Manifolds, topological space, and tangent space

1 Online Tutorials

YouTube tutorials on topology, manifolds & differential geometry by *WHYBmaths*

- Preliminary: Sets & Maps
- Topology: Introduction, Constructing Torus, Topological Spaces, Continuity, Metric Spaces, Hausdorff property
- Manifolds: Introduction, Topological Manifolds, Transition Functions, Tangent Space
- Other: Vector Fields, Differential Forms

Manifolds are **topological spaces** that are **locally homeomorphic** to Euclidean space.

2 Topological spaces

Topology space: a set M with *topology* $\{\mathcal{T}_M\}$.

2.1 Topology

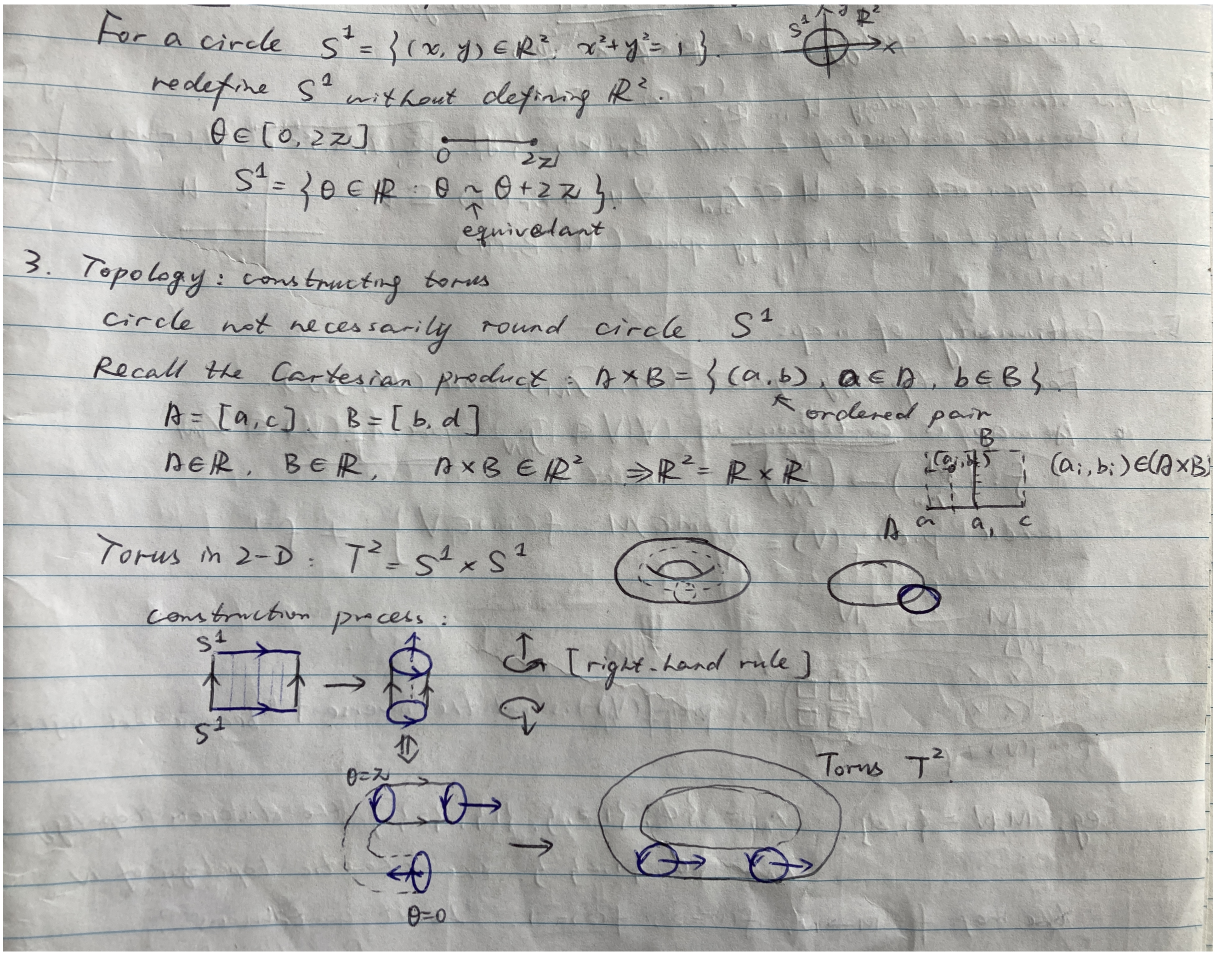
Topology is the mathematical study of the properties of a geometric object that are preserved under *continuous deformations*, such as stretching, twisting, crumpling and bending. *Wikipedia*

Topological invariant: *homeomorphic* in the property of the space.

