Differencial Geometry: Curves

1 Books

1.1 Lee, J. M., Chow, B., Chu, S. C., Glickenstein, D., and Guenther, C. (2009), "Manifolds and differential geometry," Topology. An International Journal of Mathematics, ams.org.

- Chap 1: Differentiable Manifolds
- Chap 2: The Tangent Structure
- Chap 3: Immersion and Submersion
- Chap 4: Curves and Hypersurfaces in Euclidean Space
- Chap 8: Differential Forms
- Chap 13: Riemannian and Semi-Riemannian Geometry

1.2 Kass, R. E., and Vos, P. W. (2011), Geometrical Foundations of Asymptotic Inference, John Wiley & Sons.

Appendix C: Basic Concepts in Differential Geometry

- Manifolds
- Submanifolds
- The Tangent Space
- Connections
- Riemannian Metric
- Tensors
- Tubular Neighborhoods
- The Measure Determined by a Metric
- Curvature

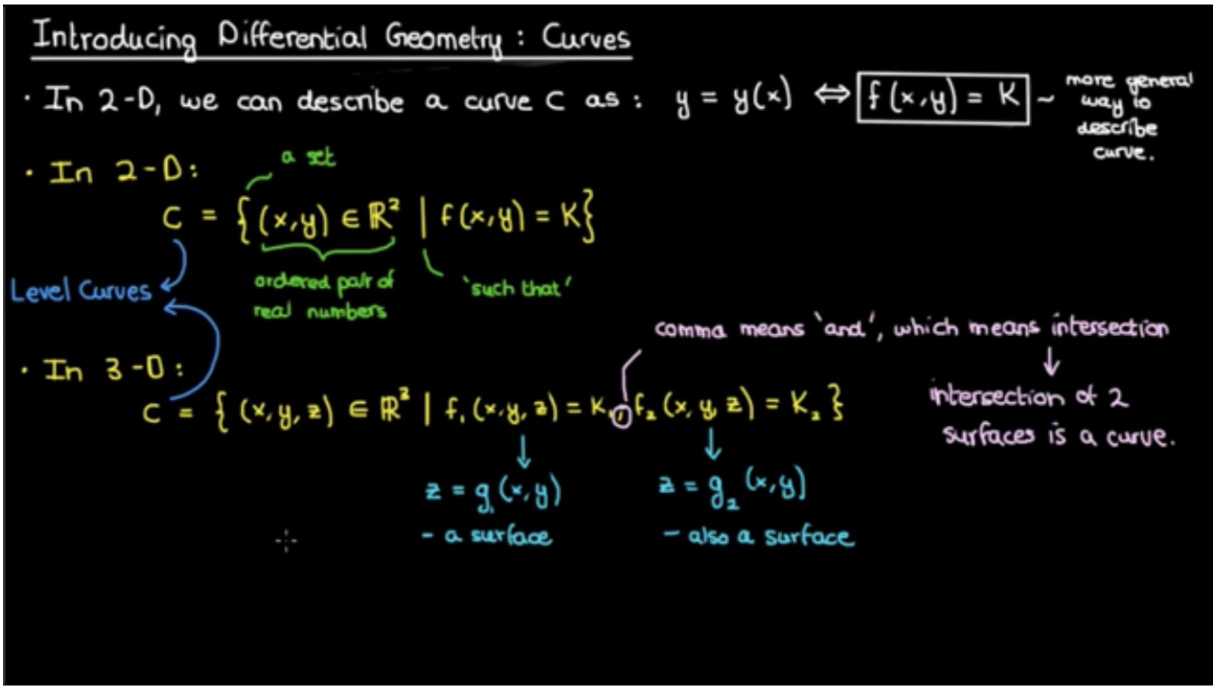
2 Online Tutorials

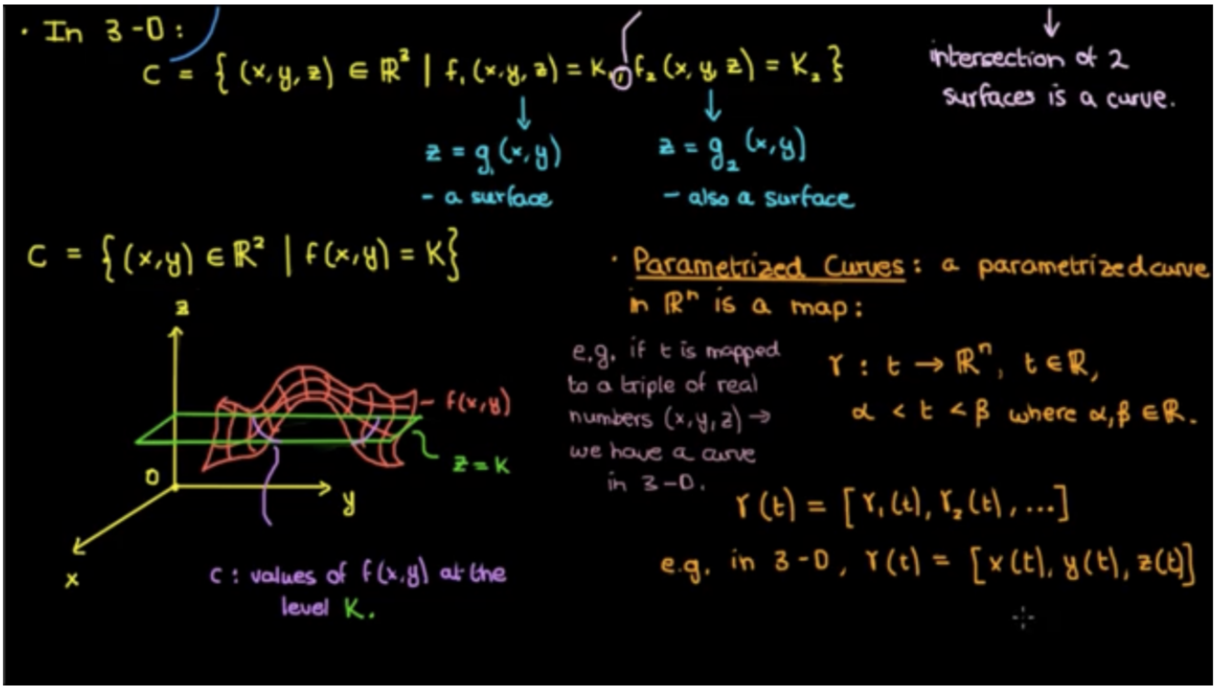
YouTube tutorials on differential geometry

