PHYS 600 HW 5

I.

(1) $\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G\bar{\rho}_m\delta_m \iff \frac{d}{da}\left(a^3 H\frac{d\delta_m}{da}\right) = 4\pi G\bar{\rho}_{m,0}\frac{\delta_m}{Ha^2}$

we start from the latter and show the expressions are equivalent.

$$\frac{d}{da}\left(a^3 H\frac{d\delta_m}{da}\right) = \frac{d}{da}\left(a^3\frac{1}{a}\frac{da}{dt}\frac{d\delta_m}{da}\right) = \frac{d}{da}\left(a^2\dot{\delta}_m\right) = 4\pi G a^3\bar{\rho}_m\frac{\delta_m}{a^2}\frac{a}{\dot{a}}$$

$$\therefore \frac{d}{da}\left(a^2\dot{\delta}_m\right) = 4\pi G\bar{\rho}_m\delta_m\frac{a^2}{\dot{a}}$$

$$\therefore \frac{\dot{a}}{a^2}\frac{d}{da}\left(a^2\dot{\delta}_m\right) = 4\pi G\bar{\rho}_m\delta_m$$

$$\frac{\dot{a}}{a^2}\left(\dot{\delta}_m 2a + a^2\ddot{\delta}_m\frac{dt}{da}\right) = 4\pi G\bar{\rho}_m\delta_m$$

$$\frac{\dot{a}}{\dot{a}}\ddot{\delta}_m + 2\frac{\dot{a}}{a}\dot{\delta}_m = 4\pi G\bar{\rho}_m\delta_m$$

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G\bar{\rho}_m\delta_m$$

Hence proven.

(2)

For a flat universe,

$$4\pi G\bar{\rho} = \frac{3}{2}H^2$$

By the Friedmann equation,

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3}$$

at equilibrium

$$\frac{H^2}{H_0^2} = \Omega_r a_{eq}^{-4} + \Omega_m a_{eq}^{-3}$$

where $\Omega_r a_{eq}^{-4} = \Omega_m a_{eq}^{-3}$

$\therefore \Omega_r = \Omega_m a_{eq}$

$$\therefore \frac{H^2}{H_0^2} = \Omega_m\left(a_{eq}a^{-4} + a^{-3}\right)$$

$$\therefore H^2 = H_0^2\Omega_m\left(a_{eq}a^{-4} + a^{-3}\right)$$

$$= \frac{H_0^2\Omega_m}{a_{eq}^3}\left(a_{eq}^4 a^{-4} + a_{eq}^3 a^{-3}\right)$$

$$= \frac{H_0^2\Omega_m}{a_{eq}^3}\left(y^{-4} + y^{-3}\right)$$

$$\therefore H = \sqrt{\frac{H_0^2\Omega_m}{a_{eq}^3}}\frac{1}{y^2}\sqrt{1+y}$$

$$H^2 = \frac{A^2}{y^4}(1+y)$$

$$= A^2\left(y^{-4} + y^{-3}\right)$$

$$= A^2\left(\frac{a_{eq}^4}{a^4} + \frac{a_{eq}^3}{a^3}\right)$$

$$= A^2 a_{eq}^3\left(a_{eq}a^{-4} + a^{-3}\right)$$

which takes the form $H = \frac{A}{y^2}\sqrt{1+y}$

At equilibrium, $y=1$, $H_{eq} = \sqrt{2}A$

$$\therefore 4\pi G\rho_{eq} = \frac{3}{2}\cdot 2A^2$$

$$A^2 = \frac{4\pi G\rho_{eq}}{3}$$

$$\therefore A = \left(\frac{4\pi G\rho_{eq}}{3}\right)^{\frac{1}{2}}$$

(3)

Let $H(y) = \frac{A}{y^2}\sqrt{1+y}$

∵ $y = \frac{a}{a_{eq}}$

∴ $dy = \frac{1}{a_{eq}}\,da \qquad da = a_{eq}\,dy$

$$\frac{1}{a_{eq}}\frac{d}{dy}\left(a_{eq}^3 y^3 \frac{A}{y^2}\sqrt{1+y}\,\frac{d\delta_m}{dy}\frac{1}{a_{eq}}\right) = 4\pi G\bar{\rho}_{m,0}\frac{\delta_m}{A\sqrt{1+y}}\,y^2\frac{1}{y^3 a_{eq}^3}$$

$$A^2 a_{eq}^3\frac{d}{dy}\left(y\sqrt{1+y}\,\frac{d\delta_m}{dy}\right) = 4\pi G\bar{\rho}_{m,0}\frac{\delta_m}{\sqrt{1+y}}$$

$$A^2\frac{a_{eq}^3}{4\pi G\bar{\rho}_{m,0}}\frac{d}{dy}\left(y\sqrt{1+y}\,\frac{d\delta_m}{dy}\right) = \frac{\delta_m}{\sqrt{1+y}}$$

$$A^2\frac{1}{4\pi G\bar{\rho}_{m,eq}}\frac{d}{dy}\left(y\sqrt{1+y}\,\frac{d\delta_m}{dy}\right) = \frac{\delta_m}{\sqrt{1+y}}$$

$$\frac{\cancel{4\pi G\rho_{eq}}}{3}\frac{1}{\cancel{4\pi G}\left(\frac{1}{2}\rho_{eq}\right)}\frac{d}{dy}\left(y\sqrt{1+y}\,\frac{d\delta_m}{dy}\right) = \frac{\delta_m}{\sqrt{1+y}}$$

$$\frac{2}{3}\frac{d}{dy}\left(y\sqrt{1+y}\,\frac{d\delta_m}{dy}\right) = \frac{\delta_m}{\sqrt{1+y}}$$

$$\left(\sqrt{1+y} + \frac{y}{2\sqrt{1+y}}\right)\frac{d\delta_m}{dy} + y\sqrt{1+y}\,\frac{d^2\delta_m}{dy^2} = \frac{3}{2}\frac{\delta_m}{\sqrt{1+y}}$$

$$\left(\frac{1}{y} + \frac{y}{2y(1+y)}\right)\frac{d\delta_m}{dy} + \frac{d^2\delta_m}{dy^2} = \frac{3}{2}\frac{\delta_m}{y(1+y)}$$

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} = \frac{3}{2}\frac{\delta_m}{y(1+y)}$$

(4)

Let $\delta_{m1} = a\left(1+\frac{3}{2}y\right)$

$\delta_{m2} = b\left[\left(1+\frac{3}{2}y\right)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y}\right.$

Plug in the differential equation derived,

$$\frac{d^2\delta_{m1}}{dy^2} = 0,\quad \frac{d\delta_m}{dy} = \frac{3}{2}a$$

∴ $$\frac{2+3y}{2y(1+y)}\,\frac{3}{2}a = \frac{3}{2}a\left(\frac{2+3y}{2y(1+y)}\right)$$

So δ_{m1} is a solution

Plug δ_{m2} in Mathematica.

It's also a solution. (See script)

(5)

At early times, we expect $D \propto \ln y$ (radiation domination, $y \ll 1$)

δ_{m2} reproduces this properly.

At late times, we expect $D \propto y$ (matter domination, $y \gg 1$)

δ_{m1} reproduces this properly.

