Q1. Let $A = \begin{pmatrix} \frac{1\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1\sqrt{3}}{2} \end{pmatrix}$. Compute A^{101}. Show all your steps.

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Answer 1.

$$A^2 = A^1 \times A^1 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$A^4 = A^2 \times A^2 = \begin{pmatrix} -\frac{1}{2} & 0 & +\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$A^8 = A^4 \times A^4 = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$A^{16} = A^8 \times A^8 = \begin{pmatrix} -\frac{1}{2} & 0 & +\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$A^{32} = A^{16} \times A^{16} = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$A^{64} = A^{32} \times A^{32} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$A^{101} = A^{64} \times A^{32} \times A^{4} \times A = \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

I have used calculator to calculate the steps.

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Q2. Let $B = \begin{pmatrix} 1 & 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & -1 & 5 \\ 2 & 1 & 0 & 7 & 10 \\ -2 & -1 & 0 & 7 & 0 \\ 0 & 0 & 3 & 0 & -1 \end{pmatrix}$. Compute $\det B$.

Answer 2. To simplfy the calculation I want to apply some row operations. After these row operations we can calculate the $\det B$ easily.

$$\begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ -2&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&0&1 \end{pmatrix}$$

Using these elementary matrices I have obtained

$$\begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ -2&1&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&1&0&3&0 \\ 0&1&2&-1&5 \\ 2&1&0&7&10 \\ -2&-1&0&7&0 \\ 0&0&3&0&-1 \end{pmatrix} = \begin{pmatrix} 1&1&0&3&0 \\ 0&1&2&-1&5 \\ 0&0&2&0&15 \\ 0&0&0&14&10 \\ 0&0&3&0&-1 \end{pmatrix} \qquad (1)$$

so far we multiply the rows scalar and then added it to another row, meaning that our determinant $\det B$ same as the determinant (1).

$$\det \begin{pmatrix} 1&1&0&3&0 \\ 0&1&2&-1&5 \\ 0&0&2&0&15 \\ 0&0&0&14&10 \\ 0&0&3&0&-1 \end{pmatrix} = \det(B)$$

if we calculate determinant of this matrix using expansion about column 1 as

$$(-1)^2[B]_{11}\det(B(1|1)) + (-1)^3[B]_{21}\det(B(2|1)) + (-1)^4[B]_{31}\det(B(3|1)) \ldots$$
$$\ldots (-1)^5[B]_{41}\det(B(4|1)) + (-1)^6[B]_{51}\det(B(5|1))$$

so we know than only contiribution come

$$(-1)^2[B]_{11}\det(B(1|1))$$

because other cases $[B]_{21}$, $[B]_{31}$, $[B]_{41}$, $[B]_{51}$ equals 0.

going in this way we have

$$1.(1)\det\begin{pmatrix}1&2&-1&5\\0&2&0&15\\0&0&14&10\\0&3&0&-1\end{pmatrix} = B_1$$

again using expansion about column 1 only contribution come

$$=(1)(1)(1)(-1)^2[B_1]_{11}\det(B_1(1|1))$$

$$=(1)(1)(1)(1)\det\begin{pmatrix}2&0&15\\0&14&10\\3&0&-1\end{pmatrix} = B_2$$

and at the end we can instanty expand this matrix and calcclate resultant determinant as;

$$=(-1)^2[B_2]_{11}\det(B_2(1|1))+(-1)^3[B_2]_{12}\det(B_2(1|2))+(-1)^4[B_2]_{13}\det(B_2(1|3))$$

$$=2\begin{vmatrix}14&10\\0&-1\end{vmatrix}-0\begin{vmatrix}0&10\\3&-1\end{vmatrix}+15\begin{vmatrix}0&14\\3&0\end{vmatrix}$$

$$=2(-14)+0+15(-14.3)$$

$$\det(B) = -658$$

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Q3. Let $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$, If $\det(A) = -3$, find $\det \begin{pmatrix} c_1 & -b_1+6c_1 & a_1 & d_1 \\ c_2 & -b_2+6c_2 & a_2 & d_2 \\ c_3 & -b_3+6c_3 & a_3 & d_3 \\ c_4 & -b_4+6c_4 & a_4 & d_4 \end{pmatrix}$

Answer 3. Using properties of determinant we can find determinant. First we can start with

$$\det(A) = \det(A^T)$$

so if we apply this we obtain

$$\det(A) = \det(A^T) = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

Second we know that swaping colums or rows affect the determinant as

$$-\det(A) = \det(B) \quad \text{(after a single swap operation)}$$

if we apply this property for colums 1 and 2

$$-\det(A) = \det \begin{pmatrix} c_1 & b_1 & a_1 & d_1 \\ c_2 & b_2 & a_2 & d_2 \\ c_3 & b_3 & a_3 & d_3 \\ c_4 & b_4 & a_4 & d_4 \end{pmatrix}$$

Third we can continue with

$$\det(B) = \alpha \det(A)$$

if we apply this for $\alpha = -1$ colum b_1 as

$$\det(A) = \begin{pmatrix} c_1 & -b_1 & a_1 & d_1 \\ c_2 & -b_2 & a_2 & d_2 \\ c_3 & -b_3 & a_3 & d_3 \\ c_4 & -b_4 & a_4 & d_4 \end{pmatrix}$$

