STAT C206B: Topics in Stochastic Processes

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Last updated: February 4, 2020

Contents

1 Lecture 4: Representation theorems

2020-01-30

Recall our current setup:

- $\mathcal{F}$ is a vector lattice of functions over Ω containing constants, i.e. $\mathbb{1} \in \mathcal{F}$
- $\mathcal{L}^+$ consists of f such that $0 \leq f_n \uparrow f < \infty$ for $f_n \in \mathcal{F}$
- $\mathcal{G} = \{G \subset \Omega : \mathbb{1}_G \in \mathcal{L}^+\}$ are the subsets whose indicators can be realized as monotone limits within $\mathcal{F}$ (i.e. can be well approximated using $\mathcal{F}$, which we can use L to measure); we use $\mathcal{G}$ as the approximating set when constructing the Munroe outer measure
- For $G \in \mathcal{G}$, define $\mu(G) = L(\mathbb{1}_G) = \lim_n L(f_n)$ and extend using **??** (??) and **??** (??) to the class $\mathfrak{M}_{\mu^*} = \mathcal{B}$ which contains $\mathcal{A} = \sigma(\mathcal{F}) = \sigma(\mathcal{G})$.

Corollary 1

Suppose that in **??** *the vector lattice $\mathcal{F}$ is closed wrt uniform convergence. Let $\mathcal{G}_{\mathcal{F}}$ be the class of sets of the form $\{f > 0\}$, $f \geq 0$, $f \in \mathcal{F}$. Then $\mathcal{G}_{\mathcal{F}}$ generates $\mathcal{A} = \sigma(\mathcal{F})$ and we have*

$$\mu(A) = \inf\{\mu(G) : A \subset G, G \in \mathcal{G}_{\mathcal{F}}\}, \qquad \forall A \in \mathcal{A}$$
$$\mu(G) = \sup\{L(f) : f \in \mathcal{F}, 0 \leq f \leq \mathbb{1}_G\}, \qquad \forall G \in \mathcal{G}_{\mathcal{F}}$$

Proof. Suffices to verify $\mathcal{G}_{\mathcal{F}}$ equals the $\mathcal{G}$ introduced during the theorem's proof, since we showed $\sigma(\mathcal{G}) = \mathcal{A}$. Taking $c = 0$ in **??** shows $\{f > 0\} \in \mathcal{G}$ for all non-negative $f \in \mathcal{F}$. On the other hand, if $G \in \mathcal{G}$ then $f_n \uparrow \mathbb{1}_G$ for some $f_n \geq 0$ in $\mathcal{F}$. Letting $f = \sum_{n=1}^{\infty} 2^{-n} f_n$, by uniform convergence of the series we have $f \in \mathcal{F}$. Clearly $f \geq 0$ and $G = \{f > 0\}$. □

A general fact about vector lattices where signed measures decompose into a positive part and negative part. If ν is a signed measure on Ω, then $\nu = \nu_+ - \nu_-$ for $\nu_\pm$ unique nonnegative meaasures with disjoint supports. Its total variation decomposes as:

$$\|\nu\| = \nu_+(\Omega) + \nu_-(\Omega)$$

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Theorem 2

Let $\mathcal{F}$ be a vector lattice of bounded functions on a set Ω such that $\mathbb{1} \in \mathcal{F}$. Suppose that we are given a linear functional L on $\mathcal{F}$ that is continuous wrt $\|f\| = \sup_\Omega |f(x)|$, i.e.

$$\|L\| = \inf\{c : \|L(f)\| \leq c\|f\| \ \forall f \in \mathcal{F}\} < \infty$$

Then L can be represented as $L = L^+ - L^-$ where $L^+ \geq 0$, $L^- \geq 0$, and for all nonnegative $f \in \mathcal{F}$ we have

$$L^+(f) = \sup_{0 \leq g \leq f} L(g), \qquad L^-(f) = -\inf_{0 \leq g \leq f} L(g)$$

In addition, letting $|L| = L^+ + L^-$, we have for $f \geq 0$

$$|L|(f) = \sup_{0 \leq |g| \leq f} |L(g)|, \qquad \|L\| = L^+(1) + L^-(1)$$

Proof. Given two nonnegative $f, g \in \mathcal{F}$ and $h \in \mathcal{F}$ such that $0 \leq h \leq f + g$, can write $h = h_1 + h_2$ where $0 \leq h_1 \leq f$, $0 \leq h_2 \leq g$, $h_1, h_2 \in \mathcal{F}$. Just let $h_1 = \min(f, g)$ and $h_2 = h - h_1$ and verify.

