Campo EM de uma Carga Pontual

Começamos com uma carga em repouso.

Sabemos que

carga

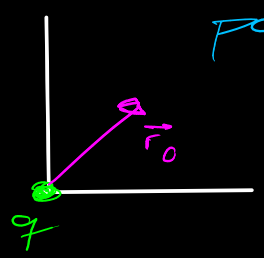
unidades

$\epsilon_0 = 1$

$c = 1$

$$\vec{E} = \frac{1}{4\pi} \frac{q}{r_0^2} \hat{r}_0$$

$$\vec{B} = \vec{0}$$



$\vec{r}_0$ é o ponto onde se mede o campo

Podemos organizar os campos no tensor de Faraday, $F_{\mu\nu}$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

No nosso caso, temos

$\vec{e}_{0_1} = x_0 \hat{x}_0$

$\vec{e}_{0_2} = y_0 \hat{y}_0$

$\vec{e}_{0_3} = z_0 \hat{z}_0$

$$F_{0i} = -F_{i0} = -\frac{1}{4\pi} \frac{q}{\sqrt{x_0^2 + y_0^2 + z_0^2}^{\,3}} \vec{e}_{0_i}$$

outras componentes zeram.

$F_{\mu\nu}$ é um tensor: se transforma sob boosts de acordo com

$$\underbrace{F_{\mu\nu} \to \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}}$$

tensor de Faraday num novo referencial

Vamos tomar Λ como um boost na direção $\hat{x}$ com velocidade v, i.e.,

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\Lambda_\mu{}^\rho = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

lembre que estamos usando $c=1$, logo $\beta = \frac{v}{c} = v$

$$\Lambda_0{}^0 = \Lambda_1{}^1 = \gamma, \quad \Lambda_2{}^2 = \Lambda_3{}^3 = 1$$

$$\Lambda_0{}^1 = \Lambda_1{}^0 = -\gamma v$$

demais zeram

Vamos então calcular $F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$

Como $F_{\mu\nu} = -F_{\nu\mu}$, $F'_{\mu\nu} = -F'_{\nu\mu}$. De fato,

$$F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$$
$$= -\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\sigma\rho}$$
$$= -\Lambda_\nu{}^\sigma \Lambda_\mu{}^\rho F_{\sigma\rho}$$
$$= -F'_{\nu\mu}$$

Logo, a diagonal de $F'_{\mu\nu}$ se anula e vale que

$F'_{0i} = -F'_{i0}$, $F'_{ij} = -F'_{ji}$ $\longrightarrow$ podemos recuperar $\vec{E}'$ e $\vec{B}'$

Calculemos F'_{0i}. Para $i=1$ (componente x), temos

$$F'_{01} = \Lambda_0{}^\rho \Lambda_1{}^\sigma F_{\rho\sigma}$$
$$= \Lambda_0{}^0 \Lambda_1{}^\sigma F_{0\sigma} + \Lambda_0{}^1 \Lambda_1{}^\sigma F_{1\sigma}$$

$F_{00} = 0$

$\Lambda_1{}^2 = \Lambda_1{}^3 = 0$

$$= \gamma \Lambda_1{}^\sigma F_{0\sigma} - \gamma v \Lambda_1{}^\sigma F_{1\sigma}$$
$$= \gamma \Lambda_1{}^1 \underbrace{F_{01}}_{-E_i} - \gamma v \Lambda_1{}^0 F_{10}$$

$F_{ij} = 0$

corresponde a $\vec{B} = \vec{0}$

$$F'_{01} = (\gamma^2 - \gamma^2 v^2) F_{01}$$

$$= F_{01} \Rightarrow F' = \begin{pmatrix} 0 & -E_x & ? & ? \\ E_x & 0 & ? & ? \\ ? & ? & 0 & ? \\ ? & ? & ? & 0 \end{pmatrix}$$

A seguir, façamos F'_{02}.

$$F'_{02} = \Lambda_0{}^{\rho} \Lambda_2{}^{\sigma} F_{\rho\sigma}$$

$$= \Lambda_0{}^{0} \Lambda_2{}^{\sigma} F_{0\sigma} + \Lambda_0{}^{1} \Lambda_2{}^{\sigma} F_{1\sigma}$$

$$= \gamma \Lambda_2{}^{\sigma} F_{0\sigma} - \gamma v \Lambda_2{}^{\sigma} F_{1\sigma}$$

$$= \gamma \Lambda_2{}^{2} F_{02} - \gamma v \Lambda_2{}^{2} \cancelto{0}{F_{12}}$$

$$= \gamma F_{02}$$

Analogamente, tem-se $F'_{03} = \gamma F_{03}$

$$\hookrightarrow F' = \begin{pmatrix} 0 & -E_x & -\gamma E_y & -\gamma E_z \\ E_x & 0 & ? & ? \\ \gamma E_y & ? & 0 & ? \\ \gamma E_z & ? & ? & 0 \end{pmatrix}$$

Resta calcularmos as componentes do campo magnético.