Last operation is multiplying a colum $\alpha = 6$ and adding it another so we know that it doesn't affect the determinant

$$\det(A) = \det\begin{pmatrix} c_1 & -b_1+6c_1 & a_1 & d_1 \\ c_2 & -b_2+6c_2 & a_2 & d_2 \\ c_3 & -b_3+6c_3 & a_3 & d_3 \\ c_4 & -b_4+6c_4 & a_4 & d_4 \end{pmatrix}$$

$$-3 = \det\begin{pmatrix} c_1 & -b_1+6c_1 & a_1 & d_1 \\ c_2 & -b_2+6c_2 & a_2 & d_2 \\ c_3 & -b_3+6c_3 & a_3 & d_3 \\ c_4 & -b_4+6c_4 & a_4 & d_4 \end{pmatrix}$$

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Q4. Compute the determinant of the 10x10 matrix M whose in row i and column j is i+j

$$\det \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \end{pmatrix} = 0$$

this matrix determinant is zero (0) because using row operations we can instantly obtain zero row or column such as multiplying row 1 with -1 and adding it second row we obtain row which consist of all 1's. Similiar manner multiplying row 4 with -1 and adding with row 3 we obtain a row which consist of all -1's. At the end using this matrix we add new row 2 with new row 3 and we obtain row which consist of all zeros

Therefore we can conclude that, because of multiying a row or column with a scalar α and adding it with row or colum doesn't change determinant's result. At the end we have a row which consist of all zeros meaning that determinant is 0.

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Q5. Compute $\det \underbrace{\begin{pmatrix} -1 & 2 & -1 & 0 & -9 & -4 \\ 2 & -1 & 8 & 22 & 4 & 2 \\ 2 & 3 & 5 & -1 & 1 & 20 \\ 1 & 8 & 1 & 7 & 7 & 6 \\ 5 & -7 & 5 & 7 & 2 & 1 \\ 2 & 2 & 1 & 1 & 1 & 2 \end{pmatrix}}_{D}$. You may use that

$$\det \underbrace{\begin{pmatrix} -1 & 2 & -1 & 0 & -9 & -4 \\ 2 & -1 & 8 & 22 & 4 & 2 \\ 2 & 3 & 5 & -1 & 1 & 20 \\ 1 & 8 & 1 & 7 & 7 & 6 \\ 5 & -7 & 5 & 7 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}}_{A} = -8691, \quad \det \underbrace{\begin{pmatrix} -1 & 2 & -1 & 0 & -9 & -4 \\ 2 & -1 & 8 & 22 & 4 & 2 \\ 2 & 3 & 5 & -1 & 1 & 20 \\ 1 & 8 & 1 & 7 & 7 & 6 \\ 5 & -7 & 5 & 7 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}}_{B} = 105990$$

$$\det \underbrace{\begin{pmatrix} -2 & 4 & -2 & 0 & -18 & -8 \\ 4 & -2 & 16 & 44 & 8 & 4 \\ 4 & 6 & 10 & -2 & 2 & 40 \\ 2 & 16 & 2 & 14 & 14 & 12 \\ 10 & -14 & 10 & 14 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}}_{C} = 6227136$$

$[A]_{61}$

Answer 5, In this question, my approach is that I analyze the determinant and try to see which determinant has which information and I saw that det(A) and det(B) has the information (if we use expansion about row 6 for det(A) and det(B))

$$\det(A) = (-1)^7 [A]_{61} \det(A(6|1)) + (-1)^8 [A]_{62} \det(A(6|2)) + (-1)^{12} [A]_{66} \det(A(6|6))$$

$$\det(B) = (-1)^9 [B]_{63} \det(B(6|3)) + (-1)^{10} [B]_{64} \det(B(6|4)) + (-1)^{11} [B]_{65} \det(B(6|5))$$

to find the det(D) we can multiply row 5 of matrix A and add with matrix B we can easily calculate the det(D) as

$$2\det(A) = 2\left[(-1)^7 [A]_{61} \det(A(6|1)) + (-1)^8 [A]_{62} \det(A(6|2)) + (-1)^{12} [A]_{66} \det(A(6|6))\right]$$

$$\det(B) = (-1)^9 [B]_{63} \det(B(6|3)) + (-1)^{10} [B]_{64} \det(B(6|4)) + (-1)^{11} [B]_{65} \det(B(6|5))$$

+

$$2\det(A) + \det(B) = \det(D)$$

$$2(-8691) + 105990 = \det(D)$$

$$\det(D) = 88\,608$$

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If we need to explain this result we can explain this like that we are able to do this because matrix A and B all rows are same except 6th row and matrix D's rows are all same except 6th row. Therefore if we expand matrix D using 6th row we obtain the result of 2det(A) + det(B).

Similiarly we can obtain the det(D) using det(C) and det(A). As you can see det(C) all rows like multipled by 2 compared to D matrix so using matrix properties and expansion we can say that

$$\det(D) = \frac{\det(C)}{2^6} + \det(A)$$

$$= \frac{6227136}{2^6} - 8691$$

$$= 97299 - 8691$$

$$= 88608$$

We can prove this result using determinant expansion properties. As you can see these two result is equal. We can prefer either way.