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EE 290
Problem Set #1
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A: Perturbation bound analysis

1.

$$\|XY\|_F^2 = \sum_{i=1}^{k} \|Xy_i\|_2^2 \leq \|X\|_2 \sum_{i=1}^{k} \|y_i\|_2^2 = \|X\|_2 \|Y\|_F^2 \tag{1}$$

Apply this to $X = L^{-1}$ and $Y = A = LL^\top$ to find

$$\|L^{-1}LL^\top\|_F = \|L^\top\|_F = \|L\|_F \leq \|L^{-1}\|_2 \|A\|_F \tag{2}$$
$$\|L^{-1}\|_2^{-1} \|L\|_F \leq \|A\|_F \tag{3}$$

This is the first inequality we wanted to show. For the second, we use the previous identity on $\|A\|_F$ followed by the original identity on $\|A^{-1}\|_2 \|L\|_F$ to find

$$\|A\|_F \|A^{-1}\|_2 \geq \|L^{-1}\|_2^{-1} \|L\|_F \|A^{-1}\|_2 \tag{4}$$
$$\geq \|L^{-1}\|_2^{-1} \|A^{-1}L\|_F \tag{5}$$
$$= \|L^{-1}\|_2^{-1} \|(L^{-1})^\top L^{-1} L\|_F = \|L^{-1}\|_2^{-1} \|L^{-1}\|_F \tag{6}$$

2. Expanding the square, applying triange inequality, and using the original inequality

$$E = -A + (A + E) = LG^\top + GL^\top + GG^\top \tag{7}$$
$$\|E\|_F \leq 2\|LG^\top\|_F + \|GG^\top\|_F \leq 2\|L\|_2 \|G\|_F + \|G\|_F^2 \tag{8}$$

Applyin the quadratic formula to solve in terms of $\|G\|_F$ and selecting solution by non-negativity gives

$$\|G\|_F \geq -\|L\|_2 + \sqrt{\|L\|_2^2 + \|E\|_F} \tag{9}$$
$$\frac{\|G\|_F}{\|L\|_2} \geq \sqrt{1 + \|E\|_F / \|L\|_2^2} - 1 \tag{10}$$

Noting that $\|A\|_2 = \|LL^\top\|_2 \geq \|L\|_2^2$ by sub-multiplicativity of spectral norm, we have

$$\frac{\|G\|_F}{\|L\|_2} = \frac{\|E\|_F / \|A\|_2}{1 + \sqrt{1 + \|E\|_F / \|A\|_2}} \tag{11}$$

The second inequality with $\kappa(A)$ comes from applying sub-multiplicativity of Frobenius norm instead to get

$$\|E\|_F \leq 2\|L\|_F\|G\|_F + \|G\|_F^2 \tag{12}$$

$$\frac{\|G\|_F}{\|L\|_F} \geq \sqrt{1 + \|E\|_F/\|L\|_F^2} - 1 \tag{13}$$

and noting by the second inequality in problem (1) applied to $\kappa(A)$ and the first inequality in problem (1)

$$\kappa(A)\|A\|_F \geq \|L\|_F\|L^{-1}\|_2\|A\|_F \geq \|L\|_F^2 \tag{14}$$

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B: Orthogonally invariant norms

1. Consider $g(s(A)) = \|\operatorname{diag}(s(A))\| = \|\Sigma\|$. This is well defined by uniqueness of SVD and orthogonal invariance of $\|\cdot\|$. To see that this is a symmetric gauge function, any permutation matrix P is unitary hence

$$g(Ps(A)) = \|P\Sigma\| = \|\Sigma\| \tag{15}$$

again by orthogonal invariance.

2. $g(\cdot)$ being an absolute vector norm gives us everything except the triangle inequality. If we let g_* denote the Fenchel dual to the vector norm g, then we have

$$\begin{aligned} g(s(A+B)) &= g_{**}(s(A+B)) && (16)\\ &= \sup_{z:g_*(z)\leq 1} \langle z, s(A+B)\rangle && (17)\\ &= \sup_{C:g_*(s(C))\leq 1} \operatorname{Tr} C^\top(A+B) && (18)\\ &\leq \sup_{C:g_*(s(C))\leq 1} \operatorname{Tr} C^\top A + \sup_{C:g_*(s(C))\leq 1} \operatorname{Tr} C^\top B && (19)\\ &= g(s(A)) + g(s(B)) && (20) \end{aligned}$$

Orthogonal invariance follows from noting that

$$g(s(A)) = g_{**}(s(A)) = \sup_{C:g_*(s(C))\leq 1} \operatorname{Tr} C^\top A = \sup_{C:g_*(s(C))\leq 1} \sum_{i=1}^{j} \sigma_i(A)\sigma_i(C) \tag{21}$$

and hence the norm $\|A\| = g(s(A))$ only depends on the singular values.

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C: Maximum Clique

The task is to generalize from $G(n, 1/2)$ to $G(n, p)$, where we show $\frac{\omega(G_n)}{\log_2 n} \to 1/p$.

The upper deviation bound

$$\Pr[\omega(G_n) \geq (1/p + \varepsilon) \log_2 n] \to 0 \tag{22}$$

follows from the first moment method using $p^{\binom{k}{2}}$ in place of $2^{-\binom{k}{2}}$.

The lower deviation bound

$$\Pr[\omega(G_n) \geq k = (1/p - \varepsilon) \log_2 n] \to 1 \tag{23}$$

similarly follows by tracing the second moment method argument and replacing $2^{-2\binom{k}{2}+\binom{l}{2}}$ with $p^{2\binom{k}{2}-\binom{l}{2}}$. ∎

D: Weyl's inequality

1.

$$A = \sum_j \lambda_j(A) u_j u_j^\top \tag{24}$$

So any V will be spanned by a $\dim(V) = i$ subset of the eigenvectors $\{u_{V(j)}\}_{j=1}^{i}$ from which we find

$$\inf_{v \in V : \|v\|_2 = 1} v^\top A v = \inf_{v \in V : \|v\|_2 = 1} \sum_j \lambda_{V(j)}(A) \langle v, u_{V(j)} \rangle = \lambda_{V(i)}(A) \tag{25}$$

where the last equality is due to the infimum being attained when all of v is weighted on the smallest eigenvalue $\lambda_{V(i)}(A)$. To maximize the smallest eigenvalue, V must contain the top i eigenvalues (i.e. $V(i) = i$) and $\sup_{V:\dim(V)=i} \lambda_{V(i)} = \lambda_i$.

2.

$$\begin{aligned}
\|A - B\|_2 &= \left| \sup_{\dim(V)=1} \inf_{v \in V : \|v\|_2 = 1} v^\top (A - B) v \right| &(26)\\
&\geq \left| \sup_{\dim(V)=1} \inf_{v \in V : \|v\|_2 = 1} v^\top A v + \sup_{\dim(W)=1} \inf_{w \in W : \|w\|_2 = 1} (-w^\top B w) \right| &(27)\\
&\geq \left| \sup_{\dim(V)=i} \inf_{v \in V : \|v\|_2 = 1} v^\top A v + \sup_{\dim(W)=n-i} \inf_{w \in W : \|w\|_2 = 1} (-w^\top B w) \right| &(28)\\
&= \left| \lambda_i(A) - \inf_{\dim(W)=n-i} \sup_{w \in W : \|w\|_2 = 1} w^\top B w \right| &(29)
\end{aligned}$$

where the inequalities follow because we used a complement of the Courant-Fischer representation in the last line.

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E: Experiments for planted clique