Let L^+ be defined by the previous theorem. We first show additivity on nonnegative functions. For $f, g \in \mathcal{F}$ nonnegative, we have

$$L^+(f+g) = \sup\{L(h) : 0 \leq h \leq f + g\} = \sup\{L(h_1) + L(h_2) : 0 \leq h_1 \leq f, 0 \leq h_2 \leq g\} = L^+(f) + L^+(g)$$

where we used the previous decomposition.

Now we show additivity on arbitrary functions. Let $f = f_1 - f_2$, where f_1, f_2 non-negative. There might be multiple decompositions for the samae f, but still

$$L^+(f) = L^+(f_1) - L^+(f_2)$$

since $f_1 + f^- = f_2 + f^+$ and we showed L^+ is additive on nonnegative functions.

Define $L^- = L^+ - L$ and since $L^+(f) \geq L(f)$ for $f \geq 0$ we have that L^- is also nonnegative.

Finally,

$$\begin{aligned}
\|L\| \leq \|L^+\| + \|L^-\| \\
&= L^+(1) + L^-(1) \\
&= 2L^+(1) - L^-(1) \\
&= \sup\{L(2\psi - 1) : 0 \leq \psi \leq 1\} \\
&\leq \sup\{L(h) : -1 \leq h \leq 1\} \\
&\leq \|L\|
\end{aligned}$$

□

Corollary 3

{corr:meas-repr-decreasing-clf} *Suppose in addition $L(f_n) \to 0$ for every $f_n \downarrow 0$. Then L^+ and L^- share this property as well, and are defined by nonnegative countably additive measures on $\sigma(\mathcal{F})$ and L has representation*

$$L(f) = \int_\Omega f \, d\mu, \qquad \forall f \in \mathcal{F}$$

with some signed countably additive measure μ on $\sigma(\mathcal{F})$.

Proof. TODO □

Here is an analogue of the Riesz representation theorem:

Theorem 4

{thm:baire-meas-repr-clf} *Let X be a topological space. The formula*

$$L(f) = \int_X f \, d\mu$$

establishes a one-to-one correspondence between Baire measures μ on X and continuous linear functionals

L on $\mathcal{C}_b(X)$ with the property

$$\lim_n L(f_n) = 0$$

for every $f_n \downarrow f$.

Proof. Any measure μ on $\mathcal{B}a(X)$ defines a continuous linear functional on $\mathcal{C}_b(X)$ through the above formula.

Converse follows from Corollary 3. □

See "Banach limit"

Theorem 5 (*Dini's theorem*)

{thm:dini} *On a compact space K, if $\{f_n\} \subset \mathcal{C}(X)$ converges pointwise decreasing to zero, then $\{f_n\}$ converges in the Banach space $\mathcal{C}(X)$ to 0, i.e. converges uniformly to zero.*

We get a Riesz representation for compact spaces:

Theorem 6 (*Riesz representation theorem*)

On a compact Hausdorff space K, every continuous linear functional L on the Banach space $C(K)$ has a unique Radon measure μ such that

$$L(f) = \int_K f d\mu, \qquad \forall f \in \mathcal{C}(K)$$

Proof. By Theorem 5, TODO □

From now, we assume S to be locally compact, second countable, and Hausdorff (lcscH). Let $\mathcal{G}, \mathcal{F}, \mathcal{K}$ denote open, closed, and compact sets in S and put $\hat{\mathcal{G}} = \{G \in \mathcal{G}, \bar{G} \in \mathcal{K}\}$. Let $\hat{C}_+ = \hat{C}_+(S)$ denote the class of continuous functions $f : S \to \mathbb{R}_+$ with compact support (i.e. closure of the set $\{x \in S; f(x) > 0\}$).

