Dead Physicists Society
Instituto de Física
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Magneto-hidrodinâmica

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22 a 31 de julho de 2019
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Magneto-hidrodinâmica - EJT2301

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Magneto-hidrodinâmica

Geodbloed
Principles of MHD
Somov
Plasma Astrophysics

Referências

1. Fundamentals of Plasma Physics
J. A. Bittencourt
2. Notas de Aula - Elisabete G. Dal Pino (site)
. Cálculo Numérico
. MHD

Introdução ao Espaço de Fase e à Equação de Boltzmann

Espaço de Fase: espaço 6D $\hat{x}, \hat{y}, \hat{z}, \hat{v}_x, \hat{v}_y, \hat{v}_z$

O elemento de volume no espaço de fase é

$$d^3r\, d^3v = d\vec{r}\, d\vec{v} = dx\,dy\,dz\,dv_x\,dv_y\,dv_z$$

Função de distribuição no EF → como se distribuem as partículas no EF

densidade das partículas

$$F_\alpha(\vec{r},\vec{v},t) = \frac{dN(\vec{r},\vec{v},t)}{d^3r\, d^3v}$$

número de partículas em um volume no espaço de fase

diferentes tipos de partícula

$F_\alpha > 0$

$F_\alpha \to 0$ para $\vec{v} \to +\infty$

tratamento não-relativístico

do contrário, $F_\alpha = 0$ para $v > c$

Algumas grandezas macroscópicas que vem de F_α:

$$n_\alpha(\vec{r},t) = \int \frac{dN(\vec{r},\vec{v},t)}{d^3r} = \int \frac{F_\alpha(\vec{r},\vec{v},t)\, d^3r\, d^3v}{d^3r} = \int F_\alpha(\vec{r},\vec{v},t)\, d^3v$$

densidade numérica média

massa da partícula

$$m_\alpha\, n(\vec{r},t) = \rho_\alpha(\vec{r},t)$$

→ densidade de massa

$$n_\alpha \cdot \vec{u}_\alpha(\vec{r},t) = \int \frac{\vec{v}_\alpha\, dN_\alpha}{d^3r} = \int F_\alpha(\vec{r},\vec{v},t)\, \vec{v}_\alpha\, d^3v$$

$$\vec{u}_\alpha(\vec{r},t) = \int \frac{F_\alpha(\vec{r},\vec{v},t)\, \vec{v}_\alpha}{n_\alpha(\vec{r},t)}\, d^3v = \frac{\int \vec{v}_\alpha\, F_\alpha(\vec{r},\vec{v},t)\, d^3v}{\int F_\alpha(\vec{r},\vec{v},t)\, d^3v}$$

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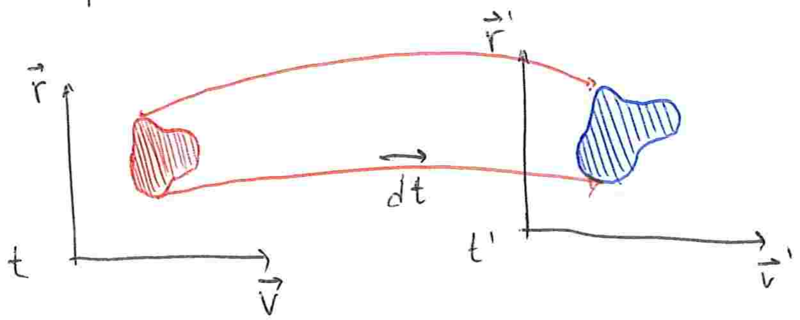
Equação de Boltzmann

↳ descreve a evolução temporal de $F_\alpha(\vec{r},\vec{v},t)$

3 variáveis nos interessam

$$\begin{cases} t' = t + dt \\ \vec{r}' = \vec{r} + \vec{v}\,dt \\ \vec{v}' = \vec{v} + \vec{a}\,dt \end{cases}$$

No espaço de fase, a evolução temporal pode ser vista na forma



O que se conserva na passagem de t para t'?

$$dN(\vec{r},\vec{v},t) = dN'(\vec{r}',\vec{v}',t')$$

Ou seja

$$F_\alpha(\vec{r},\vec{v},t)\,d^3r\,d^3v = \underbrace{F'_\alpha}_{F'_\alpha = F_\alpha}(\vec{r}',\vec{v}',t')\,d^3r'\,d^3v'$$

Podemos dizer que

$$d^3r\,d^3v = \underbrace{|J|}_{1}\,d^3r'\,d^3v'$$

jacobiano

A diferença das distribuições para diferentes tempos é então

$$\left[F_\alpha(\vec{r},\vec{v},t) - F_\alpha(\vec{r}',\vec{v}',t')\right]d^3r\,d^3v = 0$$

Abrindo $F_\alpha(\vec{r}',\vec{v}',t')$ em Taylor temos

$$F_\alpha(\vec{r}',\vec{v}',t') = F_\alpha(\vec{r}+\vec{v}\,dt,\vec{v}+\vec{a}\,dt,t+dt)$$

$$= F_\alpha(\vec{r},\vec{v},t) + \left[\frac{\partial F_\alpha}{\partial t} + \frac{\partial F_\alpha}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F_\alpha}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F_\alpha}{\partial z}\overbrace{\frac{\partial z}{\partial t}}^{v_z} + \frac{\partial F_\alpha}{\partial v_x}\frac{\partial v_x}{\partial t} + \frac{\partial F_\alpha}{\partial v_y}\frac{\partial v_y}{\partial t} + \frac{\partial F_\alpha}{\partial v_z}\underbrace{\frac{\partial v_z}{\partial t}}_{a_z}\right]dt$$

$$F_\alpha(\vec{r}',\vec{v}',t') = F_\alpha(\vec{r},\vec{v},t) + \left[\frac{\partial F_\alpha}{\partial t} + (\vec{v}\cdot\nabla)F_\alpha + (\vec{a}\cdot\nabla_{\vec{v}})F_\alpha\right]dt$$

