Fair division of mixed and indivisible items

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Fair division

Introduction

Fair division, referred to as FD, aims to find the best way to allocate a certain bundle of items to a number of participants following their subjective preferences.
The bundle to be divided is referred as the *manna*, which can be:

- A set of goods
- A set of chores
- Mixed, composed both by goods and chores
- A set of divisible items
- A set of indivisible items

Working assumptions

In the usual framework, participants have all equal rights (or responsibilities) to the manna.

The division rules will stick just to the preferences of the participants, with these assumptions:

- Agents are selfish, they only care about their particular final shares of the manna.
- The full individual responsibility for the particular preference falls on the agents; as long as their preferences are respected, they don't complain about the lowness of their shares.
- Agents do not assign values to their utilities in such a way that manipulates the division rules.

Formalisation of the problem

Indivisible case

We define an *instance* as $I = (N, O, U)$, where:

- $N = \{1, 2..., n\}$ is the set of agents.
- $O = \{o_1, o_2, ..., o_m\}$ is the set of the indivisible items in the manna (each subset is a bundle of items).
- U is a n-tuple of utility functions $u_i : 2^O \longrightarrow \mathbb{R}$.

We assume *additivity* for the utility function, i.e. $u_i(X) = \sum_{o \in X} u_i(o)$ for each bundle $X \subseteq O$.
We define an *allocation* π as a function $\pi : N \longrightarrow 2^O$ such that $\cup_{i \in N} \pi(i) = O$ and $\pi(i) \cap \pi(j) = \emptyset$ for every agent $i, j \in N$.

Mixed manna

In the case of mixed manna an item can be a *good* ($u_i(o) > 0$), a *chore* ($u_i(o) < 0$) or a *null item* ($u_i(o) = 0$) according to the personal preference of each agent:

- An item is a *subjective good* if $u_i(o) > 0$ for some agent i in N and $u_j(o) \leq 0$ for some other agent j in N (for *subjective chores*, $u_i(o) < 0$ and $u_j(o) \geq 0$ for some i, j in N)
- An item is an *objective good* if $u_i(o) > 0 \quad \forall i \in N$ (for *objective chores*, $u_i(o) < 0 \quad \forall i \in N$)

Divide and Choose

Two agents take part to the division task:

- Agent 1 decides how to partition a divisible manna, e.g., a cake
- Agent 2 decides which partition to claim

If the divider splits the manna into two equal parts, then he cannot envy the chooser, because the latter cannot pick a better share; the same holds for the chooser, because he can pick a share that is at least as good as the divider's one.

The final partition is said to be *envy-free*.

Envy-freeness

Indivisible manna

Given a certain allocation π and two agents i and j, we say that *i envies j* if $u_i(\pi(i)) < u_i(\pi(j))$.

Definition

An allocation is said to be *envy-free* (EF) if no agent envies any other agent.

We say that *i envies j by more than one item* if i envies j and $u_i(\pi(i) \setminus \{o\}) < u_i(\pi(j) \setminus \{o\}) \quad \forall o \in \pi(i) \cup \pi(j)$.

Definition

An allocation π is *envy-free up to one item* (EF1) if $\forall i, j \in N$, i does not envy j by more than one item.

Fair share

Indivisible manna

Definition

An allocation π satisfies *fair share* (FS) if it guarantees to each agent i a bundle of items $\pi(i)$ that is at least his fair share $\frac{1}{n}u_i(O)$

Definition

An allocation π satisfies *fair share up to one item* (FS1) if for each agent $i \in N$ one of the following is true:

- $u_i(\pi(i)) \geq \frac{1}{n}u_i(O)$ (FS)
- $u_i(\pi(i))+ u_i(o) \geq \frac{1}{n}u_i(O)$ for some $o \in O \setminus \pi(i)$
- $u_i(\pi(i))$- $u_i(o) \geq \frac{1}{n}u_i(O)$ for some $o \in \pi(i)$

Steinhaus's rule for FS allocation in the divisible case

Consider the case in which n agents need to divide the cake C, a measurable set in Euclidean space; let $u_i(\pi_i)$ be the utility for the agent i for consuming the share π_i, where π_i is a measurable subset of C and u_i is an atomless non-negative measure.

