Institute of Theoretical Physics
São Paulo State University

Matrix Models

IV Journeys Into Theoretical Physics
Prof. Pedro Vieira
July 6-12, 2019
Níckolas de Aguiar Alves

Matrix Models

IV Journeys Into Theoretical Physics

Professor: Pedro Vieira, Perimeter Institute

Notes by: Níckolas de Aguiar Alves

Level: Undergraduate

Period: July 6-12, 2019

São Paulo

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Matrix Models

Pedro Vieira - Perimeter / ICTP-SAIFR / IFT-UNESP

Nicholas Alves
IFUSP
alves.nicholas@usp.br
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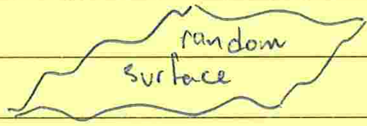
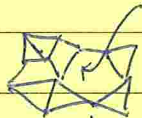
Outline

1. Simplest graphs / Gaussian Integrals
2. Gaussian integration & (Feynman) graphs
3. Graphs and maps (Topology)
4. Matrix Models and 2D Quantum Gravity

Overview



$\longleftrightarrow$ computer games



edges

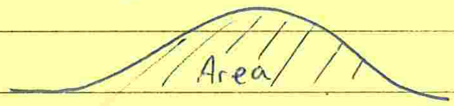
cusps

$\longrightarrow$ graphs and maps (graphs 2.0)

membrane OR 2D quantum gravity OR string theory worldsheet

$\longrightarrow$ gaussian integration

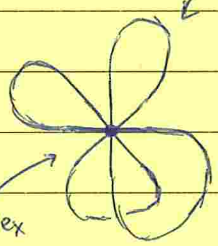
$\int$ over Matrices M_{ij}



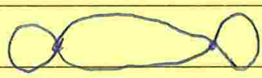
$= \sqrt{2\pi}$

$x \in \mathbb{R}$

① Outline



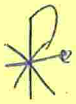
1 vertex of valency $z = 8$



$V = 2$
$E = 4$
$(z = 4)$

in how many ways can we construct this graph? For $z = n$

7 options



5 options $\rightsquigarrow$ $\#\text{ways} = (n-1)!! = (n-1)(n-3)\cdots 3 \cdot 1$

$7!! = 105$

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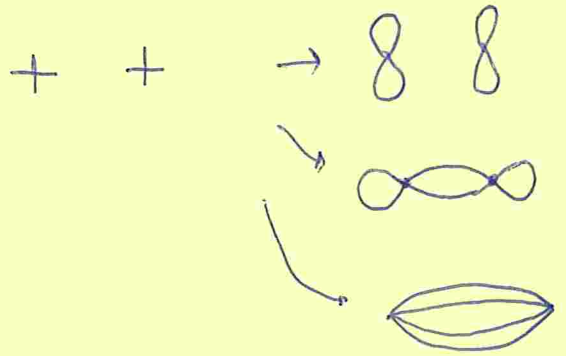


$(3!!)^2 = 9$ $\nearrow = \times$

$\underbrace{\left(\frac{4!}{2!\,2!}\right)^2}_{\text{who selfcontracts on each side}} \cdot 2 = 72$

$4! = 24$

$$\begin{array}{r} 9 \\ +\ 72 \\ 24 \\ \hline 105 \end{array}$$

way of check if you found all graphs

↳ two ways of counting the same thing

↓

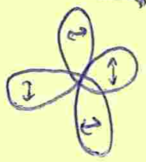
# of pairs of 8 "particles"

what if the particles are indistinguishable?

$$\frac{1}{8!}\cdot 105 = \frac{1}{4!\,2^4} = \frac{1}{\Gamma(\text{flower})}$$

reorder petals

reverse a petal



$$\mathbb{Z} = \sum_{\substack{\text{graphs of}\\ \text{valency 4 and}\\ n \text{ vertices}}} g^n \frac{1}{\underbrace{\Gamma(G)}}$$

automorphism of the graph

$88 = 9$

indistinguishable particles in each vertex

$$\frac{9}{(4!)^2\,2!}$$

change vertexes

$= 1 + \underbrace{8}_{1/8}\, g + g^2\Big(\underbrace{88}_{\frac{1}{8^2}\cdot\frac{1}{2}} +$  $+$  $\Big) + \dots$

$$\frac{1}{2(2^3)^2} = \frac{1}{\Gamma(88)}$$

exchange vertexes

reverse each petal or exchange petals

$\mathbb{Z} = e^{-\mathbb{F}}$ ↳ free energy

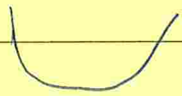
$-\mathbb{F} = \sum_{\text{connected}} \frac{g^n}{\Gamma(G)} = 8\,\frac{g}{8} + g^2\Big($  $+$  $\Big) + \mathcal{O}(g^3)$

↓ if you exponentiate

$$e^{-\mathbb{F}} = 1 + (-\mathbb{F}) + \frac{(-\mathbb{F})^2}{2} + \dots$$

$= 1 + g\,8\left(\frac{1}{8}\right) + g^2\Big[$  $+$  $+ \overbrace{88\left(\frac{1}{8^2}\cdot\frac{1}{2}\right)}^{\frac{(-\mathbb{F})^2}{2}}\Big] + \mathcal{O}(g^3)$

