Perimeter Institute for Theoretical Physics

Classical Mechanics

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Level: Graduate
Period: Does not apply (online course)

Classical Mechanics
Perimeter Scholars International Online
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Course Goals

- → Review Classical Mechanics
- → Touch advanced topics (Hamilton-Jacobi Theory, Integrability, Constraints)
- → Symplectic Geometry (Basics of Differential Geometry)

Lagrangian Mechanics

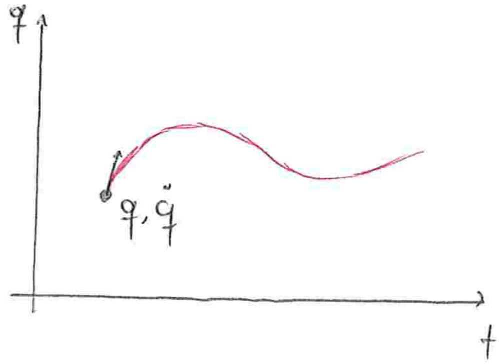
Hamilton's Principle of Least Action

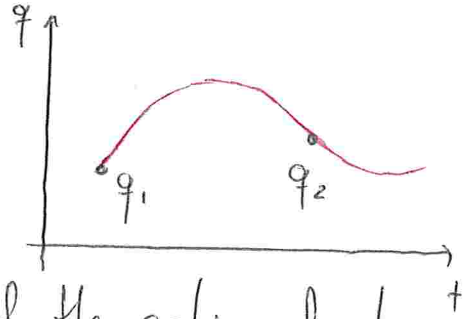
Motion: generalized coordinate $q(t)$

Two possible descriptions:

- → Initial value problem
- → Boundary value problem

$\updownarrow$ equivalent





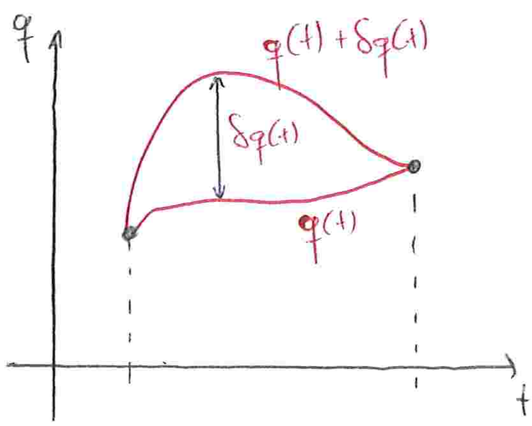
Principle of Least Action (Hamilton)

Motion of a mechanical system in $t \in (t_1, t_2)$ are given by extremals of the action functional S

$$S = S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

L → Lagrangian → determines the dynamical system

To derive the equations of motion, consider "fixed points"



$$\delta q(t_1) = \delta q(t_2) = 0$$

$\delta q(t)$ means variation of q at fixed time t

We seek an extremum

$$\delta S = S[q(t) + \delta q(t)] - S[q(t)] = 0$$

Identity:

$$\delta \frac{d}{dt} = \frac{d}{dt} \delta$$

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} \delta L \, dt$$

$$\delta \dot{q} = \frac{d}{dt} \delta q$$

$$= \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$

vanishes, for $\delta q(t_1) = \delta q(t_2) = 0$

$$= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q \, dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} = 0$$

↳ stationary action

$$\int \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i \, dt = 0 \quad \forall \ \delta q_i$$

$$\therefore \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

→ Euler-Lagrange Equations

↳ $i = 1, \dots, n$ → number of free degrees of freedom

Remarks:

→ $L(q, \dot{q}, t)$ yields 2nd order ODE → for more complicated Lagrangians, see tutorial

→ For typical mechanical systems,

$$L = T - V$$

T: kinetic energy; V: potential energy

→ the Lagrangian is not unique: $L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \dfrac{d \lambda(q, t)}{dt}$

