

# Differential Geometry: Curves

## 1 Books

**1.1 Lee, J. M., Chow, B., Chu, S. C., Glickenstein, D., and Guenther, C. (2009), “Manifolds and differential geometry,” Topology. An International Journal of Mathematics, ams.org.**

- Chap 1: Differentiable Manifolds
- Chap 2: The Tangent Structure
- Chap 3: Immersion and Submersion
- Chap 4: Curves and Hypersurfaces in Euclidean Space
- Chap 8: Differential Forms
- Chap 13: Riemannian and Semi-Riemannian Geometry

**1.2 Kass, R. E., and Vos, P. W. (2011), Geometrical Foundations of Asymptotic Inference, John Wiley & Sons.**

Appendix C: Basic Concepts in Differential Geometry

- Manifolds
- Submanifolds
- The Tangent Space
- Connections
- Riemannian Metric
- Tensors
- Tubular Neighborhoods
- The Measure Determined by a Metric
- Curvature

## 2 Online Tutorials

YouTube tutorials on differential geometry

- Curves
- Arc length and reparameterization
- Arc length as a parameter
- Closed curves and periodic curves
- Curvature: intuition and derivation

### 2.1 Curves

- 1) Level Curves: 2-D and 3-D set,  $C$ . The intersection of 2 surfaces in 3-D is a curve.
- 2) Parameterized Curves: a map  $\gamma : t \rightarrow \mathbb{R}^n, t \in \mathbb{R}$ .
- 3) Smooth function: infinite differentiable. All components of a parameterized curve  $\gamma(t)$  are differentiable.
- 4) Tangent vector: the tangent vector of  $\gamma(t)$  at the parameter value  $t$  is  $\frac{d\gamma}{dt}$ .

Introducing Differential Geometry : Curves

- In 2-D, we can describe a curve  $C$  as :  $y = y(x) \Leftrightarrow f(x, y) = K$  ~ more general way to describe curve.
- In 2-D:
 
$$C = \left\{ \underbrace{(x, y)}_{\text{ordered pair of real numbers}} \in \mathbb{R}^2 \mid f(x, y) = K \right\}$$

such that
- In 3-D :
 
$$C = \left\{ (x, y, z) \in \mathbb{R}^3 \mid f_1(x, y, z) = K_1, f_2(x, y, z) = K_2 \right\}$$

comma means 'and,' which means intersection  
 $\downarrow$   
 $\begin{aligned} z &= g_1(x, y) \\ &\text{- a surface} \end{aligned}$ 
 $\downarrow$   
 $\begin{aligned} z &= g_2(x, y) \\ &\text{- also a surface} \end{aligned}$

intersection of 2 surfaces is a curve.

• In 3-0:

$$C = \{ (x, y, z) \in \mathbb{R}^3 \mid f_1(x, y, z) = K_1, f_2(x, y, z) = K_2 \}$$

$\downarrow$                                      $\downarrow$   
 $z = g_1(x, y)$                              $z = g_2(x, y)$   
- a surface                                    - also a surface

↓  
intersection of 2  
surfaces is a curve.

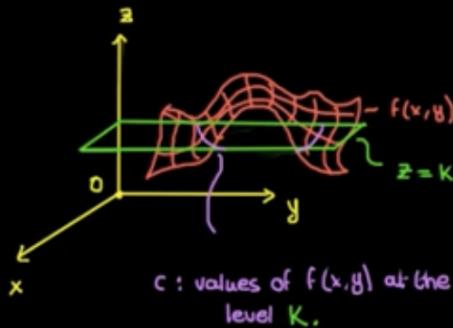
$$C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = K\}$$

- Parametrized Curves: a parametrized curve in  $\mathbb{R}^n$  is a map:

e.g., if  $t$  is mapped to a triple of real numbers  $(x, y, z) \rightarrow$  we have a curve in 3-D.

$$\mathbf{r}(t) = [r_1(t), r_2(t), \dots]$$

e.g. in 3-D,  $\mathbf{r}(t) = [x(t), y(t), z(t)]$



e.g., if it is mapped to a triple of real numbers  $(x, y, z) \rightarrow$   
we have a curve in 3-D.

$$\gamma : t \rightarrow \mathbb{R}^n, \quad t \in \mathbb{R},$$

$\alpha < t < \beta$  where  $\alpha, \beta \in \mathbb{R}$ .

e.g. in 3-D,  $\mathbf{r}(t) = [x(t), y(t), z(t)]$

- Intuitively,  $t$  is the time and as  $t$  increases, we trace out the parametric curve.

