230201057 Furkan Emre YILMAZ 15 MATH144 HW06/P.1

Q1. Give the matrix representation of the derivative $d: P_3 \to P_2$ for

1. $A \to E$;

$$R_E\left(\frac{d(x^3-2x^2+2x+1)}{dx}\right) = 3x^2-4x+2$$

$$= \alpha_1\vec{e_1} + \alpha_2\vec{e_2} + \alpha_3\vec{e_3}$$

$$= \alpha_1(x^2) + \alpha_2(x) + \alpha_3(1)$$

$$= 3(x^2) + (-4)(x) + 2(1)$$

$$= \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$$

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$$R_E\left(\frac{d(2x^3+3x+1)}{dx}\right) = 6x^2+3$$

$$= \alpha_1\vec{e_1} + \alpha_2\vec{e_2} + \alpha_3\vec{e_3}$$

$$= \alpha_1(x^2) + \alpha_2(x) + \alpha_3(1)$$

$$= 6(x^2) + 0(x) + 3(1)$$

$$= \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix}$$

$$R_E\left(\frac{d(2x^3+x^2+3x+1)}{dx}\right) = 6x^2+2x+3$$

$$= \alpha_1(x^2) + \alpha_2(x) + \alpha_3(1)$$

$$= \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$

$$R_E\left(\frac{d(-x^3+x^2+x+1)}{dx}\right) = -3x^2+2x+1$$

$$= \alpha_1(x^2)+\alpha_2(x)+\alpha_3(1)$$

$$= \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

At the end we can conclude that

$$R_{A\to E}(d) = \begin{pmatrix} | & | & | & | \\ R_E(\vec{a}_1) & R_E(\vec{a}_2) & R_E(\vec{a}_3) & R_E(\vec{a}_4) \\ | & | & | & | \end{pmatrix}$$

$$R_{A\to E}(d) = \begin{pmatrix} +3 & 6 & 6 & -3 \\ -4 & 0 & 2 & 2 \\ 2 & 3 & 3 & 1 \end{pmatrix}_{A\to E}$$

2. $A \to R$;

$$R_R(\vec{a}_1) = 3x^2-4x+2 \Rightarrow \alpha_1\vec{r}_1+\alpha_2\vec{r}_2+\alpha_3\vec{r}_3$$

$$\Rightarrow \alpha_1(x^2)+\alpha_2(x^2+x)+\alpha_3(x^2+x+1)$$

$$\Rightarrow \alpha_1(x^2)+\alpha_2x^2+\alpha_2x+\alpha_3x^2+\alpha_3x+\alpha_3$$

$$\Rightarrow x^2(\alpha_1+\alpha_2+\alpha_3)+x(\alpha_2+\alpha_3)+\alpha_3$$

$\alpha_1+\alpha_2+\alpha_3=3$ $\quad \alpha_1=7$

$\alpha_2+\alpha_3=-4$ $\quad \alpha_2=-6$ $\quad R_R(\vec{a}_1) = \begin{pmatrix} 7 \\ -6 \\ 2 \end{pmatrix}$

$\alpha_3=2$ $\quad \alpha_3=2$

$R_R(\vec{a_2}) = 6x^2+3 \Rightarrow \alpha_1 \vec{r_1} + \alpha_2 \vec{r_2} + \alpha_3 \vec{r_3}$

$\Rightarrow \alpha_1(x^2) + \alpha_2(x^2+x) + \alpha_3(x^2+x+1)$

so at this point we can say that we will obtain same coefficent matrix but different vector of constant

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3 &= 6 & \alpha_1 &= 6 \\ \alpha_2+\alpha_3 &= 0 & \alpha_2 &= -3 \\ \alpha_3 &= 3 & \alpha_3 &= 3 \end{aligned} \qquad \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} = R_R(\vec{a_2})$$

