Institute of Theoretical Physics
São Paulo State University

High Energy Physics

Summer School
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Summer 2020
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Elements of Field Theory

Brief Historical Account of QFT

Field concept: Faraday, 1845 → On the Magnetization of Light, and the Illumination of Magnetic Lines of Force

Birth of Quantum Mechanics → QM

1900 Planck Quantization of energy to explain blackbody radiation

1905 Einstein Electromagnetic radiation is quantized: photon

1913 Bohr Atoms exist in certain levels of energy and excitation (decay) involves absorption (emission) of photons with definite energy

1925/26 de Broglie, Heisenberg, Born, Jordan, Pauli, Dirac and Schrödinger Invention of QM for material particles (without radiation)

Birth of Quantum Field Theory → QFT

1926 Born, Heisenberg and Jordan First application of QM to fields
Quantization of the electromagnetic field acquires mathematical foundation

1927 Dirac First application of QFT: treatment of spontaneous emission of radiation by atoms

What is the problem with Newton's formulation of Physics based on forces among particles? → why fields?

→ special relativity: influence between particles cannot travel faster than light → forces depend on past, not present

HEP

→ Faraday: the fundamental laws of EM can be expressed more simply in terms of fields filling space and time

↳ justified mathematically by Maxwell

↳ no spooky action at a distance

→ identical particles: why are all electrons identical?

↳ they are all excitations of the same underlying field

↳ can't be explained by ordinary QM

QM + SR

"QFT arose out of our need to describe the ephemeral nature of life."

A. Zee, QFT in a Nutshell

↳ particles can be born and can die

↳ creation and annihilation

Why do we expect that?

Particle with mass m in a box of dimension L

$$\Delta x \, \Delta p \geqslant \frac{\hbar}{2}$$

↳ $\Delta x < L$ → particle is inside of the box

$$\Delta p \geqslant \frac{\hbar}{2L}$$

$E = pc$

$$\Delta E \geqslant \frac{\hbar c}{2L}$$

: if we make L small enough, we can get

$$\Delta E > 2mc^2$$

↳ a pair particle-antiparticle can be created out of vacuum

$$L \lesssim \frac{\hbar}{mc} \equiv \lambda_{\text{Compton}} = 3.86 \cdot 10^{-13}\,\text{m} \rightarrow \text{electron}$$

↳ distance in which single particle theory no longer holds and Relativistic QFT becomes essential

Early Attempts: Relativistic Version of Schrödinger Equation

$$\underbrace{E = \frac{p^2}{2m}, \quad p \rightarrow -i\hbar\nabla, \quad E \rightarrow i\hbar\frac{\partial}{\partial t}}$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

If we assume the relativistic relation $E^2 = p^2c^2 + m^2c^4$, we get

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = -c^2\nabla^2\phi + m^2c^4\phi$$

↳ doesn't explain hydrogen
negative probabilities
negative energies

↳ works for spin 0 particles, not electrons

Klein-Gordon

1928/30 Jordan, Wigner, Heisenberg, Pauli and Fermi
material particles can be understood as quanta of different fields in the same way the photon is a quantum of the EM field

↳ one field for each particle type

reality is a set of fields subject to SR and QM and everything else is derived as a consequence of field dynamics

1928 Dirac Relativistic equation for the electron, which has solution corresponding to "negative energy" electron states, which are interpreted as "negative energy" anti-electrons (positrons)

↳ can't describe electron without describing positron

↳ existence of antimatter

↳ same mass, opposite charge

1932 Anderson Discover the positron in cosmic rays

1932 Fermi Formulates a theory for beta decay. The electron is created in neutron decay as photons are created in atom decay

1934 Furry, Oppenheimer, Pauli, Weisskopf QFT can incorporate antimatter without the need of "negative energy". Describes creation and annihilation of ~~virtual~~ particles and antiparticles. Particles and antiparticles are quanta of fields.

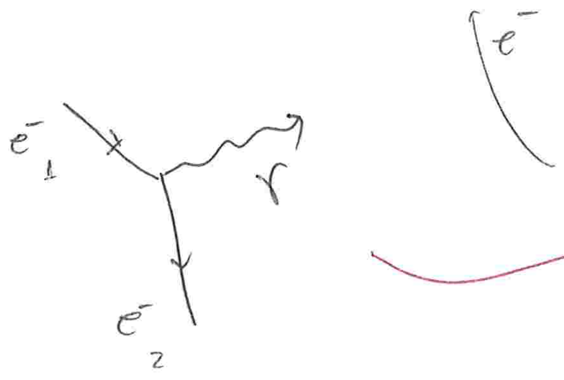
Interaction as Particle Exchange

→ charged particles interact by exchanging photons, not through classical EM fields. Photons continuously pass from a particle to the other

→ other types of forces are due to exchange of other types of particles

↳ virtual particles → $p^2 \neq m^2$

virtual particles are not observed when being exchanged, because their creation as real particles would violate conservation of energy (e.g. a free electron turning into a photon and an electron)



$$p_1^2 = (p_2 + p_\gamma)^2$$
$$m^2 = m^2 + 2 p_2 \cdot p_\gamma$$
$$2 p_2 \cdot p_\gamma = 0$$

frame with $\vec{p_2} = \vec{0}$ yields $m E_\gamma = 0$, despite $m \neq 0$, $E_\gamma \neq 0$

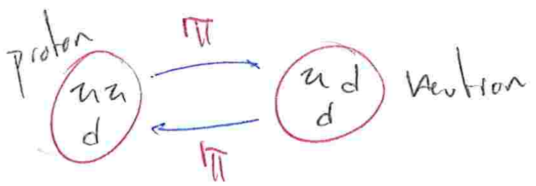
the uncertainty principle of QM states that the energy of a system which survives only for a short time must have high uncertainty

↳ virtual particles can be created in intermediate states of physical processes, but must be reabsorbed very quickly

the forces produced by a particle have a range inversely proportional to the mass of the exchange particle

1936 Yukawa predicts the existence of mesons exchanged in the interactions of nucleons and estimated their mass in 135 - 140 MeV based on range of interaction

↳ pion $\quad \pi^{\pm} = q\bar{q}$ $\quad$ quark q, antiquark $\bar{q}$



IEP

The time the π exists can be estimated from the range of strong interaction $\Delta x \simeq 1\,\text{fm} = 10^{-15}\,\text{m}$ at speed of light c

$$\Delta t = \frac{10^{-15}\,\text{m}}{3 \cdot 10^{8}\,\text{m/s}} = 3.3 \cdot 10^{-24}\,\text{s}$$

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$

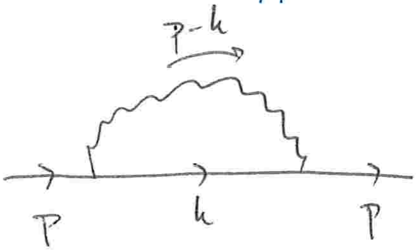
$$\Delta E \simeq \frac{\hbar}{2} \frac{1}{\Delta t} = \frac{6.6 \cdot 10^{-22}\,\text{MeV s}}{2} \frac{1}{3.3 \cdot 10^{-24}\,\text{s}} = 100\,\text{MeV}$$

$$\Delta E = m_\pi c^2 \Rightarrow m_\pi = 100\,\text{MeV} \longrightarrow c = \hbar = 1$$

The Problem with Infinities

1930 Oppenheimer Calculate the electron self-energy (γ emission and reabsorption)

$\longrightarrow$ result is infinite

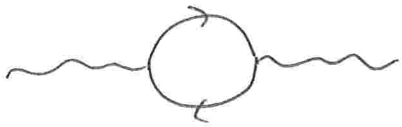


Electron and photon can share the moment p of the original electron in an infinite number of ways and k can be arbitrarily large

$$\int_0^{+\infty} d^4k = +\infty$$

1934 Weisskopf Redoes the calculation of self-energy taking into account behaviour of electron and positron. Contributions cancel and the worst part of Oppenheimer's result is solved: divergence is "just" logarithmic.

1933 Dirac Shows infinity also occurs in vacuum polarization

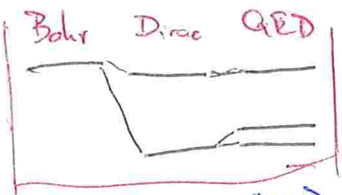


Renormalization

1936 Weisskopf — Infinities are eliminated by absorbing them in the redefinition of physical parameters

1948 Kramers

1947 Lamb — Measurement of electron self-energy effect on hydrogen atom



$g=2$

↳ Dirac predicts some pairs of excited states would have the exact same energy, but the interaction of the electron with its own EM field shifts the energy

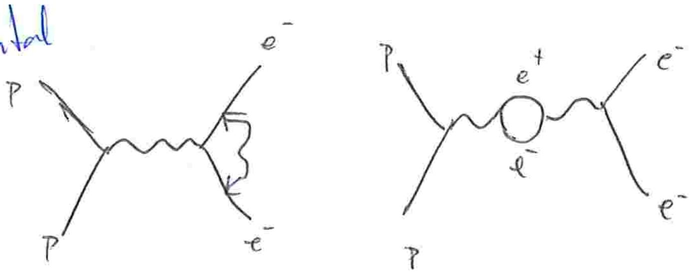
1947 Bethe — First approximate calculation of lamb shift using mass renormalization to eliminate infinity

finite physical mass

$$m_r = m_0 - \delta m$$

"self-mass" from the interaction of the electron with his virtual photon cloud

infinite bare mass from the fundamental field equations



HEP

1946/48 Tomonaga
1948/50 Schwinger, Feynman

Feyman Diagrams (pictorial rules for evaluating probability of some process to occur)
Incorporates particle - antiparticle symmetry and SR

1949 Dyson Shows the 3 formalisms are equivalent

1947 Schwinger First calculation of electron anomalous magnetic moment (g-2). Characterizes the interaction with the magnetic moment. Absorbed the effect of infinity of virtual photons on charge renormalization.

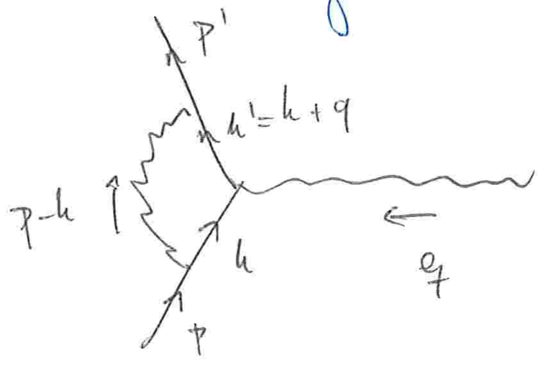
Magnetic moment

$$\mu = -g \frac{e}{2m} s = -\frac{g}{2} \frac{e}{2m}$$

$\to g = 2$

Theoretical Result g is greater than Dirac's value by 1.16 parts per thousand

Experiment at Columbia g is 1.15 - 1.21 parts per thousand larger than Dirac's



Regularization Procedure

Example: dimensional regularization

↳ integral diverges in 4 dimensions, so we calculate in $d = 4 - \varepsilon$

$$\int \frac{1}{(2\pi)^d} \frac{d^d \ell}{(\ell^2 - m^2)^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{m^2}\right)^{2-\frac{d}{2}}$$

Euler-Mascheroni

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)$$

logarithmic divergence for $d=4$

↳ infinity translates into a $1/\varepsilon$ pole, which diverges for $\varepsilon \to 0$ $(d \to 4)$

We can't calculate $\infty - \infty$, but we can calculate

$$\frac{2}{\varepsilon} - \frac{2}{\varepsilon}.$$

Another option: $\displaystyle\int_0^{+\infty} f(x)\,dx \to \lim_{\Lambda \to +\infty} \int_0^{\Lambda} f(x)\,dx$ → allows for calculations as well

A First Flavour of Feynman Diagrams

↳ the most succint representation of our present knowledge about the Physics of quantum scattering of fundamental particles

↳ Feynman diagrams are obtained from a Lagrangian by the Feynman rules

Goal: calculate cross-sections, half-lifes, et cetera

- Incoming and outgoing lines represent a particle with $E, \vec{p}, \vec{S}$
  - → electron ~~in~~ the initial state is represented by a line with $\longrightarrow\bullet$ and in the final state by $\bullet\longrightarrow$
  - → positron in the initial state is represented by a line with $\longleftarrow\bullet$ and in the final state by $\bullet\longleftarrow$
- Each internal line represents a virtual particle propagator
  - → photon: $\bullet\!\sim\!\sim\!\sim\!\sim\!\bullet$
  - → fermion: $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$
- Each vertex $\bullet$ has a factor containing the coupling constant of the interacting fields
  - ↳ QED: vertex has three lines: two ~~the~~ fermions and a photon with constant e

- Electrons as they move through space and time exchange a photon (virtual quantum)
  - ↳ also possible to exchange more photons
  - ↳ should not be interpreted literally: they show topological construction, not geometric trajectories

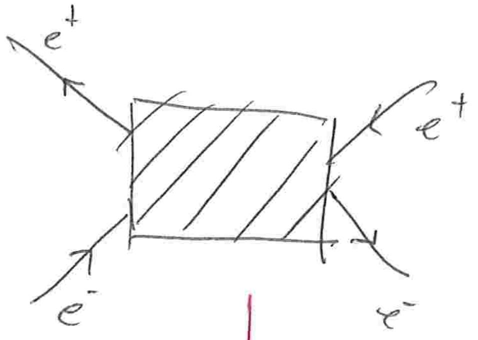
- Feynman could attach a mathematical formula to each diagram expressing the likelihood of the process the diagram depicts
  - ↳ in simple cases, he got the same answers other people found much more laboriously
- diagram representation is more useful when few "simple" diagrams supply most of the answer
  - ↳ weak coupling → each additional complicating line is relatively rare

    ↓

    almost always the case for photons in QED
- Feynman diagram: one term in the perturbation theory expansion of the S matrix element of an interaction

$S = \mathbb{1} + iT$ → found perturbatively

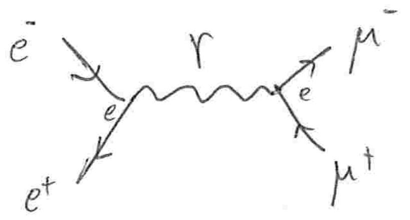
↳ if coupling constant is small, only first few diagrams are relevant



↳ interaction

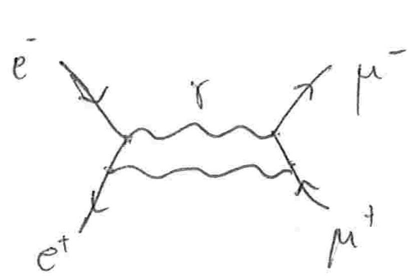
First order

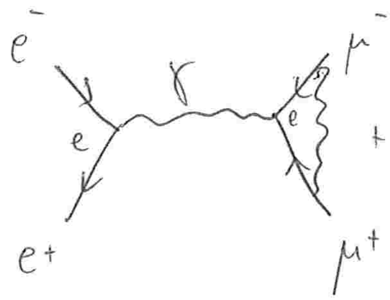
$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$$



Second Order:

$$\alpha^2 = \left(\frac{1}{137}\right)^2$$

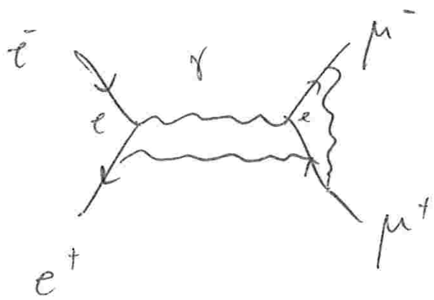




$+ \dots$

Third Order:

$$\alpha^3 = \left(\frac{1}{137}\right)^3$$



$+ \dots$

Classical Mechanics Revisited

Given a Lagrangian $L = T - V$, we know from the Principle of Least Action that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

SHO

For a simple harmonic oscillator,

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0, \quad \text{with } \omega^2 \equiv \frac{k}{m}$$

The action S is obtained integrating the Lagrangian from t_1 through t_2

$$S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt.$$

The principle of least action $(\delta S = 0)$ determines the trajectory.

The system's Hamiltonian is given by

$$H(p, q) = \sum_i p_i \dot{q}_i - L$$

Lagrangian Field Theory

$$S[\phi] = \int L\, dt = \int \mathcal{L}\, d^4x \quad \mathcal{L}(x) = \mathcal{L}[\phi(x), \partial_\mu \phi(x)]$$

The Principle of Least Action implies

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0.$$

→ the different quantum states of an elementary particle give rise to an irreducible representation of the Poincaré group. The fields assume different characteristics and transform in different ways under Poincaré depending on the particle's spin

HEP

How do we choose a Lagrangian?

→ simplest possible Lorentz invariant functional depending on fields and derivatives and yielding results confirmed by experiment

→ Lagrangian must be invariant under Poincaré, otherwise EoM wouldn't be invariant

→ a complete characterization is given in terms of fields and their derivatives (ϕ_i and $\partial_\mu \phi_i$)

→ no explicit dependence on spacetime

→ Lagrangians are hermitian

↳ probability conservation

→ at each point in spacetime, the Lagrangian depends only on $\phi_i(x)$ and $\partial_\mu \phi_i(x)$ in a neighborhood of that point

↳ locality

Natural Units

adimensional

$c = 3 \cdot 10^8 \, m/s \rightarrow 1$

$\hbar = 6.58 \cdot 10^{-25} \, GeV \, s \rightarrow 1$

$[E] = [p] = [M]$

$c = 1: \quad [L] = [T]$

$\hbar = 1: \quad [E] = [T]^{-1} = [L]^{-1} = [M]$

$\hbar c = 1 = 0.197$ GeV fm $\nearrow$ 1 GeV $= 1.6 \cdot 10^{-10}$ J

$(\hbar c)^2 = 1 = 0.389$ GeV^2 mbarn $\longrightarrow$ 1 barn $= 100$ fm^2

$$1 \text{ GeV} \simeq \frac{1}{2.0 \cdot 10^{-16}\,\text{m}} \simeq \frac{1}{6.6 \cdot 10^{-25}\,\text{s}}$$

Remark on Four-Vectors

Metric Signature: $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

D'alembertian: $\Box^2 = \partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2$

Examples of Free-Lagrangians and Equations of Movement

Scalar ($s=0$) field ϕ

Neutral scalars $\nearrow \pi^0$

Klein-Gordon Lagrangian $\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi^2\right)$

Klein-Gordon Equation $(\Box^2 + m^2)\,\phi(x) = 0$

Charged Scalars

Klein-Gordon Lagrangian $\mathcal{L}_0 = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi \phi^*$ $\downarrow$ $\pi^\pm$

Klein-Gordon Equation $\begin{cases} (\Box^2 + m^2)\,\phi(x) = 0 \\ (\Box^2 + m^2)\,\phi^*(x) = 0 \end{cases}$

Spinor ($s = 1/2$) field ψ

Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$

Dirac Equation $\begin{cases} (i\gamma^\mu \partial_\mu - m)\psi = 0 \\ \bar{\psi}(i\overleftarrow{\partial}_\mu \gamma^\mu + m) = 0 \end{cases}$

Dirac adjoint spinor: $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$

Feynman slash: $\gamma^\mu \partial_\mu \equiv \not{\partial}$

Dirac Matrices

the simplest representation of the Clifford algebra is in terms of 4×4 matrices

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Example of representation of γ: → Pauli matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

→ Weyl, or chiral, representation

$A^\mu = (\varphi, \vec{A})$

Vector ($s = 1$) field A_μ

Massless Vector Field

Maxwell Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Maxwell Equation $\partial_\mu F^{\mu\nu} = 0$

EM has two invariants: $(\vec{E}^2 - \vec{B}^2) = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu}$ and $\vec{E}\cdot\vec{B}$. The latter is a pseudo-scalar, but we want $\mathcal{L}$ to be a scalar

Massive Vector Field

Proca Lagrangian $\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$

Proca Equation $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$

Massive Charged Vector Field

Charge Vector Lagrangian $\mathcal{L}_0 = -\frac{1}{2} F^*_{\mu\nu} F^{\mu\nu} + m^2 A^*_\mu A^\mu$

Charge Vector Equation $\begin{cases} (\Box + m^2) A^\mu = 0 \\ (\Box + m^2) A^{*\mu} = 0 \end{cases}$

with $\partial_\mu A^\mu = 0$

Propagators and Virtual Particles

→ the previous Lagrangians have no interactions

↳ all of them are quadratic in the fields

They give rise to the propagators of the particles

↓

function that specifies the probability for a virtual particle to travel, without interacting, from (t, x) to (t', x') with 4-momentum $(E, \vec{p})$

This is the first set of Feynman rules that accounts for the internal (not incoming or outgoing) lines of the diagrams

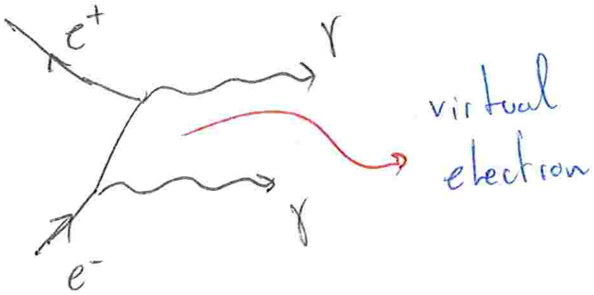
Virtual particle: $p^2 \neq m^2 \rightarrow$ off-shell

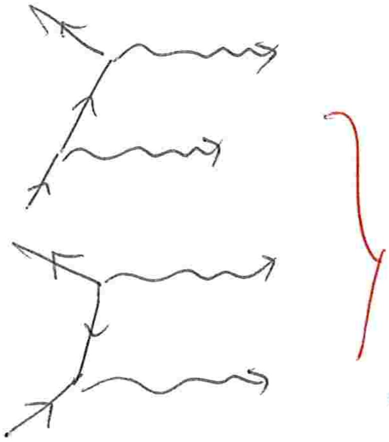
Real particle: $p^2 = m^2 \rightarrow$ on-shell

mass-shell

virtual particles conserve energy and momentum. However, the energy of an off-shell particle can even be negative

↳ the propagator takes into account particles going both ways (particle one way, antiparticle the other) and therefore carrying an opposite flow of positive energy





both are considered in the propagator

A heuristic technique to obtain the propagator of a field F is to evaluate the inverse of the operator bilinear in fields that appear in the Lagrangian, in momentum space

$$\mathcal{L}_0 = F^{conj}\, \mathcal{O}(x)\, F \Rightarrow P_F(p) = (+i)\, \mathcal{O}^{-1}(p)$$

If $F^{conj} = F$, we multiply by the combinatorial factor 2

Example: Klein-Gordon

parts

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi^2\right) = \phi\left[\frac{1}{2}\left(-\partial_\mu \partial^\mu - m^2\right)\right]\phi$$

Notice $p_\mu \to i\hbar(\partial_t, \nabla) = i\hbar\, \partial_\mu$, $p^\mu = i\hbar\, \partial^\mu$

$$\partial_\mu \to -ip_\mu \,, \quad \partial^\mu \to -ip^\mu \xrightarrow{} \hbar = 1$$

$$\Box^2 = \partial_\mu \partial^\mu = -p^2$$

$$\mathcal{O}_{KG}(x) = 2 \cdot \left[-\frac{1}{2}\left(\Box^2 + m^2 \right) \right] \Rightarrow \mathcal{O}_{KG}(p) = p^2 - m^2$$

$$P_{KG} = \frac{i}{p^2 - m^2}$$

Dirac Propagator

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m \right)\psi$$

$$\mathcal{O}_{Dirac}(x) = i\not{\partial} - m \Rightarrow \mathcal{O}_{Dirac}(p) = \not{p} - m$$

$$P_{Dirac} = +i\, \mathcal{O}^{-1}_{Dirac}(p) = \frac{+i}{\underbrace{\not{p} - m}_{\text{sort of}}} = i\,\frac{\not{p} + m}{p^2 - m^2}$$

Photon Propagator

For a massless vector field, we must add a gauge fixing term. This is because the photon has only two degrees of freedom (positive or negative helicity, or two different polarizations) but we are using a field A^μ with 4 d.o.f. We impose constraints on A^μ by writing

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2$$ → ξ fixes the gauge

$\xi = 1$ Feynman gauge

$\xi = 0$ Landau gauge

In the momentum space,

$$\mathcal{L} = -\frac{1}{2} A^*_\mu(k) \left[k^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \right] A_\nu(k)$$

the photon propagator becomes

$$D_{\mu\nu}(k) = \frac{-i}{k^2} \left[g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right]$$

In the momentum space, the field $A_\mu(x)$ is represented by $\varepsilon_\mu(k, \lambda)$ which satisfies

$$\varepsilon^\mu(k, \lambda)\, \varepsilon_\mu(k^*, \lambda') = -\delta_{\lambda\lambda'}$$ → $\lambda, \lambda' = 1, 2$

$$\varepsilon(k, \lambda) \cdot \varepsilon^*(k, \lambda') = g^{\lambda\lambda'}$$

$$\sum_\lambda g^{\lambda\lambda}\, \varepsilon^\mu(k, \lambda)\, \varepsilon^{*\nu}(k, \lambda) = g^{\mu\nu}$$

$\varepsilon^\mu(k, 1)$ and $\varepsilon^\mu(k, 2)$: transverse polarizations

$\varepsilon^\mu(k, 3)$ and $\varepsilon^\mu(k, 0)$: longitudinal and scalar polarizations

For k along the z axis

$$\varepsilon^\mu(k, 0) = (1, 0, 0, 0), \quad \varepsilon^\mu(k, 1) = (0, 1, 0, 0), \quad \varepsilon^\mu(k, 2) = (0, 0, 1, 0), \quad \varepsilon^\mu(k, 3) = (0, 0, 0, 1)$$

We can write the transverse polarizations ~~as~~ $\rightarrow$ helicity $= \pm 1$

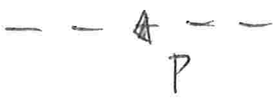
$$\varepsilon_{+\mu} = \frac{1}{\sqrt{2}}\left[\varepsilon^{\mu}(k,1) + i\,\varepsilon^{\mu}(k,2)\right] = \frac{1}{\sqrt{2}}(0,1,+i,0) = \epsilon^{*}_{-\mu}$$

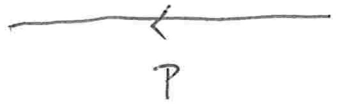
$$\varepsilon_{-\mu} = \frac{1}{\sqrt{2}}\left[\varepsilon^{\mu}(k,1) - i\,\varepsilon^{\mu}(k,2)\right] = \frac{1}{\sqrt{2}}(0,1,-i,0) = \epsilon^{*}_{+\mu}$$

Note that the propagators do not vanish for off-shell particles. In fact the propagator is basically obtained by inverting the wave equation, so it is expected to have singularities on-shell.

↳ Feynman convention: introduce a $+i\epsilon$ to circumvent the pole

Thus, we have the complete set of Feynman propagators

Scalar  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$

Dirac  $= \dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

Photon  $= \dfrac{-i\,g_{\mu\nu}}{p^2 + i\epsilon}$

after gauge fixing

Interacting Fields

Any theory must include interactions among particles which describe the very dynamics of the particular model.

Interaction terms in the Lagrangian corresponds to internal vertices where lines (external legs or particle propagators) meet. Together with the propagators, the vertices form the complete set of Feynman rules for a given theory

One can obtain the vertex factor by eliminating the fields in the Lagrangian term and multiplying by $\pm i$ and by the combinatorial factor corresponding to the identical particles in the vertex

Example: $\lambda \phi^4$ coupling of scalar fields → 4-particle interaction of scalar fields

$$\mathcal{L}_\lambda = -\frac{\lambda}{4!}\phi^4 \rightarrow \phi^4 \text{ vertex} = -i(4!)\frac{\lambda}{4!} = -i\lambda$$

4! is the number of ways the 4 ϕ's can be connected yielding the same result

Beta decay

n (or d-quark) is destroyed and a p (or u-quark) plus and electron and an electron anti-neutrino are created

Beta decay was the first place QFT was used to describe transformation processes

↳ Fermi's theory for weak interactions

Other examples of particle creation and destruction

→ several processes in QED

- radiation and absorption of photons by electrons ↗ or positrons
- creation and annihilation of electron-positron pair

There is no mechanism in ordinary QM to handle particle number changes

Attempts to construct a relativistic version of the Schrödinger equation of a particle encounter serious problems

↳ negative probabilities, negative energy states or causality breakdown

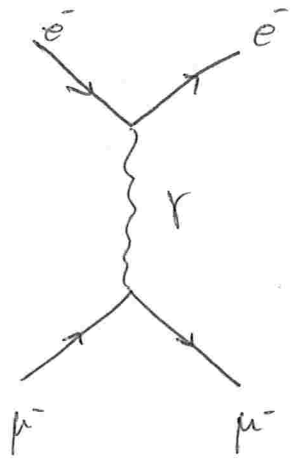
In the relativistic regime, we need a new formalism to treat states with an undetermined number of particles: QFT

Interacting Fields and the Exchange of Mediators

The association of interactions with particle exchange allows interactions to be described by the exchange of different particles

Electromagnetic Interaction (QED)

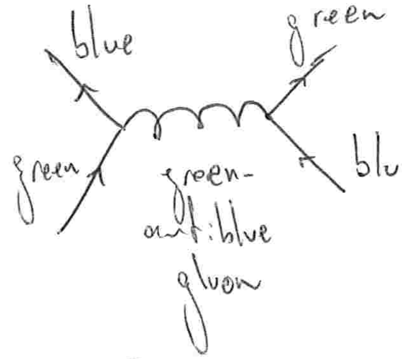
→ generated by virtual photon exchange



Strong Interaction (QCD)

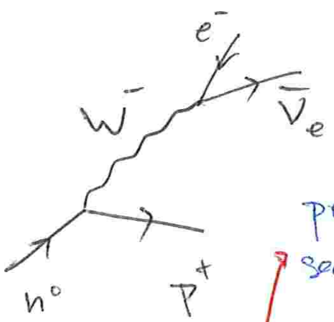
→ short distance: gluon exchange

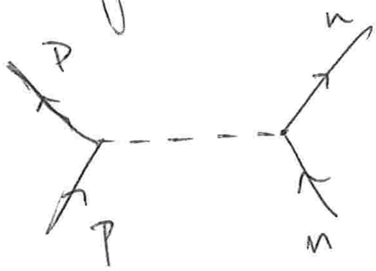
→ long distance: pion exchange



Weak Interaction (SM)

→ W and Z boson exchange





previously called second-quantization

↳ first quant.: quantize observables

↳ second quant.: quantize wave function in order to create and annihilate particles

Field Quantization

Let us take the real Klein-Gordon field and show how to quantize the theory.

momentum conjugate to ϕ

$$\pi(\vec{x}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})} = \dot{\phi}(\vec{x})$$

for KG

$$H = \int \pi(\vec{x})\,\dot{\phi}(\vec{x}) - \mathcal{L}\; d^3x = \int \mathcal{H}\, d^3x$$

or

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

We now promote ϕ and π to operators and impose suitable commutation relations

Discrete system of one or more particles: $[q_i, p_j] = i\delta_{ij}$, $[q_i, q_j] = [p_i, p_j] = 0$

Continuous: $q_i \to \phi(x)$, $p_i \to \pi(y)$, $\delta_{ij} \to \delta(\vec{x}-\vec{y})$

Continuous system: $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$$

Nota bene: in the Schrödinger picture the operators are time-independent. In the Heisenberg picture the commutation relations hold for both operators taken at the same time

To obtain the spectrum of the Hamiltonian, we make a Fourier transform

$$\phi^*(\vec{p}) = \phi(-\vec{p})$$

$$\phi(\vec{x},t) = \int \frac{1}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p},t)\, d^3p$$

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\phi(\vec{p},t) = 0$$

KG equation for $\phi(\vec{p},t)$

We can compare this form of the KGE with a SHO:

$$H_{SHO} = \frac{p^2}{2} + \frac{\omega^2 \phi^2}{2}$$

$m = 1$

ladder operators

$$\phi = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), \quad p = -i\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right), \quad [a, a^\dagger] = \mathbb{1}$$

$$\Downarrow$$

$$H_{SHO} = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$H_{SHO}|n\rangle = \omega\left(n + \frac{1}{2}\right)|n\rangle$$

$|n\rangle$ are the eigenstates of $N = a^\dagger a$

This inspires us to treat each Fourier mode of the field as an independent oscillator with its own operators a and $a^\dagger$ and write, with $\omega \to E_p$,

$$\phi(\vec{x}) = \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E_p}}\left(a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right) d^3p$$

integration by parts

$$= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E_p}}\left(a_p + a_{-p}^\dagger\right) e^{i\vec{p}\cdot\vec{x}}\, d^3p$$

$$\pi(\vec{x}) = \frac{-i}{(2\pi)^3}\int \sqrt{\frac{E_p}{2}}\left(a_p e^{i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right) d^3p$$

$$= \frac{-i}{(2\pi)^3}\int \sqrt{\frac{E_p}{2}}\left(a_p - a_{-p}^\dagger\right) e^{i\vec{p}\cdot\vec{x}}\, d^3p$$

We get

$$H = \frac{1}{(2\pi)^3} \int E_p \Big(a_p^\dagger a_p + \underbrace{\tfrac{1}{2} [a_p, a_p^\dagger]}_{\text{proportional to } \delta(0) = +\infty} \Big) d^3p$$

Expected

we are summing over all modes of the ground state energies $E_p/2$

we might ignore the divergent term, since experiments can only measure energy <u>differences</u> from the ground state of H

It holds that

$$[H, a_p^\dagger] = E_p a_p^\dagger, \quad [H, a_p] = -E_p a_p$$

The spectrum can be obtained in analogy to the SHO

Ground state: represented by $|0\rangle$, $a_p|0\rangle = 0, \ \forall p$
has $E = 0$

