UNIVERSITÀ DI PISA

DIPARTIMENTO DI FISICA

Corso di Laurea Triennale in Fisica

Characteristics and spectrum of two-dimensional turbulence

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ANNO ACCADEMICO 2021/2022

Contents

1 Introduction

The study of a turbulent flow of a fluid restricted to a two-dimensional space leads to a very different behaviour from its three-dimensional counterpart, even though the mathematical instruments used in studying it are the same.
First of all, we must admit that the world is actually three-dimensional, and the pure bidimensional turbulence doesn't actually exist in real life; however, it is still a good approximation for some aspects of fluid: for example, large-scale atmospheric flows might have a horizontal length-scale of around 1oo kilometres, while the troposphere is only around 10 kilometres deep; hence we can consider negligible the vertical motion. Other examples of applications are the fluids in which one component of the motion is suppressed. The most relevant cases are the fluids in rotations, in virtue of the Taylor-Proudman theorem, and plasmas reduced to planar motion by strong-magnetic field. Finally, laboratory experiments to study two-dimensional turbulence typically look at flowing soap films or a thin layer of conducting fluid stirred by electromagnetic forces. If this is not enough, it still seems natural to investigate this phenomenon, if not in the hope that it sheds some light on 'almost' two-dimensional turbulence, then in the interests of scientific curiosity.
The aim of this work is then to show how some properties can arise from the dynamic of the flow and from a simple spectral

analysis. In particular, the cause of all the problems is the absence in the two-dimensional case of the vortex-stretching term, whose function will be properly clarified. The main consequence is that the vorticity, in the inviscid case, has to conserve. This, associated with the conservation of a closely related quantity, the enstrophy, will lead to the two most important results: the inverse energy cascade and the direct enstrophy cascade. In both cases, we will give a simple model for the spectral density of these two quantities. Finally, our results will be analysed from a more intuitive and physical point of view, in order to give a better insight on the subject and to complete our thesis: two-dimensional turbulence is far from a particular case of its well known counterpart.

2 Conservation of vorticity

The first thing to do in order to better understand the peculiarities of a bi-dimensional flow is to find the equation for the vorticity. To do so, we start writing the momentum equation:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{v} = -\frac{\vec{\nabla}P}{\rho} + \nu\Delta\vec{v}$$

Using the identity $\frac{1}{2}\vec{\nabla}v^2 = \vec{v}\times\vec{\nabla}\times\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{v}$ we can rewrite the equation in the following way:

$$\frac{\partial \vec{v}}{\partial t} - \vec{v}\times\vec{\nabla}\times\vec{v} = -\vec{\nabla}(\frac{P}{\rho} + \frac{v^2}{2}) + \nu\Delta\vec{v}$$

Taking the curl and defining the vorticity of the flow as $\vec{w} = \vec{\nabla}\times\vec{v}$, we obtain the equation for the evolution of this quantity:

$$\frac{\partial \vec{w}}{\partial t} = \vec{\nabla}\times\vec{v}\times\vec{w} + \nu\Delta\vec{w}$$

The first term on the left can be written as :

$$\vec{\nabla}\times\vec{v}\times\vec{w} = (\vec{w}\cdot\vec{\nabla})\vec{v} - (\vec{\nabla}\cdot\vec{v})\vec{w} = (\vec{w}\cdot\vec{\nabla})\vec{v}$$

where in the last equality the hypothesis of incompressibility of the fluid was used ($\vec{\nabla} \cdot \vec{v} = 0$). Finally we have:

$$\frac{\partial \vec{w}}{\partial t} = -(\vec{w} \cdot \vec{\nabla})\vec{v} + \nu \Delta \vec{w}$$