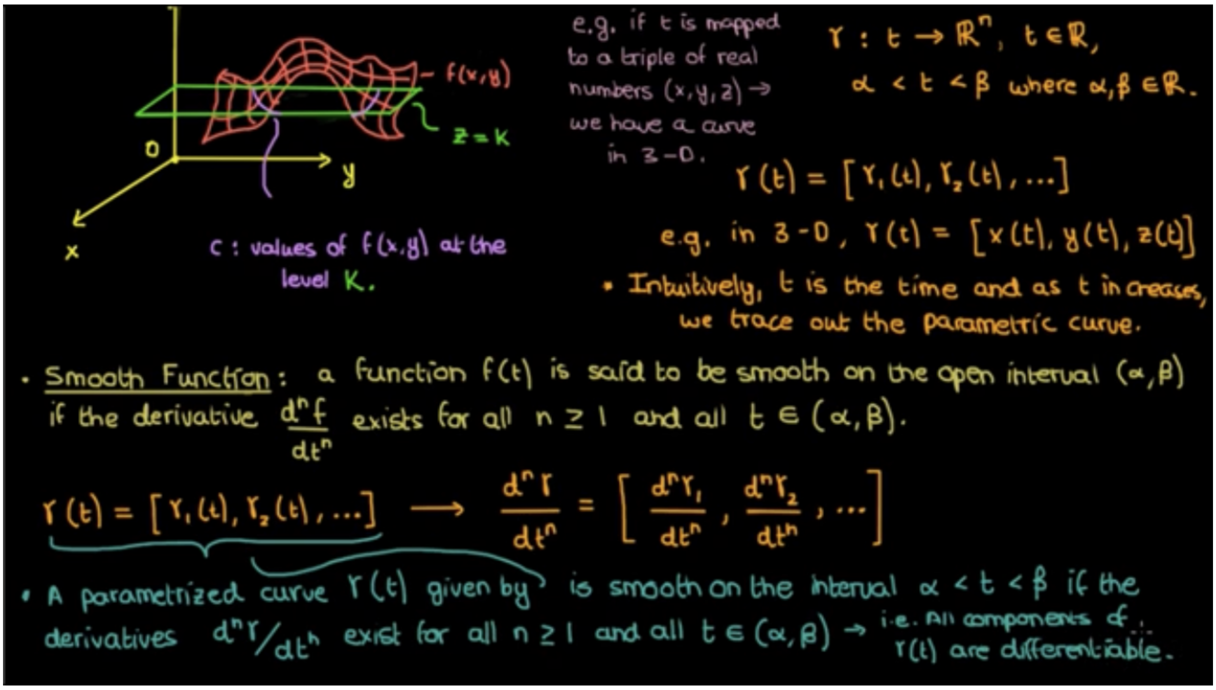
- Curves
- Arc length and reparameterization
- Arc length as a parameter
- Closed curves and periodic curves
- Curvature: intuition adn derivation

2.1 Curves

1) Level Curves: 2-D and 3-D set, C. The intersection of 2 surfaces in 3-D is a curve.
2) Parameterized Curves: a map $\gamma : t \to \mathbb{R}^n, t \in \mathbb{R}$.
3) Smooth function: infinite differentiable. All components of a parameterized curve $\gamma(t)$ are differentiable.
4) Tangent vector: the tangent vector of $\gamma(t)$ at the parameter value t is $\frac{d\gamma}{dt}$.




