STAT C206B: Topics in Stochastic Processes

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1 Lecture 1

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1.1 Ferguson distributions / Dirichlet processes

Definition 1 (*Gamma distribution*)

Random variable X supported on $(0,\infty)$ has *Gamma distribution* with shape $\alpha > 0$ and inverse scale / rate $\beta > 0$, written $X \sim \text{Gamma}(\alpha,\beta)$ if it has density

$$f_X(t) = \mathbb{1}\{t \in (0,\infty)\} \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \tag{1}$$

where $\Gamma(t) = \int_0^\infty u^{t-1} e^{-u} du$ is the Gamma function defined for all $\Re t > 0$ and analytically continued to $\mathbb{C} \setminus \{n \in \mathbb{Z} : n < 0\}$

Proposition 2 (*Gamma closed under summation*)

{prop:gamma-closed-sum} *If $Y \sim \text{Gamma}(\alpha,\beta)$ and $Z \sim \text{Gamma}(\gamma,\beta)$ are independent, then $Y + Z \sim \Gamma(\alpha+\gamma,\beta)$.*

Proof.

$$\begin{aligned}
f_{Y+Z}(t) &= \int_0^t f_Y(u) f_Z(t-u) du \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \beta^{\alpha+\gamma} e^{-\beta t} \int_0^t u^{\alpha-1} (t-u)^{\gamma-1} du \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \beta^{\alpha+\gamma} e^{-\beta t} \int_0^1 (tv)^{\alpha-1} (t-(tv))^{\gamma-1} t dv \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \beta^{\alpha+\gamma} e^{-\beta t} t^{\alpha+\gamma-1} B(\alpha,\gamma)
\end{aligned}$$

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where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the beta function □

A closely related distribution obtained from concatenating Gamma random variables into a vector and then normalizing the sum to 1 is the Dirichlet distribution.

Definition 3 (*Dirichlet distribution*)

Let $\boldsymbol{\alpha} \in (0,\infty)^K$. Random (probability) vector X taking values on the $K-1$-dimensional probability simplex $\Delta^{K-1} = \{\boldsymbol{x} \in [0,1]^K : \sum_i x_i = 1\}$ has *Dirichlet distribution* of order K and concentration parameters $\boldsymbol{\alpha}$, denoted $X \sim \mathrm{Dir}(\boldsymbol{\alpha})$, if it has density

$$f_X(\boldsymbol{x}) = \mathbb{1}\{\boldsymbol{x} \in \Delta\} \underbrace{\frac{\Gamma\left(\sum_{i=1}^K \alpha_i\right)}{\prod_{i=1}^K \Gamma(\alpha_i)}}_{=:B(\boldsymbol{\alpha})^{-1}} \prod_{i=1}^K x_i^{\alpha_i - 1}$$

Proposition 4 (*Constructing Dirichlet from Gammas*)

{prop:dirichlet-from-gamma} *Let $X_1, \ldots, X_n$ be independent* $\mathrm{Gamma}(\alpha_i, \beta)$ *distributed, $S_n = \sum_{i=1}^n X_i$. Then $(V_i)_i = (X_i/S_n)_i \sim$* $\mathrm{Dir}(\boldsymbol{\alpha})$.

Proof. $S_n \sim \Gamma(\sum_i^n \alpha_i, \beta)$ by Proposition 2 and for $\boldsymbol{v} \in \Delta^{n-1}$, we have

$$\begin{aligned}
f_V(\boldsymbol{v}) &= \int_0^\infty f_X\left(sv_1, \ldots, sv_{n-1}, sv_n\right) f_{S_n}(s) ds \\
&= \int_0^\infty e^{-\sum_{i=1}^n sv_i} \left(\prod_{i=1}^n \frac{(sv_i)^{\alpha_i - 1}}{\Gamma(\alpha_i)}\right) \frac{s^{\sum_i^n \alpha_i - 1} e^{-s}}{\Gamma(\sum_i^n \alpha_i)} ds \\
&= \frac{1}{\prod_1^n \Gamma(\alpha_i)} \prod_{i=1}^n v_i^{\alpha_i - 1} \int_0^\infty e^{-s\cancelto{1}{\sum_1^n v_i}} s^{(\sum_1^n \alpha_i) - 1} ds \\
&= \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_1^n \Gamma(\alpha_i)} \prod_{i=1}^n v_i^{\alpha_i - 1}
\end{aligned}$$

□

Similar to Proposition 2 (Gamma closed under summation), where adding two Gammas yielded another Gamma where the parameters were added, Dirichlet distributions enjoy a similar kind of closure: "clumping" coordinate axes together (described below) yields another Dirichlet distribution where the parameters of the clumped axes are summed together.

Proposition 5 (*Dirichlet clumping property*)

Suppose $X \sim$ $\mathrm{Dir}(\alpha_1, \ldots, \alpha_n)$. *For any $r \le n$, let $V_i = X_i$ for $i \in [r]$ and let $V_{r+1} = \sum_{j=r+1}^n X_j$. Then $V \sim$* $\mathrm{Dir}(\alpha_1, \ldots, \alpha_r, \sum_{j=r+1}^n \alpha_j)$.

Proof. By induction, it suffices to show this for $r = n-2$. Notice

$$\begin{aligned}
f(v_1, \ldots, v_r, s) &= B(\boldsymbol{\alpha})^{-1} \left(\prod_{i=1}^{n-1} v_i^{\alpha_i - 1}\right) \int \mathbb{1}\left\{x_{n-1} + x_n = s\right\} x_{n-1}^{\alpha_{n-1}-1} x_n^{\alpha_n - 1} dx_{n-1} dx_n \\
&= B(\boldsymbol{\alpha})^{-1} \left(\prod_{i=1}^{n-1} v_i^{\alpha_i - 1}\right) \int_0^s u^{\alpha_{n-1}-1} (s-u)^{\alpha_n - 1} du \\
&= B(\boldsymbol{\alpha})^{-1} \left(\prod_{i=1}^{n-1} v_i^{\alpha_i - 1}\right) s^{\alpha_{n-1} + \alpha_n - 1} B(\alpha_{n-1}, \alpha_n)
\end{aligned}$$

Since $\frac{B(\alpha_{n-1},\alpha_n)}{B(\boldsymbol{\alpha})} = \frac{\Gamma(\sum_1^n \alpha_i)}{\Gamma(\alpha_{n-1}+\alpha_n)\prod_1^{n-2}\Gamma(\alpha_i)}$, we are done. □

Iterating this result over coordinate axes other than the last $n-r$, we see that "clumping together" entries in a Dirichlet random vector yields another Dirichlet random vector with parameters also "clumped together." Concretely, for any mapping $\phi : [n+1] \to [m+1]$ if $U_j = \sum_{\phi(i)=j} V_i$ then U has Dirichlet distribution with parameters $\gamma_j = \sum_{\phi(i)=j} \alpha_i$.