↳ the graphs represent $\frac{1}{\Gamma(G)}$

$+\beta H(x) =$  $= \underbrace{\frac{x^2}{2}}_{\text{harmonic}} + \underbrace{g\frac{x^4}{4!}}_{\text{anharmonic}}$ $g<0$

1 d.o.f.

$$I(\alpha) \equiv \int_{-\infty}^{+\infty} e^{-\frac{\alpha x^2}{2}} dx = \sqrt{\frac{2\pi}{\alpha}}$$

$$Z \equiv \frac{1}{\sqrt{2\pi}} \int \exp\left(-\frac{x^2}{2} + g\frac{x^4}{4!}\right) dx$$

Claim

$\mathcal{Z} = \mathbb{Z}$ → zeroth order ($g=0$) check.

$$\mathbb{Z}\Big|_{g=0} = 1$$

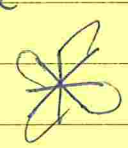
$$Z\Big|_{g=0} = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2}} dx = 1 \quad \checkmark$$

$$\langle x^k \rangle = \int_{-\infty}^{+\infty} x^k e^{-\frac{x^2}{2}} dx = \begin{cases} 0, & \text{for odd } k \\ (-2)^n \frac{\partial^n}{\partial \alpha^n} I(\alpha)\Big|_{\alpha=1}, & \text{if } k = 2n \end{cases}$$

$$\langle x^2 \rangle = -2 \frac{d}{d\alpha} \sqrt{2\pi}\, \alpha^{-1/2} = \sqrt{2\pi}\, \alpha^{-3/2}\Big|_{\alpha=1}$$

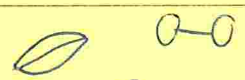
$$\langle x^4 \rangle = \sqrt{2\pi}\, 3 \cdot 1 \cdot \alpha^{-5/2}\Big|_{\alpha=1}$$

→ $\langle x^4 \rangle = -2 \frac{\partial}{\partial \alpha} \langle x^2 \rangle$

$$\int \frac{dx}{\sqrt{2\pi}} x^{2n} e^{-\frac{x^2}{2}} = (2n-1)!! = \# \text{ ways of}$$ 

$$\mathbb{Z} = Z = \sum_{n=0}^{+\infty} \frac{g^n (4n-1)!!}{n!(4!)^n}$$ → some Bessel function

$n!(4!)^n \neq (4n)!$ → $\neq \sum_{\substack{\text{graphs of} \\ \text{valency } 4n \\ \text{and 1 vertex}}} \frac{g^n}{\Gamma(G)}$



Feynman Graphs

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Rederivation Using Sources

$$Z_\alpha[j] \equiv \frac{1}{\sqrt{2\pi/\alpha}} \int e^{-\frac{\alpha x^2}{2} + jx}\, dx$$

$$Z_\alpha[j] = \underbrace{\sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}(x-\frac{j}{\alpha})^2}}_{1} + \frac{j^2}{2\alpha}\, dx = e^{j^2/2\alpha}$$

$Z_\alpha[0] = 1$ → generating functions

$$\langle x^k \rangle = \frac{\partial^k}{\partial j^k} Z_\alpha[j]\Big|_{j=0} = \frac{\partial^k}{\partial j^k} e^{\frac{j^2}{2\alpha}}\Big|_{j=0} = \begin{cases} 0, & \text{if } k \text{ odd} \\ \text{different}, & \text{for even } k \end{cases}$$

$$\langle x^{2n} \rangle = \left(\frac{\partial}{\partial j}\right)^{2n} \Big[\underbrace{\ldots}_{\text{too few } j} + \frac{j^{2n}}{(2\alpha)^n n!} + \underbrace{\ldots}_{\text{too many } j}\Big]_{j=0}$$

$$= \alpha^{-n} \underbrace{\frac{(2n)!}{2^n n!}}_{(2n-1)!!}$$

$(2n)!$ ← all = even × odd

$2^n n!$ ← even

$(2n-1)!!$ ← odd

Physical meaning of j:
- → counting parameter
- → chemical potential for vertices

$j \equiv e^{-\beta\mu}$, $j^n = e^{-\beta\mu n}$ → energy per vertex

② Multidimensional Integrals

$$Z_A[j] = \int \exp\Big(\underbrace{-\sum_{i,j=1}^{N} x_i A_{ij} x_j/2}_{-\frac{\vec{x}^T A \vec{x}}{2}} + \underbrace{\sum_i x_i j_i}_{\vec{x}\cdot\vec{j}}\Big) \underbrace{dx_1 \ldots dx_N}_{d\vec{x}}$$

A → sym. matrix, j → vector

Previously: $N = 1$, $A_{11} = \alpha$

Z is once more an important partition function, since

$$\langle x_{i_1} \cdots x_{i_n} \rangle_{\text{gaussian}} = \frac{1}{Z[0]} \frac{\partial^n}{\partial j_{i_1} \cdots \partial j_{i_n}} Z[j]\Big|_{\vec{j}=0}$$