De forma genérica temos

$$F'_{ij} = \Lambda_i{}^{\rho}\Lambda_j{}^{\sigma} F_{\rho\sigma}$$

$$= \Lambda_i{}^{0}\Lambda_j{}^{\sigma} F_{0\sigma} + \Lambda_i{}^{k}\Lambda_j{}^{\sigma} F_{k\sigma}$$

$F_{kl} = 0$ pois $\vec{B} = 0$

$F_{00} = 0$

$$= \Lambda_i{}^{0}\Lambda_j{}^{\ell} F_{0\ell} + \Lambda_i{}^{k}\Lambda_j{}^{0} F_{k0}$$

$$= \Lambda_i{}^{0}\Lambda_j{}^{k} F_{0k} - \Lambda_j{}^{0}\Lambda_i{}^{k} F_{0k}$$

$$= (\Lambda_i{}^{0}\Lambda_j{}^{k} - \Lambda_j{}^{0}\Lambda_i{}^{k}) F_{0k}$$

B_x (F_{23}) $\Lambda_2{}^{0} = 0$ $\Lambda_3{}^{0} = 0$

$$F'_{23} = (\cancel{\Lambda_2{}^{0}}\Lambda_3{}^{k} - \cancel{\Lambda_3{}^{0}}\Lambda_2{}^{k}) F_{0k}$$

$$= 0$$

B_y (F_{31}) $\Lambda_3{}^{0} = 0$

$$F'_{31} = (\cancel{\Lambda_3{}^{0}}\Lambda_1{}^{k} - \Lambda_1{}^{0}\Lambda_3{}^{k}) F_{0k}$$

$$= \gamma v \Lambda_3{}^{k} F_{0k}$$

$$= \gamma v \Lambda_3{}^{3} F_{03} = \gamma v F_{03} = -\gamma v E_z$$

$B_z \quad (F_{12})$

$$\Lambda_2^{\ 0} = 0$$

$$F'_{12} = (\Lambda_1^{\ 0}\Lambda_2^{\ \mu} - \Lambda_2^{\ 0}\Lambda_1^{\ \mu})\,F_{0\mu}$$

$$= -\gamma v F_{02}$$

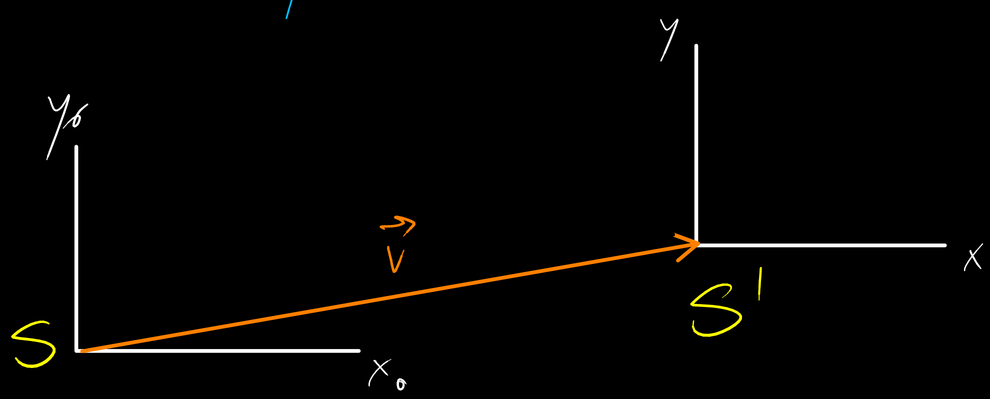
$$= \gamma v E_y$$

$$F' = \begin{pmatrix} 0 & -E_x & \gamma E_y & \gamma E_z \\ E_x & 0 & \gamma v E_y & \gamma v E_z \\ \gamma E_y & -\gamma v E_y & 0 & 0 \\ \gamma E_z & -\gamma v E_z & 0 & 0 \end{pmatrix}$$

Assim, temos, por exemplo,

$$E'_y = \frac{1}{4\pi}\,\frac{\gamma q y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}^{\,3}}\,\hat{y}_0$$

↳ expresso nas coordenadas originais



Transformação das coordenadas:

$$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$$

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \gamma t_0 - \gamma v x_0 \\ \gamma x_0 - \gamma v t_0 \\ y_0 \\ z_0 \end{pmatrix}$$

invertendo:

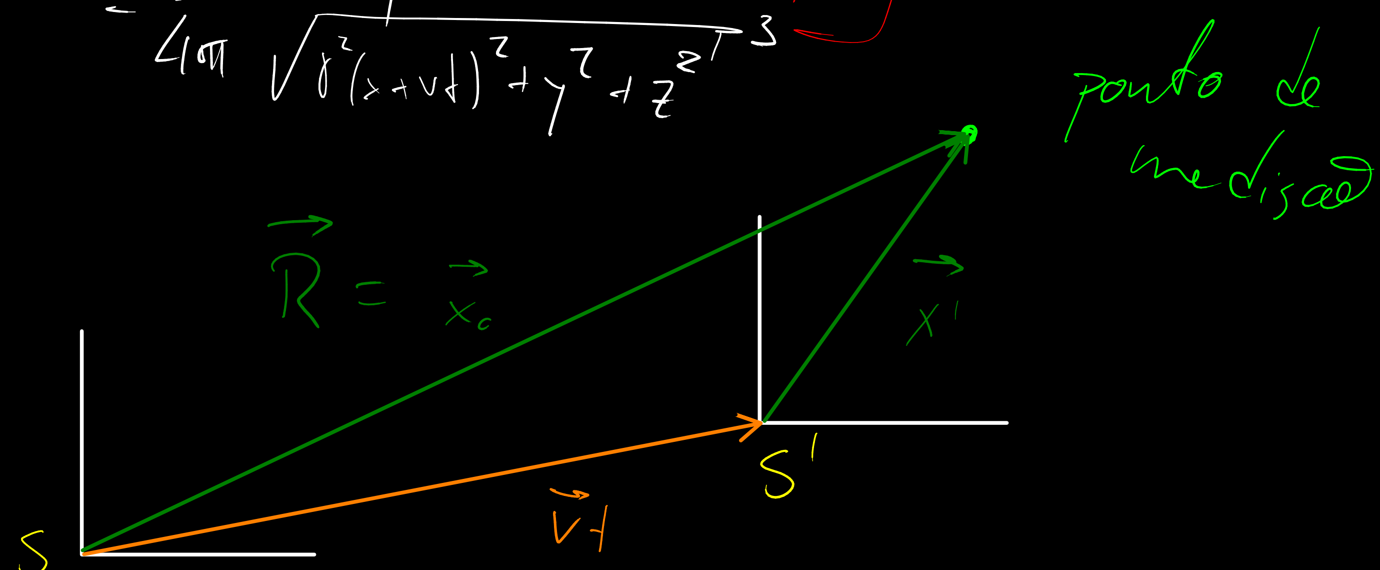
$$t_0 = \gamma t + \gamma v x = \gamma(t + vx)$$
$$x_0 = \gamma x + \gamma v t = \gamma(x + vt)$$
$$y_0 = y$$
$$z_0 = z$$

$$E'_x = \frac{1}{4\pi} \frac{q\, x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}^{\,3}} \, \hat{x}_0$$

inalterado, pois $\vec{V} \parallel \vec{x}_0$

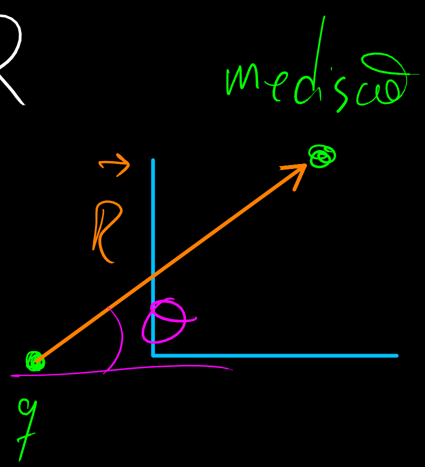
$\hat{x} = \hat{x}_0$

$$= \frac{1}{4\pi} \frac{\gamma q (x + vt)}{\sqrt{\gamma^2 (x+vt)^2 + y^2 + z^2}^{\,3}} \, \hat{x}$$



Definindo $\vec{R} = \vec{x}' + \vec{v}t = \begin{pmatrix} x'+vt \\ y \\ z \end{pmatrix}$, que aponta da partícula ao ponto de medição, teremos então

$$\vec{E} = \frac{1}{4\pi} \frac{\gamma q}{\sqrt{\gamma^2 R^2 \cos^2\theta + R^2 \sin^2\theta}^{\,3}} \vec{R}$$



$$= \frac{1}{4\pi} \frac{q(1-v^2)}{\sqrt{1 - v^2 \sin^2\theta}} \frac{\hat{R}}{R^2}$$

Resta expressar o campo magnético. Vimos que

$E'_x = E_x$ $\qquad$ $B'_x = 0$

$E'_y = \gamma E_y$ $\qquad$ $B'_y = -\gamma v E_z$

$E'_z = \gamma E_z$ $\qquad$ $B'_z = \gamma v E_y$

$\hookrightarrow \vec{B} = +\vec{v} \times \vec{E}$

De fato,

$$\vec{v} \times \vec{E} = \begin{pmatrix} \hat{x}_0 & \hat{y}_0 & \hat{z}_0 \\ v & 0 & 0 \\ E_x & \gamma E_y & \gamma E_z \end{pmatrix} = \begin{pmatrix} 0 \\ -\gamma v E_z \\ \gamma v E_y \end{pmatrix}$$

Assim,

$$\vec{B} = \frac{1}{4\pi} \frac{q(1-v^2)}{\sqrt{1-v^2 \sin^2\theta}} \frac{\vec{v} \times \hat{R}}{R^2}$$

Ver também:

D.J. Griffiths (2017) An Introduction to Electrodynamics (CUP, Cambridge).