PHYS 600 HW3.

1、

Density Parameters

$$\Omega_m = \frac{\rho_m}{\rho_{cr}} = \frac{\rho_{m,0}(1+z)^3}{\rho_{cr,0}} \frac{\rho_{cr,0}}{\rho_{cr}}$$

$$= \Omega_{m,0}(1+z)^3 \frac{1}{[\Omega_{m,0}(1+z)^3 + \Omega_\Lambda]}$$

$$= 0.3\times(1+0.5)^3 \frac{1}{[0.3\times(1+0.5)^3 + 0.7]}$$

$$\simeq 0.59$$

$$\Omega_\Lambda \simeq 0.41$$

Luminosity and Angular Diameter distances

For matter + Λ,

$$\begin{cases} a(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}\, H_0 t\right) \\ \chi(z) = c\int_{t_0}^{t} \frac{dt}{a(t)} = c\int_0^z \frac{dz}{H(z)} \end{cases}$$

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \sqrt{\Omega_{\Lambda,0}}\, H_0 \coth\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}\, H_0 t\right)$$

$$= \sqrt{\Omega_{\Lambda,0}}\, H_0 \coth\left[\operatorname{arcsinh}\left(\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{-\frac{1}{2}}(1+z)^{-\frac{3}{2}}\right)\right]$$

$$= \sqrt{\Omega_{\Lambda,0}}\, H_0 \frac{\sqrt{\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)(1+z)^{-3}+1}}{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{-\frac{1}{2}}(1+z)^{-\frac{3}{2}}}$$

$$= H(z)$$

For $\Omega_{m,0} = 1$,

$$H(z) = H_0(1+z)^{\frac{3}{2}}$$

$$\chi(z) = c\int_0^z \frac{dz}{H(z)}$$

Because $K=0$

$$d_L(z) = (1+z)\chi(z)$$

$$d_A(z) = \frac{\chi(z)}{(1+z)}$$

Plot see attached.

The $d_A(z)$ first increases with z, dominated by the increase in the coordinate distance, and then decreases with z, dominated by the effect of expansion of the universe (objects appear lager on the sky).

Plug in $\Omega_{m,0}=0.7$, $\Omega_{\Lambda,0}=0.3$, $H_0=100h=70\,km/s/Mpc \approx 2.273\times10^{-18}\,s^{-1}$

$a(t_0)=\left(\frac{0.7}{0.3}\right)^{-\frac{1}{3}}\sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{0.7}\cdot 2.273\times10^{-18}t_0\right)=1$

$t_0=4.242\times10^{17}s \approx 1.345\ yr$

$\therefore t_1=t_0-1\times10^{10}=0.345\times10^{10}yr\approx1.088\times10^{17}s$

$a(t_1)=\left(\frac{0.7}{0.3}\right)^{-\frac{1}{3}}\sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{0.7}\times2.273\times10^{-18}\times1.088\times10^{17}\right)\approx0.3493$

☆ $z\approx\frac{1}{a(t_1)}-1\approx\underline{1.86}$

A Λ-dominated Universe.