$(2n-1)!!$ before

i) $Z_A[j]/Z_A[0] = \,?\, \leftrightarrow e^{\frac{j^2}{2\alpha}}$

ii) $Z_A[0] = \,?\, \leftrightarrow \sqrt{\frac{2\pi}{\alpha}}$ before

Guess:

$$Z[j]/Z[0] = \exp\left(\vec{j}^T A^{-1} \vec{j}/2\right)$$

$$Z[0] = \frac{2\pi^{N/2}}{\sqrt{\det A}}$$

We want to complete squares with matrices:

$$-\frac{1}{2}\vec{x}^T A\vec{x} + \vec{j}\cdot\vec{x} \qquad\qquad \vec{x}-\mathbb{b}=\vec{y}$$

$$\stackrel{\text{goal}}{=} -\frac{1}{2}(\vec{x}-\mathbb{b})^T\cdot A\cdot(\vec{x}-\mathbb{b}) + \mathbb{b}^T\cdot A\cdot\mathbb{b}/2$$

$$= -\frac{1}{2}\vec{x}^T\cdot A\cdot\vec{x} + \vec{x}\cdot A\cdot\mathbb{b} \longrightarrow A \text{ is symmetric}$$

we want $A\mathbb{b}=\vec{j} \Rightarrow \mathbb{b}=A^{-1}\vec{j}$ $\qquad \underbrace{\mathbb{b}^T\cdot A\cdot\mathbb{b}/2}$

$$\vec{j}^T(A^{-1})^T A(A^{-1})\vec{j}/2$$

$$\vec{j}^T A^{-1}\vec{j}/2$$

$$Z = \underbrace{\int_{-\infty}^{+\infty}\exp(-yAy/2)}_{Z[0]}\exp\left(\frac{jA^{-1}j}{2}\right)\underbrace{dx_1\cdots dx_N}_{dy_1\cdots dy_N} \longrightarrow N=1 \quad e^{\frac{j^2}{2\alpha}}$$

too few j's $\qquad$ too many j's

$$\langle x_{i_1}\cdots x_{i_{2n}}\rangle = \frac{\partial}{\partial j_{i_1}}\cdots\frac{\partial}{\partial j_{i_{2n}}}\left[\cdots+\underbrace{\frac{1}{2^n n!}(j\cdot A^{-1}\cdot j)^n}_{j_{k_1}j_{k_2}\cdots j_{k_{2n}}(A^{-1})_{k_1k_2}\cdots(A^{-1})_{k_{2n-1}k_{2n}}}+\cdots\right]$$

a bunch of A^{-1}'s

$$\langle x_i x_k\rangle = (A^{-1})_{ik}$$

or a bunch of 2 point functions $\langle x_i x_k\rangle$ $\qquad$ propagator from i to k

Wick Theorem

$k\equiv 2n$ $\qquad \langle x_{i_1}\cdots x_{i_k}\rangle = \sum_{\text{pairs}}\prod_{a=1}^{k/2}\underbrace{A^{-1}_{j_a k_a}}_{\langle x_{j_a}x_{k_a}\rangle}$

$$\{i_1\cdots i_k\}\to(j_1k_1)\cdots(j_nk_n)$$

to avoid overcounting: $j_a<k_a,\ j_{a+1}>j_a$

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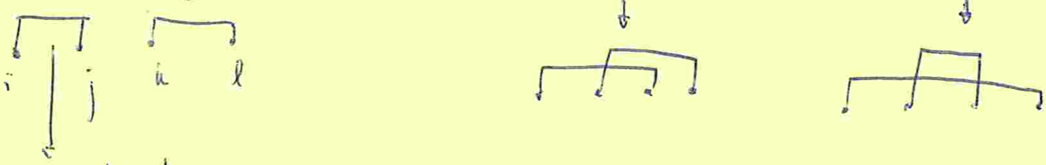
Example:

$$\langle x_i x_j x_k x_\ell \rangle = \langle x_i x_j \rangle \langle x_k x_\ell \rangle + \langle x_i x_k \rangle \langle x_j x_\ell \rangle + \langle x_i x_\ell \rangle \langle x_j x_k \rangle$$



$3!!$ terms

Wick contraction

Back to $N=1$:

$$\langle x^{2n} = \underbrace{x x x \cdots x}_{2n} \rangle = (2n-1)!! \cdot 1$$

$\langle x x \rangle = 1$

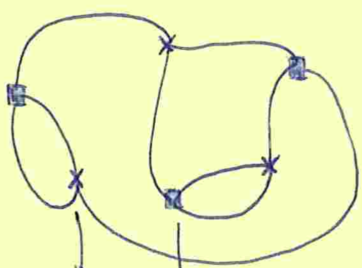
$$Z = \frac{1}{\sqrt{2\pi}} \int \exp\left(-\frac{x^2}{2} + \sum_{k \geq 3} g_k \frac{x^k}{k!} \right) dx$$

$$= \sum_{\text{graphs}} \frac{1}{\Gamma(G)} \prod_{k=3}^{\infty} g_k^{n_k} = 1 + 0\, g_4 + 0\, g_3 g_4 \cdots$$

graphs with n_k vertices of valency k

Different colored ~~gray~~ vertices that connect only to different colors

$$Z_{■x} = \sum$$



■ x

only contraction

g_x $\quad g_■ \quad g_x^3 g_■^3$

$$Z_{■x} = Z_{■x} = \frac{1}{2\pi} \int \exp\left(-xy + g_x \frac{x^3}{3!} + g_■ \frac{y^3}{3!} \right) dx\, dy$$

