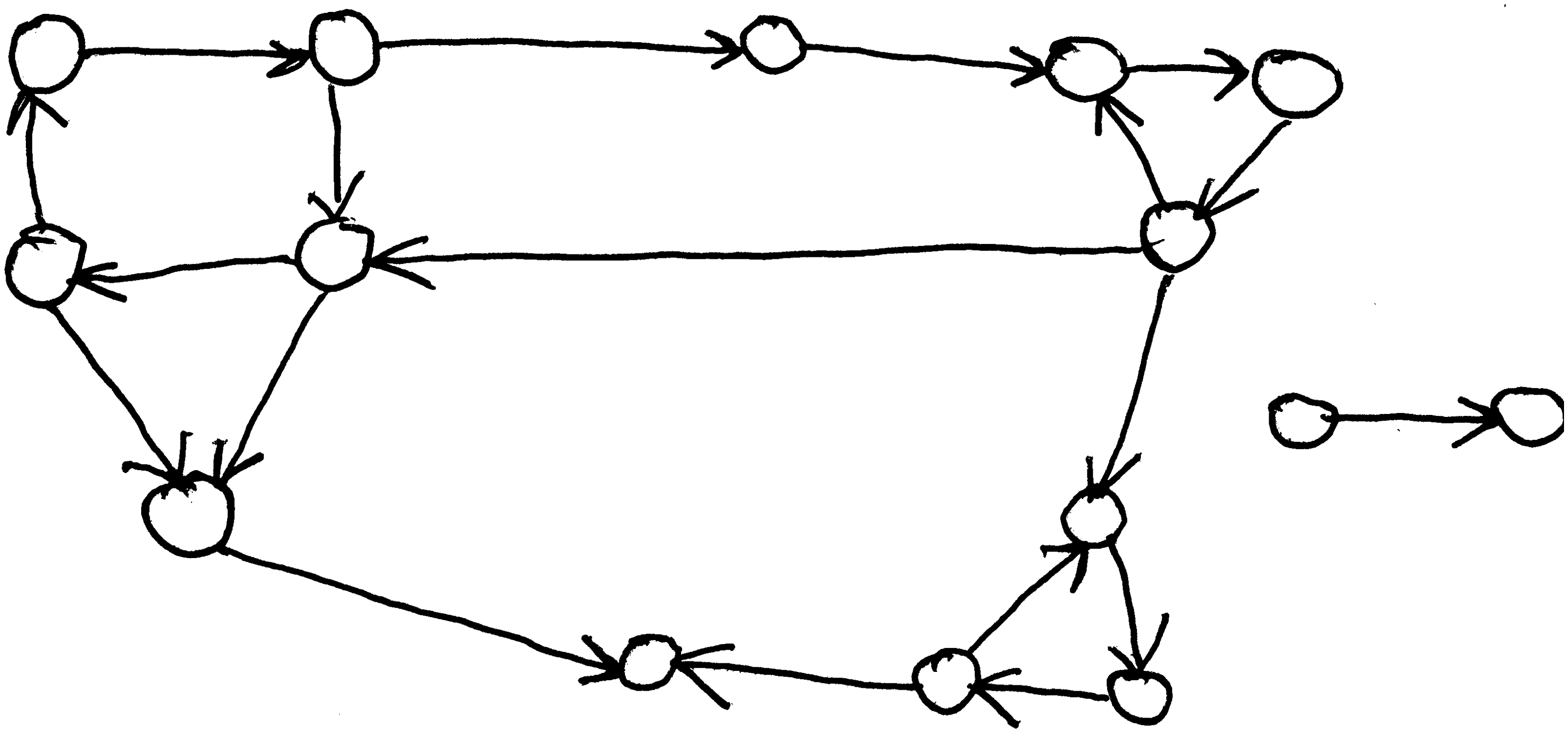
