Q1. Let $V = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$ and let

$\oplus$ for $\vec{u}, \vec{v} \in V$ define $\vec{u} \oplus \vec{v} = (v_1, v_2) \oplus (v_1, v_2) = (v_1 + v_1 - 2, v_2 + v_2 + 1)$

$\odot$: for $\alpha \in \mathbb{R}, \vec{v} \in V$ define $\alpha \odot \vec{v} = \alpha \odot (v_1, v_2) = (\alpha v_2 + \alpha - 1, \alpha v_1 - 2\alpha + 2)$

Is V a vector space over $\mathbb{R}$ with the above operations?

Solution 1. I want to approach this problem using vector space properties so, suppose that

$$\begin{aligned} (-1) \odot \vec{u} &= -\vec{u} (\text{Proof is given in the last question answers}) \\ \vec{u} \oplus (-\vec{u}) &= \vec{0} \\ \vec{u} \oplus \vec{0} &= \vec{u} \end{aligned}$$

so lets continue chosing a vector in the set such that $x_1, x_2 \in \mathbb{R}$

$$\begin{aligned} \vec{u} &= (1, 2) \\ -1 \odot (1, 2) &= ((-1)(2) + (-1) - 1, (-1)(1) - 2(-1) + 2) \\ -\vec{u} &= (-4, 3) \\ (1, 2) \oplus (-4, 3) &= (1 + (-4) - 2, 2 + 3 + 1) \\ \vec{0} &= (-5, 6) \\ (1, 2) \oplus (-5, 6) &= (1 + (-5) - 2, 2 + 6 + 1) \\ &= (-6, 9) \\ \vec{u} &\neq (-6, 9) \end{aligned}$$

so the equation doesn't satisfy our expected result. We can say that it is not a vector space over the $\mathbb{R}$

Q2. Let $V = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}, x_1 x_2 \neq 0\}$ and let

$\oplus$: for $\vec{u}, \vec{v} \in V$ define $\vec{u} \oplus \vec{v} = (u_1, u_2) \oplus (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

$\odot$: for $\alpha \in \mathbb{R}$ define $\alpha \odot \vec{v} = \alpha \odot (v_1, v_2) = (\alpha v_1, \alpha v_2)$

Is V a vector space over $\mathbb{R}$ with the above operations?

Solution 2. To be a vector space means that all the vector properties is satisfied so lets start our observations vital part of vector spaces

vector addition denoted by $\oplus$

$\oplus : V \times V \to V(1)$

scalar vector multiplication denoted by $\odot$

$\odot : \mathbb{R} \times V \to V(2)$ so lets given an example and show it doesn't satify equation 1

$$\begin{aligned}(v_1, v_2) \oplus (-v_1, -v_2) &= (0,0)\\ x_1 x_2 &= 0\\ (0,0) &\notin V\\ (1,3) \oplus (-1,4) &= (0,7)\\ x_1 x_2 &= 0\\ (0,7) &\notin V\end{aligned}$$

so $x_1, x_2 \in \mathbb{R}$ but it doesn't satisfy property 1 $\oplus : V \times V \to V$

Q3. Is the set of all invertible two by two matrices a vector space?

Solution 3. Lets start to work pointing out our vector operations.Question says invertible two by two matrices so for invertible two by two matrices we have

V={For each A_{2x2} we have B_{2x2} such that AxB=I}

$\oplus : \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \oplus \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_1 & x_2 + x_2 \\ x_3 + x_3 & x_4 + x_4 \end{pmatrix} x_1, x_2, x_3, x_4 \in \mathbb{C}$

$\odot : \alpha \odot \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} \alpha x_1 & \alpha x_2 \\ \alpha x_3 & \alpha x_4 \end{pmatrix} \alpha \in \mathbb{C}\ x_1, x_2, x_3, x_4 \in \mathbb{C}$

and we know that our vector space must statisfy all vector properties.Let's continue our observation this
vector addition denoted by $\oplus$

$\oplus : V \times V \to V(1)$

scalar vector multiplication denoted by $\odot$

$\odot : \mathbb{C} \times V \to V(2)$

so lets given an example and show it doesn't satify property 1 again

$$\begin{aligned}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &\notin V\end{aligned}$$

we can clearly see that addtion operation takes place invertible two by two matrices but result of this operation isn't element of our set so **we can say that invertible two by two matrices don't struct a vector space.** We can give lots of example doesn't satisfy this property.