Homeomorphism: a continuous function between topological spaces that has a continuous inverse function.

For example: a circle is homeomorphic to a rectangle/square by certain stretching.

Constructing torus in 2-D.



2.2 Topological space: $(M, \{\mathcal{T}_M\})$

Topology properties:

1) $M, \phi \in \{\mathcal{T}_M\}$;

2) For all topological element $\tau_i \in \{\mathcal{T}_M\}$, the finite intersection set is also in the topology,

$$\bigcap_i \tau_i \in \{\mathcal{T}_M\};$$

3) Similarly, the infinite union set is also in the topology,

$$\bigcup_i \tau_i \in \{\mathcal{T}_M\};$$

Two extreme cases:

- Chaotic topology: $\mathcal{T}_M = \{M, \phi\}$.

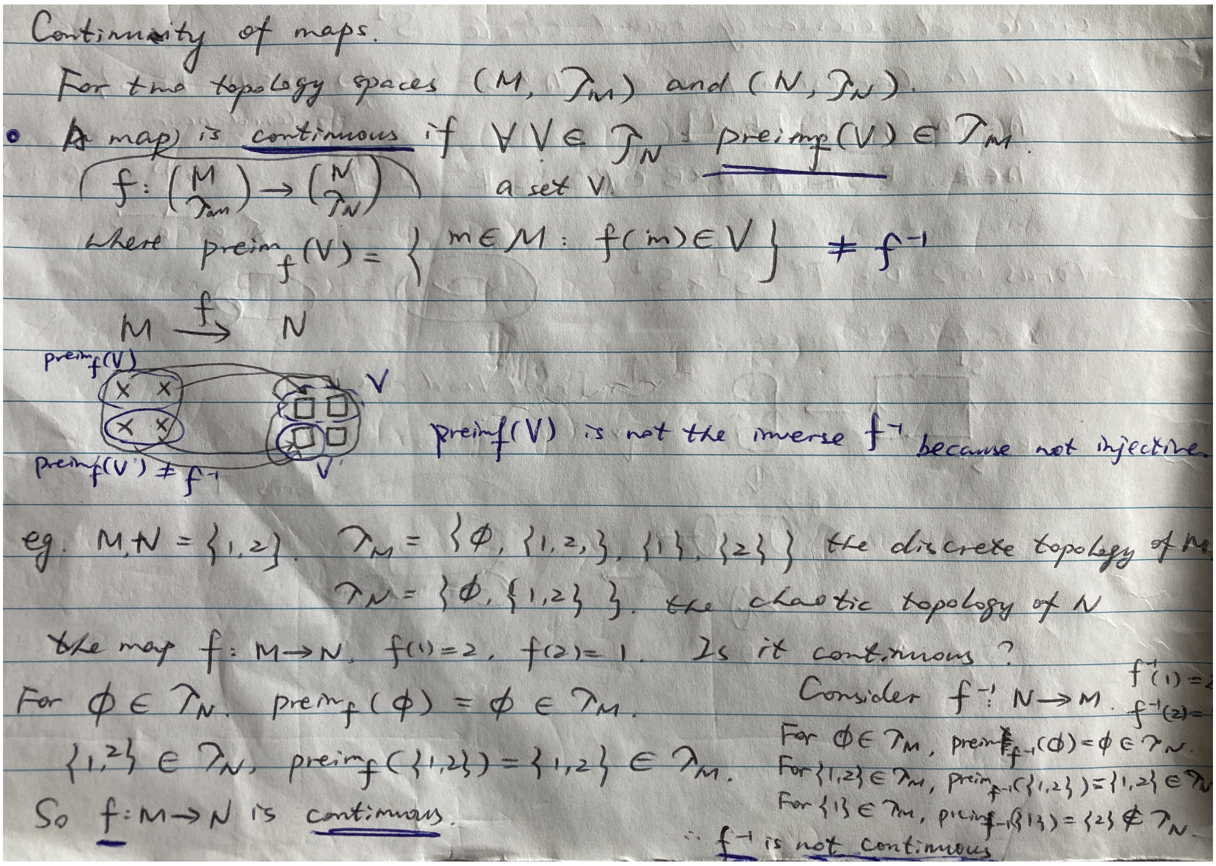
- Discrete topology: $\mathcal{T}_M = \{P(M), \phi\}$, where $P(M)$ is the power set containing all possible subset of M.

