2-1 = 1$ $\qquad$ $\Delta_2 = 4-1 = 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ \delta_1 & 2\delta_1+2\delta_2 & \delta_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 6 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's continue with

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 6 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix}$$

So using this system we can obtain c_1, c_2, c_3 as

$c_1 = 0 \qquad c_2 = 3/2 \qquad c_3 = 0$

we know that natural cubic splines in the form

$$S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$

Therefore we need to find a_i, b_i, d_i we know that a_i's

$a_1 = y_1 = 1 \qquad a_2 = y_2 = 1$

Let's find the remaining coefficients

$$d_1 = \frac{c_2-c_1}{3\delta_1} = \frac{3/2}{6} = \frac{1}{4} \qquad d_2 = \frac{c_3-c_2}{3\delta_2} = \frac{-3/2}{3} = -\frac{1}{2}$$

$$b_1 = \frac{\Delta_1}{\delta_1} - \frac{\delta_1}{3}(2c_1+c_2) = \frac{0}{\delta_1} - \frac{2}{3}\left(\frac{3}{2}\right) = -1$$

Note: We can control our work with checking properties such as $S_1(x_1) = y_1$ $S_2(x_2) = y_2$

$$b_2 = \frac{\Delta_2}{\delta_2} - \frac{\delta_2}{3}(2c_2+c_3) = \frac{3}{1} - \frac{1}{3}(3) = 2$$

$S_1(x) = 1 - (x+1) + \frac{1}{4}(x+1)^3$ on $[-1,1]$

$S_2(x) = 1 + 2(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{2}(x-1)^3$ on $[1,2]$

find the first endpoint, two control points and the last endpoint for the one-piece Bezier curve

$$x(t) = 1 + 6t^2 + 2t^3$$
$$y(t) = 1 - t + t^3$$

We know how to construct Bezier curves and their forms. They are in the form of

$$x(t) = x_1 + b_x t + c_x t^2 + d_x t^3$$
$$y(t) = y_1 + b_y t + c_y t^2 + d_y t^3$$

$$b_x = 3(x_2 - x_1)$$
$$c_x = 3(x_3 - x_2) - b_x$$
$$d_x = x_4 - x_1 - b_x - c_x$$
$$b_y = 3(y_2 - y_1)$$
$$c_y = 3(y_3 - y_2) - b_y$$
$$d_y = y_4 - y_1 - b_y - c_y$$

If we observe these equations we can see that we have a chance to recover points starting with

$(x_1, y_1) = (1, 1)$ $\quad b_x = 0 \quad b_y = -1 \quad c_x = 6 \quad c_y = 0 \quad d_x = 2 \quad d_y = 1$

$$b_x = 3(x_2 - x_1)$$
$$0 = 3(x_2 - 1)$$
$$\boxed{x_2 = 1}$$

$$b_y = 3(y_2 - y_1)$$
$$-1 = 3(y_2 - 1)$$
$$\boxed{y_2 = \frac{2}{3}}$$

$$c_x = 3(x_3 - x_2) - b_x$$
$$6 = 3(x_3 - 1) - 0$$
$$\boxed{x_3 = 3}$$

$$c_y = 3(y_3 - y_2) - b_y$$
$$0 = 3\left(y_3 - \frac{2}{3}\right) + 1$$
$$\boxed{y_3 = \frac{1}{3}}$$

$$d_x = x_4 - x_1 - b_x - c_x$$
$$= x_4 - 1 - 0 - 6$$
$$2 = x_4 - 7$$
$$\boxed{x_4 = 9}$$

$$d_y = y_4 - y_1 - b_y - c_y$$
$$= y_4 - 1 + 1 + 0$$
$$\boxed{y_4 = 1}$$

If we collect all the four points

$$\underset{\text{first endpoint}}{(\overset{x_1}{1}, \overset{y_1}{1})} \quad \underbrace{(\overset{x_2}{1}, \overset{y_2}{\tfrac{2}{3}}) \quad (\overset{x_3}{3}, \overset{y_3}{\tfrac{1}{3}})}_{\text{control points}} \quad \underset{\text{last endpoint}}{(\overset{x_4}{9}, \overset{y_4}{1})}$$