L' and L give the same EoM for any function λ

$$S' = \int L' \, dt = \int L \, dt + \int \frac{d\lambda}{dt} dt = S + \lambda \Big|_{t_1}^{t_2}$$

$$\therefore \quad \delta S' = \delta S + \left[\frac{\partial \lambda}{\partial q} \cancel{\delta q} \right]_{t_1}^{t_2}$$

$\delta q \to 0$ → fixed endpoints

$$\delta S = 0 \iff \delta S' = 0$$

Integrals of Motion

Solving the Euler-Lagrange equations:

→ numerically
→ perturbatively → corrections to a known system
→ analytically
↳ integrable systems → there are very few of them

Definition: an integral (or constant) of motion is a function $I = I(q, \dot{q}, t)$ such that $\frac{dI}{dt} = 0$ for any $q(t)$ solving the EoM

↳ I remains constant along the trajectory
↳ if you can find enough of said integrals, you can solve the EoM
↳ how does one know if there are enough integrals?
↳ Noether's Theorem → connections between symmetries and integrals of motion

Terminology → we are borrowing it from HEP

→ statement is valid on-shell is valid provided the EoM are satisfied
→ statement is valid off-shell with no conditions imposed

~~Consider~~ Types of Symmetries

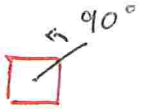
Consider transformations → not the same as in the principle of least action

$$t \to t' = t + \tilde{\delta} t$$
$$q \to q'(t') = q(t) + \tilde{\delta} q(t)$$

$$\tilde{\delta} t = \varepsilon \Delta t, \quad \tilde{\delta} q(t) = \varepsilon \Delta q$$

parameter (ε) → "how big"

generator (Δt) → "what transformation"

 90°

ε discrete

↓

no conserved quantities

lattice

a

Symmetries → discrete / continuous

 ε

continuous ↓

ε can be arbitrarily small

Symmetries → global → ε is constant

local → $\varepsilon = \varepsilon(t)$ → for fields, ε would be a function of spacetime

local ↓ gauge

Global symmetries: Noether's First Theorem → conserved quantities

Local symmetries: Noether's Second Theorem → Bianchi identities

↳ may help understand the system, but doesn't guarantee we may solve

In this course we shall only study global symmetries.

Noether's Theorem (Version I):

For every global continuous symmetry of the system, there is a corresponding on-shell integral of motion. □

Example: → the Lagrangian has no explicit time-dependence

$L \neq L(t)$

Einstein summation convention

$$E = \underbrace{\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i}_{\text{Einstein summation convention}} - L \longrightarrow \text{conserved energy}$$

on-shell

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \frac{\partial L}{\partial q}\dot{q} - \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)\dot{q} \overset{!}{=} 0$$

Example:

$L \neq L(q_c)$ → L doesn't depend on a certain coordinate q_c → cyclic coordinate

$p_c = \dfrac{\partial L}{\partial \dot{q}_c}$ → $\dfrac{dp_c}{dt} = 0$ comes directly from E-L equations

↳ generalized momentum

Noether's Theorem (Version II): → explicit formula

Let $\tilde{\delta}$ be a global continuous symmetry, i.e., off-shell we find $\tilde{\delta}q, \tilde{\delta}t$ such that $\tilde{\delta}S = 0$. Then

$$I = \frac{\partial L}{\partial \dot{q}}\tilde{\delta}q + \left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right)\tilde{\delta}t$$

is an on-shell integral of motion. □

Example:

$(\tilde{\delta}t, \tilde{\delta}q) = \epsilon(1, 0)$ → time translation → I = energy