$N=2$ multi-dim gaussian

$(x_1, x_2) = (x, y)$

$$-\frac{1}{2} (x \ y) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

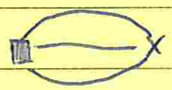
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A^{-1}$$

■—x $\langle x y\rangle = (A^{-1})_{12} = 1$ I know the rules I want, and
■⋯■ $\langle y y\rangle = (A^{-1})_{22} = 0$ thus I can find A and then
x⋯x $\langle x x\rangle = (A^{-1})_{11} = 0$ the integral representation

$$Z_{x■} = 1 + g_x g_■ \frac{1}{3!\,3!}\langle xxxyyy\rangle$$

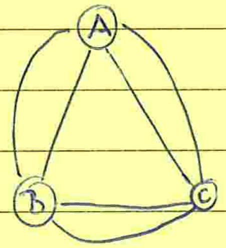
$$\underbrace{3!\langle xy\rangle\langle xy\rangle\langle xy\rangle = 3!}$$

They are distinguishable now

$\frac{1}{3!}$ 

As an another example, let's use three colors

quartic

$$\sum$$  $$= Z = \int \frac{dx\,dy\,dz}{N^3}\exp\left(-\frac{1}{2}(x\ y\ z)\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}^{-1}\begin{pmatrix}x\\y\\z\end{pmatrix} + g_A\frac{x^4}{4!} + g_B\frac{y^4}{4!} + g_C\frac{z^4}{4!}\right)$$

Parentheses

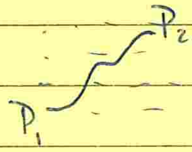
$$x \to (x,y) \to (x,y,z) \to \dots \to (\psi_{p_1} \cdots \psi_{p_\infty})$$

x ■ A B C

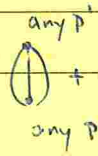
↳ point in space time

$$x^3 + y^3 \to \sum_p \psi_p^3$$ ← g Y particles splitting

ψ_p: a field that depends on space-time

$$\langle \psi_{p_1}\psi_{p_2}\rangle \equiv$$ 

propagation / prob of propagation of particle from $P_1 \to P_2$

$$Z = $$  $$+ \dots$$

↖ Quantum Field Theory

$\psi_p, \phi_p, \theta_p \dots$ for different particles

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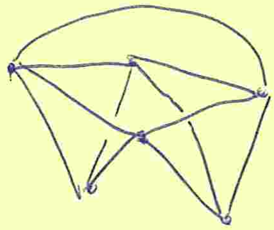
graph

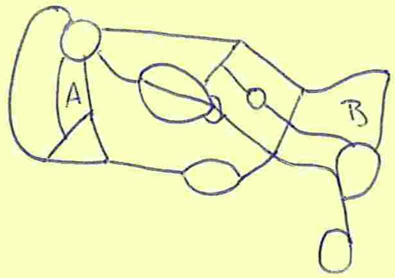


In a graph: no natural notion of distance

$\downarrow$

map

map: graph on a surface without lifting pen



planar graph or planar map

can be drawn on a plane or sphere

A is far from B & you can't connect them

Finally,
$$\mathbb{Z}[0] = \int \exp(-\vec{x}^T A \vec{x}/2)\, d\vec{x}$$

$$\uparrow$$

$$\vec{x}^T U^{-1} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_N \end{pmatrix} U \vec{x}$$

$U^{-1} = U^T$

$\hookrightarrow$ rotation

$$(U\vec{x})^T \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_N \end{pmatrix} (U\vec{x})$$

$$\mathbb{Z}[0] = \int \exp\left(-\frac{1}{2}\sum_{i=1}^{N} a_i y_i^2\right) d\vec{y}$$

$$= \prod_{i=1}^{N} \sqrt{\frac{2\pi}{a_j}} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}$$

Remark

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2} + g\frac{x^3}{3!}\right) dx$$

$\hookrightarrow$ diverges for $g \neq 0$

How to make sense of it?

$$I_\alpha = \int_{-\infty}^{+\infty} e^{-\frac{\alpha x^2}{2}} dx = \sqrt{\frac{2\pi}{\alpha}}$$

$$\alpha = \alpha_0 e^{i\phi} \qquad \alpha_0 > 0$$

$$x \to x e^{-i\phi/2} \qquad \phi : 0 \to \pi$$

$$I_\alpha = \int_{-\infty}^{+\infty} e^{-\frac{\alpha_0 x^2}{2}} dx \, e^{-i\phi/2} = \sqrt{\frac{2\pi}{\alpha_0}} e^{-i\phi/2} = \sqrt{\frac{2\pi}{\alpha}}$$

Changing the contour of integration can give sense to the integral

Two interacting atoms: $\blacksquare$ & x → interaction between atoms

$$\blacksquare\!-\!\blacksquare \equiv 0 \mapsto e^{-\beta \varepsilon_{\blacksquare\blacksquare}} \qquad g_{\blacksquare} = e^{-\beta \varepsilon_{\blacksquare}}$$

$$x\!-\!x \equiv 0 \mapsto e^{-\beta \varepsilon_{xx}} \qquad g_x = e^{-\beta \varepsilon_x}$$

$$x\!-\!\blacksquare \equiv 1 \mapsto e^{-\beta \varepsilon_{x\blacksquare}}$$

A: kinetic matrix, action
A^{-1}: propagator, correlation function
det A: related to partition function