• In 3-D:
$C = \{ (x,y,z) \in \mathbb{R}^3 \mid f_1(x,y,z) = K_1 \cap f_2(x,y,z) = K_2 \}$
intersection of 2 surfaces is a curve.
$z = g_1(x,y)$
- a surface
$z = g_2(x,y)$
- also a surface
$C = \{ (x,y) \in \mathbb{R}^2 \mid f(x,y) = K \}$
z
$f(x,y)$
$z = K$
0
y
x
C : values of $f(x,y)$ at the level K.
• Parametrized Curves: a parametrized curve in $\mathbb{R}^n$ is a map:
e.g. if t is mapped to a triple of real numbers $(x,y,z) \rightarrow$ we have a curve in 3-D.
$\gamma : t \rightarrow \mathbb{R}^n,\ t \in \mathbb{R},$
$\alpha < t < \beta$ where $\alpha, \beta \in \mathbb{R}$.
$\gamma(t) = [\gamma_1(t), \gamma_2(t), \ldots]$
e.g. in 3-D, $\gamma(t) = [x(t), y(t), z(t)]$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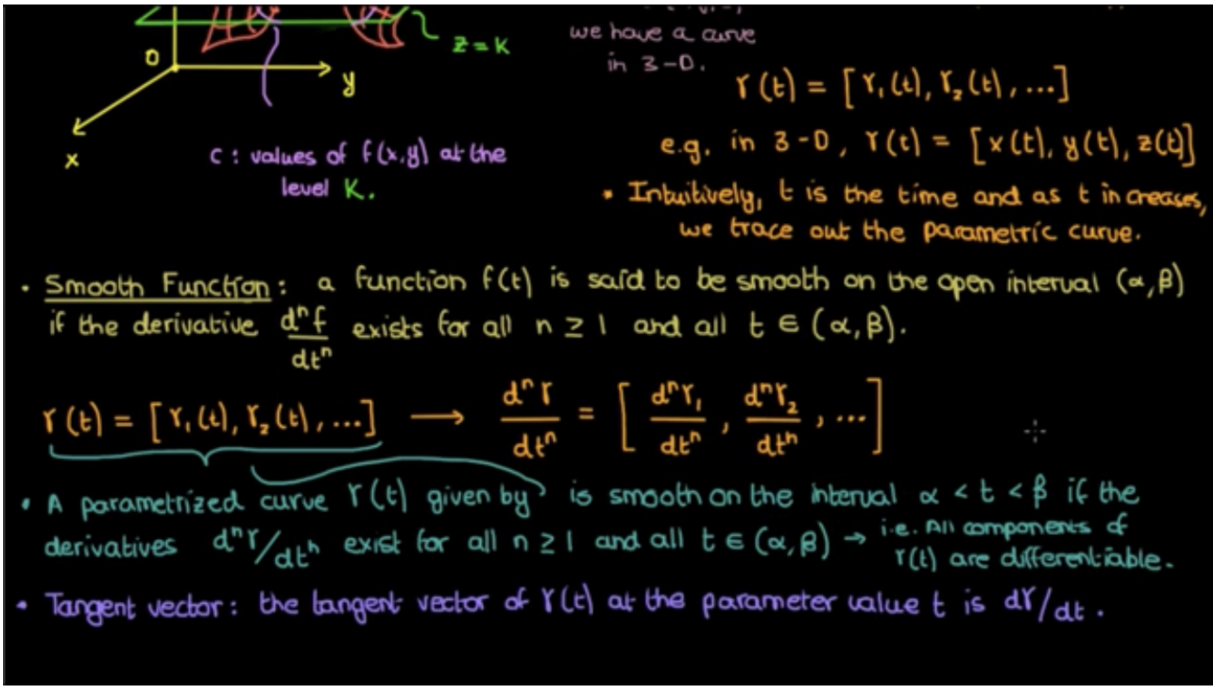
$\gamma(t) = [\gamma_1(t), \gamma_2(t), \ldots]$
e.g. in 3-D, $\gamma(t) = [x(t), y(t), z(t)]$
• Intuitively, t is the time and as t increases, we trace out the parametric curve.
• Smooth Function: a function $f(t)$ is said to be smooth on the open interval (α, β) if the derivative $\frac{d^n f}{dt^n}$ exists for all $n \geq 1$ and all $t \in (\alpha, \beta)$.
$\gamma(t) = [\gamma_1(t), \gamma_2(t), \ldots] \longrightarrow \frac{d^n \gamma}{dt^n} = \left[\frac{d^n \gamma_1}{dt^n}, \frac{d^n \gamma_2}{dt^n}, \ldots \right]$
• A parametrized curve $\gamma(t)$ given by is smooth on the interval $\alpha < t < \beta$ if the derivatives $d^n\gamma/dt^n$ exist for all $n \geq 1$ and all $t \in (\alpha, \beta) \rightarrow$ i.e. All components of $\gamma(t)$ are differentiable.



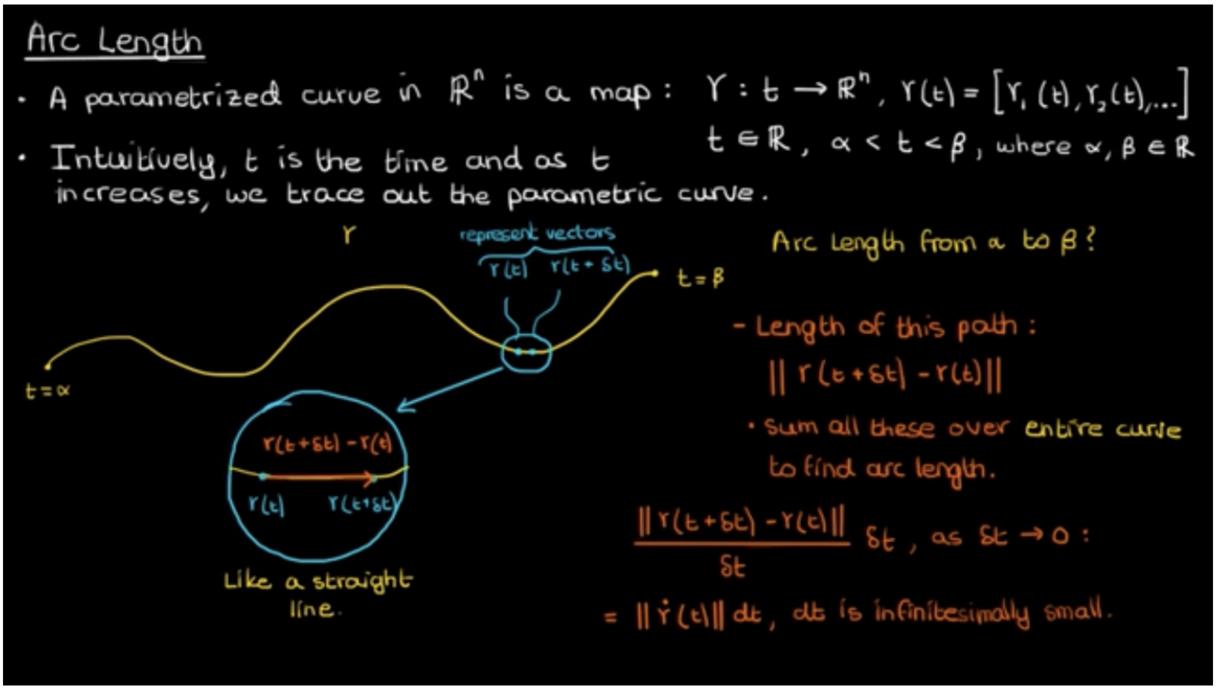
2.2 Arc length and reparameterization

1) Arc length definition: the arc length of a curve γ starting at the point $\gamma(\alpha)$ is the function $S(t)$ given by:

$$S(t) = \int_{\alpha}^{t} \|\dot{\gamma}(u)\| du.$$

2) Reparameterization: smooth bijective map exists such that the inverse map is also smooth.