Other energy eigenstates: can be built with creation operators $a_p^\dagger a_q^\dagger a_r^\dagger \dots |0\rangle$ and ~~this~~ this is an eigenstate of H with energy $E_p + E_q + E_r + \dots$ and momentum $\vec{p} + \vec{q} + \vec{r}$

$\mathcal{H}_{\vec{p}} \otimes \mathcal{H}_{\vec{q}} \otimes \mathcal{H}_{\vec{r}} \otimes \dots$

$E_{p_i} = \sqrt{|\vec{p}_i|^2 + m^2}$

Since these quantum excitations are discrete entities that have the proper relativistic energy-momentum relation, we call them particles → each particle satisfies the REM relation separately

↳ since the particle is created in the momentum space, it does not have to be localized in position space

We might also write ϕ and π in the Heisenberg picture:

$$\phi(x) = \phi(\vec{x},t) = e^{iHt}\,\phi(\vec{x})\,e^{-iHt}$$

$$i\frac{\partial \mathcal{O}}{\partial t} = [\mathcal{O}, H]$$

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)\Big|_{p^0=E_p} d^3p$$

$$\pi(\vec{x},t) = \frac{\partial\phi(\vec{x},t)}{\partial t}$$

The Dirac Field → considers first derivatives in space & time to avoid KG's complications due to second-order derivatives wrt time

We want to find $\hat{H}$ such that

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \quad \text{with} \quad \hat{H} = c\,\vec{\alpha}\cdot(-i\hbar\nabla) + \beta mc^2.$$

↳ ansatz

where β and α^i are 4×4 matrices. Thus,

$$i\hbar\left(\frac{\partial}{\partial t} + c\vec{\alpha}\cdot\nabla\right)\psi - \beta mc^2\psi = 0$$

multiplying by $\frac{\beta}{c}$, with $\beta^2 = 1$

$$i\hbar\left(\beta\frac{\partial}{\partial(ct)} + \beta\vec{\alpha}\cdot\nabla\right)\psi - mc\psi = 0$$

$\gamma^0 \equiv \beta$

$\gamma^i \equiv \beta\alpha^i$

$$i\hbar(\gamma^0\partial^0 + \gamma^i\partial^i)\psi - mc\psi = 0$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

The Dirac matrices γ^μ do not commute and respect the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

There are different representations of the Dirac matrices

Dirac-Pauli:

0 is associated with energy and γ^0 is diagonal in here

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

Chiral or Weyl:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

relevant for weak interactions

only felt by left-handed particles

Review on Pauli Matrices

$$\sigma_0 = \mathbb{1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\det \sigma_i = -1, \quad \mathrm{Tr}\, \sigma_i = 0, \quad [\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\sigma_k, \quad \{\sigma_i, \sigma_j\} = 2i\,\delta_{ij}$$

$$e^{i\theta(\hat{n}\cdot\vec{\sigma})} = \mathbb{1}\cos\theta + i(\hat{n}\cdot\vec{\sigma})\sin\theta$$

$$\det\left(i\theta(\hat{n}\cdot\vec{\sigma})\right) = \theta^2$$

4D: $\sigma^\mu \equiv (\sigma^0, \sigma^i), \quad \bar{\sigma}^\mu \equiv (\sigma^0, -\sigma^i)$

$$\sigma^\mu B_\mu = \begin{pmatrix} B_0 - B_3 & -B_1 + iB_2 \\ -B_1 - iB_2 & B_0 + B_3 \end{pmatrix}$$

$$\det \sigma^\mu B_\mu = B^\mu B_\mu$$

The definitions of σ^μ and $\bar{\sigma}^\mu$ allow us to write the Chiral or Weyl representation as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

Left-handed and right-handed Weyl spinors

Weyl spinors: $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

The Dirac Equation becomes

the equations are not coupled for $m=0$

$$(i\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

Let us write the general solution of the Dirac equation as a linear combination of plane waves

$$\psi(x) = u(p)\, e^{-ip\cdot x} \rightarrow p^2 = m^2,\ p^0 > 0, \text{ positive frequency waves}$$

$$\psi(x) = v(p)\, e^{+ip\cdot x} \rightarrow p^2 = m^2,\ p^0 > 0, \text{ negative frequency waves}$$

integrated over momenta, but we shall see the details later. For now, we focus on the core concepts

we might also interpret as both having positive frequency, but one having negative energy

There are two linearly independent solutions for $u(p)$ and $v(p)$

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} \qquad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_{-s} \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi_{-s} \end{pmatrix} \qquad s=1,2$$

square root of a matrix: square root of the eigenvalues

ξ_s and ξ_{-s}: two component normalized spinors

We can associate

$$\xi = \xi_s = (\xi_\uparrow, \xi_\downarrow)$$

$$\xi_{-s} = -i\sigma^2 \xi_s = (\xi_\downarrow, -\xi_\uparrow)$$

$$\xi_\uparrow = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix} \qquad \xi_\downarrow = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix}$$

with $\xi_s = \xi_\uparrow (\xi_\downarrow)$ for $s = 1(2)$

We assume the spin orientation along the 3 axis, ξ_s, ~~are~~ the eigenstates of σ_3 $\hookrightarrow$ $\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Some useful relations are

$$\sqrt{p\cdot\sigma} = \frac{p\cdot\sigma + m}{\sqrt{2(E+m)}} \qquad \sqrt{p\cdot\bar{\sigma}} = \frac{(p\cdot\bar{\sigma} + m)}{\sqrt{2(E+m)}}$$

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 \mathbb{1} = m^2 \mathbb{1} \qquad (p\cdot\sigma) + (p\cdot\bar{\sigma}) = 2E\mathbb{1}$$

$$\left(\sqrt{p\cdot\sigma} + \sqrt{p\cdot\bar{\sigma}}\right)^2 = 2(E+m)$$

For momentum in 3 direction $p = (E, 0, 0, p_3)$:

$$\sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p_3} & 0 \\ 0 & \sqrt{E+p_3} \end{pmatrix} \qquad \sqrt{p\cdot\bar{\sigma}} = \begin{pmatrix} \sqrt{E+p_3} & 0 \\ 0 & \sqrt{E-p_3} \end{pmatrix}$$

For positive frequency → electrons

$$u_\uparrow(p) = \begin{pmatrix} \sqrt{E-p_3}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E+p_3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \xrightarrow[\substack{\text{high energy}\\ E\sim p_3}]{} \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

$$u_\downarrow(p) = \begin{pmatrix} \sqrt{E+p_3}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{E-p_3}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \longrightarrow \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}.$$

For negative frequency (→ positrons), since $\xi_{-s} = (\xi_\downarrow, -\xi_\uparrow)$,

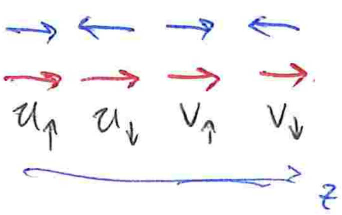
$$v_\uparrow(p) = \begin{pmatrix} \sqrt{E+p_3}\begin{pmatrix}0\\1\end{pmatrix} \\ -\sqrt{E-p_3}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \longrightarrow \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$v_\downarrow(p) = \begin{pmatrix} \sqrt{E-p_3}\begin{pmatrix}-1\\0\end{pmatrix} \\ -\sqrt{E+p_3}\begin{pmatrix}-1\\0\end{pmatrix}\end{pmatrix} \longrightarrow \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

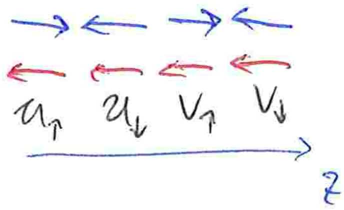
Collecting the result at the high energy limit with the spin along/opposite to the 3 axis

Right-handed: spin parallel to momentum

$\vec{p} = p\hat{z}$



$\vec{p} = -p\hat{z}$



Left handed: spin antiparallel to momentum

Note that

$$u_s(p) = -i\gamma^2 v_s^*(p), \qquad v_s(p) = -i\gamma^2 u_s^*(p)$$

If we change basis to the Dirac representation, we get the Dirac spinor in the Dirac representation

$$u_\uparrow^D(p) = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad u_\downarrow^D(p) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},$$

$$v_\uparrow^D(p) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \quad v_\downarrow^D(p) = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

Helicity and Chirality

The boost and rotation generators for a spin 1/2 particle are given by

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix} \rightarrow \text{boost}$$

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \rightarrow \text{rotations}$$

We define the helicity operator, which represents the projection of the angular momentum along the direction of momentum

$$h = \hat{p}\cdot\vec{S} = \frac{i}{2}\epsilon_{ijk}\hat{p}^i S^{jk} = \frac{1}{2}\hat{p}_i\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} = \frac{1}{2}(\hat{p}\cdot\vec{\sigma})\mathbb{1}$$

For massless fields we have

$$h_z u_\uparrow = +\frac{1}{2}u_\uparrow, \quad h_z u_\downarrow = -\frac{1}{2}u_\downarrow, \quad h_z v_\uparrow = -\frac{1}{2}v_\uparrow, \quad h_z = +\frac{1}{2}v_\downarrow$$

$u_\uparrow$ and $v_\downarrow$: helicity $h = +\frac{1}{2}$ $\longrightarrow$ spin in direction of motion

$\downarrow$

right-handed

$u_\downarrow$ and $v_\uparrow$: helicity $h = -\frac{1}{2}$ $\longrightarrow$ spin opposite to motion

$\downarrow$

left-handed

Nota bene: helicity is not, in general, Lorentz invariant

For a massive particle, $v < c$ and we may always find a frame on reference in which the particle moves in the opposite direction.

For massless particles, $v = c$ and helicity is Lorentz invariant.

We define the chirality projectors as

$$L = \frac{1}{2}(1 - \gamma^5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = \frac{1}{2}(1 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Properties:

$$R + L = \mathbb{1}, \quad R \cdot L = L \cdot R = 0, \quad R^2 = R, \quad L^2 = L$$

$\hookrightarrow$ projectors

For the spinors,

$$\psi_L = \frac{1}{2}(1 - \gamma^5)\psi, \qquad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi$$

IEP

And for the conjugate spinors

$$\bar{\psi}_L = \bar{\psi}\,\tfrac{1}{2}(1+\gamma^5), \qquad \bar{\psi}_R = \bar{\psi}\,\tfrac{1}{2}(1-\gamma^5).$$

Notice we have

$$R u_\uparrow = u_\uparrow \quad L u_\downarrow = u_\downarrow \quad L v_\uparrow = v_\uparrow$$
$$R v_\downarrow = v_\downarrow \quad \bar{u}_\uparrow L = \bar{u}_\uparrow \quad \bar{u}_\downarrow R = \bar{u}_\downarrow$$
$$\bar{v}_\uparrow R = \bar{v}_\uparrow \quad \bar{v}_\downarrow L = \bar{v}_\downarrow$$

$\cancel{R u_\uparrow = u_\uparrow \quad L u_\downarrow}$

$$R\psi_R = \psi_R \quad L\psi_L = \psi_L \quad \bar{\psi}_L = \bar{\psi}_L R \quad \bar{\psi}_R = \bar{\psi}_R L$$

Let us compare the helicity and chirality for a massive particle travelling in the $+z$ direction

$$u_\uparrow \equiv \sqrt{E}\begin{pmatrix}\begin{pmatrix}\eta_- \\ 0\end{pmatrix} \\ \begin{pmatrix}\eta_+ \\ 0\end{pmatrix}\end{pmatrix}$$

$$\eta_\pm \equiv \sqrt{1 \pm \frac{p_3}{E}}$$

For high energy, $p_3 \approx E$ and

$$\eta_+ \to \sqrt{2}, \quad \eta_- \to 0$$

$$R u_\uparrow = \frac{\eta_+}{\sqrt{2}}\sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \equiv \frac{\eta_+}{\sqrt{2}}\,u_R$$

$$L u_\uparrow = \frac{\eta_-}{\sqrt{2}}\sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \equiv \frac{\eta_-}{\sqrt{2}}\,u_L$$

Thus,

$$u_\uparrow \equiv (R+L)u_\uparrow = \frac{\eta_+}{\sqrt{2}}u_R + \frac{\eta_-}{\sqrt{2}}u_L \xrightarrow{\text{high energy}} u_R$$

Remarks

→ helicity and chirality coincide in the high energy limit
→ for antiparticles, the chirality is the opposite helicity

particle: helicity = + chirality
antiparticle: helicity = - chirality

→ the projectors L & R depend on the representation of the γ matrices. In the Dirac representation one has

$$L^{Dirac} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad R^{Dirac} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Quantization of the Dirac Field

The spinor field ψ describes fermions and antifermions. The general solution of the Dirac equation can be written as a linear combination of positive and negative frequencies

$$u(p)\, e^{-ip\cdot x} \qquad p^2 = m^2, \quad p^0 > 0$$
$$v(p)\, e^{+ip\cdot x}$$

Therefore,

$$\psi(x) = \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} a_p^s\, u^s(p)\, e^{-ip\cdot x} + b_p^{s\dagger}\, v^s(p)\, e^{+ip\cdot x}\, d^3p$$

$$\bar{\psi}(x) = \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} b_p^s\, \bar{v}^s(p)\, e^{-ip\cdot x} + a_p^{s\dagger}\, \bar{u}^s(p)\, e^{+ip\cdot x}\, d^3p$$

The creation and annihilation operators respect equal-time anticommutation relations $\longrightarrow$ Fermi-Dirac statistics

$$\{a^r_p, a^{s\dagger}_q\} = \{b^r_p, b^{s\dagger}_q\} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\, \delta^{rs}.$$

If we define $N^s_p \equiv (a^{s\dagger}_p a^s_p + b^{s\dagger}_p b^s_p)$ the Hamiltonian, momentum and charge operators are

$$H = \frac{1}{(2\pi)^3} \int \sum_s E_p N^s_p \, d^3p$$

$$\vec{P} = \frac{1}{(2\pi)^3} \int \sum_s \vec{p} N^s_p \, d^3p$$

$$Q = \frac{1}{(2\pi)^3} \int \sum_s N^s_p \, d^3p$$

Example: QED, where ψ describes electrons and positrons

$a^{s\dagger}_p$ creates electrons with E_p, $\vec{p}$, $s=\frac{1}{2}$, $Q=-1$

$\bar{\psi}(x)|0\rangle$ represents one electron at position x

$b^{s\dagger}_p$ creates positrons with E_p, $\vec{p}$, $s=\frac{1}{2}$, $Q=+1$

$\psi(x)|0\rangle$ represents one positron at position x

Quantization of the Electromagnetic Field

For massless vector fields (e.g. photon) the obtention of the propagator is more cumbersome for we need to add a gauge fixing term to the Lagrangian

Notice that

$$\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}}$$

photon

$$\pi^{0} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{0}} = 0 \,, \qquad \pi^{i} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{i}} = -\dot{A}^{i} - \partial_{i}^{\mu} A^{0} = E^{i}$$

we can't use the canonical formalism

photon has two degrees of freedom (positive and negative helicity), but A^{μ} has four components

we must impose constraints on A^{μ} to describe the photon

This can be solved if we write the Lagrangian as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_{\mu} A^{\mu} \right)^{2}.$$

gauge fixing term

Now we have

$$\pi^{0} = -\frac{1}{\xi} \partial_{\mu} A^{\mu} \,, \qquad \pi^{i} = E^{i}.$$

The commutation relations are → equal time

→ Bose-Einstein statistics

$$[A^\mu(\vec{x},t), \pi^\nu(\vec{y},t)] = i g^{\mu\nu} \delta^{(3)}(\vec{x}-\vec{y})$$

$$[A^\mu(\vec{x},t), A^\nu(\vec{y},t)] = [\pi^\mu(\vec{x},t), \pi^\nu(\vec{y},t)] = 0$$

the photon field becomes

$$A_\mu(x) = \frac{1}{(2\pi)^3} \int \frac{1}{2E_p} \sum_\lambda a_{\lambda,k} \underbrace{\varepsilon_\mu(k,\lambda)}_{} e^{ik\cdot x} + a^\dagger_{\lambda,k} \varepsilon^\dagger_\mu(k,\lambda) e^{-ik\cdot x} d^3p$$

$\varepsilon_\mu(k,\lambda)$ is the polarization vector that plays the role of the spin part of the wave function for the photon

Feynman Rules for External Leges

Scalar Boson

Initial / final state: 1

Fermion

e^- in initial state: p ⟶• $= u^s(p)$

e^- in final state: p •⟶ $= \bar{u}^s(p)$

e^+ in initial state: p ⟵• $= \bar{v}^s(p)$

e^+ in final state: p •⟵ $= v^s(p)$

Vector Boson

γ in the initial state: p 〰• $= \varepsilon_\mu(p)$

γ in the final state: p •〰 $= \varepsilon^*_\mu(p)$

General Features of Quantum Field Theory

Overview

QFT occupies a central position in our description of nature
- → it is the best description of fundamental physical laws
- → essential tools for investigating complex systems behavior

What does QFT adds to understanding beyond QM and Classical Field Theory?
- → dynamic degrees of freedom are functions of the space and time operator with appropriate commutation ratios: Quantum Fields
- → the interactions of these fields are local; one can predict the behavior of nearby objects without reference to distant ones
- → when combined with symmetry postulates (Lorentz, gauge) it becomes a powerful tool for describing particle interactions

QFT had a major impact on different areas
- → condensed matter, high energy physics, cosmology, quantum gravity, etc

New Concepts Introduced by QFT

Indistinguishable Particles: all particles of the same type are indistinguishable
- → all electrons are equal
- → how objects created so far apart in spacetime can be equal
- → why did no errors occur in electron "production"?

Both electrons are excitations of the same underlying "electron field"

↳ present throughout the Universe, occupying all of spacetime

↳ electrons are the "excitations" on the electron field

↳ not surprising all electrons are equal: they are made of the same "material"

The same is true for all fundamental particles.

Quantum Statistics: Assignment of Unique Statistics to Fermions and Bosons

→ given the indistinguishability of a class of elementary particles (and the invariance of their interactions under exchange), the general principles of QM say that

eigenstates of permutation operator are energy eigenstates, but there is no restriction on the possible states

Solutions associated with the permutation group representation retain this property over time, but do not restrict which representations are made in Nature

⇓

QFT explains the existence of indistinguishable particles and the invariance of their interactions under exchange and also restricts the symmetry of solutions

$|\uparrow\uparrow\rangle$

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$|\downarrow\downarrow\rangle$

photon

→ for bosons, only the even representation is physical

↳ symmetric wavefunctions

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

not photon

↳ one e^- can't turn into a γ, but two e^- can

→ for fermions, only the odd representation is physical

↳ antissymmetric wavefunctions

Spin-Statistic Theorem

→ objects with integer spin are bosons, objects with half-integer spin are fermions

↳ fermionic character of electrons underlies the stability of matter and the structure of the periodic table

Existence of Antiparticle Corresponding to Every Particle

QFT interprets the Dirac wave function as an operator

→ expanded in terms of Dirac equation solutions, with coefficients being operators

- positive ~~energy~~ frequency coefficients destroy electrons
- negative ~~energy~~ frequency coefficients create electrons

→ the interpretation has a simple generalization for bosons

CPT theorem: a general consequence of QFT

→ any Lorentz invariant QFT must also be invariant under the combined operation of charge conjugation C, parity P, and time inversion T, even if the individual invariances are not respected

HEP

→ the CPT product is always a symmetry of nature

→ antiparticles are the CPT conjugates of their corresponding particles

The Best Theory We Have: Quantum Electrodynamics

↳ quantized version of Electrodynamics

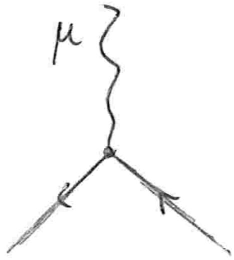
↳ one of the most important achievements of QFT

↳ enormous success of QED prediction made it into a standard to be followed by the model builders

implicit sum over flavors

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

↳ gives rise to the electromagnetic interaction vertex



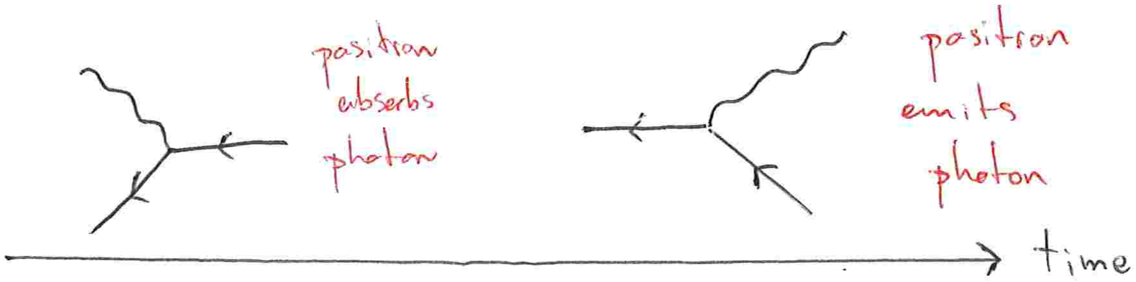
$= iQe\gamma^{\mu}$

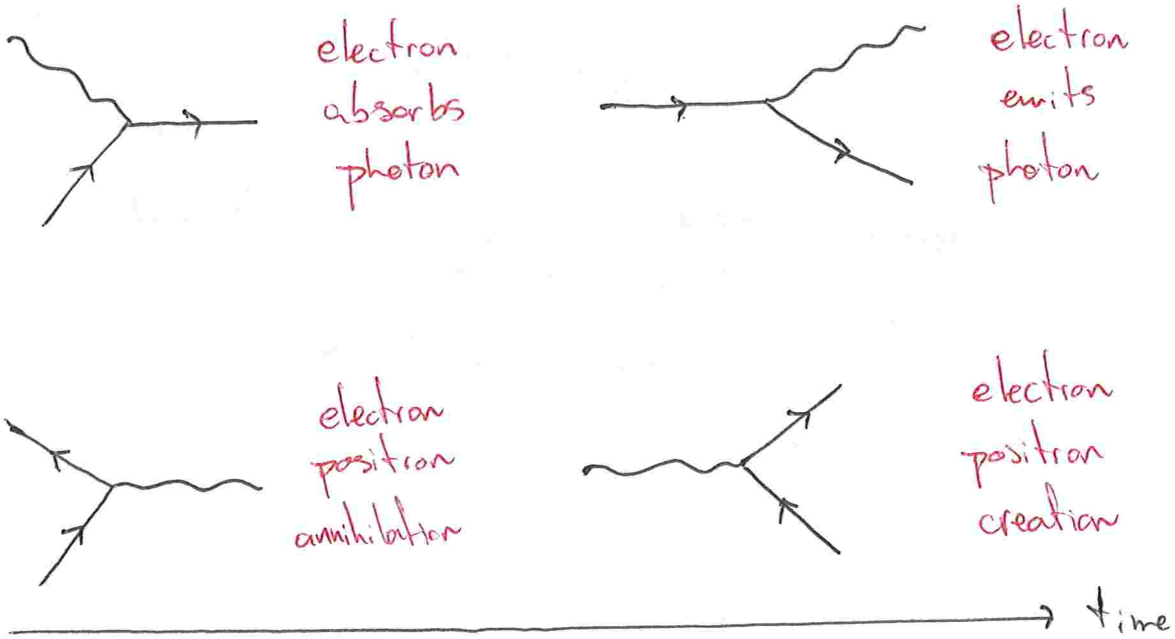
for electrons, $Q = -1$

models all possible interactions in QED

one fermion in
one fermion out

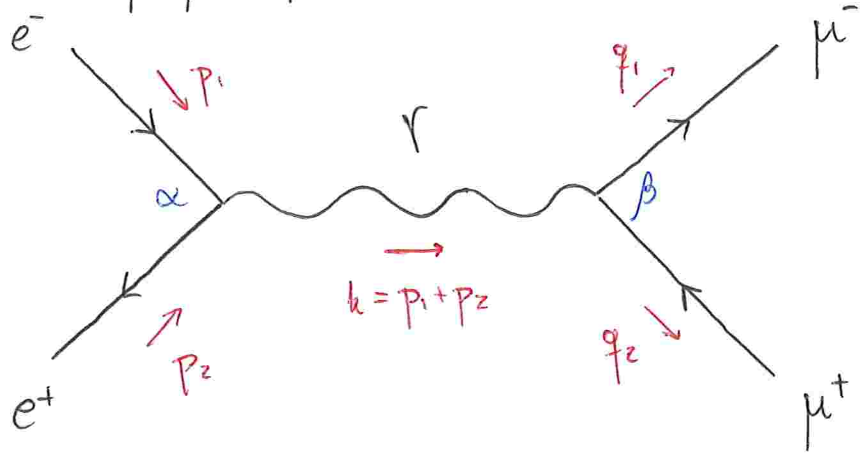
→ charge conservation

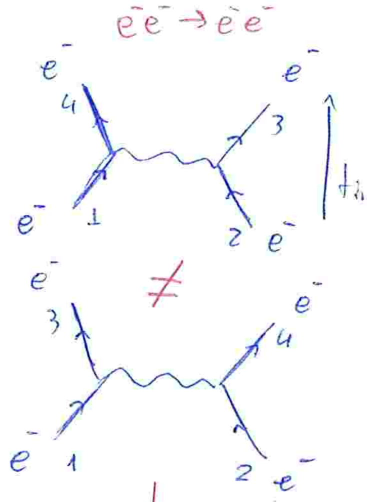




The $e^+e^- \to \mu^+\mu^-$ process in QED

but first, let us fill some theory gaps





amplitude is given by the difference of the diagrams

electrons are fermions

S-matrix

We can relate the probability of scattering in a real experiment to an idealized set of transition amplitudes between the asymptotically defined in and out states of definite momentum

$$ {}_{out}\langle \vec{p}_1 \vec{p}_2 \cdots | \vec{k}_A \vec{k}_B \rangle_{in} $$

two ~~distinct~~ incoming particles

If $|\vec{k}_A \vec{k}_B\rangle_{in}$ starts in the far past $(t \to -\infty)$ and $|\vec{p}_1 \vec{p}_2 \cdots\rangle_{out}$ is in the far future, their overlap shall be given by

HEP

$$\lim_{t_\pm \to \pm\infty} \langle \vec{p}_1 \vec{p}_2 \cdots, t_+ | U(t_+, t_-) | \vec{k}_A \vec{k}_B, t_- \rangle = \langle \vec{p}_1 \vec{p}_2 \cdots | S | \vec{k}_A \vec{k}_B \rangle$$

quotation marks are due to the fact that this is done in the Heisenberg picture

S-matrix

If we let $t_- = -T$ and $t_+ = T$, we might simply write

$$\lim_{T \to +\infty} \langle \vec{p}_1 \vec{p}_2 \cdots | U(T, -T) | \vec{k}_A \vec{k}_B \rangle = \lim_{T \to +\infty} \langle \vec{p}_1 \vec{p}_2 \cdots | e^{-i2\hat{H}T} | \vec{k}_A \vec{k}_B \rangle$$

$$= \langle \vec{p}_1 \vec{p}_2 \cdots | S | \vec{k}_A \vec{k}_B \rangle$$

For a free theory, $S = \mathbb{1}$. thus, we define the T-matrix through $S = \mathbb{1} + iT$.

Since the S-matrix must reflect 4-momentum conservation, S or T should always contain a factor $\delta^{(4)}(k_A + k_B - \sum p_f)$. we might now define the invariant matrix element $\mathcal{M}$ by

or invariant amplitude

$$\langle \vec{p}_1 \vec{p}_2 \cdots | iT | \vec{k}_A \vec{k}_B \rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f) \cdot i\mathcal{M}(k_A, k_B \to p_f)$$

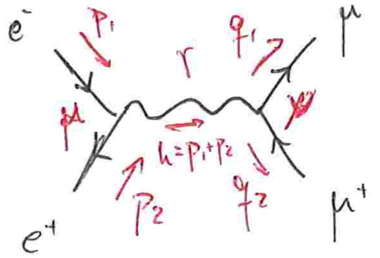
$\mathcal{M}$ plays an analogous role to the scattering amplitude f of ordinary QM

it is useful because it allows us to separate interactions (dynamics) from free propagation (kinematics)

We can obtain $\mathcal{M}$ for a certain interaction through

$i\mathcal{M}$ = sum of all connected, amputated diagrams. → evaluated via the Feynman rules

In our situation, we shall consider just the first term (diagram) in the perturbative expansion

$i\mathcal{M}$ = 

we always attribute values starting from the ← arrows, due to the matrix structure of u and v ↳ we want a number by the end of the day

$$= \bar{v}(p_2)\left[-ie\gamma_\mu\right]u(p_1)\left[\frac{-i g^{\mu\nu}}{k^2 - i\epsilon}\right]\bar{u}(q_1)\left[-ie\gamma_\nu\right]v(q_2)$$

$$= \frac{ie^2}{k^2}\left[\bar{v}(p_2)\right]_{1a}\left[\gamma_\mu\right]_{ab}\left[u(p_1)\right]_{b1}\cdot\left[\bar{u}(q_1)\right]_{1c}\left[\gamma^\mu\right]_{cd}\left[v(q_2)\right]_{d1}$$

In order to calculate the cross section, we shall need to compute $|\mathcal{M}|^2$, which means we must calculate $J_\mu^\dagger J^\mu$ for currents of the form $J^\mu = \bar{\psi}_1\gamma^\mu\psi_2$. Notice that

$$J_\mu^\dagger = \left[\bar{\psi}_1\gamma_\mu\psi_2\right]^\dagger = \psi_2^\dagger\gamma_\mu^\dagger\bar{\psi}_1^\dagger = \psi_2^\dagger\gamma^0\gamma_\mu\gamma^0\gamma^{\dagger 0}\psi_1 = \bar{\psi}_2\gamma_\mu\psi_1$$

$\bar{\psi} \equiv \psi^\dagger\gamma^0$

$\gamma^0\gamma^0 = \mathbb{1}$

$\gamma_\mu^\dagger = \gamma^0\gamma_\mu\gamma^0$

$$J_\mu^\dagger J^\mu = \left[\bar{\psi}_2\right]_{1a}\left[\gamma_\mu\right]_{ab}\left[\psi_1\right]_{b1}\cdot\left[\bar{\psi}_1\right]_{1c}\left[\gamma^\mu\right]_{cd}\left[\psi_2\right]_{d1}$$

$$= \left[\psi_2\right]_{d1}\left[\bar{\psi}_2\right]_{1a}\left[\gamma_\mu\right]_{ab}\left[\psi_1\right]_{b1}\left[\bar{\psi}_1\right]_{1c}\left[\gamma^\mu\right]_{cd}$$

$$= \left[\psi_2\bar{\psi}_2\right]_{da}\left[\gamma_\mu\right]_{ab}\left[\psi_1\bar{\psi}_1\right]_{bc}\left[\gamma^\mu\right]_{cd}$$

trace

$$J^{\dagger}_{\mu} J^{\mu} = \mathrm{Tr}\left[(\psi_2 \bar{\psi}_2)(\gamma_\mu)(\psi_1 \bar{\psi}_1)(\gamma^\mu)\right]$$

Notice we haven't specified the polarization of the incoming and outcoming beams and are treating $u^s(p)$, $v^{s'}(p')$, etc, abstractly. this comes from two facts:

→ in actual experiments, it is difficult (but possible) to polarize the incoming beams

↳ as the beams are unpolarized, we shall average over incoming spins

→ muon detectors are blind to polarization

↳ as we are seeing all muons regardless of polarisation, we must sum over outcoming spins

Thus, we want to compute the quantity

$$\frac{1}{2}\sum_{s}\frac{1}{2}\sum_{s'}\sum_{r}\sum_{r'}|\mathcal{M}(s,s'\to r,r')|^2 = \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2.$$

Using our previous results, we see that

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}^2| = \frac{e^4}{k^4}\frac{1}{4}\sum_{\text{spins}}\mathrm{Tr}\left([u(p_1)\bar{u}(p_1)]\gamma^\mu[v(p_2)\bar{v}(p_2)]\gamma^\nu\right)\mathrm{Tr}\left([v(q_2)\bar{v}(q_2)]\gamma_\mu[u(q_1)\bar{u}(q_1)]\gamma_\nu\right)$$

Completeness relations: $\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m$, $\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$

If we assume the high energy limit, under which both the electron and the muon can be considered massless, we get

$$\overline{|M|^2} \equiv \frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{e^4}{4k^4} \mathrm{Tr}\left(\not{p}_1 \gamma^\alpha \not{p}_2 \gamma^\beta\right) \mathrm{Tr}\left(\not{q}_2 \gamma_\alpha \not{q}_1 \gamma_\beta\right)$$

↳ the spinors u and v disappeared and now we have a much cleaner expression in terms of γ matrices

↳ this is a general trick: any QED amplitude involving external fermions, when squared and summed or averaged over spins, can be converted in this way to traces of products of Dirac matrices

Trace technology: $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma})$

Thus, we get

$$\mathrm{Tr}\left[\not{p}_1 \gamma^\alpha \not{p}_2 \gamma^\beta\right] = 4 p_{1\mu} p_{2\nu} \left(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}\right)$$
$$= 4\left(p_1^\alpha p_2^\beta + p_1^\beta p_2^\alpha - g^{\alpha\beta}\, p_1 \cdot p_2\right)$$

$$\overline{|M|^2} = \frac{8e^4}{k^4}\left[(p_1\cdot q_1)(p_2\cdot q_2) + (p_1\cdot q_2)(p_2\cdot q_1)\right] \nearrow \text{dynamics}$$

Now we deal with the relativistic kinematics. Let us write the momenta in the center of mass frame. ~~such~~ The initial and final 4-momenta can be written as

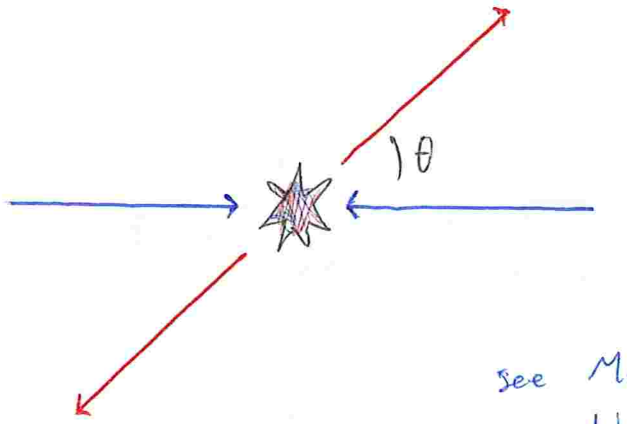
$p_1 = (E, E\hat{z})$

$p_2 = (E, -E\hat{z})$

$q_1 = (E, E\hat{q})$

$q_2 = (E, -E\hat{q})$

$\vec{q} = (0, 0, E\cos\theta)$



see Mandelstam variables

Other useful kinematic relations:

$(p_1 \cdot p_2) = (q_1 \cdot q_2) = 2E^2 \qquad k^2 = (p_1 + p_2)^2 = (q_1 + q_2)^2 \equiv s = 4E^2$

$(p_1 \cdot q_1) = (p_2 \cdot q_2) = E^2(1 - \cos\theta) \qquad (p_1 \cdot q_2) = (p_2 \cdot q_1) = E^2(1 + \cos\theta)$

$$\overline{|M|^2} = \frac{8e^4}{k^4}\left[(p_1 \cdot q_1)(p_2 \cdot q_2) + (p_1 \cdot q_2)(p_2 \cdot q_1)\right]$$

$$= \frac{8e^4}{16E^4}\left(\left[E^2(1-\cos\theta)\right]^2 + \left[E^2(1+\cos\theta)\right]^2\right)$$

$$= e^4(1 + \cos^2\theta)$$

Helicity Amplitudes: Alternative Way of Computing $\overline{|M|^2}$

Instead of summing up the invariant ampitude over the spins & using the completeness relations, we may evaluate the amplitude for all possible spin configuration and then take the sum/average over the states

16 possible configurations

$$\begin{bmatrix} \cancel{\uparrow\uparrow} & \uparrow\downarrow \\ \downarrow\uparrow & \cancel{\downarrow\downarrow} \end{bmatrix} \otimes \begin{bmatrix} \cancel{\uparrow\uparrow} & \uparrow\downarrow \\ \downarrow\uparrow & \cancel{\downarrow\downarrow} \end{bmatrix}$$

The current for two spinors with the same helicity vanishes

$$\bar{v}_\uparrow \gamma_\mu u_\uparrow = \bar{v}_\uparrow R \gamma_\mu R u_\uparrow = \bar{v}_\uparrow R L \gamma_\mu u_\uparrow = 0.$$

For opposite helicities we get

$$\bar{v}_\uparrow \gamma_\mu u_\downarrow = \bar{v}_\uparrow R \gamma_\mu L u_\downarrow = \bar{v}_\uparrow R^2 \gamma_\mu u_\uparrow = \bar{v}_\uparrow R \gamma_\mu u_\uparrow \neq 0.$$

Instead of using trace technology, we shall plug in explicitly the spinor.