Generalizing this clumping property is the motivation for *Ferguson Distributions* [Fer73].

Definition 6 (*Ferguson / Dirichlet process distribution*)

Let μ be a finite positive Borel measure on complete separable metric space E. A random probability measure μ^* on E (i.e. a stochastic process indexed by a σ-algebra on E) has *Ferguson distribution with parameter* μ if for every finite partition $(B_i)_{i\in[r]}$ of E the random vector

$$(\mu^*(B_i))_{i\in[r]} \sim \mathrm{Dir}(\mu(B_1), \dots, \mu(B_r))$$

Lemma 7 (*Preservation of Ferguson under pushforward*)

Let μ^* be Ferguson with parameter μ and $\phi : E \to F$ measurable. Then the pushforward $\mu^* \circ \phi^{-1}$ is a random probability measure on F that has Ferguson distribution with parameter $\mu \circ \phi^{-1}$.

Proof. For $(B_i)_{i\in[r]}$ a finite partition of F, $(\phi^{-1}(B_i))_i$ is a finite partition of E. Since μ^* is Ferguson

$$(\mu^*(\phi^{-1}(B_i)))_i \sim \mathrm{Dir}((\mu(\phi^{-1}(B_i)))_i)$$

Hence $\mu^* \circ \phi^{-1}$ is Ferguson with parameter $\mu \circ \phi^{-1}$. □

1.1.1 Connection to Polya urns

Next, we turn to an important relation between Ferguson distributions and a generalized Pólya urn. Our presentation draws from material originally in [BM73].

Definition 8 (*Polya sequence*)

A sequence $(X_n)_{n\in\mathbb{N}}$ with values in E is a *Polya sequence with parameter* μ if for all $B \subset E$.

$$\Pr[X_1 \in B] = \mu(B)/\mu(E)$$
$$\Pr[X_{n+1} \in B \mid X_1, \dots, X_n] = \mu_n(B)/\mu_n(E)$$

where $\mu_n = \mu + \sum_{i=1}^{n} \delta_{X_i}$.

Remark 9. When E is finite (e.g. a set of colors for the balls), (X_n) represents the result of successive draws from an urn with initially $\mu(x)$ balls of color $x \in E$ and after each draw a ball of the same color as the one drawn is added back to give an urn with color distribution $\mu_{n+1}(x)$.

[BM73] gives the following result connecting Pólya sequences and Ferguson distributions.

Theorem 10 (*Polya Urn Schemes*)

Let (X_n) be a Polya sequence with parameter μ. Then:

1. *$m_n = \mu_n/\mu_n(E)$ converges almost surely to a limiting discrete measure μ^**
2. *μ^* has Ferguson distribution with parameter μ*
3. *Given μ^*, $(X_i)_{i\geq 1}$ are independent with distribution μ^**

Proof. First consider E finite and let μ^* and $\{X_i\}$ be random variables whose joint distribution satisfies (2.) and (3.). Let π_n be empirical distribution of $(X_i)_{i\in[n]}$. $X_i \overset{\text{iid}}{\sim} \mu^*$, so by SLLN $\pi_n \overset{as}{\to} \mu^*$ and since

$$m_n = \frac{\mu + n\pi_n}{\mu(E) + n} \tag{2}$$

(1.) follows.

It remains to show (X_n) is a Polya sequence with parameter μ, i.e.

$$\Pr[A] = \prod_x \mu(x)^{[n(x)]}/\mu(E)^{[n]} \tag{3}$$

{{eq:polya-seq-meas}}

where $A = \{X_i = x_i\}_i$, $n(x) = \#\{i : x_i = x\}$, and the rising factorial $a^{[k]} = a(a+1)\cdots(a+k-1)$. Notice

$$\Pr[A] = \mathbb{E}\left[\Pr[A \mid \mu^*]\right] = \mathbb{E}\left[\prod_x \mu^*(x)^{n(x)}\right] \tag{4}$$

Since μ^* is Ferguson, viewing $E = \sqcup_{x\in E}\{x\}$ as a partition we have $(\mu^*(x))_{x\in E} \sim \mathrm{Dir}((\mu(x))_{x\in E})$ so the RHS is the $(n(x))_{x\in E}$ moment of the Dirichlet distribution, which is equal to

$$\mathbb{E}\left[\prod_x \mu^*(x)^{n(x)}\right] = \frac{\Gamma(\mu(E))}{\Gamma(\mu(E)+n)} \prod_x \frac{\Gamma(\mu(x)+n(x))}{\Gamma(\mu(x))} = \frac{1}{\mu(E)^{[n]}} \prod_x \mu(x)^{[n(x)]} \tag{5}$$

as required by Eq. (3).

General E follows from approximation argument. □

We leave the discreteness part of (1.) as an exercise, noting that similar to how Dirichlets can be defined as a set of independent Gammas normalized by their sum (Proposition 4 (Constructing Dirichlet from Gammas)) we would expect the Dirichlet process / Ferguson random measures to be definable as a gamma process with independent "increments" divided by their sum.

Exercise 11. Prove every Ferguson random measure is discrete. (Hint: argue using moments).

Remark 12. If (X_n) a Polya sequence, then it is a mixture of iid sequences, and is exchangeable i.e. $(X_i) \overset{d}{=} (X_{\sigma(i)})$ (see Eq. (3))

1.2 Invariant measures and Kakutani's Theorem

For a finite group G, the measure $\mu(g) = \frac{1}{\#G}$ is left and right translation invariant i.e. $\mu(gA) = \mu(A) = \mu(Ag)$.

In fact, all compact groups have unique translation invariant measure called the Haar measure. For example $Z_{ij} \overset{\text{iid}}{\sim} N(0,1)$ for $i,j \in [n]$ and X the Gram-Schmidt orthonormalizataino of the rows of Z. Then $XU \overset{d}{=} UX$ for all $U \in O(n)$, so X has Haar measure on compact group $O(n)$.