We want to extend the idea of invariant (Haar) measure from just groups to more general spaces such as the sphere.

Theorem 7 (*Riesz representation*)

{thm:riesz-extend-fts-to-meas} *If S is lcscH, then every positive linear functional μ on $\hat{C}_+(S)$ extends uniquely to a measure on S that assigns finite mass to compact sets.*

Proof. Kallenberg, "Foundations of modern probability" □

Theorem 8

On every lcscH group G there exists, uniquely up to normalization, a left-invariant measure $\lambda \neq 0$ that assigns finite mass to compact sets. If G is compact, then λ is also right-invariant.

Proof. Kallenberg, "Foundations of modern probability" □

Definition 9

Given group G and space S, a *left action* of G on S is a mapping $(g, s) \mapsto gs$ such that $es = s$ and $(gh)s = g(hs)$ for any $g, h \in G$ and $s \in S$, where e denotes the identity element in G.
Similarly, a *right action* is a mapping $(s, g) \mapsto sg$ satisfying similar compatibility conditions.
The action is *transitive* if for all $s, t \in S$ there exists $g \in G$ such that $gs = t$ or $sg = t$ respectively.
All actions are assumed left henceforth.

When G is a topological group, we assume the action is a continuous $G \times S \to S$ map.

Definition 10

$h : G \to S$ is *proper* if $h^{-1}K$ is compact in G for any compact $K \subset S$.

If this holds for all $\pi_s(x) = xs$, $s \in S$, we say the group action is proper.

Definition 11

A memasure μ on S is G-invariant if $\mu(xB) = \mu B$ for any $x \in G$ and $B \in \mathcal{S}$. This is clearly equivalent to

$$\int f(xs)\mu(ds) = \mu f$$

for any measurable $f : S \to \mathbb{R}_+$ and $x \in G$.

Theorem 12

If we have lcshH group G acting transively and properly on lcscH space S. Then there exists a unique (up to normalization) G-invariant measure $\mu \neq 0$ on S which assigns finite mass to compact sets.

Proof. We first show existence. Fix $p \in S$ and let $\pi = x \mapsto xp : G \to S$. Letting λ be a left Haar measure on G, define the pushforward $\mu = \lambda \circ \pi^{-1}$ on S. Since π is proper and the Haar measure on G assigns finite mass to compact sets, μ is a measure on S that assigns finite mass to compact sets. To see G-invariance, for $f \in \hat{C}_+(S)$ and $x \in G$

$$\int_S f(xs)\mu(ds) = \int_G f(xyp)\lambda(dy) = \int_G f(yp)\lambda(dy) = \mu f$$

by invariance of λ.

Now we consider uniqueness. Let μ be abitrary G-invariant measure on S assigning finite mass to compact sets. Define the subgroup

$$K = \{x \in G : xp = p\} = \pi^{-1}\{p\}$$

(the stabilizer of p, subgroup leaving p fixed) and note K is compact (since π is proper). Let ν be the normalized Haar measure on K, and define

$$\bar{f}(x) = \int_K f(xk)\nu(dk), \quad x \in G, f \in \hat{C}_+(G)$$

At each point x, $\bar{f}$ takes f and "smooths things out" using K translated to x.

If $xp = yp$ then $y^{-1}xp = p$ and so $y^{-1}x =: h \in K$ which implies $x = yh$. Hence, left invariance of ν yields

$$\bar{f}(x) = \bar{f}(yh) = \int_K f(yhk)\nu(dk) = \int_K f(yk)\nu(dk) = \bar{f}(y)$$

So the mapping $f \mapsto f^*$ given by

$$f^*(s) = \bar{f}(\pi^1\{s\}) \equiv \bar{f}(x), \quad s = xp \in S, x \in G, f \in \hat{C}_+(G)$$

is well defined, and for any $B \subset (0, \infty)$ we have

$$(f^*)^{-1}B = \pi(\bar{f}^{-1}B) \subset \pi[(\operatorname{supp} f) \cdot K]$$

where $(\operatorname{supp} f) \cdot K$ is the support of f "convolved with K" via the group action. Hence, the RHS is compact (both $\operatorname{supp} f$ and K compact) and since π and the action are continuous. Therefore f^* has compact support.