Remarks:

$$\gamma^0 \gamma^\mu = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} = \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}$$

$$u(p) = \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p}\cdot\vec{\sigma})\xi \\ \frac{1}{2}(1 + \hat{p}\cdot\vec{\sigma})\xi \end{pmatrix}, \quad v(p) = \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p}\cdot\vec{\sigma})\xi \\ -\frac{1}{2}(1 + \hat{p}\cdot\vec{\sigma})\xi \end{pmatrix}$$

For antiparticles, the handedness of the ~~particle~~ spinor is the opposite of the handedness of the particle. Since we are concerned with the particle spin, we define

$$u_{\uparrow(\downarrow)} \equiv u_{+(-)} \qquad v_{\downarrow(\uparrow)} \equiv v_{+(-)}$$

$u_{+(-)}$ are the handedness of particles while $v_{+(-)}$ are the handedness of antiparticles

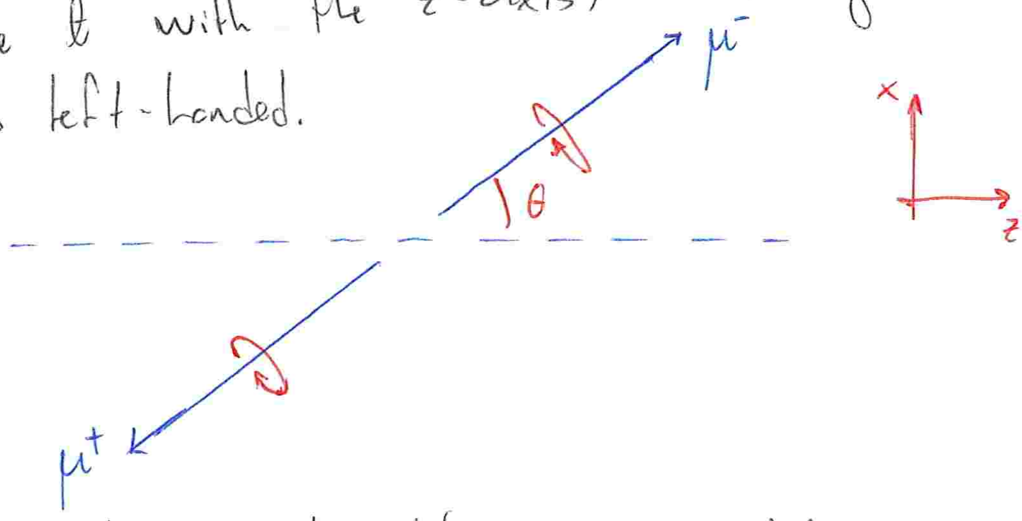
Right-handed spinor: $(\hat{p}\cdot\vec{\sigma})\xi = +\xi$ Left-handed spinor: $(\hat{p}\cdot\vec{\sigma})\xi = -\xi$

Let us consider the configuration where the electron has momentum in the +z and the positron in the -z direction. Suppose that the electron is right handed, $\xi = \begin{pmatrix}1\\0\end{pmatrix}$, and the positron is left-handed, $\xi = \begin{pmatrix}0\\1\end{pmatrix}$, also corresponding to the spin up in the $\hat{z}$ direction and both particles have the $(\hat{p}\cdot\vec{\sigma})\xi = +\xi$

$$u_+(p_1) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v_-(p_2) = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}$$

$$\bar{v}_-(p_2)\gamma^\mu u_+(p_1) = 2E\,(0\ \ -1)\,\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix} = -2E\,(0, 1, i, 0)$$

We now consider the muon current $\bar{u}_+(q_1)\gamma_\mu v_-(q_2)$ corresponding to the final state where the μ^- is emitted at an angle θ with the z-axis, and is right-handed, whilst the μ^+ is left-handed.



This current can be obtained by rotating the previous one (a four-vector) by an angle θ in the xz-plane, noting that

$$\bar{u}_+(q_1)\gamma_\mu v_-(q_2) = \left[\bar{v}_-(q_2)\gamma^\mu u_+(q_1)\right]^\dagger$$

$$R_{xz}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\bar{u}_+ \gamma_\mu v_-(q_2) = R_{xz}(\theta)\left[-2E(0,1,i,0)\right]^\dagger$$

$$= -2E\,(0, \cos\theta, -i, -\sin\theta)$$

same structure of a virtual photon with circular polarization in the $+z$ direction, $\varepsilon_{+\mu} = \frac{1}{\sqrt{2}}(0, 1, +i, 0)$

Finally,

$$\mathcal{M}_{+-\to+-}(e^-e^+\to\mu^-\mu^+) = \frac{e^2}{k^2}(2E)^2(-\cos\theta - 1) = -e^2(1+\cos\theta).$$

The complete set of non-zero amplitudes can be obtained in a similar way

$$\mathcal{M}_{+-\to+-} = \mathcal{M}_{-+\to-+} = -e^2(1+\cos\theta),$$

$$\mathcal{M}_{+-\to-+} = \mathcal{M}_{-+\to+-} = -e^2(1-\cos\theta).$$

As we could expect,

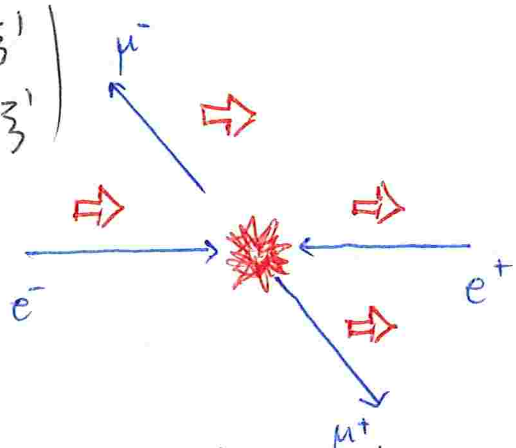
$$|\mathcal{M}_{+-\to+-}|^2 + |\mathcal{M}_{+-\to-+}|^2 + |\mathcal{M}_{-+\to-+}|^2 + |\mathcal{M}_{-+\to+-}|^2 = 4e^4(1+\cos^2\theta).$$

When we average over the incoming spins we recover the previous result for $\overline{|M|^2}$.

As a final remark, let us comment on the nonrelativistic limit $\rightarrow$ $E \gtrsim m_\mu$, but $E \gg m_e$

the electron current remains the same and the muon spinors become

$$u(q_1) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad v(q_2) = \sqrt{m}\begin{pmatrix} \xi' \\ -\xi' \end{pmatrix}$$

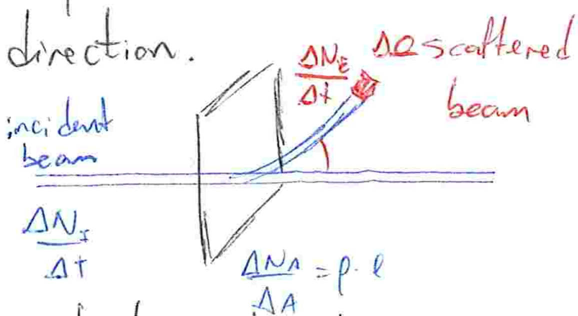


and

$$M_{+-\to\xi\xi'} = -2e^2\,\xi^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi'$$

Note that there is no angular dependence in the amplitude; it is a s-wave amplitude ($L = 0$). Since the angular momentum is conserved it requires that the total spin of the final state is equal to 1, i.e., both the spin of the muon and the antimuon must point in the $+z$ direction.

Cross Section

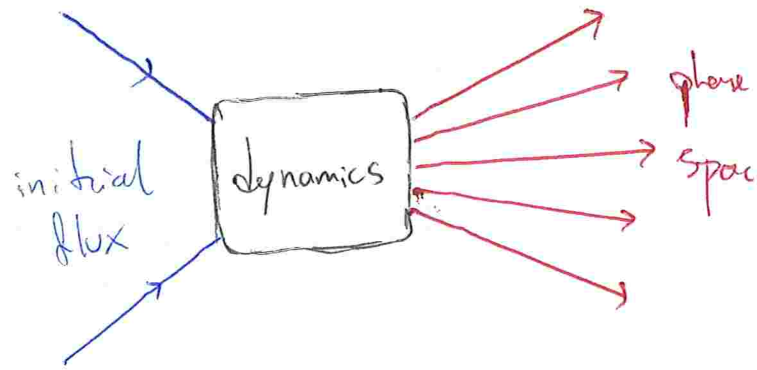


Differential cross section is defined as the probability of observing a particle scattered in a given quantum state, per unit of solid angle, when a target is hit by a flow of one particle per unit surface.

$$\frac{d\sigma}{d\Omega} = \frac{\Delta N_E / \Delta t}{(\Delta N_I / \Delta t) \cdot (\Delta N_A / \Delta A)}$$

The probability of transition from the initial state to the final state is defined by the invariant amplitude $\mathcal{M}$ that we just evaluated. It contains all the Physics of the reaction, i.e., the dynamics related to the scattering process itself and the interaction between the particles. The cross section will be given by the integration of $\overline{|\mathcal{M}|^2}$ over the phase space of all final state particles and divided by the initial flow of particles

For a process $a + b \to 1 + 2 + \dots$ we write the cross section as



$$d\sigma = \frac{1}{\mathcal{F}} \overline{|\mathcal{M}|^2}\, dR_n .$$

When there are m_i identical particles of the kind i in the final state, we must multiply by a statistical factor S

$$S = \prod_i \frac{1}{m_i!} .$$

The initial state flux is given by

$$\mathcal{F} = 2\, \lambda^{1/2}(s, m_a^2, m_b^2) \qquad \lambda(x,y,z) = (x-y-z)^2 - 4yz$$

where $s = (p_a + p_b)^2$ is the total energy in the center-of-mass reference frame.

For massless particles in the initial state,

$$\mathcal{F} = 2\sqrt{\lambda(s,0,0)} = 2s.$$

The integral over final-state momenta is manifestly Lorentz invariant and it is known as relativistically invariant n-body phase space

$$dR_n = \left(\prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}\right)(2\pi)^4\delta^{(4)}\left(p_a+p_b-\sum_{i=1}^{n}p_n\right).$$

Remark:

$$\left.\begin{array}{l}[\mathcal{F}] = E^2\\ [R_n] = E^{2n-4}\\ [\sigma] = E^{-2}\end{array}\right\} \quad [|M|^2] = E^{2-n} \rightarrow$$

in a $2\to 2$ process, the invariant amplitude is dimensionless

For a $2\to 2$ process in the c.m.s. we have

$$\int dR_2 = \frac{1}{(2\pi)^2}\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{8s}\int d\Omega.$$

If we have massless particles in the final state, we get $\sqrt{\lambda(s,0,0)} = s$ and

$$dR_2 = \frac{1}{8(2\pi)^2}d\Omega.$$

Thus, for massless particles, the cross section for the reaction $a+b \to 1+2$ in the c.m.s is given by

$$d\sigma = \frac{1}{2s} \overline{|M|^2} \frac{1}{8(2\pi)^2} d\Omega$$

$$= \frac{1}{64\pi^2 s} \overline{|M|^2} d\Omega$$

$d\Omega = \sin\theta \, d\theta \, d\phi$

Therefore, the cross section for the process $e^+e^- \to \mu^+\mu^-$ at the high energy limit is

$\alpha = \frac{e^2}{4\pi}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} e^4 (1 + \cos^2\theta) = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

integrating

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

Electron Magnetic Moment

The Dirac equation in the presence of an EM field is written as

$$(i\not\partial - e\not A - m)\psi = 0$$

minimal coupling

$\partial_\mu \to \partial_\mu + ie A_\mu$

If we "multiply" by $(i\not\partial - e\not A + m)$ we get

$$[(i\partial_\mu - eA_\mu)(i\partial_\nu - eA_\nu)\gamma^\mu\gamma^\nu - m^2]\psi = 0.$$

Defining $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, some algebra yields

$$\left[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} - m^2\right]\psi = 0.$$

Remark:

$$\sigma_{0i} = -i\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \qquad \sigma_{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

$$F_{0i} = E_i \qquad F_{ij} = -\epsilon_{ijk}B_k$$

$$\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} = -e\begin{pmatrix} (\vec{B}+i\vec{E})\cdot\vec{\sigma} & 0 \\ 0 & (\vec{B}-i\vec{E})\cdot\vec{\sigma} \end{pmatrix} \xrightarrow[\vec{E}=\vec{0}]{} -e\vec{B}\cdot\vec{\sigma} = -2e\vec{B}\cdot\vec{S}$$

represents the coupling of the electron spin with the magnetic field

When coupled to the EM field, the Dirac field no longer satisfies KG, which becomes

$$\left[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - m^2\right]\phi = 0.$$

Therefore, $\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$ expresses the difference between how scalar and spinor fields interact with an electromagnetic field.

Let us now consider the non-relativistic limit of the Dirac equation in the presence of an external magnetic field.

$$\left[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) + 2e\vec{B}\cdot\vec{S} - m^2\right]\psi = 0$$

Momentum space:

$$i\partial_\mu \to p_\mu$$

$$(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) \to (E - e\phi)^2 - (\vec{p} - e\vec{A})^2$$

The Dirac equation becomes, in momentum space,

$$\frac{1}{2m}(E - e\phi)^2\psi = \left[\frac{1}{2m}(\vec{p} - e\vec{A})^2 + \frac{m}{2} - \boxed{2}\left(\frac{e}{2m}\right)\vec{B}\cdot\vec{S}\right]\psi$$

We can identify the intrinsic magnetic moment given by

$$\vec{\mu}_{Dirac} = -\frac{e}{m}\vec{S}.$$

This is a non-trivial prediction of the Dirac theory, contrary to classical expectation. A classical particle with charge e and orbital angular momentum $\vec{L}$ generates a magnetic moment

$$\vec{\mu}_{Classic} = -\frac{e}{2m}\vec{L}.$$

the Dirac equation predicts an extra factor of $g = 2$ w.r.t. the classical replacement $\vec{L} \to \vec{S}$

$\hookrightarrow$ gyromagnetic ratio

The anomalous magnetic moment expresses the deviation of the value $g=2$ predicted by Dirac's theory

$$g = 2 + \frac{a}{2}$$

or

$$a = \frac{g-2}{2}.$$

no loops

At tree level, $a=0$. The theoretical evaluation of the magnetic moment of the electron involves a very large number of complex Feynman diagrams with several loops.

The adimensional coefficients A_n are universal → don't depend on lepton flavour

$$a = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$

There are also terms that depend on lepton mass (non-universal) and on the hadronic and weak contributions

Some Results for the calculation of A_n

$A_1 = +0.5$ 1 diagram (1948, Schwinger)

$A_2 = -0.328\,478\,965\,579\,193$ 7 diagrams (1957, Karplus, Kroll, Petermann)

$A_3 = +1.181\,241\,456\,587$ 72 diagrams (1996, Laporta, Remiddi)

$A_4 = -1.912\,245\,764$ 891 diagrams (2003, Aoyama, Hayakawa, Kinoshita, Nio)

$A_5 = +6.737(159)$ 12 672 diagrams (2012, Aoyama, Hayakawa, Kinoshita, Nio)

$$(g/2)_{exp} = 1.001\ 159\ 652\ 180\ 73(28)\ [0.28\ ppt]$$

$$(g/2)_{the} = 1.001\ 159\ 652\ 181\ 606(11)(12)(229)\ [2.3\ ppt]$$

↳ discrepancy between theory and experiment is 2.4σ.

QED's Impact on QFT

QED inspired the extension of QFT to other interactions

→ Some challenging problems

- The renormalization of physical parameters requires that these infinities arise in only a limited number of ways, such as corrections for masses, charge, etc. QED, for instance, is a renormalizable theory. The theory's coupling constant has to be small enough to allow perturbative expansion. In QED $\alpha = 1/137$ is small

→ Weak interactions

- Fermi's theory was not renormalizable: higher order infinities could not be removed
- The coupling constant is small ($\sim 10^{-5} - 10^{-7}$) but the short range required massive vector bosons

→ Strong interactions

- The coupling constant is ~ 1 and it was impossible to make reliable predictions

Those problems made many theorists in the late 1950's / early 1960's to devote themselves to the study of symmetry principles and conservation laws and S-matrix theory (Wheeler - Heisenberg)

Weak Interactions

→ several theories were proposed to describe weak interactions with different gauge groups

→ Glashow [SU(2) × U(1)]: unifies electromagnetic (A) and weak (W) interactions and predicts neutral current (Z)

- weak force is short-ranged and therefore the intermediate bosons must be very heavy (80-90 GeV)
- W and Z mass prevented gauge invariance and, consequently, renormalizability

→ Higgs Mechanism: ingenious way to restore renormalizability and lead to Standard Model

Strong Interactions

→ gauge theory [SU(3)] propose the existence of massless gluons as carrier of the interaction

→ the theory has the property of asymptotic freedom

- at short distances (high energies) the coupling constant becomes small and the theory becomes renormalizable

→ Quantum Chromodynamics (QCD) was born

Symmetries

Elements of Group Theory → the main tool to describe symmetries

Group theory plays a fundamental role in modern particle physics because it is related to symmetry. The fact that the laws of physics do not change if we apply some transformations in the system is essential to identify conservation laws or even determine the dynamics of the system. Group theory is related to this type of symmetry.

Definition [Group]:

A group G is a set of elements $\{g_1, g_2, g_3, \ldots\}$ plus an operation $\cdot$ that satisfy

1. Closure: if $g_1, g_2 \in G$, then $g_1 \cdot g_2 \in G$
2. Associativity: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$
3. Existence of Identity: $g \cdot \mathbb{1} = \mathbb{1} \cdot g = g$, with $\mathbb{1}$ being unique
4. Existence of Inverse: $\forall g \in G \ \exists g^{-1}: g \cdot g^{-1} = g^{-1} \cdot g = \mathbb{1}$ $\square$

Abelian Group: if $g_1 \cdot g_2 = g_2 \cdot g_1$ the group is called abelian, and otherwise it is nonabelian

A representation is a mapping that takes group elements $g \in G$ into linear operators R that preserve the operation rule of the group

$$R(g_1) \cdot R(g_2) = R(g_1 \cdot g_2)$$

and
$$R(\mathbb{1}) = 1.$$

A group can be a function of one or more inputs that we call parameters. Suppose that the elements $g \in G$ are specified by a finite set of n parameters: $\{\theta_1, \dots, \theta_n\}$. therefore the group elements can be written as

$$g = G(\theta_1, \dots, \theta_n)$$

and the identity is the element with $\theta_i = 0, \ \forall i$:

$$\mathbb{1} = G(0, \dots, 0).$$

Lie Groups

Lie groups represent the most useful class of groups: it depends on a finite set of continuous parameters θ_i and on the derivatives of the group elements wrt all these parameters exist.

Let us define the generators T_i by

$$T_i \propto \left.\frac{\partial g}{\partial \theta_i}\right|_{\theta_i = 0}$$

In QM we look for unitary operators and an unitary representation of the group, and thus T_i must be hermitian

$$T_i = -i \left. \frac{\partial g}{\partial \theta_i} \right|_{\theta_i = 0}$$

A representation of any Lie group can be written as

$$R_L[g(\theta)] = e^{i\theta_i T_i} \simeq 1 + i\theta_i T_i$$

↳ if T_i is hermitian, $R_L^{\dagger} R_L = 1$

The generators of a group form a vector space: we can add two generators to obtain a third and we can multiply generators by scalars and still have a generator of the group. The generators of a group can be used to represent other vector spaces, e.g., the Pauli matrices can be used to describe any 2×2 matrix.

A group can be characterized by their generators. They satisfy a commutation relation

$$[T_i, T_j] = \underbrace{f_{ijk}}_{\text{structure constants of the group}} T_k$$

Lie Algebra of the group

The rank of a group is the number of operators in the algebra that can be simultaneously diagonalized.

The Casimir operator is a nonlinear function of the generators of a group that commutes with all the generators. The number of Casimir operators for a group is given by the rank of the group. A Casimir operator is an invariant.

Examples of Lie Groups

GL(N): N×N matrices with non-vanishing determinant

Orthogonal Groups: SO(N)

$$SO(N) \equiv \{ O \in GL(N) ;\ O^T O = \mathbb{1} = O O^T,\ \det O = +1 \}$$

SO(3) is a representation of rotations in three dimensions. It has 3 parameters: the 3 angles defining rotations about the axes x ($R_x(\xi)$), y ($R_y(\varphi)$) and z ($R_z(\theta)$).

We can find the generators for each group parameter. For instance,

$$J_z = -i \left. \frac{dR_z}{d\theta} \right|_{\theta=0} = -i \frac{d}{d\theta} \left. \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or $R_z(\theta) = 1 + i\theta J_z$ with the algebra

$$[J_i, J_j] = i \epsilon_{ijk} J_k$$

Levi-Civita

Unitary Groups: SU(N)

$$SU(N) \equiv \{ U \in GL(N) ;\ U^\dagger U = \mathbb{1} = U U^\dagger,\ \det U = 1 \}$$

Unitary groups play a special role in QFT and particle physics: unitary transformations preserve the probabilities for different transitions among the states,

$$\langle \varphi | \psi \rangle = \langle \varphi | U^\dagger U | \psi \rangle = \langle \varphi | \psi \rangle .$$

The dimension of SU(N), or the number of generators, is given by N^2-1. For SU(N) the rank is $N-1$.

U(1)

Just a complex phase $U = e^{-i\theta}$. Several Lagrangians in field theory are invariant under a U(1) which is associated with the conservation of quantum numbers like the electrical charge.

SU(2) → rank 1 → only σ_3 is diagonal

The generators are the Pauli matrices σ_i. The Lie algebra is the known relation

$$[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k .$$

An element of SU(2) is $U = e^{\frac{i\sigma_i\theta_i}{2}}$ that transforms a two component spinor. When applied to a 3-component vector, the form of the generators are the matrices $\Sigma_k = -i(\epsilon_{ij})_k$, e.g.,

$$\Sigma_3 = -i\,\epsilon_{ij3} = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv J_z , \quad U = 1 + i\Sigma_3\theta = \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv R_z(\theta)$$

This is the third component of the angular momentum for a spin-1 particle: to build the N-dim irreducible representation of SU(2) use the spin $s = \frac{N-1}{2}$ angular momentum matrices.

SU(3)

It has 8 generators that are called the Gell-Mann matrices such that

$$[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$$

with the non-zero structure constants

$$f_{123} = 1$$

$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = 1/2$$

$$f_{456} = f_{678} = \sqrt{3}/2$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

A Brief Historical Account

400 BC Plato the shape of the world should be that of a perfect sphere, and all movement should be in perfect circles and with uniform velocity (Timaeu 33B-34B)

310 BC Aristarchus of Samos Archimedes in The Sand Reckoner said

"His hypotheses are that the fixed stars and the Sun remain unmoved, that the Earth revolves around the Sun on the circumference of a circle, the Sun lying in the middle of the orbit, and that the sphere of fixed stars, situated about the same center as the Sun, is so great that the circle in which he supposes the Earth to revolve bears such a proportion to the distance of the fixed stars as the center of the sphere bears to its surface."

This hypothesis was repelled in favor of Plato's circular dogma until Kepler showed that the orbits were elliptical rather than circular. The illusion and preconception of the circle has delayed the evolution of astronomy by two millenia!

1596 Kepler In Mysterium Cosmographicum suggests that the orbits of the planets were defined by platonic solids

1600/1800 Various Works on the symmetry of crystals

1687 Newton In Principia the first law establishes the conservation of momentum due to translational invariance (homogeneity of space)

1860 Pasteur Discovers the connection between optical activity and the structure of molecules

1878 Cayley Group abstract concept

1893 Lie and Engel Publish Theorie der Transformationsgruppen

1886/1904 FitzGerald, Lorentz, Larmor and Poincaré Introduce Lorentz transformations that make Maxwell's equations invariant. Lorentz group plus spacetime translations is called Poincaré group

1905 Einstein Provides the physical hypotheses behind Lorentz's transformations, creating the theory of Special Relativity

1918 Noether Demonstrates that symmetries are related to conservation laws

1918 Weyl Introduces the classic unification theory between gravitation and electromagnetism that includes the gauge invariance that leads to the conservation of electric charge

1927/28 London and Weyl Introduce gauge transformations in quantum theory

1929 Bethe Derives the separation of atomic levels that comes from the symmetry of the fields in the crystals

1930 Wigner Studies the effects of symmetry of molecular configurations on the vibration spectrum

1931 Wigner Introduces the time reversal (T) symmetry in quantum theory

1931 Pauling Studies the theory of ~~orbital~~ chemical bonds using orbital symmetry

1935 Fock Derives the hydrogen atom spectrum from the SO(4) symmetry

1936 Heisenberg Introduces charge conjugation (C) as a symmetry operation connecting particles and antiparticles

1939 Wigner Studies the unitary representations of the Poincaré group and establishes the classification of all relativistic wave equations and the transformation properties of quantum fields

1954 Yang and Mills: Introduce local isospin transformations as internal symmetries, i.e., field operator transformations that depend on the point of spacetime

1956/57 Yang and Lee: Propose that weak interactions break parity

1959/61 Heisenberg, Goldstone and Nambu: Suggest that the lowest energy (vacuum) state of QFT may break Hamiltonian symmetry: spontaneous symmetry breaking (SSB). The existence of the Goldstone bosons would be a consequence. In 1964, Higgs showed a mechanism which makes these bosons disappear and give mass to vector bosons.

1961/62 Gell-Mann and Neeman: Propose the group SU(3) as the symmetry of the particles (Eightfold Way). In 1964, Gell-Mann and Zweig proposed a new set of particles, quarks, which are organized according to SU(3)

1964 Cronin and Fitch: Showed experimentally that weak interactions can break CP (charge and parity conjugation) symmetry

1967/68 Glashow, Weinberg and Salam: Showed that electroweak interactions can be described by a theory based on $SU(2) \times U(1)$

Noether's Theorem

Symmetries are transformations that leaves the EoM invariant. If applied to a solution of the EoM, they lead to another solution with the same boundary conditions.

Amalia "Emmy" Noether
German mathematician
The main body of her work was on the creation of modern abstract algebra, but her theorem connecting the invariance of a system under a symmetry transformation and conservation laws is of most impact in Physics

A sufficient condition for the EoM to be invariant is for the Lagrangian to be invariant. When the EoM are obtained from a variational principle it is possible to establish a relantionship between the symmetries and the integrals of motion. The transformation can include changes in the coordinates and alterations in the fields itself.

The Noether theorems establish that

All continuous symmetry must correspond to a conservation law. Every conservation law must be matched by a continuous symmetry.

$$\boxed{\text{Continuous Symmetry}} \Longleftrightarrow \boxed{\text{Conservation Law}}$$

To apply to field theory, we can consider that to each transformation which leaves the action invariant corresponds a combination of the fields and their derivatives, which is also invariant.

Let us suppose that T_a, for $a = 1, 2, \dots, N$, where N is the group dimension, are the generators in some matrix representation of ϕ with ϕ_i components,

$$\phi'_i(x') = \left[e^{\omega^a T_a}\right]_{ij} \phi_j(x) \simeq \phi_i(x) + \delta\omega^a (T_a)_{ij}\, \phi_j(x)$$

where $\delta\omega^a$ are constant parameters.

Symmetry transformation $\Rightarrow \delta S = 0 \Rightarrow$ one conserved current for each group generator giving rise to a conserved charge $\rightarrow$ integral of motion

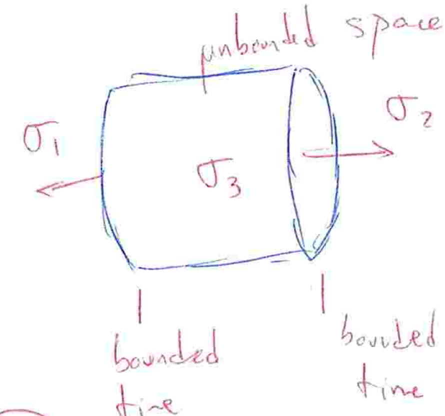
$$J_a^{\ \mu} = -\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\bar{\delta}\phi_i}{\delta\omega^a} + \mathcal{L}\frac{\delta x^\mu}{\delta\omega^a}\right]$$

with $\partial_\mu J_a^{\ \mu} = 0$, where

$$\frac{\bar{\delta}\phi_i(x)}{\delta\omega^a} = \frac{\delta\phi_i(x)}{\delta\omega^a} - \frac{\partial\phi_i(x)}{\partial x^\mu}\frac{\delta x^\mu}{\delta\omega^a} = (T_a)_{ij}\,\phi_j(x)$$

and $\frac{\delta x^\mu}{\delta\omega^a}$ depends on the particular coordinate transformation.

The Gauss Theorem in 4D reads



$$\int_\Sigma \partial_\mu J_a^{\ \mu}\, d\Sigma = \int_\sigma J_a^{\ \mu}\, d\sigma_\mu$$

$$\int_{\sigma_1} J_a^{\ \mu}\, d\sigma_\mu - \int_{\sigma_2} J_a^{\ \mu}\, d\sigma_\mu + \int_{\sigma_3} J_a^{\ \mu}\, d\sigma_\mu = 0 \rightarrow \partial_\mu J_a^{\ \mu} = 0$$

$\int_{\sigma_3} J_a^{\ \mu}\, d\sigma_\mu \rightarrow 0 \rightarrow$ if $J_a^{\ \mu}$ vanishes fastly enough at infinity

$$\int_{\sigma_1} J_a^{\ \mu}\, d\sigma_\mu = \int_{\sigma_2} J_a^{\ \mu}\, d\sigma_\mu .$$

$\therefore\ Q_a = \int_{\sigma_n} J_a^{\ \mu}\, d\sigma_\mu$ doesn't depend on σ_n, taken at different t_n $\rightarrow$ Q_a is time-independent

Alternatively, since $\partial_\mu = (\partial^0, \nabla)$,

$$0 = \int_V \partial_\mu J_a^\mu \, d^3x = \int_V (\partial_0 J_a^0 + \nabla \cdot \vec{J}_a)$$

$$= \frac{d}{dt} \int_V J_a^0 \, d^3x + \int_{\partial S} \vec{J}_a \cdot d\vec{a}$$

0

Thus, to each group generator, corresponds a charge

$$Q_a = \int J_a^0(x) \, d^3x \quad \text{which is conserved} \quad \dot{Q}_a = 0.$$

Translations and Energy-Momentum Conservation

$$x'^\mu = x^\mu + \delta x^\mu$$
$$= x^\mu + \frac{\delta x^\mu}{\delta a^\alpha} \delta a^\alpha$$

$\alpha = 0, 1, 2, 3$

$$\frac{\delta x^\mu}{\delta a^\alpha} = \delta^\mu_\alpha$$

Fields (Lorentz tensors and spinors) are unaffected by translations: $\frac{\delta \phi}{\delta a^\alpha} = 0$. Therefore,

$$\bar{\delta}\phi = -(\partial_\mu \phi)\, \delta^\mu_\alpha \, \delta a^\alpha = -(\partial_\alpha \phi)\, \delta a^\alpha.$$

The Noether current related to invariance under translations is

$$T_\alpha{}^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \delta^\mu_\alpha \mathcal{L} \right)$$

The translation algebra index α appears as a spacetime index and $J_\alpha^{\ \mu}$ is a second order tensor

$$T^{\mu\nu} = g^{\mu\alpha} T_\alpha^{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial^\mu \phi - g^{\mu\nu} \mathcal{L}$$

$\hookrightarrow$ stress-energy tensor or energy-momentum tensor of the field ϕ

The conserved charge associated with time translations is the Hamiltonian

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x$$

The conserved charges associated with spatial translations are

$$p^i = \int T^{0i} d^3x = -\int \pi \, \partial_i \phi \, d^3x$$

which is the (physical) momentum carried by the field.