$\frac{H^2}{H_0^2}=\Omega_{\Lambda,0}$

$H^2=\frac{\Lambda c^2}{3}$

$\left(\frac{\dot{a}}{a}\right)^2=\frac{\Lambda c^2}{3}$

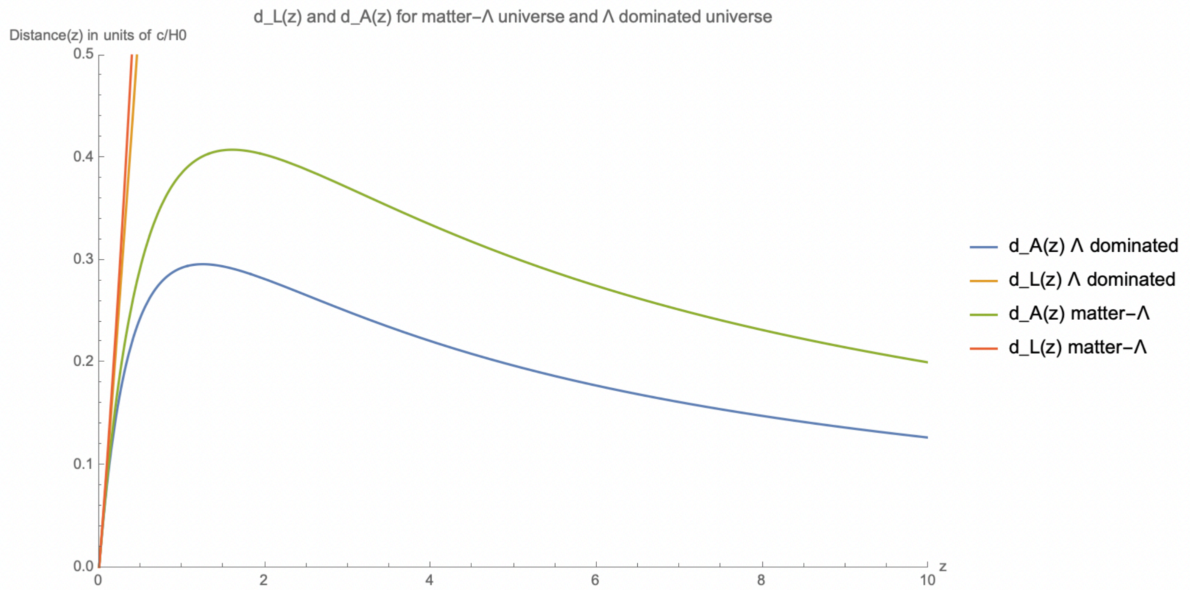
$a(t)=A\exp\left(\sqrt{\frac{\Lambda c^2}{3}}t\right)$, where A is an arbitrary constant.

$a(t_0)=A\exp\left(\sqrt{\frac{\Lambda c^2}{3}}t_0\right)=1$

$t_0=-\sqrt{\frac{3}{\Lambda c^2}}\ln A$

$\therefore$ the age of such a universe is $-\sqrt{\frac{3}{\Lambda c^2}}\ln A$, where Λ can be determined experimentally.

Out[369]=



d_L(z) and d_A(z) for matter-Λ universe and Λ dominated universe
Distance(z) in units of c/H0
0.5
0.4
0.3
0.2
0.1
0.0
0
2
4
6
8
10
z
d_A(z) Λ dominated
d_L(z) Λ dominated
d_A(z) matter-Λ
d_L(z) matter-Λ

2、Massive Neutrinos

(1)

Because we can safely assume the neutrinos to be fully relativistic,

$$\rho_\nu = \frac{g}{2\pi^2} T_\nu^4 \int_0^\infty d\xi \frac{\xi^2\sqrt{\xi^2+m_\nu^2/T_\nu^2}}{\exp[\sqrt{\xi^2+m_\nu^2/T_\nu^2}]+1}$$

$g=2$ corresponding to the 2 spin states of the neutrinos

$$\exp[\sqrt{\xi^2+m_\nu^2/T_\nu^2}]$$

$$\simeq \exp[\xi(1+\tfrac{1}{2}\tfrac{m_\nu^2/T_\nu^2}{\xi^2})]$$ in the relativistic limit, $m_\nu/T_\nu \ll 1$

$$\simeq \exp \xi$$

we drop the $\frac{1}{2}\frac{m_\nu^2/T_\nu^2}{\xi^2}$ term because when ξ is small, the denominator overall is still dominated by $\exp(0)=1$, regardless of the details of the argument of exp, while whe ξ is large, the contribution coming from $\frac{1}{2}\frac{m_\nu^2/T_\nu^2}{\xi^2}$ is completely dominated by 1.

So $$\rho_\nu = \frac{2}{2\pi^2} T_\nu^4 \int_0^\infty d\xi \frac{\xi^2\sqrt{\xi^2+m_\nu^2/T_\nu^2}}{\exp(\xi)+1}$$

$$= \frac{T_\nu^4}{\pi^2}\int_0^\infty d\xi \frac{\xi^2\sqrt{\xi^2+m_\nu^2/T_\nu^2}}{e^\xi+1}$$