II

For the spherical collapse, the equation we need to solve is just

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = E$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = 1/\left(\frac{dt}{d\theta}\right) = \frac{1}{B(1-\cos\theta)}$$

$$\therefore \dot{r} = A\sin\theta \cdot \frac{1}{B(1-\cos\theta)}$$

$$\therefore \frac{1}{2}\dot{r}^2 - \frac{GM}{r}$$

$$= \frac{1}{2}\frac{A^2\sin^2\theta}{B^2(1-\cos\theta)^2} - \frac{GM}{A(1-\cos\theta)}$$

$$= \frac{1}{2}\frac{GM\sin^2\theta}{A(1-\cos\theta)^2} - \frac{GM(1-\cos\theta)}{A(1-\cos\theta)^2}$$

$$= \frac{1}{2}\frac{2|E|\sin^2\theta}{(1-\cos\theta)^2} - \frac{2|E|(1-\cos\theta)}{(1-\cos\theta)^2}$$

$$= \frac{|E|(\cancel{1}-\cos^2\theta\cancel{-1}+2\cos\theta-1)}{(1-\cos\theta)^2}$$

$$= \frac{|E|[-(1-\cos\theta)^2]}{(1-\cos\theta)^2}$$

$$= -|E|$$

$\because$ $E<0$ inside the overdensity

$\therefore -|E| = E$

$\therefore \frac{1}{2}\dot{r}^2 - \frac{GM}{r} = E$. $\square$

III

$$k_{eq} = a_{eq}H_{eq}$$

By Friedmann equation

$$\frac{H_{eq}^2}{H_0^2} = \Omega_r a_{eq}^{-4} + \Omega_m a_{eq}^{-3}$$

where $\Omega_r a_{eq}^{-4} = \Omega_m a_{eq}^{-3}$

$$\Omega_r = \Omega_m a_{eq}$$

$$\frac{H_{eq}^2}{H_0^2} = 2\Omega_m a_{eq}^{-3}$$

$$\therefore H_{eq} = H_0\frac{1}{a_{eq}}\sqrt{\frac{2\Omega_m}{a_{eq}}}$$

$$\therefore k_{eq} = a_{eq}H_{eq} = H_0\sqrt{\frac{2\Omega_m}{a_{eq}}}$$

Plug in the standard values for the cosmological parameters

$$k_{eq} \approx 0.015\ h\,\mathrm{Mpc}^{-1}$$

IV

(1)

(a) see attached

(b) Indeed the power spectrum is given by $P(k) \propto k^{n_s}$ at large scales.
The slight deviation from the Harrison-Zel'dovich spectrum is visible.

(c) $k_{eq} = 0.0169\ h\,\mathrm{Mpc}^{-1}$ from the plot.
This is slightly larger than the $0.015\ h\,\mathrm{Mpc}^{-1}$ from the previous problem.

(d)
$$T^2(k) \propto \frac{P(k, z=0)}{k^{n_s}}$$

Plot see the attached.
This is roughly the same as what we calculated in class shapewise.

(e)
$$\sigma_8^2 = \frac{1}{2\pi^2}\int W_s^2 k^2 P(k)\,dk\ ,\quad W_s = \frac{3j_1(kR_8)}{kR_8}$$

$$R_8 = 8\ h^{-1}\,\mathrm{Mpc}$$

From the linear power spectrum,

$$\sigma_8 \approx 1.15$$

This is larger than $\sigma_8 = 0.834$ given above in problem.

(2)

(a) see attached

(b) At large scales the ratio is k independent (flat shape in curve). This is because of the $\delta \ll 1$ limit is manifest when k is small, $T(k) \simeq 1$ in this regime. At k_{eq} this linear structure growth start to break down, where k_{eq} is larger for larger z (smaller a).

(c) $P(k,t) = T^2(k)\,D(t)\,P(k,t_i)$
because $D(t)$ is only t (or a, z) dependent, I choose $k \to 0$ to do the normalization, where $T(0) = 1$, $P(k,t_i) \propto k^{n_s}$. Normalize $D(a=1) \simeq 0.78$ for convenience of comparing.
Plot see attached of part (d).

(d) Plot see the attached. The two results have the same shape, but the exact result predicts a lower intermediate D when $a<1$. They agree fairly well.

(3)

For the cosmology of the simulation, $w(a) = -1$

$$g'' + \left(\frac{5}{2} + \frac{3}{2}\Omega_{DE}(a)\right)g' + 3\Omega_{DE}(a)\,g = 0$$

