STAT 201B: Probability

Feynman Liang*
Department of Statistics, UC Berkeley

Last updated: February 4, 2020

Contents

1 Lecture 4

2020-01-30

Proposition 1 (*Simple Markov property*)

For any fixed time n

$$(X_n, X_{n+1}, \dots \mid X_n = x) \stackrel{d}{=} (X_0, X_1, \dots \mid X_0 = x)$$

Proof. To show $X \stackrel{d}{=} Y$, show $\Pr(X \in A) = \Pr(Y \in A)$ for all A in a π-system. Here π-system is FDDs, so (by Kolmogorov) its enough to take

$$A = (X_0 = x_0, X_1 = x_1, \dots, X_m = x_m)$$

for any choice of $x_0, \dots, x_m$. We need

$$\Pr[X_n = x_0, X_{n+1} = x_1, \dots, X_{n+m} = x_m \mid X_n = x] = \Pr[X_0 = x_0, X_1 = x_1, \dots, X_m = x_m \mid X_0 = x]$$

$\square$

Recall $\mathcal{F}_T$ where T is a stopping time:

$$A \in \mathcal{F}_T \Leftrightarrow A \cap \{T \le n\} \in \mathcal{F}_n$$

Proof. Key fact: $\{T = n\} \in \mathcal{F}_n$. Given $\{T = n\}$ and the $\mathcal{F}_T$ info, you basically know some $\mathcal{F}_n$ fact. $\square$

2 Lecture 5

2020-02-04

Done: 5.1 and 5.2

Today: More on 5.2

Skip 5.4 [Done in 205A], start 5.5.

*feynman@berkeley.edu

2.1 Remaining issues of 5.2

Fix countable state space S, transition matrix P on S a $S\times$ matrix row stochastic. Recall definition of transient / recurrent states:

$$x \text{ is recurrent } \Leftrightarrow \Pr_x[N_x = \infty] = 1$$
$$x \text{ is transient } \Leftrightarrow \Pr_x[N_x < \infty] = 1$$

where $N_x = \sum_{n=0}^{\infty} \mathbb{1}\{X_n = x\}$. No other option is possible.

Theorem 2 (*Theorem 5.3.2*)

If x is recurrent and $\rho_{xy} > 0$, then Y is recurrent and $\rho_{yx} = 1$ (and $\rho_{xy} = 1$).

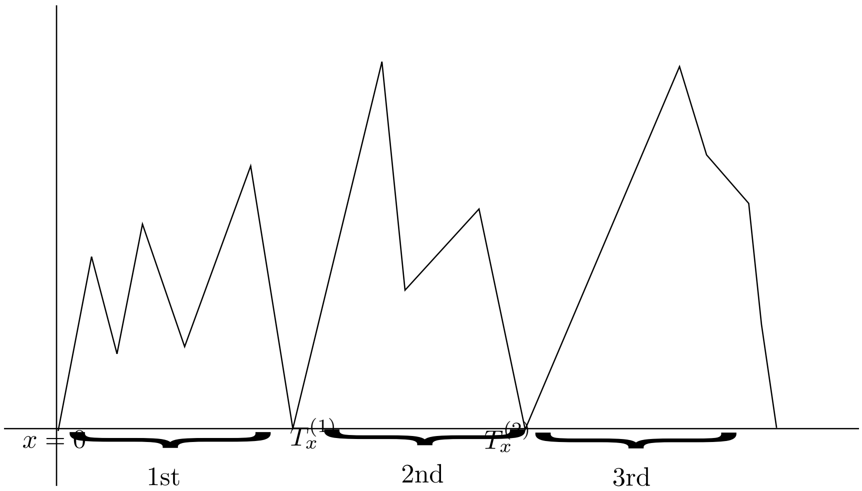

Figure 1: Revisiting recurrent state, the chain goes on excursions between visits to x. We can regard the whole chain (X_i) with $X_0 = x$ as a succession of x-blocks

{fig:revisiting-recurrent-state}

Proof. $0 < \rho_{xy} \Rightarrow \Pr_X[\text{hit } y \text{ before } T_x] > 0$

By the strong Markov property, the x-blocks are IID copies of the first x-block. So suppose $\Pr_x[\text{hit } y \text{ before } T_x] = 0$. Then $\Pr[x \text{ block contains a } y] = 0$, so $\Pr[\text{any of first 19 } x \text{ blocks contain a } y] = 0$, so $\Pr[\text{some } y \text{ before } T_x \text{ hit}] = 0$. $\square$

Can we do it without contradiction? Write

$$\rho_{xy} = \sum_{n\geq 1} \Pr_x[T_y = n]$$

Notice

$$\Pr_x[T_y = n] = \sum_{x=(x_i)_{i=1}^n, x_n = y, x_i \neq y \ \forall i<n} \underbrace{\prod_{i=1}^{n} P(x_{i-1}, x_i)}_{\Pr[\text{path}]}$$

whereas

$$\Pr_x[T_y < T_x] = \sum_{n=1}^{\infty} \sum_{x=(x_i)_{i=1}^n, x_n = y, x_i \neq x \ \forall i<n} \Pr[\text{path}]$$

Claim: each of the two above sums is > 0 iff the other is. Suppose the first sum > 0. Then exists n and some path $(x_i)_1^n$ with all $x_i \neq y$. So $\Pr[(x_i)_1^n] > 0$.

In general, most things involving Markov chains are just questions about sums over paths. The hard part is identifying which paths we are talking about.

Formalities: What is an x-block? An x-block is an element of $\cup_{n\geq 1} S^n$ such that $x_0 = x$, $x \notin x_{1:n}$

2.2 Key classes of SPs defined by symmetries

1. Independent, IID
2. $(X_i)_i$ is *exchangeable* $\Leftrightarrow (X_{\pi(i)})_i^n \overset{d}{=} (X_i)_i^n$. for all n and $\pi \in S_n$ a permutation of $[n]$.
3. (X_i) is *reversible* $\Leftrightarrow (X_n, X_{n-1}, \ldots, X_1) \overset{d}{=} (X_1, \ldots, X_n)$ for all n. Obviously exchangeable $\Rightarrow$ reversible, but reversible is much weaker.
4. $(X_1, X_2, \ldots)$ is *stationary*,

   $\Leftrightarrow (X_2, \ldots, X_{n_1}) \overset{d}{=} (X_1, \ldots, X_n)$ for all n on S^n,

   (by π-λ) $\Leftrightarrow (X_2, X_3, \ldots) \overset{d}{=} (X_1, X_2, \ldots)$ on S^∞

Exercise 3. Exchangeable $\Rightarrow$ Reversible $\Rightarrow$ Stationary