PHYS 600 HW 4

Recombination

1.

At equilibrium, for species $i \in \{e, p, H\}$,

$$n_i^{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{\frac{3}{2}} \exp\left(\frac{\mu_i - m_i}{T}\right)$$

$\because$ $e^- + p^+ \leftrightarrow H + \gamma$, $\mu_\gamma = 0$

$\therefore$ $\mu_p + \mu_e = \mu_H$

$$\therefore \frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p}\left(\frac{m_H}{m_e m_p}\frac{2\pi}{T}\right)^{\frac{3}{2}} \exp\left(\frac{m_p + m_e - m_H}{T}\right)$$

Define $E_I = m_p + m_e - m_H = 13.6\text{eV}$ as the ionization energy of hydrogen, and notice that $n_e = n_p$, $g_p = g_e = 2$ for both electrons and protons have 2 spin states, $g_H = 4$ for a $\frac{1}{2}$-$\frac{1}{2}$ spin system

$\therefore$ plug in the previous equation, we have

$$\left(\frac{n_H}{n_e^2}\right)_{eq} = \left(\frac{2\pi}{m_e T}\right)^{\frac{3}{2}} e^{E_I/T}$$

where we use $m_H \approx m_p$ in the prefactor of the RHS.

define $X_e = \frac{n_e}{n_p + n_H} = \frac{n_e}{n_e + n_H}$, $\eta = \frac{n_b}{n_\gamma}$.

Neglect the small number of helium atoms, then $n_b \approx n_p + n_H$

$$\therefore \frac{1 - X_e}{X_e^2} \approx \frac{n_H}{n_e^2} n_b$$

$$= \left(\frac{2\pi}{m_e T}\right)^{\frac{3}{2}} e^{E_I/T} \eta n_\gamma$$

$$= \left(\frac{2\pi}{m_e T}\right)^{\frac{3}{2}} e^{E_I/T} \eta \frac{2\zeta(3)}{\pi^2} T^3$$

$$\therefore \left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \frac{2\zeta(3)}{\pi^2}\eta\left(\frac{2\pi T}{m_e}\right)^{\frac{3}{2}} e^{E_I/T} \quad \text{(Saha equation)}$$

2.

Let $f(T, \eta) = \frac{2\zeta(3)}{\pi^2}\eta\left(\frac{2\pi T}{m_e}\right)^{\frac{3}{2}} e^{E_I/T}$

$$\frac{1 - X_e}{X_e^2} = f$$

$$\therefore f X_e^2 + X_e - 1 = 0$$

$$\therefore X_e = \frac{-1 \pm \sqrt{4f + 1}}{2f}$$

$\because$ $X_e > 0$ physically

$$\therefore X_e = \frac{-1 + \sqrt{1 + 4f}}{2f}$$

$\because$ $T = T_0(1+z)$, $T_0 = 2.73\text{K}$

$$\therefore f(z, \eta) = \frac{2\zeta(3)}{\pi^2}\eta\left(\frac{2\pi T_0(1+z)}{m_e}\right)^{\frac{3}{2}} \exp(E_I/[T_0(1+z)])$$

$$X_e = \frac{-1 + \sqrt{1 + 4f}}{2f}$$

plot see attached. we use $\eta \approx 6 \times 10^{-10}$.

3.

From the figure, the redshift at which $X_e = 0.1$ is approximately $z = 1260$.
The redshift at which $X_e = 0.5$ is approximately $z = 1380$.
So they are approximately $\Delta z \approx 120$ apart — not very different.

4、

From Friedmann equation,

$$a(t)=\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}\sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{\Omega_\Lambda}\,H_0 t\right), \quad \Omega_m=0.32,\ \Omega_\Lambda=0.68$$

$$H_0\approx 2.273\times10^{-18}\ s^{-1}$$

for $X_e=0.5$, $z\approx1380$

$$t\approx 1.011\times10^{13}\ s\approx 320000\ \text{yrs}$$

for $X_e=0.1$, $z\approx1260$

$$t\approx 1.159\times10^{13}\ s\approx 370000\ \text{yrs}$$

(Here I use the two-component scale factor function. There's some ambiguity in whether it's meant the universe is always matter-dominated or the universe at the epoch is matter dominated. I take the latter interpretation as it is more realistic.)

5、Assume the universe was matter dominated.