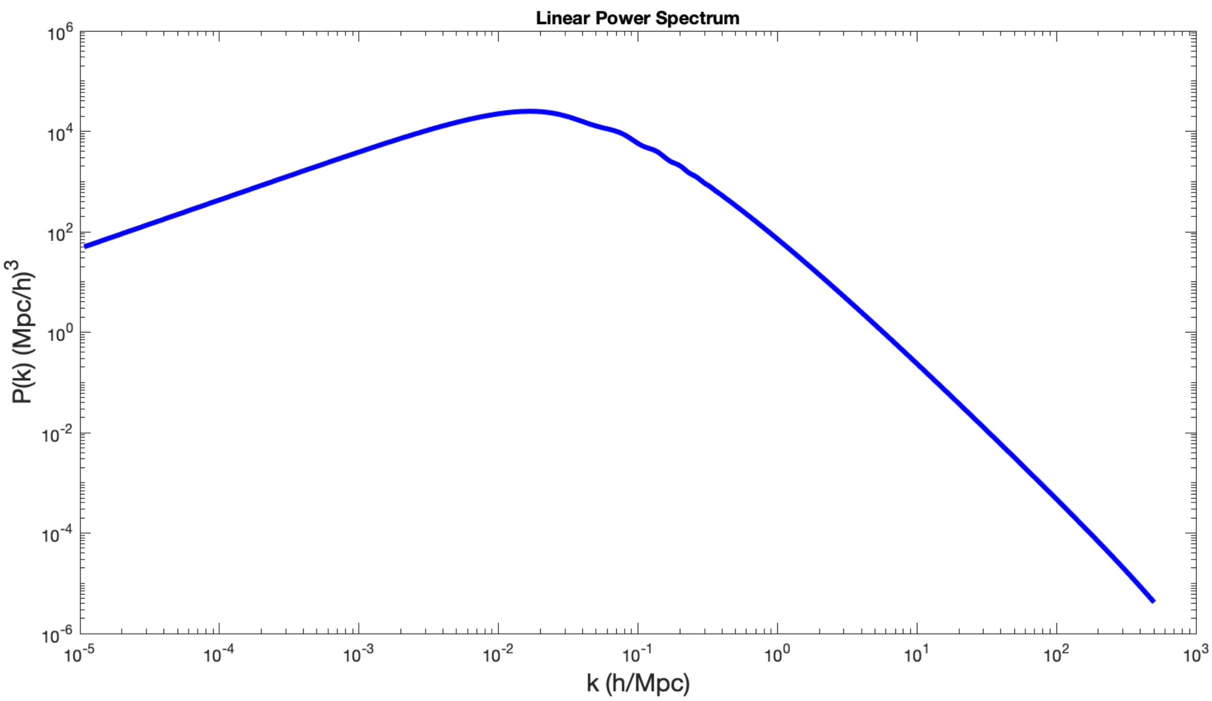
Volume $\propto a^3$

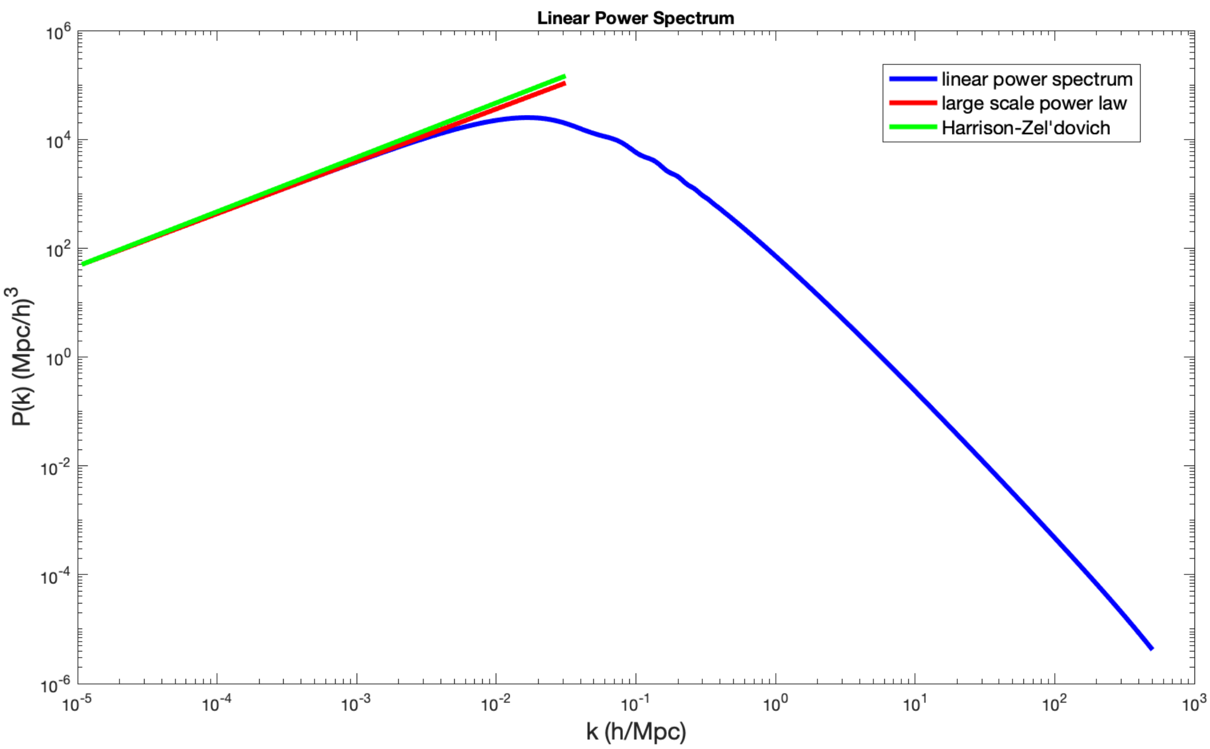
$n_{DE} \propto a^0$

$E_{DE} \propto a^3$

at $a=1$

$$\Omega_{DE}(1) \approx \frac{0.7}{0.7+0.3}\ ,\quad E_m \propto a^0$$


Linear Power Spectrum
P(k) (Mpc/h)^3
k (h/Mpc)
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}