- Smooth Function: a function  $f(t)$  is said to be smooth on the open interval  $(\alpha, \beta)$  if the derivative  $\frac{d^n f}{dt^n}$  exists for all  $n \geq 1$  and all  $t \in (\alpha, \beta)$ .

$$\Upsilon(t) = [\Upsilon_1(t), \Upsilon_2(t), \dots] \longrightarrow \frac{d^n \Upsilon}{dt^n} = \left[ \frac{d^n \Upsilon_1}{dt^n}, \frac{d^n \Upsilon_2}{dt^n}, \dots \right]$$

- A parametrized curve  $\gamma(t)$  given by is smooth on the interval  $\alpha < t < \beta$  if the derivatives  $\frac{d^n \gamma}{dt^n}$  exist for all  $n \geq 1$  and all  $t \in (\alpha, \beta)$  i.e. All components of  $\gamma(t)$  are differentiable.

we have a curve  
in 3-D.  
 $\gamma(t) = [Y_1(t), Y_2(t), \dots]$   
e.g., in 3-D,  $\gamma(t) = [x(t), y(t), z(t)]$

- Intuitively,  $t$  is the time and as  $t$  increases, we trace out the parametric curve.
- Smooth Function: a function  $f(t)$  is said to be smooth on the open interval  $(\alpha, \beta)$  if the derivative  $\frac{d^n f}{dt^n}$  exists for all  $n \geq 1$  and all  $t \in (\alpha, \beta)$ .

$$\underbrace{\gamma(t) = [Y_1(t), Y_2(t), \dots]} \longrightarrow \frac{d^n \gamma}{dt^n} = \left[ \frac{d^n Y_1}{dt^n}, \frac{d^n Y_2}{dt^n}, \dots \right]$$

- A parametrized curve  $\gamma(t)$  given by is smooth on the interval  $\alpha < t < \beta$  if the derivatives  $\frac{d^n \gamma}{dt^n}$  exist for all  $n \geq 1$  and all  $t \in (\alpha, \beta)$  → i.e. All components of  $\gamma(t)$  are differentiable.
- Tangent vector: the tangent vector of  $\gamma(t)$  at the parameter value  $t$  is  $d\gamma/dt$ .

## 2.2 Arc length and reparameterization

1) Arc length definition: the arc length of a curve  $\gamma$  starting at the point  $\gamma(\alpha)$  is the function

$S(t)$  given by:

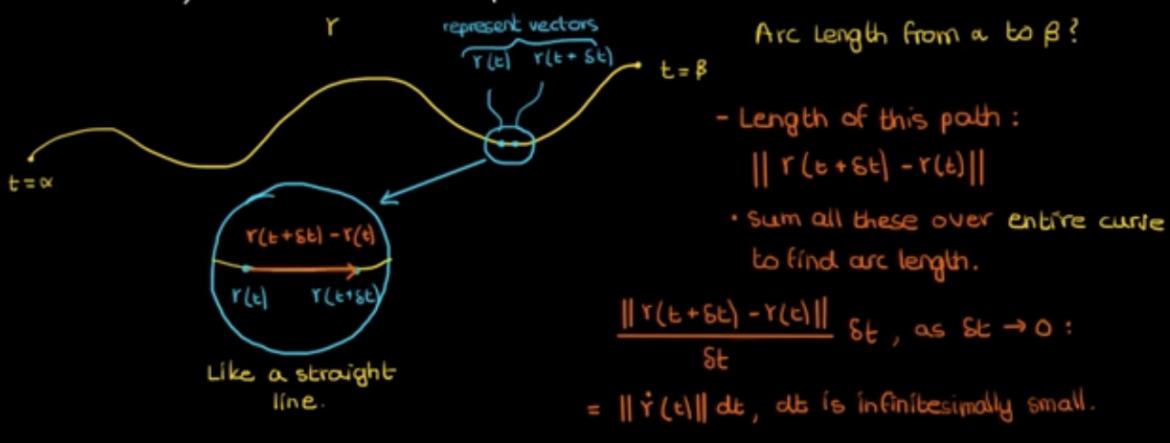
$$S(t) = \int_{\alpha}^t \|\dot{\gamma}(u)\| du.$$