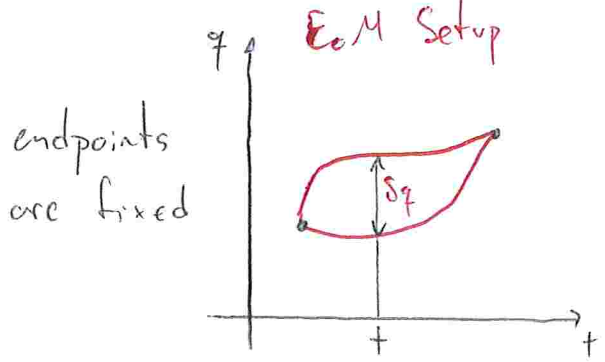
$(\tilde{\delta}t, \tilde{\delta}q) = \epsilon(1, 0)$ → space translation → I = momentum

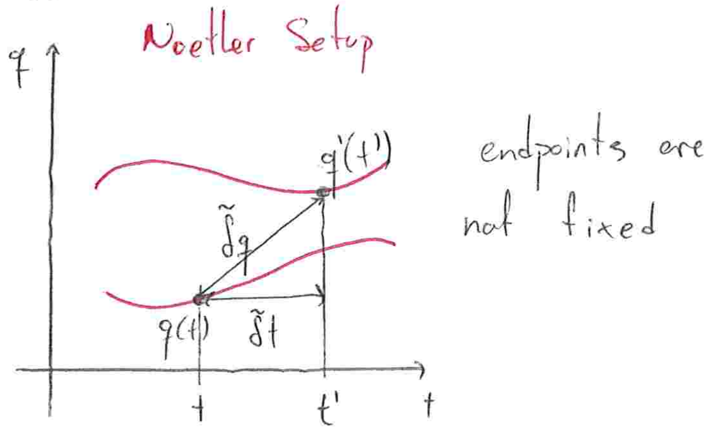
} homogeneity of space and time

$(\tilde{\delta}t, \tilde{\delta}\varphi) = \epsilon(0, 1)$ → space rotation → I = angular momentum } isotropy

Proof of Noether's Theorem

Firstly, notice the difference between the variations δ and $\tilde{\delta}$





$$\tilde{\delta} dt = dt' - dt = d\tilde{\delta}t = \frac{d\tilde{\delta}t}{dt} dt$$

$$\tilde{\delta} q(t) \equiv q'(t') - q(t) = q'(t) + \tilde{\delta}t \frac{dq'(t)}{dt} + \dots - q(t)$$

$$= \delta q(t) + \tilde{\delta}t \frac{dq'(t)}{dt} = \delta q(t) + \tilde{\delta}t \frac{dq(t)}{dt}$$ → difference between $\tilde{\delta}t \frac{dq'}{dt}$ and $\tilde{\delta}t \frac{dq}{dt}$ is second-order

In general, $\tilde{\delta} = \delta + \tilde{\delta}t \frac{d}{dt}$ for any function $f(q, \dot{q}, t)$

$$\tilde{\delta} S = \int \tilde{\delta} L\, dt + \int L\, \tilde{\delta} dt = 0$$

$$= \int \delta L + \tilde{\delta}t \frac{dL}{dt} + L \frac{d\tilde{\delta}t}{dt}\, dt = \int \delta L + \frac{d}{dt}\left(L \tilde{\delta}t\right) dt$$

$$= \int \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{d}{dt}\left(L \tilde{\delta}t\right) dt$$

doesn't vanish, for the endpoints ain't fixed

$$= \int \frac{\partial L}{\partial q} \delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \delta q + \frac{d}{dt}\left(L \tilde{\delta}t\right) dt$$

$$= \int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q + L \tilde{\delta}t\right) + \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) \delta q\, dt$$

vanishes on-shell

Therefore,

$$I = \frac{\partial L}{\partial \dot{q}} \delta q + L \tilde{\delta}t$$

$$= \frac{\partial L}{\partial \dot{q}} \left(\tilde{\delta} q - \tilde{\delta}t \frac{dq}{dt}\right) + L \tilde{\delta}t = \frac{\partial L}{\partial \dot{q}} \tilde{\delta} q + \left(L - \dot{q} \frac{\partial L}{\partial \dot{q}}\right) \tilde{\delta}t$$

is an on-shell integral of motion.