$\nabla_{\vec{v}} = \hat{v}_x\frac{\partial}{\partial v_x} + \hat{v}_y\frac{\partial}{\partial v_y} + \hat{v}_z\frac{\partial}{\partial v_z}$

Como $F_\alpha(\vec{r},\vec{v},t) - F_\alpha(\vec{r}',\vec{v}',t') = 0$, temos

$$\frac{\partial F_\alpha}{\partial t} + (\vec{v}\cdot\nabla)F_\alpha + (\vec{a}\cdot\nabla_{\vec{v}})F_\alpha = 0$$

caso particular da Equação de Boltzmann

Equação de Boltzmann Não-Colisional

Outra forma de escrever a mesma equação é

$$\frac{D}{Dt}F_\alpha(\vec{r},\vec{v},t) = 0$$

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla + \vec{a}\cdot\nabla_{\vec{v}}$

A Equação de Boltzmann geral é

provém de usarmos $F_\alpha d^3r\, d^3v \neq F_\alpha' d^3r' d^3v'$

$$\frac{\partial F_\alpha}{\partial t} + (\vec{v}\cdot\nabla)F_\alpha + (\vec{a}\cdot\nabla_{\vec{v}})F_\alpha = \left[\frac{\partial F_\alpha}{\partial t}\right]_{Col}$$

Velocidades total, média e aleatória

$$\vec{v}_\alpha = \vec{u}_\alpha + \tilde{\vec{v}}_\alpha$$

Energia interna

$$E = \int \frac{\tilde{v}_\alpha^2}{2} f_\alpha(\vec{r},\vec{v},t)\, d^3v$$

densidade de energia por massa

Comentário

$$f_\alpha(\vec{r},\vec{v},t)d^3v = n(\vec{r},t)\, e^{-\frac{m\tilde{v}^2}{2k_BT}}\left[\frac{m}{2\pi k_B T}\right]^{3/2}$$

$$E = \int \frac{\tilde{v}_\alpha^2 m}{2} f_\alpha(\vec{r},\vec{v},t)\, d^3v = \frac{3}{2}k_BT$$

Equações de Conservação e Momentos da Equação de Boltzmann

$$\int v_\alpha^k \left(\frac{\partial f_\alpha}{\partial t} + (\vec{v}\cdot\nabla)f_\alpha + (\vec{a}\cdot\nabla)f_\alpha\right)d^3v$$

O momento de ordem zero

$$\left[\frac{\partial f_\alpha}{\partial t}\right]_{col} = 0$$

Temos

$$0 = \underbrace{\int \frac{\partial f}{\partial t} dv_i}_{\mathrm{I}} + \underbrace{\int v_i \frac{\partial f}{\partial r_i} dv_i}_{\mathrm{II}} + \underbrace{\int a_i \frac{\partial f}{\partial v_i} dv_i}_{\mathrm{III}}$$

Parte 1:

$$\frac{\partial}{\partial t}\int f_\alpha \, dv_i = \frac{\partial n_\alpha}{\partial t}(\vec{r}, t)$$

Parte 2:

$$v_i \frac{\partial f}{\partial r_i} = \frac{\partial}{\partial r_i}(v_i f)$$

$$\frac{\partial}{\partial r_i}\int v_i f \, dv_i = \frac{\partial}{\partial r_i}(u_i n)$$

Parte 3:

$$\int a_i \frac{\partial f}{\partial v_i} dv_i = \int \frac{\partial (f a_i)}{\partial v_i} dv_i = \left[f a_i\right]_{-\infty}^{+\infty} = 0$$

Logo,

$$0 = \frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial r_i}(n_\alpha u_i).$$

Em forma vetorial,

termo fonte ou sorvedouro

$$0 = \frac{\partial n}{\partial t} + \nabla \cdot (\vec{u} \cdot n)$$

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{u} \cdot \rho)$$

O momento de ordem um

$$\int v_j \left[\frac{\partial f_\alpha}{\partial t} + v_i \frac{\partial f_\alpha}{\partial r_i} + a_i \frac{\partial f_\alpha}{\partial v_i} \right] d^3v = 0$$

$$\underbrace{\int v_j \frac{\partial f_\alpha}{\partial t} d^3v}_{\mathrm{I}} + \underbrace{\int v_i v_j \frac{\partial f_\alpha}{\partial r_i} d^3v}_{\mathrm{II}} + \underbrace{\int a_i v_j \frac{\partial f_\alpha}{\partial v_i} d^3v}_{\mathrm{III}} = 0$$

Parte 1:

$$\int v_j \frac{\partial f_\alpha}{\partial t} d^3v = \frac{\partial}{\partial t} \int v_j f_\alpha d^3v = \frac{\partial}{\partial t}(u_j n)$$

Parte 2:

$$\int v_i v_j \frac{\partial f_\alpha}{\partial r_i} d^3v = \frac{\partial}{\partial r_i} \int v_j v_i f \, d^3v$$

$$= \frac{\partial}{\partial r_i} \int (u_j + \tilde{v}_j)(u_i + \tilde{v}_i) f \, d^3v$$

$$= \frac{\partial}{\partial r_i} \left[\int u_i u_j f d^3v + \int (\tilde{v}_j u_i + u_j \tilde{v}_i) f d^3v + \overbrace{\int \tilde{v}_i \tilde{v}_j f d^3v}^{\frac{1}{m} P_{ij}} \right]$$