3) Regular point, singular point and regular curve.



Arc Length Definition: The arc length of a curve γ starting at the point $\gamma(\alpha)$ is the function $S(t)$ given by:

$$S(t) = \int_{\alpha}^{t} \underbrace{\|\underbrace{\dot{\gamma}(u)}_{\text{velocity}}\|}_{\text{speed}}\, du$$

Speed: If $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is a parametrized curve, its speed at the point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$. γ is said to be a unit-speed curve if $\dot{\gamma}(t)$ is a unit vector for all $t \in (\alpha, \beta)$. (i.e. $\|\dot{\gamma}(t)\| = 1$)

Theorem 1: Suppose $\gamma(t)$ is a smooth unit speed curve. Then, $\dot{\gamma}(t) \cdot \ddot{\gamma}(t) = 0$.

Proof: Since $\gamma(t)$ is unit speed, $\|\dot{\gamma}(t)\| = 1 \longrightarrow \|\dot{\gamma}(t)\|^2 = 1$

$\dot{\gamma}(t) \cdot \dot{\gamma}(t) = 1$, differentiate w.r.t. t:

$$\ddot{\gamma}(t) \cdot \dot{\gamma}(t) + \dot{\gamma}(t) \cdot \ddot{\gamma}(t) = 0$$

$$2\dot{\gamma}(t) \cdot \ddot{\gamma}(t) = 0$$

$$\rightarrow \dot{\gamma}(t) \cdot \ddot{\gamma}(t) = 0, \text{ QED.}$$

Thus, for a particle travelling at constant speed, the acceleration is either 0 or perpendicular to the particle's velocity.

Reparametrization

$\gamma(t),\ t \in (\alpha, \beta) \xrightarrow{\text{change to}} \tilde{\gamma}(\tilde{t}),\ \tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$

Definition: A parametrized curve $\tilde{\gamma} : \tilde{t} \in (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ is a reparametrization of a parametrized curve $\gamma : t \in (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth bijective map $\phi : \tilde{t} \in (\tilde{\alpha}, \tilde{\beta}) \to t \in (\alpha, \beta)$ (the reparametrization map) such that the inverse map $\phi^{-1} : t \in (\alpha, \beta) \to \tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$ is also smooth and $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$ for all $\tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$.

- Similarly, since ϕ has a smooth inverse, γ is also a reparametrization of $\tilde{\gamma}$.

Example: $x^2 + y^2 = 1$, $\gamma(t) = [\cos t, \sin t]$, $t \in [0, 2\pi]$

$\tilde{\gamma}(\tilde{t}) = [\sin \tilde{t}, \cos \tilde{t}]$, $\tilde{t} \in [0, 2\pi]$

$\underbrace{t = \phi(\tilde{t}) = \pi/2 - \tilde{t}}_{\text{bijective \& smooth}}$ $\quad \tilde{t} = \phi^{-1}(t) = \pi/2 - t$: smooth

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Definitions: Suppose γ is a parametrized curve.

A) $\gamma(t)$ is a regular point if $\dot{\gamma}(t) \neq 0$.

B) $\gamma(t)$ is a singular point if $\dot{\gamma}(t) = 0$.

C) γ is a regular curve if $\dot{\gamma}(t) \neq 0$ at all points of the curve.

2.3 Arc length as a parameter

1-3) Proof of the reparameterization with regard to arc length is a valid option.

4) Theorem about unit speed curve.

Arc Length as a Parameter

- Suppose $\gamma = \gamma(t)$ is a smooth, regular curve on the interval I in $\mathbb{R}^n$.

Definition: A parametrized curve $\tilde{\gamma} : \tilde{t} \in (\tilde{\alpha}, \tilde{\beta}) \longrightarrow \mathbb{R}^n$ is a reparameterization of the parametrized curve $\gamma : t \in (\alpha, \beta) \longrightarrow \mathbb{R}^n$ if there is a smooth, bijective map $\phi : \tilde{t} \in (\tilde{\alpha}, \tilde{\beta}) \to t \in (\alpha, \beta)$ (i.e. the reparameterization map) such that the inverse map $\phi^{-1} : t \in (\alpha, \beta) \to \tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$ is also smooth and $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$ for all $\tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$.

- To reparametrize w.r.t. arc length:

1) The reparameterization map from t to arc length must be smooth.

2) The reparameterization map must be bijective (one-to-one) with a smooth inverse.

Definition: The arc length S of a curve γ starting at the point $\gamma(t_0)$ is the function $S(t)$ given by:

$$S(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| \, du \quad \text{— (1)}$$

If $t \geq t_0$, $S \geq 0$ and is equal to the arc length of γ from t_0 to t. If $t < t_0$, S is negative of the arc length of γ between t_0 and t.

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$$\frac{dS}{dt} = \frac{d}{dt}\int_{t_0}^{t} \|\dot{\gamma}(u)\|\, du \longrightarrow \frac{dS}{dt} = \|\dot{\gamma}(t)\| \quad — (2)$$

If $\gamma(t) = [\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)]$, then $\dot{\gamma}(t) = [\dot{\gamma}_1(t), \dot{\gamma}_2(t), \ldots, \dot{\gamma}_n(t)]$.

$$\therefore \|\dot{\gamma}(t)\| = \sqrt{\underbrace{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \ldots + \dot{\gamma}_n^2}_{\text{since } \gamma \text{ is smooth, this expression is always smooth}}} \underline{\underline{> 0}}$$

↳ lies b/w 0 and ∞

* The square root of a smooth function is itself smooth on the interval $(0, \infty)$: $y = \sqrt{x}$ is infinitely differentiable b/w 0 and ∞.

$\therefore$ since $\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \ldots + \dot{\gamma}_n^2$ lies between 0 & ∞, $\sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \ldots + \dot{\gamma}_n^2}$ is smooth. Thus, dS/dt is smooth $\longrightarrow S(t)$ is smooth.