Standard topology T_{std} of a set $U \subset \mathbb{R}^2$:

1) Consider constructing a ball $B_r(x_i)$;
2) For any open set $U \in T_{std}$, there exists a ball $B_r(U) \subset U$.

1)&2) give a 2-d topology space $(U, B_r(U))$. Example

3 Continuity of maps



Continuity of maps.

For two topology spaces $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$.

- A map is continuous if $\forall V \in \mathcal{T}_N$: preimp$(V) \in \mathcal{T}_M$.

$f: \binom{M}{\mathcal{T}_M} \to \binom{N}{\mathcal{T}_N}$ a set V.

where $\text{preim}_f(V) = \{ m \in M : f(m) \in V \} \neq f^{-1}$

$M \xrightarrow{f} N$

preimf(V) is not the inverse f^{-1} because not injective.

eg. $M, N = \{1, 2\}$. $\mathcal{T}_M = \{\phi, \{1,2\}, \{1\}, \{2\}\}$ the discrete topology of M

$\mathcal{T}_N = \{\phi, \{1,2\}\}$. the chaotic topology of N

the map $f: M \to N$, $f(1) = 2$, $f(2) = 1$. Is it continuous?

For $\phi \in \mathcal{T}_N$, $\text{prem}_f(\phi) = \phi \in \mathcal{T}_M$.

$\{1,2\} \in \mathcal{T}_N$, $\text{preim}_f(\{1,2\}) = \{1,2\} \in \mathcal{T}_M$.

So $f: M \to N$ is continuous.

Consider $f^{-1}: N \to M$. $f^{-1}(1) = 2$, $f^{-1}(2) = 1$

For $\phi \in \mathcal{T}_M$, $\text{preim}_{f^{-1}}(\phi) = \phi \in \mathcal{T}_N$.

For $\{1,2\} \in \mathcal{T}_M$, $\text{preim}_{f^{-1}}(\{1,2\}) = \{1,2\} \in \mathcal{T}_N$

For $\{1\} \in \mathcal{T}_M$, $\text{preim}_{f^{-1}}(\{1\}) = \{2\} \notin \mathcal{T}_N$.

$\therefore f^{-1}$ is not continuous

4 Metric Spaces

6. Metric spaces: a topology space with the metric $g(x, y) \to \mathbb{R}$ distance
$(M, \mathcal{T}_M, g)$

metric $g(x, y)$ properties:

1) symmetric: $g(x, y) = g(y, x)$
2) positive definite: $g(x, y) \geq 0$, "=" iff $x = y$.

eg. $(\mathbb{R}, \mathcal{T}_{std}, g)$, where $g(x, y) := |y - x|$

1) $B_r(x_i) = \{ y_i : g(x_i, y_i) < r \}$.
2) open set U: $U \subset \mathcal{T}_M : B_{r>0}(x) \subseteq U$. neighborhood = open set $U \subset \mathcal{T}_M$

Hausdorff (T2) space: $U \cap V = \phi$ U V
no interaction between open sets

T1: $v \in U$.
point on the boundary of the openset U v

5 Manifolds

7. Manifolds — Manifolds are topological spaces that are locally homeomorphic to Euclidean space.

a topological space which ~~can~~ be covered in charts ~~with coordinates~~ used to give coordinates to a local subset of the space

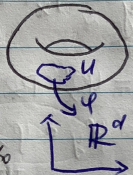
S^1. homeomorphic, topologically equivelant

S^2 $(\mathbb{R}^3, \mathcal{T}_{std})$ $\simeq$ $\neq$ not torus

cover the topological space in charts.