Comentário

$$P_{ij} = \int \tilde{v}_i \tilde{v}_j f \, d^3v \, m_\alpha$$

se a pressão for isotrópicas podemos dizer que os únicos termos não-nulos são $i \neq j$

$$P = \delta_{ij} P_{ij}$$

$$P_{ii} = m \int \tilde{v}_1 \tilde{v}_1 f d^3v = m \int \tilde{v}_2 \tilde{v}_2 f d^3v = m \int \tilde{v}_3 \tilde{v}_3 f d^3v$$

$$= m \frac{1}{3} \int \tilde{v}_i \tilde{v}_i f d^3v = P_{ij} \delta_{ij} = P$$

$$P_{ii} = \int \frac{\tilde{v}_i^2}{3} m f d^3v = \frac{2}{3} E m$$

A integral II se torna

$$\frac{\partial (P_{ij}\delta_{ij})}{\partial r_i} + \frac{\partial}{\partial r_i}\left[\underbrace{\int u_i u_j f\, d^3v}_{u_i u_j n} + \underbrace{\int (\tilde{v}_j u_i + u_j \tilde{v}_i) f d^3v}_{2u_i \int \tilde{v}_j f\, d^3v = 0}\right]$$

$$\frac{\partial (P_{ij}\delta_{ij})}{\partial r_i} + \frac{\partial}{\partial r_i}\left(u_i u_j n\right)$$

Para a parte 3,

$$\int v_j a_i \frac{\partial f}{\partial v_i} d^3v = a_i \int v_j \frac{\partial f}{\partial v_i} d^3v = - a_i \delta_{ij} \cdot n$$

$$\frac{\partial (f v_j)}{\partial v_i} + \frac{\partial f}{\partial v_i} v_j + f \frac{\partial v_j}{\partial v_i}$$

Nova equação de conservação → momento

$$\frac{\partial}{\partial t}(n u_j) + u_i \frac{\partial}{\partial r_i}(u_j n) = - \frac{\partial (P_{ij}\delta_{ij})}{\partial r_i} + n a_i \delta_{ij}$$

$$\frac{\partial}{\partial t}(\rho \vec{u}) + (\vec{u}\cdot\nabla)(\vec{u}\cdot\rho) = -\nabla P + \vec{a}\rho$$

Momento de Boltzmann de ordem dois

$$\int \frac{v_j^2}{2}\left[\frac{\partial f_\alpha}{\partial t} + v_i \frac{\partial f_\alpha}{\partial r_i} + a_i \frac{\partial f_\alpha}{\partial v_i}\right] d^3v = 0$$

$$\underbrace{\frac{1}{2}\int v_j^2 \frac{\partial f_\alpha}{\partial t} d^3v}_{\mathrm{I}} + \underbrace{\frac{1}{2}\int v_j^2 v_i \frac{\partial f_\alpha}{\partial r_i} d^3v}_{\mathrm{II}} + \underbrace{\frac{1}{2}\int a_i v_j^2 \frac{\partial f_\alpha}{\partial v_i}}_{\mathrm{III}} = 0$$

Para a parte 1

$$\frac{1}{2}\int v_j^2 \frac{\partial f_\alpha}{\partial t} d^3v = \frac{1}{2}\frac{\partial}{\partial t}\int v_j^2 f_\alpha\, d^3v$$

$$\frac{1}{2}\int v_j^2 \frac{\partial f_\alpha}{\partial t}\, d^3v = \frac{1}{2}\frac{\partial}{\partial t}\int (u_j+\tilde{v}_j)(u_j+\tilde{v}_j)\, f_\alpha\, d^3v$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\int \Big(u_j^2 + \overbrace{2u_j\tilde{v}_j}^{0} + \tilde{v}_j^2\Big) f_\alpha\, d^3v$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\left[n u_j^2 + 2\mathcal{E}\right]$$

Quanto a parte 2,

$$\frac{1}{2}\int v_j^2 v_i \frac{\partial f_\alpha}{\partial r_i}\, d^3v = \frac{1}{2}\frac{\partial}{\partial r_i}\int\Big(\underbrace{u_j^2 v_i}_{i} + 2\underbrace{u_j\tilde{v}_j v_i}_{ii} + \underbrace{\tilde{v}_j^2 v_i}_{iii}\Big) f_\alpha\, d^3v$$

Para a parte i,

$$\frac{1}{2}\frac{\partial}{\partial r_i}\int u_j^2 v_i f\, d^3v = \frac{1}{2}\frac{\partial}{\partial r_i}\left(u_j^2 u_i n\right).$$

Para a parte ii,

$$\frac{2}{2}\frac{\partial}{\partial r_i}\int u_j\tilde{v}_j v_i f_\alpha\, d^3v = \frac{2}{2}\frac{\partial}{\partial r_i}\left(u_j\int(\tilde{v}_j u_i + \tilde{v}_j\tilde{v}_i) f\, d^3v\right),$$

$$= \frac{2}{2}\frac{\partial}{\partial r_i}\left(u_j u_i \underbrace{\int \tilde{v}_j f\, d^3v}_{0} + u_j\underbrace{\int \tilde{v}_j\tilde{v}_i f\, d^3v}_{\frac{1}{m}P_{ij}}\right),$$