Linear Power Spectrum
linear power spectrum
large scale power law
Harrison-Zel'dovich
P(k) (Mpc/h)^3
k (h/Mpc)
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
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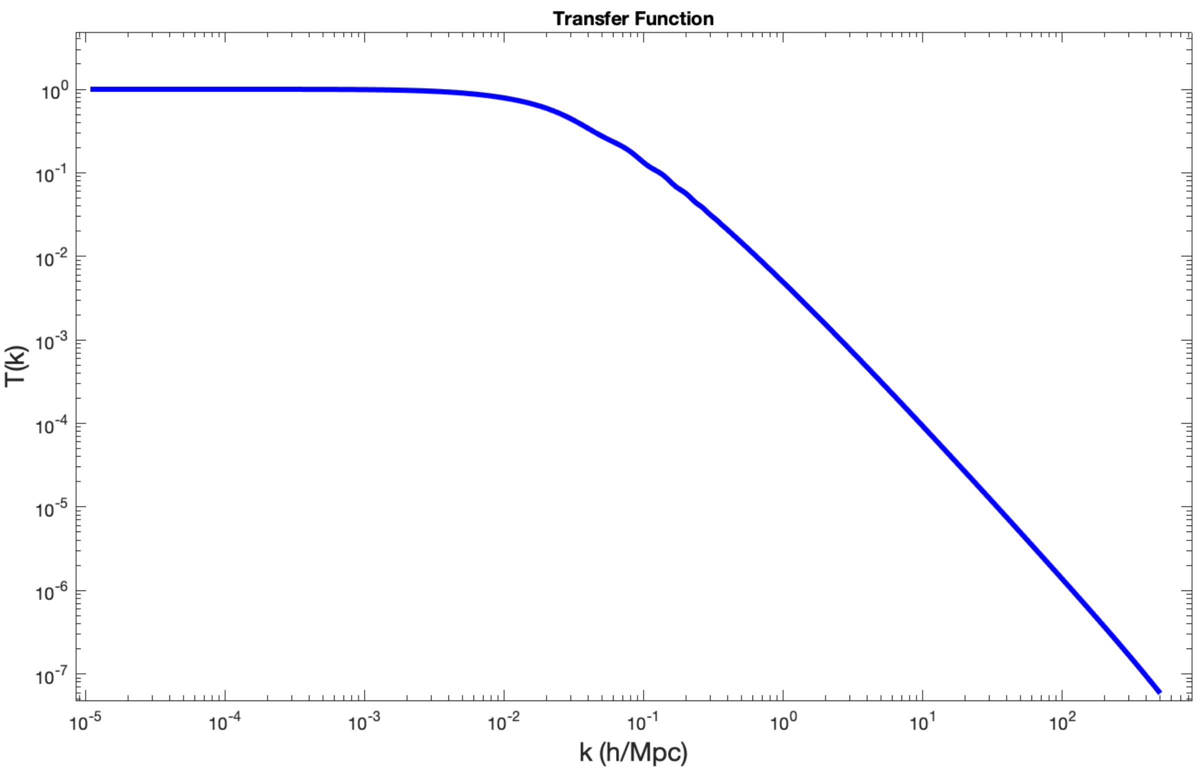
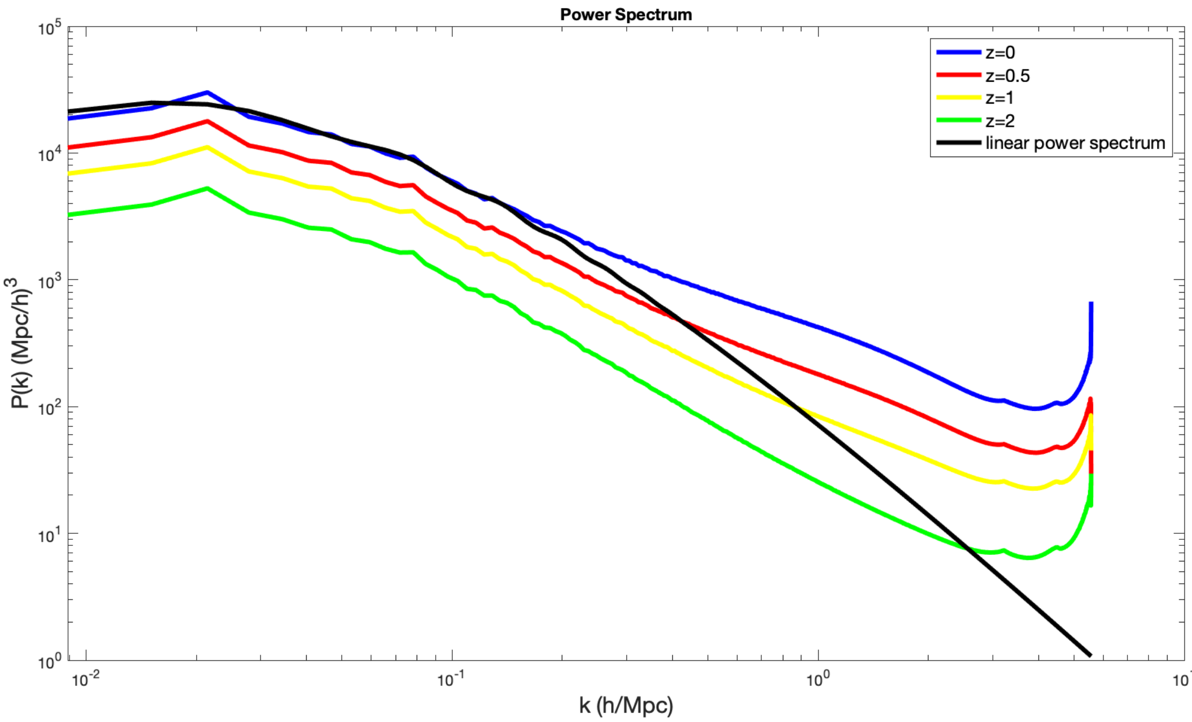
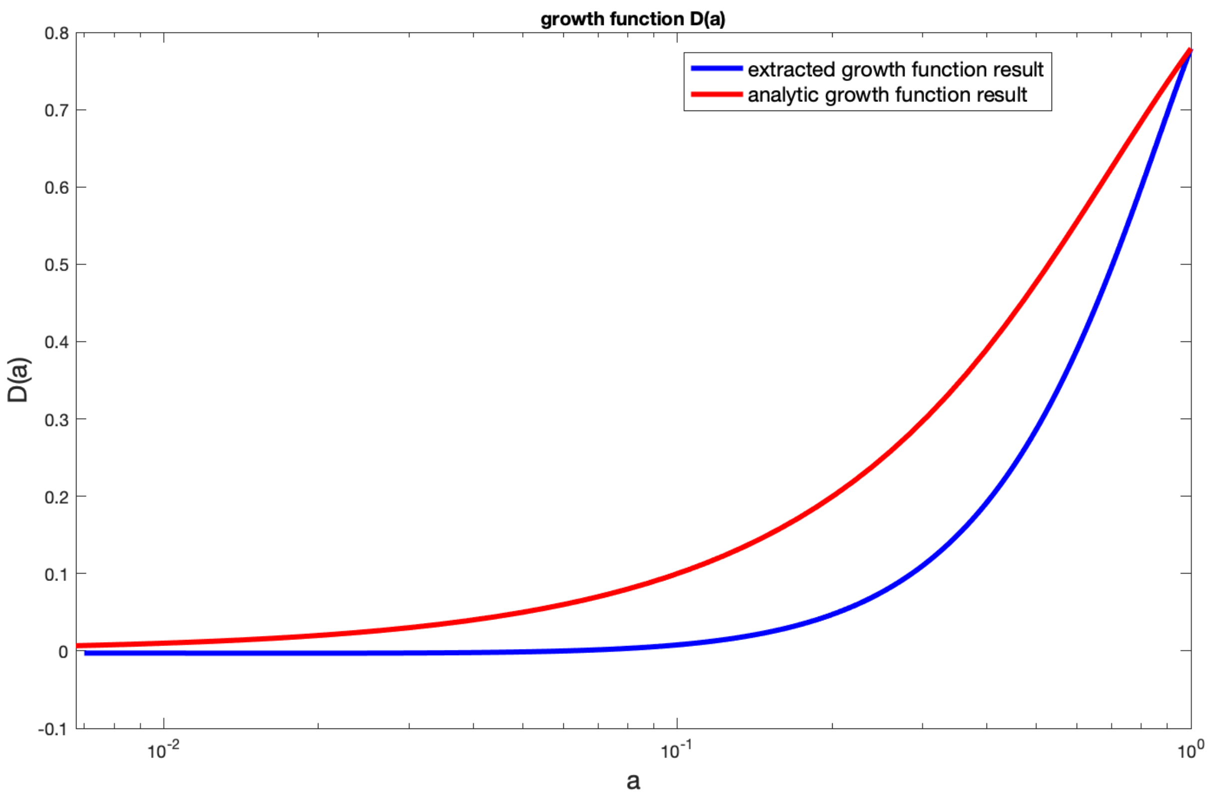
10^{0}
10^{1}
10^{2}
10^{3}