1. Agent 1 cuts a fair share z_1 for him according to his utility u_1, such that $u_1(z_1) = \frac{1}{n}u_1(C)$. He offers z_1 to agent 2.
2. If $u_2(z_1) \geq \frac{1}{n}u_2(C)$, agent 2 cuts a smaller share $z_1' \subset z_1$ such that $u_2(z_1') = \frac{1}{n}u_2(C)$ and offers it to agent 3; otherwise he passes z_1 to agent 3.
3. The round repeats for all the agents; the last agent to cut gets the share.
4. The rule then repeats with the remaining cake and n-1 agents.

Relations among FS, FS1, EF, EF1

By definition, EF implies EF1, and FS implies FS1.
What about the relation between FS and EF?

Proposition

An EF allocation π satisfies FS.

Proof.

By additivity, summing up the EF condition for all the agents j:

$$\sum_{j \in N} u_i(\pi(i)) = n u_i(\pi(i)) \geq \sum_{j \in N} u_i(\pi(j)) = u_i(O) \quad \forall i \in N$$

□

The vice versa is not true.

Proposition

An EF1 allocation π satisfies FS1.

Finding EF1 allocations

Case with only goods

Round-robin rule

In an order which is established before the beginning of the procedure, each agent, in his round, picks the most preferred item from the remaining ones in the manna; the rounds repeat until no item is left to pick.

In the more general case of a mixed manna, the round-robin rule fails to find an EF1 allocation.

Theorem

For additive utilities, the double round-robin algorithm returns an EF1 allocation.

Finding EF1 allocations

Double round-robin rule

1: $\pi(i) = \emptyset \quad \forall i \in N$
2: Partition O in $O^+ = \{o \in O : \exists i \in N \quad \text{s.t.} \quad u_i(o) > 0\}$ and $O^- = \{o \in O : \quad u_i(o) \leq 0 \quad \forall i \in N\}$. Suppose $|O^-| = an - k$ with $a \in \mathbb{N}$ and $k \in \{0, ..., n-1\}$
3: Add k objective (dummy) null items that were not in the original manna to O^- in order to have $|O^-| = an$
4: Agents pick their most preferred items (one for round) in O^- according to the round-robin rule, in the order $(1, 2, ..., n)$, until O^- is completely allocated
5: An inverse round-robin rule with order $(n, n-1, ..., 1)$ allocates all the items in O^+ (each agent picks the most preferred item in his round). If there are no items with strictly positive utility for a particular agent, he picks a dummy null item that was not in the original manna
6: Remove the dummy items from the allocation
7: **return** the resultant allocation π

Efficiency

Definition

Given an allocation π, another allocation π' is a *Pareto improvement* of π if $u_i(\pi'(i)) \geq u_i(\pi(i)) \quad \forall i \in N$ and $u_j(\pi'(j)) > u_j(\pi(j))$ for some $j \in N$.

Definition

An allocation π is *Pareto optimal* (PO) if there is no allocation that is a Pareto improvement of π.