Nota Bene: not to be confused with the canonical momentum

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Discrete Symmetries

Quantum Numbers

$\hookrightarrow$ quantized property of a particle that is conserved in particle reactions like collision, disintegration, etc

$\hookrightarrow$ there are additive and multiplicative quantum numbers

Let us consider a process (i.e. scattering, decay) where the initial states I transform in the final state(s) F. Let us suppose that the quantum numbers before the reaction is given by $N_1^{(I)}, N_2^{(I)}, \dots,$ and after the reaction by $N_1^{(F)}, N_2^{(F)}, \dots$.

If an additive q. n. is conserved in a given reaction, it means that

$$\sum_i N_i^{(I)} = \sum_i N_i^{(F)}.$$

$\sum_i N_i$ is conserved

For a multiplicative q. n.,

$$\prod_i N_i^{(I)} = \prod_i N_i^{(F)}$$

$\prod_i N_i$ is conserved

Parity → inversion of all 3 spatial coordinates of a system

$$\vec{x} = (x, y, z) \xrightarrow{P} -\vec{x} = (-x, -y, -z)$$

In order to explore the meaning of this operation, let us put the system in interaction with a measuring system. We suppose the system is described by

$$\mathcal{L}_{free} + J_\mu(x) A^\mu(x)$$

EM current of a given field, For the Dirac field, $e\bar{\psi}\gamma_\mu\psi$

A^μ is an external classical field that, under parity,

$$A^{\mu}(\vec{x},t) \xrightarrow{P} \left(A^0(-\vec{x},t), -\vec{A}(-\vec{x},t)\right) = A_{\mu}(-\vec{x},t).$$

If Parity is conserved, the EoM and the commutation relations remain the same to preserve the quantum dynamics ($[P, \mathcal{L}]=0$).

This requires

$$P\,\mathcal{L}_0(\vec{x},t)\,P^{-1} = \mathcal{L}_0(-\vec{x},t),$$

$$P\,J_{\mu}(\vec{x},t)\,P^{-1} = J^{\mu}(-\vec{x},t).$$

For a scalar field,

$$P\,\phi(\vec{x},t)\,P^{-1} = \pm\,\phi(-\vec{x},t).$$

in QFT, the eigenvalue of parity is a property of particles called the intrinsic parity of the particle, and is a multiplicative quantum number

↳ the sign $\pm$ represents the intrinsic ~~particle~~ parity of the scalar field

→ + for scalar

→ − for pseudo-scalar

↳ bosons have the same intrinsic parity for both particles and antiparticles

Vector Algebra and Physical Examples

	Vector Algebra	Physics	Parity
Scalar	M	m, φ, T, q, t	$+$
Pseudo-Scalar	$\vec{A}\cdot(\vec{B}\times\vec{C})$	$h=\vec{S}\cdot\hat{p},\ \Phi=\vec{B}\cdot d\vec{\sigma}$	$-$
Vector	$\vec{V}$	$\vec{E}, \vec{v}, \vec{p}, \vec{F}$	$-$
Pseudo-Vector	$\vec{A}\times\vec{B}$	$\vec{B}, \vec{L}=\vec{r}\times\vec{p}, \vec{S}$	$+$

In the case of the spinor field,

$$P\psi(\vec{x},t)P^{-1} = \gamma^0\psi(-\vec{x},t)$$

convenient to work in the Dirac representation, where

$$\gamma_0^{Dirac} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

For a massless ~~particle~~ fermion travelling in the $\pm z$ direction, the Dirac spinors are

$$u_\uparrow = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix},\quad u_\downarrow = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \mp 1 \end{pmatrix},\quad v_\uparrow = \sqrt{E}\begin{pmatrix} 0 \\ \mp 1 \\ 0 \\ 1 \end{pmatrix},\quad v_\downarrow = \sqrt{E}\begin{pmatrix} \pm 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

where u and v represent particles and antiparticles, respectively, and $\uparrow$ and $\downarrow$ correspond to spin up (right-handed) and spin down (left handed) respectively.

Thus,

$$\gamma_0 u_\uparrow(-z) = +u_\uparrow(z), \quad \gamma_0 u_\downarrow(-z) = +u_\downarrow(z), \quad \gamma_0 v_\uparrow(-z) = -v_\uparrow(z), \quad \gamma_0 v_\downarrow(-z) = -v_\downarrow(z).$$

→ particles with spin ½ have positive parity
→ antiparticles with spin ½ have negative parity

A vector boson has spin 1 and negative parity (1^-). The most famous example is the photon. A pseudo vector has spin 1 and positive parity (1^+).

↳ usual notation: $J^P \equiv \text{Spin}^{\text{Parity}}$

Parity is not always conserved. It is conserved in the EM and strong interactions, but broken (i.e., not conserved) in the weak interaction. When we write down Lagrangians to describe the different interaction terms it is very important to know how the various Dirac field bilinears transform under parity

	Scalar	Pseudo-Scalar	Vector	Pseudo-Vector	Tensor
Bilinear	$\bar\psi\psi$	$i\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$\bar\psi\sigma^{\mu\nu}\psi$
Parity	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Charge Conjugation

↳ the operation that converts particles in antiparticles and vice-versa

Charge conjugation reverses the sign of all quantum numbers including charges, lepton number, hypercharge and all the "charge-like" numbers which characterize the particle

The conditions for charge conjugation invariance are

$$C \mathcal{L}_0(x)\, C^{-1} = \mathcal{L}_0 ,$$
$$C J_\mu C^{-1} = -J_\mu .$$

Charged scalar field:

$$C \phi(x)\, C^{-1} = \phi^*(x)$$
$$C \phi^*(x)\, C^{-1} = \phi(x)$$

Dirac field:

$$C \psi_a(x)\, C^{-1} = \underbrace{C_{ab}}_{\text{charge conjugation matrix}} \bar{\psi}_b(x)$$
$$C \bar{\psi}_a(x)\, C^{-1} = -\psi_b(x)\, C^{-1}_{ba}$$

charge conjugation is conventionally defined to take a fermion with a given spin orientation into an antifermion with the same spin orientation

We can implement C as a unitary linear operator

$$C \psi(x) C = -i\gamma^2 \psi^*(x) = -i\gamma^2 \left(\psi^\dagger\right)^T - i\left(\bar{\psi}\gamma^0\gamma^2\right)^T$$

$$-i\gamma^2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \rightarrow -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Using the Dirac spinors for a massless fermion travelling in the +z direction

$$-i\gamma_2 u_\uparrow^* = u_\downarrow, \quad -i\gamma_2 u_\downarrow^* = u_\uparrow, \quad -i\gamma_2 v_\uparrow^* = -v_\downarrow, \quad -i\gamma_2 v_\downarrow^* = -v_\uparrow,$$

and the bilinears become

	Scalar	Pseudo-Scalar	Vector	Pseudo-Vector	Tensor
Bilinear	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$
Charge Conjugation	$+1$	$+1$	-1	$+1$	-1

Time Reversal

↳ classically time reversal invariance means that if we change the sign of time, the velocities change direction and the system goes from what was the final state to the initial state

In QM, the corresponding operator must be anti-linear or anti-unitary. In fact, $\langle F|I\rangle = \langle I|F\rangle^*$ and if we want $\langle T\psi_F|T\psi_I\rangle = \langle\psi_I|\psi_F\rangle$, T must involve the complex conjugation operation.

For a spinor field,

$$T\psi(t,\vec{x})T = -\gamma^1\gamma^3\psi(-t,\vec{x}),$$

and the bilinears become

	Scalar	Pseudo-Scalar	Vector	Pseudo-Vector	Tensor
Bilinear	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$
Time Reversal	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$

CPT

It is a fundamental theorem in QFT that the product TCP is an invariance of any theory that satisfies the following general conditions

→ the theory is local and covariant for the Lorentz transformations

→ the theory is quantized using the usual relation between spin and statistics, i.e., commutators for bosons and anti-commutators for fermions

This theorem due to Lüdus, Zumino, Pauli and Schwinger has an important consequence that if one of the discrete symmetries is not preserved then another one must also be violated to preserve the invariance of the product.

Review of the discrete symmetries

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

Internal Symmetries $\longrightarrow$ gauge, vector, axial

Gauge Transformation

Electrodynamics Background

We can make some changes in the scalar and vector potentials φ and $\vec{A}$ without changing the field equations and the physical fields $\vec{E}$ and $\vec{B}$.

$$\vec{B} = \nabla \times \vec{A} \implies B^i = \epsilon^{ijk} \partial^j A^k$$

we can add any gradient of a scalar function to $\vec{A}$ without changing $\vec{B}$ nor the Maxwell equations

χ is a scalar function

$$A^i \to A'^i = A^i + \partial^i \chi$$

gauge transformation

$$B'^i = \epsilon^{ijk} \partial^j (A^k + \partial^k \chi)$$

$$= \epsilon^{ijk} \partial^j A^k + \underbrace{\epsilon^{ijk}}_{\text{antisymmetric}} \underbrace{\partial^j \partial^k}_{\text{symmetric}} \chi = B^i$$

We can make different choices of gauge by implementing a gauge transformation. If we impose $\nabla \cdot \vec{A} = \partial^i A^i = 0$, we call this the Coulomb gauge. If we consider the four vector $A^\mu = (\varphi, A^i)$, the Lorenz condition becomes $\partial_\mu A^\mu = \frac{\partial \varphi}{\partial t} + \partial^i A^i = 0$.

EM admits different configurations of the potential which yield identical observables (EM fields and Maxwell equations). Our physical description thus contains some inherent freedom which allows us to make different choices accordingly to a given situation (choice of gauge). A transformation from one description to another is called a gauge transformation, and the underlying invariance is called a gauge invariance.

Question: can we make a unique choice of χ for all points of space (i.e., the whole Universe)?

We know that the zero of the EM potential is arbitrary, but is this choice valid for the whole Universe? If zero volts is set differently in São Paulo and in Rio de Janeiro, then problems will occur if you run a conducting wire between their respective electrical grounds. Therefore, $A(x)$ must be chosen with a function $\chi(x)$ which varies from place to place.

Thus we need to distinguish between global (unique everywhere in the Universe) and local (different for particular points) gauge transformations.

Gauge Invariance in QM

we know EM can be described by the potentials $A^\mu = (\varphi, \vec{A})$,

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}.$$

If we make a gauge transformation G,

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi$$

$$\varphi \to \varphi' = \varphi - \frac{\partial \chi}{\partial t}$$

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla \chi$$

the EM fields remain the same

$$\vec{E} \to \vec{E}' = \vec{E}, \qquad \vec{B} \Rightarrow \vec{B}' = \vec{B}.$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

The Lorentz force can be derived from the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi,$$

and, with the usual prescription $\vec{p} \to -i\nabla$, we get the Schrödinger equation for a particle in an EM field

$$\left[\frac{1}{2m}(-i\nabla - q\vec{A})^2 + q\varphi\right]\psi(\vec{x},t) = i\frac{\partial\psi}{\partial t}(\vec{x},t),$$

which can be written as

$$\frac{1}{2m}(-i\vec{D})^2\psi = iD_0\psi$$

$$\frac{\partial}{\partial t} \to D_0 = \frac{\partial}{\partial t} + iq\varphi$$

$$\nabla \to \vec{D} = \nabla - iq\vec{A}$$

Question: if we make the transformation

$$(\varphi, \vec{A}) \xrightarrow{G} (\varphi', \vec{A}')$$

does ψ', which is the solution of

$$\frac{1}{2m}(-i\vec{D}')^2\psi' = iD_0'\psi'$$

describes the same Physics?

Answer: No, just if, at the same time, we make the transformation

$$\psi' = e^{iq\chi}\psi,$$

with the same χ used for the EM fields.

As we can see,

$$\vec{D}'\psi' = \left[\nabla - iq(\vec{A} + \nabla\chi)\right] e^{iq\chi}\psi,$$

$$= e^{iq\chi}\nabla\psi + \cancel{iq\nabla\chi\, e^{iq\chi}\psi}$$

$$- iq\vec{A}\, e^{iq\chi}\psi - \cancel{iq\nabla\chi\, e^{iq\chi}\psi},$$

$$= e^{iq\chi}\vec{D}\psi,$$

and, in the same way,

$$D_0'\psi' = e^{iq\chi}D_0\psi.$$

Now ψ, $\vec{D}\psi$ and $D_0\psi$ all transform in the same way.

Therefore,

$$\frac{1}{2m}(-i\vec{D}')^2\psi' = \frac{1}{2m}(-i\vec{D}')(-i\vec{D}'\psi'),$$

$$= \frac{1}{2m}(-i\vec{D}')\left[-i\, e^{iq\chi}\vec{D}\psi\right],$$

$$= e^{iq\chi}\frac{1}{2m}(-i\vec{D})^2\psi,$$

$$= e^{iq\chi}(iD_0)\psi = iD_0'\psi'.$$

Now ψ' and ψ describe the same Physics, since $|\psi|^2 = |\psi'|^2$.

Thus, in order to guarantee the observables, we just have to make sure to substitute

$$\nabla \to \vec{D}, \quad \frac{\partial}{\partial t} \to D_0$$

in all observables. For example, the current

$$\vec{J} \propto \psi^* (\nabla \psi) - (\nabla \psi)^* \psi$$

which becomes also gauge invariant with this substitution,

$$\psi^{*\prime} (\vec{D}' \psi') = \psi^* e^{-iq\chi} e^{iq\chi} (\vec{D} \psi)$$

$$= \psi^* (\vec{D} \psi).$$

Internal Vector and Axial Symmetries

The Dirac Lagrangian is invariant under rotating the phase of the spinor: $\psi \to e^{-i\alpha} \psi$. This gives rise to the current

$$J_V^\mu = \bar{\psi} \gamma^\mu \psi.$$

The left and right-handed components of the spinor transform in the same way under this symmetry. The current is conserved ($\partial_\mu \partial_V^\mu = 0$) and the conserved quantity arising from this symmetry is

$$Q = \int \bar{\psi} \gamma^0 \psi \, d^3x = \int \psi^\dagger \psi \, d^3x$$

interpreted as electric charge, or particle number, for fermions

When $m=0$, the Dirac Lagrangian admits an extra internal symmetry which rotates left and right-handed fermions in opposite directions,

$$\psi \to e^{i\alpha\gamma^5}\psi \quad \text{and} \quad \bar{\psi} \to \bar{\psi}\, e^{i\alpha\gamma^5}.$$

This gives rise to the axial-vecter current

$$J^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

It is conserved only when $m=0$, since

$$\partial_\mu J^\mu_A = 2im\,\bar{\psi}\gamma^5\psi.$$

When the theory is coupled to gauge fields the axial transformation remains a symmetry of the classical Lagrangian. However, it does not survive the quantization process. It is the archetypal example of an anomaly: a symmetry of the classical theory that is not preserved in the quantum theory.

Particles and Interactions

what we achieved in 100 years of Particle Physics

Periodic Table $\Rightarrow$ Standard Model

$\sim 10^{-9}$ m $\quad\quad$ $\sim 10^{-18}$ m

ν_e	ν_μ	ν_τ	W	H
e	μ	τ	Z	
u	c	t	γ	
d	s	b	g	

leptons, quarks and ~~gluons~~ bosons

Leptons

→ electron (1897)
Thomson

→ muon (1937)
Neddermeyer and Anderson

→ tau (1975)
Perl

→ electron neutrino (1956)
Cowan and Reines

→ muon neutrino (1962)
Lederman

→ tau neutrino (2000)
DONUT Collab., Fermilab

$\boxed{\nu_e}$ $\boxed{\nu_\mu}$ $\boxed{\nu_\tau}$

$\boxed{e}$ $\boxed{\mu}$ $\boxed{\tau}$

Quarks

→ up, down, strange (1964) → theoretical prediction
Gell-Mann, Zweig

→ charm (1974)
Ting, Richter

→ bottom (1977)
Herb et al.

→ top (1995)
CDF and DØ Collab., Fermilab

$\boxed{u}$ $\boxed{c}$ $\boxed{t}$

$\boxed{d}$ $\boxed{s}$ $\boxed{b}$

Intermediate Vector Bosons

$\boxed{W}$ $\boxed{Z}$ $\boxed{\gamma}$ $\boxed{g}$

- → photon (1923)
  Compton
- → W and Z (1983)
  UA1/2 Collab., CERN
- → gluon (1979)
  DESY Collaborations

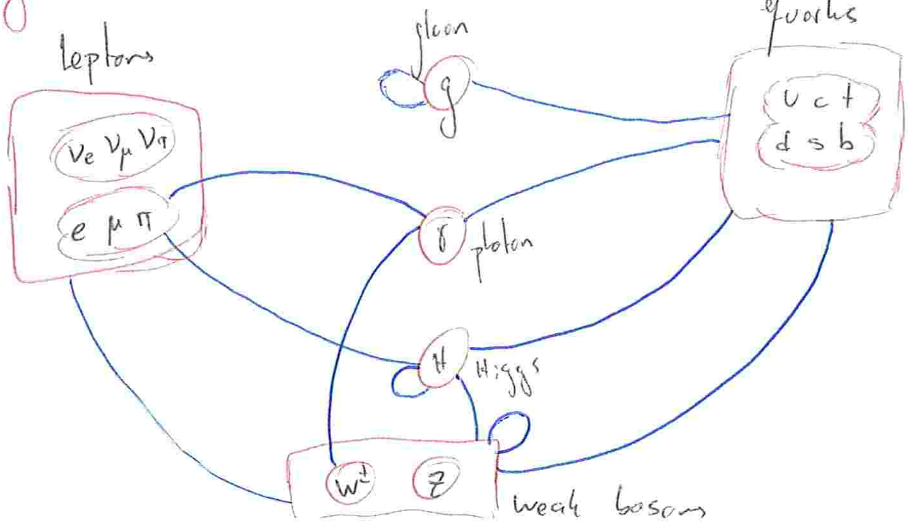
Higgs Boson

$\boxed{H}$

- → theoretical proposal by Higgs et al. (1964)
- → employed in a particle physics model by Weinberg (1967)
- → discovered by Atlas and CMS Collab., CERN (2012)

Interactions

	Strength	Range	Behaviour	A/R (attractive / repulsive)	Carrier
Gravitational	1	∞	$1/r^2$	A	"Graviton"
Electromagnetic	10^{36}	∞	$1/r^2$	A & R	Photon
Weak	10^{25}	$< 10^{-18}$ m	$\frac{1}{r} e^{-M_W r}$	A & R	W^+ W^- Z^0
Strong	10^{38}	$< 10^{-15}$ m	$\sim r$	A & A	Gluons



A Historical Perspective

Nobel

N 1896 Becquerel evidence for spontaneous radioactivity effect

N 1897 Thomson discovery of the electron (cathode rays)

N 1899 Rutherford evidence for α and β components of uranium radiation

N 1900 Planck quantum hypothesis and explanation of the blackbody radiation. The beginning of the Quantum Era

N 1905 Einstein explanation of the photoelectric effect, invention of the special relativity. The beggining of the Relativistic Era

N 1911 Milikan measurement of the electron charge

1911 Rutherford evidence for the atomic nucleus (α + Atom)

N 1913 Bohr invention of the quantum theory of atonic spectra

1914 Chadwick first observation that the β spectrum is continuous (see below); indirect evidence on the existence of neutral penetrating particles (neutrino)

1919 Rutherford discovery of the proton, constituent of the nucleus (α + Atom $\to p + X$)

N 1923 Compton the photon is an elementary particle ($\gamma + C \to \gamma + C$)

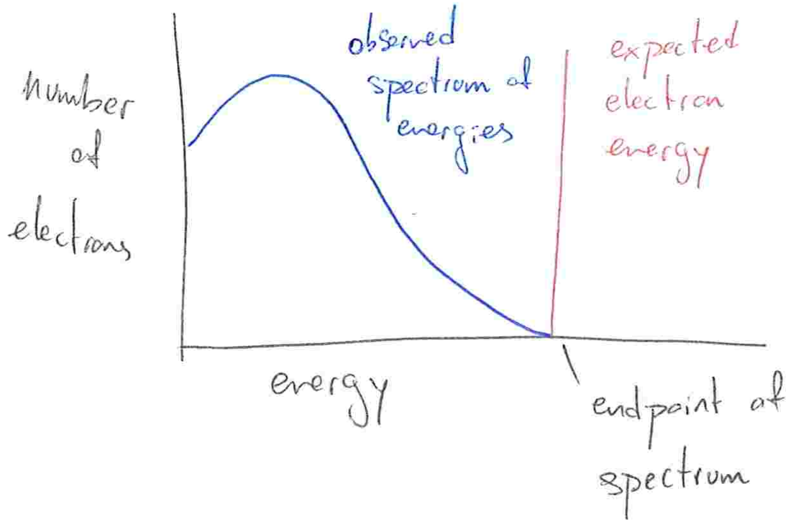
N 1923 de Broglie corpuscular-wave dualism for electrons

N 1925 Pauli discovery of the exclusion principle

N 1925 Heisenberg foundation of quantum mechanics

N 1926 Schrödinger creation of quantum mechanics

1927 Ellis & Woosler confirmation that the β spectrum is continuous



$E_{e^-} = \underbrace{\Delta M \cdot c^2}$

no other possible energy (no energy available)

if two particles are emmitted, a continuous spectrum is possible

1927 Dirac foundations of quantum electrodynamics (QED)

discovery of the relativistic wave equations for electrons

N 1928 Dirac discovery of the magnetic moment of the electron and prediction of the magnetic moment of the electron

1929 Skobelzyn the birth of cosmic rays particle physics

1930 Pauli proposed existence of a neutral fermion emitted in β decay (neutrino)

1930 Oppenheimer self-energy of the electron - first UV divergence in QED

max energy for infinity treatments

1931 Dirac prediction of positron and anti-proton

↳ theory breaks after some energy scale

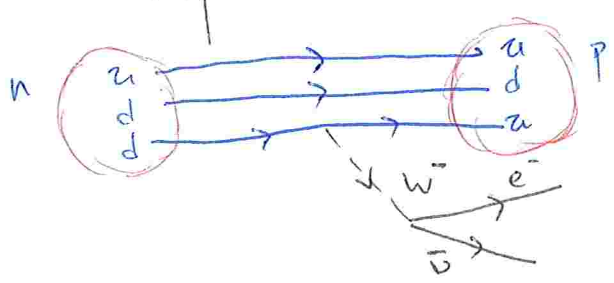
N 1932 Anderson first evidence for the positron (cosmic rays)

N 1932 Chadwick first evidence for the neutron ($\alpha + Be \rightarrow N + n$)

1932 Heisenberg suggestion that nuclei are composed of protons and neutrons

1934 Pauli explanation of continuous electron spectrum of β decay

Proposal for the neutrino: $n \rightarrow p + e^- + \bar{\nu}_e$



1934 Fermi field theory for β decay, assuming the neutrino. In analogy with "the theory of radiation that describes the emission of a quantum of light from an excited atom"

$$\mathcal{L}_{QED} = e J_\mu A^\mu \longrightarrow \mathcal{L}_{weak} = \frac{G_F}{\sqrt{2}} \left(\bar{\psi}_p \gamma_\mu \psi_n \right) \left(\bar{\psi}_e \gamma^\mu \psi_\nu \right)$$

1936 Gamov and Teller extension of Fermi theory for transition with $\Delta J^{nuc} \neq 0$

Fermi can't describe all interactions, so we extend the theory

$$\mathcal{L}_{weak} = \frac{G_F}{\sqrt{2}} \sum_i C_i \left(\bar{\psi}_p \Gamma^i \psi_n \right) \left(\bar{\psi}_e \Gamma^i \psi_\nu \right)$$

with Γ^i representing all Dirac bilinears

$$\Gamma^S = \mathbb{1}, \quad \Gamma^P = \gamma_5, \quad \Gamma^V_\mu = \gamma_\mu, \quad \Gamma^A_\mu = \gamma_\mu \gamma_5, \quad \Gamma^T_{\mu\nu} = \sigma_{\mu\nu}.$$

Nuclear transitions

$(A+B)^2$

$A^2 + 2AB + B^2$

$\Delta J = 0$: $S \cdot S$ and/or $V \cdot V$

we must have in order to explain all nuclear transitions

$\Delta J = 0, \pm 1 \; (0 \not\to 0)$: $A \cdot A$ and/or $T \cdot T$

However interference between them ($\propto \frac{m_e}{E_e}$) increase emission of low energy electrons, which was not observed. Therefore,

$(S \cdot S$ or $V \cdot V)$ and $(A \cdot A$ or $T \cdot T)$

axial vector or pseudo-vector

($\Gamma^P \sim 0$ in the non-relativistic limit).

1937 Neddermeyer and Andersen first evidence for the muon

1937 Majorana Majorana neutrino theory

1937 Bloch and Nordsieck treatment of IR divergences → infinite number of low energy photons

1940 Williams and Roberts first observation of muon decay: $\mu^- \to e^- + (\bar{\nu}_e + \nu_\mu)$

1943 Heisenberg invention of the S-matrix formalism

N 1943 Tomonaga creation of the covariant quantum electrodynamic theory

1947 Bethe first theoretical calculation of the Lamb shift in non-relativistic QED

N 1947 Kusch and Foley first measurement of g-2 for the electron

$$g_e = 2(1 + 1.19 \cdot 10^{-3})$$

N 1947 Lattes, Occhialini & Powell confirmation of the π^- and first evidence for pion decay

$$\pi^\pm \to \mu^\pm + (\nu_\mu)$$

1947 Pontecorvo hint on the universality of the Fermi weak interactions (decay ~ capture)

1947 Rochester & Butler first evidence for V events (strange particles)

1948 Schwinger first theoretical calculation of g-2 for the electron

N 1948 Feynman, Schwinger, Tati & Tomonaga creation of the covariant theory of QED

1949 Dyson equivalence of Tomonaga, Schwinger and Feynman methods

1949 Wheeler, Tiomno, Lee, Rosenbluth & Yang: proposal of the universality of the Fermi weak interactions

β decay: $n \to p + e^- + \bar{\nu}_e$

μ decay: $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$

μ capture: $\mu^- + p \to \nu_\mu + n$

same nature with same coupling constant

$$G_F = 1.03 \cdot 10^{-5} / M_p^2$$

1950's particles, particles and particles

$\pi^0, K^{\pm}, \Lambda, K^0, \Delta^{++}, \Xi^-, \Sigma^{\pm}, \bar{\nu}_e, \bar{p}, K_{L,S}, \bar{n}, \Sigma^0 \ldots$

1950 Ward Ward identity in QED

1953 Stückelberg, Gell-Mann invention and exploration of renormalization group

1954 Yang & Mills introduction of local gauge isotopic invariance in QFT

1955 Alvarez, Goldhaber; Birge et al. $\theta-\tau$ puzzle

decay lenght $\Gamma_\theta = \Gamma_\tau$ and $M_\theta = M_\tau$, so θ and τ seem to be the same state, however

$$\theta^+ \to \pi^+ + \pi^0 \longrightarrow J^P = 0^+$$

$$\tau^+ \to \pi^+ + \pi^+ + \pi^- \longrightarrow J^P = 0^-$$

this suggests that parity is being violated

1955 Lehmann, Symanik & Zimmermann beginnings of the axiomatic field theory of the S-matrix

1955 Nishijima classification of strong particles and prediction of Σ^0 and Ξ^-

N 1956 Lee and Yang proposals to test spatial parity conservation in weak interactions

1957 Wu et al. first evidence for parity nonconservation in weak decays

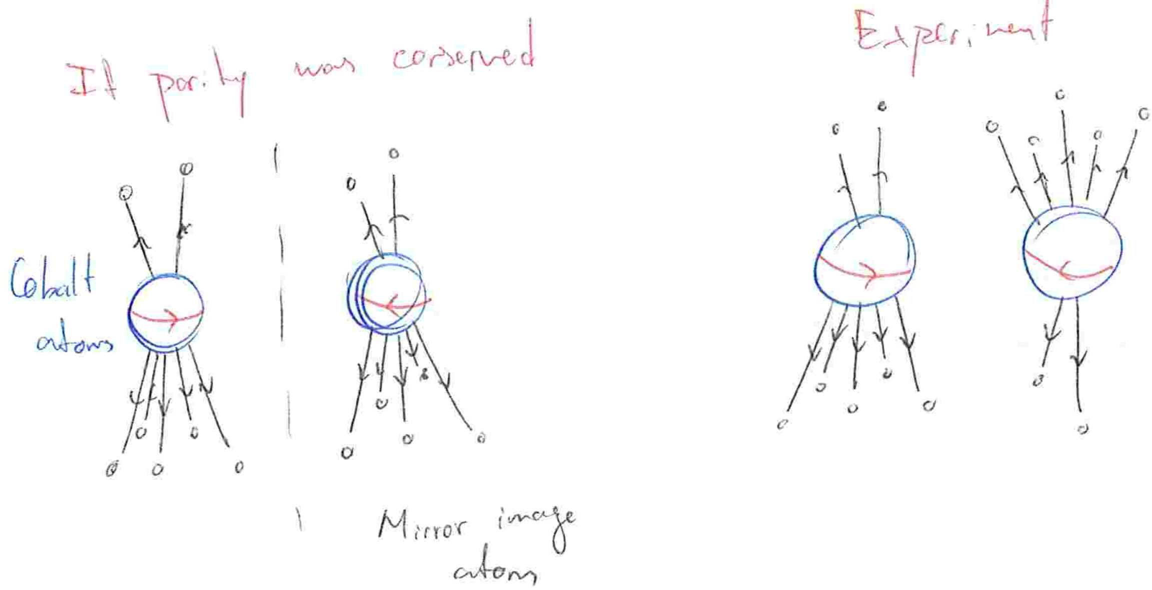
Measurementa of the angular distribution of the electrons in

$${}^{60}_{27}Co\ (\text{polarized}) \to {}^{60}_{28}Ni + e^- + \bar{\nu}_e \quad \begin{cases} J_{Co} = 5 \\ J_{Ni} = 4 \end{cases}$$

The strong magnetic field polarizes the Co ($J^z_{Co} = +5$). Since $J_{Ni} = 4$ and the angular momentum is conserved

$$J^z_{Co} = J^z_{Ni} + J^z_e + J^z_{\bar{\nu}} = +5 \Rightarrow J^z_e = J^z_{\bar{\nu}} = 1/2$$

The concentration of the electrons in the negative-z direction indicated a preference for left-handed electrons. Parity would be conserved only if, in the decay of nuclei, equal numbers of electrons were emitted in both directions



1957 Garwin, Lederman and Weinrich, Friedman and Telegdi confirmation of parity violation in weak decays. Measurement of the electron asymmetry (muon polarization) in the decay

$$\pi^+ \to \mu^+ + \nu_\mu$$

muon decay $\hookrightarrow e^+ + \nu_e + \bar{\nu}_\mu$

1957 Frauenfelder et al. Further conservation of parity non conservation in weak decays. Measurement of the longitudinal polarization of the electron ($\vec{\sigma}_e \cdot \vec{p}_e$) emitted in β decay.

$$^{60}Co \to e^- \,(\text{long. polar.}) + \bar{\nu}_e + X$$

Electrons are mostly left-handed

Conclusion: $\not P$ and $\not C$, but CP is conserved, and we need a γ^5

$$\mathcal{L}_{weak} \to \frac{G_F}{\sqrt{2}} \sum_i C_i \left(\bar{\Psi}_p \Gamma^i \Psi_n\right) \left[\bar{\Psi}_e \Gamma^i (1 \pm \gamma^5) \Psi_\nu\right]$$

1957 Salam, Lee and Yang, Landau two-component theory of neutrino. The neutrino is <u>either</u> right or left-handed

Since it was know that electrons (positrons) involved in weak decays are left (right) handed

$$J^i_{lept} = \left[\bar{\Psi}_e \Gamma^i (1 \pm \gamma^5) \Psi_\nu\right] \to \left[\bar{\Psi}_e \frac{(1+\gamma^5)}{2} \Gamma^i (1 \pm \gamma^5) \Psi_\nu\right]$$

→ if $\Gamma^i = V$ or A, then $\{\gamma_5, \Gamma^i\} = 0$ and ν is left

→ if $\Gamma^i = S$ or T, then $[\gamma_5, \Gamma^i] = 0$ and ν is right

condition for a non-vanishing Lagrangian

The neutrino helicity defines the structure of the weak current

1957 Schwinger, Lee and Yang development of the idea of the intermediate vector boson in weak interaction

0 angular momentum in partial wave expansion

Since the Fermi interaction is "point-like" it is a s-wave interaction.