2) Reparameterization: smooth bijective map exists such that the inverse map is also smooth.

3) Regular point, singular point and regular curve.

### Arc Length

- A parametrized curve in  $\mathbb{R}^n$  is a map:  $\gamma: t \rightarrow \mathbb{R}^n$ ,  $\gamma(t) = [Y_1(t), Y_2(t), \dots]$
- Intuitively,  $t$  is the time and as  $t \in \mathbb{R}$ ,  $\alpha < t < \beta$ , where  $\alpha, \beta \in \mathbb{R}$  increases, we trace out the parametric curve.



Arc Length Definition: The arc length of a curve  $\gamma$  starting at the point  $\gamma(\alpha)$  is the function  $S(t)$  given by: 
$$S(t) = \int_{\alpha}^t \underbrace{\|\dot{\gamma}(u)\| du}_{\begin{array}{l} \text{velocity} \\ \text{speed} \end{array}}$$

Speed: If  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^n$  is a parametrized curve, its speed at the point  $\gamma(t)$  is  $\|\dot{\gamma}(t)\|$ .  $\gamma$  is said to be a unit-speed curve if  $\dot{\gamma}(t)$  is a unit vector for all  $t \in (\alpha, \beta)$ . (i.e.  $\|\dot{\gamma}(t)\| = 1$ )

Theorem 1: Suppose  $\gamma(t)$  is a smooth unit speed curve. Then,  $\ddot{\gamma}(t) \cdot \ddot{\gamma}(t) = 0$ .

Proof: Since  $\gamma(t)$  is unit speed,  $\|\dot{\gamma}(t)\| = 1 \rightarrow \|\dot{\gamma}(t)\|^2 = 1$   

$$\ddot{\gamma}(t) \cdot \dot{\gamma}(t) = 1, \text{ differentiable w.r.t. } t;$$

Thus, for a particle travelling at constant speed, the acceleration is either 0 or perpendicular to the particle's velocity.

$$\begin{aligned} \ddot{\gamma}(t) \cdot \dot{\gamma}(t) + \dot{\gamma}(t) \cdot \ddot{\gamma}(t) &= 0 \\ \therefore 2\dot{\gamma}(t) \cdot \ddot{\gamma}(t) &= 0 \\ \rightarrow \dot{\gamma}(t) \cdot \ddot{\gamma}(t) &= 0, \text{ QED.} \end{aligned}$$

### Reparametrization

$\gamma(t), t \in (\alpha, \beta) \xrightarrow{\text{change to}} \tilde{\gamma}(\tilde{t}), \tilde{t} \in (\bar{\alpha}, \bar{\beta})$

Definition: A parametrized curve  $\tilde{\gamma} : \tilde{t} \in (\bar{\alpha}, \bar{\beta}) \rightarrow \mathbb{R}^n$  is a **reparametrization** of a parametrized curve  $\gamma : t \in (\alpha, \beta) \rightarrow \mathbb{R}^n$  if there is a smooth bijective map  $\phi : \tilde{t} \in (\bar{\alpha}, \bar{\beta}) \rightarrow t \in (\alpha, \beta)$  (the **reparametrization map**) such that the inverse map  $\phi^{-1} : t \in (\alpha, \beta) \rightarrow \tilde{t} \in (\bar{\alpha}, \bar{\beta})$  is also smooth and  $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$  for all  $\tilde{t} \in (\bar{\alpha}, \bar{\beta})$ .