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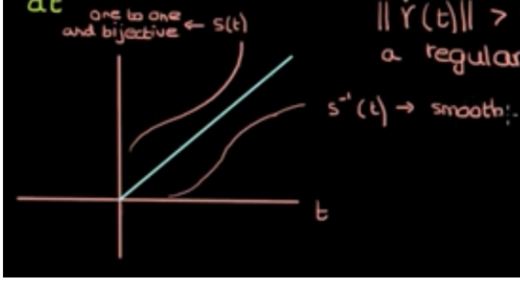
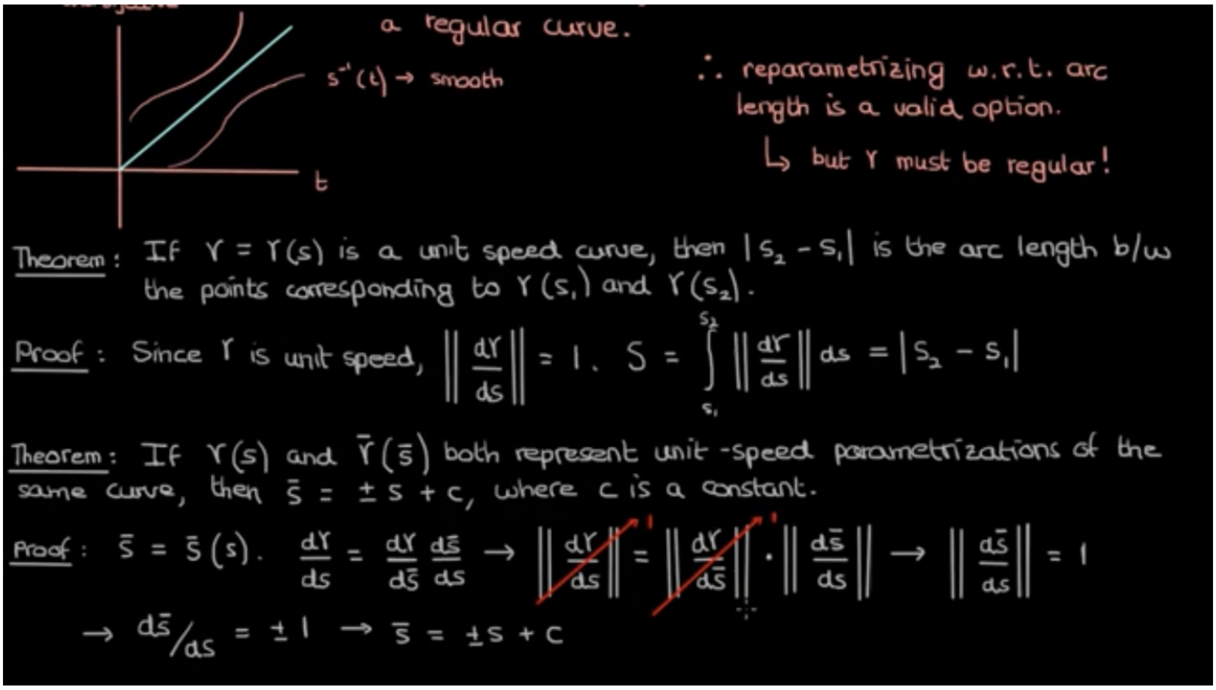
$\frac{dS}{dt} = \|\dot{\gamma}(t)\|$ — (2). Since γ is regular, $\dot{\gamma}$ is never 0. Thus, $\|\dot{\gamma}(t)\| > 0$ always, so dS/dt is always positive for a regular curve.



$\therefore$ reparametrizing w.r.t. arc length is a valid option.