If we are considering the flow for high Reynolds number, we can neglect the viscosity of the fluid and the second term of the previous equation. Last, if the flow is two-dimensional as in the case we are going to treat, the equation takes the easy form (the first equivalence is obvious since $\vec{v} \cdot \vec{\nabla} \vec{w} = 0$ in two dimensions):

$$\frac{\partial \vec{w}}{\partial t} = \frac{D\vec{w}}{Dt} = 0$$

In fact, a flow field is said to be two-dimensional when the velocity is everywhere at right angles to a certain direction and independent of displacements parallel to that direction. Supposing the motion on the x-y plane, the only non-vanishing component of $\vec{w}$ is w_z, but the ∇-operator has a null component on this direction, so that $(\vec{w} \cdot \vec{\nabla})\vec{v} = 0$. This term is named "vortex stretching and tilting", and it is responsible of the difference between the three-dimensional case and the two-dimensional. Its function is more evident in the tensorial form $w_i \frac{\partial v_j}{\partial x_i}$. In particular, it is possible to distinguish two different behaviour:

- if i=j, the variation of the i-component of velocity on the i-direction increases the i-component of vorticity, without changing its direction ("stretching" it). There is then a mechanism of self-amplification of the vorticity (if the velocity gradient in this direction is positive), with no need of external sources.

- if $i \neq j$, the variation of the j-component of velocity on the i direction causes a variation of the j-component of vorticity. This means that a part of the i-component of vorticity turns in the j-component. There is then a redistribution of the vorticity among the different components, so that the total vorticity changes its direction ("tilting").

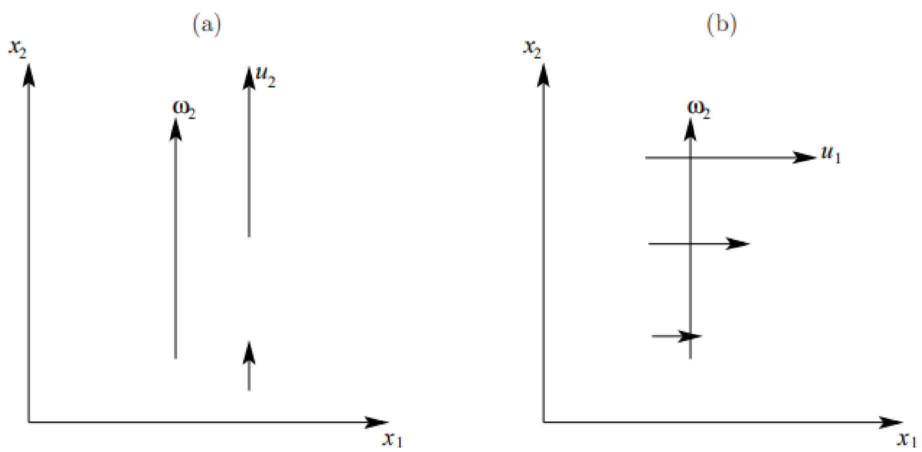

Figure 1: a)vortex stretching, b) vortex tilting

The "stretching" mechanism, which is prevalent in three-dimensional turbulence, causes the lengthening of the fluid elements, which in the incompressible case, where the volume of the fluid elements is constant, implies thinning in the directions perpendicular to the stretching direction, reducing the radial length of the vorticity. In this way energy is transferred to smaller and smaller scale, until the viscosity is not negligible and causes dissipation. This description is not valid in the two-dimensional case, where the vortex stretching term is null, and the vorticity is then preserved.

3 Spectral behaviour of kinetic energy for large scale

In this section we introduce some statistical quantities and we work out their relations with the spectral distribution of energy.
First the second order tensor of the velocity fluctuations correlations, in the case of null mean flow (where the velocity and its fluctuations are identical), is defined as

$$Q_{ij}(\vec{x}, \vec{x}') = \langle v_i(\vec{x}) v_j(\vec{x}') \rangle$$

where $\vec{x}' = \vec{x} + \vec{r}$, $\vec{r}$ being a general displacement. This quantity physically measures how the fluctuations of different components of velocity in different points in space are related. In the homogeneous turbulence, where the properties are independent by the position considered, $Q_{ij}(\vec{x}, \vec{x}')$ depends only on the difference between the vectors $\vec{x}$ and $\vec{x}'$, hence $Q_{ij}(\vec{x}, \vec{x}') = Q_{ij}(\vec{r})$
Then we define the turbulent kinetic energy as:

$$E_{turb} = \frac{1}{2} \langle v^2 \rangle$$

and is linked with the the second order tensor of the velocity fluctuations correlations by the easy relation $E_{turb} = Q_{ii}(0)$.
To compute the energy spectrum in the high scale limit it is necessary to find $E(k)$, representing the kinetic energy which

is contained in the wave-number interval $[k, k + dk]$ and defined as:

$$E_{turb} = \int_0^{+\infty} E(k)dk$$

Before continuing the discussion, it is important to pay attention to the fact that actually, since the spatial space is not infinite, the wave-number spectrum is not actually continuous, and the integration on dk is not mathematically correct. However, when considering homogeneous and isotropic turbulence we are usually restricting to a portion of space and, in particular, we assume to be far from the boundaries of the fluid. In this approximation we consider the portion of the fluid unbounded, so that a Fourier series can be replaced by a Fourier transform. Moreover calculations in discrete space give the same results, so, for simplicity, we can carry on our derivation in continuous space .

Now, defining $\phi_{ij}(\vec{k}, t)$ as the bi-dimensional Fourier transform of $Q_{ij}(\vec{r})$ and considering the isotropic turbulence such that $Q_{ij}(\vec{r})$ depends only on $r = |\vec{r}|$, it is possible to write the following relation:

$$E_{turb} = \frac{1}{2}Q_{ii}(0) = \frac{1}{2}\int \phi_{ii}(\vec{k})d\vec{k} = \frac{1}{2}\int_0^{+\infty}\int_0^{+2\pi} k\phi_{ii}(k)dkd\theta =$$

$$= \pi \int_0^{+\infty} k\phi_{ii}(k)dk$$

Hence, $E(k) = \pi k \phi_{ii}(k)$.

Conversely, using the inverse Fourier transform:

$$\phi_{ii}(k) = \frac{1}{(2\pi)^2} \int_0^{+\infty} rQii(k)dr \int_0^{+2\pi} e^{-ikr\cos\theta} d\theta =$$

$$= \frac{1}{\pi} \int_0^{+\infty} rQ_{ii}(r) J_0(kr) dr$$

Substitution in the expression for $E(k)$ in function of ϕ_{ii} leads to: :

$$E(k) = k \int_0^{+\infty} rQ_{ii}(r) J_0(kr) dr$$

Thus, it is easy to work out the behaviour of the spectrum at the very large scales ($\lim k \to 0$) by the property of the zero-order Bessel function. In particular, $E(k) \sim k \int_0^{+\infty} rQ_{ii}(r)dr$ as k approaches 0, showing that the spectral kinetic energy density grows like k, contrary to the k^2 behaviour of three dimensional turbulence. The result is independent of the dynamics of turbulence, and shows how the simple change to two-dimensional or three-dimensional polar coordinates leads to very different behaviour.

4 Spectral dynamic

An higher degree of knowledge of how energy is exchanged and behave at different wave-number can be reached analysing the dynamic equation in spectral space. Applying then the Fourier transform to the momentum equation and to the continuity equation for an incompressible fluid we obtain:

$$\frac{\partial \hat{v}_i}{\partial t} + ik_j\widehat{v_j v_i} = -i\frac{k_j\hat{P}}{\rho} - \nu k^2 \hat{v}_i$$

$$k_i v_i = 0$$

Now, we define the projector tensor $Pij = \delta_{ij} - \frac{k_i k_j}{k^2}$, so that if $\vec{a}$ is a vector, $P_{ij}a_j$ is the component of $\vec{a}$ perpendicular to $\vec{k}$. It is then possible to project the previous equation on the plane perpendicular to $\vec{k}$, in order to eliminate the pressure gradient's contribution:

$$\frac{\partial \hat{v}_i}{\partial t} + \nu k^2 \hat{v}_i = -iP_{ij}k_n\widehat{v_j v_n} = -\frac{i}{2}P_{ijn}\widehat{v_j v_n}$$

where $P_{ijn} = P_{ij}k_n + P_{in}k_j$

Hence the evolution of the mode $\hat{v}_i$ is driven by the dumping of the viscosity and by a forcing of all components which verify the condition $\vec{k} = \vec{p} + \vec{q}$. In fact, the term $\widehat{v_j v_n}$ can be explicited :

$$\widehat{v_j v_n} = \int \hat{v}_j(\vec{p})\hat{v}_n(\vec{k}-\vec{p})d\vec{p} = \int \hat{v}_j(\vec{p})\hat{v}_n(\vec{q},t)\delta(\vec{p}+\vec{q}-\vec{k},t)d\vec{p}d\vec{q}$$