(2)

$$\therefore \rho_\nu = \frac{T_\nu^4}{\pi^2}\int_0^\infty d\xi \frac{\xi^2 \xi\sqrt{1+\frac{m_\nu^2/T_\nu^2}{\xi^2}}}{e^\xi+1}$$

$$\approx \frac{T_\nu^4}{\pi^2}\int_0^\infty d\xi \left(\frac{\xi^3}{e^\xi+1} + \frac{1}{2}\frac{m_\nu^2\xi/T_\nu^2}{e^\xi+1}\right)$$

$$= \frac{\pi^2 T_\nu^4}{15}\cdot\frac{7}{8} + \frac{T_\nu^4}{\pi^2}\cdot\int_0^\infty d\xi \frac{1}{2}\frac{m_\nu^2\xi/T_\nu^2}{e^\xi+1}$$

$$= \frac{\pi^2 T_\nu^4}{15}\cdot\frac{7}{8} + \frac{1}{2}m_\nu^2 T_\nu^2\cdot\frac{1}{12}$$

$$= \frac{\pi^2 T_\nu^4}{15}\cdot\frac{7}{8}\left(1+\frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2}\right) = \rho_{\nu,0}\left(1+\frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2}\right)$$

(3)

We take the assumption that ρ_ν is significantly larger than $\rho_{\nu 0}$ if $\rho_\nu - \rho_{\nu 0} \gtrsim 0.001\, \rho_{\nu 0} = 0.1\%\, \rho_{\nu 0}$

Under this criterion, the approximation is still reasonably good at leading order.

Because the neutrinos are now decoupled, their temperature evolves as $T_\nu \propto a^{-1}$.

At the decoupling temperature ~ 0.8 MeV, $z \sim \frac{1}{5\times10^{-9}} - 1 \Rightarrow$ $a_1 \simeq 5\times10^{-10}$

At the recombination, $z \approx 1270 \Rightarrow a_2 \simeq \frac{1}{1+1270} \simeq 7.9\times10^{-4}$.

$\therefore T_{\nu,rec} = \frac{5\times10^{-10}}{7.9\times10^{-4}}\, 0.8\ \text{MeV} \approx 6.3\times10^{-7} \times 0.8 \simeq 5.1\times10^{-7}\ \text{MeV}$ $= 0.51\ \text{eV}$

$$\therefore \frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2} = 0.001$$

$$m_\nu^2 = \frac{7\pi^2}{5}\cdot 0.001 \cdot T_\nu^2$$

$$= \frac{7\pi^2}{5}\cdot 0.001 \cdot 0.51^2$$

$$= 0.0036\ \text{eV}^2$$

$$m_\nu = \sqrt{0.0036}\ \text{eV}$$

$$= 0.06\ \text{eV}$$

So to have $(\rho_\nu - \rho_{\nu 0})/\rho_{\nu 0} \gtrsim 0.1\%$ as a threshold for some observable effects in the CMB, $m_\nu \gtrsim 0.06$ eV.

Of course the lowerbound on m_ν is lower if the precision of measurement is lower.

(4)

When the neutrinos become non-relativistic,

$T_\nu \approx m_\nu$

Take $T_{\nu,dec} \approx 0.8\,MeV$, $a_{dec} \approx 5\times10^{-10}$

$\therefore T_{\nu,non\text{-}rel}\, a_{non\text{-}rel} = T_{\nu,dec}\, a_{dec}$

$$a_{non\text{-}rel} = \frac{T_{\nu,dec}\, a_{dec}}{T_{\nu,non\text{-}rel}} = \frac{T_{\nu,dec}\, a_{dec}}{m_\nu}$$

$$\therefore z = \frac{1}{a_{non\text{-}rel}} - 1$$

$$= \frac{m_\nu}{T_{\nu,dec}\, a_{dec}} - 1 \quad \left(= \frac{m_\nu (1+z_{dec})}{T_{\nu,dec}} - 1 \right)$$

$$= \frac{m_\nu}{4\times10^{-10}\,MeV} - 1 \quad \text{general form}$$

If $m_\nu \approx 0.02\,eV$

$$z \approx \frac{0.02}{4\times10^{-4}} - 1 \approx 49 \quad \text{(special example)}$$

(5)

At the neutrino decoupling, the neutrinos are relativistic, so we can use the relativistic limit to calculate the number density.

