$$Y_i = \beta_1 x_i + \varepsilon_i$$

1. Considere o modelo $\underline{Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i}$, $i = 1, 2, \ldots, n$, com $\varepsilon \sim N(0, \sigma^2 I)$

Suponha que vc ajusta o modelo, mas o verdadeiro valor de $\beta_0 = 0$.

\> Compare as variâncias do estimador de β_1 nos dois cenários, i.e, quando $\beta_0 = 0$ e $\beta_0 \neq 0$.

Estimador de MQO $\quad Y = X\beta + \varepsilon$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n \times 1} \qquad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$\uparrow$

O estimador MQO é dado por $\hat{\beta} = (X'X)^{-1}X'Y$ $\quad cov(\hat{\beta}) = \sigma^2 (X'X)^{-1}$

$$\boxed{\hat{\beta}_1' = \frac{\sum (Y_i - \bar{Y})(x_i - \bar{x})}{\sum (x_i - \bar{x})^2}}$$

$$\boxed{Var(\hat{\beta}_1') = \frac{\sigma^2}{\sum (x_i - \bar{x})^2}}$$

Quando consideramos o β_0.

O EMQO sem o β_0 é dado por

$Y_i = \beta_1 x_i + \varepsilon_i$ ou $Y = \beta_1 X_1 + \varepsilon$

X_1 → vetor coluna

$$\hat{\beta}_1 = (X_1' X_1)^{-1} X_1' Y = \frac{\sum x_i Y_i}{\sum x_i^2}$$

$\langle X_1, X_1 \rangle = \|X_1\|^2$

$$\boxed{\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}}$$

A variância do $\hat{\beta}_1$ é dada por

$$var(\hat{\beta}_1) = \sigma^2 (X'X)^{-1} = \frac{\sigma^2}{\sum x_i^2}$$

Calculando a Eficiência relativa dos estimadores:

$$\frac{Var(\hat{\beta}_1)}{Var(\hat{\beta}_1')} = \frac{\dfrac{\sigma^2}{\sum x_i^2}}{\dfrac{\sigma^2}{\sum (x_i - \bar{x})^2}} = \frac{\sum (x_i - \bar{x})^2}{\sum x_i^2} = \frac{\sum x_i^2 - n\bar{x}^2}{\sum x_i^2} = 1 - \underbrace{\frac{n\bar{x}^2}{\sum x_i^2}}_{\text{não-negativo}} < 1$$

2. Seja $Y = X\beta + \varepsilon$, sem termo constante. Considere K regressores.

Calcule a estatística F para avaliar a significância dos parâmetros.

$$F = \frac{QM_{Reg}}{QM_{Res}}$$

$$F = \frac{\chi^2_v / v}{\chi^2_u / u}$$

Independentes

$$QM_{Reg} = \frac{\hat{\beta}' X' X \hat{\beta}}{k} \sim \chi^2_k$$

$$QM_{Res} = \frac{e'e}{n-k} \sim \chi^2_{n-k}$$

Independentes

$$R^2 = \frac{SQ_{Reg}}{SQ_{Total}}$$

$$\frac{SQ_{Total}}{SQ_{Total}} = \frac{SQ_{Reg}}{SQ_{Total}} + \frac{SQ_{Res}}{SQ_{Total}}$$

$$1 = R^2 + \frac{SQ_{Res}}{SQ_T}$$

$$\frac{SQ_{Res}}{SQ_{Total}} = 1 - R^2$$

sob H_0,

$$F_{k,n-k} = \frac{\overbrace{\frac{SQ_{Reg}}{k}}^{QM_{Reg}}}{\underbrace{\frac{SQ_{Res}}{n-k}}_{QM_{Res}}} = \frac{\frac{SQ_{Reg}}{k \cdot SQ_{Total}}}{\frac{SQ_{Res}}{(n-k)\, SQ_{Total}}} = \frac{R^2/k}{\frac{1-R^2}{n-k}}$$

$$= \frac{n-k}{k} \cdot \frac{R^2}{1-R^2} = F_{k,n-k}$$

// ✓

$$F_{k,n-k} = \frac{\hat\beta' X' X \hat\beta / k}{\dfrac{e'e}{n-k}}$$

$$e = \underbrace{(I-H)Y}_{Y-\hat{Y}} = (I-H)(X\beta+\varepsilon) = \cancel{(I-H)X}\beta + (I-H)\varepsilon = \underline{(I-H)\varepsilon}$$