Assume the standard values for the cosmological parameters.

$$\Gamma_\gamma\approx H$$

$$\Gamma_\gamma=n_b X_e\sigma_T=\frac{2\zeta(3)}{\pi^2}\eta\sigma_T X_e T^3$$

$$H=H_0\sqrt{\Omega_m}\left(\frac{T_{dec}}{T_0}\right)^{\frac{3}{2}}$$

$$\therefore X_e T^{\frac{3}{2}}\approx\frac{\pi^2}{2\zeta(3)}\frac{H_0\sqrt{\Omega_m}}{\eta\sigma_T T_0^{\frac{3}{2}}}$$

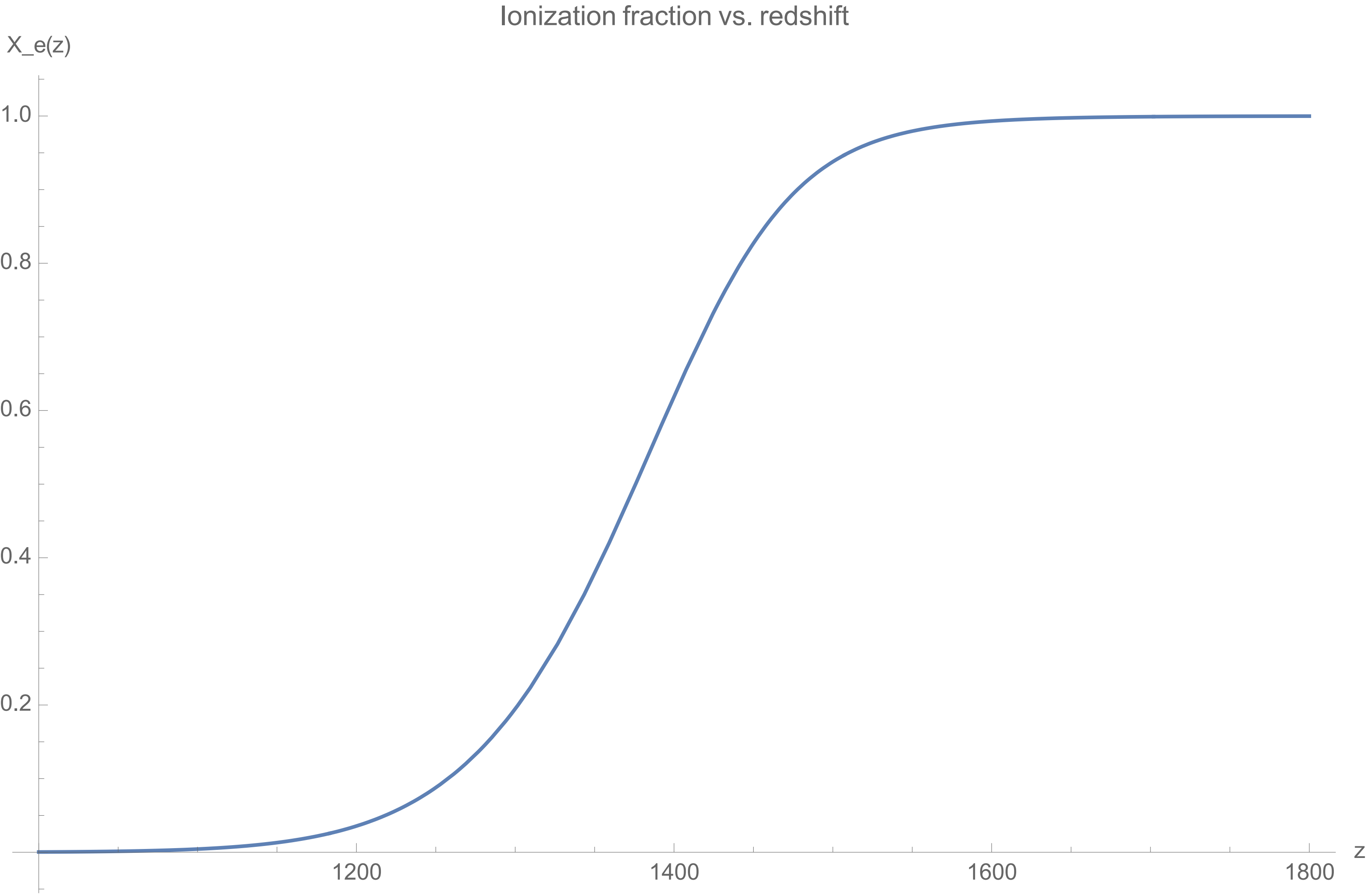
$T=T_0(1+z)$, $\Omega_m=0.32$, $\Omega_\Lambda=0.68$