Noether's Theorem (Version III): → sneaky recipe

a) Observe that S is invariant under a global continuous transformation (characterized by a constant parameter ϵ): $\tilde{\delta} S = 0$.

b) Promote ϵ to $\epsilon(t)$ with fixed endpoints. Then we must have

$$\tilde{\delta} S = \int \dot{\epsilon} I \, dt,$$

which is the most general form linear in ϵ one can write.

c) Integrating by parts we find

see remark below

$$\tilde{\delta} S = - \int \frac{dI}{dt} \epsilon \, dt = 0,$$

where the last equality follows from the fact that ϵ is an "arbitrary variation δq" and it is valid on-shell (we used the action principle). This then implies that $\frac{dI}{dt} = 0$. So I is an on-shell integral of motion and can be read off from the expression of $\tilde{\delta} S$ in terms of $\dot{\epsilon}$ and I. $\square$

For explicit examples, see tutorial.

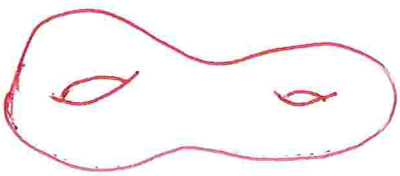
Note that

$$\tilde{\delta} S = \delta S + \int \frac{d}{dt}\left(L \, \tilde{\delta} t \right) dt.$$

Thus the two variations are equal up to a total derivative and it follows that on-shell we have $\tilde{\delta} S = 0 = \delta S$.

Introduction to Differential Geometry

Manifolds and Tensors



A manifold has the local differential structure of $\mathbb{R}^n$, but not its global properties.

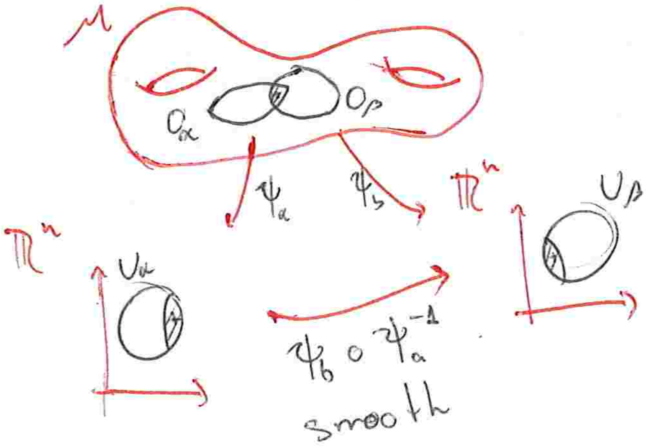
Every manifold is embedded in some $\mathbb{R}^m$ for sufficiently high m, but this is not useful for us.

Definition: An ~~manifold~~ n-dimensional manifold M is a "set of points" together with a colection of subsets $\{O_\alpha\}$ satisfying:

i) for each $p \in M$ lies in at least one O_α → the O_α cover the whole manifold

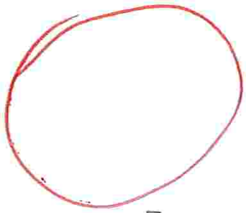
ii) for each α, there is an one-on-one, onto map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$ → locally, M looks like $\mathbb{R}^n$ → ψ_α is a "nice map"

iii) If any two sets O_α and O_β, $\alpha \neq \beta$, overlap, i.e., $O_\alpha \cap O_\beta \neq \emptyset$, then the map $\psi_\beta \circ \psi_\alpha^{-1}$ is smooth → in Physics, C^∞

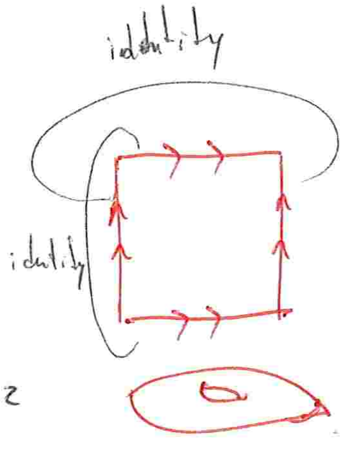


A Manifold is made of pieces that look like open sets of $\mathbb{R}^n$ which are sewn together smoothly