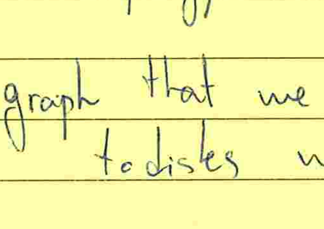
On a 2D surface, perhaps we would like $\sum_{\substack{\cancel{\text{graphs}} \\ \text{maps}}} e^{-\beta H}$

③ Maps and Topology

map: graph that we draw on a surface whose faces are isomorphic to disks without lifting the pen

plane ≅ sphere ≅ cube

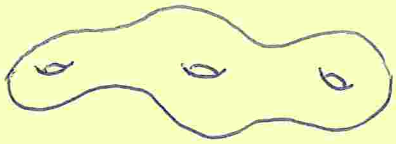
genus 0 surface
$g = 0$



$\cong$ mug $\not\cong$ sphere

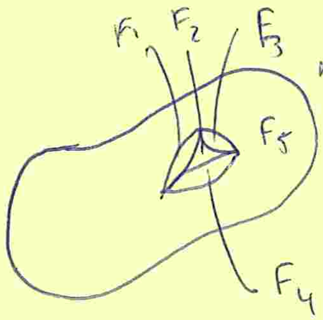
donut

$g=1$



$g=3$ surface

etc



map ✓

F_2 & F_5 are in/outside of Δ and are $\cong$

map ✓

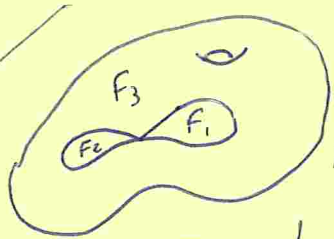
$F_j \cong$ 

$\cong$

not a map

map ✓

1 face



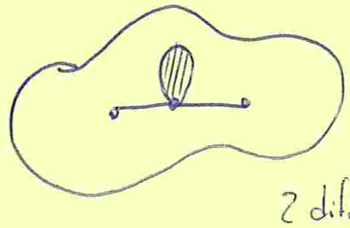
not a map

F_3 not $\cong$ to disk

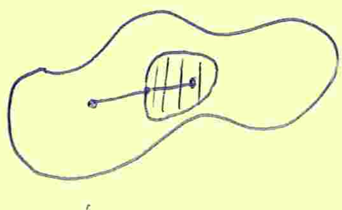
$\langle x^4 \rangle = 3!! = 3 =$ 

$= 2 + 1$

$g=0$ $\quad$ $g=1$



vs

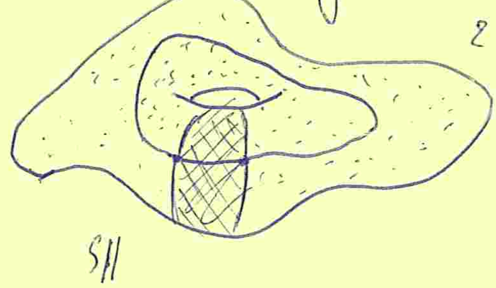


2 different maps

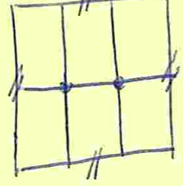
b/c $\neq$ # of vertices inside face

$g=1$ map

2 faces



$\cong$



Take a genus g map with V vertices, F faces and E edges and define

$\chi \equiv V + F - E$ ⤳ Euler Characteristic

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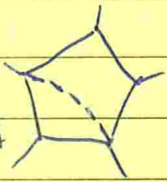
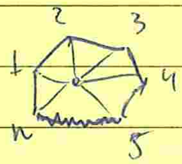
Claim: χ is a topological invariant, $\chi = \chi(g) = 2-2g$ $\rightsquigarrow$ experiment for previous examples

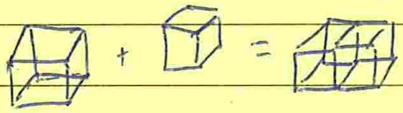


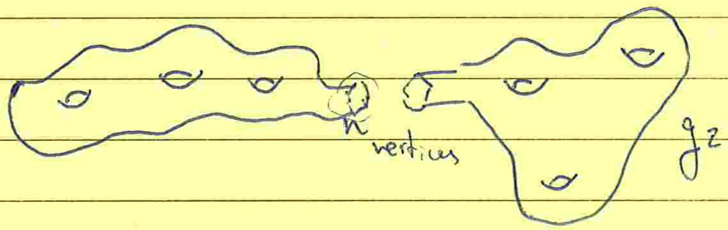
Local face at a larger map

$$\begin{array}{r} \delta V = 0 \\ \delta F = 1 \\ -\delta E = -1 \\ \hline 0 \end{array}$$



$$\begin{array}{r} \delta V = 1 \\ \delta F = n-1 \\ -\delta E = -n \\ \hline 0 \end{array}$$