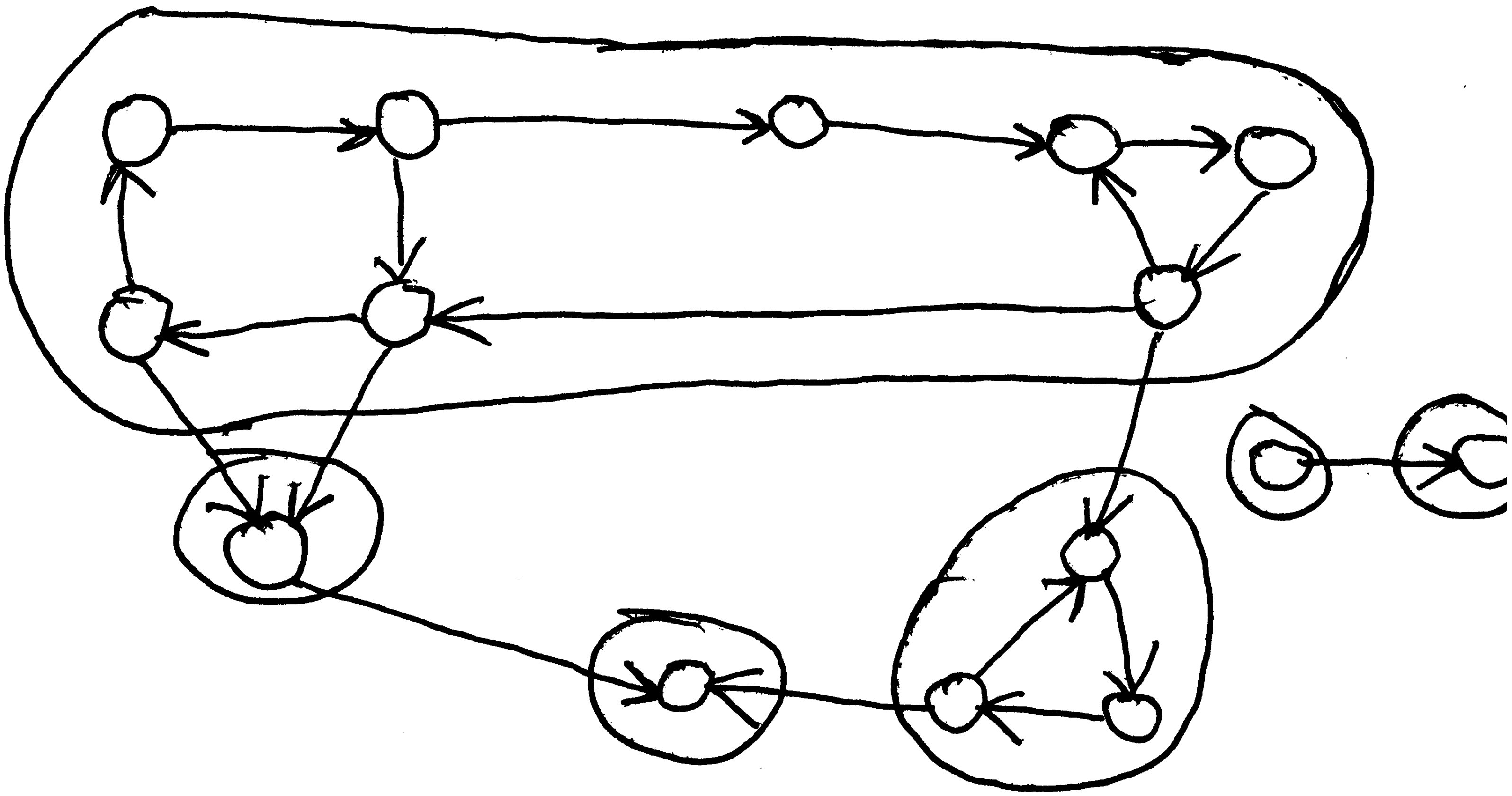
(More) Graphs