There are then two different kinds of interaction, the local one, when $|\vec{k}| \sim |\vec{p}| \sim |\vec{q}|$, and the non-local one, when $|\vec{k}| \ll |\vec{p}| \sim |\vec{q}|$. in both cases, $\vec{k}$, $\vec{p}$ and $\vec{q}$ form an interacting triad.

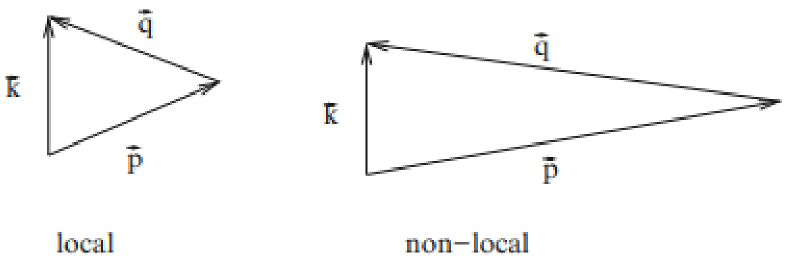

Figure 2: The two different kinds of triad interaction

Now we multiply the spectral equation of momentum by $\hat{v}_i^*$ (where the $*$ means the complex conjugate), obtaining:

$$\frac{1}{2}\frac{\partial \hat{v}_i \hat{v}_i^*}{\partial t} + \nu k^2 \hat{v}_i \hat{v}_i^* = -\frac{i}{2} P_{ijn} v_j \hat{v}_n \hat{v}_i^*$$

Taking the average and using the definition of $\phi_{ii}(\vec{k}, t) = \hat{v}_i \hat{v}_i^*$, we get:

$$\frac{\partial \phi_{ii}(\vec{k}, t)}{\partial t} + 2\nu k^2 \phi_{ii}(\vec{k}, t) = -\frac{i}{2}\langle P_{ijn} v_j \hat{v}_n \hat{v}_i^* \rangle$$

where, using the property that the Fourier transform of a real function $(v(\vec{r}))$ is even, we can rewrite the last term as :

$$-\frac{i}{2}P_{ijn}\int\langle\hat{v_j}(\vec{p},t)\hat{v_n}(\vec{q},t)\hat{v_i^*}(\vec{k},t)\rangle\delta(\vec{p}+\vec{q}-\vec{k})d\vec{p}d\vec{q}=$$

$$-\frac{i}{2}P_{ijn}\int\langle(\hat{v_j}^*(\vec{p},t)\hat{v_n}^*(\vec{q},t)\hat{v_i}(\vec{k},t)\rangle\delta(\vec{p}+\vec{q}-\vec{k})d\vec{p}d\vec{q}=$$

$$=-\frac{i}{2}P_{ijn}\int\langle(\hat{v_j}(-\vec{p},t)\hat{v_n}(-\vec{q},t)\hat{v_i}(\vec{k},t)\rangle\delta(\vec{p}+\vec{q}-\vec{k})d\vec{p}d\vec{q}=$$

$$=-\frac{i}{2}P_{ijn}\int\langle(\hat{v_j}(\vec{p},t)\hat{v_n}(\vec{q},t)\hat{v_i}(\vec{k},t)\rangle\delta(\vec{p}+\vec{q}+\vec{k})d\vec{p}d\vec{q}$$

In particular, in the isotropic case, using $E(k,t)=\pi k\phi_{ii}(k,t)$:

$$\frac{\partial E(k,t)}{\partial t}+2\nu k^2E(k,t)=T_E(k,t)$$

$T_E(k,t)$ is called the "spectral kinetic energy transfer function". Its existence is due to the non-linear terms of Navier-Stokes equation and it is the direct manifestation of the energy exchange between the modes that are compatible with the condition $\vec{k}+\vec{q}+\vec{p}=0$. The term thus express a reciprocal pumping of energy among different scales, caused by the different instabilities of turbulence flow. In fact, the energy transfer is in both direction, and the dominant one is different in two-dimensional and three-dimensional turbulence. In the former the energy transfer towards the large scales is predominant, in the latter the energy flows mainly from large scales to smaller ones.