⇒ give coordinates ~~in charts~~ to the space

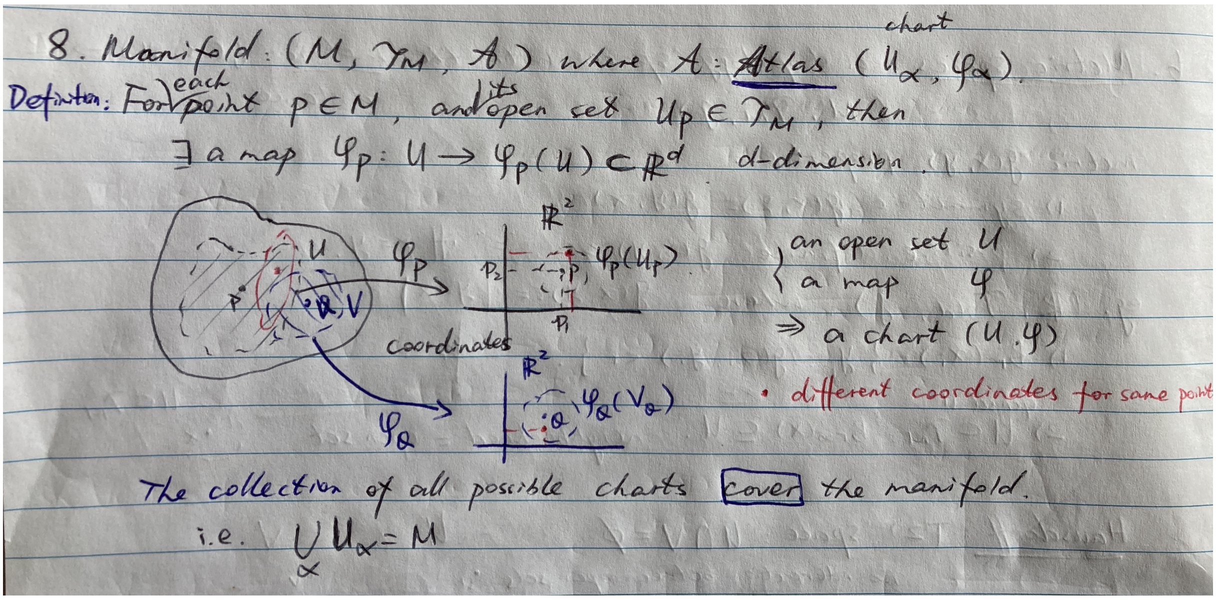
by mapping a small portion of the manifold into a subset of real numbers in $\mathbb{R}^d$.



Locally, the manifold (looks like) subset of real numbers $\mathbb{R}^d$. homeomorphic to $\mathbb{R}^d$.

- by putting all the charts together, we are able to reconstruct the manifold.

Wiki:
- In math, a manifold is a $\boxed{\text{topological space}}$ that $\boxed{\text{locally}}$ resembles Euclidean space near each point.
- An n-dimensional manifold is a topological space with the property that each point has a neighborhood that is homomorphic to the Euclidean space of dimension n. $\mathbb{R}^n$