Also, $\bar{f}$ is continuous (by group operation cts and dominated convergence), so $\bar{f}^{-1}[t, \infty)$ is closed and hence compact for every $t > 0$.

??? So f^* is something we can integrate against μ.

We may now define functional λ on $\hat{C}_+(G)$ by $\lambda f = \mu f^*$ for $f \in \hat{C}_+(G)$. Linearity and positivity of λ are clear from the corresponding properties of the mapping $f \mapsto f^*$ and the measure μ. We note that λ is finite on $\hat{C}_+(G)$ since μ is locally finite, so by Theorem 7 we can extend λ to a measure on G that assigns fintie mass to compact sets.

To see λ left invariant, for $f \in \hat{C}_+(G)$ and define $f_y(x) = f(yx)$. THen for $s = xp \in S$ and $y \in G$ we have

$$f_y^*(s) = \bar{f}_y(x) = \int_K \bar{f}_y(xk)\nu(dk) = \bar{f}(yx) = f^*(ys)$$

Hence by invariance of μ we have

$$\int_G f(yx)\lambda(dx) = \lambda f_y = \mu f_y^* = \int_S f^*(ys)\mu(ds) = \mu f^* = \lambda f$$

So λ is the Haar measure.

Now fix $g \in \hat{C}_+(S)$ and put $f(x) = g(xp) = g \circ \pi(x)$ for $x \in G$. THen $f \in \hat{C}_+(G)$ because $\{f > 0\} \subset \pi^{-1} \operatorname{supp} g$ which is compact since π is proper. By definitino of K, for $s = xp \in S$ we have

$$f^*(s) = \bar{f}(x) = \int_K f(xk)\nu(dk) = \int_K g(xkp)\nu(dk) = \int_K g(xp)\nu(dk) = g(s)$$

so we've found an inverse for the $*$ operation, so

$$\mu g = \mu f^* = \lambda f = \lambda(g \circ \pi) = (\lambda \circ \pi^{-1})g$$

which shows $\mu = \lambda \circ \pi^{-1}$. Since λ is unique up to normalization, the same thing is true for μ. □

2 Lecture 5: Extreme point representation of measures

2020-02-04

Throughout, let E be a real TVS.

Definition 13

For convex $A \subset E$, an *open segment* is a subset of type

$$\{(1-\lambda)a + \lambda b : \lambda \in (0,1)\}, \qquad a \neq b \in A$$

$x_0 \in A$ is an *extreme point* of A if it belongs to no open segment of A, i.e.

$$\begin{aligned}&(\exists \lambda \in [0,1] : x_0 = (1-\lambda)a + \lambda b)\\ &\Rightarrow x_0 = a \text{ or } x_0 = b\end{aligned}$$

Closed $B \subset A$ is an *extreme subset* of A if

$$\begin{aligned}&(\exists \lambda \in (0,1) : (1-\lambda)a + \lambda b) \in B)\\ &\Rightarrow \{a, b\} \subset B\end{aligned}$$

Lemma 14

{lem:cvx-subset-has-extreme-subset} *Let E real Hausdorff TVS, $A \subset E$ nonempty compact convex, f continuous linear functional on E, $\beta = \inf_{x \in A} f(x)$. Then $B = A \cap f^{-1}(\beta)$ is nonempty, compact, extreme subset of A.*

Proof. β exists because A is compact.

Continuity and linearity of f ensure B is closed and convex. Check nonempty and compact.