$\chi_1 = 2-2g_1$, $\chi_2 = 2-2g_2$

$$\begin{aligned} \chi_{tot} = & \; V_1 + V_2 - n \\ & + F_1 + F_2 - 2 \\ & - E_1 - E_2 + n \quad = \chi_1 + \chi_2 - 2 \end{aligned}$$

$$\chi(g_1+g_2) = \chi(g_1) + \chi(g_2) - 2 \implies \chi(g) = Ag + 2$$

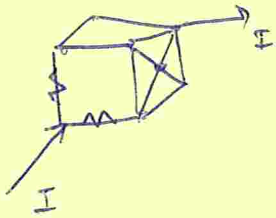
$\hookrightarrow$ plug some g and A comes out by calculating an example

$F = 2-2g-V+E$

If we could count F, we could promote graph tech $\mapsto$ maps

$\hookrightarrow$ e.g. $105 \mapsto \underset{F}{(g=0)} + \underset{F'}{(g=1)}$

Proof: first at $j=0$



want I_j at edge j

$j = 1 \dots E$

-1: last is automatic

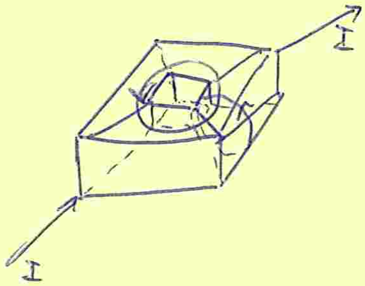
$E = (V-1) + (F-1)$

E ↙ # current

$(V-1)$ ↓ charge conservation eqs

$(F-1)$ ↳ voltage drop around a face is zero eqs

$j=1$



$E = (V-1) + (F-1+2)$ → two new possible loops

Example 1.
Soccer ball

$F = P + H$

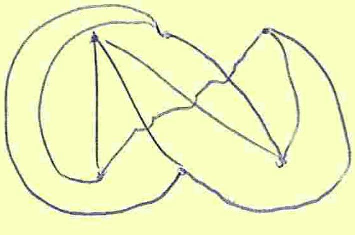
$E = \dfrac{5P + 6H}{2}$

$V = \dfrac{2E}{3}$ → each edge ends at two vertices and each vertex connects 3 edges ↳ check angles for example

$$V + F - E = \frac{5P + 6H}{3} + P + H - \frac{5P + 6H}{2} = \frac{P}{6} \Rightarrow P = 12$$

Example 2.
3 Houses + 3 Utilities

or



If it could with $g=0$

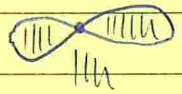
$$2 = V + F - E$$
$$= 6 - 9 + F$$

but possible on genus 1

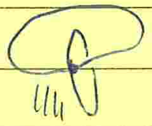
$F = 5 \Rightarrow 10$ edges > 9 impossible

no triangles

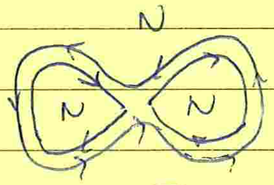
How can I count faces?



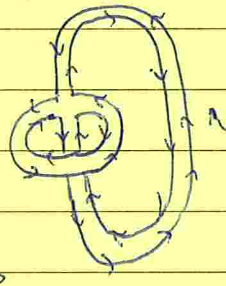
$F=3$



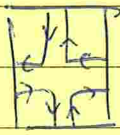
$F=1$



$\sim g^1 N^3$

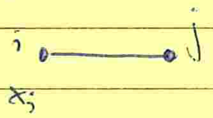


$\sim g^1 N^1$



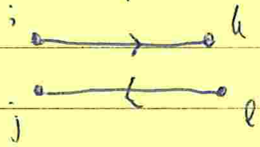
$3g \to 2gN^3 + gN$

Idea:



$x_j \to$ vector

$\Downarrow$ matrix



$i = 1, \dots, N$

x_i vector $\to$ M_{ij} matrix $\to$ Matrix Models

④ Matrix Models

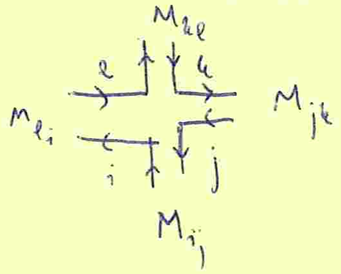
$$Z = \frac{1}{(\dots|_{g=0})} \int \exp\Big(-\frac{\mathrm{tr}\, M^2}{2} + g \overbrace{\mathrm{tr}\, M^4}^{M_{ij} M_{jk} M_{kl} M_{li}}\Big)\, DM$$

$\to$ hermitian $N \times N$ matrices

$$M^*_{jk} = M_{kj}$$
$$M_{kj} = R_{kj} + i I_{kj}$$
$$M_{jk} = R_{kj} - i I_{kj}$$

