1. Suponha $\hat{\theta}_1$ e $\hat{\theta}_2$ dois estimadores não-viesados para θ, independentes. sejam $Var(\hat{\theta}_1) = v_1$ e $Var(\hat{\theta}_2) = v_2$.

$\hat{\theta} = c_1 \hat{\theta}_1 + c_2 \hat{\theta}_2$ Qual proporciona um estimador não-viesado com EQM mínimo.

$$EQM(\hat{\theta}) = \underbrace{Var(\hat{\theta})}_{②} + \underbrace{viés^2(\hat{\theta})}_{①}$$

a. Calculando o viés do $\hat{\theta}$

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}[c_1 \hat{\theta}_1 + c_2 \hat{\theta}_2] = c_1 \mathbb{E}\hat{\theta}_1 + c_2 \mathbb{E}\hat{\theta}_2 = c_1 \theta + c_2 \theta = (c_1 + c_2)\theta$$

para $\hat{\theta}$ ser não-viesado, então $c_1 + c_2 = 1$.

b. Minimizando o $EQM(\hat{\theta})$.

b1. viés: $\hat{\theta}$ é não-viesado quando $c_1 + c_2 = 1$ ou $c_2 = 1 - c_1$

b2. variância.

$$Var(\hat{\theta}) = Var(c_1\hat{\theta}_1 + c_2\hat{\theta}_2) = c_1^2 \, Var(\hat{\theta}_1) + c_2^2 \, Var(\hat{\theta}_2) = c_1^2 v_1 + c_2^2 v_2$$

$$= c_1^2 v_1 + (1-c_1)^2 v_2 \quad //$$

$$\frac{\partial Var(\hat{\theta})}{\partial c_1} = 2c_1 v_1 + 2(1-c_1)(-1)v_2 = 2c_1 v_1 - 2(1-c_1)v_2 \leftarrow$$

$$2c_1^* v_1 - 2(1-c_1^*)v_2 = 0 \rightarrow \boxed{c_1^* = \frac{v_2}{v_1+v_2}}$$

Calculando a segunda derivada

$$\frac{\partial^2 \text{Var}(\hat{\theta})}{\partial c_1^2} = 2v_1 + 2v_2 > 0$$

, Daí o c_1^* é um <u>mínimo</u>!

Dessa forma

$$\hat{\theta} = c_1\hat{\theta}_1 + c_2\hat{\theta}_2 = \frac{v_2}{v_1+v_2}\hat{\theta}_1 + \left(1 - \underbrace{\frac{v_2}{v_1+v_2}}\right)\hat{\theta}_2 = \frac{v_2}{v_1+v_2}\hat{\theta}_1 + \frac{v_1}{v_1+v_2}\hat{\theta}_2 \qquad \blacksquare$$

$$\langle c, X \rangle = \langle X, c \rangle = \underline{\sum c_i X_i}$$

2. Considere $y_i = \beta x_i + \varepsilon_i, \quad i = 1, 2, \dots, n$

$$\underset{n\times 1}{Y} = \underset{n\times 1}{X}\,\underset{1\times 1}{\beta} + \underset{n\times 1}{\varepsilon}, \quad \mathbb{E}(\varepsilon \mid X) = \underline{0} \quad \text{e} \quad \text{Cov}(\varepsilon \mid X) = \underline{\sigma^2 I}$$

a. Qual o estimador linear com EQM mínimo para β?

Considere $\hat{\beta} = c'Y = c'(X\beta + \varepsilon) = c'X\beta + c'\varepsilon$

a1. Viés

$$\mathbb{E}(\hat{\beta}) = \mathbb{E}(c'X\beta + c'\varepsilon) = \mathbb{E}(\beta c'X) + \mathbb{E}(c'\varepsilon) = \beta\,\mathbb{E}(c'X) + \mathbb{E}(c'\varepsilon)$$

$$= \beta c'X + c'\cancel{\mathbb{E}(\varepsilon)}^{\,0} = \beta c'X. \quad \rightarrow \quad \boxed{\mathbb{E}(\hat{\beta}) = \beta c'X} \leftarrow \text{número}$$

$$\mathbb{E}(\hat{\beta} - \beta) = \beta c'X - \beta = \beta\,\underline{(c'X - 1)} \leftarrow \text{viés de } \hat{\beta}.$$

a_2. Variância

$$\mathrm{Var}(\hat{\beta}) = \mathrm{Var}(\beta c'X + c'\varepsilon) = \mathrm{Var}(c'\varepsilon) = \sigma^2 c'c$$

$$\mathrm{Cov}(\hat{\beta}) = \mathrm{Cov}(c'\varepsilon) = \mathbb{E}[c'\varepsilon][c'\varepsilon]' = \mathbb{E}[c'\varepsilon\varepsilon' c] = c'\,\mathbb{E}(\varepsilon\varepsilon')c = c'\sigma^2 I c$$

$$= \sigma^2 c'c$$

Daí,

$$EQM(\hat{\beta}) = \mathrm{Var}(\hat{\beta}) + [\mathrm{Viés}(\hat{\beta})]^2 = \sigma^2 c'c + \beta^2 (c'X - 1)^2$$

$$\boxed{EQM(\hat{\beta}) = \sigma^2 c'c + \beta^2 (c'X - 1)^2} \longrightarrow \text{número}$$

$X'X$ número.