To show B extreme, suppose $(1-\lambda)a + \lambda b \in B$ for $a, b \in A$ and $\lambda \in (0,1)$. If, for example, $a \notin B$, then $f(a) > \beta$ by definition of B and so by linearity

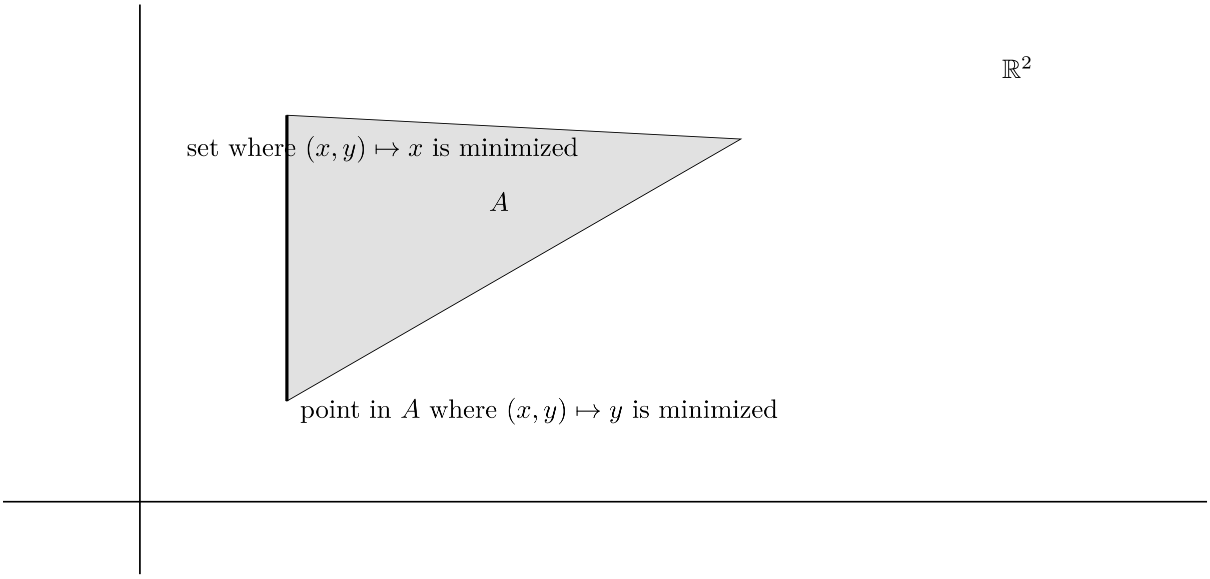
$$f((1-\lambda)a + \lambda b) = (1-\lambda)f(a) + \lambda f(b) > (1-\lambda)\beta + \lambda\beta = \beta$$

contradicting $(1-\lambda)a + \lambda b \in B$

TODO: finish □

Theorem 15

E real Hausdorff LCTVS, A nonempty compact convex subset of E, then A is the closed convex hull in



{fig:extreme-points} Figure 1: On a triangle, the extreme points are the vertices and the extreme subsets are the sides

E of the set of extreme points of A.

This is saying something along the same lines as Caratheodory's theorem for convex things.

Proof. We first show each nonempty extreme subset $X \subset A$ contains an extreme point of A. Let $\mathfrak{X}$ consist of extreme subsets of A contained in X. $\mathfrak{X}$ is nonempty (by Lemma 14), so partially order $\mathfrak{X}$ by inclusion. Notice the intersection of any chain is a non-empty compact set $\in \mathfrak{X}$ because it is the intersection of nonempty compact sets (c.f. Hausdorff's theorem), hence by Zorn's lemma $\mathfrak{X}$ possesses a minimal element, say Y.

It remains to show Y is a singleton. Otherwise, Y would contain $x \neq y$ and since E is Hausdorff and LC, (by Hahn-Banach separating hyperplane version) there exists continuous linear function f on E such that $f(x) < f(y)$. By Lemma 14, $Z = Y \cap f^{-1}(\inf f(Y))$ is a nonempty extreme subset of A that does not contain y. Thus $Z \subsetneq Y$, contradicting minimality of Y.