R and I are symmetric

Vieira



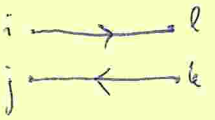
$$\mathrm{tr}\, M^2 = M_{ij} M_{jk}$$

$$= \sum_{i=1}^{N} M_{ii}^2 + 2 \sum_{\underset{k<l}{j<i}} R_{ji}^2 + I_{ji}^2$$

$$\langle \underbrace{(R_{ij} + i I_{ij})}_{M_{ij}} \underbrace{(R_{kl} \pm i I_{kl})}_{M_{kl}} \rangle = 0 \quad \text{for } k<l$$

$$\underset{k>l}{} = (1/2 + 1/2)\, \delta_{il} \delta_{jk} \qquad \text{otherwise as wanted}$$

$$\langle M_{ij} M_{kl} \rangle =$$



$$= \delta_{il} \delta_{jk}$$

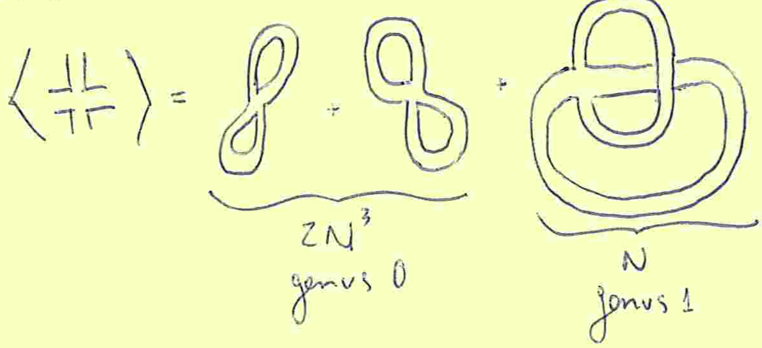
$$\langle \mathrm{tr}\, M^4 \rangle = \langle M_{ij} M_{jk} M_{kl} M_{li} \rangle = \overbrace{MM}\overbrace{MM} + M\overbrace{MM}M + \text{MMMM}$$

$$= (\delta_{ik} \underbrace{\delta_{jj}}_{N})(\delta_{ki} \underbrace{\delta_{ll}}_{N}) + \cdots + (\delta_{il} \delta_{jk})(\delta_{ji} \delta_{kl})$$

$$= N^2 \delta_{ii} + N^3 + \delta_{ii} = 2N^3 + N$$

our much wanted $3!! = 2 + 1$!

in pictures



$$Z = \cdots + g^V N^F \# + \cdots \qquad \text{but } V + F - E = V + F - \frac{4V}{2} = 2 - 2g$$

$$F = 2 - 2g + V$$

$$Z = \cdots + N^{2-2g} (\overbrace{gN}^{\lambda})^V + \cdots$$

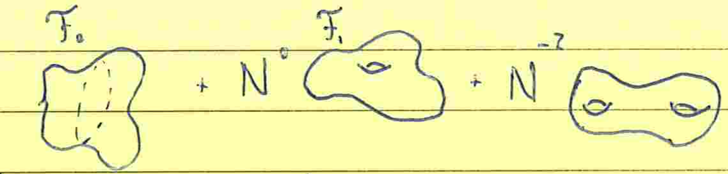
λ: 't Hooft coupling

↳ graph of genus g and V vertices

$$\log Z = \sum_g N^{2-2g} \left[\sum_{\substack{\text{connected} \\ \text{graphs of} \\ \text{genus } g \text{ and} \\ V \text{ vertices}}} \lambda^V = \mathcal{F}_g(\lambda) \right]$$

Vieira

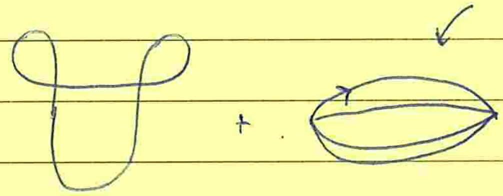
$\log Z = N^2 \, \mathcal{F}_0 + N^0 \, \mathcal{F}_1 + N^{-2} \, \cdots + \cdots$



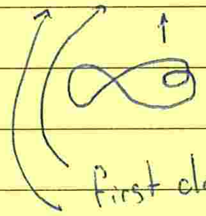
sum surfaces like a string theory

$V=2, \quad g=0$

who this one connects to



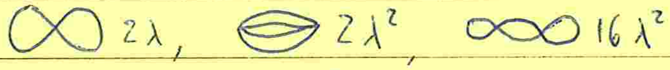
$4\cdot 4\cdot 2 \quad + \quad 4 \quad = \boxed{36} \longrightarrow 18$



first clockwise to pair in each vertex

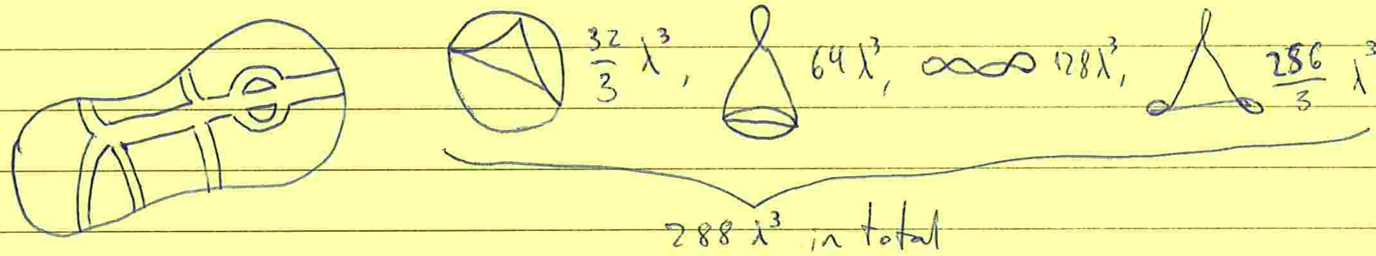
$\frac{1}{2!}$ from exponent

Amazingly one can often compute $\mathcal{F}_g$!