Partial wave unitarity requires that

$$\sigma < \frac{4\pi}{p^2_{cm}}$$

Since the dimension of the Fermi constant is $[G_F] = M^{-2}$ one must have

$$\sigma \sim G_F^2 \, p_{cm}^2 .$$

Therefore, the Fermi theory violates unitarity for $p_{cm} \simeq 300$ GeV. This behaviour can be improved supposing that the interaction is transmitted by a intermediate vector boson

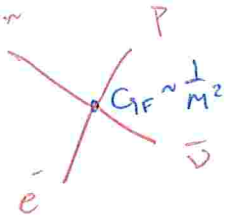
→ charged: β decay requires charge-changing currents

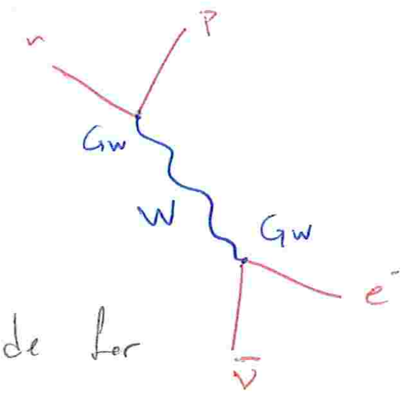
→ large mass: ~ "point-like" $\left[\Delta E \, \Delta t = M_W^2 c^2 \frac{R}{c} > \hbar \right]$

→ indefinite parity: to allow, e.g., a $(V-A)$ structure

The Fermi Lagrangian becomes

$$\mathcal{L}_{weak} = \frac{G_F}{\sqrt{2}} \left(J^\alpha J_\alpha^\dagger + h.c. \right) \rightarrow \mathcal{L}_{weak}^{W} = G_W \left(J^\alpha W_\alpha^+ + J^{\dagger\alpha} W_\alpha^- \right)$$





Comparing the invariant amplitude for μ-decay in the low energy limit

$$\mathcal{M}_{weak} = i \frac{G_F}{\sqrt{2}} J^\alpha(\mu) \, J_\alpha(e)$$

$$\mathcal{M}_{weak}^{W} = \left[i \, G_W \, J^\alpha(\mu) \right] \left[\frac{-i}{k^2 - M_W^2} \left(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{M_W^2} \right) \right] \left[i \, G_W \, J^\beta(e) \right]$$

At low energies

$$\mathcal{M}_{weak}^{W} \longrightarrow i \frac{G_W^2}{M_W^2} J^\alpha(\mu) \, J_\alpha(e) \implies G_W^2 = \frac{M_W^2 \, G_F}{\sqrt{2}} \text{ is dimensionless}$$

$\frac{1}{M^2}$ coupling constant: non-renormalizable

dimensionless coupling constant: could be renormalizable

M coupling constant: super renormalizable

However, at high energies the theory of IVB still violates unitarity. For instance, the cross section for $\nu\bar{\nu} \to W^+W^-$. Let us consider the W^+ polarizations

Rest frame $\qquad$ z-boost: $p^\mu = (E, 0, 0, p)$

$$\epsilon^\mu_{T_1}(0) = (0,1,0,0) \to \epsilon^\mu_{T_1}(0)$$

$$\epsilon^\mu_{T_2}(0) = (0,0,1,0) \to \epsilon^\mu_{T_2}(0)$$

$$\epsilon^\mu_{L}(0) = (0,0,0,1) \to \epsilon^\mu_{L}(p) = \left(\frac{|\vec{p}|}{M_W}, \frac{E}{M_W}\hat{p}\right) \simeq \frac{p^\mu}{M_W}$$

At high energies

$$\sigma(\nu\bar{\nu} \to W^+_T W^-_T) \to \text{constant}$$

$$\sigma(\nu\bar{\nu} \to W^+_L W^-_L) \to \frac{G_F^2 s}{3\pi}$$

which still violates unitarity!

1958 Goldhaber, Grodzins and Sunyar first evidence for the negative ν_e helicity

→ beta decays of $_{63}Eu^{152m}$ via e capture

- $_{62}Sm^{152*}$ and neutrino must have opposite momenta and opposite spin directions

→ Sm^{152*} decays emitting a gamma ray
  - the helicity of the recoil is transferred to γ

→ γ ressonantly excite the Sm^{152} in target

→ ν helicity is determined from rate of γ as a function of $\vec{B}$ direction

Weak interaction is V-A

1958 Feynman, Gell-Mann, Marshak & Sudarshan, Sakurai universal V-A weak interactions

$$J^{\mu}_{lept} = \left[\bar{\psi}_e \gamma^{\mu} (1 - \gamma_5) \psi_{\nu} \right]$$

1958 Leite Lopes estimates the mass of the hypothetical vector boson to be about 50 proton masses assuming the coupling with fermions has the same strenght as the electric charge

N 1959 Reines and Cowan confirmation of the detection of the $\bar{\nu}_e$ in

$$\bar{\nu}_e + p \rightarrow e^+ + n$$

$\hookrightarrow + e^- \rightarrow \gamma + \gamma$ and $n + {}^{108}Cd \rightarrow {}^{109m}Cd \rightarrow {}^{109}Cd + \gamma$

Towards the Standard Model

1961 Goldstone prediction of unavoidable massless bosons if global symmetry of the Lagrangian is spontaneously broken

1961 Salam and Ward invention of the gauge principle as basis to construct QFTs of interacting fundamental fields

N 1961 Glashow first introduction of the neutral intermediate weak boson (Z^0)

N 1962 Danby et al. first evidence of ν_μ from $\pi^{\pm} \rightarrow \mu^{\pm} + (\nu / \bar{\nu})$

1963 Cabibbo introduction of the Cabibbo angle and hadronic weak currents

$$J^{\mu}_{hadr} = \cos\theta_c \, J^{\mu}_{hadr}(d \to u) + \sin\theta_c \, J^{\mu}_{hadr}(s \to u) + h.c.$$

1964 Bjorken & Glashow proposal for the existence of a charmed fundamental fermion

N 1964 Higgs; Englert & Brout; Guralnik, Hagen & Kibble example of a field theory with spontaneous symmetry breakdown, no massless Goldstone boson and massive vector boson

N 1964 Christenson, Cronin, Fitch & Turlay first evidence of CP violation in the decay of K^0 mesons

N 1964 Salam et Ward Lagrangian for the electroweak synthesis estimation of the W mass

N 1964 Gell-Mann; Zweig introduction of quarks as fundamental building blocks for hadrons

1964 Greenberg; Han et Nambu introduction of color quantum number and colored quarks and gluons

1967 Kibble extension of the Higgs mechanism of mass-generation for non-abelian gauge field theories

N 1967 Weinberg Lagrangian for the electroweak synthesis and estimation of W and Z ~~synthesis~~ masses

1967 Faddeev et Popov method for construction of Feynman rules for Yang-Mills gauge theories

N 1968 Salam Lagrangian for the electroweak synthesis

1969 Bjorken invention of the Bjorken scaling behaviour

1969 Feynman birth of the partonic picture of hadron collisions

1970 Glashow, Iliopoulos et Maiani introduction of lepton-quark symmetry and the proposal of charmed quark (GIM mechanism)

1971 't Hooft rigorous proof of renormalizability of the Yang-Mills QFT with spontaneous broken gauge invariance

1973 Kobayashi et Maskawa CP violation is accommodated in the SM with six flavours

1973 Hasert et al. (CERN PS) first experimental indication of the existence of weak neutral currents

$$\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$$

$$\nu_\mu + N \to \nu_\mu + X$$

N 1973 Gross et Wilczek; Politzer discovery of asymptotic freedom property of interacting Yang-Mills field theories

1973 Fritzsch, Gell-Mann et Leutwyler invention of the QCD Lagrangian

1974 Bennenuti et al. (FNAL) confirmation of the existence of weak neutral currents ($\nu_\mu + N \to \nu_\mu + X$)

N 1974 Aubert et al. (Brookhaven, p + Be); Augustin et. al. (SLAC, e^+e^-) evidence for the J/ψ ($c\bar{c}$)

N 1975 Perl et al (SLAC) first indication of the τ lepton

1977 Herb et al. (FNAL) first evidence of Υ ($b\bar{b}$)

1979 Barber et al.; Brandelik et. al.; Berger et. al. (DESY, PETRA) evidence for the gluon jet in $e^+e^- \to 3$ jet

N 1983 Arnison et al. (CERN, UA1); Banner et. al. (CERN, UA2) evidence for the neutral intermediate boson Z^0

$$p + \bar{p} \to Z (\to \ell^+ + \ell^-) + X$$

N 1986 Van Dyck, Schwinberg et Dehmelt high precision measurement of the electron g-2

1987 Albajar et al. (CERN, UA1); Albrecht et al. (DESY, ARGUS) first evidence of $B^0 - \bar{B}^0$ mixing

1987 Koshiba et al. (Kamiokande II) observation of neutrino burst from SN1987a

1989 Abrams et al. (SLAC, MARK-II) first evidence that the number of light neutrinos is 3

1991 Adeva et al. (CERN, L3); Alexander et al. (CERN, OPAL); Abreu et al. (CERN, DELPHI); Decamp et al. (CERN, ALEPH) precise determination of the Z^0 parameters

1992 LEP Collaborations (ALEPH, DELPHI, L3 and OPAL) precise determination of the Z parameters

1992 Gallex Collaboration solar neutrino deficit at low energy

1995 Abe et. al. (FNAL, CDF); Abachi et al. (FNAL, DØ) confirmation of the top quark production

1997 LEP experiments measurements of the vector-boson triple couplings

N 1998 Koshiba et al. (Super Kamiokande Collaboration) solar and atmospheric neutrino experiments, discovery of neutrino oscillations

2001 Ahmad et al. (SNO Collaboration) direct evidence for neutrino flavor transformation from neutral current interactions

2001 Kodama et al. (DONUT Collaboration, Fermilab) discovery of the tau neutrino

2012 CERN, ATLAS et CMS Collaborations discovery of the Higgs boson

The Gauge Principle $\rightarrow$ from symmetries to interactions

Noether: Continuous Symmetry $\Longleftrightarrow$ Conservation Law

Is it possible that imposing a symmetry constraint the form of the force laws (interactions) are essentially determined?

$\hookrightarrow$ Symmetry $\overset{?}{\Longleftrightarrow}$ Dynamics

This happens in QED, a paradigmatic example of a successful QFT, where the existence and some properties of the gauge field (photon) follow from a principle of invariance under local gauge transformations of the $U(1)$ group

"Our basic postulate is that it should be possible to generate strong, weak and electromagnetic interaction terms by making local gauge transformations on the kinetic-energy terms in the free Lagrangian for all particles"

Salam et Ward (1961)

This was accomplished after some new ingredients were introduced

→ Weak Interactions

- spontaneous breakdown of the gauge symmetry and Higgs mechanism
  - ↳ incorporates heavy massive weak gauge boson without breaking the gauge invariance

→ Strong Interactions

- asymptotic freedom
  - ↳ allows to describe perturbatively the strong interaction ~~at~~ short distance

QED: A Prototype

Dirac free-Lagrangian

$$\mathcal{L}_{\psi} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

→ not invariant under local $U(1)$ gauge transformation

IEP

$$\psi \to \psi' = e^{-i\alpha(x)} \psi$$

$$\mathcal{L}_\psi \to \mathcal{L}'_\psi = \mathcal{L}_\psi + \bar{\psi} \gamma_\mu \psi (\partial^\mu \alpha)$$

However, if we introduce the gauge field A_μ through minimal coupling

$$D_\mu \equiv \partial_\mu + i e A_\mu$$

and require that A_μ transforms like

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e} \partial_\mu \alpha$$

we get

$$\mathcal{L}_\psi \to \mathcal{L}'_\psi = \bar{\psi}' [(i\partial\!\!\!/ - eA\!\!\!/') - m] \psi'$$

$$= \bar{\psi} e^{i\alpha} [i\partial\!\!\!/ - e(A\!\!\!/ + \frac{1}{e} \partial\!\!\!/ \alpha) - m] e^{-i\alpha} \psi$$

$$= \mathcal{L}_\psi - e \bar{\psi} \gamma_\mu \psi A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Since the strenght tensor is invariant under the gauge transformations, so is the Lagrangian for the free field

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

e.g. electrons

photon

Note that the coupling between ψ and A_μ arises naturally when we require the invariance under local gauge transf. of the kinetic terms in the free fermion Lagrangian.

Caution: a possible mass term is not invariant

$$\mathcal{L}^m_A \to \mathcal{L}^{m\,\prime}_A = -\frac{1}{2} A'_\mu A^{\mu\prime} \neq \mathcal{L}^m_A$$

↳ something else is necessary to describe massive vector bosons

Gauge Invariance for Non-Abelian Groups

1932 Heisenberg suggestion that nuclei are composed of protons and neutrons. Theory of nuclear exchange forces. Invention of nucleon isotopic spin.

Under nuclear interactions, protons and neutrons can be regarded as degenerated ↳ $M_p \simeq M_n$, EM interaction is negligible

Thus, any arbitrary combination of their wave function would be equivalent

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \to \psi' = U\psi$$

U is unitary to preserve probability

with $\det U = 1$, U represents the $SU(2)$ Lie group

$$U = \exp\left(-i\frac{\sigma^a}{2}\alpha^a\right) \simeq \mathbb{1} - i\frac{\sigma^a \alpha^a}{2}$$

↳ σ^a, $a = 1,2,3$ are the Pauli matrices

1954 Yang et Mills introduction of local gauge isotopic invariance in QFT

"The differentiation between a neutron and a proton is then a pure arbitrary process. As usually conceived, however, this arbitrariness is subject to the following limitation: once one chooses what to call a proton, what to call a neutron, at one space-time point, one is then not free to make any choices at other space-time points. It seems that this is not consistent with the localized field concept that underlies the usual physical theories" Yang et Mills (1954)

Therefore, we should preserve our freedom to choose no matter when or where we are

$$\alpha^a \rightarrow \alpha^a(x)$$

This procedure can be generalized to any non-Abelian group G with generators t_a satisfying the Lie algebra

$$[t_a, t_b] = i\, C_{abc}\, t_c$$ structure constants

$\mathcal{L}_\psi$ should be invariant under the matter field transformations

$$\psi \rightarrow \psi' = \Omega\, \psi \qquad \Omega = e^{-i T^a \alpha^a(x)}$$

T^a is a convenient representation of the generators t_a

Introducing one gauge field for each generator and the covariant derivative,

$$D_\mu \equiv \partial_\mu - i g T^a A^a_\mu ,$$

which transforms just like the matter field $D_\mu \psi \to \Omega (D_\mu \psi)$

This will ensure the invariance under the local non-Abelian gauge transformation for the terms containing the fields and its gradients as long as the gauge field transformation is

$$T^a A^a_\mu \to \Omega \left(T^a A^a_\mu + \frac{i}{g} \partial_\mu \right) \Omega^{-1}$$

or, infinitesimal, i.e. for $\Omega \simeq \mathbb{1} - i T^a \alpha^a (x)$

$$A^{a\,\prime}_\mu = A^a_\mu - \frac{1}{g} \partial_\mu \alpha^a + \underbrace{C_{abc}}_{\text{vanishes for Abelian groups}} \alpha^b A^c_\mu$$

Finally, we must generalize the strength tensor

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \underbrace{C_{abc}}_{\text{vanishes for Abelian groups}} A^b_\mu A^c_\nu$$

transforms like

$$F^{a\,\prime}_{\mu\nu} = F^a_{\mu\nu} + C_{abc} \alpha^b F^c_{\mu\nu}$$

Thus, the invariant kinetic term for bosons is invariant under the local gauge transformation

$$\mathcal{L}_A = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

Mass terms are not invariant!

$$A^a_\mu A^{a\mu} \to \left(A^a_\mu - \frac{1}{g}\partial_\mu \alpha^a + C_{abc}\alpha^b A^c_\mu\right)\left(A^{a\mu} - \frac{1}{g}\partial_\mu \alpha^a + C_{ade}\alpha^d A^{e\mu}\right)$$

↳ $F \propto (\partial A - \partial A) + g AA$ provides a new feature

↳ self-coupling of the gauge fields

$$\mathcal{L}_A \propto \underbrace{(\partial A - \partial A)^2}_{\text{propagator}} + g\underbrace{(\partial A - \partial A) AA}_{\text{triple}} + \underbrace{g^2 AAAA}_{\text{quartic}}$$

Spontaneous Symmetry Breaking → the relevance of the vacuum

Exact symmetries ⟹ exact conservation laws

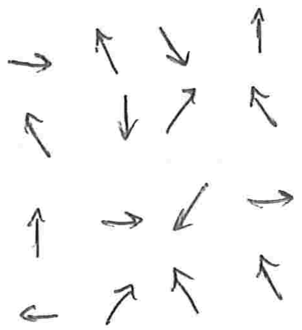
↳ both Lagrangian and vacuum are invariant

Some conservation laws are not exact → isospin, strangeness, etc

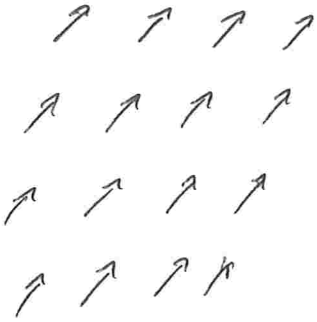
↳ can be described by adding to the invariant Lagrangian ($\mathcal{L}_{sym}$) a small term that breaks this symmetry ($\mathcal{L}_{SB}$) → $\mathcal{L} = \mathcal{L}_{sym} + \mathcal{L}_{SB}$

Another situation occurs when the system has a Lagrangian invariant but the vacuum is not invariant

Example: ferromagnet described by rotationally-invariant Lagrangian undergoing a spontaneous magnetization for $T < T_c$



paramagnetic $(T < T_c)$
disordered
rotation invariance



ferromagnetic $(T > T_c)$
ordered
no rotation invariance in 3D
↳ symmetry broken from $SO(3)$ to $SO(2)$

Nota bene: ordering is loss of symmetry

SSB for a Scalar Self-Interacting Real Field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \qquad V(\phi) = \underbrace{\frac{1}{2} \mu^2 \phi^2}_{\text{mass term}} + \frac{1}{4} \lambda \phi^4$$

In the theory of the phase transition of a ferromagnet, the Gibbs free energy density is analogous to $V(\phi)$ with ϕ being the spontaneous magnetization M. Notice that $\mathcal{L}$ is invariant under the transformation

$$\phi \to -\phi$$

↳ is the vacuum invariant as well?

The vacuum ϕ_0 can be obtained from the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left[(\partial_0 \phi)^2 + (\nabla \phi)^2 \right] + V(\phi)$$

Zero derivatives

$\phi_0 = \text{cte}$ corresponds to the minimum of $V(\phi)$ and consequently of the energy

$$\phi_0 \left(\mu^2 + \lambda \phi_0^2 \right) = 0.$$

$\lambda > 0$ to ensure $\mathcal{H}$ is bounded

minimum depends on sign of μ^2

$\mu^2 > 0$ usual mass term

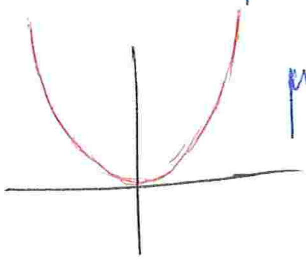
unique vacuum at $\phi_0 = 0$

$\mu^2 < 0$

two vacua at $\phi_0^{\pm} = \pm \sqrt{-\frac{\mu^2}{\lambda}}$

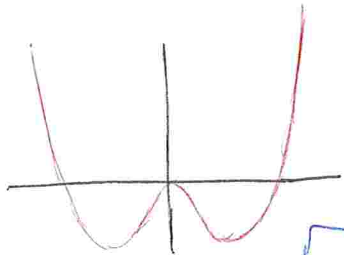
First case: vacuum is invariant under $\phi \to -\phi$

Second case: one of the vacua must be chosen and the choice spontaneously breaks the symmetry



$\mu^2 > 0$

$\phi_0 = 0$



$\mu^2 < 0$

$\phi_0^{\pm} = \pm \sqrt{-\frac{\mu^2}{\lambda}}$

As $\mathcal{L}$ is invariant under $\phi \to -\phi$, the choice between ϕ_0^+ and ϕ_0^- is irrelevant. However, once we choose one (e.g. $v = \phi_0^+$) the symmetry is spontaneously broken

$$\boxed{\mathcal{L} \text{ is invariant, but the vacuum is not}}$$

We can define new fields

$$\phi' \equiv \phi - v$$

in such a way that the vacuum is again at $\phi_0' = 0$. The Lagrangian in terms of the new fields become

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi' \partial^\mu \phi' - \frac{1}{2}\left(\sqrt{-2\mu^2}\right)^2 \phi'^2 - \lambda v \phi'^3 - \frac{1}{4}\lambda \phi'^4$$

→ ϕ' real and positive mass $M_{\phi'} = \sqrt{-2\mu^2}$

→ $\mathcal{L}$ lost the symmetry due to the ϕ'^3

Question: what if the symmetry was continuous?

Charged self-interacting scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^*, \phi), \quad V(\phi^*, \phi) = \mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2$$

↳ invariant under global phase transformation

$$\phi \to e^{-i\theta}\phi$$

Redefining the fields like

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

the Lagrangian becomes

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu\phi_1\,\partial^\mu\phi_1 + \partial_\mu\phi_2\,\partial^\mu\phi_2\right) - V(\phi_1,\phi_2)$$

$\hookrightarrow$ invariant under SO(2) rotations

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

For $\mu^2>0$, the vacuum is at $\phi_1=\phi_2=0$ and for small oscillations

$$\mathcal{L}_0 = \sum_{i=1}^{2}\frac{1}{2}\left(\partial_\mu\phi_i\,\partial^\mu\phi_i - \mu^2\phi_i^2\right)$$

which corresponds to a system with two scalar fields ϕ_1 and ϕ_2 with mass $m^2=\mu^2>0$. For $\mu^2<0$ there is a continuum of distinct vacua

$$\langle|\phi|^2\rangle = \frac{\langle\phi_1\rangle^2+\langle\phi_2\rangle^2}{2} = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}$$

$\mu^2<0$

The continuum of vacua is also invariant under SO(2), but this symmetry is spontaneously broken when we choose a particular vacuum. Let us choose $\phi_1=v$, $\phi_2=0$.

The new fields, suitable for small perturbations, are

$$\phi_1' = \phi_1 - v \quad \text{and} \quad \phi_2'=\phi_2$$

and the Lagrangian becomes

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi_1' \partial^\mu \phi_1' - \frac{1}{2}\underbrace{(-2\mu^2)}\,\phi_1'^{\,2} + \frac{1}{2}\partial_\mu \phi_2' \partial^\mu \phi_2' \quad + \text{interactions}$$

ϕ_1' is a real massive boson with mass $M_{\phi_1'} = \sqrt{-2\mu^2}$

ϕ_2' is a massless scalar boson

↓

Goldstone Boson

1961 Goldstone prediction of unavoidable massless bosons if global symmetry of the Lagrangian is spontaneously broken

Goldstone Theorem

When an exact continuous global symmetry is spontaneously broken, i.e., it is not a symmetry of the physical vacuum, the theory contains one massless scalar particle for each broken generator of the original group

Let us ~~cond~~ consider N_G real scalar fields ϕ_i belonging to a N_G-dimensional vector Φ

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - V(\Phi).$$

Suppose G is a continuous group that leaves the Lagrangian invariant with Φ transforming as

$$\delta\Phi = -i\,\alpha^a T^a\,\Phi.$$

α^a → arbitrary gauge parameters

Since the potential is invariant under G,

$$\delta V(\Phi) = \frac{\partial V}{\partial \phi_i}\delta\phi_i = -i\frac{\partial V}{\partial \phi_i}\alpha^a (T^a)_{ij}\,\phi_j = 0$$

$$\frac{\partial V}{\partial \phi_i}(T^a)_{ij}\,\phi_j = 0 \quad \rightsquigarrow \; N_G \text{ equations}$$

...

Taking a second derivative of this equation one gets

$$\frac{\partial^2 V}{\partial \phi_k \partial \phi_i} (T^a)_{ij} \phi_j + \frac{\partial V}{\partial \phi_i} (T^a)_{ik} = 0 .$$

If we evaluate this result at the vacuum state, Φ_0, which minimizes the potential, we get

$$\left.\frac{\partial^2 V}{\partial \phi_k \partial \phi_i}\right|_{\Phi = \Phi_0} (T^a)_{ij} \phi^0_j = 0$$

or, in terms of the mass matrix,

mass is the second derivative of the potential wrt field

$$M^2_{ki} (T^a)_{ij} \phi^0_j = 0$$

If, after we choose a ground state, a subgroup g of G, with dimension n_g, remains a symmetry of the vacuum, then

$$(T^a)_{ij} \phi^0_j = 0, \quad \forall\ a \leq n_g$$

$n_g \leq N_g$

for each generator of g, while for the $N_G - n_g$ generators that break the symmetry

$$(T^a)_{ij} \phi^0_j \neq 0, \quad \forall\ a > n_g$$

thus, $M^2_{ki} (T^a)_{ij} \phi^0_j = 0$ shows that there are $N_G - n_g$ zero eigenvalues of the mass matrix: the massless Goldstone bosons

The Higgs Mechanism → getting rid of the unobserved Goldstone bosons

"The controversy over the Goldstone theorem did not end with the publication of my two letters. [...] The resulting preprint led to an invitation from Dyson to give a talk at the Institute, Princeton in March 1966; there I confronted an audience containing axiomatic field theorists whose belief in the Goldstone theorem was based on the rigorous algebraic proof by Kastler, Robinson and Swieca. The next day Stanley Deser had arranged for me to talk at Harvard, where an equally skeptical audience awaited and, Sidney Coleman told me (in 1989) that they 'had been looking forward to tearing apart this idiot who thought he could get around the Goldstone theorem'. [...] Meanwhile, Brout, Englert and I tried fruitless to find an application in hadronic flavour symmetry breaking."

Higgs, IJMPA (2002)

"In the summer of 1965, I gave a talk at a small conference outside of Munich, that was sponsored by Heisenberg. He and the many other senior physicists at the conference thought these ideas were junk, and let me know with much enthusiasm that they felt that way. This evaluation was made very clear to me by Heisenberg, who arguably had discovered spontaneous symmetry breaking in the first place. This contributed considerably to my fear that I could not survive in Physics. Ken Wilson also spoke at this conference on his ideas of doing calculations on space-time lattices. He also got beaten up pretty badly."

Guralnik, IJMPA (2009)

"At some point in the fall of 1967, I think while driving to my office at MIT, it occurred to me that I had been applying the right ideas to the wrong problem. It is not the ρ meson that is massless: it is the photon. And its partner is not the A_1, but the massive intermediate bosons, which since the time of Yukawa had been suspected to be the mediators of the weak interactions."

Weinberg, Nobel Lecture (1980)

Question: is there a way out to the Goldstone Theorem and Goldstone bosons, which we do not have any evidence?

Yes.

1964 Higgs; Englert et Brout; Guralnik, Hagen et Kibble example of a field theory with spontaneous symmetry breakdown without massless Goldstone boson and with massive vector boson

$$\mathcal{L} \text{ with SSB} \oplus \text{Local Gauge Invariance}$$
$$\Downarrow$$
$$\text{GB disappears} \oplus \text{Massive Gauge Boson}$$

The Higgs, Englert et Brout and Guralnik, Hagen et Kibble Mechanism

$$\mathcal{L}_e = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - V(\phi^*, \phi) \qquad V(\phi^*, \phi) = \mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2$$

Let us require invariance under the local phase transformation

$$\phi \to e^{iq\alpha(x)}\phi .$$

In order to make the Lagrangian invariant, we introduce a gauge boson and the covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu + iq A_\mu \qquad A_\mu \to A'_\mu = A_\mu - \partial_\mu \alpha(x)$$

SSB for $\mu^2 < 0$, with vacuum $\langle |\phi|^2 \rangle = \frac{-\mu^2}{2\lambda} = \frac{v^2}{2}$.

the new fields, suitable for small perturbations, can be parametrized as

$$\phi = e^{\frac{i}{v}\phi'_2} \frac{(\phi'_1 + v)}{\sqrt{2}},$$

$$\simeq \frac{1}{\sqrt{2}}\left(\phi'_1 + v + i\phi'_2\right) = \phi' + \frac{v}{\sqrt{2}}.$$

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi'_1 \partial^\mu \phi'_1 - \frac{1}{2}(-2\mu^2)\phi_1'^2 + \frac{1}{2}\partial_\mu \phi'_2 \partial^\mu \phi'_2 + \text{interact}$$

$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^2 v^2}{2} A_\mu A^\mu + q v A_\mu \partial^\mu \phi'_2$$

particle content

- ϕ'_1 massive scalar $M_{\phi'_1} = \sqrt{-2\mu^2}$
- ϕ'_2 massless scalar (Goldstone boson)
- A_μ massive vector boson $M_A = qv$

To eliminate $qvA_\mu \partial^\mu \phi_2'$ we can choose the gauge parameter as

$$\alpha(x) = -\frac{1}{qv}\phi_2'$$

such that

$$\phi = e^{iq\left(-\frac{1}{qv}\phi_2'\right)} e^{\frac{i}{v}\phi_2'} \frac{\phi_1' + v}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\phi_1' + v).$$

In this gauge (unitary) the Goldstone boson disappears

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi_1' \partial^\mu \phi_1' - \frac{1}{2}(-2\mu^2)\phi_1'^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^2v^2}{2}A_\mu' A^{\mu'}$$

$$+ \frac{1}{2}q^2(\phi_1' + 2v)\phi_1' A_\mu' A^{\mu'} - \frac{\lambda}{4}\phi_1'^3(\phi_1' + 4v)$$

ϕ_1' massive scalar $(M_{\phi_1'} = \sqrt{-2\mu^2})$

A_μ' massive vector boson $(M_A = qv)$

↳ where is ϕ_2', the Goldstone boson?

Let us count the degrees of freedom

the degree of freedom correspond to the Goldstone boson was eaten by the new massive vector boson

↳ it corresponds to the longitudinal component of A_μ'

Initial			Final		
$\phi^{(x)}$	charged scalar	2	ϕ_1'	neutral scalar	1
A_μ	massless vector	2	A_μ'	massive vector	3
		4			4

The Non-Abelian Mechanism

For a non-Abelian group G of dimension N and generators T^a, we introduce N gauge bosons such that

$$\partial_\mu \to D_\mu = \partial_\mu - i g T^a B^a_\mu .$$

After the SSB, a sub-group g of dimension n remains as a symmetry of the vacuum

$$T^a_{ij} \phi^0_i = 0, \qquad \forall \ a \leq n .$$

We would expect $N-n$ massless Goldstone bosons. Parametrising the field as

$$\phi = (\tilde{\phi} + v)\, e^{\frac{i \phi^a_{GB} T^a}{v}}$$

where T^a are the $N-n$ broken generators that do not annihilate the vacuum.

Choose $\alpha^a(x)$ to eliminate ϕ^a_{GB} $\longrightarrow$ $N-n$ gauge bosons become massive

ϕ	massless scalar	S
B^a_μ	massless vector	$2N$
		$S+2N$

$\tilde{\phi}$	massive scalar	$S-(N-n)$
$\tilde{B}^a_\mu$	massive vector	$3(N-n)$
B^a_μ	massless vector	$2n$
		$S+2N$

Construction of the Model for Leptons

General Principles to Construct a Gauge Theory

→ choose the gauge group G (N generators)
→ add N vector fields (gauge bosons) in a specific representation
→ choose the representation of the matter ~~gauge~~ fields (elementary particles)
→ add scalar fields to give mass to (some) vector bosons
→ define the covariant derivative and write the most general renormalizable Lagrangian invariant under G which couples all these fields
→ shift the scalar fields such that the minimum of the potential is at zero
→ apply the usual techniques of QFT to verify the renormalizability
→ make predictions with the model
→ check with Nature if the model has anything to do with reality
→ make history or throw away the model and restart from the beginning!

Early Attempts

There were several attempts to construct a gauge theory for the (electroweak interaction

1957 Schwinger $G = O(3)$, gauge fields triplet (V^+, V^-, V^0)

$V^\pm$: weak bosons

V^0: photon

1958 Bludman $G = SU(2)$, $V^{\pm}, V^0$ weak bosons (V-A)
No unification. Anticipate neutral current

1961 Glashow
1964 Salam, Ward

$G = SU(2) \otimes U(1)$
4 gauge bosons: W^1, W^2, W^3 et B^0
$W^{\pm} = (W^1, W^2)$ charged weak bosons and
Z^0 et $\gamma = (W^3, B^0)$. Mass terms for
$W^{\pm}, Z^0$ put by hand.

1967 Weinberg
1968 Salam

Same gauge structure
Spontaneous symmetry breaking ⊕ Higgs mechanism
$W^{\pm}, Z^0$ ~~mass~~ mass preserve gauge invariance

"A model similar to ours was discussed by S. Glashow (1961); the chief difference is that Glashow introduces symmetry-breaking terms into the Lagrangian, and therefore gets less definite predictions. [...] Of course our model has too many arbitrary features for these predictions to be taken very seriously, but [...]"