8. Manifold: $(M, \mathcal{T}_M, \mathcal{A})$ where $\mathcal{A}$: Atlas (chart) $(U_\alpha, \varphi_\alpha)$.

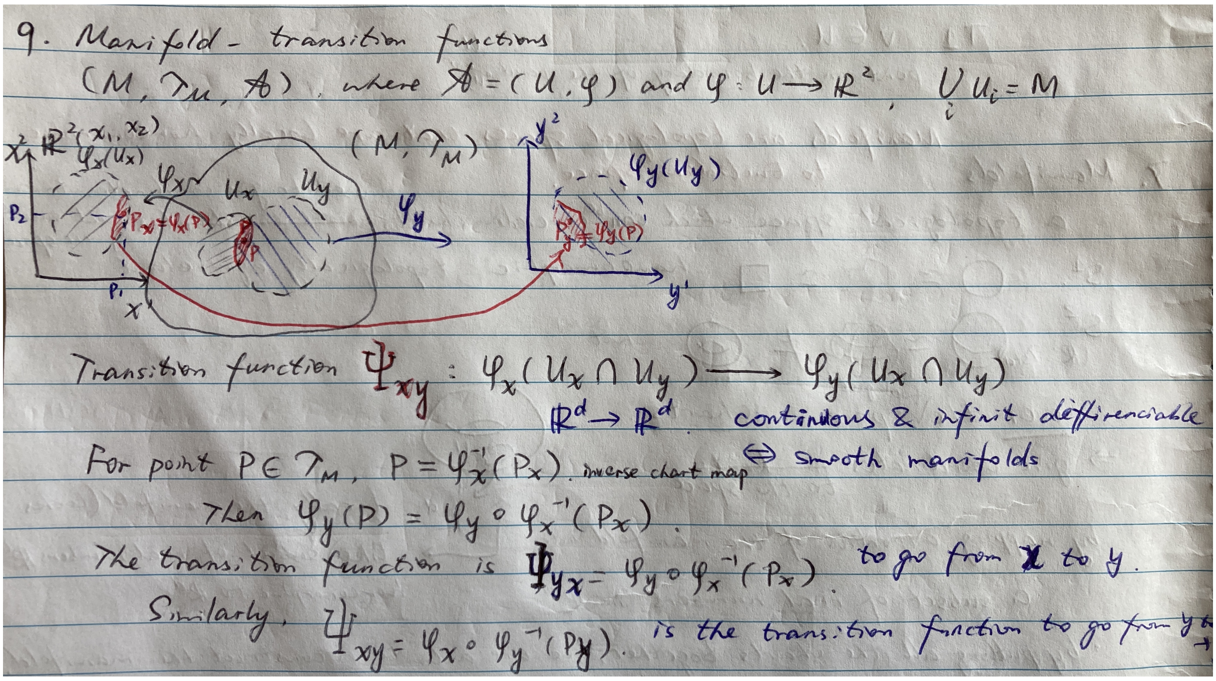
Definition: For each point $p \in M$, and its open set $U_p \in \mathcal{T}_M$, then $\exists$ a map $\varphi_p : U \to \varphi_p(U) \subset \mathbb{R}^d$ d-dimension.

an open set U, a map φ $\Rightarrow$ a chart (U, φ)

• different coordinates for same point

The collection of all possible charts $\boxed{\text{cover}}$ the manifold. i.e. $\bigcup_\alpha U_\alpha = M$

6 Transition functions



9. Manifold - transition functions

$(M, \mathcal{T}_M, \mathcal{A})$, where $\mathcal{A} = (U, \varphi)$ and $\varphi : U \to \mathbb{R}^2$, $\bigcup_i U_i = M$

Transition function $\Psi_{xy} : \varphi_x(U_x \cap U_y) \longrightarrow \varphi_y(U_x \cap U_y)$

$\mathbb{R}^d \to \mathbb{R}^d$. continuous & infinit differenciable $\Leftrightarrow$ smooth manifolds

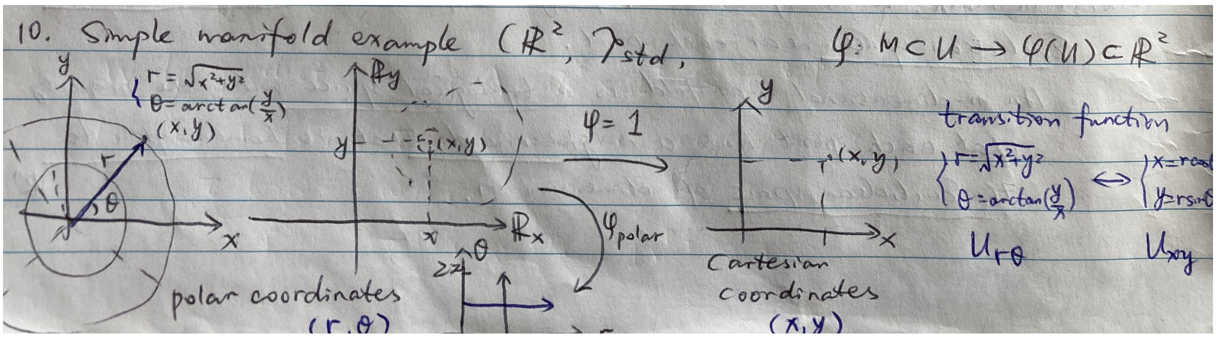
For point $P \in \mathcal{T}_M$, $P = \varphi_x^{-1}(P_x)$. inverse chart map

Then $\varphi_y(P) = \varphi_y \circ \varphi_x^{-1}(P_x)$.