$\therefore z\approx1150$

$$a(t)=\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}\sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{\Omega_\Lambda}\,H_0 t\right)\approx\frac{1}{1150}$$

$$t\approx 420000\ \text{yrs}$$

$$X_e\approx0.01$$


Ionization fraction vs. redshift
X_e(z)
1.0
0.8
0.6
0.4
0.2
1200
1400
1600
1800
z

"What-if" BBN

Define the neutron fraction as $X_n = \frac{n_n}{n_n+n_p}$

(1)

Consider reactions $n + \nu_e \leftrightarrow p^+ + e^-$

$n + e^+ \leftrightarrow p^+ \ \bar{\nu}_e$

Because the electrons and neutrinos are very light in comparison, we can assume that their chemical potentials are negligibly small

$\therefore$ at equilibrium $\mu_n = \mu_p$

By the first problem,

$$(n_i)_{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{\frac{3}{2}} \exp\left(\frac{\mu_i - m_i}{T}\right)$$

$$\therefore \left(\frac{n_n}{n_p}\right)_{eq} = \left(\frac{m_n}{m_p}\right)^{\frac{3}{2}} \exp[-(m_n - m_p)/T]$$

$\because m_n \simeq m_p$

$\therefore$ we can take $m_n \simeq m_p$ in the prefactor

$$\therefore \left(\frac{n_n}{n_p}\right)_{eq} = \exp[-Q/T]$$

with $Q = m_n - m_p$

$$\therefore X_n = \frac{n_n}{n_n+n_p} = \frac{1}{1+e^{Q/T}} = \frac{e^{-Q/T}}{1+e^{-Q/T}}$$

(2)

$\because$ the freeze out temperature is 0.8 MeV

$$\therefore X_n = \frac{1}{1+e^{Q/T}} = \frac{1}{1+e^{1.3/0.8}} \simeq 0.165$$

(3)

Because all of these neutrons are converted to helium 4

$n_{He} = \frac{1}{2} n_n$

$n_H = n_p - n_n$

$$\therefore Y_p = \frac{4 n_{He}}{n_H} = \frac{4 \cdot \frac{1}{2} n_n}{n_p - n_n} = \frac{2 n_n}{n_p - n_n}$$

$$= \frac{2}{\frac{1}{X_n} - 2}$$

$$\simeq 0.49$$

(4)

$$X_n' = \frac{1}{1+e^{Q/T}} = \frac{1}{1+e^{2.6/0.8}} \simeq 0.0373$$

$$Y_p = \frac{2}{\frac{1}{X_n'} - 2} \simeq 0.081$$

THE helium abundance would be smaller.

Freeze-in DM

(1)

$$\frac{1}{a^3}\frac{d(na^3)}{dt} = 2\Gamma h(t)\, n_{\sigma,eq}(t)$$

let $Y=\frac{n}{T^3}$, $x=\frac{m_\sigma}{T}$, $\frac{dx}{dt}=Hx$

$$\therefore LHS = \frac{1}{a^3}\frac{d(YT^3a^3)}{dt} = \frac{1}{a^3}T^3a^3\frac{dY}{dt} = T^3\frac{dY}{dx}\frac{dx}{dt} = T^3Hx\frac{dY}{dx}$$