2λ, $2\lambda^2$, $16\lambda^2$

$\frac{32}{3}\lambda^3$, $64\lambda^3$, $128\lambda^3$, $\frac{286}{3}\lambda^3$



$288\lambda^3$ in total

$$= \mathcal{F}_0(\lambda) = \frac{(u-1)(9-u)}{24} - \frac{\log u}{2}, \quad \text{where } u = \frac{1-\sqrt{1-48\lambda}}{24\lambda}$$

$$= 1 + 12\lambda + 288\lambda^2 + \cdots$$

previous page

$$-\mathcal{F}_0(\lambda) = 2\lambda + 18\lambda^2 + 288\lambda^3 + \cdots$$

$$= \sum \lambda^v \underbrace{\frac{12^v (2v-1)!}{v!(v+2)!}}_{\#\text{ graphs with } v \text{ vertices}}$$

$$\mathcal{F}_1 = \log \frac{(2-u)}{12}$$

etc

How are such beautiful results derived?

i) $S[M] = S[\Lambda M \Lambda^{-1}]$ (kind of gauge sym)

$$\downarrow$$

$$S[M] = S[z_j] = \frac{1}{2}\sum z_j^2 - \frac{\lambda}{N}\sum z_j^4$$

$\hookrightarrow$ eigenvalues of M $\qquad \mathcal{O}(N^2)$ variables

ii) $$\mathcal{D}M = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}\, M_{ij} \prod_{i<j} d\,\mathrm{Im}\, M_{ij}$$

$$= \underbrace{(\text{Jacobian})} \prod_{i=1}^{N} dz_i \leftarrow \mathcal{O}(N) \text{ var}$$

$$= \Delta^2(z) = \prod_{i<j}(z_i - z_j)^2$$

$\uparrow$ Vandermond Determinant

iii) Action is huge $\Rightarrow$ extremum dominates

$\mathcal{Z} \sim e^{-S_{ce}}$ where

$$S_{ce} = -\sum \frac{z_i^2}{2} + \sum \frac{\lambda}{N} z_i^4 + \sum_{i \neq j} \log|z_i - z_j|$$

where z_i obey extremum condition $\frac{\partial S}{\partial z_i} = 0$ or

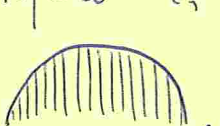
$$\frac{1}{2} z_i - 2\frac{\lambda}{N} z_i^3 = \sum \frac{1}{z_i - z_j}$$

2D Coulomb like problem

repulsion

external force

iv) replace $z_i \to \rho(z)$ since N is huge



$-2\sqrt{u}$ $\qquad 2\sqrt{u}$

previous eq $\to$ int eq for ρ with solution

$$\rho(z) = \left[1 - 8\lambda u - 4\lambda^2\right]\sqrt{4u - \lambda^2}$$

v) plug this density into S_{ce} to get

$S_{ce} \simeq -N^2 F_0(\lambda)$ with F_0 as given above

This is it about 1 Matrix Model (1MM). We can get more interesting decorated graphs with 2MM, 3MM, ... like in previous lectures with colors

$$Z = \int \exp\left(-\mathrm{tr}\,[M\,N]\begin{bmatrix}'' & ''\\ '' & ''\end{bmatrix}\begin{bmatrix}M\\N\end{bmatrix} + V(M)+V(N)\right) DM\,DN$$

still solvable

$$Z_{line} = \int e^{-S}\, DM_1 DM_2 \cdots DM_N$$

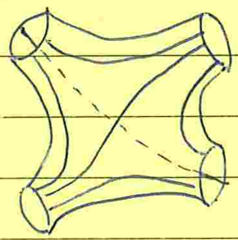
harder, since $\mathrm{tr}(MN) \neq f(z_i, w_j)$, but doable

$$S = \mathrm{tr}(M_1M_2)+\mathrm{tr}(M_2M_3)+\cdots+\mathrm{tr}(M_{n-1}M_n) + \sum V(M_j) \rightarrow \text{solvable}$$

$+\mathrm{tr}(M_nM_1) \rightarrow$ not yet

↳ Z_{ring}

String theory



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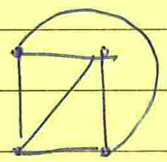
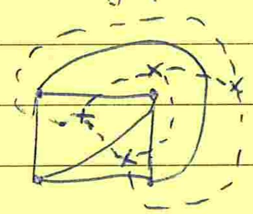


← triangulation

↕ duality

$\mathrm{tr}\, M^3$ model

Dual graphs

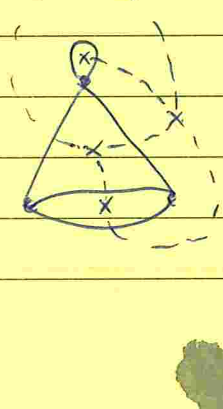


cubic vertices ⤳

triangle or hexagon coupling

∃ few such couplings ↳ Z_{ring}
exactly the (still) unsolved one
There is work to do! :)

original graph



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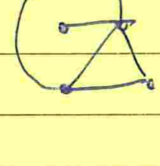

dual graph

Vieira