PHYS 600 HW2

1.
Derivation of Friedmann equation II.

Friedmann I : $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{R_0^2 a^2}$

continuity : $\dot{\rho} + 3H(\rho+P) = 0$

Differentiate Friedmann I equation,

$$2\left(\frac{\dot{a}}{a}\right)\cdot\frac{\ddot{a}a-\dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} - \frac{K}{R_0^2}\cdot\frac{(-2)}{a^3}\dot{a}$$

$\because$ Plug in $\dot{\rho} = -3H(\rho+P)$

$$\therefore 2\frac{\ddot{a}}{a} - \cancel{2H^2} = -8\pi G(\rho+P) + \frac{16\pi G\rho}{3} - \cancel{2H^2}$$

$$\frac{\ddot{a}}{a} = -4\pi G(\rho+P) + \frac{8\pi G\rho}{3}$$

$$= \frac{-12\pi G\rho + 8\pi G\rho - 12\pi GP}{3}$$

$$= \frac{-4\pi G(\rho+3P)}{3}$$

$$\therefore \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho+3P)}$$

2.
Cosmological Dimming

(1) $f = \frac{L}{4\pi D_L^2}$, where D_L is the luminosity distance

L is fixed for a source,

$$M = -2.5\log_{10}\left(\frac{L/4\pi(10pc)^2}{f_0}\right)$$

$$M + DM(z) = -2.5\log_{10}\left(\frac{L/4\pi(10pc)^2}{f_0}\right) + 5\log_{10}\left(\frac{D_L(z)}{10pc}\right)$$

$$= -2.5\log_{10}\left(\frac{L/4\pi(10pc)^2}{f_0}\right) + 2.5\log_{10}\left(\frac{D_L(z)}{10pc}\right)^2$$

$$= -2.5 \log_{10}\left(\frac{L/4\pi(10pc)^2}{f_0} \cdot \frac{(10pc)^2}{D_L(z)^2}\right)$$

$$= -2.5 \log_{10}\left(\frac{L/(4\pi D_L(z)^2)}{f_0}\right)$$

$$= -2.5 \log_{10}\left(\frac{f}{f_0}\right)$$

$$= m$$

$\therefore \; m = M + DM(z)$

(2)

$\because$ the redshift contracts the entire spectrum in ν-direction by $(1+z)$, $\therefore$ the differential S_ν is scaled up by $(1+z)$ to keep the total energy constant.

$\because$ the observed ν is corresponding to the emitted $\nu(1+z)$, hence $L_{\nu(1+z)}$ in the expression.

The general expression for flux $f = \frac{L}{4\pi D_L^2}$ is the same.

Adding the modification, we have

$$S_\nu = (1+z)\frac{L_{\nu(1+z)}}{4\pi D_L^2} = (1+z)\frac{L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{4\pi D_L^2}$$

$\therefore \; M + DM + K$

$$= -2.5 \log_{10}\left(\frac{(1+z)L_{\nu(1+z)}/4\pi(10pc)^2}{f_0}\right)$$

$$+ 2.5 \log_{10}\left(\frac{D_L(z)}{10pc}\right)^2$$

$$+ 2.5 \log_{10}\left((1+z)\frac{L_{\nu(1+z)}}{L_\nu}\right)$$

$$= -2.5 \log_{10}\left[\frac{L_\nu/4\pi(10pc)^2}{f_0} \cdot \frac{(10pc)^2}{D_L(z)^2} \frac{(1+z)L_{\nu(1+z)}}{L_\nu}\right]$$

$$= -2.5 \log_{10}\left[\frac{(1+z)L_{\nu(1+z)}/4\pi D_L(z)^2}{f_0}\right]$$

$$= -2.5 \log_{10}\left[\frac{f_\nu}{f_0}\right]$$

$$= m$$

A static universe

(1)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$= \frac{8\pi G}{3}\left(\rho_m + \frac{\Lambda}{8\pi G}\right) - \frac{k}{a^2}$$

$\therefore$ referring to the first problem,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \frac{\Lambda}{8\pi G} + 3P\right)$$

$$P_m = 0, \quad P_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m + \rho_\Lambda - 3\rho_\Lambda)$$

$$= -\frac{4\pi G}{3}(\rho_m - 2\rho_\Lambda)$$

(2)

$\because$ $\dot{a} = 0, \ddot{a} = 0$

$\therefore$ $\frac{8\pi G}{3}\left(\rho_m + \frac{\Lambda}{8\pi G}\right) - \frac{k}{a^2} = 0$

$$\rho_m + \frac{\Lambda}{8\pi G} + 3P = 0$$

$\therefore$ $\rho_m - \frac{\Lambda}{4\pi G} = 0$

$$\Lambda = 4\pi G\rho_m$$

$$\frac{8\pi G}{3}\left(\rho_m + \frac{\rho_m}{2}\right) = \frac{k}{a^2}$$

$$\frac{8\pi G}{3}\frac{3\rho_m}{2} = \frac{k}{a^2}$$

$$k = 4\pi G\rho_m a^2$$

Because $\dot{a} = 0$, $a = \text{const}$.