XX' matriz

Minimizando $EQM(\hat\beta)$

$$\frac{\partial EQM(\hat\beta)}{\partial c} = 2\sigma^2 \underline{\underline{c}} + 2\beta^2(c'x - 1)\underline{\underline{X}} \rightarrow \text{vetor}$$

igualando a zero

vetor

$$2\sigma^2 c^* + 2\beta^2(c^{*\prime}x - 1)X = 0 \quad \text{ou} \quad \boxed{c^* = -\frac{\beta^2}{\sigma^2}(c^{*\prime}x - 1)X}$$

vetor

$$\beta^2(c^{*\prime}x - 1)X = -\sigma^2 c^*$$

$$\beta^2(c^{*\prime}x - 1)X'X = -\sigma^2 X'c^*$$

$$x'c^* = c^{*\prime}x$$

$$\langle x, c^* \rangle = \langle c^*, x \rangle$$

$$\beta^2 (c^{*\prime}x - 1) x'x = -\sigma^2 x'c^* = -\sigma^2 c^{*\prime}x$$

$$\beta^2 c^{*\prime}x \cdot x'x - \beta^2 x'x = -\sigma^2 c^{*\prime}x$$

$$\beta^2 c^{*\prime}x \cdot x'x + \sigma^2 c^{*\prime}x = \beta^2 x'x$$

$$(\beta^2 x'x + \sigma^2)\, c^{*\prime}x = \beta^2 x'x$$

$$\boxed{c^{*\prime}x = \frac{\beta^2 x'x}{\beta^2 x'x + \sigma^2}}$$

Reescrevendo

$$\boxed{c^* = -\frac{\beta^2}{\sigma^2}(c^{*\prime}x - 1)x}$$

$$c^* = -\frac{\beta^2}{\sigma^2}\left(c^{*\prime}x - 1\right)x = -\frac{\beta^2}{\sigma^2}\left[\frac{\beta^2 x'x}{\beta^2 x'x + \sigma^2} - 1\right]x = -\frac{\beta^2}{\sigma^2}\left[\frac{\cancel{\beta^2 x'x} - \cancel{\beta^2 x'x} - \sigma^2}{\sigma^2 + \beta^2 x'x}\right]x$$

$$= \underbrace{-\frac{\beta^2}{\sigma^2}\left[-\frac{\sigma^2}{\sigma^2 + \beta^2 x'x}\right]}x = \frac{\beta^2}{\sigma^2}\left[\frac{\sigma^2}{\sigma^2 + \beta^2 x'x}\right]x$$

$$c^* = \frac{\beta^2}{\sigma^2}\left[\frac{1}{1 + \frac{\beta^2}{\sigma^2}x'x}\right]x \rightarrow \boxed{c^* = \left[\frac{1}{\frac{\sigma^2}{\beta^2} + x'x}\right]x}$$ ponto crítico.

calculando a 2ª derivada:

$$\frac{\partial^2 EQM(\hat{\beta})}{\partial c' \partial c} = 2\sigma^2 + 2\beta^2 \underline{x'x} > 0$$, daí c^* é mínimo

Finalmente

$$\hat{\beta} = c^{*\prime} Y = \left[\frac{1}{\frac{\sigma^2}{\beta^2} + X'X}\right]^{\prime} X'Y \quad \hookrightarrow \text{Eis o } \hat{\beta}$$

b) $$\frac{EQM(\hat{\beta})}{EQM(\hat{\beta}_{MQ})} = \frac{\tau^2}{1+\tau^2}, \quad \tau^2 = \frac{\beta^2}{\sigma^2/X'X} = \frac{X'X\,\beta^2}{\sigma^2} = \frac{X'X}{\frac{\sigma^2}{\beta^2}}$$

$$\downarrow \qquad \uparrow$$

$$\mathrm{cov}(\hat{\beta}_{MQ}) = \sigma^2 (X'X)^{-1} \qquad \frac{\beta^2}{\sigma^2}(X'X)$$