show strongly connected comps

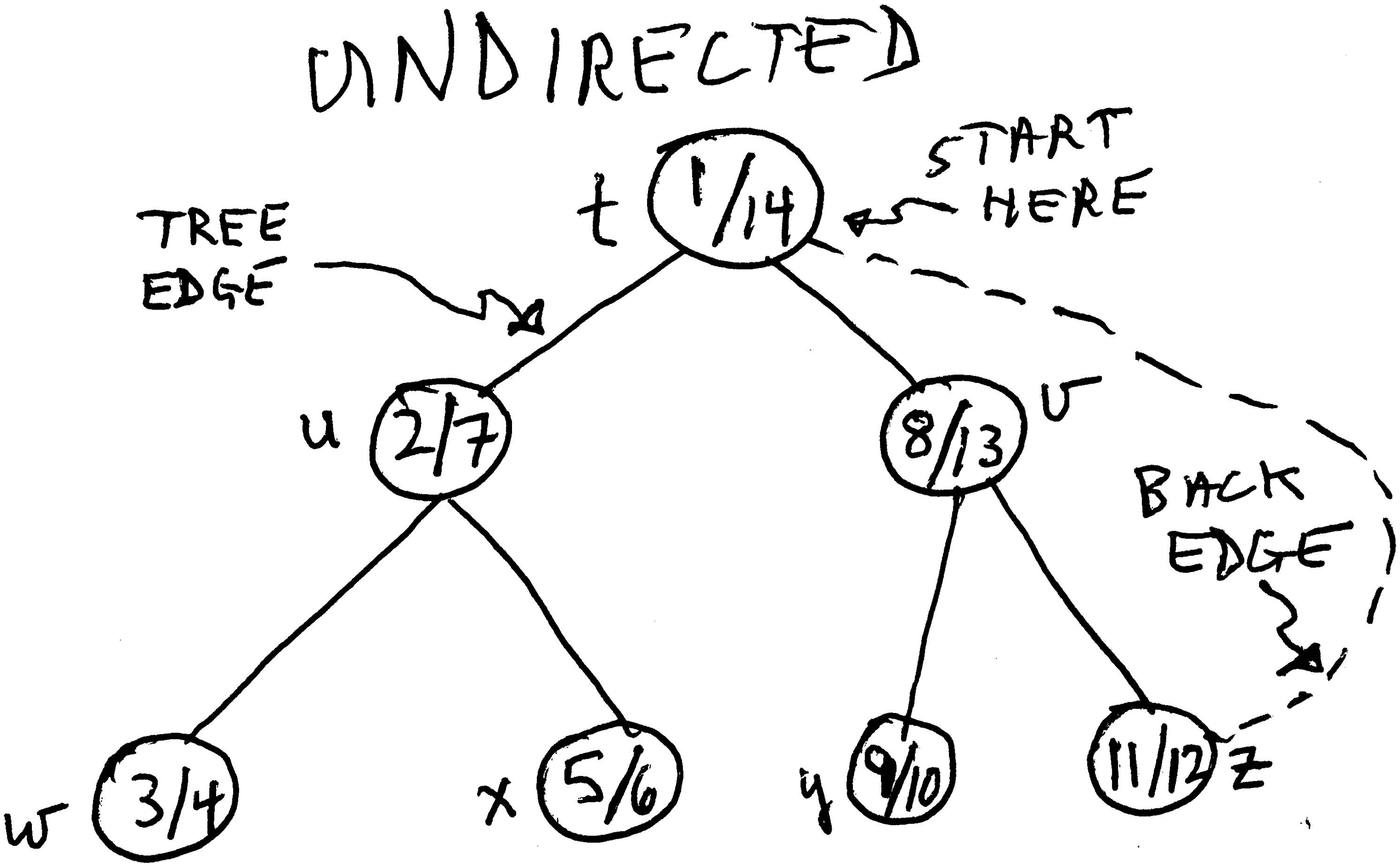


see next slide

from prev slide

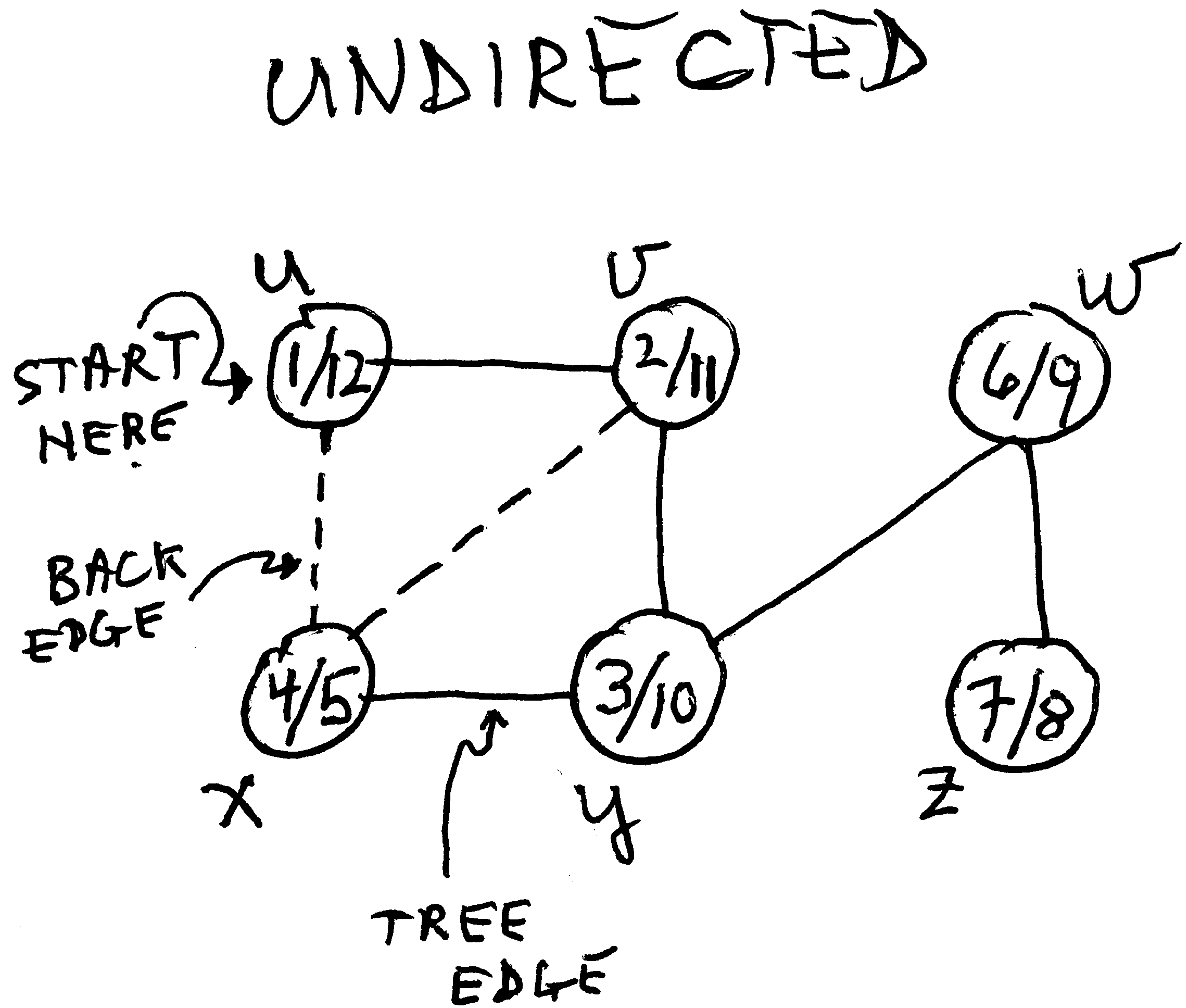


DFS examples

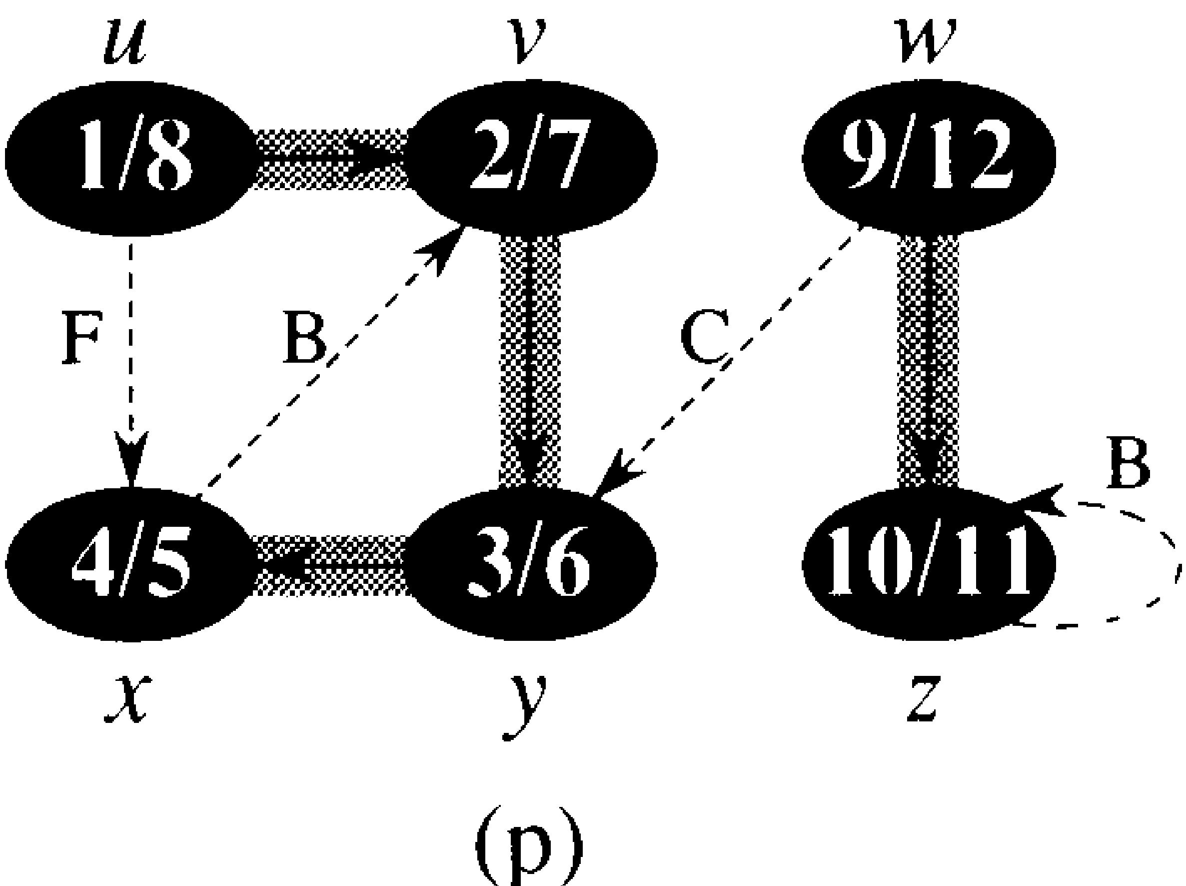


On an ***undirected*** graph, any edge that is not a “tree” edge is a “back” edge (from descendant to ancestor).

DFS Examples



DFS Example: digraph



Here, we get a ***forest*** (two trees).

B = back edge (descendant to ancestor, or self-loop)

F = forward edge (ancestor to descendant)