Definition 13

A *topological vector space* (TVS) is a vector space equipped with a topology such that vector space operations are jointly continuous.

Example 14

$\mathbb{R}^n$ with standard topology, any Banach space.

Definition 15

{def:equicontinuous}

A family $\mathfrak{G}$ of linear transformations on TVS $\mathfrak{X}$ is *equicontinuous on subset* $K \subset \mathfrak{X}$ if for every neighborhood V of the origin, there exists a neighborhood U of the origin such that

$$\forall k_1, k_2 \in K : k_1 - k_2 \in U \Rightarrow \mathfrak{G}(k_1 - k_2) \subset V \tag{6}$$

That is, $T(k_1 - k_2) \in V$ for all $T \in \mathfrak{G}$.

Definition 16

A *locally convex topological vector space* (LCTVS) is a TVS with a local base of absolutely convex absorbing sets at the origin.

To construct Haar measure for any compact group, we will need a fix point theorem due to Kakutani.

Theorem 17 (*Kakutani Fix Point Theorem*)

{thm:kakutani}

K compact convex subset of LCTVS $\mathfrak{X}$, $\mathfrak{G}$ group of linear transforms equicontinuous on K and such that $\mathfrak{G}(K) \subset K$, then $\exists p \in K$ such that

$$\mathfrak{G}(p) = \{p\} \tag{7}$$

Proof.

- By Zorn's lemma applied to chains $(K_\lambda)_\lambda$ (note $K_a \subset K_b$ for $a < b$), $\exists$ minimal $K_1 \subset K$ such that $K_1 \neq \emptyset$ and $\mathfrak{G}(K_1) \subset K_1$.
- If K_1 is a single point, then proof is complete.
- Otherwise, by minimality the compact (becacuse $-$ is continuous) set $K_1 - K_1$ contains a point other than the origin, so exists $V \in N(0)$ such that $\bar{V} \not\supset K_1 - K_1$.
- For some $|\alpha| \leq 1$, there is a convex neighborhood $V_1 \in N(0)$ such that $\alpha V_1 \subset V$.
- By equicontinuity of $\mathfrak{G}$ on $K \supset K_1$, there is $U_1 \in N(0)$ such that for $k_1, k_2 \in K$ and $k_1 - k_2 \in U_1$ we have $\mathfrak{G}(k_1 - k_2) \subset V_1 >$
- Because $T \in \mathfrak{G}$ is invertible ($\mathfrak{G}$ is a group), T maps open sets to open sets and $T(A \cap B) = TA \cap TB$ for sets A, B.
- Since T is linear, for any A
$$T \operatorname{conv}(A) = \operatorname{conv}(TA) \tag{8}$$
- So
$$U_2 := \operatorname{conv}(\mathfrak{G}U_1 \cap (K_1 - K_1)) = \operatorname{conv}(\mathfrak{G}(U_1 \cap (K_1 - K_1))) \subset V$$
is relatively open in $K_1 - K_1$ and satisfies $\mathfrak{G}U_2 = U_2 \not\supset K_1 - K_1$.
- By continuity, $\mathfrak{G}U_2 = \overline{\mathfrak{G}U_2}$.
- Let $\delta = \inf\{a : a > 0, aU_2 \supset K - 1 - K_1\} \geq 1$, by compactness $\delta < \infty$. Let $U := \delta U_2$.
- For each $\varepsilon \in (0, 1)$
$$(1 + \varepsilon)U \supset K_1 - K_1 \not\subset (1 - \varepsilon)\overline{U}$$
- Because $(1 - 1/4n)\bar{U} \not\supset K_1 - K_1$, we have $K_2 \neq K - 1$.
- K_2 is closed and convex
- Further, since $T(a\bar{U}) \subset a\bar{U}$ for $T \in \mathfrak{G}$, we have
$$T(a\bar{U} + k) \subset a\bar{U} + Tk \qquad \text{forall } T \in \mathfrak{G}, k \in K_1$$
- Recalling $TK_1 \subset K_1$ for $T \in \mathfrak{G}$ and that $\mathfrak{G}$ is a group, we find that $TK_1 = K_1 \mathfrak{G} K_2 =$ a contradiction Finish

∎

2 Lecture 2: Measure theory

2020-01-23

Throughout our discussion, all topological spaces are assumed Hausdorff unless explicitly noted otherwise.

2.1 Construction of Haar measure

Definition 18

A *topological group* is a group equipped with a topology such that the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous.

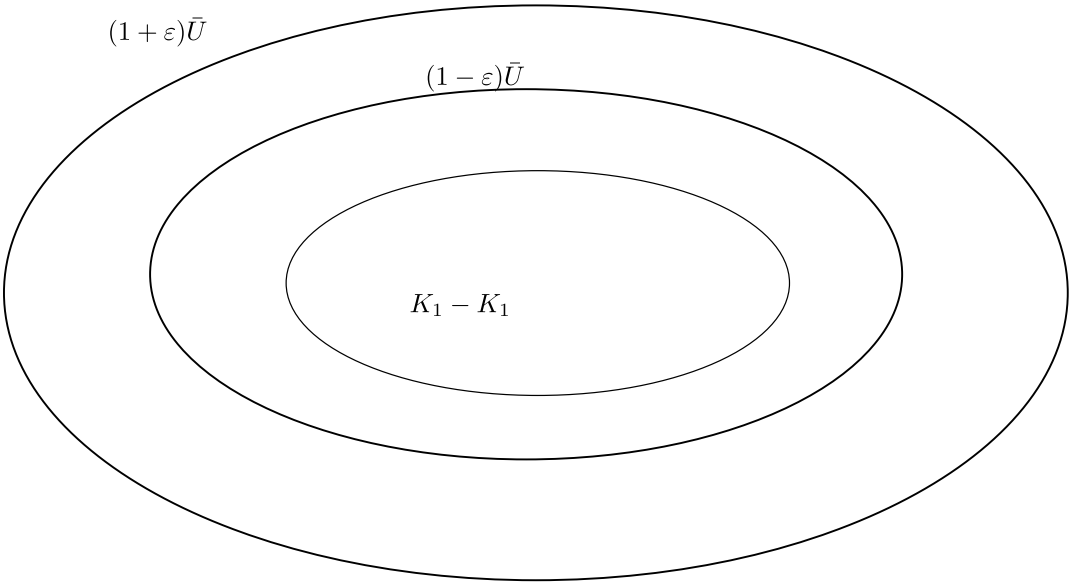


Figure 1: Sketch of proof of Kakutani's theorem

Theorem 19 (*Existence of Haar Measure*)