viés de $\hat{\beta}$

$$\mathbb{E}\hat{\beta} = \mathbb{E}\left[\frac{1}{\frac{\sigma^2}{\beta^2} + X'X} X'Y\right] = \frac{1}{\frac{\sigma^2}{\beta^2} + X'\cancel{X}} X' \mathbb{E} Y = \frac{1}{\frac{\sigma^2}{\beta^2} + X'X} X'X\beta$$

$$\mathbb{E}\hat{\beta} = \frac{\beta X'X}{\frac{\sigma^2}{\beta^2} + X'X}$$

viés: $$\mathbb{E}\hat{\beta} - \beta = \frac{\beta X'X}{\frac{\sigma^2}{\beta^2} + X'X} - \beta = \frac{\cancel{\beta X'X} - \beta\frac{\sigma^2}{\beta^2} - \cancel{\beta X'X}}{\frac{\sigma^2}{\beta^2} + X'X} = -\frac{\sigma^2}{\beta} \cdot \left[\frac{1}{\frac{\sigma^2}{\beta^2} + X'X}\right]$$

Variância $\hat{\beta}$

$$Var(\hat{\beta}) = Var\left[\frac{X'Y}{\frac{\sigma^2}{\beta^2} + X'X}\right] = Var\left[\frac{X'(X\beta + \varepsilon)}{\frac{\sigma^2}{\beta^2} + X'X}\right] = Var\left[\frac{X'X\beta + X'\varepsilon}{\frac{\sigma^2}{\beta^2} + X'X}\right]$$

$$= Var\left[\frac{X'\varepsilon}{\frac{\sigma^2}{\beta^2} + X'X}\right] = \frac{1}{\left(\frac{\sigma^2}{\beta^2} + X'X\right)^2} Var(X'\varepsilon)$$

$$\mathbb{E}(X'\varepsilon)(X'\varepsilon)' = \mathbb{E}(X'\varepsilon\varepsilon'X) = X'\sigma^2 I X = \sigma^2 X'X$$

$$\rightarrow Var(\hat{\beta}) = \frac{1}{\left(\frac{\sigma^2}{\beta^2} + X'X\right)^2} \cdot \sigma^2 X'X \quad //$$

$$EQM(\hat{\beta}) = Var(\hat{\beta}) + viés^2(\hat{\beta})$$

$$= \frac{\sigma^2(x'x)}{\left(\frac{\sigma^2}{\beta^2} + x'x\right)^2} + \frac{\sigma^4}{\beta^2} \cdot \frac{1}{\left(\frac{\sigma^2}{\beta^2} + x'x\right)^2} = \frac{1}{\left(\frac{\sigma^2}{\beta^2} + x'x\right)^2}\left(\sigma^2(x'x) + \frac{\sigma^4}{\beta^2}\right)$$

$$\frac{EQM(\hat{\beta})}{EQM(\hat{\beta}_{MQ})} = \frac{\left(\sigma^2(x'x) + \frac{\sigma^4}{\beta^2}\right) \cdot \frac{1}{\left(\sigma^2/\beta^2 + x'x\right)^2}}{\underbrace{\sigma^2(x'x)^{-1}}_{\sigma^2/x'x}} = \frac{(x'x)^2 + \frac{\sigma^2}{\beta^2}(x'x)}{\left(\frac{\sigma^2}{\beta^2} + x'x\right)^2} = \frac{(x'x)\cancel{\left[x'x + \sigma^2/\beta^2\right]}}{\cancel{\left(\frac{\sigma^2}{\beta^2} + x'x\right)}^2}$$

$$= \frac{x'x}{\frac{\sigma^2}{\beta^2} + x'x} = \frac{EQM(\hat{\beta})}{EQM(\hat{\beta}_{MQ})}$$

$$\frac{EQM(\hat{\beta})}{EQM(\hat{\beta}_{MQ})} = \frac{X'X}{\frac{\sigma^2}{\beta^2} + X'X} = \frac{\frac{\beta^2}{\sigma^2} X'X}{1 + \frac{\beta^2}{\sigma^2} X'X} = \frac{\tilde{\tau}^2}{1 + \tilde{\tau}^2} \quad \blacksquare$$

$$\tau \longrightarrow \infty, \quad \frac{EQM(\hat{\beta})}{EQM(\hat{\beta}_{MQ})} \longrightarrow 1.$$

EXERCÍCIO PARA SEXTA

Considere o seguinte modelo de regressão.

$$\mathbb{E}Y_i = \beta_0 + \beta_1 x_i + \beta_2(3x_i^2 - 2), \quad i = 1, 2, 3$$

onde $x_1 = -1$, $x_2 = 0$ e $x_3 = 1$.

Encontre os estimadores MQO para β_0, β_1 e β_2.