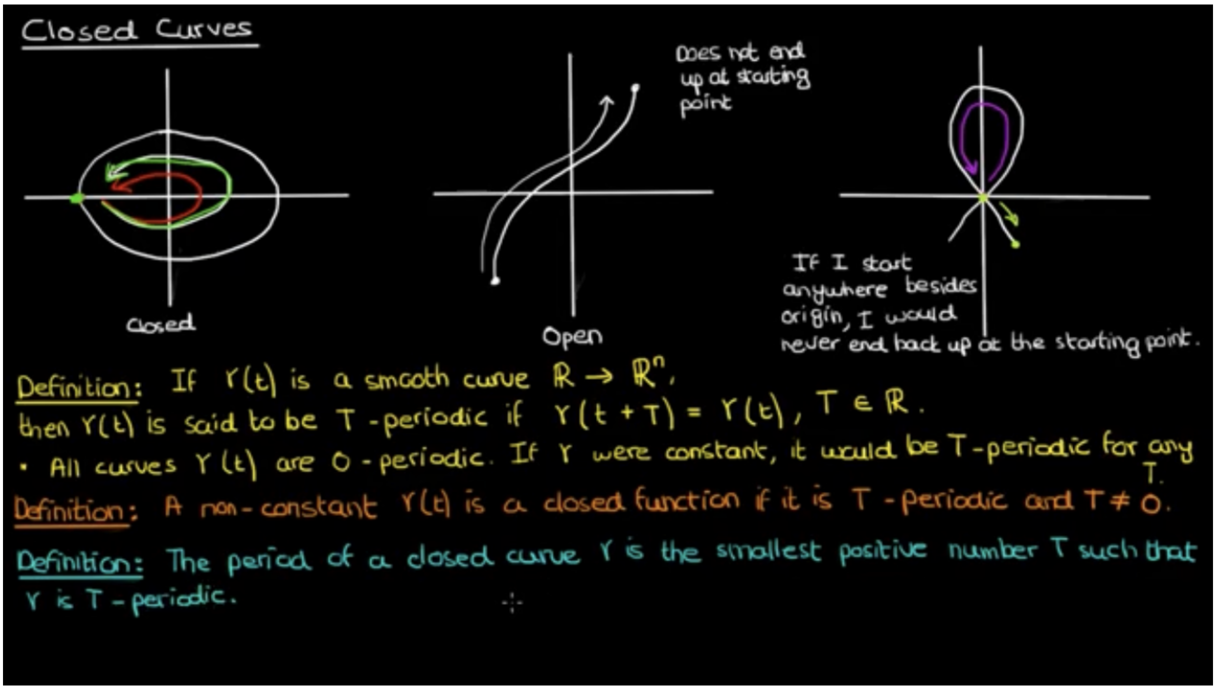


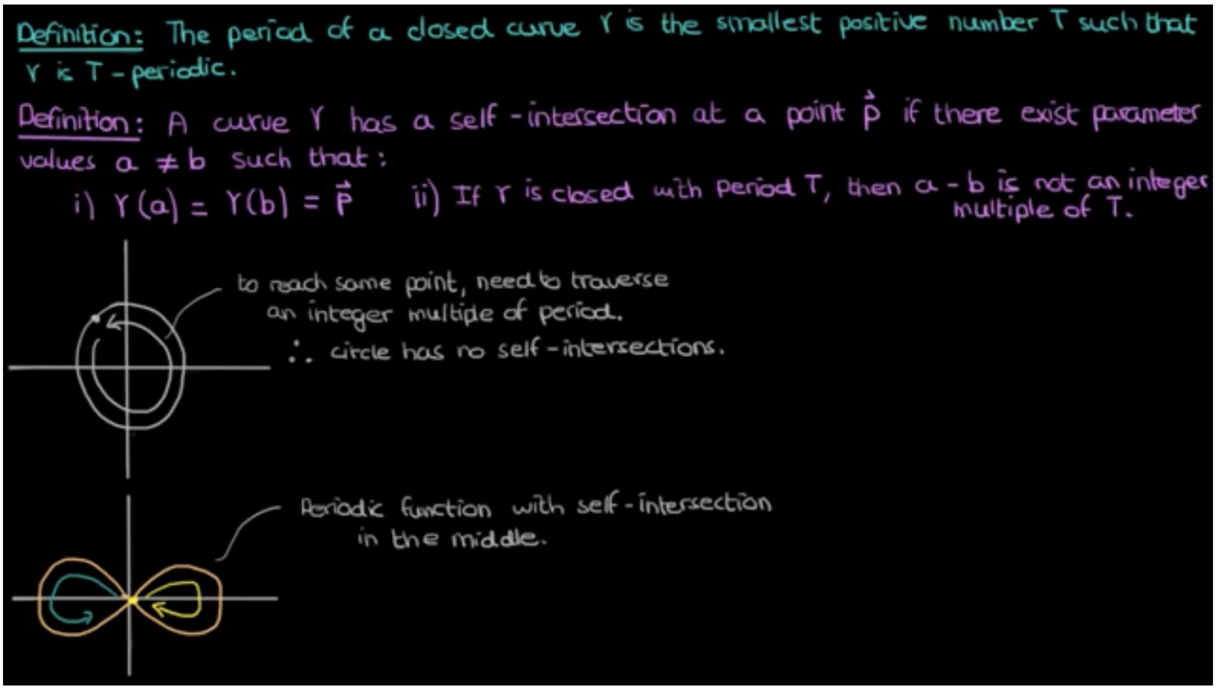
2.4 Closed curves and periodic curves

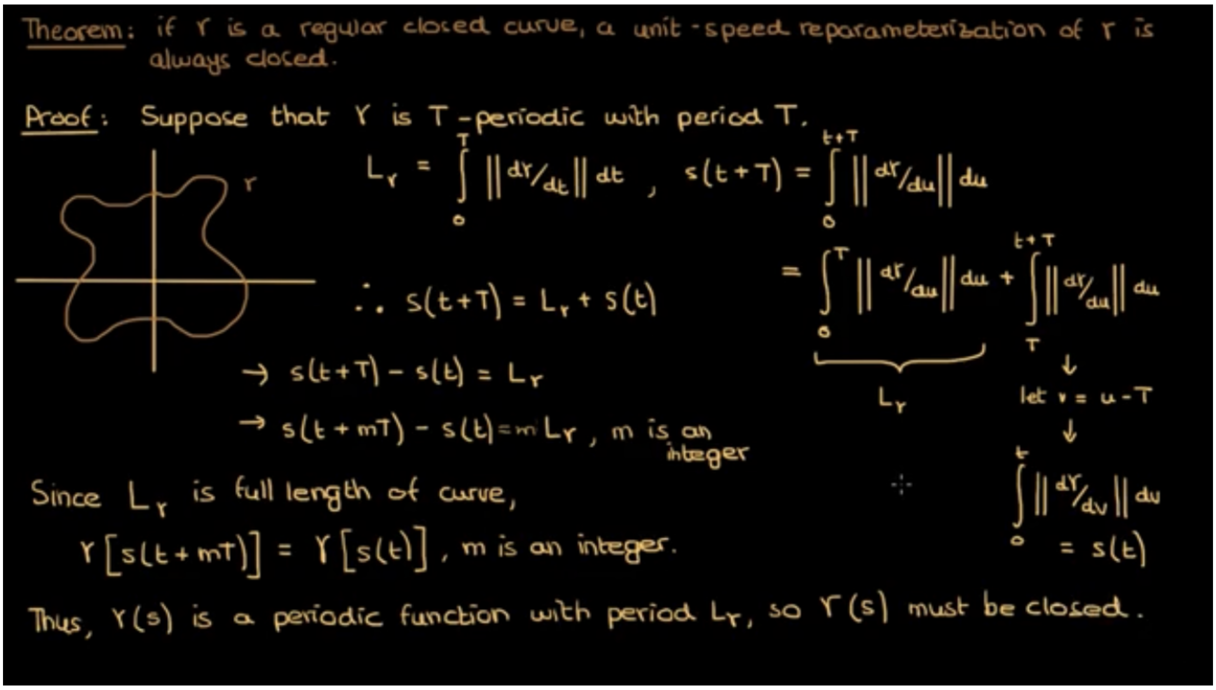
1) T-periodic: $Y(t+T) = Y(t), T \in \mathbb{R}$.