\* Similarly, since  $\phi$  has a smooth inverse,  $\tilde{\gamma}$  is also a **reparametrization** of  $\gamma$ .

Example:  $x^2 + y^2 = 1, \gamma(t) = [\cos t, \sin t], t \in [0, 2\pi]$

$$\tilde{\gamma}(\tilde{t}) = [\sin \tilde{t}, \cos \tilde{t}], \tilde{t} \in [0, 2\pi]$$

$$\begin{aligned} t &= \phi(\tilde{t}) = \underbrace{\frac{\pi}{2} - \tilde{t}}_{\text{bijective} \neq \text{smooth}} & \tilde{t} &= \phi^{-1}(t) = \frac{\pi}{2} - t : \text{smooth} \\ \therefore & & \end{aligned}$$

Example:  $x^2 + y^2 = 1$ ,  $\gamma(t) = [\cos t, \sin t]$ ,  $t \in [0, 2\pi]$

$$\tilde{\gamma}(\bar{t}) = [\sin \bar{t}, \cos \bar{t}], \bar{t} \in [0, 2\pi]$$

$$t = \phi(\bar{t}) = \frac{\pi}{2} - \bar{t}$$

bijection  $\neq$  smooth

$$\bar{t} = \phi^{-1}(t) = \frac{\pi}{2} - t : \text{smooth}$$

Definitions: Suppose  $\gamma$  is a parametrized curve.

- A)  $\gamma(t)$  is a regular point if  $\dot{\gamma}(t) \neq 0$ .
- B)  $\gamma(t)$  is a singular point if  $\dot{\gamma}(t) = 0$ .
- C)  $\gamma$  is a regular curve if  $\dot{\gamma}(t) \neq 0$  at all points of the curve.

⋮

## 2.3 Arc length as a parameter

1-3) Proof of the reparameterization with regard to arc length is a valid option.

4) Theorem about unit speed curve.

### Arc Length as a Parameter

Suppose  $\gamma = \gamma(t)$  is a smooth, regular curve on the interval  $I$  in  $\mathbb{R}^n$ .

Definition: A parametrized curve  $\tilde{\gamma} : \bar{t} \in (\bar{\alpha}, \bar{\beta}) \rightarrow \mathbb{R}^n$  is a reparameterization of the parametrized curve  $\gamma : t \in (\alpha, \beta) \rightarrow \mathbb{R}^n$  if there is a smooth, bijective map

$\phi : \bar{t} \in (\bar{\alpha}, \bar{\beta}) \rightarrow t \in (\alpha, \beta)$  (i.e. the reparameterization map) such that the inverse map  $\phi^{-1} : t \in (\alpha, \beta) \rightarrow \bar{t} \in (\bar{\alpha}, \bar{\beta})$  is also smooth and  $\tilde{\gamma}(\bar{t}) = \gamma(\phi(\bar{t}))$  for all  $\bar{t} \in (\bar{\alpha}, \bar{\beta})$ .

To reparametrize w.r.t. arc length:

- 1) The reparameterization map from  $t$  to arc length must be smooth.
- 2) The reparameterization map must be bijective (one-to-one) with a smooth inverse.

Definition: The arc length  $S$  of a curve  $\gamma$  starting at the point  $\gamma(t_0)$  is the function  $S(t)$  given by:

$$S(t) = \int_{t_0}^t \|\dot{\gamma}(u)\| du — (1)$$

If  $t \geq t_0$ ,  $S \geq 0$  and is equal to the arc length of  $\gamma$  from  $t_0$  to  $t$ . If  $t < t_0$ ,  $S$  is negative of the arc length of  $\gamma$  between  $t_0$  and  $t$ .