{thm:haar-measure} *Let G be a compact topological group and $\mathcal{C}(G)$ the set of continuous maps $G \to \mathbb{R}$. Then there is a unique linear form $m : \mathcal{C}(G) \to \mathbb{R}$ such that*

1. *$m(f) \geq 0$ for $f \geq 0$ (positive)*
2. *$m(\mathbb{1}) = 1$ (normalized)*
3. *$m({}_s f) = m(f)$ where ${}_s f(g) = f(s^{-1}g)$ for $s, g \in G$ (left invariant)*
4. *$m(f_s) = m(f)$ where $f_s(g) = f(gs)$ (right invariant)*

m is called the Haar measure *on G.*

We will need the following theorem to relate compactness with equicontinuity:

Theorem 20 (*Generalization of Arzela-Ascoli*)

{thm:arzela-ascoli} *Let X be a compact Hausdorff space. A subset of $\mathbb{R}$-valued continuous functions $F \subset \mathcal{C}(X)$ is relatively compact in topology induced by uniform norm $\|\cdot\|_\infty \Leftrightarrow F$ is equicontinuous and pointwise bounded.*

Proof of Theorem 19. Fix $f \in \mathcal{C}(G)$ and let $\mathcal{C}_f$ denote the convex hull of all left translates of f, i.e. $g \in \mathcal{C}_f$ are finite sums of form

$$g(x) = \sum_{\text{finite}} a_i f(s_i x), \qquad a_i > 0, \sum_{\text{finite}} a_i = 1, s_i \in G$$

Clearly $\|g\|_\infty \leq \|f\|_\infty < \infty$, thus $\mathcal{C}_f(x) = \{g(x) : g \in \mathcal{C}_f\}$ is bounded for all $x \in G$ hence $\mathcal{C}_f$ is pointwise bounded.

As f is a continuous function on compact G, it is uniformly continuous hence for $\varepsilon > 0$ there exists a neighborhood V_ε of the identity $e \in G$ such that

$$y^{-1}x \in V_\varepsilon \Rightarrow |f(x) - f(y)| \leq \varepsilon$$

Since $(s^{-1}y)^{-1}s^{-1}x = y^{-1}x$, we also have

$$y^{-1}x \in V_\varepsilon \Rightarrow |{}_s f(y) - {}_s f(x)| < \varepsilon$$

Since $g \in \mathcal{C}_f$ are convex combinations of ${}_s f$, by the triangle inequality

$$y^{-1}x \in V_\varepsilon \Rightarrow |g(y) - g(x)| < \varepsilon$$

As this works for any $g \in \mathcal{C}_f$, we have that $\mathcal{C}_f$ is equicontinuous.

By Theorem 20 (Generalization of Arzela-Ascoli), $\mathcal{C}_f$ is relatively compact in $\mathcal{C}(G)$, so its closure $K_f \coloneqq \overline{\mathcal{C}_f}$ is compact (and still convex).

Consider G acting on $\mathcal{C}(G)$ by left translation $f \mapsto {}_sf$. Notice $G\mathcal{C}_f \subset \mathcal{C}_f$ (as $\mathcal{C}_f$ already contains all finite convex combinations of all left translations of f) and hence $GK_f \subset K_f$ as well.

Furthermore, $\| {}_sf - {}_sg\|_\infty = \|f - g\|_\infty$ so G acts as a group of isometries on $\mathcal{C}(G)$. In particular, this group is equicontinuous (with the same $U = V$ in Definition 15).

Taking $\mathfrak{G} = G$ and $K = K_f$ in Theorem 17 (Kakutani Fix Point Theorem), there is a fixed point $g \in K_f$ of this action of G on K_f which satisfies

$$ {}_sg = g \ (\forall s \in G) \quad \Rightarrow \quad g(s^{-1}) = {}_sg(e) = g(e) = c \ (\forall s \in G) $$

for some constant $c \in \mathbb{R}$ (which we will later use to define $m(f) \coloneqq c$).

We first show there is only one constant function in K_f, so the fix point $Gg = \{g\} = \{c\mathbb{1}\}$ is unique and $m(f) = c$ is well defined. For any constant function $c\mathbb{1} \in K_f$ and $\varepsilon > 0$, we can (because $K_f = \overline{\mathcal{C}_f}$) find $\{s_1, \dots, s_n\} \subset G$ and $a_i > 0$ such that

$$ \sum_{i=1}^{n} a_i = 1, \quad \text{and} \quad \left| c - \sum_{i=1}^{n} a_i f(s_i x) \right| < \varepsilon \qquad (\forall x \in G) \tag{9} $$ {{eq:combo-close-to-constant}}

for any $\varepsilon > 0$.

Similarly, consider the same construction as before expect now use right translations of f (i.e. using the opposite group G' of G, or the function $f' = f(x^{-1})$, obtaining relatively compact set $\mathcal{C}'_f$ with compact convex closure K'_f with fix point $g' = c'\mathbb{1}$). Approximating $c'\mathbb{1}$ using $\mathcal{C}'_f$, we have

$$ \left| c' - \sum_j b_j f(xt_j) \right| < \varepsilon \qquad (\text{for some } t_j \in G,\, b_j > 0 \text{ with } \sum_j b_j = 1) \tag{10} $$

Opposite group

The opposite group g' of the group G is the group that coincides with G as a set but has group operation $(x, y) \mapsto y^{-1}x^{-1}$

Summing over i

$$ \left| c' - \sum_{i,j} a_i b_j f(s_i t_j) \right| < \varepsilon \sum_i a_i = \varepsilon $$

Operating symmetrically on Eq. (9) (multiply by b_i and put $x = t_i$) shows

$$ \left| c - \sum_{i,j} a_i b_j f(s_i t_j) \right| < \varepsilon $$

Together, we have $|c' - c| < 2\varepsilon$ so taking $\varepsilon \to 0$ shows $c' = c$. Since $c\mathbb{1} \in K_f$ was an arbitrary constant function, we have that the constant function in K_f is actually unique and so the function $m(f) \coloneqq c \in K_f$ is well defined. Moreover, $m(f)\mathbb{1}$ is the *only* constant function which can be arbitrary well approximated by convex combinations of left or right translates of f.