C = cross edge (between branches of a tree, or between trees)

DFS running time is Θ(V+E)

we visit each vertex once; we traverse each edge once

```
DFS(G)
for each vertex u ∈ V[G]
    do color[u] ← WHITE
       π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
    do if color[u] = WHITE
          then DFS-VISIT(u)
```

$\Theta(V)$

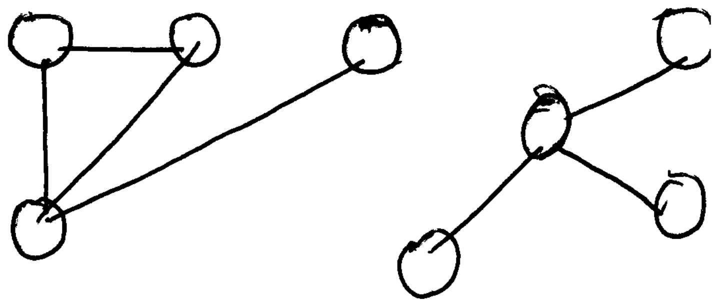
CALLED ONLY ON WHITE VERTICES ($\Theta(V)$)

```
DFS-VISIT(u)
color[u] ← GRAY                ▷ White vertex u has just been discovered.
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]            ▷ Explore edge (u, v).
    do if color[v] = WHITE
          then π[v] ← u
               DFS-VISIT(v)
color[u] ← BLACK               ▷ Blacken u; it is finished.
f[u] ← time ← time + 1
```

$$\sum_{u \in V} |ADJ[u]| = \Theta(E)$$

applications of DFS

Connected components of an **undirected** graph. Each call to DFS_VISIT (from DFS) explores an entire connected component (see ex. 22.3-11).