In the second step, let B be the closed convex hull in E of the set of all extreme points of A. B is compact, convex, and contained in A. To show $B = A$, it suffices to show $A \subset B$ is empty. Suppose towards contradiction $x_0 \in A \setminus B$, then by Hahn-Banach theorem there exists (separating) continuous linear functional f on E such that $\inf_B f(x) > f(x_0)$. Then by Lemma 14 $W = A \cap f^{-1}(\inf_A f(x))$ is a nonemmpty extreme subset of A disjoint from B. However, by the previous part W would contain an exterme point of A, which is a contradiction since $W \cap B = \emptyset$. ☐

Proposition 16

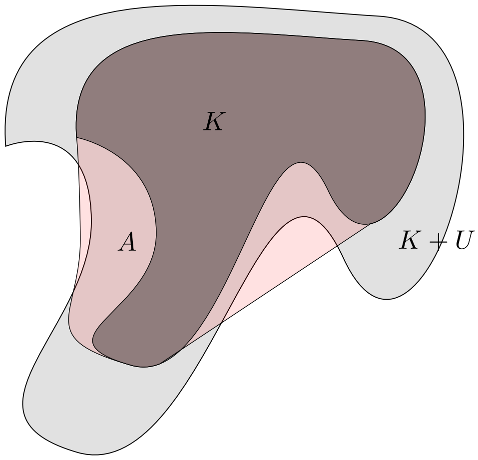
Suppose E is Hausdorff real LCTVS, $K \subset E$ compact whose closed convex hull A is compact. THen each extreme point of A belongs to K

Proof. Let $x \in A$ be extreme point. For any closed convex nbd $0 \in U \subset E$, by compactness of K there exists fintely many $a_i \in K$ such that $a_i + U$ cover K. Let $A_i = \mathrm{conv}(K \cap (a_i + U))$, each A_i is compact because $A_i \subset K$ and K is compact. Then $\mathrm{conv}(\cup_i^n A_i)$ is compact and $K \subset \mathrm{conv}(\cup_i^n A_i) \subset A$, so we must have $A = \mathrm{conv}(\cup_i^n A_i)$.

Hence, $x = \sum_i^n \lambda_i x_i$ with $x_i \in A_i$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$. As $x \in A$ is extreme point, x must coincide with some x_i. Thus, $x \in A_i \subset a_i + U$, so $x \in K + U$. Since K is closed and U is an arbitrary nbd of 0, $x \in K$ as desired. ☐

Example 17

A compact convex set A need not be the convex hull of its extreme points. Take $E = \ell^\infty$, $e_n = \delta_n \in E$,



{fig:cvx-hull-contains-extreme-pts}

Figure 2: cvx-hull-contains-extreme-pts

A the closed convex hull in E of e_n/n for $n \in \mathbb{N}$. By (TODO: prop), the extreme points of A are 0 and the points $\{e_n/n\}$. A is compact and contains all points $x = \sum_{n\geq 1} \lambda_n e_n/n$, λ_n a convex combinations. TODO: finish

Definition 18

For a Banach space $(X, \|\cdot\|_X)$, let X' be the set of bounded linear functionals $L : X \to \mathbb{R}$, i.e. $\sup_{x\in X}|L(x)|/\|x\| \leq c$ for some $c \geq 0 \Leftrightarrow L$ is linear and continuous.
Given $L \in X'$, write $\|L\|_{X'}$ for smallest c that works $\|L\|_{X'} = \inf\{|l(x)| : \|x\| = 1\}$.
FACT: $(X', \|\cdot\|_{X'})$ is a Banach space. Each $x \in X$ defines a linear map $X' \to \mathbb{R}$ via evaluation

$$e_x = L \mapsto L(x)$$

The *weak-$*$ topology* on X' is the weakest/coarsest (i.e. initial topology) $\tau(X', \{e_x\}_{x\in X})$ that makes all of these maps continuous.

Example 19 (*Riesz representation theorem*)

Let S be a compact Hausdorff space, $X = \mathcal{C}(S)$ continuous functions from S to $\mathbb{R}$. X is a Banach space with the sup-norm $\|x\| = \sup_{s\in S}|x(s)|$. Then X' finite signed memasures on S. associated with any finite signed measure μ. is the continuous linear functional $L(x) = \int_S x(s)\mu(ds)$.
We want to know what is $\|\cdot\|_{X'} = \|\cdot\|_{\mu(S)}$? If $L(x) = \int_S x(s)\mu(ds)$ then by definition

$$\begin{aligned}\|L\|_{X'} &= \sup\{|L(x)| : \|x\|_X = 1\}\\ &= \sup\left\{\left|\int_S x(s)\mu(ds)\right| : \sup_{s\in S}|x(s)| = 1\right\}\\ &= \sup\left\{\left|\int_S x(s)\mu^+(ds) - \int_S x(s)\mu^-(ds)\right| : -1 \leq |x(s)| \leq 1\right\}\end{aligned}$$