let $Y_{eq}=\frac{n_{\sigma,eq}}{T^3}$

$$\therefore RHS = 2\Gamma h\, Y_{eq}T^3$$

$$LHS=RHS \Rightarrow T^3Hx\frac{dY}{dx} = 2\Gamma h(x)\, Y_{eq}(x)\, T^3$$

$$\frac{H(m_\sigma)}{x^2}x\frac{dY}{dx} = 2\Gamma h(x)\, Y_{eq}(x)$$

$$\frac{dY}{dx} = \frac{2\Gamma x}{H(m_\sigma)}h(x)\, Y_{eq}(x)$$

let $\lambda_1 = \frac{2\Gamma}{H(m_\sigma)}$

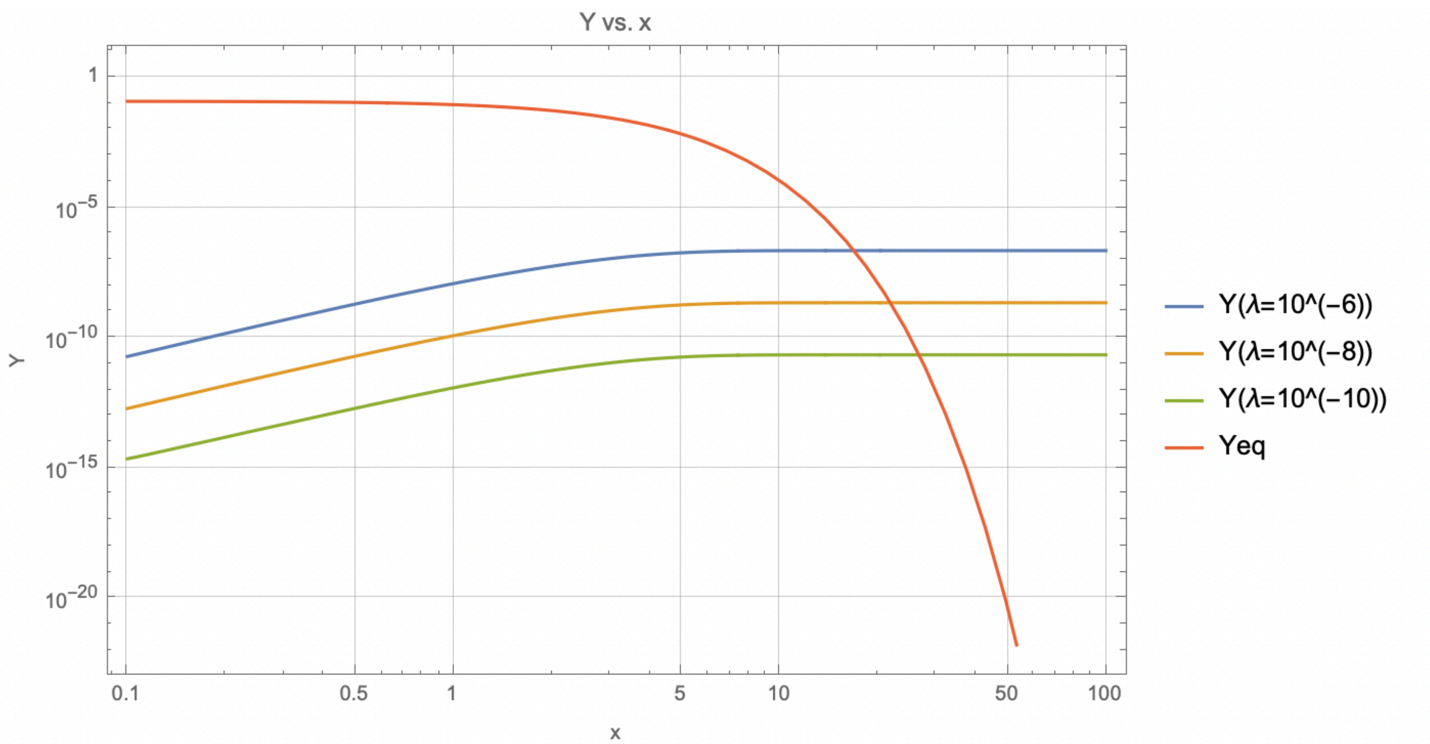
$$\frac{dY}{dx} = \lambda_1 x h(x)\, Y_{eq}(x)$$

(2)

Plot see attached.

(3)

For freeze-out mechanism, the relic abundance scales inversely with Γ;
for freeze-in mechanism, the relic abundance scales positively with Γ.
This is because for freeze-out, the higher Γ is, the more dark matter gets annihilated before "freezing out", whereas for freeze-in, the higher Γ is, the more dark matter gets produced by SM particle before n_σ gets suppressed.



Y vs. x
1
10^−5
10^−10
10^−15
10^−20
Y
0.1
0.5
1
5
10
50
100
x
Y(λ=10^(−6))
Y(λ=10^(−8))
Y(λ=10^(−10))
Yeq