Q1 With the usual matrix matrix addition and scalar matrix multiplication is D a linear combination of A,B and C where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 4 & -1 & 1 \\ 0 & 0 & -8 \\ 1 & 6 & 2 \end{pmatrix}, B = \begin{pmatrix} 5 & 2 & 0 \\ 4 & 3 & 1 \\ 3 & 0 & -2 \\ 1 & 5 & 1 \end{pmatrix}, C = \begin{pmatrix} 3 & 2 & 9 \\ 1 & 0 & 8 \\ 0 & -1 & 0 \\ 7 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} -8 & 4 & 1 \\ 11 & -11 & -5 \\ -6 & 1 & -36 \\ -4 & 20 & 8 \end{pmatrix}$$

Solution 1. Let's start to work with simple equations to find a linear combination.I want to point out

$$\alpha_1 \begin{pmatrix} (1) & 2 & 2 \\ 4 & -1 & 1 \\ (0) & (0) & -8 \\ 1 & 6 & 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} (5) & 2 & 0 \\ 4 & 3 & 1 \\ (3) & (0) & -2 \\ 1 & 5 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} (3) & 2 & 9 \\ 1 & 0 & 8 \\ (0) & (-1) & 0 \\ 7 & 0 & 0 \end{pmatrix} = \begin{pmatrix} (-8) & 4 & 1 \\ 11 & -11 & -5 \\ (-6) & (1) & -36 \\ -4 & 20 & 8 \end{pmatrix}$$

If we write down each representing equation and solve separetly we obtain the coefficents.Lets continue with

$$\begin{aligned} 0\alpha_1 + 3\alpha_2 + 0\alpha_3 &= -6 \\ 3\alpha_3 &= -6 \\ \alpha_3 &= -2 \end{aligned}$$

$$\begin{aligned} 0\alpha_1 + 0\alpha_2 + -1\alpha_3 &= 1 \\ \alpha_2 &= -1 \end{aligned}$$

$$\begin{aligned} 1\alpha_1 + 5\alpha_2 + 3\alpha_3 &= -8 \\ 1\alpha_1 - 10 - 3 &= -8 \\ \alpha_1 &= 5 \end{aligned}$$

so we found $\alpha_1, \alpha_2, \alpha_3$ lets try to subsitude and see whether satisfy or not

$$5 \begin{pmatrix} 1 & 2 & 2 \\ 4 & -1 & 1 \\ 0 & 0 & -8 \\ 1 & 6 & 2 \end{pmatrix} + (-2) \begin{pmatrix} 5 & 2 & 0 \\ 4 & 3 & 1 \\ 3 & 0 & -2 \\ 1 & 5 & 1 \end{pmatrix} + (-1) \begin{pmatrix} 3 & 2 & 9 \\ 1 & 0 & 8 \\ 0 & -1 & 0 \\ 7 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -8 & 4 & 1 \\ 11 & -11 & -5 \\ -6 & 1 & -36 \\ -4 & 20 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 10 & 10 \\ 20 & -5 & 5 \\ 0 & 0 & -40 \\ 5 & 30 & 10 \end{pmatrix} + \begin{pmatrix} -10 & -4 & 0 \\ -8 & -6 & -2 \\ -6 & 0 & 4 \\ -2 & -10 & -2 \end{pmatrix} + \begin{pmatrix} -3 & -2 & -9 \\ -1 & 0 & -8 \\ 0 & 1 & 0 \\ -7 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -8 & 4 & 1 \\ 11 & -11 & -5 \\ -6 & 1 & -36 \\ -4 & 20 & 8 \end{pmatrix}$$

$$\begin{pmatrix} -8 & 4 & 1 \\ 11 & -11 & -5 \\ -6 & 1 & -36 \\ -4 & 20 & 8 \end{pmatrix} = \begin{pmatrix} -8 & 4 & 1 \\ 11 & -11 & -5 \\ -6 & 1 & -36 \\ -4 & 20 & 8 \end{pmatrix}$$

Q2 In CVS are the vectors $\begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{pmatrix}2\\2\end{pmatrix}$ linearly dependent?