Weinberg, PRL (1967)

The Gauge Structure

Right and Left-Handed Fermions

At high energies ($E \gg m$) the Dirac spinor is an eigenstate of γ^5. The helicity $+1/2$ (R) and helicity $-1/2$ (L) states satisfy $u(p,s)$ and

$$v(p,s) = C\,\bar{u}^T(p,s) = i\gamma_2 u^*(p,s)$$

$$u_{\substack{R\\L}} = \frac{1}{2}(1 \pm \gamma_5)u \qquad v_{\substack{R\\L}} = \frac{1}{2}(1 \mp \gamma_5)v$$

helicity projectors:

$$L = \frac{1}{2}(\mathbb{1} - \gamma_5), \qquad R = \frac{1}{2}(\mathbb{1} + \gamma_5)$$

$$L + R = \mathbb{1}, \quad L^2 = L, \quad R^2 = R, \quad RL = LR = 0$$

For the conjugate spinors

$$\bar{\psi}_L = (L\psi)^\dagger \gamma_0 = \psi^\dagger L^\dagger \gamma_0 = \psi^\dagger L \gamma_0 = \psi^\dagger \gamma_0 R = \bar{\psi} R$$

$$\bar{\psi}_R = \bar{\psi} L.$$

For the currents,

Mass term — mixes R et L

$$\bar{\psi}\psi = \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R$$

V−A — only L plays a role in weak interactions

$$\bar{\psi}_L \gamma^\mu \psi_L = \bar{\psi} R \gamma^\mu L \psi = \bar{\psi}\gamma^\mu L^2 \psi = \bar{\psi}\gamma^\mu L \psi = \frac{1}{2}\bar{\psi}\gamma^\mu(1-\gamma_5)\psi$$

V — R + L

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_R \gamma^\mu \psi_R + \bar{\psi}_L \gamma^\mu \psi_L$$

Let us investigate which gauge group would be able to unify the electromagnetic and weak interactions. Starting with the charged weak current for leptons,

$$J_\mu^+ = \bar{\ell}\gamma_\mu(1-\gamma_5)\nu$$
$$= 2\,\bar{\ell}_L\gamma_\mu\nu_L.$$

If we introduce the left-handed isospin doublet $(T=1/2)$,

$$\mathbb{L} \equiv \begin{pmatrix}\nu\\ \ell\end{pmatrix}_L = \begin{pmatrix}L\nu\\ L\ell\end{pmatrix} = \begin{pmatrix}\nu_L\\ \ell_L\end{pmatrix} \begin{matrix}\longrightarrow T_3 = +1/2\\ \longrightarrow T_3 = -1/2\end{matrix}$$

and the right-handed singlet $(T=0)$ → notice we are not considering ν_R

$$\mathbb{R} \equiv R\ell = \ell_R \longrightarrow T_3 = 0$$

The charged weak current can be written in terms of leptonic isospin currents

$$J_\mu^i = \bar{\mathbb{L}}\gamma_\mu\frac{\sigma^i}{2}\mathbb{L}$$

$$\begin{cases} J_\mu^1 = \frac{1}{2}(\bar{\nu}_L\ \bar{\ell}_L)\gamma_\mu\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\nu_L\\ \ell_L\end{pmatrix} = \frac{1}{2}\left(\bar{\ell}_L\gamma_\mu\nu_L + \bar{\nu}_L\gamma_\mu\ell_L\right)\\ J_\mu^2 = \frac{1}{2}(\bar{\nu}_L\ \bar{\ell}_L)\gamma_\mu\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\nu_L\\ \ell_L\end{pmatrix} = \frac{i}{2}\left(\bar{\ell}_L\gamma_\mu\nu_L - \bar{\nu}_L\gamma_\mu\ell_L\right)\\ J_\mu^3 = \frac{1}{2}(\bar{\nu}_L\ \bar{\ell}_L)\gamma_\mu\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\nu_L\\ \ell_L\end{pmatrix} = \frac{1}{2}\left(\bar{\nu}_L\gamma_\mu\nu_L - \bar{\ell}_L\gamma_\mu\ell_L\right)\end{cases}$$

Thus, the weak current that couples with W_μ^- becomes

$$J_\mu^+ = 2\left(J_\mu^1 - iJ_\mu^2\right)$$

Question: what do we do with the third (neutral) current J^3_μ?

If we define the hypercharge current by

$$J^Y_\mu = -\left(\bar{\mathbb{L}}\gamma_\mu \mathbb{L} + 2\bar{\mathbb{R}}\gamma_\mu \mathbb{R}\right),$$
$$= -\left(\bar{\nu}_L \gamma_\mu \nu_L + \bar{l}_L \gamma_\mu l_L + 2\bar{l}_R \gamma_\mu l_R\right).$$

The electromagnetic current can be written as

$$J^{EM}_\mu = -\bar{l}\gamma_\mu l,$$
$$= -\left(\bar{l}_L \gamma_\mu l_L + \bar{l}_R \gamma_\mu l_R\right),$$
$$= J^3_\mu + \frac{1}{2} J^Y_\mu.$$

Neither T_3 nor Q commute with $T_{1,2}$.

However, the "charges" associated to these currents,

$$T^i = \int J^i_0 \, d^3x, \qquad Y = \int J^Y_0 \, d^3x$$

satisfy the algebra of the $SU(2)\otimes U(1)$ group

$$\hookrightarrow [T^i, T^j] = i\epsilon^{ijk} T^k, \quad [T^i, Y] = 0$$

and the Gell-Mann - Nishijima relation between Q and T_3 emerges

$$Q = T_3 + \frac{1}{2} Y$$

which defines the weak hypercharge of the doublet and singlet,

$$Y_{\mathbb{L}} = -1, \quad Y_{\mathbb{R}} = -2.$$

Following our previous receipt for model building, the gauge group was already chosen,

$$SU(2)_L \otimes U(1)_Y.$$

We introduce the gauge fields corresponding to each generator

$$SU(2)_L \longrightarrow W_\mu^1,\ W_\mu^2,\ W_\mu^3$$

$$U(1)_Y \longrightarrow B_\mu$$

Defining the strenght tensors for the gauge fields,

$$W_{\mu\nu}^i \equiv \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g\,\epsilon^{ijk} W_\mu^j W_\nu^k,$$

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu.$$

We might now write the free Lagrangian for the gauge fields

$$\mathcal{L}_{gauge} = -\frac{1}{4} W_{\mu\nu}^i W^{i\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

For the fermions (massless leptons) we write the free Lagrangian

$$\begin{aligned}\mathcal{L}_{fermions} &= \bar{\mathbb{R}}\, i\,\partial\!\!\!/\,\mathbb{R} + \bar{\mathbb{L}}\, i\,\partial\!\!\!/\,\mathbb{L} \\ &= \bar{\ell}_R\, i\,\partial\!\!\!/\,\ell_R + \bar{\ell}_L\, i\,\partial\!\!\!/\,\ell_L + \bar{\nu}_L\, i\,\partial\!\!\!/\,\nu_L \\ &= \bar{\ell}\, i\,\partial\!\!\!/\,\ell + \bar{\nu}\, i\,\partial\!\!\!/\,\nu\end{aligned}$$

Fermions and bosons are then coupled via the covariant derivative

$$\mathbb{L}: \quad \partial_\mu + i\frac{g}{2}\sigma^i W^i_\mu + i\frac{g'}{2} Y B_\mu$$

$$\mathbb{R}: \quad \partial_\mu + i\frac{g'}{2} Y B_\mu$$

$$\mathcal{L}_{\text{fermions}} \longrightarrow \mathcal{L}_{\text{fermions}} + \underbrace{\bar{\mathbb{L}}\, i\gamma^\mu\left(i\frac{g}{2}\sigma^i W^i_\mu + i\frac{g'}{2} Y B_\mu\right)\mathbb{L}} + \bar{\mathbb{R}}\, i\gamma^\mu\left(i\frac{g'}{2} Y B_\mu\right)\mathbb{R}$$

$$\mathcal{L}^{L}_{\text{fermions}} = \underbrace{-g\,\bar{\mathbb{L}}\gamma^\mu\left(\frac{\sigma^1}{2}W^1_\mu + \frac{\sigma^2}{2}W^2_\mu\right)\mathbb{L}}_{\text{charged}} - \bar{\mathbb{L}}\gamma^\mu\left(\frac{g}{2}\sigma^3 W^3_\mu + \frac{g'}{2} Y B_\mu\right)\mathbb{L}$$

$$\mathcal{L}^{L(\pm)}_{\text{fermions}} = -\frac{g}{2}\bar{\mathbb{L}}\gamma^\mu\begin{pmatrix} 0 & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & 0 \end{pmatrix}\mathbb{L}$$

suggests the definition of the charged gauge bosons

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp W^2_\mu\right)$$

$$\mathcal{L}^{L(\pm)}_{\text{fermions}} = -\frac{g}{2\sqrt{2}}\left(\bar{\nu}\gamma^\mu(1-\gamma_5)\ell\, W^+_\mu + \bar{\ell}\gamma^\mu(1-\gamma_5)\nu\, W^-_\mu\right)$$

reproduces the V-A structure of the weak charged current.

The low-energy phenomenology ($G_W = \frac{1}{2\sqrt{2}} g$) requires that

$$\frac{g}{2\sqrt{2}} = \left(\frac{M_W^2 G_F}{\sqrt{2}} \right)^{1/2}$$

Let us now consider the neutral piece of the fermion Lagrangian

$$\mathcal{L}^{L+R\,(0)}_{\text{fermions}} = -g\, \bar{\mathbb{L}} \left(\gamma^\mu \frac{\sigma^3}{2} \right) \mathbb{L}\, W^3_\mu - \frac{g'}{2} \left(\bar{\mathbb{L}} \gamma^\mu Y \mathbb{L} + \bar{\mathbb{R}} \gamma^\mu Y \mathbb{R} \right) B_\mu$$

$$= -g\, J^\mu_3 W^3_\mu - \frac{g'}{2} J^\mu_Y B_\mu$$

$$J^\mu_3 = \frac{1}{2} \left(\bar{\nu}_L \gamma^\mu \nu_L - \bar{\ell}_L \gamma^\mu \ell_L \right)$$

$$J^\mu_Y = - \left(\bar{\nu}_L \gamma^\mu \nu_L + \bar{\ell}_L \gamma^\mu \ell_L + 2 \bar{\ell}_R \gamma^\mu \ell_R \right)$$

Notice that the 'charges' and currents satisfy

$$\begin{array}{ccccc} Q & = & T_3 & + & \frac{1}{2} Y \\ \downarrow & & \downarrow & & \downarrow \\ J_{em} & = & J_3 & + & \frac{1}{2} J_Y \end{array}$$

Let us make the rotation in the neutral fields

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix}$$

with

$$\sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}.$$

The neutral part becomes

$$\mathcal{L}^{L+R\,(0)}_{\text{fermions}} = -\left(g \sin\theta_W J_3^\mu + \frac{1}{2} g' \cos\theta_W J_Y^\mu\right) A_\mu$$
$$-\left(g \cos\theta_W J_3^\mu - \frac{1}{2} g' \sin\theta_W J_Y^\mu\right) Z_\mu$$
$$= -g \sin\theta_W (\bar{\ell}\gamma^\mu \ell) A_\mu - \frac{g}{2\cos\theta_W} \sum_{\psi_i = \nu, \ell} \bar{\psi}_i \gamma^\mu (g_V^i - g_A^i \gamma_5) \psi_i Z_\mu$$

and we identify the electromagnetic charge

$$e = g \sin\theta_W = g' \cos\theta_W$$

and define the V and A couplings of the Z bosons

$$g_V^i \equiv T_3^i - 2 Q_i \sin^2\theta_W \quad \text{and} \quad g_A^i \equiv T_3^i .$$

Up to now we have

→ 4 massless gauge fields
- $W_\mu^\pm$
- Z_μ
- A_μ

→ 2 massless fermions
- neutrino ν
- charged lepton ℓ

Now we need

→ SSB ⊕ Higgs mechanism
- $W^\pm$ and Z massive weak bosons
- A massless bosons (photon)

The Higgs et al. Mechanism

In order to use the Higgs mechanism to give mass to $W^\pm$ and Z^0, let us introduce the scalar doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \begin{matrix} \rightarrow T_3 = +1/2, \quad Y = 2(1 - 1/2) = 1 \\ \rightarrow T_3 = -1/2, \quad Y = 2(0 + 1/2) = 1 \end{matrix}$$

and write the Lagrangian

$$\mathcal{L}_{scalar} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - V(\Phi^\dagger \Phi)$$

where the potential is given by

$$V(\Phi^\dagger \Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$$

To preserve gauge invariance, we define the covariant derivative

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + ig \frac{\sigma^i}{2} W^i_\mu + i \frac{g'}{2} Y B_\mu$$

We can choose the vacuum expectation value of the Higgs fields as

$$\langle \Phi \rangle_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

where

$$v = \sqrt{\frac{-\mu^2}{\lambda}}.$$

Since we want to preserve the exact eletromagnetic symmetry to maintain the electric charge conserved,

$$SU(2)_L \otimes U(1)_Y \rightsquigarrow U(1)_{EM}$$

Id est, after the SSB, the subgroup $U(1)_{EM}$, with dimension 1, should remain as a symmetry of the vacuum. In this case, the corresponding gauge boson (photon) will remain massless

We can see that our choice leaves the vacuum invariant under $U(1)_{EM}$

$$e^{i\alpha Q} \langle \Phi \rangle_0 \simeq (1 + i\alpha Q) \langle \Phi \rangle_0 = \langle \Phi \rangle_0$$

$$Q \langle \Phi \rangle_0 = 0$$

This is exactly what happens

$$Q \langle \Phi_0 \rangle = \left(T_3 + \frac{1}{2} Y \right) \langle \Phi \rangle_0$$

$$= \frac{1}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = 0.$$

The other gauge bosons, corresponding to the broken generators T_1, T_2 and

$$T_3 - \frac{1}{2} Y = 2T_3 - Q$$

should aquire mass.

Let us parametrize the Higgs doublet as

$$\Phi \equiv \exp\left(i \frac{\sigma^i}{2} \frac{\chi^i}{v} \right) \begin{pmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{pmatrix}$$

$$\simeq \langle \Phi \rangle_0 + \frac{1}{2\sqrt{2}} \begin{pmatrix} \chi_2 + i\chi_1 \\ 2H - i\chi_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i\sqrt{2}\, \omega^+ \\ v + H - i z^0 \end{pmatrix}$$

where $\omega^\pm$ and z^0 are the Goldstone bosons

If one make a $SU(2)_L$ transformation with $\alpha_i = \frac{\chi_i}{v}$ (unitary gauge) the fields become

$$W^i_\mu \to W^{i\,\prime}_\mu , \qquad B_\mu \to B_\mu$$

$$\mathbb{L} \to \mathbb{L}' = e^{-i \frac{\sigma^i}{2} \frac{\chi_i}{v}} \mathbb{L} \qquad \mathbb{R} \to \mathbb{R}$$

$$\Phi \to \Phi' = e^{-i \frac{\sigma^i}{2} \frac{\chi_i}{v}} \Phi = \frac{v+H}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathcal{L}_{scalar} = \left| \left(\partial_\mu + i g \frac{\sigma^i}{2} W^i_\mu + i \frac{g'}{2} Y B_\mu \right) \frac{(v+H)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2$$

$$- \mu^2 \frac{(v+H)^2}{2} - \lambda \frac{(v+H)^4}{4}$$

scalar Lagrangian written in terms of the new fields

In terms of the physical fields, $W^\pm$ and Z^0, the first term is

$$\left|\begin{pmatrix} 0 \\ \partial_\mu H/\sqrt{2} \end{pmatrix} + i\frac{g}{2}(v+H)\begin{pmatrix} W_\mu^+ \\ -\frac{1}{\sqrt{2}\cos\theta_W} Z_\mu \end{pmatrix}\right|^2 = \frac{1}{2}\partial_\mu H \partial^\mu H + \frac{g^2}{4}(v+H)^2\left(W_\mu^+ W^{-\mu} + \frac{1}{2\cos^2\theta_W} Z_\mu Z^\mu\right)$$

The terms quadratic in the vector fields are → mass terms

$$\frac{g^2 v^2}{4} W_\mu^+ W^{-\mu} + \frac{g^2 v^2}{8\cos^2\theta_W} Z_\mu Z^\mu$$

When compared with the mass terms for the vector boson,

$$M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} M_Z^2 Z_\mu Z^\mu$$

$$M_W = \frac{gv}{2}$$

$$M_Z = \frac{gv}{2\cos\theta_W} = \frac{M_W}{\cos\theta_W}$$

→ the photon remains massless

The other terms give the self-interaction of the H scalar

$$-\frac{1}{2}(-2\mu^2)H^2 + \frac{1}{4}\mu^2 v^2\left(\frac{4}{v^3}H^3 + \frac{1}{v^4}H^4 - 1\right)$$

→ $M_H = \sqrt{-2\mu^2}$

Higgs mass

→ Higgs self-couplings

Some General Comments on the Standard Model

On the mass matrix of the neutral bosons

Note that the mass term of W^3_μ and B_μ can be written as

$$\mathcal{L}_M^{W^3-B} = \frac{v^2}{2}\left|\left(g\frac{\sigma^3}{2}W^3_\mu + \frac{g'}{2}YB_\mu\right)\begin{pmatrix}0\\1\end{pmatrix}\right|^2$$

$$= \frac{v^2}{8}\left[\left(B_\mu \; W^3_\mu\right)\begin{pmatrix}g'^2 & -gg'\\ -gg' & g^2\end{pmatrix}\begin{pmatrix}B^\mu\\ W^{3\mu}\end{pmatrix}\right]$$

the mass matrix has eigenvalues

$0 \rightarrow$ photon mass

$\left(\frac{1}{2}\right)\frac{(g^2+g'^2)v^2}{4} = \frac{1}{2}M_Z^2 \rightarrow Z$ mass

Notice that the same matrix used to define the physical fields diagonalizes the mass matrix for the neutral gauge bosons, i.e.

$$R_W = \begin{pmatrix}\cos\theta_W & \sin\theta_W\\ -\sin\theta_W & \cos\theta_W\end{pmatrix}$$

$$R_W\frac{v^2}{4}\begin{pmatrix}g'^2 & -gg'\\ -gg' & g^2\end{pmatrix}R_W^T = \begin{pmatrix}0 & 0\\ 0 & M_Z^2\end{pmatrix}$$

On the ρ Parameter

We may define a dimensionless parameter ρ through

$$\rho = \frac{M_W^2}{\cos^2\theta_W \, M_Z^2}$$

represents the relative strength of the neutral and charged effective Lagrangians $\frac{J^{0\mu} J^0_\mu}{J^{+\mu} J^-_\mu}$

$$\rho = \frac{g^2}{8\cos^2\theta_W \, M_Z^2} = \frac{g^2}{8 M_W^2}$$

In the SM, at tree level, $\rho = 1$

not a general consequence of the gauge invariance of the model, but a successful prediction of the SM

In a model with an arbitrary number of Higgs multiplets ϕ_i with isospin T_i and third component T_i^3, and v.e.v. v_i

$$\rho = \frac{\sum_i \left[T_i(T_i+1) - (T_i^3)^2\right] v_i^2}{2\sum_i (T_i^3)^2 v_i}$$

1 for an arbitrary number of doublet

Thus, ρ represents a good test for the isospin structure of the Higgs sector. It is also sensitive to radiative corrections.

On the Gauge Fixing Term

Unitary gauge $\Longrightarrow$ physical spectrum is clear: massive $W^\pm$ and Z^0 and no Goldstone bosons

$$P^U_{\mu\nu}(V) = \frac{-i}{q^2 - M_V^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_V^2}\right)$$

$P^U_{\mu\nu}(V) \underset{q\to\infty}{\not\to} \frac{1}{q^2}$ leads to complicated cancellations at high energies and it is hard to prove renormalizability

power counting analysis

Add a gauge-fixing Lagrangian (R_ξ gauge):

$$\mathcal{L}_{gf} = -\frac{1}{2}\left(2G_W^+ G_W^- + G_Z^2 + G_A^2\right)$$

$$G_W^\pm = \frac{1}{\sqrt{\xi_W}}\left(\partial^\mu W_\mu^\pm \mp i\,\xi_W M_W \omega^\pm\right)$$

$$G_Z = \frac{1}{\sqrt{\xi_Z}}\left(\partial^\mu Z_\mu - \xi_Z M_Z z^0\right)$$

Goldstone bosons

$$G_A = \frac{1}{\sqrt{\xi_A}}\partial^\mu A_\mu$$

$$-\frac{1}{2}G_V^2 = -\frac{1}{2\xi_V}\left(\partial_\mu V^\mu - \xi_V M_V v\right)^2 = \frac{1}{2}V_\mu\left(\frac{1}{\xi_V}\partial^\mu\partial^\nu\right)V_\nu - \frac{1}{2}\xi_V M_V^2 v^2 + \underbrace{M_V v\,\partial^\mu V_\mu}$$

cancelled by identical term in scalar Lagrangian

The vector boson propagators now become

$$P^{R_\xi}_{\mu\nu}(V) = \frac{-i}{q^2 - M_V^2}\left(g_{\mu\nu} - (1-\xi_V)\frac{q_\mu q_\nu}{q^2 - \xi_V M_V^2}\right)$$

The Goldstone bosons, with mass $\sqrt{\xi_V}\, M_V$, remain in the spectrum and have propagator

$$P^{R_\xi}(GB) = \frac{i}{q^2 - \xi_V M_V^2}.$$

The physical Higgs propagator remains the same.

In the limit $\xi_V \to +\infty$, the Goldstone bosons disappear and the unitary gauge is recovered

$\xi_V \to 0$: Landau gauge

$\xi_V \to 1$: Feynman gauge

all physical processes should not depend on ξ_V

How to Measure $\sin^2\theta_W$ at Low Energies

Neutrino-lepton scattering

the cross-section for elastic scattering

$$\frac{\nu_\mu}{\bar\nu_\mu} + e \to \frac{\nu_\mu}{\bar\nu_\mu} + e$$

involves a t-channel Z^0 exchange and is given by

$$\sigma = \frac{G_F^2 M_e E_\nu}{2\pi} \left[\left(g_V^e \pm g_A^e \right)^2 + \frac{1}{3} \left(g_V^e \mp g_A^e \right)^2 \right]$$

$$g_A^e = -\frac{1}{2}$$

$$g_V^e = -\frac{1}{2} + 2\sin^2\theta_W$$

For ν_e: $g^e_{V,A} \to \left(g^e_{V,A} + 1 \right)$ due to the W exchange

Measuring the ratio $\frac{\sigma(\nu_\mu e)}{\sigma(\bar{\nu}_\mu e)}$ the systematic uncertainties cancel out

Deep inelastic neutrino scattering $\longrightarrow$ from isoscalar target N

$$R_{\nu(\bar{\nu})} \equiv \frac{\sigma^{NC}[\nu(\bar{\nu})N]}{\sigma^{CC}[\nu(\bar{\nu})N]}, \quad R_{\nu(\bar{\nu})} \simeq \frac{1}{2} - \sin^2\theta_W + \frac{5}{9}\left[1 + r\left(\frac{1}{r}\right)\right]\sin^4\theta_W$$

$$r \equiv \frac{\sigma^{CC}(\bar{\nu}N)}{\sigma^{CC}(\nu N)} \simeq 0.44$$

Atomic parity violation in Cs, Tl, Pb, Bi is given by Z^0 mediated e-nucleus interaction

$$\mathcal{H} = \frac{G_F}{2\sqrt{2}} Q_W \gamma^5 \rho_{nuc}$$

"weak charge"

$$Q_W \simeq Z(1 - 4\sin^2\theta_W) - N$$

$Z(N) = \#p(n)$

The Yukawa Couplings and the Lepton Mass

the charged lepton is still massless, since

$$M_\ell \bar{\ell}\ell = M_\ell(\bar{\ell}_R \ell_L + \bar{\ell}_L \ell_R)$$

mixes L and R components and breaks gauge invariance

A way to give mass to leptons in a gauge invariant way is ~~to~~ via the Yukawa coupling of the leptons with the Higgs field

$$\mathcal{L}^\ell_{Yuk} = - G_e \left[\bar{\mathbb{R}} (\Phi^+ \mathbb{L}) + (\bar{\mathbb{L}} \Phi) \mathbb{R} \right]$$

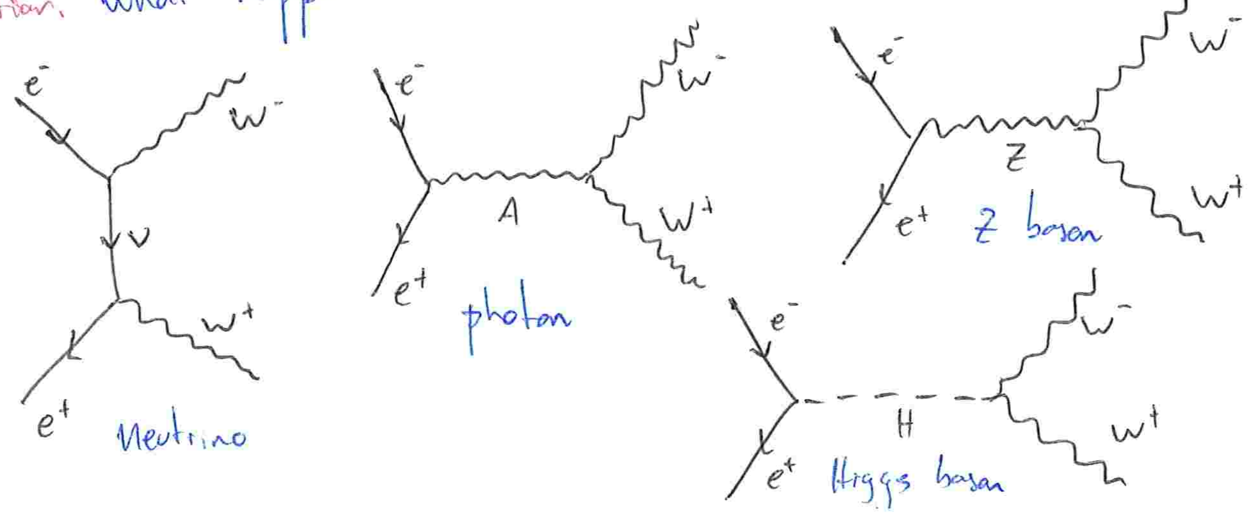
$$= - G_e \frac{(v+H)}{\sqrt{2}} \left[\bar{\ell}_R \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} + \begin{pmatrix} \bar{\nu}_L & \bar{\ell}_L \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ell_R \right]$$

$$= - \frac{G_e v}{\sqrt{2}} \bar{\ell}\ell - \frac{G_e}{\sqrt{2}} \bar{\ell}\ell H$$

charged lepton mass: $M_\ell = \frac{G_e v}{\sqrt{2}}$

Higgs-lepton coupling, with strength $C_{\bar{\ell}\ell H} = \frac{M_\ell}{v}$

Question: what happens with cross sections like $e^+e^- \to W^+W^-$?



Besides the first two graphs (ν, A), that come from any IVB theory, the Standard Model introduces

→ neutral current contribution (Z)

→ Higgs boson exchange (H)

The leading p wave divergence ($\propto s$) of the neutrino is cancelled by the contributions of A and Z exchange. ↳ consequence of the gauge structure of the couplings

The s wave scattering amplitude is also divergent ($\propto m_f \sqrt{s}$). It is cancelled by the Higgs exchange

↳ the existence of a scalar (s wave) that couples with m_f is an essential ingredient of the theory

"If the Higgs boson did not exist, we should have to invent something very much like it" Quigg in his book

General Features of the Standard Model

→ reproduces very well the low-energy phenomenology

→ predicts the existence of weak interaction via neutral current

→ predicts the vector boson masses

$$M_W^2 = \frac{g^2 v^2}{4}, \quad M_Z^2 = \frac{M_W^2}{c_W^2}$$

$c_W \doteq \cos\theta_W$

$s_W \doteq \sin\theta_W$

From low-energy data, $g^2 = 4\sqrt{2}\, G_F M_W^2$.

$$v = \left(\sqrt{2}\, G_F\right)^{1/2} \simeq 246 \text{ GeV}$$

$$M_W^2 = \frac{e^2 v^2}{4 s_W^2} \simeq \left(\frac{37.2}{s_W}\ GeV\right)^2 \sim (80\ GeV)^2$$

$$M_Z^2 \simeq \left(\frac{37.2}{s_W c_W}\ GeV\right)^2 \sim (90\ GeV)^2$$

→ predicts the existence of (at least) one scalar boson (H)
→ predicts the couplings of the Higgs with fermions and bosons
→ fermion gets (arbitrary) mass in a gauge invariant way
→ amplitudes are well-behaved at high-energies
→ the theory is renormalizable

Introducing the Quarks → hadrons also interact weakly

Weak Interaction of Quarks

Weak decays with $\Delta S = 1$ are strongly suppressed

↳ $\Gamma_{\Delta S=0}(n_{udd} \to p_{uud}\, e \bar{\nu}) \gg \Gamma_{\Delta S=1}(\Lambda_{uds} \to p_{uud}\, e \bar{\nu})$

to make the hadronic current

$$J_\mu^H = \underbrace{\bar{d}\, \gamma_\mu (1-\gamma_5) u}_{\Delta S = 0} + \underbrace{\bar{s}\, \gamma_\mu (1-\gamma_5) u}_{\Delta S = 1}$$

universal, Cabibbo (1963) introduced a rotation angle that mixes the mass eigenstates (d, s) to obtain the interaction states (d', s')

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

$\bar{d} \leftrightarrow u$: $G_F \cos\theta_c \simeq 0.97\, G_F$ $\quad$ $\bar{s} \leftrightarrow u$: $G_F \sin\theta_c \simeq 0.24\, G_F$

Therefore, the hadronic current should be written in terms of the new interaction eigenstates

$$J_\mu^H = \bar{d}' \gamma_\mu (1-\gamma_5) u$$
$$= \cos\theta_c \, \bar{d} \gamma_\mu (1-\gamma_5) u + \sin\theta_c \, \bar{s} \gamma_\mu (1-\gamma_5) u$$

Question: what happens with the neutral current?

$$J_\mu^H(0) = \bar{u} \gamma_\mu (1-\gamma_5) u + \bar{d}' \gamma_\mu (1-\gamma_5) d'$$
$$= \bar{u} \gamma_\mu (1-\gamma_5) u + \cos^2\theta_c \, \bar{d} \gamma_\mu (1-\gamma_5) d$$
$$+ \sin^2\theta_c \, \bar{s} \gamma_\mu (1-\gamma_5) s + \cos\theta_c \sin\theta_c \left[\bar{d} \gamma_\mu (1-\gamma_5) s + \bar{s} \gamma_\mu (1-\gamma_5) d \right]$$

Flavour changing neutral current

However, FCNC is "forbidden"

$$BR(K^+_{u\bar{s}} \to W^+ \to \mu^+ \nu) \simeq 0.6356, \quad BR(K^+_{u\bar{s}} \to \pi^+_{u\bar{d}} (Z^0 \to) \nu \bar{\nu}) \simeq 1.73 \cdot 10^{-10}$$

$$BR(K^L_{d\bar{s}} \to \pi^\pm_{u\bar{d}} (W^\mp \to) e^\mp \nu) \simeq 0.4055, \quad BR(K^L_{d\bar{s}} \to Z^0 \to \mu^+ \mu^-) \simeq 6.84 \cdot 10^{-9}$$

1970 Glashow, Iliopoulos et Maiani: introduction of lepton-quark symmetry and the proposal of charmed quark (GIM mechanism)

In order to avoid FCNC, a new quark flavour must be introduced: the charm (c)

$$\mathbb{L}_u \equiv \begin{pmatrix} u \\ d' \end{pmatrix}_L, \quad \mathbb{L}_c \equiv \begin{pmatrix} c \\ s' \end{pmatrix}_L, \quad \mathbb{R}_u, \mathbb{R}_d, \mathbb{R}_c, \mathbb{R}_s.$$

As in the leptonic case, we start with the free massless Dirac Lagrangian

$$\mathcal{L}_{quarks} = \bar{\mathbb{L}}_u i \not\partial \mathbb{L}_u + \bar{\mathbb{L}}_c i \not\partial \mathbb{L}_c + \bar{\mathbb{R}}_u i \not\partial \mathbb{R}_u + \dots + \bar{\mathbb{R}}_s i \not\partial \mathbb{R}_s$$

After introducing the gauge bosons via the covariant derivative for the quarks with

$$Y_{L_Q} = \frac{1}{3}, \quad Y_{R_U} = \frac{4}{3}, \quad Y_{R_D} = -\frac{2}{3},$$

we obtain the charged weak couplings quark-gauge bosons

$$\mathcal{L}^{(\pm)}_{quarks} = \frac{g}{2\sqrt{2}}\left[\bar{u}\gamma^\mu(1-\gamma_5)d' + \bar{c}\gamma^\mu(1-\gamma_5)s'\right]W^+_\mu + h.c.$$

The neutral current becomes diagonal in the flavours, avoiding the FCNC

$$\mathcal{L}^{(0)}_{quarks} = -\frac{g}{2c_W}\sum_{\psi_q = u,\dots,c}\bar{\psi}_q\gamma^\mu(g^q_V - g^q_A\gamma_5)\psi_q Z_\mu,$$

with coupling similar to the leptonic situation

$$g^q_V = T^q_3 - 2Q_q\sin^2\theta_W$$
$$g^q_A = T^q_3$$

Anomaly Cancellation

Loop corrections can violate a classical local conservation law, derived from gauge invariance via Noether's theorem (anomaly). When this happen, it

→ breaks Ward-Takahashi identities

→ invalidate the proofs of renormalizability

The V-A gauge theories give rise to VVA triangle loops that can generate anomaly.