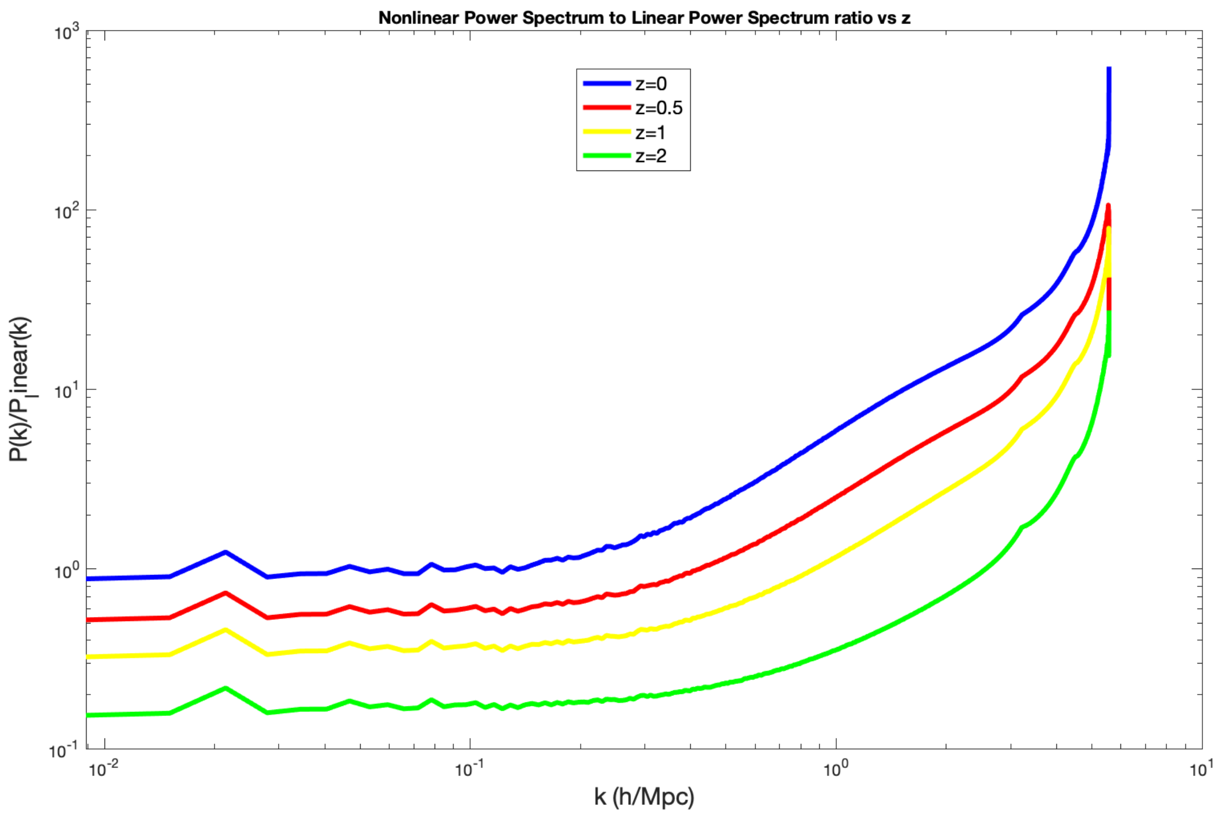


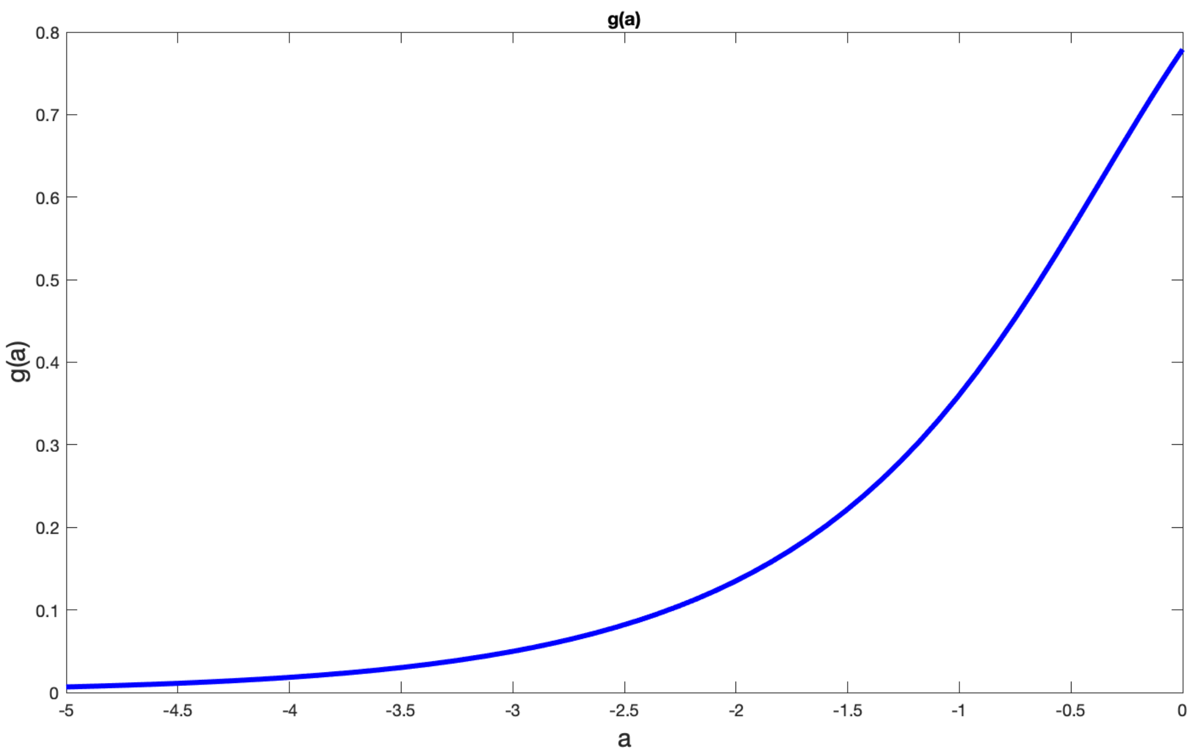
Transfer Function
T(k)
k (h/Mpc)
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
10^2


Power Spectrum
z=0
z=0.5
z=1
z=2
linear power spectrum
P(k) (Mpc/h)^3
k (h/Mpc)
10^5
10^4
10^3
10^2
10^1
10^0
10^{-2}
10^{-1}
10^0
10^1



growth function D(a)
extracted growth function result
analytic growth function result
D(a)
a
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
10^{-2}
10^{-1}
10^{0}


Nonlinear Power Spectrum to Linear Power Spectrum ratio vs z
z=0
z=0.5
z=1
z=2
P(k)/P_linear(k)
k (h/Mpc)
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-1
10^0
10^1


g(a)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
a
g(a)

$$\Omega_{tot}(a) \approx \frac{\frac{7}{3}a^3}{\frac{7}{3}a^3+1} = \frac{\frac{7}{3}e^{3\ln a}}{\frac{7}{3}e^{3\ln a}+1}$$

$$\therefore\ g'' + \left(\frac{5}{2}+\frac{3}{2}\left(\frac{\frac{7}{3}e^{3\ln a}}{\frac{7}{3}e^{3\ln a}+1}\right)\right)g' + \frac{7e^{3\ln a}}{\frac{7}{3}e^{3\ln a}+1}\,g = 0$$

Integrate numerically; at $\ln a = -5$, $g(a)=1$.
Plot see the attached.

(4)
I use the first column (M_{vir}).

(a)
For a flat universe

$$4\pi G\bar{\rho} = \frac{3}{2}H^2$$

At $z=0$, $\bar{\rho}_m \approx 0.32\bar{\rho} = \frac{1}{4\pi G}\cdot 0.32\cdot\frac{3}{2}H_0^2$

$$M = \bar{\rho}_m \cdot 1^3 h^{-3}\,\mathrm{Gpc}^3$$

$$m = \frac{M}{1024^3} = \frac{\bar{\rho}_m\cdot 1^3 h^{-3}\,\mathrm{Gpc}^3}{1024^3} \approx 1.82\times10^{10}\ M_\odot/h$$

determined numerically

(b)
both $z=0$, $z=1$ plots see attached.
There is indeed a turnover at low masses.
This is due to the suppression at larger mass due to the critical overdensity.
I don't think the question "how many particles are in halos at this mass" is a well defined question. Instead, I will answer how many particles are in halos below this mass. The solution is to find the total mass of halos below this mass and divide by mass per particle.
For $z=0$, by numerical calculation, $N_{<m_c} \approx 8.92\times10^7$
For $z=1$, by numerical calculation, $N_{<m_c} \approx 1.81\times10^8$
Since the total number of particles is conserved, this tells us that as the universe evolves, more particles live in large mass halos, as expected from class.

(c)

$$\sigma^2(M) = \int d\ln k\,\Delta^2_{lin}(k)\,|W(k,M)|^2 = \int dk\,\frac{1}{k}\Delta^2_{lin}(k)\,|W(k,M)|^2$$

$$M = \frac{4}{3}\pi R^3\bar{\rho}_m$$

$$R = \left(\frac{3M}{4\pi\bar{\rho}_m}\right)^{\frac{1}{3}}$$

$$W(k,M) = \frac{3}{(kR)^3}\left[\sin(kR) - kR\cos(kR)\right]$$

$$= \frac{3}{k^3\, 3M} 4\pi \bar{\rho}_m \left[\sin\left(k\left(\frac{3M}{4\pi\bar{\rho}_m}\right)^{\frac{1}{3}}\right) - k\left(\frac{3M}{4\pi\bar{\rho}_m}\right)^{\frac{1}{3}} \cos\left(k\left(\frac{3M}{4\pi\bar{\rho}_m}\right)^{\frac{1}{3}}\right)\right]$$

$$\bar{\rho}_m = \frac{M_{tot}}{V_{tot}} = \frac{1.9537\times10^{19} M_\odot/h}{1\,h^{-3}\,Gpc^3} = 1.9537\times10^{19}\, h^2 M_\odot Gpc^{-3}$$

$$\Delta^2_{lin}(k) = \frac{k^3}{2\pi^2} P_{lin}(z=0,k)$$

$$\therefore \sigma^2(M) = \int dk \frac{k^2}{2\pi^2} P_{lin}(z=0,k)\,|W(k,M)|^2$$

For $z=1$ case,

since $P(k,z) \propto D^2(z)$

$$\therefore \sigma(M, z=1) = \sigma(M, z=0)\frac{D(z=1)}{D(z=0)}$$

$\because a = \frac{1}{1+z}$

$\therefore a(z=1) = \frac{1}{2}$

Numerically we found $\frac{D(z=1)}{D(z=0)} \approx 0.3679$

Plots see the attached.