$$= \frac{1}{m}\frac{\partial}{\partial r_i}\left(u_j P_{ij}\right).$$

Para a parte iii,

$$\frac{1}{2}\frac{\partial}{\partial r_i}\int \tilde{v}_j^2 v_i f_\alpha\, d^3v = \frac{1}{2}\frac{\partial}{\partial r_i}\left(u_i\int \tilde{v}_j^2 f_\alpha\, d^3v + \overbrace{\int \tilde{v}_j^2\tilde{v}_i f_\alpha\, d^3v}^{0}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial r_i}\left(\tilde{v}_j^2 u_i n\right)$$

também podemos escrever como $\mathcal{E}$

Finalmente, quanto à parte 3

$$\frac{1}{2}\int a_i v_j^2 \frac{\partial f_\alpha}{\partial v_i}\, d^3v$$

$v_j^2 = v_j v_j = v_i v_i = v_i^2$

$$\frac{\partial v_j^2}{\partial v_i} = 2v_i$$

$$\frac{\partial f_\alpha}{\partial v_i} v_j^2 = \frac{\partial (f v_j^2)}{\partial v_i} - f\frac{\partial v_j^2}{\partial v_i}$$

$$\frac{1}{2}\int a_i \left(\underbrace{\frac{\partial (f v_j^2)}{\partial v_i}}_{0} - 2 f v_i \right) d^3v = -\frac{1}{2} a_i u_i n$$

Assim, a equação de conservação de energia fica

$$\frac{\partial}{\partial t}\left(\frac{u^2 n}{2} + E\right) + \frac{\partial}{\partial r_i}\left(n u_i \left(\frac{u_j^2}{2} + \frac{\tilde{v}_j^2}{2}\right)\right) = a_i n u_i - \frac{\partial}{\partial r_i}\left(u_j \frac{P_{ij}}{m}\right)$$

$$\frac{\partial}{\partial t}\left(\rho\left(\frac{u^2}{2} + E\right)\right) + \frac{\partial}{\partial r_i}\left(\rho u_i \left(\frac{u^2}{2} + E\right)\right) = a_i \rho u_i - \frac{\partial}{\partial r_i}\left(u_j P_{ij}\right)$$

$$\mathcal{E} \equiv \rho\left(\frac{u^2}{2} + E\right)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot(\vec{u}\,\mathcal{E}) = \vec{a}\cdot\vec{u}\,\rho - \nabla\cdot(\vec{u}\,P)$$

Comentário: Referenciais de Euler e Lagrange

$$\underbrace{\frac{\partial}{\partial t} + \vec{u}\cdot\nabla}_{\text{visão Euleriana}} = \underbrace{\frac{D}{Dt} = \frac{d}{dt}}_{\text{visão Lagrangeana}}$$

derivada material

referencial do laboratório

referencial do fluido

Exemplo: eq. de conservação de massa

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\vec{u}\rho) = \frac{\partial \rho}{\partial t} + \rho\nabla\cdot\vec{u} + (\vec{u}\cdot\nabla)\rho$$

$\hookrightarrow (\vec{u}\cdot\nabla)\rho = \vec{u}\cdot\nabla\rho$

$$\frac{\partial \rho}{\partial t} + (\vec{u}\cdot\nabla)\rho = -\rho\nabla\cdot\vec{u}$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\vec{u} \rightarrow \text{forma Lagrangeana}$$

Comentário: forças no plasma → ênfase em plasmas astrofísicos

$$\vec{F} = \begin{cases} -\nabla\Phi\cdot\rho & \rightarrow \text{gravidade} \\ \dfrac{\vec{J}\times\vec{B}}{4\pi} & \rightarrow \text{Lorentz} \\ +\rho\nu\left(\nabla^2\vec{u} + \frac{1}{3}\nabla(\nabla\cdot\vec{u})\right) & \rightarrow \text{viscosidade} \end{cases}$$

em geral, o próprio plasma compensa efeitos elétricos

↳ vamos ignorar

A equação para o momento se torna então

$$\frac{\partial}{\partial t}(\vec{u}\rho) + (\vec{u}\cdot\nabla)(\vec{u}\rho) = -\nabla P - \rho\nabla\Phi + \frac{\vec{J}\times\vec{B}}{4\pi}$$

Já a equação para a energia

$$\frac{\partial \varepsilon}{\partial t} + \nabla\cdot\left(\vec{u}(\varepsilon + P)\right) = \vec{F}\cdot\vec{u}.$$

densidade de energia: $\frac{E^2}{8\pi}$, $\frac{B^2}{8\pi}$

fluxo de energia: $\frac{c}{4\pi}(\vec{B}\times\vec{E})$

$\Downarrow$ considerando $\vec{E}$ e $\vec{B}$

$$\frac{\partial \varepsilon}{\partial t}\left(\varepsilon + \frac{E^2}{8\pi} + \frac{B^2}{8\pi}\right) + \nabla\cdot\left(\vec{u}[\varepsilon + P] + \frac{c}{4\pi}(\vec{B}\times\vec{E})\right) = \vec{F}\cdot\vec{u}$$

$$\vec{F} = e\left(\vec{E} + \frac{\vec{u}\times\vec{B}}{c}\right)\cdot\frac{1}{V} \rightarrow \text{volume}$$

$$\frac{e}{V}\cdot\vec{u}\cdot\left(\vec{E} + \frac{\vec{u}\times\vec{B}}{c}\right) = \frac{e(\vec{u}\cdot\vec{E})}{V}$$

Vamos analisar as forças sobre os elétrons no sistema

$$n_e m_e \frac{d\vec{u}_e}{dt} = -\nabla p_e + n_e m_e \vec{g} - n_e e\left(\vec{E} + \frac{\vec{u} \times \vec{B}}{c}\right) + \vec{P}_{ei}$$