The following properties are obvious:

- $m(\mathbb{1}) = 1$ since $K_f = \{1\}$ for $f = \mathbb{1}$
- $m(f) \geq 0$ if $f \geq 0$
- $m({}_sf) = m(f) = m(f_s)$ (since $K_{{}_sf} = K_f$, $K'_f = K'_{f_s}$, and uniqueness of $m(f)\mathbb{1}$ being the only constant function approximable by both K_f and K'_f)

- $m(af) = am(f)$ for any $a \in \mathbb{R}$ (since $K_{af} = K_f$)

To show m is linear, it suffices (due to the last bullet above) to show that m is additive. Fix $f, g \in \mathcal{C}(G)$. Approximate $m(f)$ using K_f to get

$$\left| m(f) - \sum_{\text{finite}} a_i f(s_i x) \right| \tag{11}$$

{{eq:approx-mf-using-Kf}}

Define $h(x) = \sum_{\text{finite}} a_i g(s_i x)$ using the same a_i and s_i and approximate $m(h)$ using $\mathcal{C}_h$ to get

$$\left| m(h) - \sum_{\text{finite}} b_j h(t_j x) \right| < \varepsilon$$

Since $h \in \mathcal{C}_g$, we have $\mathcal{C}_h \subset \mathcal{C}_g$ hence $K_h \subset K_g$. But $m(g)\mathbb{1} \in K_g$ is the only constant function so $m(h) = m(g)$ and (after expanding the definition of h) we have

$$\left| m(g) - \sum_{i,j<\infty} a_i b_j g(s_i t_j x) \right| < \varepsilon$$

On the other hand, multiplying Eq. (11) by b_j replacing x with $t_j x$, summing over j, and finally adding with the above ineqeuality gives

$$|m(f) + m(g) - \sum_{i,j} a_i b_j (f+g)(s_i t_j x)| < 2\varepsilon$$

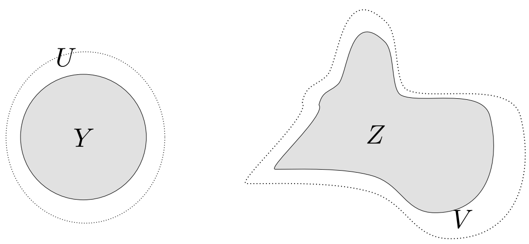
Thus $m(f) + m(g) \in K_{f+g}$, establishing additivity. Note that the only constant in K_{f+g} is $(m(f) + m(g))\mathbb{1}$. □

2.2 Facts from topology

We now want to head towards some integration against probability measures defined on spaces more abstract than $\mathbb{R}^n$.

Definition 21

A topological space X is *normal* if for any disjoint closed sets Y and Z there exists disjoint open sets U and V such that $Y \subset U$ and $Z \subset V$.



{fig:normal-topological-space}

Figure 2: Normal topological spaces admit separating closed sets with two disjoint open sets

Definition 22

X is *completely regular* (*Tychonoff* if X is also Hausdorff) if for all $y \in X$ and every closed $Z \subset X \setminus \{y\}$ there exists $f : X \to [0,1]$ continuous such that $f(y) = 0$ and $f(z) = 1$ for all $z \in Z$. We say y and Z are separated by a (Urysohn) function.

Corollary 23 (*Urysohn's Lemma*)

Every normal space is completely regular.

Lemma 24

A compact (Hausdorff) space is normal hence completely regular.

Proof. Fix disjoint closed Y and Z and let $y \in Y$. Consider the open cover of Z given by $\{V_{y,z} : z \in Z\}$ where each $V_{y,z} \in N(z)$ is disjoint from some $U_{y,z} \in N(y)$ (existence ensured by Hausdorff). By compactness, there exists a finite subcover $\{V_{y,z_i}\}_{i=1}^n$. For each of these V_{y,z_i}, let $U_{y,z_i} \in N(y)$ denote the corresponding disjoint neighborhood of y and consider

$$U'_y = \bigcap_{i=1}^n U_{y,z_i} \in N(y)$$

U'_y is open because it is the intersection of finitely many open sets. It is also disjoint from

$$V'_y \coloneqq \bigcup_{i=1}^n V_{y,z_i}$$

which contains B and is also open.

Now consider the open cover $\{U'_y : y \in Y\}$, let $\{U'_{y_i}\}_{i=1}^n$ be a finite subcover, and let $U = \cup_{i=1}^n U'_{y_i}$. Analogously, let $V = \cap_{i=1}^n V'_{y_i}$ where V'_y is given above (open cover of B and disjoint from U'_y). Then $U \supset Y$ and $V \supset Z$ provide two disjoint separating open sets. □

Lemma 25

{lem:completely-regular-equals-initial-topo-cts} *A topological space (X, τ) is completely regular (i.e. Tychonoff) space iff the original topology coincides with the initial topology $\sigma(X, \mathcal{C}(X))$ i.e. the smallest topology that makes every function in $\mathcal{C}(X)$ continuous.*

Proof. We only show $\Rightarrow$. Let U be τ-open and for $x \in U$ pick an Urysohn function $f \in \mathcal{C}(X)$ such that $f(x) = 0$ and $f(U^c) = 1$. Then $V_x = \{y : f(y) < 1\} = f^{-1}((-\infty, 1))$ is a $\sigma(X, \mathcal{C}(X))$-open neighborhood of x contained in U, so $U = \cup_{x \in U} V_x$ is $\sigma(X, \mathcal{C}(X))$-open. Since $\sigma(X, \mathcal{C}(X))$ is minimal, we have $\tau = \sigma(X, \mathcal{C}(X))$. □

2.3 Radon, Borel, and Baire measures

Definition 26

A non-negative set function $m : 2^X \to [0, +\infty]$ on X is an *outer measure on* X (or Carathéodory outer measure) if:

1. $m(\emptyset) = 0$
2. $A \subset B \Rightarrow m(A) \leq m(B)$ (monotone)
3. $m\left(\cup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty m(A_n)$ for all $A_n \subset X$. (countable subadditivity)

Definition 27

Let $m : 2^X \to [0, +\infty]$ be a non-negative set function satisfying $m(\emptyset) = 0$. A set $A \subset X$ is *Carathéodory measurable wrt* m (Carathéodory m-measurable) if for any $E \subset X$

$$m(E) = m(E \cap A) + m(E \setminus A)$$

We use $\mathfrak{M}_m$ to denote the class of all Carathéodory m-measurable sets.