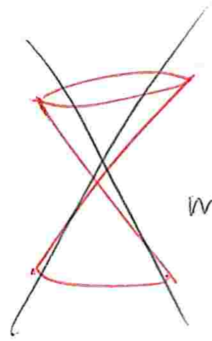
Examples:



sphere S^2



torus T^2



not a manifold → not smooth

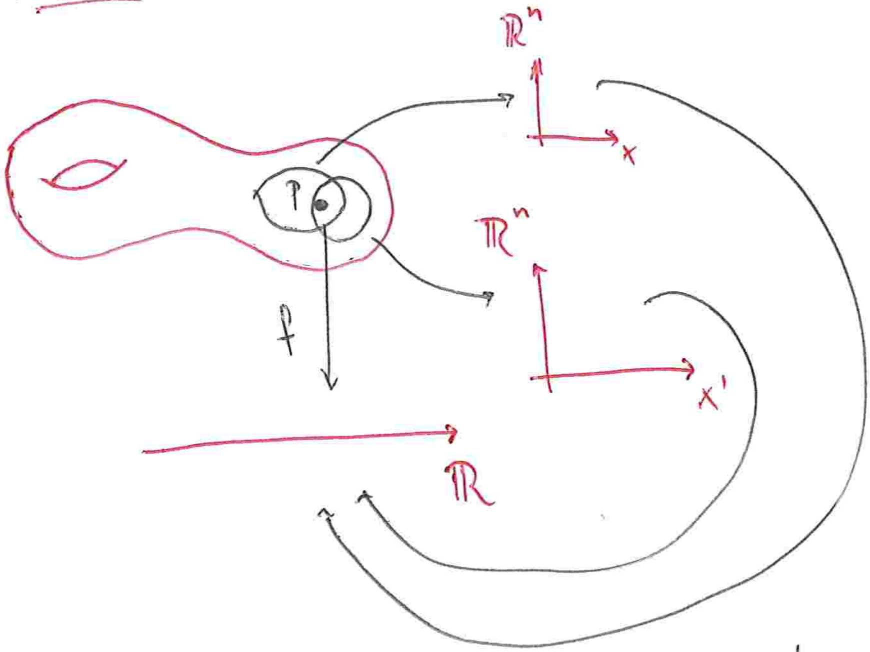
Terminology:

ψ_α: charts (for mathematicians) or coordinate system (for physicists) → usually denoted by x

$\{O_\alpha, \psi_\alpha\}$: atlas

Remark:
We need manifolds for description of point-particles in GR, but this ain't necessary for strings → String Theory works on orbifolds

A scalar function f is a map $f: M \to \mathbb{R}$ → must not depend on the coordinate system



