230201057 Furkan Emre YILMAZ 12 MATH 144 / HW8

Q1. Compute the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{pmatrix} 6 & -4 & 1 \\ 5 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Answer 1.

We start to work with finding characterictic polynomial and eigenvectors

$$\det(A - \lambda I_n) = 0$$

$$\det\left(\begin{pmatrix} 6 & -4 & 1 \\ 5 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\det \begin{pmatrix} 6-\lambda & -4 & 1 \\ 5 & -3-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0$$

$$6-\lambda \begin{vmatrix} -3-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} - (-4) \begin{vmatrix} 5 & 1 \\ 0 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 5 & -3-\lambda \\ 0 & 0 \end{vmatrix} = 0$$

$$(6-\lambda)((-3-\lambda)(1-\lambda) - 0) + 4(5(1-\lambda)) + 0 = 0$$

$$(6-\lambda)(3-\lambda)(1-\lambda) + 20(1-\lambda) = 0$$

$$(-18-6\lambda+3\lambda+\lambda^2)(1-\lambda) + 20(1-\lambda) = 0$$

$$(2-3\lambda+\lambda^2)(1-\lambda) = 0$$

$$(\lambda-2)(-1+\lambda)(1-\lambda) = 0$$

$$-(\lambda-2)(1-\lambda)^2 = 0$$

$$\lambda = 2 \qquad \lambda = 1$$

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Lets find corresponding eigenvectors

$$(A - \lambda I_n) = \vec{0}$$

$\lambda = 2$ $$\begin{pmatrix} 6 & -4 & 1 \\ 5 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 4 & -4 & 1 \\ 5 & -5 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \vec{0}$$

If we row-reduce this matrix obtain

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we solve for eigen vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad x_2 \in \mathbb{C}$$

Using this set we can instantly create an eigenvector which is eigenvalue $\lambda = 2$ such as $x_2 = 1$

$$\begin{pmatrix} 6 & -4 & 1 \\ 5 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{u}} = \underbrace{2}_{\lambda} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}}$$

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Lastly

$$(A - \lambda I_n) = \vec{0}$$

$\lambda = 1$

$$\begin{pmatrix} 6 & -4 & 1 \\ 5 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 5 & -4 & 1 \\ 5 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \vec{0}$$

Row-reduced to

$$\begin{pmatrix} 1 & -4/5 & 1/5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if we solve this for $\lambda = 1$ we create a set;

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} +4/5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1/5 \\ 0 \\ 1 \end{bmatrix} \qquad x_2, x_3 \in \mathbb{C}$$

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Q2. Find all the eigenvalues and their corresponding eigenvectors for the matrix B where

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$A\vec{x} = \lambda\vec{x}$

$Ax - \lambda x = \vec{0}$

$(A - \lambda I_n)\vec{x} = \vec{0}$

$\det(A - \lambda I_n) = 0$

Answer 2.

$$\det\left(\begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\det \begin{pmatrix} 2-\lambda & 0 & 0 & 0 \\ 2 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{pmatrix} = 0$$

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$$[B]_{11} \det(B(1|1)) + [B]_{12} \det(B(1|2)) + \underbrace{[B]_{13}}_{=0} \det(B(1|3)) + \underbrace{[B]_{14}}_{=0} \det(B(1|4))$$

$$(2-\lambda) \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^3(3-\lambda) + 2 \cdot 0 = 0$$

$\lambda = 2 \quad \lambda = 3$

Lets find $\lambda = 2$ and $\lambda = 3$ corresponding eigenvectors

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$$A - \lambda I_n = \vec{0}$$

$\lambda = 2$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -2 \end{pmatrix} = \vec{0}$$

Row reduced to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \vec{0}$$

If we solve we create a solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad x_2, x_3 \in \mathbb{C}$$

and lastly lets solve for $\lambda = 3$

$$A - \lambda I_n = \vec{0}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \vec{0}$$

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$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \vec{0}$$

Row reduced to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \vec{0}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad x_4 \in \mathbb{C}$$

so using these solution set we both $\lambda = 2$ and $\lambda = 3$ instantly create a eigenvector.

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Q3. Compute the eigenvalue and corresponding eigenvectors for the matrix

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Answer 3.

$$\det(O - \lambda I_n) = 0$$

$$\det\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 = 0 \qquad \lambda = 0$$

so if we solve for corresponding eigenvectors we obtain

$\lambda = 0$ $\qquad O - \lambda I_n = \vec{0}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and for this system solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad x_1, x_2 \in \mathbb{C}$$

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Q4 Let $U = \begin{pmatrix} -1 \\ 1 \\ 5 \\ -1 \\ 1 \\ 5 \\ 7 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & -2 & 3 & -4 & 1 & 5 & 3 \end{pmatrix}$. Find two distinct eigenvalues and corresponding eigenvectors of the UV.

Answer 4, If we think about UV matrix multiplication we must realize something we are taking linear combinations so if we inspect carefully multiplication

$$UV = \begin{array}{c} \\ \\ -1 \end{array}\begin{array}{ccccccc} -1 & 1 & 5 & -1 & 1 & 5 & 7 \\ \end{array}$$

$$UV = \begin{pmatrix} -1\times 1 & -1 \times -2 & -1 \times 3 & -1 \times -4 & -1 \times 1 & -1 \times 5 & -1 \times 3 \\ 1 \times 1 & 1 \times -2 & 1 \times 3 & 1 \times -4 & 1 \times 1 & 1 \times 5 & 1 \times 3 \\ 5 \times 1 & 5 \times -2 & 5 \times 3 & 5 \times -4 & 5 \times 1 & 5 \times 5 & 5 \times 3 \\ -1 \times 1 & -1 \times -2 & -1 \times 3 & -1 \times -4 & -1 \times 1 & -1 \times 5 & -1 \times 3 \\ 1 \times 1 & 1 \times -2 & 1 \times 3 & 1 \times -4 & 1 \times 1 & 1 \times 5 & 1 \times 3 \\ 5 \times 1 & 5 \times -2 & 5 \times 3 & 5 \times -4 & 5 \times 1 & 5 \times 5 & 5 \times 3 \\ 7 \times 1 & 7 \times -2 & 7 \times 3 & 7 \times -4 & 7 \times 1 & 7 \times 5 & 7 \times 3 \end{pmatrix}$$

and when we think $UV\vec{v} = \lambda\vec{v}$ what left side of equation doing is that it takes the linear combination of colums UV so

$$UV \begin{pmatrix} -1 \\ 1 \\ 5 \\ -1 \\ 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 63 \\ -126 \\ 189 \\ -252 \\ 63 \\ 315 \\ 189 \end{pmatrix} = \underset{\lambda}{\underset{\|}{63}} \begin{pmatrix} -1 \\ 1 \\ 5 \\ -1 \\ 1 \\ 5 \\ 7 \end{pmatrix}$$

↳ eigenvector

↳ eigenvalue

2/

-1
-2
15
4
1
25
21

4 51

-1.1
-2.1
5.7
-7
7
25
21

46
51
66

21
25
1
4
15

-2
1

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Q5. Suppose 10×10 matrix has eigenvalue zero of multiplicity ten (in other words the set of eigenvalues contains only one value which is zero). Is this matrix neccessarily the 10×10 zero matrix? If yes prove it, no give a counterexample.

Answer 5.

It isn't neccessarily be zero matrix such as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{10 \times 10}$$

If we think about this matrix

$$\det(A - \lambda I_{10}) = 0$$

we have

$$\lambda^{10} = 0 \qquad \lambda = 0$$

only eigenvalue is $\lambda = 0$ but matrix isn't a zero matrix.