For a generic gauge theory with interaction of the form

$$\mathcal{L}_{int} = -g\left(\bar{R}\gamma^\mu T_+^a R + \bar{L}\gamma^\mu T_-^a L\right) V_\mu^a$$

group generators in the R\L representation

gauge bosons

Such a theory is anomaly free if

$$A^{abc} = A_+^{abc} - A_-^{abc} = 0$$

$$A_\pm^{abc} \equiv Tr\left(\{T_\pm^a, T_\pm^b\} T_\pm^c\right)$$

The only possible anomalies of the SM come from $SU(2)^2 U(1)$ and $U(1)^3$ triangles

$SU(2)^2 U(1)$ $\quad Tr\left(\{\sigma^a, \sigma^b\} Y\right) = Tr\left(\{\sigma^a, \sigma^b\}\right) Tr\, Y \propto \sum_{\text{doublets}} Y$

$U(1)^3$ $\quad Tr\left(Y^3\right) \propto \sum_{\text{fermions}} Y^3$

Recalling that $Y_{L_e} = -1$, $Y_{R_e} = -2$, $Y_{L_Q} = 1/3$, $Y_{R_U} = 4/3$, $Y_{R_D} = -2/3$,

$$A^{abc} \propto -\sum_{\text{doublets}} Y = -\left[-1 + 3\cdot\frac{1}{3}\right] = 0$$

$$A^{abc} \propto \sum_{\text{fermions}} Y_+^3 - Y_-^3 = \left((-2)^3 + 3\left(\left(\frac{4}{3}\right)^3 + \left(\frac{-2}{3}\right)^3\right)\right) - \left((-1)^3 + (-1)^3 + 3\left(\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^3\right)\right)$$

$$= 0$$

The SM is free from anomalies if the fermions appears in complete multiplets with the structure (generations):

$$\left\{ \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \; e_R, \; \begin{pmatrix} u \\ d \end{pmatrix}_L, \; u_R, d_R \right\}$$

and that structure must repeat itself for the other particles

$$\left\{ \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \; \mu_R, \; \begin{pmatrix} c \\ s \end{pmatrix}_L, \; c_R, \; s_R \right\}$$

$$\left\{ \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \; \tau_R, \; \begin{pmatrix} t \\ b \end{pmatrix}_L, \; t_R, \; b_R \right\}$$

The Quark Masses

To generate mass for U and D quarks, define a conjugate doublet Higgs doublet $(Y = -1)$

$$\tilde{\Phi} = i\,\sigma_2\,\Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$$

we can write for the three generations

$$\mathcal{L}^q_{Yuk} = -\sum_{i=1}^{3}\sum_{j=1}^{3} \tilde{G}_{ij}\,\overline{\mathbb{R}}_{U_i}\left(\tilde{\Phi}^\dagger \mathbb{L}_j\right) + G_{ij}\,\overline{\mathbb{R}}_{D_i}\left(\Phi^\dagger \mathbb{L}_j\right) + h.c.$$

From the vacuum expectation value of Φ and $\tilde{\Phi}$ we obtain the mass terms

$$\overline{(u', c', t')}_R \, M^U \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_L + h.c. \qquad \overline{(d', s', b')}_R \, M^D \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L + h.c.$$

The $M_{ij}^{U(D)}$ are non-diagonal matrices. The weak eigenstates (q') are linear superpositions of the mass eigenstates (q) given by the unitary transformations

$$\begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_{L,R} = U_{L,R} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R}, \qquad \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_{L,R} = D_{L,R} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R},$$

where $U(D)_{L,R}$ diagonalize the mass matrices.

$$U_R^{-1} M^U U_L = \mathrm{diag}(m_u, m_c, m_t)$$

$$D_R^{-1} M^D D_L = \mathrm{diag}(m_d, m_s, m_b)$$

The V-A charged weak current will be proportional to

$$\overline{(u', c', t')}_L \gamma_\mu \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L = \overline{(u, c, t)}_L \underbrace{(U_L^\dagger D_L)} \gamma_\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$$

$V \equiv U_L^\dagger D_L$ describes the generation mixing of mass eigenstates

As for the neutral current,

$$\overline{(u', c', t')}_L \gamma_\mu \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_L = \overline{(u, c, t)}_L \underbrace{(U_L^\dagger U_L)} \gamma_\mu \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L$$

no mixing in the neutral sector, for

$$U_L^\dagger U_L = \mathbb{1}$$

By convention, the mixing is restricted to the down quarks ($T_3^q = -1/2$), i.e.,

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L = V \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$$

Cabibbo - Kobayashi - Maskawa matrix

$$V = R_1(\theta_{23}) R_2(\theta_{13}, \delta_{13}) R_3(\theta_{12})$$

rotation matrices

$R_i(\theta_{jk})$ are rotation matrices around i and θ_{jk} describes the mixing of generations j and k. δ_{13} is a phase

$$V = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix}$$

In the limit of $\theta_{23} = \theta_{13} \to 0$, we associate $\theta_{12} \to \theta_c$ and

$$V \to \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\lambda \equiv s_{12} \gg s_{23} \gg s_{13}$,

$$d' \simeq d + \lambda s + e^{-i\delta_{13}} \lambda^3 b$$

$$s' \simeq s - \lambda d + \lambda^2 b$$

$$b' \simeq b - \lambda^2 s - e^{i\delta_{13}} \lambda^3 d$$

$\hookrightarrow$ weak transactions $\downarrow$ as separation $\uparrow$ and mass difference $\uparrow$

The Standard Model Lagrangian

Gauge Bosons and Scalar

$$\mathcal{L}_{gauge} + \mathcal{L}_{scalar} = \mathcal{L}_A^{free} + \mathcal{L}_W^{free} + \mathcal{L}_Z^{free} + \mathcal{L}_H^{free}$$
$$+ W^+W^-A + W^+W^-Z + W^+W^-AA + W^+W^-ZZ$$
$$+ W^+W^-AZ + W^+W^-W^+W^- + HHH + HHHH$$
$$+ W^+W^-H + W^+W^-HH + ZZH + ZZHH$$

Leptons + Yukawa

$$\mathcal{L}_{leptons} + \mathcal{L}^{\ell}_{Yukawa} = \sum_{\ell = e,\mu,\tau} \mathcal{L}_\ell^{free} + \mathcal{L}_{\nu_e}^{free} + \bar{\ell}\ell A + \bar{\nu}_e \ell W^+ + \bar{\ell}\nu_e W^-$$
$$+ \bar{\ell}\ell Z + \bar{\nu}_e \nu_e Z + \bar{\ell}\ell H$$

Quarks + Yukawa

$$\mathcal{L}_{quarks} + \mathcal{L}^{q}_{Yukawa} = \sum_{q = u,\dots,t} \bar{q}(i\not{\partial} - m_q)q + \bar{q}qA + \bar{u}d'W^+ + \bar{d}'uW^- + \bar{q}qZ + \bar{q}qH$$

we will deal with the strong force later

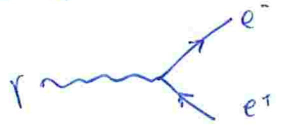
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

EM and nuclear forces

Same diagram coming from this term

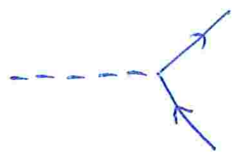
$$+ i \bar{\Psi} \not{D} \psi + h.c.$$

fermion motion and interactions

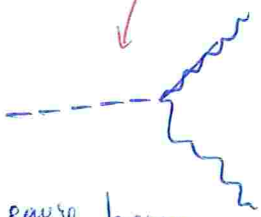


$$+ \psi_i y_{ij} \psi_j \phi + h.c.$$

fermion masses

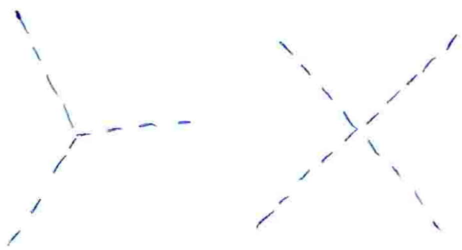


$$+ |D_\mu \phi|^2 - V(\phi)$$



gauge boson masses

Higgs sector $\rightsquigarrow$ SSB



Radiative Corrections $\rightsquigarrow$ beyond the trees

General Scheme of Renormalization

Standard Model
↓
Renormalizable Field Theory
↓
Loop diagrams (UV divergent)
↓
Regularization (e.g. dimensional regularization)

Divergences absorbed in couplings & masses
↓
Renormalization condition (e.g. on-shell scheme)
↓
Counterterms
↓
Finite results for S-matrix elements

Predictions of the SM for (derived) observables
↓
Falsification of the theory

Given the SM Lagrangian, $\mathcal{L}_{SM}(g, g', v, \dots)$ → M_H, m_t, that influence just via radiative corrections

We can determine precision observables in terms of g, g' and v. However, we need 3 precisely measured observables to determine the basic input parameters (tree level ~~relations~~ relations)

→ the electromagnetic fine structure constant

$$\alpha(0) = \frac{1}{137.035999084(21)}$$

measured from $g-2$ or quantum Hall effect

$$\alpha_0(0) = \frac{g^2 s_W^2}{4\pi}$$

s_W^2 depends only on g and g'

→ the Fermi constant

$$G_F(\mu) = 1.1663787(6) \cdot 10^{-5}\ \mathrm{GeV}^{-2}$$

from the muon lifetime

$$G_{F_0} = \frac{1}{\sqrt{2}\, v^2}$$

→ the Z boson mass

$$M_Z = 91.1876(21)\ \mathrm{GeV}$$

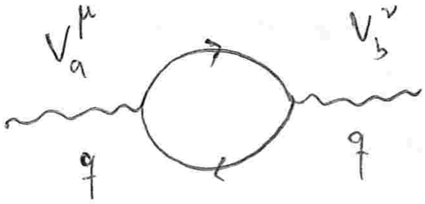
from LEP data at the Z pole

$$M_{Z_0}^2 = \frac{g^2 v^2}{4 c_W^2}$$

One Loop Calculations

Introducing the vacuum polarization amplitude (self-energy) for vector bosons $(a, b = \gamma, W, Z)$:

$$\Pi^{\mu\nu}_{ab}(q^2) \equiv g^{\mu\nu}\,\Pi_{ab}(q^2) + (q^\mu q^\nu \text{ terms})$$



can be dropped since

$$q^\mu \bar{\psi}\gamma_\mu \psi \propto m_f \sim 0$$

Relevant Quantities for the Loop Corrections

→ V and A form factors of the Z^0 coupling at $q^2 = M_Z^2$ (vertex of fermion self-energy)

$$V^\mu_{Zf\bar{f}} = -i\,\frac{g}{2\cos\theta_W}\,\bar{\psi}_f\gamma^\mu\left(g_V^f - g_A^f\gamma_5\right)\psi_f$$

$$g_V^f = \sqrt{\rho}\left(T_3^f - 2k_f Q_f \sin^2\theta_W\right) \qquad g_A^f = \sqrt{\rho}\,T_3^f$$

→ correction to μ-decay amplitude at $q^2 = 0$ (Box, Vertex of fermion self-energy)

$$\mathcal{M}(\mu) = -i\,\delta G_{(B,V)}\left[\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_{\nu_e}\right]\left[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu\right]$$

The corrections to the input parameters are

$$\alpha = \alpha_0 + \delta\alpha \quad\rightarrow\quad \frac{\delta\alpha}{\alpha} = \Pi_{\gamma\gamma}(0) - 2\,\frac{s_W}{c_W}\,\frac{\Pi_{\gamma Z}(0)}{M_Z^2}$$

$$M_Z^2 = M_{Z_0}^2 + \delta M_Z^2 \quad\rightarrow\quad \delta M_Z^2 = \Pi_{ZZ}(M_Z^2)$$

$$G_F = G_{F_0} + \delta G_F \quad\rightarrow\quad \frac{\delta G_F}{G_F} = \frac{\Pi_{WW}(0)}{M_W^2} + \frac{\delta G_{(B,V)}}{G_F}$$

Correction to the Derived Observables

Starting from the tree-level input variables $I_0^i \equiv \{\alpha_0, G_{F_0}, M_{Z_0}\}$ we compute the radiative corrections

$$I_0^i \to I^i(I_0^i) = I_0^i + \delta I^i(I_0^i)$$

the relation for the renormalized input variables I^i can be inverted to write

$$I_0^i = I_0^i(I^i)$$

The same holds for any derived observable $\mathcal{O}$ or any S-matrix element

$$\begin{aligned}
\mathcal{O}\left[I_0^i(I^i)\right] &= \mathcal{O}_0(I_0^i) + \delta\mathcal{O}(I_0^i) \\
&= \mathcal{O}_0(I^i - \delta I^i) + \delta\mathcal{O}(I^i - \delta I^i) \\
&= \mathcal{O}_0(I^i) - \sum_i \frac{\partial \mathcal{O}_0}{\partial I^i}\delta I^i + \delta\mathcal{O}(I^i) \\
&\equiv \mathcal{O}_0(I^i) + \Delta\mathcal{O}(I^i)
\end{aligned}$$

At one loop it is enough to renormalize the input variables I^i. At two loops it is necessary to renormalize all other parameters that intervene at one loop, like m_t and M_H.

W boson mass: an example

Tree level:

$$M_{W_0}^2 = c_{W_0}^2 M_{Z_0}^2$$

in terms of input variables

$$c_W^2 = 1 - s_W^2 = 1 - \frac{4\pi\alpha}{g^2} = 1 - \frac{\pi\alpha}{\sqrt{2}\, G_F M_Z^2 c_W^2}$$

Solving for c_W^2,

$$M^2_{W_0}(I^i) = \frac{1}{2}\left[1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}\,G_F M_Z^2}}\right] M_Z^2 .$$

Since

$$\frac{\partial M^2_{W_0}}{\partial \alpha} = \frac{s_W^2 c_W^2}{s_W^2 - c_W^2}\frac{M_Z^2}{\alpha}, \qquad \frac{\partial M^2_{W_0}}{\partial G_F} = -\frac{s_W^2 c_W^2}{s_W^2 - c_W^2}\frac{M_Z^2}{G_F}$$

$$\frac{\partial M^2_{W_0}}{\partial M_Z^2} = -\frac{c_W^4}{s_W^2 - c_W^2}$$

$$M_W^2 = M^2_{W_0}(I^i) - \sum_i \frac{\partial M^2_{W_0}}{\partial I^i}\delta I^i + \delta M_W^2(I^i)$$

$$= c_W^2 M_Z^2 - \frac{c_W^2 M_Z^2}{s_W^2 - c_W^2}\left(s_W^2\frac{\delta\alpha}{\alpha} - s_W^2\frac{\delta G_F}{G_F} - c_W^2\frac{\delta M_Z^2}{M_Z^2}\right) + \delta M_W^2$$

$$\delta M_W^2 = \Pi_{WW}(M_W^2)$$

The ρ, κ and r_W parameters

It is useful to define the (in principle) independent parameters

$$\rho = 1 + \Delta\rho + \tilde{\Delta}\rho$$

$$\kappa = 1 + \Delta\kappa + \tilde{\Delta}\kappa$$

Δ does not depend on the external fermions (universal)
Dominated by the top contribution

$$\Delta\rho = \frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\Pi_{WW}(0)}{M_W^2}$$

$$\simeq (\Delta\rho)_t \overset{SM}{=} \frac{s_w^2}{c_w^2}(\Delta k)_t \simeq \frac{3G_F}{8\sqrt{2}\pi^2} m_t^2$$

$\tilde{\Delta}$ depends on the external fermion flavor (non-universal)
e.g. $Zb\bar{b}$ vertex correction

$$(\tilde{\Delta}\rho)_b = -2(\tilde{\Delta}k)_b$$

$$\simeq -\frac{4}{3}(\Delta\rho)_t + \frac{\alpha}{4\pi s_w^2}\left(\frac{8}{3} + \frac{1}{6c_w^2}\right)\log\left(\frac{m_t^2}{M_W^2}\right)$$

The relation between M_W and M_Z, in terms of α and G_F, is also modified by the radiative corrections

$$\left(1 - \frac{M_W^2}{M_Z^2}\right)\frac{M_W^2}{M_Z^2} = \frac{\pi\alpha(M_Z^2)}{\sqrt{2}G_F M_Z^2(1-\Delta r_w)}$$

$\Delta r_w \overset{SM}{=} -\frac{c_w^2}{s_w^2}\Delta\rho$

Δr_w does not contain QED corrections

→ photonic effects are contained in $\alpha(M_Z^2)$ and G_F

→ the Higgs boson mass just appears as

$$\log\left(\frac{M_H^2}{M_Z^2}\right)$$

The Higgs Boson Physics

almost 50 years hunting for the goddamn particle

General Features of the SM Higgs

SM: 4 scalars $\begin{cases} 3 \text{ eaten by the gauge boson} \\ 1 \text{ remains: } \dfrac{\phi^0 + \bar{\phi}^0}{\sqrt{2}} = H + v \end{cases}$

$$M_H = \sqrt{-2\mu^2} = -\sqrt{2\lambda}\, v = \sqrt{\frac{\sqrt{2}}{G_F}}\, \sqrt{\lambda} \qquad v = \sqrt{\frac{-\mu^2}{\lambda}}$$

Couples to all particles that get mass ($\propto v$) through the SSB of $SU(2) \otimes U(1)$

$Hf\bar{f}$	$\dfrac{M_f}{v}$	HHW^+W^-	$\dfrac{M_W^2}{v^2}$
HW^+W^-	$\dfrac{2M_W^2}{v}$	HHZ^0Z^0	$\dfrac{M_Z^2}{2v^2}$
HZ^0Z^0	$\dfrac{M_Z^2}{v}$	HHH	$\dfrac{M_H^2}{2v}$
		$HHHH$	$\dfrac{M_H^2}{8v^2}$

Therefore,

→ produced in association with heavy particles
→ decays into the heaviest accessible particles

At one loop level, the Higgs can also couple to

gluons strong ↔ quarks ↔ weak → Higgs

photons e.m. ↔ $\begin{pmatrix} \text{leptons} \\ \text{quarks} \\ W^\pm \end{pmatrix}$ ↔ weak → Higgs

The Spirit of the Times - 36 BH → Before Higgs

"We should perhaps finish with an apology and a caution. We apologize to experimentalists for having no idea what is the mass of the Higgs boson, unlike the case with charm and for not being sure of its couplings to other particles, except that they are probably all very small. For these reasons we do not want to encourage big experimental searches for the Higgs boson, but we do feel that people performing experiments vulnerable to the Higgs boson should know how it may turn up."

Ellis, Gaillard, Nanopoulos (1976)

The very existence of the Higgs boson was far from certain. In the following years there were several attempts to find a mechanism for generating the mass of the vector bosons which keep the successful features of the SM and, at the same time, did not introduce any spurious particle. All attempts failed and the Higgs mechanism remained as the unique viable alternative.

Decay Modes of the Higgs Boson

LEP results on Higgs mass

→ direct search: $M_H > 114.4$ GeV with 95% of C.L.

→ global fitting of 18 parameters:

$$M_H = 129^{+74}_{-49} \text{ GeV}$$

$$M_H < 285 \text{ GeV with } 95\% \text{ of C.L.}$$

$115 \text{ GeV} < M_H < 130 \text{ GeV}$

→ dominated by $b\bar{b}$

→ the Higgs is narrow: $\Gamma_H < 10$ MeV

$M_H > 130$ GeV

→ dominated by W^+W^-/ZZ channels

→ the Higgs is wide: $\Gamma_H \sim \frac{M_H^3}{2}$ (TeV)

Production Mechanisms of the Higgs Boson

Electron - Positron Colliders

→ Bjorken $e^+ + e^- \to Z \to ZH$

→ WW Fusion $e^+ + e^- \to \nu\bar{\nu}(WW) \to \nu\bar{\nu}H$

→ ZZ Fusion $e^+ + e^- \to e^+e^-(ZZ) \to e^+e^-H$

→ Association with top $e^+ + e^- \to (\gamma, Z) \to t\bar{t}H$

CERN: LEP I ($\sqrt{s} = M_Z$) and LEP II ($\sqrt{s} = 2M_W$)

→ dominated by the Bjorken or bremsstrahlung process ($\sigma \propto \frac{1}{s}$) with the Z on (off)-shell

→ LEP were able to rule out from very small Higgs masses up to 114.4 GeV

Hadron Colliders

→ Gluon Fusion $p\bar{p} \to gg \to H$

→ VV Fusion $p\bar{p} \to VV \to H$

→ Association with V: $p\bar{p} \to q\bar{q}' \to VH$

→ Association with top: $p\bar{p} \to gg(q\bar{q}') \to t\bar{t}H$

Fermilab: Tevatron ($\sqrt{s} = 1.8(2)$ TeV)

→ produced in association with V: $VH(\to b\bar{b})$

CERN: Large Hadron Collider ($\sqrt{s} = 13$ TeV)

→ dominant production mechanism is the gluon fusion

→ best signal $H \to ZZ \to 4\ell^{\pm}$ for $M_H > 130$ GeV

→ for $M_H < 130$ GeV should rely on the small $BR(H \to \gamma\gamma) \sim 10^{-3}$

→ LHC could explore up to $M_H \sim 700$ GeV ($\mathcal{L} \sim 100\, fb^{-1}$)

The Goldstone Boson Equivalence Theorem

At high energies, the amplitude for emission or absorption of a longitudinally polarized gauge boson is equal to the amplitude for emission or absorption of the corresponding Goldstone boson

$$\hookrightarrow \mathcal{M}(W_L^{\pm}, Z_L^{\circ}) = \mathcal{M}(\omega^{\pm}, z^{\circ}) + \mathcal{O}(M_{W,Z}^2 / E^2)$$

We may use an effective Lagrangian describing the Goldstone boson interactions

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} i\sqrt{2}\,\omega^+ \\ v + H - i z^{\circ} \end{pmatrix}$$

The Higgs potential becomes

$$V(\Phi^{\dagger}\Phi) = \mu^2 \Phi^{\dagger}\Phi + \lambda(\Phi^{\dagger}\Phi)^2$$

$$= \frac{1}{2} M_H^2 H^2 + \frac{g}{4}\frac{M_H^2}{M_W} H(H^2 + 2\omega^+\omega^- + z^{\circ 2}) + \frac{g^2}{32}\frac{M_H^2}{M_W^2}(H^2 + 2\omega^+\omega^- + z^{\circ 2})^2$$

The result for the amplitude $W_L^+ W_L^- \to W_L^+ W_L^-$ is

$$\mathcal{M}(\omega^+\omega^- \to \omega^+\omega^-) = -i\frac{g^2}{4}\frac{M_H^2}{M_W^2}\left(2 + \frac{M_H^2}{s - M_H^2} + \frac{M_H^2}{t - M_H^2}\right)$$

$$\to -i\frac{g^2}{2}\frac{M_H^2}{M_W^2} = -i\,2\sqrt{2}\,G_F M_H^2$$

Therefore, for s-wave, unitarity requires

$$A_0 = \frac{1}{16\pi}|M| = \frac{2G_F}{8\pi\sqrt{2}} M_H^2 < 1$$

When combined with the other channels (Z^0Z^0, Z^0h, hh) we obtain an upper bound for the Higgs boson

$$M_H < 1.23 \text{ TeV}$$

The SM Higgs Boson Discovery

2012 Higgs Discovery

Observed independently by both ATLAS and CMS Collaborations

→ $L_0 = 5.1\ fb^{-1}$ (7 TeV) + $5.3\ fb^{-1}$ (8 TeV)

→ excess of events (c.f. "no Higgs boson" hypothesis) in invariant mass distribution around 125 GeV in several channels

- mass from different channels ($\gamma\gamma$, $ZZ \to 4\ell$, ...) were compatible
- data from different data-taking runs (7 and 8 TeV) were compatible
- combination made under hypothesis that the new particle was the SM Higgs boson

36th International Conference for High Energy Physics (ICHEP 2012)

CMS Collaboration

↳ a new boson was observed with a mass of $125.3 \pm 0.4(\text{stat}) \pm 0.5(\text{syst})$ GeV at significance level of 5σ

ATLAS Collaboration

↳ clear evidence for the production of a neutral boson with mass of $\underline{126.0 \pm 0.4\,(stat) \pm 0.4\,(syst)\ GeV}$ and significance of $\underline{5.9\sigma}$

What does 5σ mean? → the golden standard in Physics for a discovery

$$Pr(\mu - 5\sigma \leq X \leq \mu + 5\sigma) \approx 99.99994267\%$$

"the probability of the background alone fluctuating up by this amount or more is about one in three million" CMS

"[...] meaning that only one experiment in three million would see an apparent signal this strong in an universe without a Higgs." ATLAS

"the Nobel Prize in Physics 2013 was awarded jointly to François Englert and Peter W. Higgs 'for the theoretical discovery of a mechanism that contributes to our understanding of the origin of mass of subatomic particles, and which recently was confirmed through the discovery of the predicted fundamental particle, by the ATLAS and CMS experiments at CERN's Large Hadron Collider'."

The Higgs Details

Mass recoil energy of the Z in the Bjorken process (2 body), invariant mass of fully visible decay ($4\ell, \gamma\gamma, \ldots$)

Width and Couplings combination of several channels

Spin isotropic distribution of the decay products

Parity angular distribution in the Bjorken process

$$\frac{d\sigma}{d\cos\theta} \begin{cases} 0^+ : \vec{\epsilon}_{Z^*} \cdot \vec{\epsilon}_Z & \Rightarrow \sim \sin^2\theta \\ 0^- : \vec{\epsilon}_{Z^*} \times \vec{\epsilon}_Z \cdot \vec{p}_Z & \Rightarrow \sim (1+\cos^2\theta) \end{cases}$$

Full angular distribution of visible ~~distribution~~ products in gluon fusion process

Higgs self-coupling needs high energy and luminosity

→ HHH from subprocesses $VV \to HH$ or $gg \to HH$

→ HHHH only from loop contributions

Higgs Boson Mass

$M_H \propto \sqrt{\lambda}$ determines the high-energy behaviour of the electroweak model

→ measure the high-energy peak directly from the high-resolution channels

- $\gamma\gamma$, 4ℓ

→ determine M_H from a global fit to the electroweak model parameters

- most relevant $\sin^2(\theta^\ell_{eff})$, M_W and A's

→ good agreement between the directly observed values of M_W and M_t and those from a fit with $M_H = 125$ GeV

$$M_H = 125.09 \pm \underbrace{0.21}_{\text{Stat}} \pm \underbrace{0.11}_{\text{syst}} \text{ GeV}$$

Vacuum Stability

RGE corrections to λ depends on M_H, m_t and α_s. This can cause an instability of the EW potential at $\sim 10^{11}$ GeV with lifetime much longer than the age of the Universe

Higgs Mass Quantum Corrections

One-loop corrections to M_H

The loop integrals diverge with the cutoff scale

$$\int^{\Lambda_{NP}} \frac{d^4k}{k^2 - m^2} \sim \Lambda_{NP}^2$$

The correction to M_H at one-loop level

$$\delta M_H^2 = \frac{3\Lambda_{NP}^2}{8\pi^2 v^2}\left[M_H^2 + 2M_W^2 + M_Z^2 - 4m_t^2\right] + \cdots$$

nota bene: the Standard Model is renormalizable and there is no problem with the Higgs corrections

Higgs Boson Width

For $M_H = 125.09$ GeV, the theoretical Higgs total width is ~ 4.1 MeV, far lower than experimental resolutions, so it is impossible to measure directly. An alternati[ve] strategy is to leverage two regimes of the $H \to ZZ$ channel

→ write the gluon fusion production dependance on Γ_H through the Higgs boson propagator

$$\frac{d\sigma_{gg\to H\to ZZ}}{dM^2_{ZZ}} \sim \frac{g^2_{ggH}\, g^2_{HZZ}}{(m^2_{ZZ}-M^2_H)^2 + M^2_H\,\Gamma^2_H}$$

→ integrate either in a small window around M_H (on-shell Higgs), or above the mass threshold where $M_{ZZ} > 2M_Z$ (off-shell Higgs, but on-shell Z)

$$\sigma^{\text{on-shell}}_{gg\to H\to ZZ^*} \sim \frac{g^2_{ggH}\, g^2_{HZZ}}{M_H\,\Gamma_H} \quad \text{and} \quad \sigma^{\text{off-shell}}_{gg\to H^*\to ZZ} \sim \frac{g^2_{ggH}\, g^2_{HZZ}}{(2m_Z)^2}$$

Measure the off-shell/on-shell ration to estimate Γ_H

→ can use both $ZZ\to 4\ell$ and $ZZ\to 2\ell 2\nu$

Experimental results:

$\Gamma_H < 22$ MeV at 95% CL, and best fit result $\Gamma_H = 1.8^{+7.7}_{-1.8}$ MeV

Higgs Boson Couplings

Parametrize the H couplings to fermions λ_f and massive bosons g_V as

$$\lambda_f = \left(\frac{M_f}{M}\right)^{1+\epsilon}, \qquad g_V = 2\left(\frac{M_V^{2(1+\epsilon)}}{M^{1+2\epsilon}}\right)$$

which reproduce to the couplings of the SM Higgs boson in the double limit

$\epsilon\to 0$ and $M\to v = 246$ GeV

The Experimental measurements are

$$\epsilon = 0.01^{+0.041}_{-0.036} \quad \text{and} \quad M = 245 \pm 15 \text{ GeV}.$$

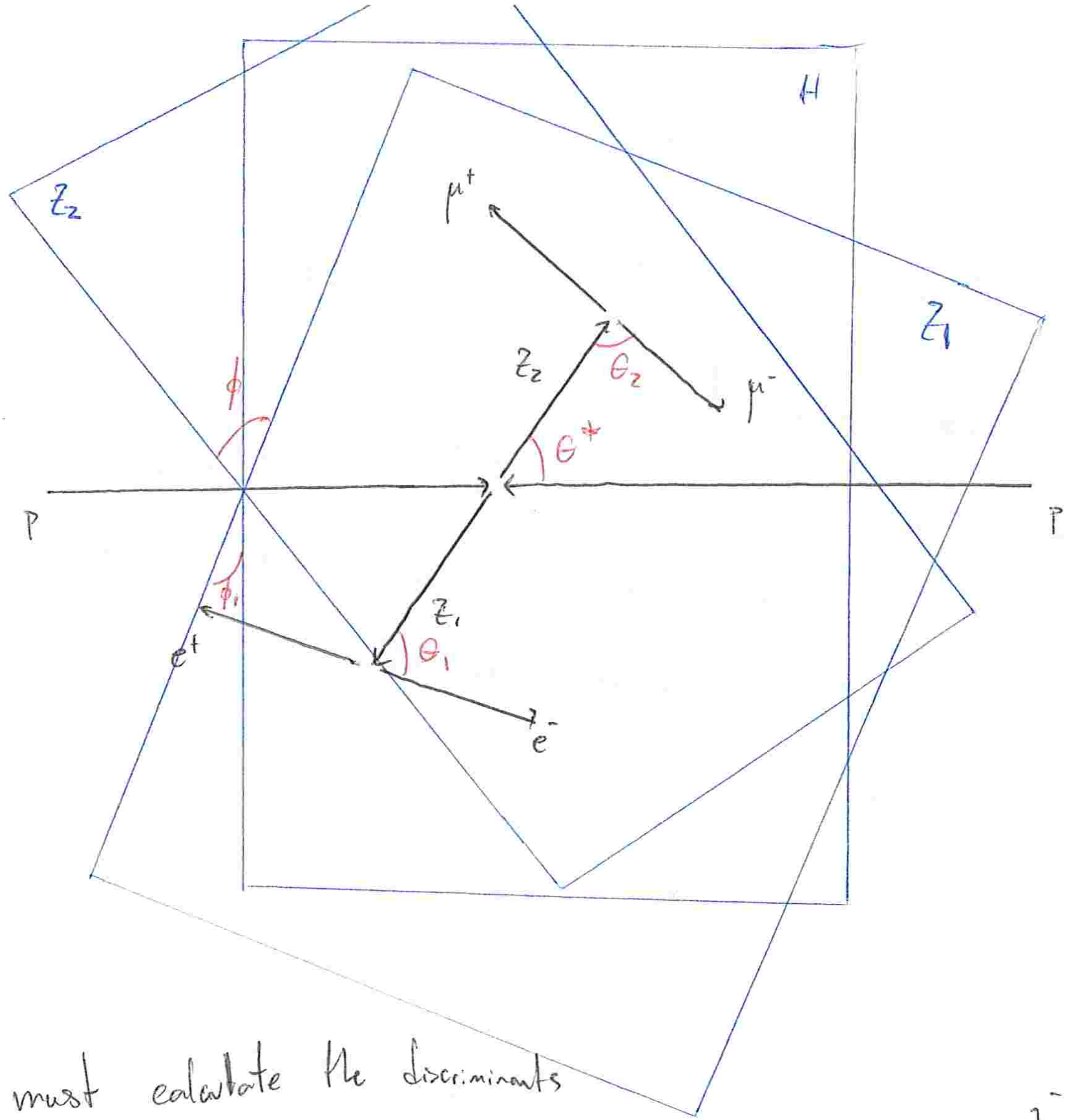
Landau-Yang theorem

Higgs Boson Spin and Parity

$H \to \gamma\gamma$ decay means that the new particle must not have spin 1. To infer spin and parity use the full angular information $\vec{\Omega}$ of the $H \to ZZ \to 4\ell$ decay. This is the Matrix Element Likelihood Approach (MELA). There are five angles in total:

- θ^*: angle of the $H \to ZZ$ decay in the rest frame of the Higgs boson
- θ_1, θ_2: the two decay angles of the leptons in the corresponding Z boson rest frames
- ϕ: azimutal angle between the decay planes of the two Z bosons
- ϕ_1: azimutal angle between the flight direction of one Z boson and the $H \to ZZ$ decay plane

We can obtain probability densities P for the four leptons to lead to a given $\vec{\Omega}$, using leading-order matrix element calculators, for both the $gg \to H$ signal ($P^{kin}_{0^+}$) and $q\bar{q}(gg) \to ZZ$ backgrounds (P^{kin}_{bkg}). It is very important to consider interference effects.