2) Self-intersection of a curve.

3) Theorem: a unit-speed reparameterization of a regular closed curve γ is always closed.







2.5 Curvature: intuition adn derivation

1) Curvature: the extend to which γ deviates from a straight line.

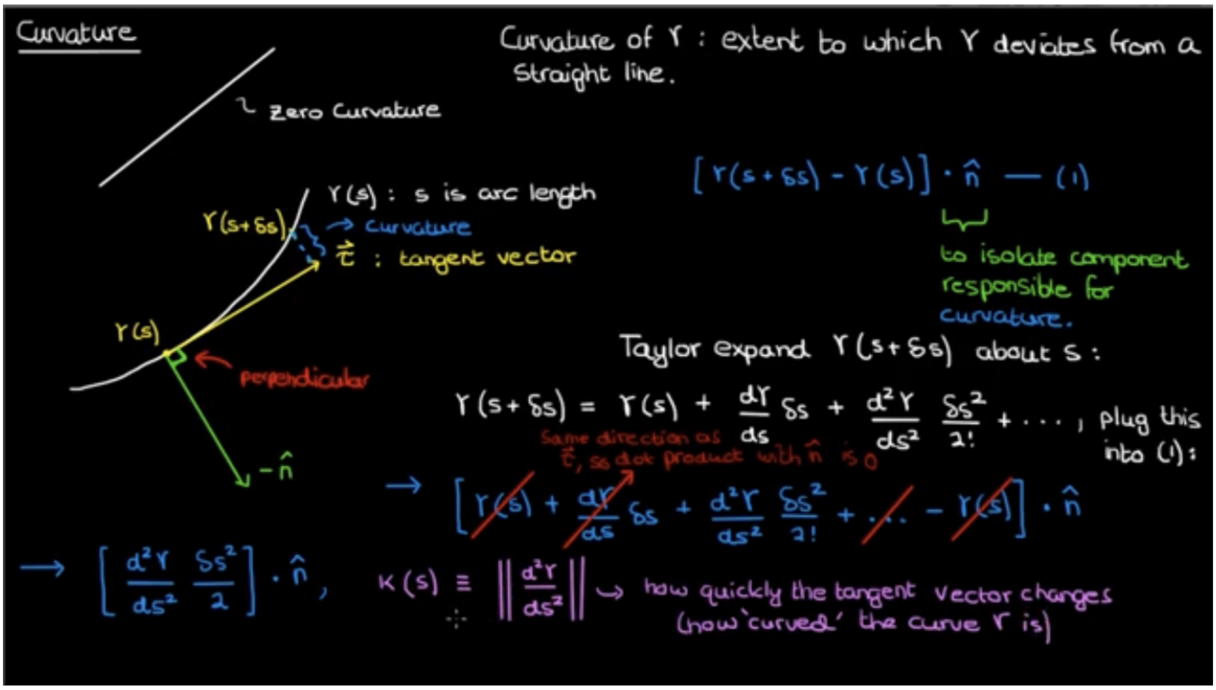
$$K(s) = \|\frac{d^2\gamma}{ds^2}\|,$$

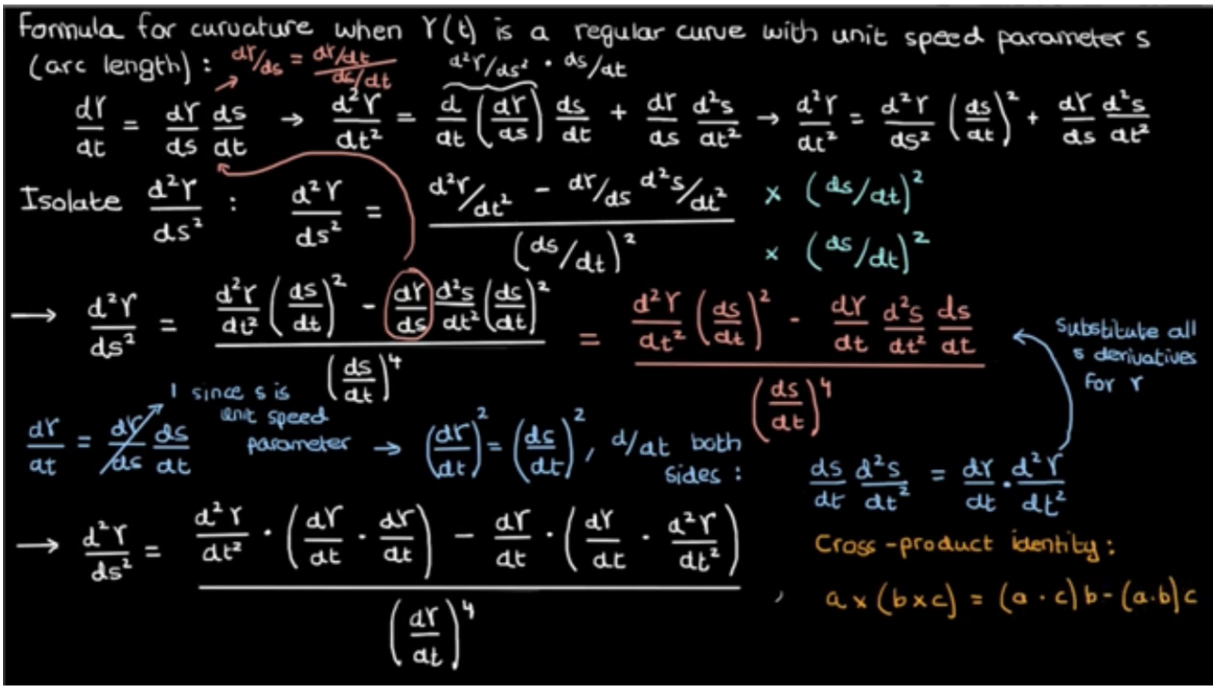
which shows how quickly the tangent vector changes.