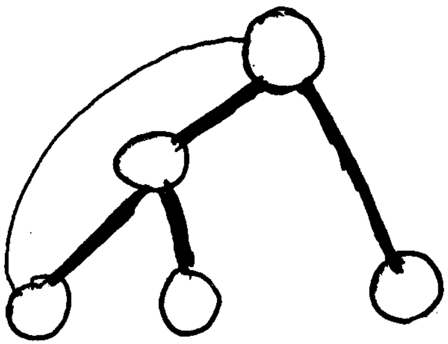
So modify DFS to count the number of times it calls DFS_VISIT:

```
5   for each vertex u Є V[G]
6       do if color[u] = WHITE
6.5          then  cc_counter ← cc_counter + 1
7                  DFS_VISIT(u)
```

Note: it would be easy to label each vertex with its cc number, if we wanted to (i.e. add a field to each vertex that would tell us which conn comp it belongs to).

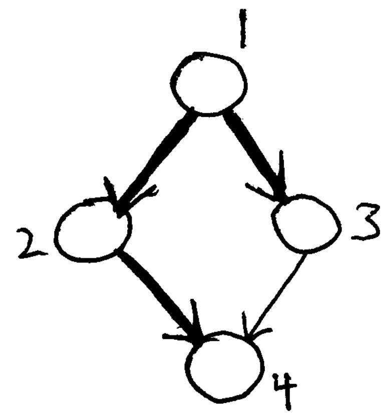
Applications of DFS

Cycle detection: Does a given graph G contain a cycle?



Idea #1: If DFS ever returns to a vertex it has visited, there is a cycle; otherwise, there isn't.

OK for **undirected** graphs, but what about:



No cycles, but a DFS from 1 will reach 4 twice.
Hint: what kind of edge is (3, 4)?

Cycle detection theorem

Theorem: A graph G (directed or not) contains a cycle if and only if a DFS of G yields a back edge.

→: Assume G contains a cycle. Let ***v*** be the first vertex reached on the cycle by a DFS of G. All the vertices reachable from ***v*** will be explored from ***v***, including the vertex ***u*** that is just "before" ***v*** in the cycle. Since ***v*** is an ancestor of ***u***, the edge (***u,v***) will be a **back edge**.

←: Say the DFS results in a back edge from ***u*** to ***v***. Clearly, ***u***→***v*** (that should be a ***wiggly arrow***, which means, "there is a path from ***u*** to ***v***", or "***v*** is reachable from ***u***"). And since ***v*** is an ancestor of ***u*** (by def of back edge), ***v***→***u*** (again should be wiggly). So ***v*** and ***u*** must be part of a **cycle**. QED.