5 Enstrophy

It is possible to define an another characteristic quantity, the enstrophy:

$$Z = \frac{1}{2}\langle w^2 \rangle$$

and, in analogy with the turbulent kinetic energy, we define $Z(k)$ as:

$$Z = \int_0^{+\infty} Z(k)dk$$

Recalling the results obtained in the first section, it is evident that the equation governing vorticity and velocity are the same, so without any calculation we can state:

$$\frac{\partial Z(k,t)}{\partial t} + 2\nu k^2 Z(k,t) = T_Z(k,t)$$

It is obvious that vorticity conservation implies enstrophy conservation. Thus, we can state that for a bi-dimensional flow, when viscosity is negligible, enstrophy is conserved.

6 Consequence of the enstrophy conservation

The conservation of enstrophy, associated with the energy conservation , in the length scale where viscosity is negligible, causes the main difference between two-dimensional and three-dimensional turbulence, where the first quantity is not conserved. In order to evaluate these different behaviours, we start by considering three Fourier modes of wave-numbers $k_1 < k_2 < k_3$ and energy E_1, E_2 and E_3. The mode are assumed to be in non-linear interaction in order to consider the effect of $T(k,t)$, so $k_1 + k_2 + k_3 = 0$. The conservation laws stated above, expressed in terms of the variation of the quantities of the single terms are:

$$dE_1 + dE_2 + dE_3 = 0$$

$$k_1^2 dE_1 + k_2^2 dE_2 + k_3^2 dE_3 = 0$$

Solving for dE_1 and dE_2 we find:

$$dE_1 = -\frac{k_3^2 - k_2^2}{k_3^2 - k_1^2} dE_2$$

$$dE_3 = -\frac{k_2^2 - k_1^2}{k_3^2 - k_1^2} dE_2$$

One first result that can be obtained from the previous solution is that no single of the three components can represent a source or a sink for the both two remaining ones unless this is represented by a scale intermediate between the scales of the two other components. So we proceed considering this situation. If the energy of the intermediate mode k_2 decreases $dE_2 < 0$, the energy of the other modes increases ($dE_3 > 0$, $dE_1 > 0$). If we suppose that $k_2 = \lambda k_1$, keeping in mind that $k_1 + k_2 = k_3 = (1+\lambda)k_1$, we find that, for $= 1 < \lambda < 1+\sqrt{3}$, $dE_1 > dE_3$. This means that if the wave-numbers are close enough the energy is mainly transferred to the smaller one, associated with the largest length scale. This mechanism is called "inverse cascade of energy". On the other hand we may observe that simultaneously, enstrophy would rather tend to cascade to the small scales for, generally, $Z_3 = k_3^2 dE_3 > Z_1 = k_1^2 dE_1$.

The same conclusion could be obtained by a more solid argument introducing the energy centroid of the spectrum:

$$k_c = \frac{\int_0^\infty kE(k,t)dk}{\int_0^\infty E(k,t)dk}$$

The conservation of energy and enstrophy in the inertial range (where viscosity is negligible) means that:

$$\frac{d}{dt}\int_0^\infty E(k,t)dk = 0$$

$$\frac{d}{dt}\int_0^\infty k^2 E(k,t)dk = 0$$

A measure for the fluctuation of the spectrum is given by:

$$\Delta k = \int_0^{\infty} (k - k_c)^2 E(k,t) dk$$

We can reasonably assume that the spreading of wave-number increases with time, so $\frac{d\Delta k}{dt} > 0$. Then, using the conservation listed above:

$$\frac{d\Delta k}{dt} = \frac{d}{dt} \int_0^{\infty} k^2 E(k,t) dk + \frac{d}{dt}(k_c^2) \int_0^{\infty} E(k,t) dk +$$

$$-2\frac{d}{dt} k_c^2 \int_0^{\infty} E(k,t) dk = -2k_c \frac{dk_c}{dt} \int_0^{\infty} E(k,t) dk$$

Finally, since for definition $k_c > 0$:

$$\frac{dk_c}{dt} < 0$$

So, in the inertial range, the energy on average moves upscale. More complicate calculation shows that enstrophy, instead, tends to move to the lower scales, leading to a direct cascade.