↳ termo associado a pressão colisional entre elétrons e íons

$$\frac{d\vec{u}_e}{dt} \approx \vec{0} \quad e \quad n_e m_e \vec{g} \approx \vec{0}$$

$$\vec{0} = -\nabla p_e - n_e e\left(\vec{E} + \frac{\vec{u} \times \vec{B}}{c}\right) + \vec{P}_{ei}$$

$$\vec{E} = \frac{-\nabla p_e}{n_e e} - \frac{\vec{u} \times \vec{B}}{c} + \frac{\vec{P}_{ei}}{n_e e}$$

Abrindo P_{ei}:

$$\vec{P}_{ei} = n_e n_i \sigma_{ei} u_{th} (\vec{u}_i - \vec{u}_e) m_e$$

Lembrando que

$$\vec{J} = \underbrace{Z n_i e}_{q_i} \vec{u}_i - \underbrace{n_e e}_{q_e} \vec{u}_e$$

Assumimos que há neutralidade de carga

$$Z n_i e = n_e e$$

$$\therefore \vec{J} = Z n_i e (\vec{u}_i - \vec{u}_e)$$

$$\vec{u}_i - \vec{u}_e = \frac{\vec{J}}{Z n_i e}$$

$$\vec{P}_{ei} = \frac{n_e n_i \sigma_{ei} u_{th} m_e \vec{J}}{Z n_i e}$$

$$\frac{\vec{P}_{ei}}{n_e e} = \underbrace{\boxed{\frac{\sigma_{ei} u_{th} m_e}{Z e^2}}}_{\eta} \cdot \vec{J}$$

$$\vec{E} = \eta \vec{J} \longrightarrow 1^{\underline{a}} \text{ Lei de Ohm}$$

Logo,

$$\vec{E} = -\frac{\nabla p_e}{n_e e} + \eta \vec{J} - \frac{\vec{u} \times \vec{B}}{c}.$$

A equação de energia se torna

$$\frac{\partial}{\partial t}\left(\mathcal{E} + \frac{E^2}{8\pi} + \frac{B^2}{8\pi}\right) + \nabla \cdot \left(\vec{u}(\mathcal{E} + P) + \frac{c}{4\pi}(\vec{E} \times \vec{B})\right) = \vec{J} \cdot \left(\frac{\nabla p_e}{n_e e} + \eta \vec{J} - \frac{\vec{u} \times \vec{B}}{c}\right)$$

A equação de indução do campo magnético

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \nabla \times \left[-\cancel{\frac{\nabla p_e}{n_e e}} + \eta \vec{J} - \frac{\vec{u} \times \vec{B}}{c} \right]$$

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \left[-\frac{u \times \vec{B}}{c} + \eta \vec{J} \right]$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left[\frac{\vec{u} \times \vec{B}}{c} - \eta \vec{J} \right]$$

Temos então as quatro equações de MHD

- $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$
- $$\frac{\partial}{\partial t}(\rho \vec{u}) + (\vec{u} \cdot \nabla)(\rho \vec{u}) = -\nabla P + \nabla \Phi - \frac{\vec{J} \times \vec{B}}{4\pi}$$
- $$\frac{\partial}{\partial t}\left(\mathcal{E} + \frac{E^2}{8\pi} + \frac{B^2}{8\pi}\right) + \nabla \cdot \left[\vec{u}(\mathcal{E} + P) + \frac{c}{4\pi}(\vec{B} \times \vec{E})\right] = \vec{J} \cdot \left(\frac{\nabla p_e}{n_e e} + \eta \vec{J} - \frac{\vec{u}_e \times \vec{B}}{c}\right)$$
- $$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) - c \nabla \times (\nabla \times (\vec{B} \eta))$$

Na equação de energia, podemos fazer

$$\vec{u}_e = \vec{u}_i - \vec{u}_i + \vec{u}_e$$
$$= \vec{u}_i - \frac{\vec{J}}{n_e e}$$

$$\vec{J}\cdot(\vec{u}_e \times \vec{B}) = \vec{J}\cdot\left(\vec{u}_i \times \vec{B} - \frac{\vec{J}\times\vec{B}}{n_e e}\right)$$

Para reescrever $\vec{J}\times\vec{B}$, podemos usar a conservação do momento

$$\rho\frac{d\vec{u}_i}{dt} = -\nabla p_i - \rho\nabla\Phi + \frac{1}{c}\vec{J}\times\vec{B}.$$

$$\vec{J}\times\vec{B} = c\left[\rho\frac{d\vec{u}_i}{dt} + \nabla p_i + \rho\nabla\Phi\right].$$

Juntando à equação de energia,

$$\frac{\rho}{2}\frac{du^2}{dt} + \vec{u}\cdot\nabla p - \rho\nabla\phi - \frac{\vec{J}\cdot\nabla p_e}{n_e e} + \eta J^2$$

$$\rho\frac{df}{dt} = \frac{\partial(\rho f)}{\partial t} + \nabla\cdot(\rho\vec{u}f)$$

$$\frac{\partial}{\partial t}\left(\rho(u^2+\phi)\right) + \nabla\cdot\left(\rho\vec{u}\left(\frac{u^2}{2}+\phi+\frac{P}{\rho}\right)\right) - P\frac{dV}{dt} - \vec{J}\frac{\nabla p_e}{n_e e} + \eta\vec{J}$$