The transition function is $\Psi_{yx} = \varphi_y \circ \varphi_x^{-1}(P_x)$. to go from x to y.

Similarly, $\Psi_{xy} = \varphi_x \circ \varphi_y^{-1}(P_y)$. is the transition function to go from y to

6.1 Example



10. Simple manifold example $(\mathbb{R}^2, \mathcal{T}_{std},$ $\varphi : M \subset U \to \varphi(U) \subset \mathbb{R}^2$

7 Tangent spaces

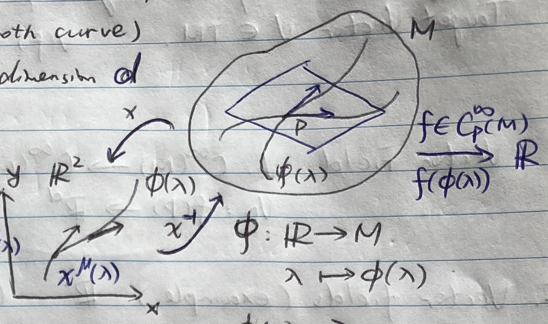
11. Tangent space (intro) $(V, +, \cdot)_{\mathbb{R}}$ — a set, addition, multiplication

Consider a manifold M of dimension d.

For point $P \in M$, the curve that crosses point P on the manifold is denoted as $C_P^\infty(M)$. (smooth curve)

The tangent space is $T_P(M)$ with dimension d

$d(T_P(M)) = d(M)$



$(x \circ \phi): \mathbb{R} \to M \to \mathbb{R}^2$

$\lambda \mapsto \phi(\lambda) \mapsto x^\mu(\lambda)$

$\phi: \mathbb{R} \to M$

$\lambda \mapsto \phi(\lambda)$

$\phi(0) = P$

12. Tangent space (details)

The tangent vector V to curve ϕ act on the function f is defined as

$$V_\phi(f) = \frac{\partial}{\partial \lambda} f(\phi(\lambda))\Big|_{\lambda=0} \qquad \phi(\lambda) = x^{-1} \circ x^\mu(\lambda)$$

$$f(\phi(\lambda)) = f(x^{-1} \circ x^\mu(\lambda)) = (f \circ x^{-1}) \circ (x^\mu(\lambda))$$

Then
$$V_\phi(f) = \frac{\partial}{\partial \lambda}\Big|_{\lambda=0} (f \circ x^{-1}) \circ (x^\mu(\lambda))$$

$$= \underbrace{\frac{\partial}{\partial x^\mu}(f \circ x^{-1})}_{f(x^\mu)} \underbrace{\frac{\partial x^\mu}{\partial \lambda}\Big|_{\lambda=0}}_{V^\mu}$$

V^μ: tangent of the curve

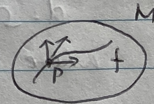
$$= \left(\frac{\partial x^\mu}{\partial \lambda}\Big|_{\lambda=0} \underbrace{\frac{\partial}{\partial x^\mu}}_{\partial_\mu}\right) f$$

basis of the tangent space $\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu}$.

$$= \boxed{(V^\mu \cdot \partial_\mu)} f \quad , \ f \text{ is arbitrary}$$

$V^\mu \cdot \partial_\mu = V$. coordinates free, holds for any coordinate system

13. Tangent space: vectors are tangent to C^∞ (smooth) curves



- $T_P M$: Basis $\frac{\partial}{\partial x^\mu} = \partial_\mu$ is the coordinates basis
- Vector $V = V^\mu \partial_\mu = \cancel{V_\mu \frac{\partial x}{\partial x^\mu}}$ $V^\mu \frac{\partial}{\partial x^\mu} = \hat{V}^\nu \frac{\partial}{\partial y^\nu}$ $\quad (U, x^\mu) \Leftrightarrow (U', y^\mu)$

Consider the chain rule $\frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \cdot \frac{\partial}{\partial y^\nu}$. and substitute

$$V^\mu \cdot \frac{\partial y^\nu}{\partial x^\mu} \cdot \frac{\partial}{\partial y^\nu} = \hat{V}^\nu \frac{\partial}{\partial y^\nu} \Rightarrow \boxed{V^\mu \cdot \underbrace{\frac{\partial y^\nu}{\partial x^\mu}}_{\text{Jacobian}} = \hat{V}^\nu} \quad y^\nu(x^\mu)$$