Definition: The arc length  $S$  of a curve  $\gamma$  starting at the point  $\gamma(t_0)$  is the function  $S(t)$  given by:

$$S(t) = \int_{t_0}^t \|\dot{\gamma}(u)\| du \quad (1)$$

If  $t \geq t_0$ ,  $S \geq 0$  and is equal to the arc length of  $\gamma$  from  $t_0$  to  $t$ . If  $t < t_0$ ,  $S$  is negative of the arc length of  $\gamma$  between  $t_0$  and  $t$ .

$$\frac{dS}{dt} = \frac{d}{dt} \int_{t_0}^t \|\dot{\gamma}(u)\| du \implies \frac{dS}{dt} = \|\dot{\gamma}(t)\| \quad (2)$$

If  $\gamma(t) = [\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)]$ , then  $\dot{\gamma}(t) = [\dot{\gamma}_1(t), \dot{\gamma}_2(t), \dots, \dot{\gamma}_n(t)]$ .

$$\therefore \|\dot{\gamma}(t)\| = \sqrt{\underbrace{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dots + \dot{\gamma}_n^2}_{\text{since } \gamma \text{ is smooth, this expression is always smooth}}} > 0$$

$\hookrightarrow$  lies b/w 0 and  $\infty$

\* The square root of a smooth function is itself smooth on the interval  $(0, \infty)$ :  $y = \sqrt{x}$  is infinitely differentiable b/w 0 and  $\infty$ .

$\therefore$  since  $\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dots + \dot{\gamma}_n^2$  lies between 0 &  $\infty$ ,  $\sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dots + \dot{\gamma}_n^2}$  is smooth. Thus,  $dS/dt$  is smooth  $\rightarrow S(t)$  is smooth.  $\therefore$

If  $\gamma(t) = [\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)]$ , then  $\dot{\gamma}(t) = [\dot{\gamma}_1(t), \dot{\gamma}_2(t), \dots, \dot{\gamma}_n(t)]$ .

$$\therefore \|\dot{\gamma}(t)\| = \sqrt{\underbrace{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dots + \dot{\gamma}_n^2}_{\text{since } \gamma \text{ is smooth, this expression is always smooth}}} > 0$$

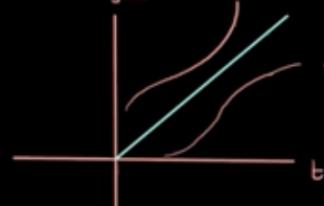
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\* The square root of a smooth function is itself smooth on the interval  $(0, \infty)$ :  $y = \sqrt{x}$  is infinitely differentiable b/w 0 and  $\infty$ .

$\therefore$  since  $\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dots + \dot{\gamma}_n^2$  lies between 0 &  $\infty$ ,  $\sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dots + \dot{\gamma}_n^2}$  is smooth. Thus,  $dS/dt$  is smooth  $\rightarrow S(t)$  is smooth.

$$\frac{dS}{dt} = \|\dot{\gamma}(t)\| \quad (2)$$

Since  $\gamma$  is regular,  $\dot{\gamma}$  is never 0. Thus,  $\|\dot{\gamma}(t)\| > 0$  always, so  $dS/dt$  is always positive for a regular curve.



$$s^{-1}(t) \rightarrow \text{smooth.}$$

$\therefore$  reparametrizing w.r.t. arc length is a valid option.

Theorem: If  $Y = Y(s)$  is a unit speed curve, then  $|s_2 - s_1|$  is the arc length b/w the points corresponding to  $Y(s_1)$  and  $Y(s_2)$ .

Proof: Since  $Y$  is unit speed,  $\left\| \frac{dY}{ds} \right\| = 1$ .  $S = \int_{s_1}^{s_2} \left\| \frac{dY}{ds} \right\| ds = |s_2 - s_1|$

Theorem: If  $Y(s)$  and  $\bar{Y}(\bar{s})$  both represent unit-speed parametrizations of the same curve, then  $\bar{s} = \pm s + c$ , where  $c$  is a constant.