$V = \frac{1}{\rho}$

$P_i \approx P$

Primeira Lei da Termodinâmica

$dU = dQ - PdV$ ~~PdV~~ ~~[illegible]~~

$dQ = TdS$

~~[illegible]~~

~~[illegible]~~

Tomando a variação temporal

$$\frac{dU}{dt} = T\frac{dS}{dt} - \frac{PdV}{dt}$$

Definição de entalpia

$$H = U + \frac{P}{\rho}.$$

Aplicando a primeira lei da termodinâmica na relação para a energia

$$\frac{\partial}{\partial t}\left(\rho\left(\frac{1}{2}u^2 + \phi + \mathcal{U}\right)\right) + \nabla\cdot\left(\rho\vec{u}\left(\frac{1}{2}u^2 + \phi + H\right)\right) - \rho T\frac{dS}{dt} - \frac{\vec{J}\cdot\nabla p_e}{n_e e} + \eta J^2.$$

A equação de energia total fica

$$\frac{\partial}{\partial t}\left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi} + \rho\left(\frac{1}{2}u^2 + \phi + \mathcal{U}\right)\right) + \nabla\cdot\left(\frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right) + \rho\vec{u}\left(\frac{1}{2}u^2 + \phi + H\right)\right)$$
$$= \rho T\frac{ds}{dt} - \eta J^2 + \frac{\vec{J}\cdot\nabla p_e}{n_e e}$$

Tomando que

$$U = \frac{3}{2}\frac{k_B T}{m} \quad e \quad H = \frac{3k_B T}{2m} + \frac{k_B T}{m}$$

$$\frac{\partial}{\partial t}\left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi} + \rho\left(\frac{1}{2}u^2 + \phi + \frac{3k_B T}{2m}\right)\right) + \nabla\cdot\left(\overbrace{\frac{B^2\vec{u}_e}{4\pi} + \frac{\vec{u}_e\cdot\vec{B}}{4\pi}\vec{B}}^{\mathrm{I}} + \frac{cn}{4\pi}\vec{J}\times\vec{B}\right.$$
$$\left. + \rho\vec{u}\left(\frac{1}{2}u^2 + \phi + \frac{5k_B T}{2m}\right)\right) - \frac{\vec{J}\cdot\nabla p_e}{n_e e} - \rho\frac{Tds}{dt} + \eta J^2 = 0$$

Para chegarmos em I

$$\frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{1}{4\pi}\left(-\vec{u}_e\times\vec{B} + \eta\vec{J}\times\vec{B}\right)\times\vec{B}$$

A equação de aquecimento $\xrightarrow{}$ ~~[illegible]~~ energia irradiada

$$\rho\frac{dT}{dS} = \eta J^2 + K\nabla^2 T - \rho\mathcal{L}(\rho, T)$$

$$S = \frac{1}{m}k_B\log\left(\frac{T^{\frac{1}{\gamma-1}}}{\rho}\right)$$

A equação do calor assume a forma

$$\underbrace{\rho T\frac{d}{dt}}_{\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{u}\cdot\nabla}\left(\frac{k_B}{m}\log\left(\frac{T^{\frac{1}{\gamma-1}}}{\rho}\right)\right) = K\nabla^2 T + \eta J^2 - \rho\mathcal{L}(\rho, T)$$

Temos assim

$$0 = \frac{k_B}{m}\left(\frac{1}{\gamma-1}\rho\frac{\partial T}{\partial t} + \frac{1}{\gamma-1}\rho(\vec{u}\cdot\nabla)T - T\frac{\partial\rho}{\partial t} - T(\vec{u}\cdot\nabla)\rho\right) - K\nabla^2 T - \eta J^2 - \rho\mathcal{L}(\rho, T)$$

Perturbando as equações $\longrightarrow$ $\vec{B}$, T, $\vec{u}$ e ρ

As perturbações são da ~~[illegible]~~ forma

$$f = \underbrace{f_0}_{\text{constante}} + f_1 \longrightarrow f_1 = f_1(t, x, y, z)$$

$$\vec{B} = \vec{B}_0 + \vec{B}_1$$
$$T = T_0 + T_1$$
$$\rho = \rho_0 + \rho_1$$
$$\vec{u} = \vec{u}_0 + \vec{u}_1$$

Conservação de massa

$$\frac{\partial}{\partial t}(\rho_0 + \rho_1) + \nabla\cdot((\rho_0 + \rho_1)\vec{u}_1) = 0$$

$\rightarrow$ ρ_0 é constante $\downarrow$ ao ser derivado se anula

Equação de Indução

$$\frac{\partial}{\partial t}(\vec{B}_0 + \vec{B}_1) = \nabla \times (\vec{u} \times \vec{B}) - c\, \nabla \times (\nabla \times \vec{B}\eta)$$

$$\frac{\partial \vec{B}_1}{\partial t} - (\vec{B}_0 \cdot \nabla)\vec{u}_1 + \vec{B}_0 \nabla \cdot \vec{u}_1 - c\eta \nabla^2 \vec{B}_1 = 0$$

A equação de momento

$$\frac{\partial}{\partial t}\left[(\rho_0 + \rho_1) \cdot \vec{u}_1\right] + (\vec{u}_1 \cdot \nabla)((\rho_0 + \rho_1)\vec{u}_1) = \frac{k_B}{m} \nabla\left[(\rho_0 + \rho_1)(T_0 + T_1)\right]$$

$$+ \rho \nabla \phi + \frac{\vec{J}_1 \times (\vec{B}_0 + \vec{B}_1)}{4\pi}$$

$$0 = \frac{\partial u}{\partial t} + \frac{k_B \nabla T}{m} + \frac{k_B T_0 \nabla \rho_1}{\rho_0} + \frac{1}{4\pi\rho_0} \vec{B}_0 \times (\nabla \times B_1) + \nabla \phi$$