Example

Instance definition and utilities

Consider an instance with $n = 4$ agents and the indivisible manna $O = \{o_1, o_2, ..., o_9\}$. The utilities are:

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
Agent 1	1	-1	2	1	-2	-4	-6	-1	-1
Agent 2	4	-3	6	2	-2	-2	-2	-1	-1
Agent 3	0	11	8	11	0	0	0	10	0
Agent 4	0	11	8	11	0	0	0	0	10

$$u_1(\pi(1)) = 0; \quad u_2(\pi(2)) = 4;$$

$$u_3(\pi(3)) = 10; \quad u_4(\pi(4)) = 10$$

Example

Fair share and FS1

Consider an instance with $n = 4$ agents and the indivisible manna $O = \{o_1, o_2, ..., o_9\}$. The utilities are:

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
Agent 1	1	-1	2	1	-2	-4	-6	-1	-1
Agent 2	4	-3	6	2	-2	-2	-2	-1	-1
Agent 3	0	11	8	11	0	0	0	10	0
Agent 4	0	11	8	11	0	0	0	0	10

Allocation π satisfies FS:

$$u_1(\pi(1)) = 0 \geq -\frac{11}{4} = \frac{1}{n}u_1(O), \quad u_2(\pi(2)) = 4 \geq \frac{1}{4} = \frac{1}{n}u_2(O),$$

$$u_3(\pi(3)) = 10 \geq 10 = \frac{1}{n}u_3(O), \quad u_4(\pi(4)) = 10 \geq 10 = \frac{1}{n}u_4(O)$$

Example

Envy freeness and EF1

Consider an instance with $n = 4$ agents and the indivisible manna $O = \{o_1, o_2, ..., o_9\}$. The utilities are:

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
Agent 1	1	-1	2	1	-2	-4	-6	-1	-1
Agent 2	4	-3	6	2	-2	-2	-2	-1	-1
Agent 3	0	11	8	11	0	0	0	10	0
Agent 4	0	11	8	11	0	0	0	0	10

Allocation π does not satisfy neither EF nor EF1:

$$u_3(\pi(3)) = 10 < u_3(\pi(1)) = 22, \quad u_4(\pi(4)) = 10 < u_4(\pi(1)) = 22$$

$$u_3(\pi(3)) = 10 < u_3(\pi(1) \setminus o_2) = 11, \quad u_3(\pi(3)) = 10 < u_4(\pi(1) \setminus o_4) = 11$$

$$u_4(\pi(4)) = 10 < u_4(\pi(1) \setminus o_2) = 11, \quad u_4(\pi(4)) = 10 < u_4(\pi(1) \setminus o_4) = 11$$

Example

Efficiency

Consider an instance with $n = 4$ agents and the indivisible manna $O = \{o_1, o_2, ..., o_9\}$. The utilities are:

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
Agent 1	1	-1	2	1	-2	-4	-6	-1	-1
Agent 2	4	-3	6	2	-2	-2	-2	-1	-1
Agent 3	0	11	8	11	0	0	0	10	0
Agent 4	0	11	8	11	0	0	0	0	10

Allocation π is not Pareto optimal, since we can have a Pareto improvement by moving the items o_5, o_6, o_7 from agent 1 and 2 to agent 3 or 4.

Finding PO and EF1 allocations

Adjusted Winner

1. Let agent 1 be the *winner* and agent 2 the *loser*
2. For every item o in the manna, if $u_1(o) > u_2(o)$, o is given to the winner, otherwise is given to the loser.
3. The items given to the winner are ordered increasingly by the ratio $\frac{u_1(o)}{u_2(o)}$
4. Fraction of items in the winner allocation are continuously given to the the loser, in the sequence specified above (from smallest to greatest), until an allocation which gives equal utility to both agents is obtained.

Drawbacks:

- The Adjusted Winner works only with goods
- The Adjusted Winner returns a fractional allocation

Finding PO and EF1 allocations

Generalised Adjusted Winner

$\pi(i) = \emptyset \quad \forall i \in \{w, l\}$

Let $O_w^* = \{o \in O : u_w(o) \geq 0 \quad \text{and} \quad u_l(o) \leq 0\}$ and $O_l^* = \{o \in O : u_l(o) \geq 0 \quad \text{and} \quad u_w(o) < 0\}$

Let $O^+ = \{o \in O : u_i(o) > 0, \quad i \in \{w, l\}\}$ and $O^- = \{o \in O : u_i(o) < 0, \quad i \in \{w, l\}\}$