We know μ^+ and μ^- are perpendicular; their supports are disjoint, so

$$\|L\|_{X'} = \sup\left\{\left|\int_{S^+} x(s)\mu^+(ds) - \int_{S^-} x(s)\mu^-(ds)\right| : -1 \leq |x(s)| \leq 1\right\}$$

Take $x = \mathbb{1}_{S^+} - \mathbb{1}_{S^-}$ to conclude $\|L\|_{X'} = \mu^+(S^+) + \mu^-(S^-) = \mu^+(S) + \mu^-(S) = |\mu|(S) = \|\mu\|_{TV}$ Hence, $(\mathcal{C}(S), \|\cdot\|_\infty)$ has dual $(M(S), \|\cdot\|_{TV})$ and the weak-$*$ topology on $M(S)$ is the weakest/coarsest/smallest topology that makes continuous all maps $\mu \mapsto \int_S x(s)\mu(ds)$ for $x \in \mathcal{C}(S)$. Notice that this is strictly

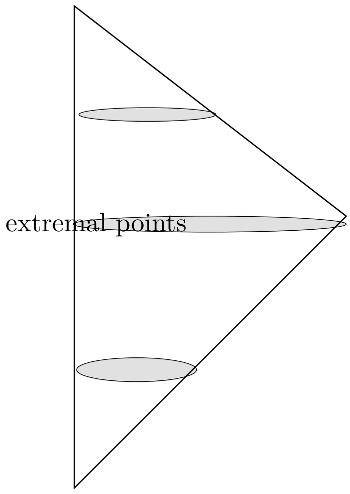


Figure 3: higher-dim-extreme-pt

{fig:higher-dim-extreme-pt}

weaker than the TV topology, because for example two unit point masses have TV norm 2 but $\int x(s)\delta_{s'}(ds) = x(s') \approx x(s'') = \int x(s)\delta_{s''}(ds)$ when $s' \approx s''$.

This is the story for compact spaces. What about for only locally compact spaces?

Let T Hausdorff LC, $M(T)$ real bounded signed Radon measures on T.

Definition 20

$\mathcal{C}_0(T)$ are the continuous functions vanishing at infinity, i.e. $f \in cC(T)$ such that $\lim_{x\to\infty} f(\pm x) = 0$.

View $M(T)$ as the (Banach) dual of $C_0(T)$ equipped with weak-$*$ topology. The set $M_+^1(T)$ of positive measures in $M(T)$ having total mass at most one is compact and convex, because:

Theorem 21 (*Banach-Alaoglu*)

The unit ball in $(X', \|\cdot\|_{X'})$ is compact in the weak-$*$ topology.

We will show that the extreme points of $M_+^1(T)$ of are 0 and the Dirac measures δ_t for $t \in T$.

It is clear that 0 is an extreme point of $M_+^1(T)$.

Suppose $\mu \neq 0$ is another element in $M_+^1((T)$, then it suffices show $K = \operatorname{supp}\mu$ is a single point. If $t_1 \neq t_2 \in K$, then by Hausdorffness choose $U_1 \ni t_1$ and $U_2 \ni t_2$ disjoint. Then $m = \mu(U_1)$ satisfies $0 < m < 1$.

Define two measures $\alpha = (\mu|U_1)/m$ (μ restricted and renormalized) and $\beta = (\mu - m\alpha)/(1-m)$ what's left over after subtracting $m\alpha$. Then $\alpha, \beta \in M_+^1(T)$, $\alpha \neq \beta$, and $\mu = m\alpha + (1-m)\beta$, contradicting μ extremal. Hence, K is a single point.

A similar argument shows that if T is compact, then the set of positive measures of unit total mass is compact and convex, and that its extreme points are the Dirac measures δ_t.

Theorem 22 (*Stone-Weierstrass*)

Let E be a subalgebra of $C(S)$. Suppose E separates points and contains constants, then E is dense in $C(S)$.