Abrindo $\mathcal{L}(\rho, T)$ em série

$$\rho \mathcal{L}(\rho, T) = (\rho_0 + \rho_1)\left[\mathcal{L}(\rho_0, T_0) + \left.\frac{\partial \mathcal{L}}{\partial T}\right|_{\rho_0}(T - T_0) + \frac{\partial \mathcal{L}}{\partial \rho}(\rho - \rho_0)\right]$$

$$\rho \mathcal{L}(\rho, T) = (\rho_0 + \rho_1)\left[\mathcal{L}_T(T_1) + \mathcal{L}_\rho(\rho_1)\right]$$

$$\rho \mathcal{L}(\rho, T) = \rho_0\left[\mathcal{L}_T T_1 + \mathcal{L}_\rho \rho_1\right]$$

Assim a equação de energia perturbada é:

$$0 = \frac{k_B}{m}\left(\frac{\rho_0}{\gamma - 1}\frac{\partial T_1}{\partial t} - T_0 \frac{\partial \rho_1}{\partial t}\right) + \rho_0(T_1 \mathcal{L}_T + \rho_1 \mathcal{L}_\rho) - K \nabla^2 T_1$$

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A perturbação de forma geral

$$f_1 = f_1^* \cdot e^{i(\omega t + \vec{k}\cdot\vec{p})}$$

ERRATA

$$f_1 = f_1^* e^{\omega t + i\vec{k}\cdot\vec{p}}$$

$$\frac{\partial f_1}{\partial t} = \omega f_1$$

etc

Temos

$$\frac{\partial f_1}{\partial t} = i\omega f_1$$

$$\nabla f_1 = i\vec{k} f_1$$

$$\nabla \cdot \vec{f}_1 = i\vec{k}\cdot\vec{f}_1$$

$$\nabla \times \vec{f}_1 = i\vec{k} \times \vec{f}_1$$

$$\nabla^2 f_1 = -k^2 f_1$$

Aplicando a definição de f_1 nas equações teremos

- Conservação de massa

$$\omega \frac{\rho_1}{\rho_0} + i\vec{k}\cdot\vec{u} = 0$$

- Equação de indução

$$\omega \vec{B}_1 - i(\vec{k}\cdot\vec{B}_0)\vec{u} + i\vec{B}_0(\vec{k}\cdot\vec{u}) + c\eta k^2 \vec{B}_1 = 0$$

- Equação de calor

$$\frac{k_B}{m}\left(\frac{\rho_0}{\gamma-1} i\omega T_1 - i\omega T_0 \rho_1\right) + \rho_0(\rho_1 \mathcal{L}_\rho + T_1 \mathcal{L}_T) + K T_1 k^2 = 0$$

$$\left[\frac{k_B}{m} i\omega \frac{\rho_0}{\gamma-1} + \mathcal{L}_T \rho_0 + K k^2\right] T_1 - \frac{T_0 \rho_1 k_B i\omega}{m} + \rho_0 \rho_1 \mathcal{L}_\rho = 0$$

$$\frac{T_1}{T_0} = \frac{\rho_1}{\rho_0}\left[\frac{\frac{k_B i\omega}{m} - \frac{\rho_0 \mathcal{L}_\rho}{T_0}}{\frac{k_B i\omega}{m(\gamma-1)} + \mathcal{L}_T \rho_0 + \frac{K k^2}{\rho_0}}\right]$$

Lembrando $P = \rho \frac{k_B T}{m}$

$$\frac{dP}{dt} = \frac{k_B}{m}\left[T\frac{d\rho}{dt} + \rho\frac{dT}{dt}\right]$$

$$= \frac{k_B}{m}\left[T_0\rho_1 i\omega + \rho_0 T_1 i\omega\right]$$

$$i\omega P_1 = \frac{i\omega k_B}{m}\left[T_0\rho_1 + \rho_0 T_1\right]$$

$$P_1 = \frac{k_B}{m}\left[\rho_1 + \rho_0\frac{T_1}{T_0}\right]T_0$$

$$= \frac{\rho_0 k_B T_0}{m}\left[\frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}\right]$$

$$\frac{P_1}{P_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}$$

$$\frac{P_1}{P_0} = \frac{\gamma\rho_1}{\rho_0}\overbrace{\left[\frac{\frac{k_B\omega}{m(\gamma-1)} + \frac{1}{\gamma}\left(\mathcal{L}_T + \mathcal{L}_\rho\frac{\rho_0}{T_0}\right) + \frac{K h^2}{\gamma\rho_0}}{\frac{k_B\omega}{m}\frac{1}{\gamma-1} + \mathcal{L}_T + \frac{K h^2}{\rho_0}}\right]}^{\alpha(\omega, h)}$$

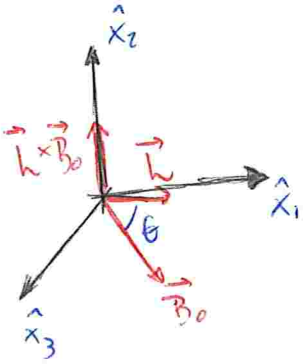
- Equação de momento

$$\omega\vec{u} + i\gamma\underbrace{\frac{P_0}{\rho_0}}_{c_s^2}\alpha\vec{h}\frac{\rho_1}{\rho_0} + \frac{i}{4\pi\rho_0}\vec{B}_0\times(\vec{h}\times\vec{B}_1) = 0$$