Add each item $o \in O^+ \cup O_w^*$ to $\pi(w)$, and each item $o \in O^- \cup O_l^*$ to $\pi(l)$

Sort items in $O^+ \cup O^-$ in increasing order of $\frac{|u_w(o)|}{|u_l(o)|}$. We denote the ordered items with $o_1, ..., o_k$ $(k = |O^+ \cup O^-|)$

set t=1

while agent l envies agent w by more then one item, **do**:

  if $o_t \in O^+$, **then**

    $\pi(w) = \pi(w) \setminus \{o_t\}$ and $\pi(l) = \pi(l) \cup \{o_t\}$

  else if $o_t \in O^-$, **then**

    $\pi(w) = \pi(w) \cup \{o_t\}$ and $\pi(l) = \pi(l) \setminus \{o_t\}$

  t=t+1

return π

Finding PO and EF1 allocations

Generalised Adjusted Winner

Lemma

In the Generalised Adjusted Winner, the allocation π is Pareto optimal during each step of reallocation.

Theorem

For two agents with additive utilities, a PO and EF1 allocation always exists.

Connected FS1 allocations

Definition

An allocation π is said to be *connected* if for each agent $i \in N$, $\pi(i)$ is connected in the sequence (*path*) of items $(o_1, o_2, ..., o_m)$

Definition

A connected allocation π is FS1$_{outer}$ if $\forall i \in N$:

- $u_i(\pi(i)) \geq \frac{1}{n} u_i(O)$ (the allocation is FS), or
- $u_i(\pi(i)) + u_i(o) \geq \frac{1}{n} u_i(O)$ for some $o \in O \setminus \pi(i)$ such that $\pi(i) \cup \{o\}$ is connected, or
- $u_i(\pi(i)) - u_i(o) \geq \frac{1}{n} u_i(O)$ for some $o \in \pi(i)$ such that $\pi(i) \setminus \{o\}$ is connected.

Finding connected FS allocations

Dubins-Spanier moving knife procedure for positive utility

Moving knife procedure

1. A knife is slowly moved at constant speed parallel to itself over the top of the cake, covering an increasing sub-interval of the cake.
2. As soon as a slice that is equal to the fair share of any of the agents is reached by the knife, that agent stops the knife and receives that slice.
3. The process is repeated with the other $n - 1$ agents and with what remains of the cake.
4. The last agent receives the remaining slice of cake.

Finding connected FS allocations

Generalised moving knife $\mathcal{A}$ for mixed cake

$\pi(i) = \emptyset \quad \forall i \in N'$

$N^+ = \{i \in N' : \hat{u}_i([l, r]) > 0\}$

if $N^+ \neq \emptyset$ **then**

    if $|N^+| = 1$ **then**

        Allocate $[l, r]$ to the unique agent in N^+

    else

        Let x_i be the minimum point where $\hat{u}_i([l, x_i]) = \frac{1}{|N'|}\hat{u}_i([l, r])$, $i \in N^+$

        Let agent j be the one with $x_j = \min_{i \in N^+} x_i$

        $\pi(j) = [l, x_j]$ and $\pi|_{N' \setminus \{j\}} = \mathcal{A}([x_j, r], N' \setminus \{j\}, (\hat{u}_i)_{i \in N' \setminus \{j\}})$

        return π

else

    Let x_i be the maximum point where $\hat{u}_i([l, x_i]) = \frac{1}{|N'|}\hat{u}_i([l, r])$, $i \in N'$

    Let agent j be the one with $x_j = \max_{i \in N'} x_i$

    $\pi(j) = [l, x_j]$ and $\pi|_{N' \setminus \{j\}} = \mathcal{A}([x_j, r], N' \setminus \{j\}, (\hat{u}_i)_{i \in N' \setminus \{j\}})$

    return π

Finding FS1$_{outer}$ connected allocations

Translating the indivisible problem into a divisible one

Consider the set $(o_1, o_2, ..., o_m)$; we want to find an FS1 allocations. We map the task into a divisible one with:

- A mixed cake $[0, m]$
- Uniform utilities with values $u_i(o_j)$ in the intervals $[j-1, j]$ for each $j \in (1, 2, ..., m)$

Using the generalised moving knife, we get:

Theorem

For additive utilities, a contiguous FS allocation of a mixed cake exists.