(d)

$$n_{ps}(M) = -\sqrt{\frac{2}{\pi}}\, \nu \exp\left(-\frac{\nu^2}{2}\right)\frac{\bar{\rho}}{M^2}\frac{d\ln\sigma(M)}{d\ln M}$$

$$= -\sqrt{\frac{2}{\pi}}\frac{\delta_c}{\sigma}\exp\left(-\frac{\delta_c^2}{2\sigma^2}\right)\frac{\bar{\rho}}{M^2}\frac{M}{\sigma}\frac{d\sigma}{dM}$$

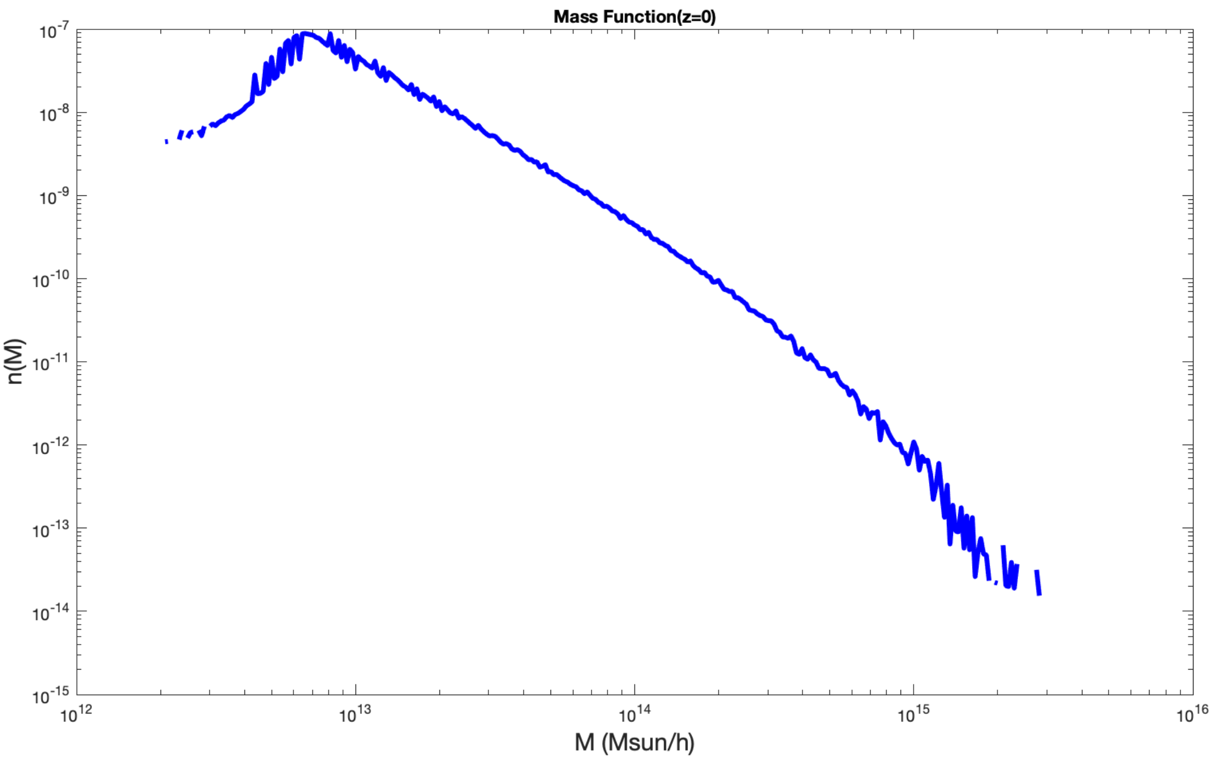
$$= -\sqrt{\frac{2}{\pi}}\frac{\delta_c}{\sigma^2}\frac{\bar{\rho}}{M}\exp\left(-\frac{\delta_c^2}{2\sigma^2}\right)\frac{d\sigma}{dM}$$

Calculate numerically. Note this is number density in volume too. To compare, multiply it by $1\,h^{-3}\,Gpc^3 \sim 10^9\, h^{-3}\,Mpc^3$. Each $\frac{dn}{dM}$ is also multiplied by dM to get a number, where dM is the mass step chosen numerically.

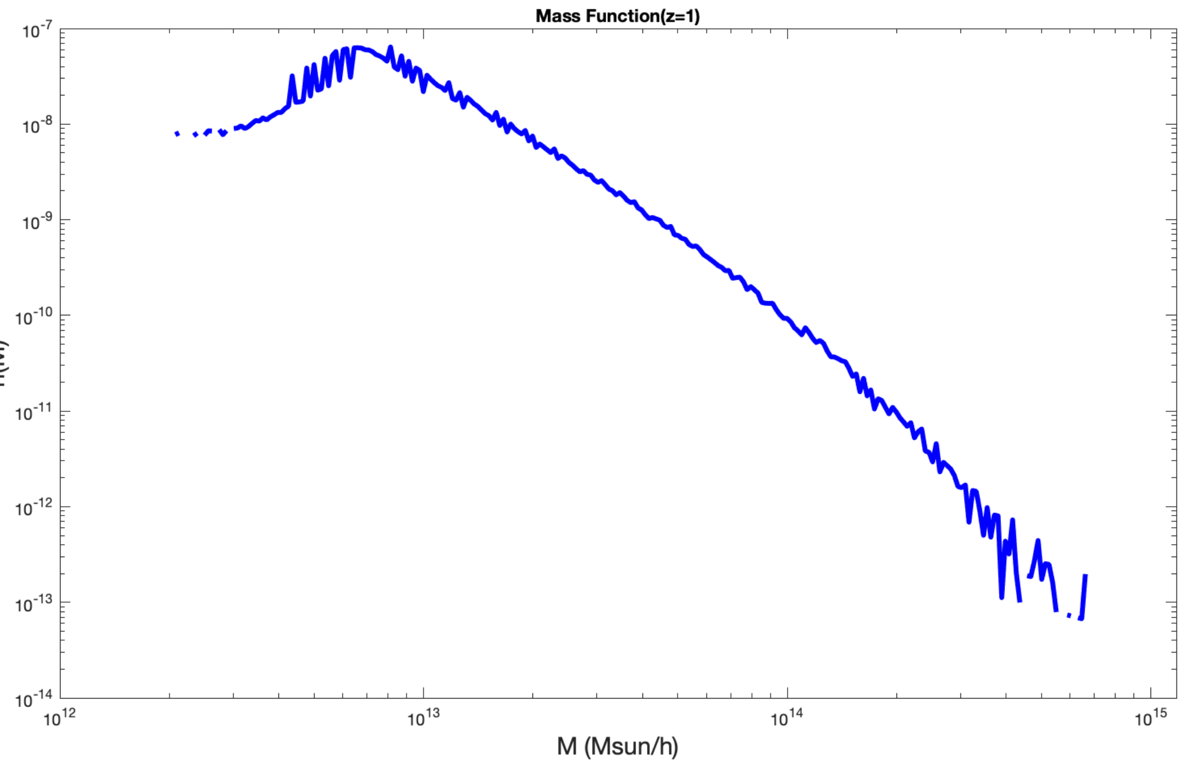
They agree fairly well.