$\cancel{(I-H)X} \to 0$

sabemos que $\hat\beta = \beta + \underbrace{(X'X)^{-1}X'\varepsilon}$. Assumindo $\underline{\beta = 0}$, então

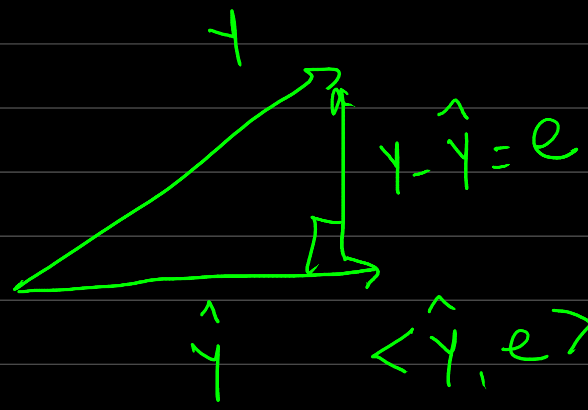
$$F_{k,n-k} = \frac{\varepsilon' X (X'X)^{-1} \cancel{X'X} \cancel{(X'X)^{-1}} X'\varepsilon / k}{\varepsilon'(I-H)'(I-H)\varepsilon/(n-k)} = \frac{\varepsilon' X (X'X)^{-1} X'\varepsilon / k}{\varepsilon'(I-H)\varepsilon/(n-k)} = \frac{n-k}{k} \cdot \frac{\varepsilon' H \varepsilon}{\varepsilon'(I-H)\varepsilon}$$

$$H \cdot (I-H) = 0$$

$$\mathbb{E}\, F_{k,n-k} = \frac{n-k}{n-k-2}$$ // Valor esperado da estatística de teste sob H_0.

Seja $Y = X\beta + \varepsilon$. Qual o valor $\underbrace{\sum_{i=1}^{n} \hat{Y}_i (Y_i - \hat{Y}_i)}_{\hat{Y}'(Y-\hat{Y})}$?

$$\underbrace{\hat{Y}' \cdot e}_{} = 0$$



$\langle \hat{Y}, e \rangle = 0.$

3. Seja $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$. Assuma A1 - A5

1. Mostre que $\hat{\beta}_0$ e $\hat{\beta}_1$ são não-correlacionados see $\bar{x} = 0$.

$$Y = X\beta + \varepsilon$$

$$\mathbb{1}'\mathbb{1} = \sum \mathbb{1} = n.$$

$$X = [\mathbb{1} \quad X_1]$$

$$X'X = \begin{bmatrix} \mathbb{1} \\ X_1 \end{bmatrix} [\mathbb{1} \quad X_1] = \begin{bmatrix} \mathbb{1}'\mathbb{1} & \mathbb{1}'X_1 \\ X_1'\mathbb{1} & X_1'X_1 \end{bmatrix} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} = \begin{bmatrix} n & n\bar{x} \\ n\bar{x} & \sum x_i^2 \end{bmatrix}$$

$2\times 1 \quad 1\times 2$

$$(X'X)^{-1} = \frac{1}{\underbrace{n\sum x_i^2 - (n\bar{x})^2}_{n\sum(x_i-\bar{x})^2}} \begin{bmatrix} \sum x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{bmatrix} = \underbrace{\frac{1}{\sum(x_i-\bar{x})^2} \begin{bmatrix} \frac{1}{n}\sum x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix}}$$

$$\operatorname{Cov}(\hat{\beta}) = \sigma^2 (X'X)^{-1} = \frac{\sigma^2}{\sum(x_i-\bar{x})^2} \begin{bmatrix} \frac{1}{n}\sum x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix}$$

$$\operatorname{Var}(\hat{\beta}_0) = \frac{\sigma^2 \cdot \frac{1}{n}\sum x_i^2}{\sum(x_i-\bar{x})^2} \qquad \operatorname{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum(x_i-\bar{x})^2} \qquad \operatorname{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\sigma^2\bar{x}}{\sum(x_i-\bar{x})^2}$$