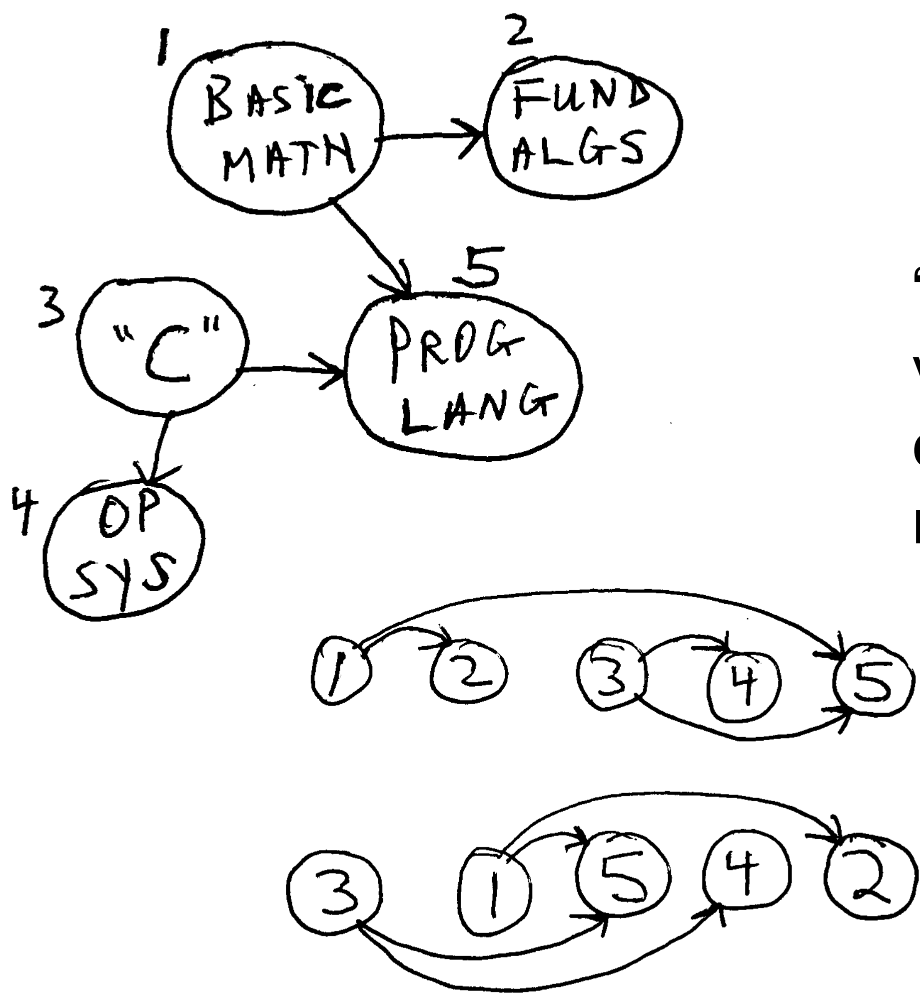
Back Edge Detection

How can we detect back edges with DFS? For **undirected** graphs, easy: see if we've visited the vertex before, i.e. *color* ≠ WHITE.

For **directed** graphs: Recall that we color a vertex GRAY while its adjacent vertices are being explored. If we re-visit the vertex while it is still GRAY, we have a back edge.

We blacken a vertex when its adjacency list has been examined completely. So any edges to a BLACK vertex cannot be back edges.

TOPOLOGICAL SORT



"Sort" the vertices so all edges go left to right.

TOPOLOGICAL SORT

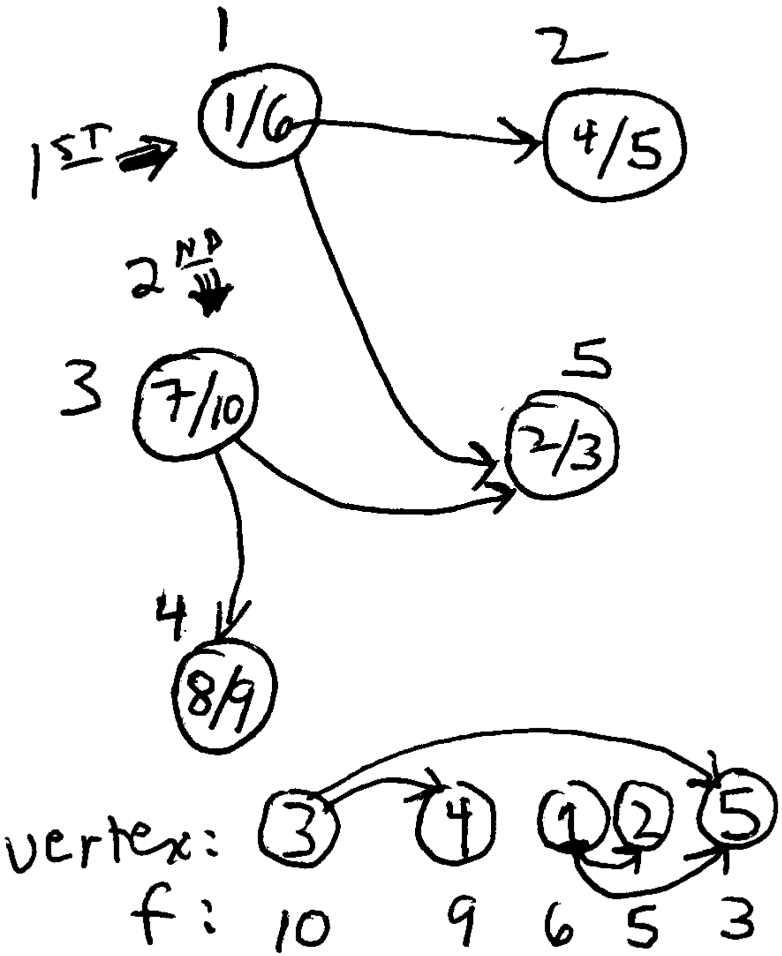
For topological sort to work, the graph ***G*** must be a ***DAG*** (directed acyclic graph). ***G***'s undirected version (i.e. the version of ***G*** with the "directions" removed from the edges) need not be connected.