The missing turnover is according to the inadequacy of linear power spectrum.

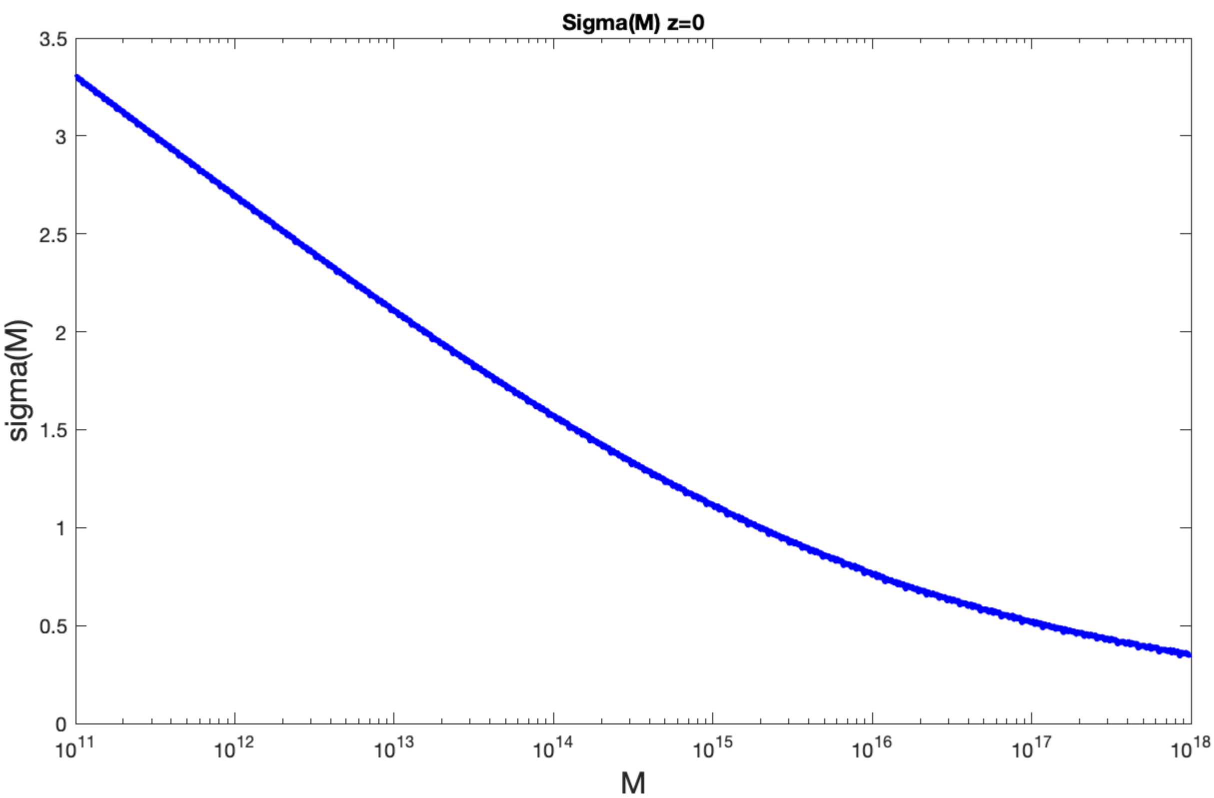
Plots see the attached.

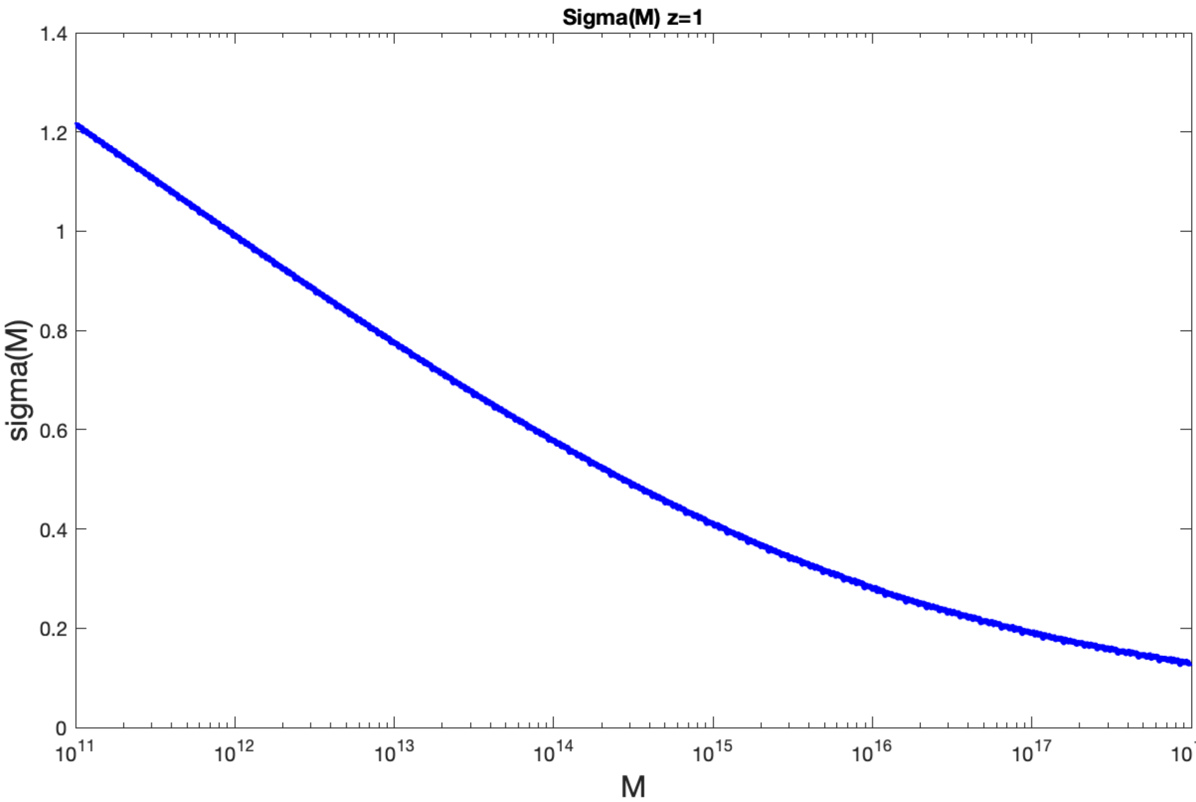


Mass Function(z=0)
n(M)
M (Msun/h)