$R_R(\vec{a_3}) = 6x^2+2x+3 \Rightarrow \alpha_1 \vec{r_1} + \alpha_2 \vec{r_2} + \alpha_3 \vec{r_3}$

so same case we can directly use

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3 &= 6 & \alpha_1 &= 4 \\ \alpha_2+\alpha_3 &= 2 & \alpha_2 &= -1 \\ \alpha_3 &= 3 & \alpha_3 &= 3 \end{aligned} \qquad R_R(\vec{a_3}) = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

$R_R(\vec{a_4}) = -3x^2+2x+1 \Rightarrow \alpha_1 \vec{r_1} + \alpha_2 \vec{r_2} + \alpha_3 \vec{r_3} + \alpha_4 \vec{r_4}$

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3 &= -3 & \alpha_1 &= -5 \\ \alpha_2+\alpha_3 &= 2 & \alpha_2 &= 1 \\ \alpha_3 &= 1 & \alpha_3 &= 1 \end{aligned} \qquad R_R(\vec{a_4}) = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$

At the and we can conclude

$$R_{A\to R} = \begin{pmatrix} 7 & 6 & 4 & -5 \\ -6 & -3 & -1 & 1 \\ 2 & 3 & 3 & 1 \end{pmatrix}$$

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3. $A \to C$;

$$R_C(\vec{a_1}) = 3x^2-4x+2 \Rightarrow \alpha_1\vec{c_1}+\alpha_2\vec{c_2}+\alpha_3\vec{c_3}$$

$$\Rightarrow \alpha_1(4x^2+3x+1)+\alpha_2(2x^2+2x+1)+\alpha_3(3x^2+x)$$

$$\Rightarrow 4\alpha_1x^2+3\alpha_1x+\alpha_1+2\alpha_2x^2+2\alpha_2x+\alpha_2+3\alpha_3x^2+\alpha_3x$$

$$\Rightarrow x^2(4\alpha_1+2\alpha_2+3\alpha_3)+x(3\alpha_1+2\alpha_2+\alpha_3)+\alpha_1+\alpha_2$$

$$4\alpha_1+2\alpha_2+3\alpha_3=3$$

$$3\alpha_1+2\alpha_2+\alpha_3=-4$$

$$\alpha_1+\alpha_2=2$$

Let's solve this equation using augmented matrix

$$\left(\begin{array}{ccc|c} 4 & 2 & 3 & 3 \\ 3 & 2 & 1 & -4 \\ 1 & 1 & 0 & 2 \end{array}\right)$$

so to solve this system I created a matrix which is when we multiply this with them we will obtain row-reduced echelon form of this matrix

$$\begin{pmatrix} -1 & 3 & -4 \\ 1 & -3 & 5 \\ 1 & -2 & 2 \end{pmatrix}\left(\begin{array}{ccc|c} 4 & 2 & 3 & 3 \\ 3 & 2 & 1 & -4 \\ 1 & 1 & 0 & 2 \end{array}\right)=\begin{pmatrix} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & 25 \\ 0 & 0 & 1 & 15 \end{pmatrix}$$

So our result for $R_C(\vec{a_1})$

$$R_C(\vec{a_1}) = \begin{pmatrix} -23 \\ 25 \\ 15 \end{pmatrix}$$

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As we can see in the previous examples our coefficent matrix stay same only vector of constant is charging so,

$$R_c(\vec{a_2}) = 6x^2 + 3 \Rightarrow \alpha_1 \vec{c_1} + \alpha_2 \vec{c_2} + \alpha_3 \vec{c_3}$$

$$\begin{pmatrix} -1 & 3 & -4 \\ 1 & -3 & 5 \\ 1 & -2 & 2 \end{pmatrix} \left(\begin{array}{ccc|c} 4 & 2 & 3 & 6 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{pmatrix}$$

so our result for $R_c(\vec{a_2})$ =

$$R_c(\vec{a_2}) = \begin{pmatrix} -18 \\ 21 \\ 12 \end{pmatrix}$$

and continue with

$$R_c(\vec{a_3}) = 6x^2 + 2x + 3$$

$$\begin{pmatrix} -1 & 3 & -4 \\ 1 & -3 & 5 \\ 1 & 2 & 2 \end{pmatrix} \left(\begin{array}{ccc|c} 4 & 2 & 3 & 6 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$

$$R_c(\vec{a_3}) = \begin{pmatrix} -12 \\ 15 \\ 8 \end{pmatrix}$$

and doing same operations

$$R_c(\vec{a_4}) = \begin{pmatrix} 5 \\ -4 \\ -5 \end{pmatrix}$$ at the we can conclude with

$$R_{A \to C} = \begin{pmatrix} -23 & -18 & -12 & 5 \\ 25 & 21 & 15 & -4 \\ 15 & 12 & 8 & -5 \end{pmatrix}$$