7 Energy and enstrophy spectrum

Integrating in dk the equation of evolution for $E(k,t)$ and $Z(k,t)$, between 0 and $+\infty$, we obtain the evolution of turbulent kinetic energy and enstrophy:

$$\frac{dE_{turb}}{dt} = \frac{d}{dt}\int_0^{+\infty} E(k,t)dk = -2\nu \int_0^{+\infty} k^2 E(k,t)dk =$$

$$= -2\nu \int_0^{+\infty} Z(k,t)dk \equiv -\epsilon(t)$$

$$\frac{dZ}{dt} = \frac{d}{dt}\int_0^{+\infty} Z(k,t)dk = -2\nu \int_0^{+\infty} k^4 E(k,t)dk =$$

$$= -2\nu \int_0^{+\infty} k^2 Z(k,t)dk \equiv -\xi(t)$$

The latter equation shows that enstrophy is a decreasing function, since $\xi(t) > 0$. Hence, the initial value is an upper limit and enstrophy is bounded from above.
Now we analyse the stationary case, such that $E(k)$, $Z(k)$, ξ and ϵ are all independent of time. In the inertial range, viscosity is negligible; in other words, the inertial range is characterized by the limit $\nu \to 0$. in this range, from the firs equation we see that $\epsilon \to 0$, which means that the spectrum for the kinetic energy is independent by the dissipation rate of energy. The only parameter left is the dissipation rate of enstrophy $\xi(t)$. We can now state the bidimensional analogue

of Kolmogorov hypothesis for homogeneous and isotropic turbulence

- First similarity hypothesis: The structure functions for the velocity within an isotropic homogeneous turbulence just depend on ϵ, ξ and ν.
- Second similarity hypothesis: if the distance between the points is large compared to the dissipation scale, then the structure functions depend only on ξ

According to the second hypothesis, $E(k)$ depends only k and ξ. By simple dimensional analysis, and defining v_k and τ_k as the characteristic velocity and time scales for the wavenumber k, we obtain:

$$\xi \sim k^2 v_k^2 \tau_k^{-1} = k^3 v_k^3$$

where the dimensional relation $\tau_k \sim k^{-1} v_k^{-1}$ was used.
Conversely, $v_k^2 \sim E(k)k$, hence:

$$\xi \sim k^3 k^{3/2} E(k)^{3/2} \sim k^{9/2} E(k)^{3/2}$$

Finally, inverting this relation we get:

$$E(k) = \xi^{2/3} k^{-3} f(\xi, k)$$

where f is an adimensional function; thus f must have an adimensional argument. However, there is not an adimensional combination of the parameter ξ and k, so the function must be a constant. We have then work out an expression for the energy spectrum using only dimensional reasoning:

$$E(k) = C_E \xi^{2/3} k^{-3}$$

The expression agrees with the previous discussion on the energy cascade, showing that energy tends to accumulate in large scale. In this range, the dissipation occurs, since from the evolution equation for energy the dissipation term is proportional to $k^2 E(k) \sim k^{-1}$. With the same reasoning applied to the enstrophy spectrum, we find that:

$$Z(k) = C_Z \xi^{2/3} k^{-1}$$

This expression is compatible with the definition of the enstrophy cascade, since the enstrophy dissipation term is proportional to $k^2 Z(k) \sim k$. It is then evident that energy and enstrophy dissipation occurs at two opposite limit of the spectrum, the latter being dominant at smaller scales. Finally, we can evaluate a lower limit for the inertial range considered above combining ξ and ϵ. Thus, the inertial range contains the wave-number such that $k \ll \xi^{1/6} \nu^{-1/2}$.