$$n_{dec} = \frac{\zeta(3)}{\pi^2} \cdot 2\, T_{dec}^3 \cdot \frac{3}{4}$$

$$= \frac{3\zeta(3)}{2\pi^2} T_{dec}^3$$

$$n_{dec}\, a_{dec}^3 = n_0 a_0^3 = n_0$$

$$\therefore \quad n_0 = n_{dec}\, a_{dec}^3$$
$$\approx \frac{3\zeta(3)}{2\pi^2} T_{dec}^3 \cdot a_{dec}^3$$
$$= \frac{3\zeta(3)}{2\pi^2} T_\nu^3$$
$$= \frac{3\zeta(3)}{2\pi^2}\left(\left(\frac{4}{11}\right)^{\frac{1}{3}} T_\gamma\right)^3$$
$$\approx \frac{3\zeta(3)}{2\pi^2}\left(\left(\frac{4}{11}\right)^{\frac{1}{3}}\times 2.725\right)^3 \frac{k^3}{c^3\hbar^3}$$
$$\approx 112\ cm^{-3}$$

Thus for 3 species its approximately $336\ cm^{-3}$.

(6)

$$\because \quad n_{\nu 0} = \frac{3\zeta(3)}{2\pi^2}\left(\left(\frac{4}{11}\right)^{\frac{1}{3}} T_\gamma\right)^3$$
$$= \frac{3}{4}\times\frac{4}{11}\, n_\gamma$$
$$= \frac{3}{11} n_{\gamma 0}$$
$$\rho_\nu = m_\nu n_{\nu 0}$$
$$= \frac{3}{11} n_{\gamma 0} m_\nu$$
$$\Omega_\nu h^2 = \frac{\rho_\nu}{\rho_\gamma}\Omega_\gamma h^2$$
$$= \frac{3}{11} n_{\gamma 0} m_\nu \frac{1}{\rho_\gamma}\Omega_\gamma h^2$$
$$= \frac{3}{11} m_\nu \frac{2\zeta(3)}{\pi^2} T_0^3 \cdot \frac{15}{\pi^2 T_0^4}\Omega_\gamma h^2$$
$$= \frac{90\zeta(3)}{11\pi^4}\frac{m_\nu}{T}\Omega_\gamma h^2$$

$$\approx \frac{90\zeta(3)}{11\pi^4}\frac{m_\nu}{2.725}\, 2.5\times10^{-5}\cdot 11605\left(\frac{K}{eV}\right)$$

$$\approx \frac{m_\nu}{94eV}$$

3、

(1)

$$\chi = \int dz \frac{c}{H_0\left[\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}+(1-\Omega_{m,0}-\Omega_{\Lambda,0})(1+z)^2\right]^{\frac{1}{2}}}$$

$$\chi'(z) = \frac{c}{H_0}\frac{1}{\left[\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}+(1-\Omega_{m,0}-\Omega_{\Lambda,0})(1+z)^2\right]^{\frac{1}{2}}}$$

$$\chi''(z) = -\frac{3\Omega_{m,0}(1+z)^2+2(1+z)(1-\Omega_{m,0}-\Omega_{\Lambda,0})}{2\left[\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}+(1-\Omega_{m,0}-\Omega_{\Lambda,0})(1+z)^2\right]^{\frac{3}{2}}}$$

$$\chi^{(3)}(z) = -\frac{15\left[2(1+z)(1-\Omega_{m,0}-\Omega_{\Lambda,0})+3\Omega_{m,0}(1+z)^2\right]^3}{8\left[(1+z)^2(1-\Omega_{m,0}-\Omega_{\Lambda,0})+\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}\right]^{\frac{7}{2}}}$$
$$+\frac{9\left[2(1-\Omega_{m,0}-\Omega_{\Lambda,0})+6\Omega_{m,0}(1+z)\right]\left[2(1+z)(1-\Omega_{m,0}-\Omega_{\Lambda,0})+3\Omega_{m,0}(1+z)^2\right]}{4\left[(1-\Omega_{m,0}-\Omega_{\Lambda,0})(1+z)^2+\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}\right]^{\frac{5}{2}}}$$
$$-\frac{3\Omega_{m,0}}{\left[(1-\Omega_{m,0}-\Omega_{\Lambda,0})(1+z)^2+\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}\right]^{\frac{3}{2}}}$$