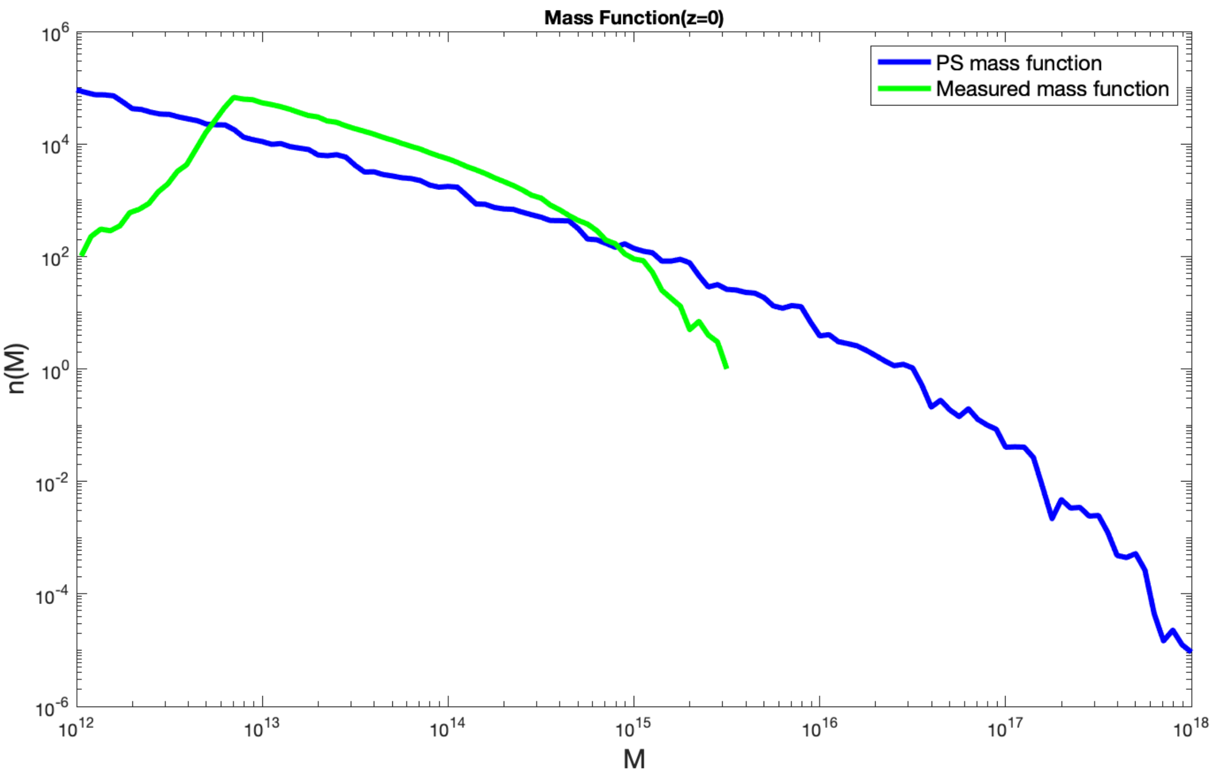


Mass Function(z=1)
n(M)
M (Msun/h)
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{12}
10^{13}
10^{14}
10^{15}

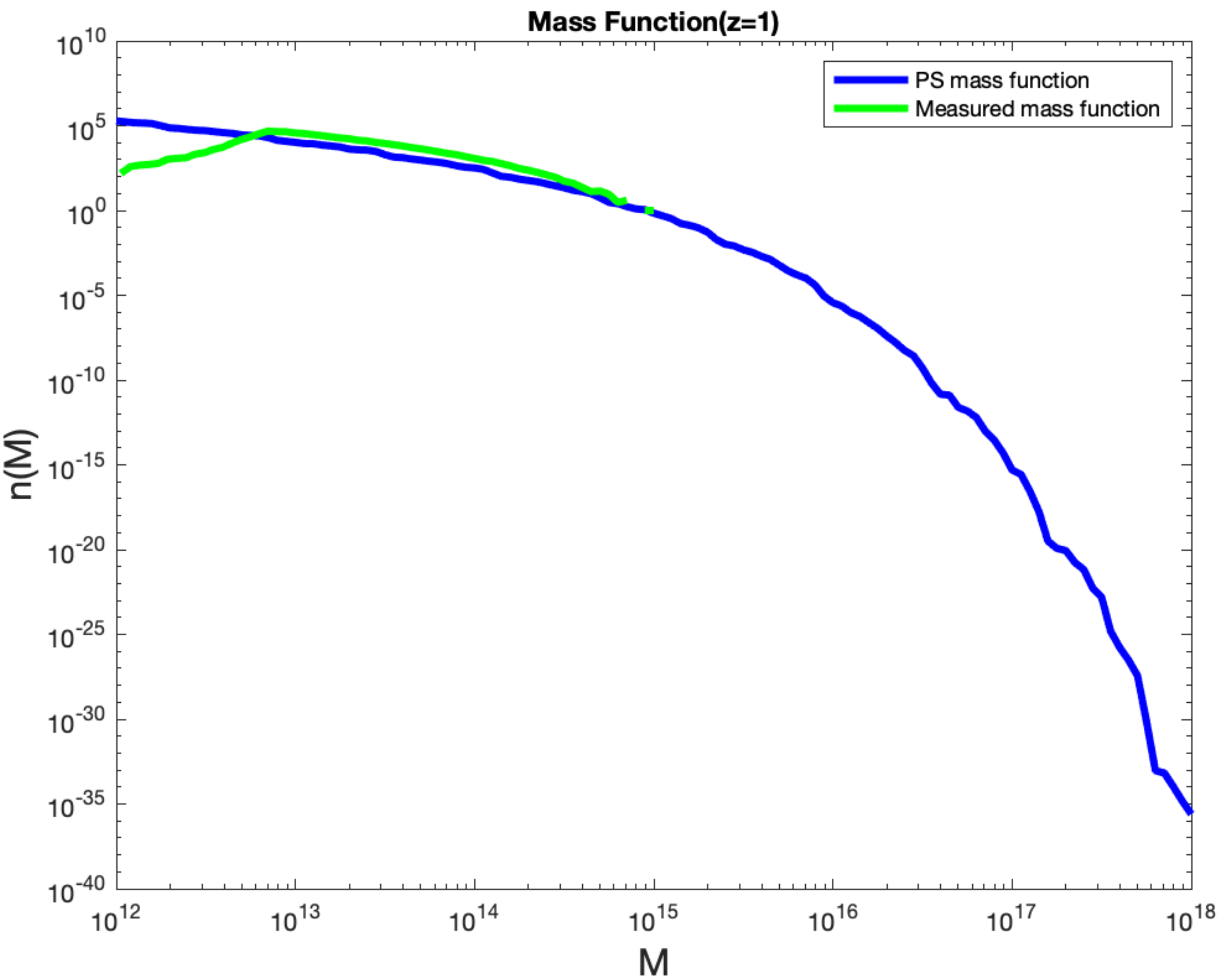

Sigma(M) z=0
sigma(M)
M
3.5
3
2.5
2
1.5
1
0.5
0
10^{11}
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
10^{18}



Sigma(M) z=1
sigma(M)
M
1.4
1.2
1
0.8
0.6
0.4
0.2
0
10^{11}
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
10^{18}



Mass Function(z=0)
PS mass function
Measured mass function
n(M)
M
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
10^{18}



Mass Function(z=1)
PS mass function
Measured mass function
n(M)
M
10^{10}
10^{5}
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-30}
10^{-35}
10^{-40}
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
10^{18}