Theorem 28 (*Carthéodory construction*)

1. *$\mathfrak{M}_m$ is an algebra, m is additive on $\mathfrak{M}_m$*

2. *(Finite additivity) For all sequences of pairwise disjoint $A_i \in \mathfrak{M}_m$ and any $E \subset X$*

$$m\left(E \cap \bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(E \cap A_i)$$
$$m\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(E \cap A_i) + \lim_{n\to\infty} m\left(E \cap \bigcup_{i=n}^{\infty} A_i\right)$$

3. *If m is an outer measure on X, then $\mathfrak{M}_m$ is a σ-algebra, m is countably additive on $\mathfrak{M}_m$, and m is complete on $\mathfrak{M}_m$*

Remark 29. The outer measure is constructed such that it satisfies countable additivity on the measurable sets $\mathfrak{M}_m$.

Example 30

Let $\mathfrak{X}$ be a family of subsets of X such that $\emptyset \in \mathfrak{X}$. Given $\tau : \mathfrak{X} \to [0, +\infty]$ with $\tau(\emptyset) = 0$, set

$$m(A) = \inf\left\{\sum_{n=1}^{\infty} \tau(X_n) : X_n \in \mathfrak{X}, A \subset \cup_{n=1}^{\infty} X_n\right\}$$

where $\mathfrak{M}(A) = \infty$ in the absence of such sets X_n. Then m is an outer measure, denoted τ^*.
This is where the "outer" comes from: $\cup_n X_n \supset A$ is an outer approximation to A using (potentially overlapping) sets from $\mathfrak{X}$ hence $\sum_{n=1}^{\infty} \tau(X_n)$ is an overapproximation to the "size" of A. $m(A)$ is the best (i.e. smallest) overapproximation.

Recall the *Borel σ-algebra*, denoted $\mathcal{B}(X)$, is generated by all open sets.

Definition 31

The *Baire σ-algebra*, denoted by $\mathcal{B}a(X)$, is generated by sets of the form

$$\{x \in X : f(x) > 0\} \tag{12}$$

{{eq:functionally-open}}

where $f \in \mathcal{C}(X)$ (called *functionally open sets*).

Remark 32. $\mathcal{B}a(X)$ is the smallest σ-algebra where every $f \in \mathcal{C}(X)$ is measurable. It coincides (via a truncation and monotonicity argument) to the smallest one making every $f \in \mathcal{C}_b(X)$ measurable. Contrast this to Lemma 25, which shows that completely regular spaces are those with the smallest topology where every $f \in \mathcal{C}(X)$ is continuous.

Remark 33. Since the functionally open sets can be written as $f^{-1}((0, \infty))$ for continuous f, they are also Borel sets. Therefore, the class of Baire sets are contained in the class of Borel sets.

Lemma 34

{lem:metric-space-closed-set-variety}

In a metric space (X, d), any closed set S is the set of zeros of a continuous function (namely $d_S(x) = \inf_{s\in S} d(x, s)$). Hence, $\mathcal{B}(X) = \mathcal{B}a(X)$.

Lemma 35 (*Baire sets are countably determined*)

Every $A \in \mathcal{B}a(X)$ is determined by some countable family of functions, i.e. has the form

$$A = \{x : (f_i(x))_{i=1}^{\infty} \in B\} \quad \text{for some } f_i \in \mathcal{C}(X), B \in \mathcal{B}(\mathbb{R}^{\aleph_0})$$

Moreover, every set of this form is Baire and we can take $f_i \in \mathcal{C}_b(X)$.

Proof. We first show every set of the same form as A is Baire. True if B is closed, since Lemma 34 allows us to write $B = \phi^{-1}(0)$ for some continuous function $\phi : \mathbb{R}^{\aleph_0} \to \mathbb{R}$ so $\psi = x \mapsto \phi((f_n(x))_{n\geq 1})$ is continuous hence $A = \psi^{-1}(0)$ is also closed. But this is the converse.

For any fixed $\{f_n\}_{n\geq 1}$, the class of sets $B \in \mathcal{B}(\mathbb{R}^{\aleph_0})$ satisfying

$$\{x : (f_i(x))_{i\geq 1} \in B\} \in \mathcal{B}a(X)$$

is a σ-algebra containing $B = \prod_i(-\infty, a_i)$ where $a_i \neq \infty$ for only finitely many i. This is a basis for $\mathcal{B}(\mathbb{R}^{\aleph_0})$, thus $\mathcal{B}a(X)$ contains it and the two coincide (recall $\mathcal{B}a \subset \mathcal{B}$ since functionally determined sets are $\mathcal{B}$-open).

On the other hand??? □

A consequence of the monotone class theorem ???

Throughout, we consider (signed) measures of *bounded variation* unless explicitly denoted otherwise.

Definition 36

Let X be a topological space.

- A countably additive measure on $\mathcal{B}(X)$ is called a *Borel measure*
- A countably additive measure on $\mathcal{B}a(X)$ is called a *Baire measure*
- A Borel measure μ on X is called *Radon measure* if every $B \in \mathcal{B}(X)$ can be approximated from the inside by compact sets: for $\varepsilon > 0$ exists $K_\varepsilon \subset B$ such that $|\mu|(B \setminus K_\varepsilon) < \varepsilon$.

When are two Borel measures equal?

Lemma 37

If two Borel measures coincide on all open sets, thne they coincide on all Borel sets.

Proof. Split $\mu = \mu^+ - \mu^-$ and notice that each of the two components are nonnegative and coincide on open sets. By monotone class theorem, $\mu^+ = \mu^-$. □ finish

μ is Radon iff $|\mu|$ is Radon iff both μ^+ and μ^- are Radon.

Inner and outer approximatino of measures on $\mathbb{R}^n$:

Theorem 38

$\mu \geq 0$ on $\mathcal{B}(\mathbb{R}^n)$, then any Borel set $B \subset \mathbb{R}^n$ and any $\varepsilon > 0$ exists U_ε open and F_ε closed such that $F_\varepsilon \subset A \subset U_\varepsilon$ and $\mu(U_\varepsilon \setminus F_\varepsilon) < \varepsilon$.

Proof. Let $\mathcal{A}$ the class of all sets $Ain\mathcal{B}$ such that $F_\varepsilon \subset A \subset U_e$ and $\mu(U_\varepsilon \setminus F_\varepsilon) < \varepsilon$ for some closed set F_ε and open set U_ε.