Proof:  $\bar{s} = \bar{s}(s)$ .  $\frac{d\bar{s}}{ds} = \frac{d\bar{s}}{d\bar{s}} \frac{d\bar{s}}{ds} \rightarrow \left\| \frac{d\bar{s}}{ds} \right\| = \left\| \frac{d\bar{s}}{d\bar{s}} \right\| \cdot \left\| \frac{d\bar{s}}{ds} \right\| \rightarrow \left\| \frac{d\bar{s}}{ds} \right\| = 1$   
 $\rightarrow \frac{d\bar{s}}{ds} = \pm 1 \rightarrow \bar{s} = \pm s + c$

## 2.4 Closed curves and periodic curves

- 1) T-periodic:  $Y(t+T) = Y(t), T \in \mathbb{R}$ .
- 2) Self-intersection of a curve.
- 3) Theorem: a unit-speed reparameterization of a regular closed curve  $\gamma$  is always closed.

Closed Curves

Definition: If  $Y(t)$  is a smooth curve  $\mathbb{R} \rightarrow \mathbb{R}^n$ , then  $Y(t)$  is said to be T-periodic if  $Y(t+T) = Y(t), T \in \mathbb{R}$ .

- All curves  $Y(t)$  are 0-periodic. If  $Y$  were constant, it would be T-periodic for any  $T$ .

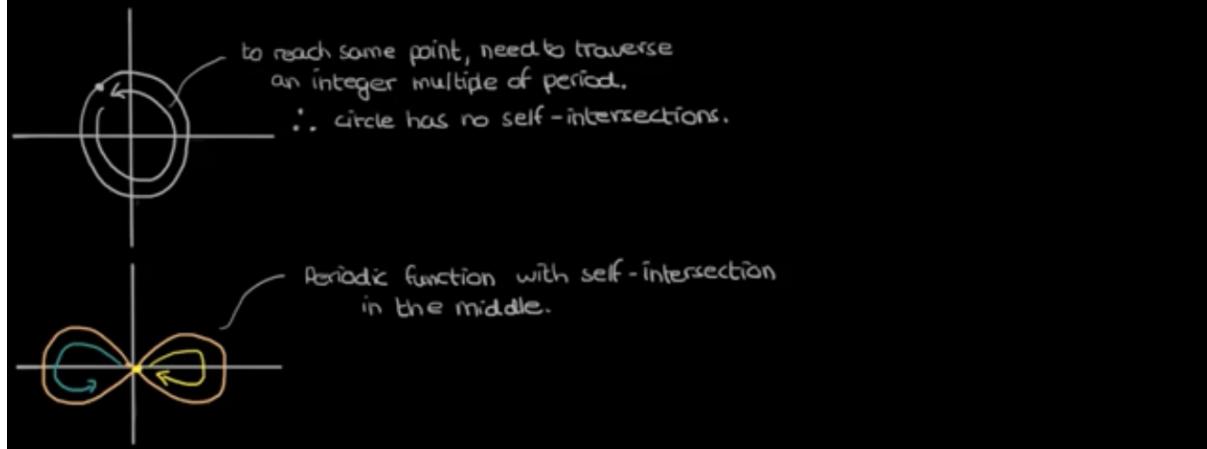
Definition: A non-constant  $Y(t)$  is a closed function if it is T-periodic and  $T \neq 0$ .

Definition: The period of a closed curve  $Y$  is the smallest positive number T such that  $Y$  is T-periodic.

Definition: The period of a closed curve  $\gamma$  is the smallest positive number  $T$  such that  $\gamma$  is  $T$ -periodic.

Definition: A curve  $\gamma$  has a self-intersection at a point  $\vec{p}$  if there exist parameter values  $a \neq b$  such that:

- i)  $\gamma(a) = \gamma(b) = \vec{p}$
- ii) If  $\gamma$  is closed with period  $T$ , then  $a - b$  is not an integer multiple of  $T$ .



Theorem: If  $\gamma$  is a regular closed curve, a unit-speed reparameterization of  $\gamma$  is always closed.