8 Physical interpretation

It is useful to give a more physical and intuitive insight on the difference between bidimensional and three-dimensional turbulence. First, as seen in the section above,the dissipation term, which can be written as $\epsilon = \nu\langle w^2\rangle$, vanish in the two-dimensional case, since $\langle w^2\rangle$ is bounded from above. In three dimensional turbulence, instead, this value is the same among different scales, arising from the fact that if ν is very small, the vorticity squared can compensate with greater values to make ϵ constant. Then, in the former case, we have a near conservation of energy, which leads to another important property of bidimensional turbulence: it is long lived. Obviously, this process is still dissipative and can't last forever, but it lasts longer than the its three dimensional counter part.

Now, let's consider the equation seen in section 1 for the z-component of vorticity for a bidimensional flow with $\vec{w} = (0, 0, w)$:

$$\frac{Dw}{Dt} = \nu\Delta w$$

This is the same equation governing the evolution of a scalar, so, in the limit of $\nu \to 0$, we can say that the z-component of vorticity is materially conserved, and the iso-vortical lines (points where the vorticity has the same value) coincide with material lines, acting in their same way. In particular, they are teased out by the flow, leading to thin sheets of fluid.

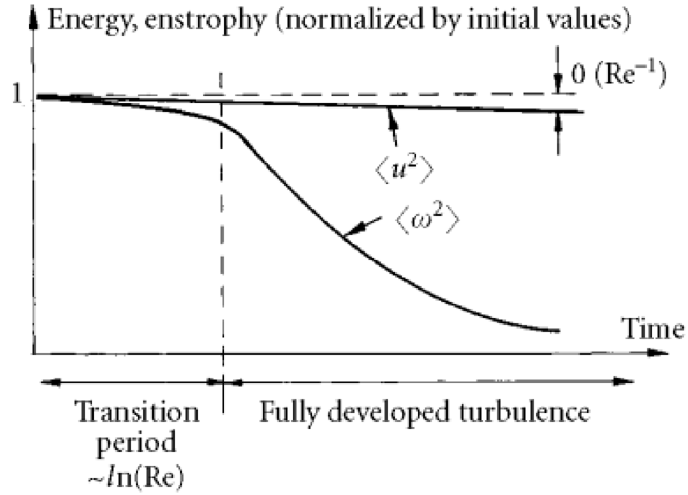

Figure 3: Evolution of mean kinetic energy and enstrophy with time

Then, if we imagine the turbulence to start with velocity and vorticity gradients of the same scale, the increasing filamentation of material lines (and isovortical ones) amplifies the vorticity gradients (they scale like $\sim l^{-1}$, being l the characteristic length), rising the value of $(\nabla w)^2$ and leading to the process we are going to describe.

If we go back to the equation above in this section, multiplying by w and using the relation $\vec{\nabla} \cdot (w\vec{\nabla} w) = \vec{\nabla} w \cdot \vec{\nabla} w + w\Delta w$ we get:

$$\frac{1}{2}\frac{Dw^2}{Dt} = -\nu[(\vec{\nabla} w)^2 - \vec{\nabla} \cdot (w\vec{\nabla} w)]$$

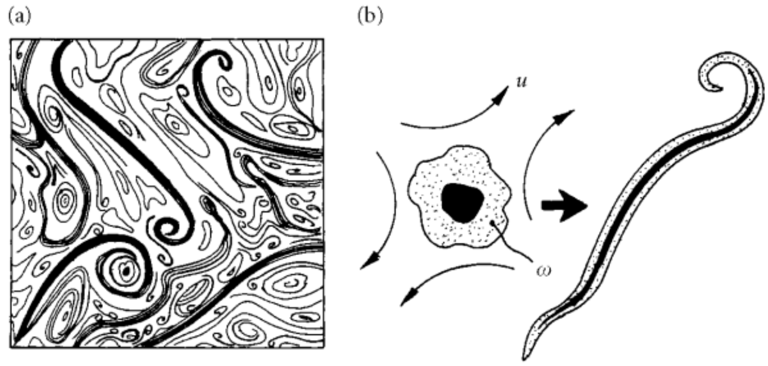

Figure 4: (a) Schematic of isovorticity lines in two-dimensional turbulence. (b) Filamentation of a vorticity patch.