We must calculate the discriminants

$$D_{bkg} = \left[1 + \frac{P^{kin}_{bkg}(m_{Z_1}, m_{Z_2}, \vec{\Omega} | m_{4\ell}) \cdot P_{bkg}(m_{4\ell})}{P^{kin}_{0^+}(m_{Z_1}, m_{Z_2}, \vec{\Omega} | m_{4\ell}) \cdot P_{0^+}(m_{4\ell})} \right]^{-1}$$

$$D_{J^P} = \left[1 + \frac{P^{kin}_{J^P}(m_{Z_1}, m_{Z_2}, \vec{\Omega} | m_{4\ell})}{P^{kin}_{0^+}(m_{Z_1}, m_{Z_2}, \vec{\Omega} | m_{4\ell})} \right]^{-1}$$

and create a 2D likelihood function

$$\mathcal{L}^{J^P}_{2D}(D_{bkg}, D_{J^P})$$

where we can make a hypothesis test:

$$q = -2 \log\left(\frac{\mathcal{L}^{J}_{2D}}{\mathcal{L}^{0}_{2D}} \right).$$

Alternative hypothesis: $J^P = 0^-$

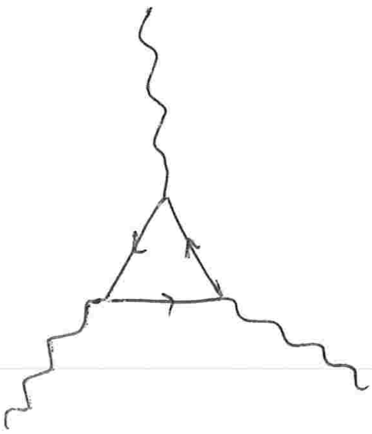
All hypotheses excluded w.r.t. $J^P = 0^+$, i.e., the Higgs boson is a scalar particle.

Fundamentals of Quantum Chromodynamics → QCD

Evidences for Color Quantum Number

Anomaly Cancellation

The cancellation of the anomalies is so important that have been used as a guide for constructing realistic theories. In the SM, the only possible anomalies come from VVA triangle loops and are proportional to



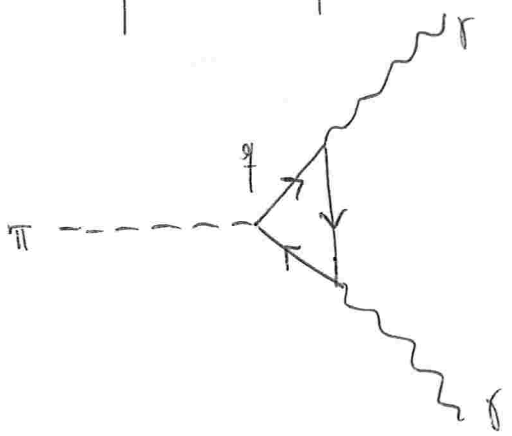
$$A^{abc} \propto -\sum_{doub} Y = -\left[-1 + N_c \cdot \frac{1}{3}\right] = 0$$

$$A^{abc} \propto \sum_{ferm} Y_+^3 - Y_-^3 = \left((-2)^3 + N_c\left[\left(\frac{4}{3}\right)^3 + \left(\frac{-2}{3}\right)^3\right]\right)$$
$$- \left((-1)^3 + (-1)^3 + N_c\left[\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^3\right]\right) = 0$$

the number of colors of the quarks ($N_c = 3$) is essential to cancel the anomaly

Pion Decay to 2 Photons

The decay occurs via the coupling of the pion to the quark loop and the width is given by



$$\Gamma(\pi^0 \to \gamma\gamma) = \left[N_c\left(Q_u^2 - Q_d^2\right)\right]^2 \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2}$$

$$= \left[N_c\left(Q_u^2 - Q_d^2\right)\right]^2 \cdot 7.6 \text{ eV}$$

experimental value:

$$\Gamma_{exp}(\pi^0 \to \gamma\gamma) = 7.82 \pm 0.14 \pm 0.17 \text{ eV}$$

suggests that

$$N_c\left(Q_u^2 - Q_d^2\right) = 3 \cdot \left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right] = 1$$

Wave Function of Δ^{++}

The baryon $\Delta(1232)^{++}$ is the lowest-mass quantum excitations of the proton. It is a $Q = +2$ and $J^P = 3/2^+$ state which decays very fast ($\sim 10^{-24}$ s) into a $p\pi^+$ mode. It is composed of three u-quarks which, of course, must have all spins aligned in order to give rise to its spin of 3/2. We expect that its wave function must be given by

$$\psi_\Delta = \psi^S_{flavor}(uuu) \times \psi^S_{spin}(\uparrow\uparrow\uparrow).$$

Δ^{++} is a fermion, and thus it must follow Fermi-Dirac statistics, i.e., its wave function must be antisymmetric. However, both flavor and spin wavefunctions are symmetric. We must then introduce a new wavefunction corresponding to a new quantum number (color) in such a way that

$$\psi^{s}_{color}(rgb)$$

in such a way that $\Delta^{++} = u_r\uparrow u_g\uparrow u_b\uparrow$ and

$$\psi^{A}_{\Delta} = \psi^{s}_{flavor}(uuu) \times \psi^{s}_{spin}(\uparrow\uparrow\uparrow) \times \psi^{A}_{color}(r\,g\,b).$$

Constructing a Gauge Theory for the Strong Interactions

The gauge group G must be a compact Lie group with finite unitary representation

N_c generators

- $N_c = 3$: quarks belong to triplet representation
- quark and anti-quark are different states: triplet is complex
- mesons and baryons are singlets ("colorless"): no proliferation of states (qq, $qqqq$, etc)

→ possible choices:

- $SO(3)\,[\simeq SU(2) \simeq S_p(1)]$
  ↳ triplet representation is real and particle and antiparticle are identical
  ↳ theory is not asymptotically free

- $U(3)$
  - ↳ there is a singlet gauge boson
  - ↳ long range strong interaction
- $SU(3)$
  - ↳ triplet representation is complex
  - ↳ matter fields: fundamental representation, 3
  - ↳ gauge fields: adjoint representation, 8

Gauge Bosons and Matter Representations

The N_G gauge bosons must belong to the adjoint representation of the gauge group. For $SU(N_c=3)$ the number of generators (number of gluons) is

$$N_G = N_c^2 - 1 = 3^2 - 1 = 8.$$

We choose the fundamental representation for the matter fields (quarks)

$$SU(N_c=3) \Rightarrow N_c = 3 = R, G, B$$

red
green
blue

→ quarks (q) belong to the fundamental representation of $SU(3)$ of dimension 3

→ gluons (G) belong to the adjoint representation of $SU(3)$ of dimension 8

We must define the covariant derivative and write the most general renormalizable Lagrangian, invariant under G, which couples all these fields. In non-Abelian gauge theories self-coupling of the gauge bosons is present $\to$ $\bar{\psi}\psi G$, GGG, $GGGG$

Lagrangian of QCD

The classical part of the Lagrangian is

$$\mathcal{L}_0 = \sum_{\text{flavors}} \bar{\psi}_i (i\not{D} - m)_{ij} \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

$i, j, k, \ldots = 1, 2, 3$
$a, b, c, \ldots = 1, \ldots, 8$
$\mu, \nu, \sigma, \ldots = 0, \ldots, 3$

where the sum runs over the N_F quark flavors. For each flavor the quark spinor is written as

$$\psi_i = \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix}.$$

the covariant derivative acting on the quark triplet field is

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + ig(T^a G^a_\mu)_{ij}$$

The kinetic term of the gluons is written in terms of the stress tensor

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g f^{abc} G^b_\mu G^c_\nu.$$

It is impossible to define a gluon propagator from this Lagrangian without making a choice of gauge. Therefore we should add a gauge fixing term

$$\mathcal{L}^{\text{color}}_{\text{gauge-fixing}} = -\frac{1}{2\xi}\left(\partial_\mu G^\mu\right)^2$$

Fixes the class of covariant gauge with gauge parameter ξ

The structure constants of the SU(3) color group f^{abc} is defined through

$$\left[T^a, T^b\right] = i f^{abc} T^c .$$

It is totally antisymmetric and given by

$$f^{123} = 1$$
$$f^{147} = f^{246} = f^{257} = f^{345} = f^{516} = f^{637} = \frac{1}{2}$$
$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

The SU(3) generators T^a are given in the Gell-Mann representation as

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$T_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The SU(3) generators T^a satisfy

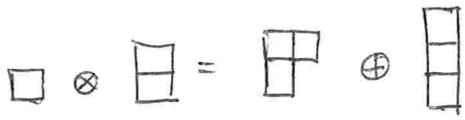
$$\rightarrow \quad \mathrm{Tr}\left[T^a T^b\right] = T_R\, \delta^{ab}, \qquad T_R = 1/2$$

$$\rightarrow \quad \sum_a T^a_{ik} T^a_{kj} = C_F\, \delta_{ij}, \qquad C_F = \frac{N^2-1}{2N} = \frac{4}{3}$$

$$\rightarrow \quad \sum_{cd} f^{acd} f^{bcd} = C_A\, \delta^{ab}, \qquad C_A = N = 3$$

Note that the gluon has a color and an anticolor. For SU(3) the product of the representation 3 (quarks) and $\bar{3}$ (antiquarks) is

$$3 \otimes \bar{3} = 8 \oplus 1$$



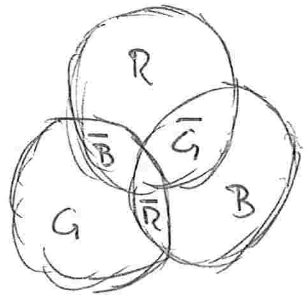
Young tableaux

We have an octet of color states

$$G_1 = R\bar{G}, \qquad G_2 = R\bar{B}, \qquad G_3 = G\bar{R},$$
$$G_4 = G\bar{B}, \qquad G_5 = B\bar{R}, \qquad G_6 = B\bar{G},$$

$$G_7 = \frac{1}{\sqrt{2}}\left(R\bar{R} - G\bar{G}\right), \qquad G_8 = \frac{1}{\sqrt{6}}\left(R\bar{R} + G\bar{G} - 2B\bar{B}\right),$$

and one singlet (white) state proportional to $R\bar{R} + G\bar{G} + B\bar{B}$ which can be ignored.



Quark Gluon Interaction

The quark-gluon interaction term of the Lagrangian is given by

$$\mathcal{L}_{int} = \bar{\psi}_i \left[i\gamma^\mu \left(i g T^a_{ij} G^a_\mu \right) \right] \psi_j$$

which gives rise to the Feynman rule

$$-ig T^a_{ij} \gamma_\mu .$$

QED: EM interaction between 2 fermions can be written in terms of the electric charge

$$q_1 q_2 \alpha, \quad q = \pm 1, \pm 1/3, \pm 2/3$$

The strength of the color interaction in QCD can be written as

$$\frac{1}{2} c_1 c_2 \alpha_s = C_F \alpha_s$$

c_i is the coefficient of the gluon-quark interaction

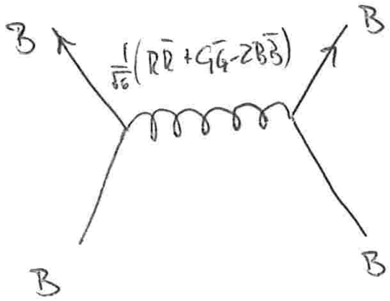
$$\alpha_s = \frac{g_s^2}{4\pi}$$

Two Quark Interaction

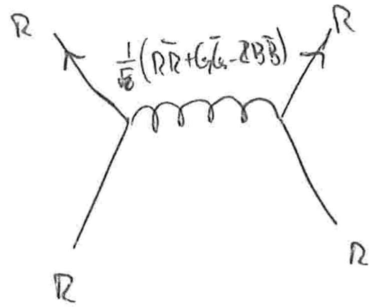
→ two blue quarks can interact just via the G_8 gluon and gives $c_1c_2 = 2/3$

→ two red quarks can occur both via G_8 and G_7 gluons giving $c_1c_2 = 1/6$ and $c_1c_2 = 1/2$

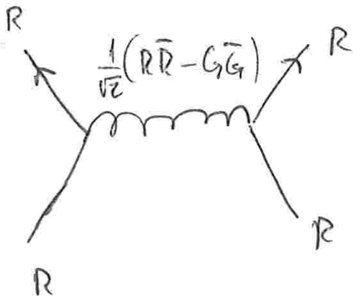
→ a red and a blue quark interact via G_8 and G_2 gluon with $c_1c_2 = -1/3$ and $c_1c_2 = 1$



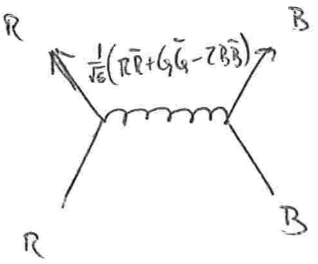
$c_1c_2 = \left(-\frac{2}{\sqrt{6}}\right)\left(-\frac{2}{\sqrt{6}}\right) = \frac{2}{3}$



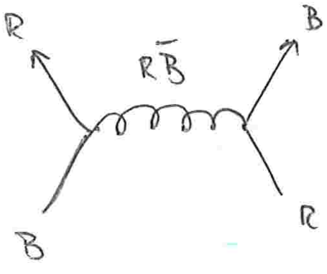
$c_1c_2 = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) = \frac{1}{6}$



$c_1c_2 = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$



$c_1c_2 = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{-2}{\sqrt{6}}\right) = -\frac{1}{3}$



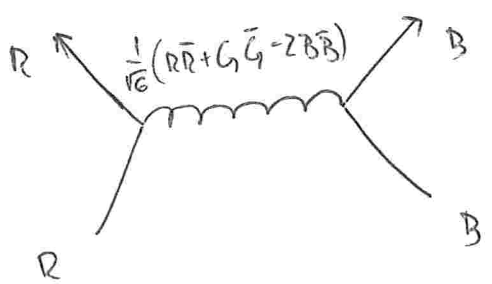
$c_1c_2 = 1$

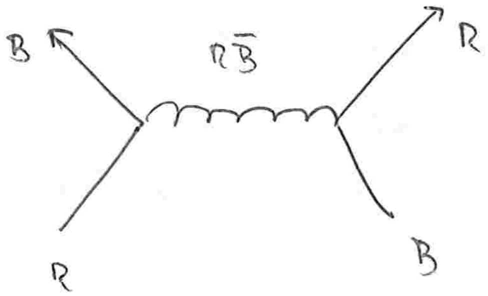
The amplitudes must be added (subtracted) if the quarks form a symmetric (antisymmetric) state

A meson is composed of a pair quark-antiquark in a color singlet state, i.e.

$$\langle q\bar{q}\rangle_{singlet} = \sqrt{\frac{1}{3}}\left(R\bar{R} + G\bar{G} + B\bar{B}\right)$$

The color factor for the gluon exchange between these quarks is (NB: the minus signal for the antiquark color factor):





$\hookrightarrow c_1 c_2 = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{-2}{\sqrt{6}}\right) = -\frac{1}{3}$ $\qquad$ $\hookrightarrow c_1 c_2 = 1$

Taking into account all possible colors (R, G, B) in the initial state ($\times 3$) we see that the strong force is attractive (negative):

$$c_1 c_2 = 3\sqrt{\frac{1}{3}}\sqrt{\frac{1}{3}}\left(-\frac{2}{3}-1-1\right) = -\frac{8}{3} \implies C_F = -\frac{4}{3}$$

Now, for a color octet quark-antiquark state we have a repulsive interaction

$$C_F = +\frac{1}{6}$$

For a color singlet three quark state (a baryon),

$$3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1,$$

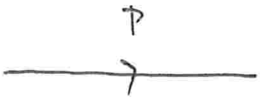
$$\square \otimes \square \otimes \square = (\square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}) \otimes \square = \square\square\square \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},$$

the interaction is very attractive:

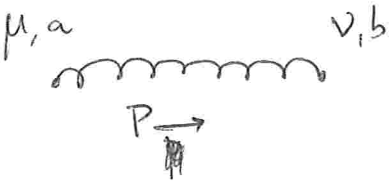
$$C_F = -2$$

this makes a natural assumption that the $q\bar{q}$ and qqq composites only exist in a color singlet state

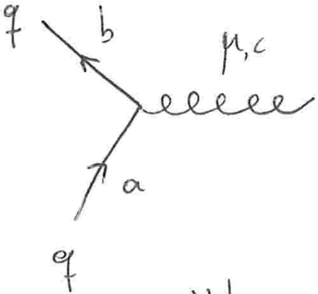
Feynman Rules for QCD



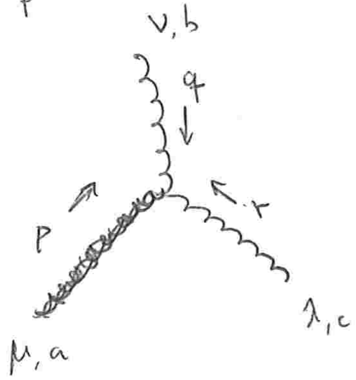
$$= \frac{i}{\not p + m + i\epsilon}$$



$$= \frac{-i\,\delta_{ab}\left[g_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2}\right]}{p^2 + i\epsilon}$$

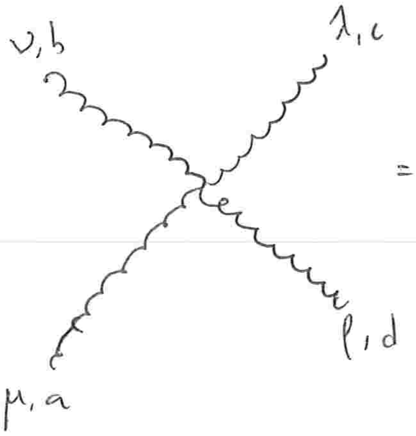


$$= -i g_s T^c_{ab} \gamma^\mu$$



$$= -g_s\left[(p-q)_\lambda g_{\mu\nu} + (q-r)_\mu g_{\nu\lambda} + (r-p)_\nu g_{\lambda\mu}\right]$$

$\hookrightarrow$ $p + q + r = 0$



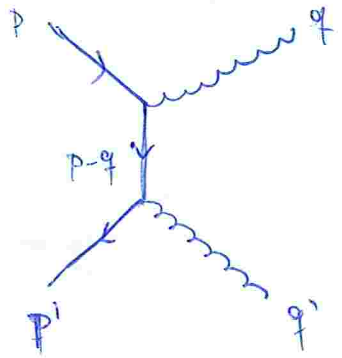
$$= -i g_s^2\Big[f^{abe} f^{cde}\left(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}\right)$$
$$+ f^{ace} f^{dbe}\left(g_{\mu\rho} g_{\nu\lambda} - g_{\mu\nu} g_{\lambda\rho}\right)$$
$$+ f^{ade} f^{bce}\left(g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho}\right)\Big]$$

Some Features of QCD

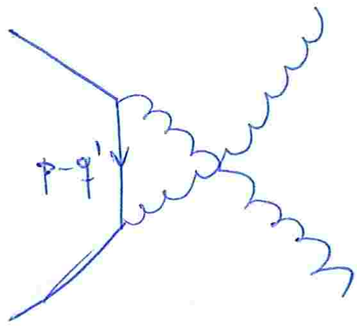
The Need for Ghosts

Consider the process

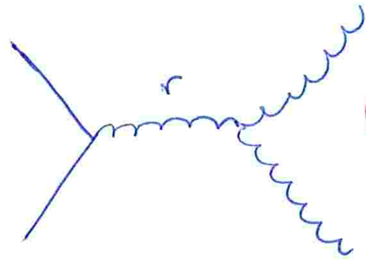
$$q_i(p) + \bar{q}_j(p') \to g_a(q) + g_b(q')$$



1



2



3

The invariant amplitude is

$$|\mathcal{M}|^2 = \sum_{\text{polar.}} |\mathcal{M}^{\alpha\beta}\epsilon_\alpha(q)\epsilon_\beta(q')|^2$$

$$= \sum_{\text{polar.}} \left[\left(\mathcal{M}^{+\alpha'\beta'}\epsilon^*_{\alpha'}(q)\epsilon^*_{\beta'}(q')\right)\left(\mathcal{M}^{\alpha\beta}\epsilon_\alpha(q)\epsilon_\beta(q')\right)\right]$$

$$= \left(\mathcal{M}^{+\alpha'\beta'}\mathcal{M}^{\alpha\beta}\right)\sum_{\text{polar.}}\left[\epsilon^*_{\alpha'}(q)\,\epsilon_{\alpha}(q)\right]\left[\epsilon^*_{\beta'}(q')\epsilon_\beta(q')\right]$$

$$\mathcal{M}^{\alpha\beta} = \mathcal{M}_1^{\alpha\beta} + \mathcal{M}_2^{\alpha\beta} + \mathcal{M}_3^{\alpha\beta}$$

The amplitude 1 is

$$\mathcal{M}_1^{\alpha\beta} = \bar{v}(p')\left(-ig\,T^b_{ik}\,\gamma^\beta\right)\frac{i\,\delta^{kl}}{\not{p}-\not{q}}\left(-ig\,T^a_{lj}\,\gamma^\alpha\right)u(p)$$

Notation: $T^aT^b \equiv T^a_{ik}T^b_{kj}$

$$\mathcal{M}_1^{\alpha\beta} = -ig^2\,\bar{v}(p')\left[\gamma^\beta T^b \frac{1}{\not{p}-\not{q}}\gamma^\alpha T^a\right]u(p).$$

$$\mathcal{M}_2^{\alpha\beta} = -ig^2\,\bar{v}(p')\left[\gamma^\alpha T^a \frac{1}{\not{p}-\not{q}}\gamma^\beta T^b\right]u(p).$$

Using the Feynman gauge for the gluon propagator, we have

$$\mathcal{M}_3^{\alpha\beta} = -g f^{abc} \underbrace{V^{\alpha\beta r}(-q,-q',r)} \frac{-i\,\delta^{cd} g_{rs}}{r^2} \bar{v}(p')\left[-ig T^d_{ij} \gamma^s\right] u(p)$$

$$V^{\alpha\beta r}(p,q,r) = g^{\alpha\beta}(p-q)^r + g^{\beta s}(q-r)^\alpha + g^{r\alpha}(r-p)^\beta$$

Using $[T^a,T^b] = i f^{abc} T^c$, the gluon exchange amplitude becomes

$$\mathcal{M}_3^{\alpha\beta} = -ig^2 V^{\alpha\beta r}(-q,-q',r)\frac{1}{r^2}\bar{v}(p')\gamma^s[T^a,T^b]u(p).$$

Now we must sum over the gluon polarization. Remember that in QED we write for the sum of the photon polarization:

$$E_{\mu\nu} = \sum_{\text{polar.}} \epsilon^*_\mu(q)\epsilon_\nu(q) = -g_{\mu\nu}.$$

Due to the EM gauge invariance $A_\alpha \to A_\alpha + \partial_\alpha \Lambda$ or, in momentum space, $\epsilon_\alpha \to \epsilon_\alpha + \lambda q_\alpha$, the fermionic electromagnetic current, ~~is conserved~~

$$\int e J^\mu A_\mu d^4x = \int e\bar{\psi}\gamma^\mu\psi A_\mu d^4x$$

is conserved: $\partial_\mu J^\mu = 0$.

$$\begin{aligned}\bar{v}(p')\gamma^\mu u(p) q_\mu &= \bar{v}(p')(\not{p}' + \not{p})u(p)\\ &= \bar{v}(p')(-m+m)u(p)\\ &= 0\end{aligned}$$

the Ward identity follows:

$$q_\alpha \mathcal{M}^{\alpha\beta} = 0$$

Question: is that all true for a non-Abelian gauge theory such as QCD?

Let us analyze with more detail the sum over the gluon polarization.

For a gluon with momentum $q = \frac{\sqrt{s}}{2}(1,0,0,1)$, we can write the transverse polarizations (helicity $= \pm 1$) as

$$\epsilon_{+\mu} = \frac{1}{\sqrt{2}}(0,1,+i,0) = \epsilon^*_{-\mu},$$

$$\epsilon_{-\mu} = \frac{1}{\sqrt{2}}(0,1,-i,0) = \epsilon^*_{+\mu}.$$

The sum over polarization states,

$$E^{\mu\nu} = \sum_{\lambda=\pm} \epsilon^{*\mu}_{\lambda}(q)\, \epsilon^{\nu}_{\lambda}(q),$$

becomes

$$E^{\mu\nu} = \frac{1}{2}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix} \otimes (0\ \ 1\ \ i\ \ 0) + \frac{1}{2}\begin{pmatrix}0\\1\\+i\\0\end{pmatrix} \otimes (0\ \ 1\ \ -i\ \ 0)$$

$$= \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&+i&0\\0&-i&1&0\\0&0&0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&-i&0\\0&+i&1&0\\0&0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

$$\hookrightarrow E^{\mu\nu}(q,q') = -g^{\mu\nu} + \frac{q'^{\mu}q^{\nu} + q^{\mu}q'^{\nu}}{q\cdot q'}$$

$\nearrow q' = \frac{\sqrt{s}}{2}(1,0,0,-1)$

Notice this sum runs only over the physical gluon states (transverse polarization states). That wouldn't be true if we discarded the second term and kept just the metric tensor: in this case there are some non-physical contributions. In QED, the last term has no effect, for $q_\mu \mathcal{M}^{\mu\nu} = 0$.

Let us check what happens for our QCD amplitude. In this case:

$$q_\alpha \left[\mathcal{M}_1^{\alpha\beta} + \mathcal{M}_2^{\alpha\beta}\right] = \cancel{-ig^2 \bar{v}(p')\left(\gamma^\beta [T^a, T^b]\right) u(p)}$$

$$q_\alpha \left[\mathcal{M}_3^{\alpha\beta}\right] = ig^2 \frac{1}{r^2} \bar{v}(p')\left(\not{q}' [T^a, T^b]\right) u(p)\, q' + \cancel{ig^2 \bar{v}(p')\left(\gamma^\beta [T^a, T^b]\right) u(p)}$$

terms cancel when summed

$$q_\alpha \mathcal{M}^{\alpha\beta} = H(q')\, q'^\beta$$

$$H(q') = ig^2 \frac{1}{r^2} \bar{v}(p')\left(\not{q}' [T^a, T^b]\right) u(p)$$

By Bose symmetry and taking into account the color commutator,

$$q'_\alpha \mathcal{M}^{\alpha\beta} = -H(q)\, q^\beta.$$

Therefore,

$$|\mathcal{M}|^2 = \left(\mathcal{M}^{\dagger\alpha'\beta'} \mathcal{M}^{\alpha\beta}\right) E_{\alpha'\alpha}(q,q') E_{\beta',\beta}(q,q'),$$

$$= \mathcal{M}^\dagger_{\alpha\beta} \mathcal{M}^{\alpha\beta} + H^\dagger(q') H(q) + H^\dagger(q) H(q'),$$

where we have used the so called physical (or axial) gauge,

$$E^{\mu\nu}(q,q') = -g^{\mu\nu} + \frac{q'^{\mu} q^{\nu} + q^{\mu} q'^{\nu}}{q \cdot q'} .$$

The first term comes from the metric tensor part of $E_{\mu\nu}$. Note that the second and third terms are proportional to $[T_a, T_b]$ and would vanish for Abelian gauge theories. These terms are essential to subtract the unphysical polarizations that were included in the first term. Thus, we are not allowed (in general) to use

$$E_{\mu\nu} \equiv \sum_{\text{polar.}} \epsilon^*_\mu(q)\, \epsilon_\nu(q) = -g_{\mu\nu}$$

in a non-Abelian gauge theory.

If we insist on using $E_{\mu\nu} = -g_{\mu\nu}$, we should include the ghost term in the Lagrangian

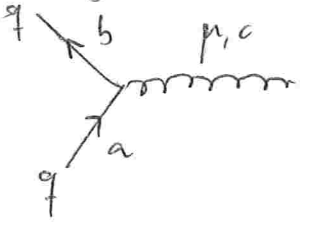
$$\mathcal{L}_{\text{ghost}} = \partial_\mu \eta^{a\dagger} (D^\mu_{ab} \eta^b)$$

η is a complex scalar field that obeys the Fermi statistics ("wrong" spin-statistics)

the covariant derivative for the octet (gluons) is

$$D^\mu_{ab} = \partial^\mu \delta_{ab} + i g (T^c G^c_\mu)_{ab} = \partial^\mu \delta_{ab} + i g f^{abc}$$

this Lagrangian gives rise to a new Feynman rule



$$= g f^{abc} q^\mu$$

we should include also the ghost external legs to the process. In this case, we can add the unphysical gluon states since they will be canceled by the ghost contribution.

Choice of Gauge

↳ instead of taking ghosts into account, we might work in the axial gauge

New gauge-fixing term, written in terms of another vector, denoted by n:

$$\mathcal{L}^{\text{axial}}_{\text{gauge-fixing}} = \frac{-1}{2\xi}(n\cdot A)^2 .$$

Compare this term with the one that fixes the class of covariant gauges:

$$\mathcal{L}^{\text{covar.}}_{\text{gauge-fixing}} = \frac{-1}{2\xi}(\partial_\mu A^\mu)^2 .$$

The gluon 2-point function in the axial gauge is given by

$$\Gamma^{(2)}_{ab,\alpha\beta} = i\delta_{ab}\left(p^2 g_{\alpha\beta} - p_\alpha p_\beta + \frac{1}{\xi} n_\alpha n_\beta\right),$$

which gives rise to the gluon propagator in the axial gauge:

$$\Delta_{ab,\alpha\beta}(p) = \frac{i\delta_{ab}}{p^2}\left[-g_{\alpha\beta} + \frac{n_\alpha p_\beta + p_\alpha n_\beta}{n\cdot p} - \frac{(n^2+\xi p^2)}{(n\cdot p)^2} p_\alpha p_\beta\right]$$

In the light-cone gauge $\xi = 0$ and $n^2 = 0$

$$\Delta_{ab,\alpha\beta}(p) = \frac{i\,\delta_{ab}}{p^2}\left[-g_{\alpha\beta} + \frac{n_\alpha p_\beta + n_\beta p_\alpha}{n\cdot p}\right].$$

This corresponds to the sum of gluon polarizations:

$$\sum_{\text{polar.}} \epsilon^*_\alpha(p)\,\epsilon_\beta(p) = E_{\alpha\beta}(p,n)$$

$$= -g_{\alpha\beta} + \frac{n_\alpha p_\beta + n_\beta p_\alpha}{n\cdot p}$$

Asymptotic Freedom

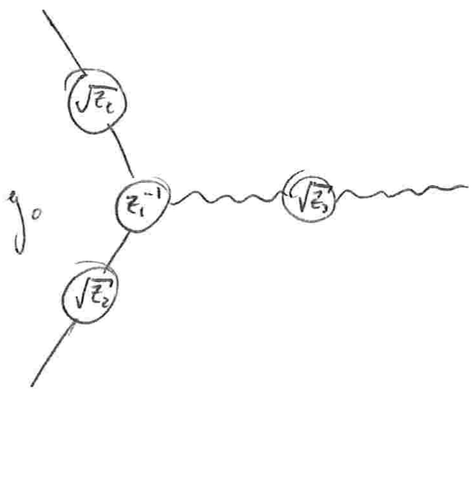
the running of the coupling constant is determined by the renormalization group equation

$$\beta_{QCD}(\alpha) = Q^2 \frac{d\alpha_{QCD}}{dQ^2} \longrightarrow \alpha_{QCD} = g_s^2 \cdot \frac{1}{4\pi}$$

↳ the dependence on Q^2 of the physical constant arises from the renormalization constant

$$g_s = Z_g(Q^2)\, g_0 \longrightarrow \text{bare charge}$$

Z_g depends on the vertex (Z_1), the fermion self-energy (Z_2) and the vacuum polarization (Z_3) renormalization constants through



$$Z_g = \frac{Z_2\sqrt{Z_3}}{Z_1} = -\frac{1}{2}\left(11 - \frac{2}{3}N_f\right)$$

Solving the equation for $\beta(\alpha)$ we obtain the QCD running coupling constant

$$\alpha_{QCD}(Q^2) = \frac{\alpha(\mu^2)}{1 \underset{\sqcup}{+} b\,\alpha_{QCD}(\mu^2)\log(Q^2/\mu^2)}$$

$$b = \frac{33 - 2N_F}{12\pi}$$

For $N_F < 17$, b is positive and the coupling constant decreases at small distances (high energies). QCD is asymptotically free. Perturbation theory could be used for high Q^2 (small distances), but ~~not~~ breaks down for small Q^2 (large distances).

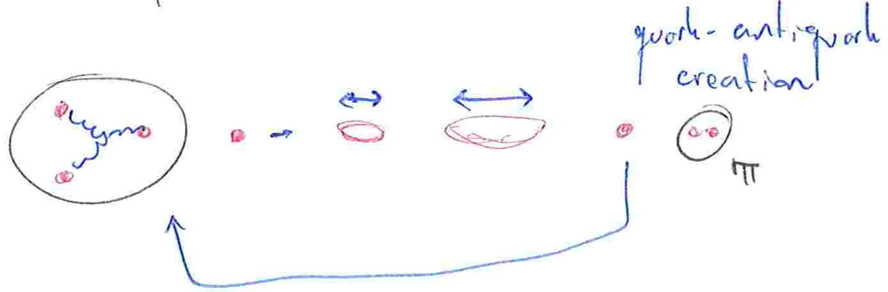
What about the QED running constant?

$$\alpha_{QED}(Q^2) = \frac{\alpha(\mu^2)}{1 \underset{\sqcup}{-} \left[\alpha_{QED}(\mu^2)/3\pi\right]\log(Q^2/\mu^2)}$$

α_{QED} increases with the increase of energy (or decrease of distance)

Confinement

Color confinement is the phenomenon that color charged particles cannot be isolated, and thus can't be directly observed in normal conditions. Quarks and gluons must remain together forming the hadrons. They cannot be separated from their parent hadron without producing new hadrons



Lattice QCD is able to evaluate the static quark-antiquark potential as a function of the distance (in lattice units). The existence of the linearly rising term ($\sigma > 0$) is a signal for confinement since it costs infinite energy to separate quarks to an infinite distance.