Suppose that in such allocation $\hat{\pi}$ each agent $i = 1, 2..., n$ receives the sub-interval $[x_{i-1}, x_i]$ of the mixed cake ($x_0 = 0, x_n = m$), and that no agent gets an empty bundle ($x_{i-1} < x_i$ for all $i = 1, 2..., n$).

Finding FS1$_{outer}$ connected allocations

Translating the divisible solution into an indivisible one

- Item o_j is assigned to agent i if $x_{i-1} \leq j-1 \leq j \leq x_i$ for some $i \in N$;

If instead $j-1 \leq x_l \leq x_{l+1} \leq ... \leq x_r \leq j$, with $x_l = \min\{x_i : x_i \geq j-1\}$ (*left-most* agent) and $x_r = \max\{x_i : x_i \leq j\}$ (*right-most agent*):

- if $\min\{u_l(o_j), u_r(o_j)\} < 0 < max\{u_l(o_j), u_r(o_j)\}$, then item o_j is allocated to the agent with positive utility
- if $\min\{u_l(o_j), u_r(o_j)\} \geq 0$, o_j is assigned to the left-agent l
- if $\max\{u_l(o_j), u_r(o_j)\} < 0$, o_j is assigned to the right-agent r

Theorem

For additive utilities, a connected FS1$_{outer}$ allocation of a path always exists.

Fractional allocations and FPO

In *integral* allocations each item is allocated to a single agent.

Definition

A *fractional* allocation is an allocation $x = (x_1, x_2, ..., x_n)$, where x_i is the allocation of agent i and $x_{i,o}$ is the fraction of item o allocated to agent i ($\sum_{i \in N} x_{i,o} = 1$).

The utility that an agent i gets from a fractional allocation is $u_i(x_i) = \sum_{o \in O} u_i(o) x_{i,o}$

Definition

An allocation satisfies *fractional Pareto optimality* (FPO) if it cannot be improved by any fractional allocation.

Consumption graph

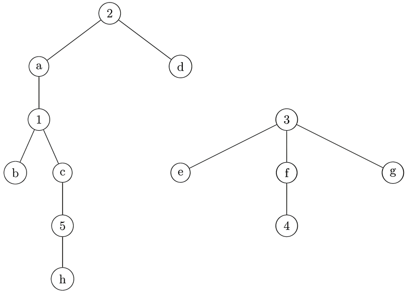
Definition

Given a fractional or integral allocation x, the corresponding *consumption graph* G_x is a bipartite graph with vertices $N \cup O$ and edge set $E = \{\{i, o\} : x_{i,o} > 0\}$.

If an agent i shares an item with an agent j, we say that j is agent i's *neighbor*.

Consumption graph

example



	a	b	c	d	e	f	g	h
agent 1	$\oplus$	$\oplus$	$\ominus$	$+$	-	$+$	$+$	$+$
agent 2	$\oplus$	-	-	$\oplus$	-	$+$	$+$	$+$
agent 3	$+$	$+$	-	-	$\ominus$	$\oplus$	$\oplus$	-
agent 4	-	$+$	-	-	-	$\oplus$	$+$	-
agent 5	-	-	$\ominus$	$+$	-	-	$+$	$\oplus$