Taking the mean of the equation, and using the fact that for a statistical homogeneous turbulence all divergences integrate to 0 we obtain the following equation of enstrophy:

$$\frac{d}{dt}(\frac{1}{2}\langle w^2\rangle) = \nu\langle(\nabla w)^2\rangle$$

Hence, increasing values of $(\nabla w)^2$ lead to a decrease of the enstrophy at the length scale considered. But, since enstrophy is conserved, it must be transferred to smaller scales, leading to the cascade described above.

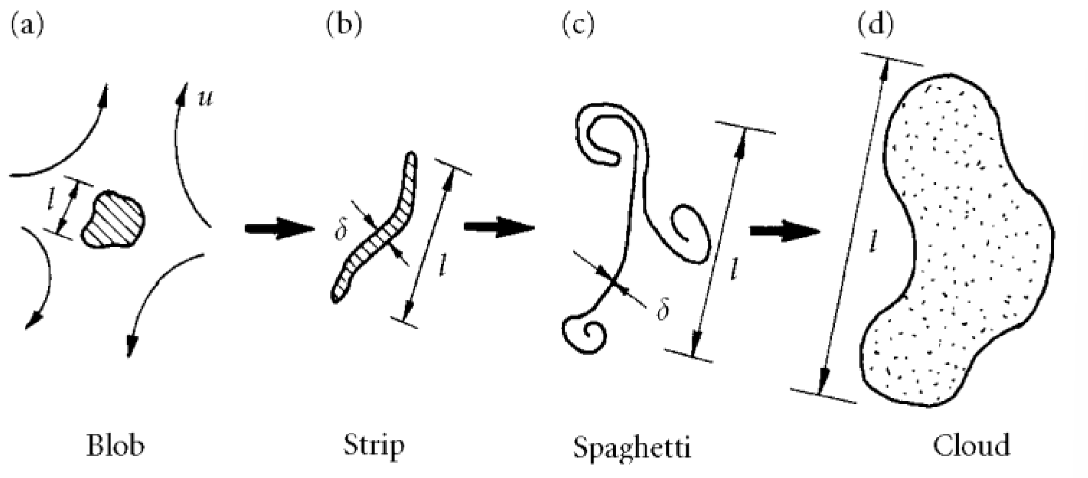

Figure 5: evolution of a vorticity patch described below

An analogue way of seeing this process is considering a vortex patch, a portion of fluid with same value of vorticity. As stated above, this is teased out by the flow, but since vorticity is materially conserved, the flux through the patch, and thus its area must be conserved, increasing the length and decreasing the thickness. The enstrophy cascade is associated with the decreasing thickness, reaching smaller scales and finally the the viscous regime, where diffusion become relevant. At the same time, since l could be considered as a length scale for the eddies, it is evident that their size is increasing; this is an evidence of how energy is transferred to larger scales, according to the description of the inverse energy cascade.

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Acknowledgements

Ringrazio il professor Del Pozzo per aver collaborato e per aver suscitato in me l'interesse per la fluidodinamica, punto di partenza fondamentale per questo lavoro. Ringrazio i miei genitori per avermi supportato in tutte le mie scelte, per avermi consigliato, compreso e sopportato nei momenti di difficoltà, oltre che per essere il principale motivo per cui sono arrivato a questo importante traguardo della mia vita, sia per motivi pratici che ideologici. Ringrazio Camilla per non disprezzare completamente quello che faccio, nonostante sia la letterata di casa. Ringrazio i nonni per avermi sostenuto incondizionatamente, senza avere bisogno di sapere quello che stessi facendo. Ringrazio poi i miei compagni di corso, e in particolare il mio gruppo di studio, perchè da solo in questi tre anni non ce l'avrei di certo fatta ad andare avanti. Ringrazio gli amici, vecchi e nuovi, per avermi aiutato a capire chi sono e perchè non si raggiungono gli obiettivi senza divertirsi nel tempo libero.
Infine ringrazio chiunque si sia interessato, si sia fatto vedere nelle difficoltà e mi abbia aiutato a superare, con ironia o con pazienza, i momenti di stress, tensione e tristezza che si sono presentati in questi anni.