Q4. Compute

1. $\left\{(-1)\odot\left[\begin{pmatrix}8\\1\end{pmatrix}\oplus\begin{pmatrix}0\\1\end{pmatrix}\right]\right\}\oplus\left\{3\odot\left[\begin{pmatrix}2\\0\end{pmatrix}\oplus\begin{pmatrix}0\\0\end{pmatrix}\right]\right\}$

2. $\left\{\left[0\odot\begin{pmatrix}2\\3\end{pmatrix}\right]\oplus\begin{pmatrix}4\\-1\end{pmatrix}\right\}\oplus\left\{\left[3\odot\begin{pmatrix}1\\2\end{pmatrix}\right]\oplus\left[2\odot\begin{pmatrix}9\\1\end{pmatrix}\right]\right\}$

3. $\left\{0\odot\left[\begin{pmatrix}2\\3\end{pmatrix}\oplus\begin{pmatrix}4\\-1\end{pmatrix}\right]\right\}\oplus\left\{7\odot\left[\begin{pmatrix}1\\2\end{pmatrix}\oplus\begin{pmatrix}9\\1\end{pmatrix}\right]\right\}$

Solution 4. Let start to solve equations step by step

1. $$\underbrace{\underbrace{\left\{(-1)\odot\underbrace{\left[\begin{pmatrix}8\\1\end{pmatrix}\oplus\begin{pmatrix}0\\1\end{pmatrix}\right]}_{\text{Step 1.}}\right\}}_{\text{Step 2.}}\oplus\underbrace{\left\{3\odot\underbrace{\left[\begin{pmatrix}2\\0\end{pmatrix}\oplus\begin{pmatrix}0\\0\end{pmatrix}\right]}_{\text{Step 3.}}\right\}}_{\text{Step 4.}}}_{\text{Step 5.}}$$

Step 1.

$$\begin{aligned}\begin{pmatrix}8\\1\end{pmatrix}\oplus\begin{pmatrix}0\\1\end{pmatrix} &= \begin{pmatrix}8+0-4\\1+1-3\end{pmatrix}\\ &= \begin{pmatrix}4\\-1\end{pmatrix}\end{aligned}$$

Step 2.

$$\begin{aligned}-1\odot\begin{pmatrix}4\\-1\end{pmatrix} &= \begin{pmatrix}(-1)4-4(-1)+4\\(-1)(-1)-3(-1)+3\end{pmatrix}\\ &= \begin{pmatrix}4\\7\end{pmatrix}\end{aligned}$$

Step 3.

$$\begin{aligned}\begin{pmatrix}2\\0\end{pmatrix}\oplus\begin{pmatrix}0\\0\end{pmatrix} &= \begin{pmatrix}2+0-4\\0+0-3\end{pmatrix}\\ &= \begin{pmatrix}-2\\-3\end{pmatrix}\end{aligned}$$

Step 4.

$$\begin{aligned} 3 \odot \begin{pmatrix} -2 \\ -3 \end{pmatrix} &= \begin{pmatrix} 3(-2) - 4(3) + 4 \\ 3(-3) - 3(3) + 3 \end{pmatrix} \\ &= \begin{pmatrix} -14 \\ -15 \end{pmatrix} \end{aligned}$$

Step 5.

$$\begin{aligned} \begin{pmatrix} 4 \\ 7 \end{pmatrix} \oplus \begin{pmatrix} -14 \\ -15 \end{pmatrix} &= \begin{pmatrix} 4 + (-14) - 4 \\ 7 + (-15) - 3 \end{pmatrix} \\ &= \begin{pmatrix} -14 \\ -11 \end{pmatrix} \end{aligned}$$

Step 5 is our final step at it is our result as well.

2. $$\underbrace{\underbrace{\left\{ \underbrace{\left[0 \odot \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right]}_{\text{Step 1.}} \oplus \begin{pmatrix} 4 \\ -1 \end{pmatrix} \right\}}_{\text{Step 2.}} \oplus \underbrace{\left\{ \underbrace{\left[3 \odot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]}_{\text{Step 3.}} \oplus \underbrace{\left[2 \odot \begin{pmatrix} 9 \\ 1 \end{pmatrix} \right]}_{\text{Step 4.}} \right\}}_{\text{Step 5.}}}_{\text{Step 6.}}$$

Step 1.

$$\begin{aligned} 0 \odot \begin{pmatrix} 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} (0)2 - 4(0) + 4 \\ (0)3 - 3(0) + 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \end{aligned}$$

Step 2.

$$\begin{aligned} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} 4 \\ -1 \end{pmatrix} &= \begin{pmatrix} 4 + 4 - 4 \\ 3 + (-1) - 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -1 \end{pmatrix} \end{aligned}$$

Step 3.

$$\begin{aligned} 3 \odot \begin{pmatrix} 1 \\ 2 \end{pmatrix} &= \begin{pmatrix} (3)1 - 4(3) + 4 \\ (3)2 - 3(3) + 3 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 0 \end{pmatrix} \end{aligned}$$

Step 4.

$$\begin{aligned} 2 \odot \begin{pmatrix} 9 \\ 1 \end{pmatrix} &= \begin{pmatrix} (2)9 - 4(2) + 4 \\ (2)1 - 3(2) + 3 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ -1 \end{pmatrix} \end{aligned}$$

Step 5.

$$\begin{aligned} \begin{pmatrix} -5 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 14 \\ -1 \end{pmatrix} &= \begin{pmatrix} -5 + 14 - 4 \\ 0 + (-1) - 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -4 \end{pmatrix} \end{aligned}$$

Step 6.

$$\begin{aligned} \begin{pmatrix} 4 \\ -1 \end{pmatrix} \oplus \begin{pmatrix} 5 \\ -4 \end{pmatrix} &= \begin{pmatrix} 4 + 5 - 4 \\ (-1) + (-4) - 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -8 \end{pmatrix} \end{aligned}$$

Step 6 is our result for the whole operations and system.

