Solution 1.

a. Using three point difference formula we are asked to approximate $f'(0)$, $f(x) = cos(x)$, $h = 0.01$

Start with writing three point difference formula for second derivative aproximation with error term included:

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} - \underbrace{\frac{h^2}{12} f^{\text{iv}}(c)}_{\text{Error Term}} \tag{1}$$

where we know that c in the interval of $x - h < c < x + h$ and continue with substituting values to the formula (1) without error term included:

$$\begin{aligned} f''(0) &= \frac{f(-0.01) - 2f(0) + f(0.01)}{(0.01)^2} \\ &= \frac{cos(-0.01) - 2cos(0) + cos(0.01)}{(0.01)^2} \\ &= -0.999991666 \end{aligned}$$

for c we can say $-0.01 < c < 0.01$ and error is

$$\frac{(0.01)^2}{12} cos(c)$$

b. We are asked to find the error term for the approximation formula:

$$f'(x) = \frac{4f(x+h) - 3f(x) - f(x-2h)}{6h}$$

Using Taylor's Theorem we can make two expansion to reach our goal approximation:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(c_1) \tag{2}$$

$$f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4}{3} h^3 f'''(c_2) \tag{3}$$

applying Generalized Intermediate Value Theorem we can combine $c_1 = c_2 = c$ and $x - 2h < c < x + h$ and continue with multiplying (2) equation with 4:

$$4f(x+h) = 4f(x) + 4hf'(x) + 2h^2 f''(x) + \frac{2}{3} h^3 f'''(c_1) \tag{4}$$

end substracting equation (4)(3) we obtain:

$$\begin{aligned} 4f(x+h) &= 4f(x) + 4hf'(x) + 2h^2 f''(x) + \frac{2}{3} h^3 f'''(c_1) \\ f(x-2h) &= f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4}{3} h^3 f'''(c_2) \end{aligned}$$

result is:

$$\begin{aligned} 4f(x+h) - f(x-2h) &= 3f(x) + 6hf'(x) + 2h^3 f'''(c) \\ 6hf'(x) &= 4f(x+h) - 3f(x) - f(x+2h) - 2h^3 f'''(c) \\ f'(x) &= \frac{4f(x+h) - 3f(x) - f(x+2h)}{6h} - \underbrace{\frac{h^2}{3} f'''(c)}_{\text{Error Term}} \end{aligned}$$

We obtained the error term as

$$\frac{h^2}{3} f'''(c)$$

where c in the interval of $x - 2h < c < x + h$

Solution 2.a For the composite Trapezoid Rule, h is equal to $\frac{(b-a)}{m}$ and $a < c < b$

$$\int_a^b f(x)dx = \frac{h}{2}(y_0 + y_m + 2\sum_{i=1}^{m-1} y_i) - \frac{(b-a)h^2}{12} f''(c)$$

Taking $m = 1$ and $h = \frac{\pi}{2}$ the approximation is

$$\int_0^{\frac{\pi}{2}} cosxdx \approx \frac{\pi}{4}(1 + 0 + 2[0]) \approx 0,785$$

The error is at most

$$\frac{(\frac{\pi}{2})^3}{12} |-1| = 0.322982$$

where the exact value is 1.

Taking $m = 2$ and $h = \frac{\pi}{4}$ the approximation is

$$\int_0^{\frac{\pi}{2}} cosxdx \approx \frac{\pi}{8}(1 + 0 + 2[0.7071]) \approx 0,948$$

The error is at most

$$\frac{(\frac{\pi}{2}).(\frac{\pi}{4})^2}{12} |-1| = 0.080746$$

where the exact value is 1.

Taking $m = 4$ and $h = \frac{\pi}{8}$ the approximation is

$$\int_0^{\frac{\pi}{2}} cosxdx \approx \frac{\pi}{16}(1 + 0 + 2[0.9238 + 0.7071 + 0.3826]) \approx 0,987$$

The error is at most

$$\frac{(\frac{\pi}{2}).(\frac{\pi}{8})^2}{12}\left|-1\right| = 0.020187$$

where the exact value is 1.

Solution 2.b For the composite Simpson's Rule, h is equal to $\frac{(b-a)}{2m}$ and $a < c < b$

$$\int_a^b f(x)dx = \frac{h}{3}(y_0 + y_{2m} + 4\sum_{i=1}^{m} y_{2i-1} + 2\sum_{i=1}^{m-1} y_{2i}) - \frac{(b-a)h^4}{180} f''''(c)$$

Taking $m = 1$ and $h = \frac{\pi}{4}$ the approximation is

$$\int_0^{\frac{\pi}{2}} cosxdx \approx \frac{\pi}{12}(1 + 0 + 4[0.7071] + 2[0]) \approx 1.002273$$

The error is at most

$$\frac{(\frac{\pi}{2}).(\frac{\pi}{4})^4}{180}\left|-1\right| = 0.003321$$

where the exact value is 1.

Taking $m = 2$ and $h = \frac{\pi}{8}$ the approximation is

$$\int_0^{\frac{\pi}{2}} cosxdx \approx \frac{\pi}{24}(1 + 0 + 4[0.9238 + 0.3826] + 2[0.7071]) \approx 1.000126$$

The error is at most

$$\frac{(\frac{\pi}{2}).(\frac{\pi}{8})^4}{180}\left|-1\right| = 0.0002075$$

where the exact value is 1.

Taking $m = 4$ and $h = \frac{\pi}{16}$ the approximation is

$$\int_0^{\frac{\pi}{2}} cosxdx \approx \frac{\pi}{48}(1+0+4[0.9807+0.8314+0.5555+0.1950]+2[0.9238+0.7071+0.3826]) \approx 1.0000081$$

The error is at most

$$\frac{(\frac{\pi}{2}).(\frac{\pi}{16})^4}{180}\left|-1\right| = 0.000013$$

where the exact value is 1.

Solution 2.c Approximating the integral using Trapezoid Rule with

m = 32 results 6.89456

m = 64 results 6.88927

m = 128 results 6.88795

Solution 2.d Approximating the integral using Simpson's Rule with

m = 32 results 6.88750936

m = 64 results 6.88751044

m = 128 results 6.88751058