- Suponha que vc quer testar $H_0: \beta_1 = c$.

sabemos que $\hat{\beta} = (X'X)^{-1}X'Y$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{\sum (Y_i - \bar{Y})(x_i - \bar{x})}{\sum (x_i - \bar{x})^2} \quad \checkmark$$

$$H_0: \beta_1 = c \quad \text{ou} \quad H_0: \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = c$$

$$\underline{k'\beta = c}$$

$$\frac{1}{\sum(x_i-\bar{x})^2}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{n}\sum x_i^2 & -\bar{x} \\ -\bar{x} & 1\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix} = \frac{1}{\sum(x_i-\bar{x})^2}\begin{bmatrix}-\bar{x} & 1\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix} = \frac{1}{\sum(x_i-\bar{x})^2}$$

$$K'(X'X)^{-1}K$$

$$SQ_{HLG} = (\hat{\beta}_1 - c)' \sum(x_i-\bar{x})^2 (\hat{\beta}_1 - c) = (\hat{\beta}_1 - c)^2 \cdot \sum(x_i-\bar{x})^2$$

1 g.l

1

$$\bar{Y} = \bar{\hat{Y}}$$

$$SQ_{Res} = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \underbrace{\sum (Y_i - \bar{Y})^2}_{SQ_{Total}} - \underbrace{\sum (\hat{Y}_i - \bar{Y})^2}_{SQ_{Reg}}$$

$$R^2 = 1 - \frac{SQ_{Res}}{SQ_{Total}} \Rightarrow \boxed{SQ_{Res} = (1 - R^2)\, SQ_{Total}}$$

Sob H_0.

$$F_{1,n-2} = \frac{QM_{Reg}}{QM_{Res}} = \frac{(\hat{\beta}_1 - c)^2 \sum (x_i - \bar{x})^2}{\frac{(1 - R^2)\, SQ_{Total}}{n-2}}$$

Se $c = 0$

$$F_{1,n-2} = \frac{\hat{\beta}_1^2 \sum (x_i - \bar{x})^2}{(1-R^2)\, SQ_{Total} / (n-2)} = \frac{\hat{\beta}_1^2 \sum (x_i - \bar{x})^2}{(1-R^2) \sum (y_i - \bar{y})^2 / (n-2)} = \frac{\left[\frac{\sum (y_i - \bar{y})(x_i - \bar{x})}{\sum (x_i - \bar{x})^2}\right]^2 \sum (x_i - \bar{x})^2}{(1-R^2) \sum (y_i - \bar{y})^2 / (n-2)}$$

$$\frac{\hat{\beta}_1 \sum (x_i - \bar{x})^2}{\sum (y_i - \bar{y})^2} = \frac{\sum (\cancel{x_i - \bar{x}})^2 \cdot \left[\sum (y_i - \bar{y})(x_i - \bar{x})\right]^2}{\sum (y_i - \bar{y})^2 \left[\sum (\cancel{x_i - \bar{x}})^2\right]^2} = \frac{\left[\sum (y_i - \bar{y})(x_i - \bar{x})\right]^2}{\sum (y_i - \bar{y})^2 \sum (x_i - \bar{x})^2} = R^2$$

$$F_{1,n-2} = \frac{R^2}{1 - R^2} \cdot (n-2).$$

> O que acontece com $H_0: \beta_0 = 0$.

Mostre que $F = \dfrac{QM_{REG}}{QM_{res}} = \dfrac{\hat{\beta}_0^2 \cdot \dfrac{n\sum(x_i-\bar{x})^2}{\sum x_i^2}}{\sum(Y_i-\hat{Y}_i)^2/(n-2)}$ $\triangleright$ Mostre!

> Dado $\bar{x}=0$, verifique que a estatística F fica da seguinte forma:

$$F = \frac{(\hat{\beta}_1 - \bar{Y})^2}{\dfrac{\sum(Y_i-\hat{Y})^2}{n-2}\left[\dfrac{1}{n} + \dfrac{1}{\sum(x_i-\bar{x})^2}\right]}.$$