$c_s^2 \rightarrow$ velocidade do som

Aplicando as eqs. de massa e indução perturbadas, temos

$$0 = \vec{u}\omega + \frac{\gamma\alpha c_s^2}{\omega}\vec{h}\cdot(\vec{h}\cdot\vec{u}) - \frac{\vec{B}_0\times\left[(\vec{h}\times\vec{u})(\vec{h}\cdot\vec{u}) - (\vec{h}\cdot\vec{u})(\vec{h}\times\vec{B}_0)\right]}{4\pi\rho_0(\omega + c\eta h^2)}$$



Tomando que

$$\vec{h}\cdot\vec{u} = h u_1$$
$$\vec{h}\times\vec{u} = u_2 \hat{x}_3 - u_3 \hat{x}_2$$
$$\vec{h}\cdot\vec{B}_0 = h B_0 \cos\theta$$
$$\vec{h}\times\vec{B}_0 = h B_0 \sin\theta \, \hat{x}_2$$

$$u_A = \frac{B_0}{\sqrt{4\pi\rho}}$$ velocidade Alfvén

(A) $$\omega u_1 + \frac{\gamma\alpha c_s^2 h}{\omega} u_1 + \frac{h^2 u_A^2 \sin\theta}{\omega + c\eta h^2}\left[\sin\theta\, u_1 - \cos\theta\, u_3\right] = 0$$

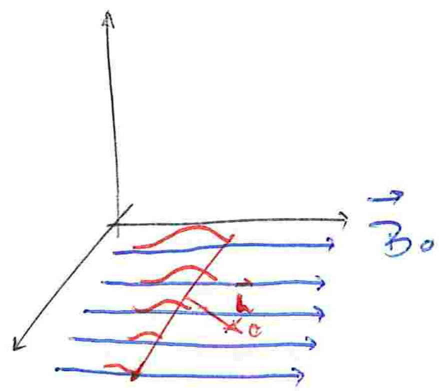
(B) $$\omega u_2 + \frac{h^2 u_A^2}{\omega + c\eta h^2}\cos^2\theta\, u_2 = 0$$

(C) $$\omega u_3 + \frac{h^2 u_A^2}{\omega + c\eta h^2}\cos\theta\left[\cos\theta\, u_3 - \sin\theta\, u_1\right] = 0$$

Eq. (B)

Primeiramente $\eta = 0$

$$\omega^2 + h^2 u_A^2 \cos^2\theta = 0$$
$$\omega = \pm i h u_A \cos\theta$$



Onda Alfven

$$u_\phi = \frac{i\omega}{h} = u_A \cos\theta$$

Força Restauradora

$$\frac{(\vec{B}\cdot\nabla)\vec{B}}{4\pi}$$

Observação

$$\vec{J}\times\vec{B} = \frac{(\nabla\times\vec{B})}{4\pi}\times\vec{B} = \frac{(\vec{B}\cdot\nabla)\vec{B}}{4\pi} + \nabla\left(\frac{B^2}{8\pi}\right)$$

Eq. (A) e (C)

$$\begin{cases} (\omega + \gamma h^2 c_s^2 + h^2 u_A^2 \sin^2\theta)\, u_1 - (h^2 u_A^2 \sin\theta \cos\theta)\, u_3 = 0 \\ (\omega^2 + h^2 u_A^2 \cos^2\theta)\, u_3 - (h^2 u_A^2 \sin\theta\cos\theta)\, u_1 = 0 \end{cases}$$

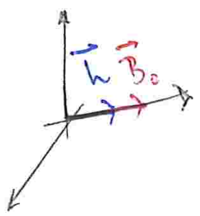
Para que o determinante das equação seja 0, é preciso que

$$\text{(D)} \quad \omega^4 + h^2\omega^2 \underbrace{(u_A^2 + \gamma c_s^2)}_{u_{MS}^2 \,\to\, \text{velocidade magneto-sônica}} + \gamma h^4 u_A^2 c_s^2 \cos^2\theta = 0$$

Como $u_\phi = \frac{i\omega}{h}$, obtemos ao dividir por h^4

$$u_\phi^2 - (u_A^2 + \gamma c_s^2)\, u_\phi^2 + \gamma c_s^2 u_A^2 \cos^2\theta = 0$$

Caso em que $\theta = 0$

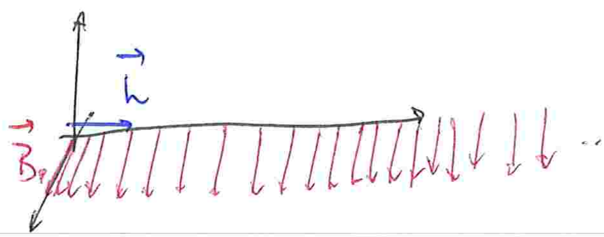


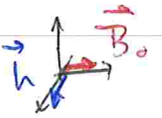
$$0 = u_\phi^4 - (u_A^2 + \gamma c_s^2)\, u_\phi^2 + \gamma c_s^2 u_A^2$$

$$0 = (u_\phi^2 - u_A^2)(u_\phi^2 - \gamma c_s^2)$$

$$\begin{cases} u_\phi = \pm u_A \to \text{onda Alfvén} \\ u_\phi = \pm\sqrt{\gamma}\, c_s \to \text{onda acústica} \end{cases}$$

Caso em que $\theta = \frac{\pi}{2}$





$$u_\phi(u_\phi - (u_A^2 + \gamma c_s^2)) = 0$$

$$\begin{cases} u_\phi = 0 \\ u_\phi = \pm\sqrt{u_A^2 + \gamma c_s^2} \end{cases} \to \text{ondas magnetoacústicas}$$

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