Every closed A is in $\mathcal{A}$, since we can take $F_\varepsilon = A$ and U_ε some open δ-nbd and consider $\delta \to 0$.

It suffices to show that $\mathcal{A}$ is a σ-algebra, since the closed sets generate $\mathcal{B}$. $\mathcal{A}$ is closed wrt complements, so it remains to verify closure under countable union.

Let $A_j \in \mathcal{A}$, $\varepsilon > 0$. Then exists closed F_j and open $U - j$ such that $F_j \subset A_j \subset U - j$ and $\mu(U_j \setminus F_j) < \varepsilon 2^{-j}$ for $j \in \mathbb{N}$.

The set $U = U_{j=1}^{\infty} U_j$ is open, and $Z_k = U_{j=1}^{k} F_j$ is closed.

Observe $Z_k \subset \cup_{j=1}^{\infty} A_j \subset U$ and for sufficiently large k $\mu(U \setminus Z_k) < \varepsilon$.

Indeed, $\mu(\cup_j^{\infty} \mu_j \setminus F_j) < 2$ □ ??

Definition 39

Set function $\mu \geq 0$ defined on $\mathcal{A} \subset 2^X$ is *tight* on $\mathcal{A}$ if $\forall \varepsilon > 0$ exists compact $K_\varepsilon \subset X$ such that $\mu(A) < \varepsilon$ for all $A \in \mathcal{A}$ that does not meet K_ε.
Additive set function μ of bounded variation on an algebra is *tight* if its total variation $|\mu|$ is tight.

A Borel measure is tight iff $\forall \varepsilon > 0$ exists compact K_ε such that $|\mu|(X \setminus K_\varepsilon) < \varepsilon$ (the "total variation measure").

The second definition is necessary to handle Baire sets.

Definition 40

μ is *regular* if $\forall A \in \mathcal{A}, \varepsilon > 0$, $\exists F_\varepsilon$ closed such that $F_\varepsilon \subset A$, $A \setminus F_\varepsilon \in \mathcal{A}$, and

??

Theorem 27 implies any Borel measure on $\mathbb{R}_n$ is regular, and the same proof works to show any Borel measure on metric space is regular.

Corollary 41 (*Baire measures are regular*)

{corr:baire-measure-regular}

Every Baire measure μ on topological space X is regular. Moreover, for every Baire set E and $\varepsilon > 0$, there exists a continuous function f on X such that $f^{-1}(0) \subset E$ and $|\mu|(E \setminus f^{1}(0)) < \varepsilon$.

3 Lecture 3

Theorem 42 (*Extension to Radon measure*)

{thm:extend-tight-to-radon}

Suppose an algebra $\mathcal{A}$ of subsets of topological space X contains a base of the topology. Let μ be a regular additive set function of bounded variation on $\mathcal{A}$. If μ is tight, then it admits a unique extension to a Radon measure on X.

Proof. V.I. Bogachev, "Measure Theory" Theorem 7.3.2 □

Tightness is important because it says the whole space is inner approximable by a compact set.

Corollary 43

Let X be a completely regular spaace. Then every tight Baire measure μ on X admits a unique extensino to a Radon measure.

Proof. Every Baire measure is regular by Corollary 41, and since X is completely regular, functionally open sets form a base of the topology. Apply Theorem 42. □

This allows us to extend measures on the Baire σ-field to measures on the Borel σ-field.

Definition 44

A *vector lattice of functions* is a linear space of real functions on a nonempty set Ω such that $\max(f, g) \in \mathcal{F}$ for all $f, g \in \mathcal{F}$.
Notice $\min(f, g) = \max(-f, -g) \in \mathcal{F}$ and $|f| \in \mathcal{F}$.

Theorem 45 (*Daniell integration*)

Let $\mathcal{F}$ be a vector lattice of functions on a set Ω such that $\mathbb{1} \in \mathcal{F}$. Let L be a linear functional on $\mathcal{F}$ with:

- *$L(f) \geq f$ for all $f \geq 0$ (positive)*
- *$L(\mathbb{1}) = 1$*
- *$L(f_n) \to 0$ for every $f_n \downarrow 0$*

Then there exists a unique probability measure μ on $\mathcal{A} = \Sigma(\mathcal{F})$ generated by $\mathcal{F}$ such that $\mathcal{F} \subset \mathcal{L}^1(\boldsymbol{U})$ and

$$L(f) = \int_\Omega f d\mu, \qquad \forall f \in \mathcal{F}$$

Compare this with Riesz representation theorem

For X a compact space, L linear functional on $\mathcal{C}(X)$ with $L(\mathbb{1}) = 1$ and $L(f) \geq 0$ for $f \geq 0$ (positive linear functional), then $L(f) = \int_X f d\mu$ with unique regular Borel probability measure μ on X.
The relation is through Dini's theorem: If $\{f_n\} \subset \mathcal{C}(X)$, X compact, and $f_n(x) \downarrow 0$, then $\lim\limits_{n\to\infty} \sup_{x \in X} f_n(x) = 0$.

Proof. Denote $\mathcal{L}^+$ the set of all bounded functions f of the form $f(x) = \lim_{n\to\infty} f_n(x)$, where $f_n \in \mathcal{F}$ are nonnegative and increasing. $\{f_n\}$ is uniformly bounded, hence $\{L(f_n)\}$ is increasing and bounded by properties of L

Let $L(f) = \lim_n L(f_n)$. We show that the extended functional is well-defined, coincides on bounded nonnegative functions in $\mathcal{F}$ with the original functional, and possesses the following properties:

1. $L(f) \leq L(g)$ for all $f, g \in \mathcal{L}^+$ with $f \leq g$
2. $L(f+g) = L(f) + L(g)$, $L(cf) = cL(f)$ for all $c \in [0, +\infty)$
3. $\min(f, g) \in \mathcal{L}^+$, $\max(f, g) \in \mathcal{L}^+$, and
$$L(f) + L(g) = L(\min(f, g)) + L(\max(f, g))$$
4. $\lim_n f_n \in \mathcal{L}^+$ for every uniformly bounded increasing sequence of functions $f_n \in \mathcal{L}^+$, and $L(\lim_n f_n) = \lim_n L(f_n)$.