Solution 2. Lets start work with

$$\begin{aligned}
\alpha_1 \odot \begin{pmatrix}0\\0\end{pmatrix} \oplus \alpha_2 \odot \begin{pmatrix}2\\2\end{pmatrix} &= \begin{pmatrix}4\\3\end{pmatrix}\\
\begin{pmatrix}0\alpha_1 - 4\alpha_1 + 4\\0\alpha_1 - 3\alpha_1 + 3\end{pmatrix} \oplus \begin{pmatrix}2\alpha_2 - 4\alpha_2 + 4\\2\alpha_2 - 3\alpha_2 + 3\end{pmatrix} &= \begin{pmatrix}4\\3\end{pmatrix}\\
\begin{pmatrix}-4\alpha_1 + 4\\-3\alpha_1 + 3\end{pmatrix} \oplus \begin{pmatrix}-2\alpha_2 + 4\\-\alpha_2 + 3\end{pmatrix} &= \begin{pmatrix}4\\3\end{pmatrix}\\
\begin{pmatrix}-4\alpha_1 - 2\alpha_2 + 4\\-3\alpha_1 - \alpha_2 + 3\end{pmatrix} &= \begin{pmatrix}4\\3\end{pmatrix}
\end{aligned}$$

then if we solve last step using equations as

$$\begin{aligned}
-4\alpha_1 - 2\alpha_2 = 0\\
-3\alpha_1 - \alpha_2 = 0
\end{aligned}$$

and this matrix row reduced echelon form is an identity matrix means that we have only have trival solution and this is

$$\begin{aligned}
\begin{pmatrix}-4 & -2\\-3 & -1\end{pmatrix} &= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\
\alpha_1 &= 0\\
\alpha_2 &= 0
\end{aligned}$$

so at the end we can say they are **linearly independent**, not **dependent**.

Q3 Are the continuous functions $e^{2\ln x}, x^2, \cos x$ linearly dependent or independent, under the standart operations on functions?

Solution 3. We are searching,

$$\alpha_1 e^{2\ln x} + \alpha_2 x^2 + \alpha_3 \cos x = 0$$

we can instantly substitude

$$\begin{aligned}
\alpha_1 &= -1\\
\alpha_2 &= 1\\
\alpha_3 &= 0
\end{aligned}$$

and this satisfy our equation. We found a non-trivial solution to our system, we conclude that this functions are **linearly dependent**.

Q4 Suppose the following two equalities hold:

$$\begin{aligned}\vec{v} &= \alpha_1\vec{u_1} + \alpha_2\vec{u_2} + ... + \alpha_k\vec{u_k}\\ \vec{v} &= \beta_1\vec{u_1} + \beta_2\vec{u_2} + ... + \beta_k\vec{u_k}\end{aligned}$$

such that for at least one index i, we have $\alpha_i \neq \beta_i$. Does it mean that $\vec{u_i} = \vec{0}$ where $\vec{0}$ is the zero vector?

Solution 4. At this point I want to emphasize "at least one" and start to observe

Suppose that $\alpha_1 \neq \beta_1$ and all others index i $\alpha_i = \beta_i$ Lets compute

$$\begin{aligned}\vec{v} - \vec{v} &= \vec{0}\\ \vec{v} - \vec{v} &= (\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + ... + \alpha_k\vec{u_k}) - (\beta_1\vec{u_1} + \beta_2\vec{u_2} + ... + \beta_k\vec{u_k})\\ \vec{0} &= \alpha_1\vec{u_1} - \beta_1\vec{u_1}\\ \vec{0} &= (\alpha_1 - \beta_1)\vec{u_1}\ (\alpha_1 \neq \beta_1)\end{aligned}$$

in this situation only option is the

$$\vec{u_1} = \vec{0}$$

as a result if we are talking about at least one index i $\alpha_i \neq \beta_i$ at least one condition only hold when one vector such that $\vec{u_i} = \vec{0}$ exist in linear combination.

Q5 Suppose that $\vec{u_1}, \vec{u_2}, \vec{u_3}$ are linearly dependent. What can you say about the linear dependence or independence of the vectors $\vec{v_1} = \vec{u_1} - 2\vec{u_2}$ and $\vec{v_2} = 3\vec{u_2} + 2\vec{u_3}$?

Solution 5. I want to show two different and possible situation based on this we conclude that **we can't say anything about linear dependence or independence of $\vec{v_1}, \vec{v_2}$**

First one is, lets assume

$$\begin{aligned}\vec{u_1} &= \begin{pmatrix}1\\0\\0\end{pmatrix}\\ \vec{u_2} &= \begin{pmatrix}0\\0\\0\end{pmatrix}\\ \vec{u_3} &= \begin{pmatrix}0\\0\\0\end{pmatrix}\end{aligned}$$

$\vec{u_1}, \vec{u_2}, \vec{u_3}$ are now linearly dependent, lets compute $\vec{v_1}, \vec{v_2}$

$$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\vec{v_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

in this case $\vec{v_1}, \vec{v_2}$ are **linearly dependent**

In other case is

$$\vec{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\vec{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\vec{u_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

again our $\vec{u_1}, \vec{u_2}, \vec{u_3}$ vectors are linearly dependent. Lets compute $\vec{v_1}, \vec{v_2}$

$$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\vec{v_2} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

and as you can see in this case they are **linearly independent**.