$\hat{V}^\nu = V^\mu \cdot \frac{\partial y^\nu}{\partial x^\mu}$. transform between coordinates,

8 Others

8.1 Vector fields

14. Vector Fields (schematic)

tangent space of a point P: T_pM "tangent fibre"

~~Tang~~ Collection of all tangent fibre: TM "tangent bundle"

Tangent Vector $V \in TM$. $V(f) = \frac{\partial}{\partial \lambda}\Big|_{\lambda=0} f(\phi(\lambda))$

$= \frac{\partial f}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial \lambda}\Big|_{\lambda=0}$ for single point P

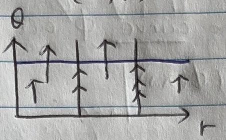
evaluated at P

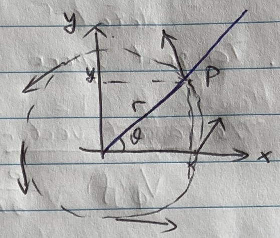
Vector Field: $V: C^\infty(M) \to C^\infty(M)$. $V(f) = \frac{\partial f}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial \lambda}$.

$V_p: f(p) \to \mathbb{R}^d$

15. Vector fields (example) $\mathbb{R}^2$ (x, y)

(r, θ)





$$\begin{cases} V = 1 \cdot \frac{\partial}{\partial \theta} = 1 \cdot \partial_\theta & \text{basis vector} \\ \quad \parallel \\ V = V^x \partial_x + V^y \partial_y \end{cases}$$

Recall that $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\theta \mapsto (x, y)$

Then use the chain rule. $\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}$

$= -r\sin\theta \cdot \partial_x + r\cos\theta\, \partial_y$

$= -y\partial_x + x\partial_y$.

16. Differential Forms $\mathbb{R}^d$

- 0-form: a function $\Omega^0(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)$ is a space of smooth functions
- Exterior derivative $\boxed{df = \frac{\partial f}{\partial x^\mu} dx^\mu}$ 1-form df

$d: \Omega^q \to \Omega^{q+1}$

where $dx^\mu = \frac{\partial x^\mu}{\partial x^\nu} dx^\nu = \delta^\mu_\nu dx^\nu = dx^\mu$.

dx^μ: basis of space

coordinate basis 1-fo

For any component $w \in \Omega^1$, we can write as $w_\mu dx^\mu = w$ linear combination

Wedge product: a map $\wedge: \Omega^q \times \Omega^q \to \Omega^{q+p}$

For two 1-forms τ and w, the wedge product is

$\tau \wedge w = -w \wedge \tau$ anti-symmetric

$\tau^{(p)} \wedge w^{(q)} = (-1)^{p \cdot q}\, w \wedge \tau$

8.2 Differential forms

17. Constructing 2-forms   wedge product antisymmetric

$w = w_\mu dx^\mu$. $w \wedge \tau = -\tau \wedge w$.

$J = w \wedge \tau = w_\mu dx^\mu \wedge \tau_\nu dx^\nu$.

$= w_\mu \tau_\nu dx^\mu \wedge dx^\nu = J_{\mu\nu}$

2-forms : $\Omega^2(\mathbb{R}^2)$

$\Omega^2(\mathbb{R}^2)$: $J = \underline{J_{11} dx^1 \wedge dx^1} + J_{12} dx^1 \wedge dx^2 + J_{21} dx^2 \wedge dx^1 + \cdots$

J_{11} J_{12} 0

$$\begin{array}{cc} J_{11} & \\ J = \begin{pmatrix} 0 & J_{12} \\ J_{21} & 0 \end{pmatrix} & \\ & J_{22} \end{array}$$

is diagonal, antisymmetric $J_{21} = -J_{12}$

$\Rightarrow \zeta = \frac{1}{2} J_{\mu\nu} dx^\mu dx^\nu$

$\sum_{\mu<\nu} J_{\mu\nu} dx^\mu dx^\nu$.

space of 2-forms on $\mathbb{R}^2$ is one dimensional, with one basis element $(dx^1 \wedge dx^2$.