Let $\{f_n\}$ and $\{g_k\}$ be two increasing sequences of nonnegative functions in $\mathcal{F}$ with $\lim_n f_n \leq \lim_k g_k$. By linearity and positivity of L, $\lim_n L(f_n) \leq \lim_n L(g_k)$ proving (1).

From the hypotheses of the theorem if $\psi_m \uparrow \psi \in \mathcal{F}$ are all nonnegative functions then $L(\psi_m) \to L(\psi)$ again through linearity: $L(\psi) - L(\psi_m) = L(\psi - \psi_m) \to L(0) = 0$ proving (2).

To verify (4), suppose $f_{k,n} \uparrow f_n \in \mathcal{L}^+$. Let $g_m = \max_{n \leq m} f_{m,n}$, so $g_m \in \mathcal{F}$ increasing and $f_{m,n} \leq g_m \leq f_m$ for $n \leq m$. Therefore $\lim_m f_m = \lim_m g_m \in \mathcal{L}^+$ and by linearity for $n \leq m$

$$L(g_m) \leq L(g_{m+1}), \qquad L(f_{m,n}) \leq L(g_m) \leq L(f_m)$$

Hence $\lim_m L(f_m) = \lim_m L(g_m) = L(\lim_m g_m) = L(\lim_m f_m)$

$$\begin{array}{cccc}
g_1 & f_{1,1} & f_{2,1} & f_{1,3} \\
 & \rotatebox{-90}{$\leq$} & \rotatebox{-90}{$\leq$} & \rotatebox{-90}{$\leq$} \\
g_2 & f_{1,2} & f_{2,2} & f_{2,3} \\
 & \rotatebox{-90}{$\leq$} & \rotatebox{-90}{$\leq$} & \rotatebox{-90}{$\leq$} \\
 & \vdots & \vdots & \vdots \\
 & f_1 & f_2 & f_3
\end{array}$$

{fig:sketch-of-proof-of-4} Figure 3: sketch of proof of 4

Denote by $\mathcal{G}$ the class of all sets G with $\mathbb{1}_G \in \mathcal{L}^+$, and for $G \in \mathcal{G}$ define $\mu(G) = L(\mathbb{1}_G)$. Observe that min/max convert to indicators, so by (3) the class $\mathcal{G}$ is closed wrt finite intersection/union, and hence by countable union by (4).

Furthermore, μ is nonnegative monotone additive function on $\mathcal{G}$, with inclusion-exclusion, and $\mu(G_n) \uparrow \mu(G)$.

According to (TODO: Ref: thm 20) and closure of $\mathcal{G}$ under countable union, the function

$$\mu^*(A) = \inf\{\mu(G) : G \in \mathcal{G}, A \subset G\}$$

is countably additive measure on the class

$$\mathcal{B} = \{B \subset \Omega : \mu^*(B) + \mu^*(\Omega \setminus B) = 1\}$$

Uncertain about above theorem

Should check details of section 1.5 Borgachev

For (iii) we verify $\mathcal{A} = \sigma(\mathcal{F}) \subset \mathcal{B}$. If $f \in \mathcal{L}^+$, then $\{f > g\} \in \mathcal{G}$ for all c since

$$\mathbb{1}\{f > c\} = \lim_n \min(1, n \max(f - c, 0))$$

Hence $f \in \mathcal{L}^+$ are measurable wrt $\sigma(\mathcal{G})$, but they are also measurable wrt $\sigma(\mathcal{F})$ (since they are monotone limits of things in $\mathcal{F}$), so $\mathcal{G} \subset \sigma(\mathcal{L}^+) = \sigma(\mathcal{F})$ and by Dynkin π-λ we have $\sigma(\mathcal{G}) = \sigma(\mathcal{F}) = \mathcal{A}$. Thus, it suffices to show $\mathcal{G} \subset \mathcal{B}$.

For $G \in \mathcal{G}$, let $f_n \uparrow \mathbb{1}_G$ so

$$\mu^*(G) = \mu(G) = \lim_{n \to \infty} L(f_n)$$

and since $\mu^*(G) + \mu^*(\Omega \setminus G) \geq 1$, to show $G \in \mathcal{B}$ it suffices to prove $\mu^*(G) + \mu^*(\Omega \setminus G) \leq 1$ i.e.

$$\mu^*(\Omega \setminus G) \leq \lim_n L(\mathbb{1} - f_n)$$

But since $\mathbb{1} - f_n \downarrow \mathbb{1}_{\Omega \setminus G}$ and $U_c = \{1 - f_n > c\}$ contains $\Omega \setminus G$ hence belongs to $\mathcal{G}$, therefore

$$\mathbb{1}_{U_c} \leq c^{-1}(\mathbb{1} - f_n)$$
$$\mu^*(\Omega \setminus G) \leq \mu(U_c) = L(\mathbb{1}_{U_c}) \leq c^{-1} L(1 - f_n)$$

Take $c \to 1$ and $n \to \infty$.

It remains to prove $\mathcal{F} \subset \mathcal{L}^1(\mu)$ and that $L(f) = \int_\Omega f d\mu$. Approximate $f \in \mathcal{L}^+$ with $f \leq 1$ by simple functions

$$f_n = \sum_{j=1}^{2^n - 1} j 2^{-n} \mathbb{1}\{j 2^{-n} < x < (j+1) 2^{-n}\}$$
$$L(f_n) = \sum_{j=1}^{2^n - 1} j 2^{-n} \mu\{j 2^{-n} < x < (j+1) 2^{-n}\}$$

Hence

$$L(f_n) = \int_\Omega f_n d\mu$$

TODO: finish

The uniqueness of μ satisfying ?? follows from the fact that it is uniquely determined on the class $\mathcal{G}$, which is closed wrt finite intersections and generates $\mathcal{A}$. ☐

References

{blackwell1973} [BM73] David Blackwell and James B. MacQueen. "Ferguson Distributions Via Polya Urn Schemes". In: *Ann. Statist.* 1.2 (Mar. 1973), pp. 353–355. DOI: `10.1214/aos/1176342372`. URL: `https://doi.org/10.1214/aos/1176342372`.

{ferguson1973} [Fer73] Thomas S. Ferguson. "A Bayesian Analysis of Some Nonparametric Problems". In: *Ann. Statist.* 1.2 (Mar. 1973), pp. 209–230. DOI: `10.1214/aos/1176342360`. URL: `https://doi.org/10.1214/aos/1176342360`.