$\therefore$ For $\Omega_{m,0}=0.3,\ \Omega_{\Lambda,0}=0.7$

$$\therefore \chi(z) = \chi(0)+\chi'(0)z+\frac{1}{2}\chi''(0)z^2+\frac{1}{6}\chi^{(3)}(0)z^3+\cdots$$
$$\approx \frac{c}{H_0}\Big[z-\frac{3\Omega_{m,0}+2-2\Omega_{m,0}-2\Omega_{\Lambda,0}}{2}\cdot\frac{1}{2}z^2$$
$$+\frac{1}{6}\Big[-\frac{15(2-2\Omega_{m,0}-2\Omega_{\Lambda,0}+3\Omega_{m,0})^3}{8}$$
$$+\frac{9(2-2\Omega_{m,0}-2\Omega_{\Lambda,0}+6\Omega_{m,0})(2-2\Omega_{m,0}-2\Omega_{\Lambda,0}+3\Omega_{m,0})}{4}$$
$$-\frac{3\Omega_{m,0}}{1}\Big]z^3\Big]$$

$$= \frac{c}{H_0}\left\{ z - \frac{2+\Omega_{m,0}-2\Omega_{\Lambda,0}}{4} z^2 + \frac{1}{6}\left(-\frac{15(2+\Omega_{m,0}-2\Omega_{\Lambda,0})^3}{8} + \frac{9(2+4\Omega_{m,0}-2\Omega_{\Lambda,0})(2+\Omega_{m,0}-2\Omega_{\Lambda,0})}{4} - 3\Omega_{m,0} \right) z^3 \right\}$$

☆

$$\chi(z) = \frac{c}{H_0}\left\{ z - \frac{0.9}{4} z^2 + \frac{1}{6}(1.37813) z^3 + \cdots \right\}$$

(2)

For $\frac{|\chi(z)_{apr} - \chi(z)_{full}|}{\chi(z)} = 10\%$

$z \simeq 0.589$

(3)

For very low redshift measurements,

$$\chi(z) \approx \frac{c}{H_0} z$$

So only H_0 enters into the parametrization.

Hence the only parameter that can be measured is H_0.

(4)

The combination is $2+\Omega_{m,0}-2\Omega_{\Lambda,0}$.

Let $2+\Omega_{m,0}-2\Omega_{\Lambda,0} = a$, at $O(z^2)$,

$$\chi(z) = \frac{c}{H_0} z - \frac{c}{4H_0} a z^2 + \cdots$$

∴ after determining H_0 at low redshift, the one independent parameter that can be measured is $a = 2+\Omega_{m,0}-2\Omega_{\Lambda,0}$, up to constant factor.

(5)

$$\chi(z) = \int dz \frac{c}{H_0[\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0} + (1-\Omega_{m,0}-\Omega_{\Lambda,0})(1+z)^2]^{\frac{1}{2}}}$$

Let the measurements be Gaussian distributions with width of 1% of the expectation value.

up to some normalization $\sim \frac{c}{100\,km/s/Mpc}$

$\chi(0.01) = 0.01425$

$\chi(0.1) = 0.13997$

$\chi(0.2) = 0.27548$

$\chi(0.3) = 0.4085$

$$L(\chi|\vec{p}) = \frac{1}{0.01\langle\chi\rangle\sqrt{2\pi}} \exp\left[-\frac{(\chi-\langle\chi\rangle)^2}{2(0.01\langle\chi\rangle)^2}\right]$$

$$\ln L(\chi|\vec{p}) = \frac{1}{0.01\langle\chi\rangle\sqrt{2\pi}} - \frac{(\chi-\langle\chi\rangle)^2}{2(0.01\langle\chi\rangle)^2}$$

$$\therefore \sigma_i^2 = (F^{-1})_{ii} = \left\langle -\frac{\partial^2 \ln L}{\partial p_i \partial p_i} \right\rangle$$

I used the $O(z^3)$ expansion to do the following calculation.
Implement this in Mathematica, we have

For $z=0.01$, The Fisher matrix is highly singular, which tells us we have too much freedom in parameter fitting. The error is very large.

For $z=0.1$, the relative errors are

$r_{\Omega m} \approx 1.06$, $r_{\Omega\Lambda} \approx 0$, $r_h \approx 0.093$

For $z=0.2$

$r_{\Omega m} \approx 0.687$ $r_{\Omega\Lambda} \approx 0$, $r_h \approx 0.16$

For $z=0.3$

$r_{\Omega m} \approx 0.516$ $r_{\Omega\Lambda} \approx 0$, $r_h \approx 0.21$