Finding FS1 and FPO allocations

Find a fractional allocation x_{FS} that guarantees a share $\frac{1}{n}$ of each item to each agent i
Find an FPO fractional allocation x, with acyclic consumption graph G_x, that Pareto dominates x_{FS}
if some agent j shares an item o for which $u_j(o) = 0$ **then**
    o is fully given to an agent i who shares o in the allocation x
Consider $Q = \emptyset$, an empty FIFO queue of agents
while there is an agent i sharing at least one item o with others **do**
    Add agent i to Q
    while $Q \neq \emptyset$ **do**
        Take the first agent j out of Q
        Add all the neighbours of j to the end of Q
        for each o shared by j **do**
            if $u_j(o) > 0$ **then**
                give o fully to j, i.e. $x_{j,o} = 1$ and $x_{k,o} = 0 \quad \forall k \neq j$
            else if $u_j(o) < 0$ **then**
                give o to a neighbour with whom o is shared
return the updated allocation $x^* = x$

Rounding procedure

Fractional consumption graph

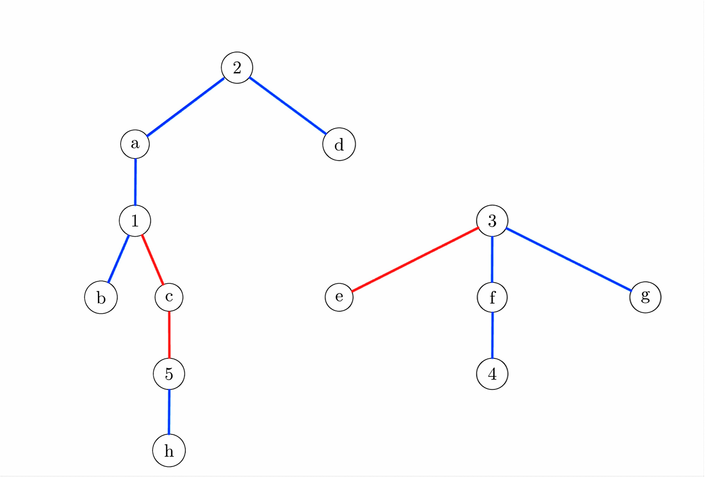


Figure: Consumption graph before the rounding procedure. Red edges stand for chores, blue edges for goods.

Rounding procedure

Integral consumption graph

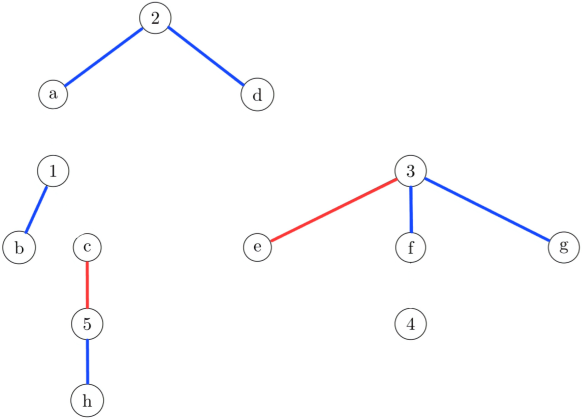


Figure: Consumption graph after the rounding procedure. Red edges for chores, blue edges for goods.

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Ladies! Gentlemen! Everybody stop what you're doing for a sec.
I have gained access to a cake!!
Let us divide it among us so we each get a fair amount, yes? I will divide exactly into thirds!
But T-Rex, maybe I like cherries and you like icing. It doesn't have to be equal to be fair!
FINE. What if I divided it so that no-one would want to trade their piece with someone else? That's fair!
That's ENVY-FREE, baby!
But that's not really fair either!
HOW IS THAT NOT FAIR??
It's greedy. Let's cut it such that there's no other way it could be cut that would make ANY of us better off, without making at least one of us worse off. That's OPTIMAL for all of us, T-Rex, and that's optimal for SOCIETY.
That's one person keeping the whole cake! Nice try, Utahraptor!!
LATER:
F XXX, it's strawberry cake
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