3. $$\underbrace{\underbrace{\left\{0 \odot \underbrace{\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} 4 \\ -1 \end{pmatrix}\right]}_{\text{Step 1.}}\right\}}_{\text{Step 2.}} \oplus \underbrace{\left\{7 \odot \underbrace{\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 9 \\ 1 \end{pmatrix}\right]}_{\text{Step 3.}}\right\}}_{\text{Step 4.}}}_{\text{Step 5.}}$$

Step 1.

$$\begin{aligned} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} 4 \\ -1 \end{pmatrix} &= \begin{pmatrix} 2 + 4 - 4 \\ 3 + (-1) - 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{aligned}$$

Step 2.

$$\begin{aligned} 0 \odot \begin{pmatrix} 2 \\ -1 \end{pmatrix} &= \begin{pmatrix} (0)2 - 4(0) + 4 \\ (0)(-1) - 3(0) + 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \end{aligned}$$

Step 3.

$$\begin{aligned}\begin{pmatrix}1\\2\end{pmatrix} \oplus \begin{pmatrix}9\\1\end{pmatrix} &= \begin{pmatrix}1+9-4\\2+1-3\end{pmatrix}\\ &= \begin{pmatrix}6\\0\end{pmatrix}\end{aligned}$$

Step 4.

$$\begin{aligned}7 \odot \begin{pmatrix}6\\0\end{pmatrix} &= \begin{pmatrix}(7)6-4(7)+4\\(7)0-3(0)+3\end{pmatrix}\\ &= \begin{pmatrix}18\\3\end{pmatrix}\end{aligned}$$

Step 5.

$$\begin{aligned}\begin{pmatrix}4\\3\end{pmatrix} \oplus \begin{pmatrix}18\\3\end{pmatrix} &= \begin{pmatrix}4+18-4\\3+3-3\end{pmatrix}\\ &= \begin{pmatrix}18\\3\end{pmatrix}\end{aligned}$$

Last step our result for whole operations

Q5. Find the additive inverse of

1. $\begin{pmatrix}0\\0\end{pmatrix}$

2. $\begin{pmatrix}8\\1\end{pmatrix}$

3. $(-4) \odot \begin{pmatrix}4\\3\end{pmatrix}$

Solution 5. To find additive inverses I want to use Theorem AISM

Theorem AISM Additive Inverses from Scalar Multiplication

Suppose that V is a vector space and $\boldsymbol{u} \in V$. Then **-u**=*(-1)***u**.

Proof

$$\begin{aligned}
-u &= -u + \vec{0} \\
&= -u + \vec{0}u \\
&= -u + (1 + (-1))u \\
&= -u + (1u + (-1)u) \\
&= -u + (u + (-1)u) \\
&= (-u + u) + (-1)u \\
&= \vec{0} + (-1)u \\
&= (-1)u
\end{aligned}$$

Using this theorem we can easily find additive inverse of the vectors let's start working with first one

1.

$$\begin{aligned}
\vec{u} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
-\vec{u} &= -1 \odot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} (-1)0 - 4(-1) + 4 \\ (-1)0 - 3(-1) + 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 8 \\ 6 \end{pmatrix}}_{\text{-}\vec{u}}
\end{aligned}$$

Lets test our result using vector properties $\vec{u} - \vec{u} = \vec{0}$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 + 8 - 4 \\ 0 + 6 - 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\vec{0} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

As you can see it satisfy property of vector space

2.

$$\begin{aligned}
\vec{u} &= \begin{pmatrix} 8 \\ 1 \end{pmatrix} \\
-\vec{u} &= (-1) \odot \begin{pmatrix} 8 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (-1)8 + (-4)(-1) + 4 \\ (-1)1 + (-3)(-1) + 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 5 \end{pmatrix}}_{\text{- }\vec{u}}
\end{aligned}$$

Lets test our result using vector properties $\vec{u} - \vec{u} = \vec{0}$

$$\begin{aligned} \begin{pmatrix} 8 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 8+0-4 \\ 1+5-3 \end{pmatrix} &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ \vec{0} &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \end{aligned}$$

3. Result of this scalar multiplication is zero vector because $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is zero vector for this vector space we already came this result subuqestion of 1,2 so,

$$(-4) \odot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Additive inverse of zero vector is itself because it satisfy $\vec{u} - \vec{u} = \vec{0}$

$$-1 \odot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$