Q2: Give the matrix representation of the identity map $id: P_2 \to P_2$ for

1. $C \to R$

Let's start to work with

$$Rep_R(\vec{c}_1) = 4x^2+3x+1 \Rightarrow \alpha_1\vec{r}_1 + \alpha_2\vec{r}_2 + \alpha_3\vec{r}_3$$

$$\Rightarrow \alpha_1(x^2) + \alpha_2(x^2+x) + \alpha_3(x^2+x+1)$$

$$\Rightarrow x^2(\alpha_1+\alpha_2+\alpha_3) + x(\alpha_2+\alpha_3) + \alpha_3$$

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3 &= 4 & \alpha_1 &= 1 \\ \alpha_2+\alpha_3 &= 3 & \alpha_2 &= 2 \\ \alpha_3 &= 1 & \alpha_3 &= 1 \end{aligned} \qquad Rep_R(\vec{c}_1) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

coeffiecent matrix stays same only vector of constant charging for the system. Let's continue with

$$Rep_R(\vec{c}_2) = 2x^2+2x+1 \Rightarrow \alpha_1\vec{r}_1 + \alpha_2\vec{r}_2 + \alpha_3\vec{r}_3$$

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3 &= 2 & \alpha_1 &= 0 \\ \alpha_2+\alpha_3 &= 2 & \alpha_2 &= 1 \\ \alpha_3 &= 1 & \alpha_3 &= 1 \end{aligned} \qquad Rep_R(\vec{c}_2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and for last

$$Rep_R(\vec{c}_3) = 3x^2+x \Rightarrow \alpha_1\vec{r}_1 + \alpha_2\vec{r}_2 + \alpha_3\vec{r}_3$$

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3 &= 3 & \alpha_1 &= 2 \\ \alpha_2+\alpha_3 &= 1 & \alpha_2 &= 1 \\ \alpha_3 &= 0 & \alpha_3 &= 0 \end{aligned} \qquad Rep_R(\vec{c}_3) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

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At the end we can conclude with

$$R_{C\to B}(id) = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

2. $R \to C$

$$R_C(\vec{r_1}) = x^2 \Rightarrow \alpha_1 \vec{c_1} + \alpha_2 \vec{c_2} + \alpha_3 \vec{c_3}$$

$$\Rightarrow \alpha_1(4x^2+3x+1) + \alpha_2(2x^2+2x+1) + \alpha_3(3x^2+x)$$

$$\Rightarrow x^2(4\alpha_1+2\alpha_2+3\alpha_3) + x(3\alpha_1+2\alpha_2+\alpha_3) + \alpha_1+\alpha_2$$

$4\alpha_1+2\alpha_2+3\alpha_3=1 \quad \alpha_1=-1$

$3\alpha_1+2\alpha_2+\alpha_3=0 \quad \alpha_2=1 \qquad R_C(\vec{r_1}) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\alpha_1+\alpha_2=0 \quad \alpha_3=1$

$$R_C(\vec{r_2}) = x^2+x \Rightarrow \alpha_1 \vec{c_1} + \alpha_2 \vec{c_2} + \alpha_3 \vec{c_3}$$

$4\alpha_1+2\alpha_2+3\alpha_3=1 \quad \alpha_1=2$

$3\alpha_1+2\alpha_2+\alpha_3=1 \quad \alpha_2=-2 \qquad R_C(\vec{r_2}) = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