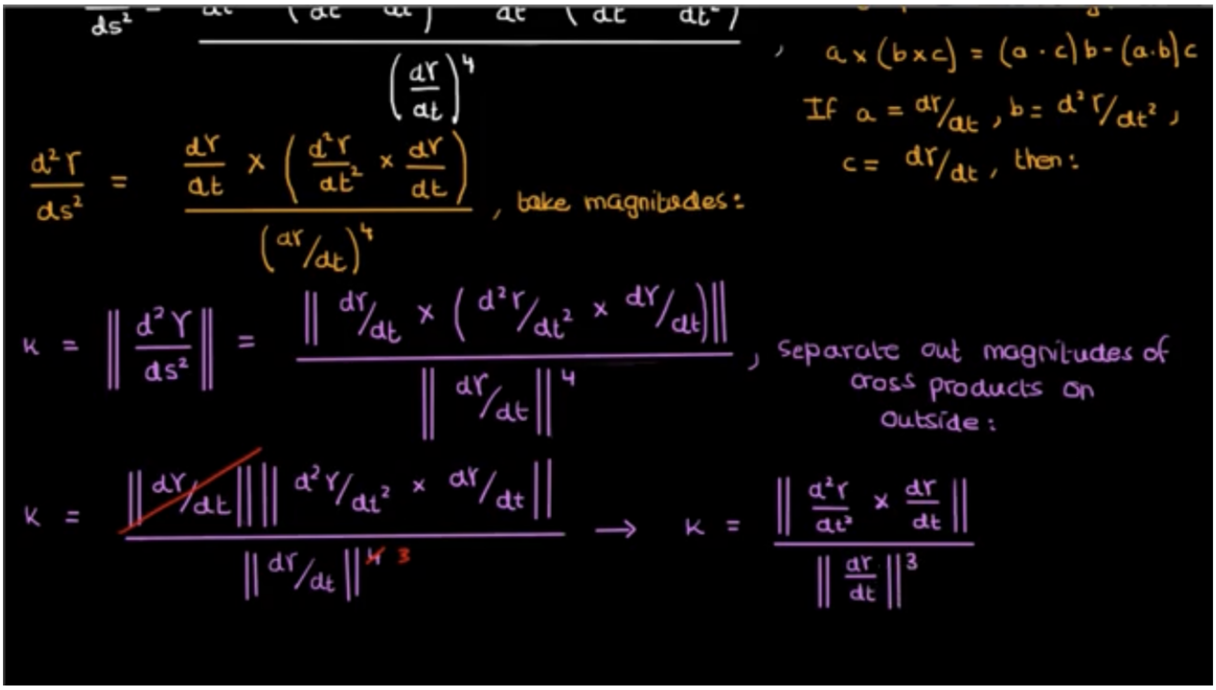
2-4) Formula for curvature when $\gamma(t)$ is a regular curve with unit speed parameter s:

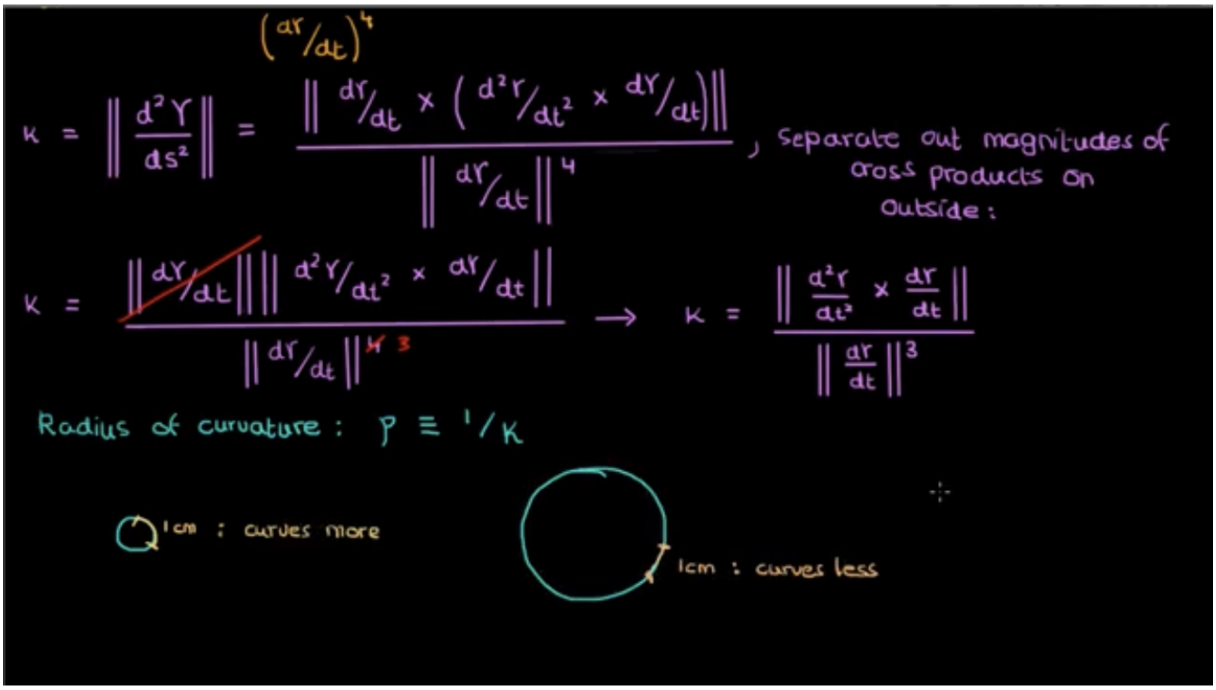
$$k = \frac{\left\| \frac{d^2r}{dt^2} \times \frac{dr}{dt} \right\|}{\left\| \frac{dr}{dt} \right\|^3}$$

4) Radius of curvature: $p = 1/K$.









3 Discussion