$\therefore$ $\Lambda = 4\pi G\rho_m$, $k = 4\pi G\rho_m a^2$

At present time, $a = 1$

$$\Lambda = 4\pi G\rho_m, \quad k = 4\pi G\rho_m$$

(3)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \frac{\Lambda}{8\pi G} + 3P\right)$$

$$= -\frac{4\pi G}{3}\left(\rho_m - \frac{\Lambda}{4\pi G}\right)$$

$\because a(t) = 1 + \delta a(t)$

$\therefore \ddot{a}(t) = \delta\ddot{a}(t)$

$$\therefore \frac{\delta\ddot{a}(t)}{1+\delta a(t)} = -\frac{4\pi G}{3}\left(\rho_m - 3\rho_m\delta a(t) - \frac{\Lambda}{4\pi G}\right)$$

$$\frac{\delta\ddot{a}(t)}{1+\delta a(t)} - 4\pi G\rho_m\delta a(t) = -\frac{4\pi G\rho_m}{3} + \frac{\Lambda}{3}$$

$\because \rho_m = \frac{\Lambda}{4\pi G}$

$$\therefore \frac{\delta\ddot{a}}{1+\delta a} - \Lambda\delta a = -\frac{\Lambda}{3} + \frac{\Lambda}{3} = 0$$

$$\therefore \delta\ddot{a} = \Lambda(1+\delta a)\delta a \xrightarrow{\text{lowest order}} \delta\ddot{a} = \Lambda\delta a$$

(4)

$$\delta\ddot{a} = \Lambda\delta a + \Lambda\delta a^2$$

To lowest order,

$$\delta\ddot{a} = \Lambda\delta a$$

$$\therefore \delta a(t) = A\cos\left(\sqrt{\frac{1}{\Lambda}}t + \phi\right)$$

$\because \delta a(0) = \delta a_0$, $\left.\frac{d(\delta a)}{dt}\right|_{t=0} = 0$

$\therefore A\cos\phi = \delta a_0$

$-A\sqrt{\frac{1}{\Lambda}}\sin\phi = 0$

$\therefore \phi = 0$, $A = \delta a_0$

$$\therefore \delta a(t) = \delta a_0 \cos\left(\sqrt{\frac{1}{\Lambda}}t\right).$$

Redshift drift

$\because$ the frequency $\frac{\nu(t_1)}{\nu(t_0)} = 1+z$

$$\therefore \frac{dt_1}{dt_0} = \frac{\nu(t_0)}{\nu(t_1)} = \frac{1}{1+z}$$

$$\therefore \frac{dz}{dt_0} = \frac{\frac{da(t_0)}{dt_0}a(t_1) - \frac{da(t_1)}{dt_1}\frac{dt_1}{dt_0}a(t_0)}{a(t_1)^2}$$

$$= \frac{\frac{da(t_0)}{dt_0}a(t_1) - \frac{da(t_1)}{dt_1}\frac{1}{1+z}a(t_0)}{a(t_1)^2}$$

$$= \frac{\frac{da(t_0)}{dt_0}a(t_1) - \frac{da(t_1)}{dt_1}a(t_1)}{a(t_1)^2}$$

$$= \left(\frac{da(t_0)}{dt_0} - \frac{da(t_1)}{dt_1}\right)\frac{1}{a(t_1)}$$

$$= \frac{da(t_0)/dt_0}{a(t_0)}\frac{a(t_0)}{a(t_1)} - \frac{da(t_1)/dt_1}{a(t_1)}$$

$$= H_0(1+z) - H(t_1)$$

$$\therefore \frac{dz}{dt_0} = (1+z)H_0 - H(t_1)$$

As shown in class, $H(z) = H_0(1+z)^{\frac{3}{2}}$. when, $z=1$,

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{\frac{3}{2}}$$

$$= 2H_0 - 2\sqrt{2}H_0$$

$$= (2-2\sqrt{2})H_0$$

$$\approx -82.8h \text{ km/s/Mpc}$$

Cosmological Dimming

The surface brightness is defined by $\frac{f}{a}$, where f is the flux and a is the area of the source.
We know the observed flux is $\frac{1}{(1+z)^2}$ of the emitted because ① photon energy gets redshifted; ② arrival rate gets reduced by $\frac{1}{(1+z)}$.
Now, since $a \sim \theta^2$, where θ is the subtended angle, $\theta^2 \sim \frac{1}{(1+z)^2} d^2$ where d is the fixed size of the source.
$\therefore$ in summary, $I_0 = \frac{1}{(1+z)^4} I_e$.