Proof: Suppose that  $\gamma$  is  $T$ -periodic with period  $T$ .

$$\begin{aligned}
 L_\gamma &= \int_0^T \left\| \frac{d\gamma}{dt} \right\| dt, \quad s(t+T) = \int_0^{t+T} \left\| \frac{d\gamma}{du} \right\| du \\
 &= \int_0^T \left\| \frac{d\gamma}{du} \right\| du + \underbrace{\int_T^{t+T} \left\| \frac{d\gamma}{du} \right\| du}_{L_\gamma} \\
 &\therefore s(t+T) = L_\gamma + s(t) \\
 &\rightarrow s(t+T) - s(t) = L_\gamma \\
 &\rightarrow s(t+mT) - s(t) = m L_\gamma, \quad m \text{ is an integer}
 \end{aligned}$$

Since  $L_\gamma$  is full length of curve,

$$\gamma[s(t+mT)] = \gamma[s(t)], \quad m \text{ is an integer.}$$

Thus,  $\gamma(s)$  is a periodic function with period  $L_\gamma$ , so  $\gamma(s)$  must be closed.

## 2.5 Curvature: intuition and derivation

- 1) Curvature: the extent to which  $\gamma$  deviates from a straight line.

$$K(s) = \left\| \frac{d^2\gamma}{ds^2} \right\|,$$

which shows how quickly the tangent vector changes.

2-4) Formula for curvature when  $\gamma(t)$  is a regular curve with unit speed parameter  $s$ :

$$k = \frac{\left\| \frac{d^2r}{dt^2} \times \frac{dr}{dt} \right\|}{\left\| \frac{dr}{dt} \right\|^3}$$

4) Radius of curvature:  $r = 1/k$ .

Curvature

Curvature of  $\gamma$ : extent to which  $\gamma$  deviates from a straight line.

$\gamma(s)$ :  $s$  is arc length

$\vec{t}$ : tangent vector

$\gamma(s+ss)$

perpendicular

$\hat{n}$

$\left[ \gamma(s+ss) - \gamma(s) \right] \cdot \hat{n} \quad \text{--- (1)}$

to isolate component responsible for curvature.

Taylor expand  $\gamma(s+ss)$  about  $s$ :

$\gamma(s+ss) = \gamma(s) + \frac{d\gamma}{ds} ss + \frac{d^2\gamma}{ds^2} \frac{ss^2}{2!} + \dots$ , plug this into (1):

$\rightarrow \left[ \gamma(s) + \cancel{\frac{d\gamma}{ds} ss} + \cancel{\frac{d^2\gamma}{ds^2} \frac{ss^2}{2!}} + \dots - \gamma(s) \right] \cdot \hat{n}$

$\rightarrow \left[ \frac{d^2\gamma}{ds^2} \frac{ss^2}{2!} \right] \cdot \hat{n}$ ,  $K(s) \equiv \left\| \frac{d^2\gamma}{ds^2} \right\|$  ↳ how quickly the tangent vector changes (how 'curved' the curve  $\gamma$  is)

Formula for curvature when  $\gamma(t)$  is a regular curve with unit speed parameter  $s$  (arc length):  $\frac{dr}{ds} = \frac{dr/dt}{ds/dt} = \frac{d^2\gamma/ds^2}{ds/dt} \cdot ds/dt$

$$\frac{d\gamma}{dt} = \frac{d\gamma}{ds} \frac{ds}{dt} \rightarrow \frac{d^2\gamma}{dt^2} = \frac{d}{dt} \left( \frac{d\gamma}{ds} \right) \frac{ds}{dt} + \frac{d\gamma}{ds} \frac{d^2s}{dt^2} \rightarrow \frac{d^2\gamma}{dt^2} = \frac{d^2\gamma}{ds^2} \left( \frac{ds}{dt} \right)^2 + \frac{d\gamma}{ds} \frac{d^2s}{dt^2}$$