$\alpha_1+\alpha_2=0 \quad \alpha_3=-1$

$$R_C(\vec{r_3}) = x^2+x+1$$

$4\alpha_1+2\alpha_2+3\alpha_3=1 \quad \alpha_1=-2$

$3\alpha_1+2\alpha_2+\alpha_3=1 \quad \alpha_2=3 \qquad R_C(\vec{r_3}) = \begin{pmatrix} -2 \\ +3 \\ 1 \end{pmatrix}$

$\alpha_1+\alpha_2=1 \quad \alpha_3=1$

At the end we can conclude with

$$R_{B\to C}(id) = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$$

3. $C \to E$;

$$R_E(\vec{c_1}) = 4x^2 + 3x + 1 \Rightarrow \alpha_1 \vec{e_1} + \alpha_2 \vec{e_2} + \alpha_3 \vec{e_3}$$

$$\Rightarrow \alpha_1 (x^2) + \alpha_2 (x) + \alpha_3 (1)$$

$$R_E(\vec{c_1}) = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

$$R_E(\vec{c_2}) = 2x^2 + 2x + 1 \Rightarrow \alpha_1 \vec{e_1} + \alpha_2 \vec{e_2} + \alpha_3 \vec{e_3}$$

$$R_E(\vec{c_2}) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$R_E(\vec{c_3}) = 3x^2 + x \Rightarrow \alpha_1 \vec{e_1} + \alpha_2 \vec{e_2} + \alpha_3 \vec{e_3}$$

$$R_E(\vec{c_3}) = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

at the end we can conclude that

$$R_{C\to E}(id) = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Q3. How are the matrices in Q1.2, Q1.3 and Q2.2 related?

Q1.2 $\Rightarrow$ $A \to B$

Q2.2 $\Rightarrow$ $B \to C$

Q1.3 $\Rightarrow$ $A \to C$

If we denote

$T: A \to B$

$S: B \to C$

and $(S \circ T)(x) = S(T(u))$ is the $A \to C$ meaning that

S composition T is $A \to C$

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Q4. How are the matrices in Q2.1, Q2.2 related?

Q 2.1 $\Rightarrow$ $C \to R$

Q 2.2 $\Rightarrow$ $R \to C$

if we denote $C \to R$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}_{C \to R}$$

$$A^{-1} = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}_{R \to C}$$

meaning that they are inverse each other.

Q5, Let $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}_A$. Compute representation of the derivative of $\vec{v}$ in basis

R, C and E

In question two we already prepred needed ingridients so only job we must do is multiplying

$$R_E(d(\vec{a})) = R_{A\to E}(\phi)\, R_A(\vec{a})$$

$$= \begin{pmatrix} 3 & 6 & 6 & -3 \\ -4 & 0 & 2 & 2 \\ 2 & 3 & 3 & 1 \end{pmatrix}_{A\to E} \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}_A$$

$$= \begin{pmatrix} 24 \\ -2 \\ 10 \end{pmatrix}_E$$

$$= 24x^2 - 2x + 10$$

$$R_R(d(\vec{a})) = R_{A\to R}(\phi)\, R_A(\vec{a})$$

$$= \begin{pmatrix} 7 & 6 & 4 & 5 \\ -6 & -3 & -1 & 1 \\ 2 & 3 & 3 & 1 \end{pmatrix}_{A\to R} \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}_A$$

$$= \begin{pmatrix} 26 \\ -12 \\ 10 \end{pmatrix}_R = 26\vec{r_1} - 12\vec{r_2} + 10\vec{r_3}$$

$$= 24x^2 - 2x + 10$$

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$$R_C(d(\vec{a})) = R_{A\to C}(\phi)\, R_A(\vec{a})$$

$$= \begin{pmatrix} -23 & -18 & -12 & 5 \\ 25 & 21 & 15 & -4 \\ 15 & 12 & 8 & -5 \end{pmatrix}_{A\to C} \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}_A$$

$$= \begin{pmatrix} -70 \\ 80 \\ 48 \end{pmatrix}_C$$

$$= -70\vec{c}_1 + 80\vec{c}_2 + 48\vec{c}_3$$

$$= 24x^2 - 2x + 10$$