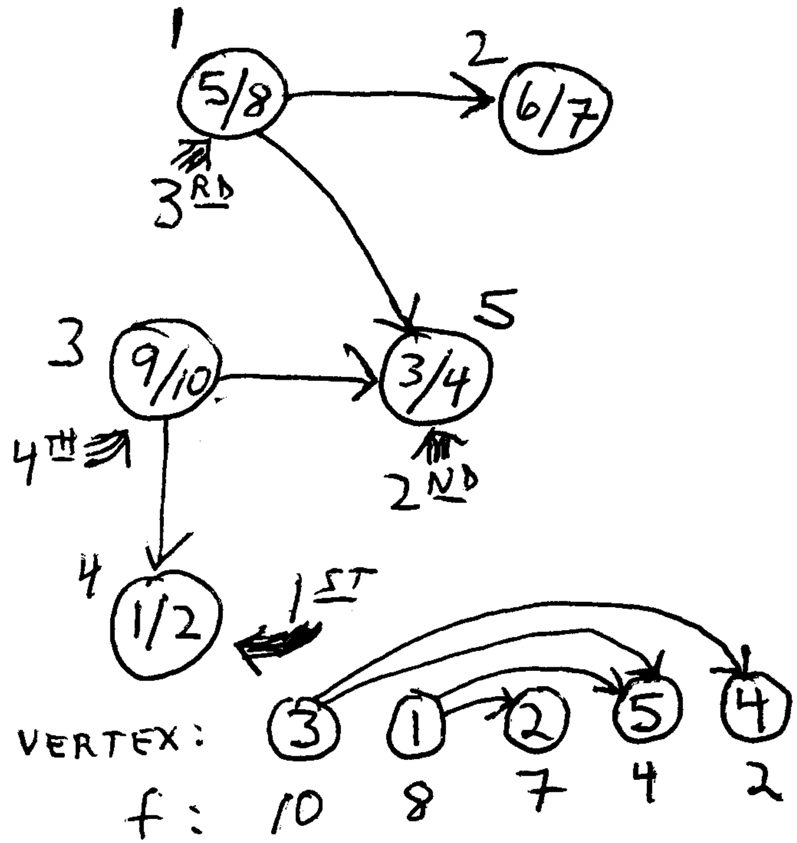
Theorem: Listing a dag's vertices in reverse order of finishing time (i.e. from highest to lowest) yields a topological sort.

Implementation: modify DFS to stick each vertex onto the front of a linked list as the vertex is finished.

see examples next slide....

Topological Sort Examples





More on Topological Sort

Theorem (again): Listing a dag's vertices in order of highest to lowest finishing time results in a topological sort. Putting it another way: If there is an edge (***u,v***), then f[***u***] > f[***v***].

Proof : Assume there is an edge (***u,v***).

Case 1: DFS visits ***u*** first. Then ***v*** will be visited and finished before ***u*** is finished, so f[***u***] > f[***v***].

Case 2: DFS visits ***v*** first. There cannot be a path from ***v*** to ***u*** (why not?), so ***v*** will be finished before ***u*** is even discovered. So again, f[***u***] > f[***v***].

QED.