Isolate  $\frac{d^2\gamma}{ds^2}$ :

$$\frac{d^2\gamma}{ds^2} = \frac{\frac{d^2\gamma}{dt^2} - \frac{d\gamma}{ds} \frac{d^2s}{dt^2}}{\left( \frac{ds}{dt} \right)^2} \times \left( \frac{ds}{dt} \right)^2$$

$$\rightarrow \frac{d^2\gamma}{ds^2} = \frac{\frac{d^2\gamma}{dt^2} \left( \frac{ds}{dt} \right)^2 - \cancel{\frac{d\gamma}{ds} \frac{d^2s}{dt^2} \left( \frac{ds}{dt} \right)^2}}{\left( \frac{ds}{dt} \right)^4} = \frac{\frac{d^2\gamma}{dt^2} \left( \frac{ds}{dt} \right)^2 - \frac{d\gamma}{dt} \frac{d^2s}{dt^2} \frac{ds}{dt}}{\left( \frac{ds}{dt} \right)^4} \quad \leftarrow \begin{array}{l} \text{substitute all} \\ \text{s derivatives} \end{array} \text{for } \gamma$$

$\frac{d\gamma}{dt} = \frac{d\gamma}{ds} \frac{ds}{dt}$  since  $s$  is unit speed parameter  $\rightarrow \left( \frac{d\gamma}{dt} \right)^2 = \left( \frac{ds}{dt} \right)^2$ ,  $d/dt$  both sides:

$$\frac{ds}{dt} \frac{d^2s}{dt^2} = \frac{d\gamma}{dt} \cdot \frac{d^2\gamma}{dt^2}$$

$$\rightarrow \frac{d^2\gamma}{ds^2} = \frac{\frac{d^2\gamma}{dt^2} \cdot \left( \frac{ds}{dt} \cdot \frac{ds}{dt} \right) - \frac{d\gamma}{dt} \cdot \left( \frac{ds}{dt} \cdot \frac{d^2\gamma}{dt^2} \right)}{\left( \frac{ds}{dt} \right)^4}, \quad a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

$$\frac{ds^2}{ds^2} = \frac{\alpha \cdot (\alpha \cdot \omega) - \alpha \cdot (\alpha \cdot \omega)}{\left(\frac{d\gamma}{dt}\right)^4}, \quad \alpha \times (\beta \times \gamma) = (\alpha \cdot \beta) \gamma - (\alpha \cdot \gamma) \beta$$

$$\frac{d^2\gamma}{ds^2} = \frac{\frac{d\gamma}{dt} \times \left( \frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt} \right)}{\left(\frac{d\gamma}{dt}\right)^4}, \text{ take magnitudes:}$$

$$\kappa = \left\| \frac{d^2\gamma}{ds^2} \right\| = \frac{\left\| \frac{d\gamma}{dt} \times \left( \frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt} \right) \right\|}{\left\| \frac{d\gamma}{dt} \right\|^4}, \text{ separate out magnitudes of cross products on outside:}$$

$$\kappa = \frac{\left\| \frac{d\gamma}{dt} \right\| \left\| \frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt} \right\|}{\left\| \frac{d\gamma}{dt} \right\|^3} \rightarrow \kappa = \frac{\left\| \frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt} \right\|}{\left\| \frac{d\gamma}{dt} \right\|^3}$$

$$\kappa = \left\| \frac{d^2\gamma}{ds^2} \right\| = \frac{\left\| \frac{d\gamma}{dt} \times \left( \frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt} \right) \right\|}{\left\| \frac{d\gamma}{dt} \right\|^4}, \text{ separate out magnitudes of cross products on outside:}$$

$$\kappa = \frac{\left\| \frac{d\gamma}{dt} \right\| \left\| \frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt} \right\|}{\left\| \frac{d\gamma}{dt} \right\|^3} \rightarrow \kappa = \frac{\left\| \frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt} \right\|}{\left\| \frac{d\gamma}{dt} \right\|^3}$$

Radius of curvature:  $\rho \equiv 1/\kappa$

### 3 Discussion