$$f(p) \equiv f(x(p)) = f'(x'(p))$$

transformation law for f

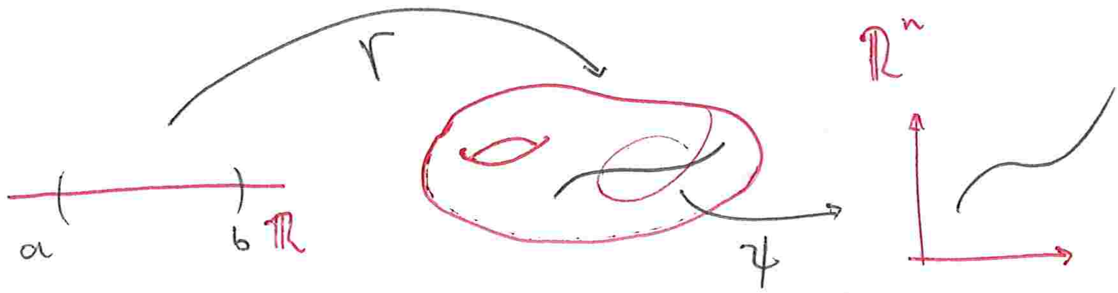
$$f(x) = f'(x')$$

assigns the same value regardless of coordinate system

Example of a scalar function: temperature on Earth

A curve γ on M is a map $\gamma: I \to M$, $I \subseteq \mathbb{R}$, such that

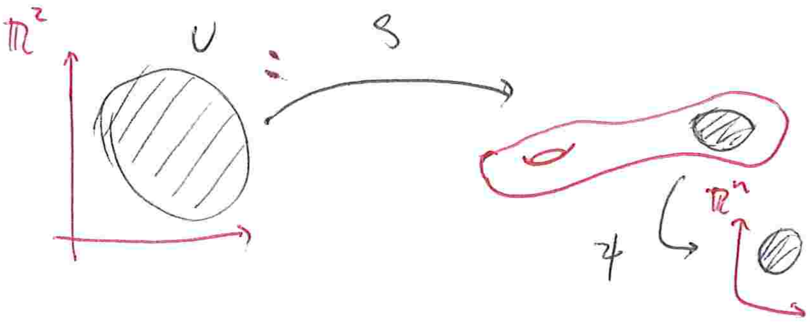
$(\psi \circ \gamma)(t) = [x^1(t), x^2(t), \dots, x^n(t)]$ is smooth



Examples: river, particle worldline

A surface S is a map $S: U \to M$, $U \subseteq \mathbb{R}^2$, such that

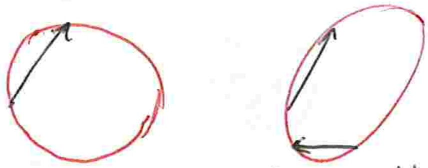
$(\psi \circ S)(\tau, \sigma) = [x^1(\tau, \sigma), x^2(\tau, \sigma), \dots, x^n(\tau, \sigma)]$ is smooth



Examples: lake, worldsheet of a string

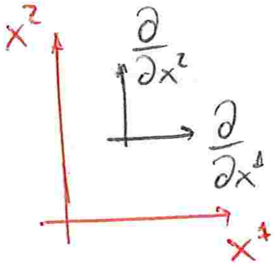
How can we define a vector in a manifold?

You can't !!. But we can define something which is almost as good.



A tangent vector is associated with the "direction of a derivative at a point"

In $\mathbb{R}^n$, $v^\mu = (v^1, v^2, \dots, v^n) \longleftrightarrow$ directional derivative operator $v^\mu \frac{\partial}{\partial x^\mu} \equiv \hat{v}$



$\hat{v} f = v^\mu \frac{\partial f}{\partial x^\mu} \longrightarrow$ at a point, it is just a number. If you run through points, it is a new function in $\mathbb{R}^n$

takes a function, gives a number

Directional derivative operator: characterized by linearity and the Leibnitz rule

$\rightarrow$ this can be generalized to a manifold

Definition: let $\mathcal{F}$ be a collection of C^∞ scalar functions. A tangent vector v at a point $p \in M$ is a map $v: \mathcal{F} \to \mathbb{R}$ such that

i) linear: $v(af + bg) = a\,v(f) + b\,v(g)$, $a, b \in \mathbb{R}$, $f, g \in \mathcal{F}$,

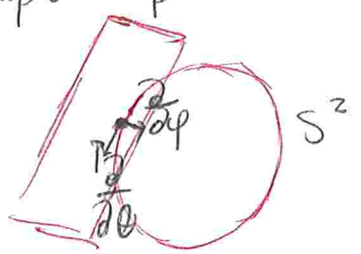
ii) Leibnitz rule: $v(fg) = f(p)\,v(g) + g(p)\,v(f)$.

Theorem: the set of tangent vectors at a point $p \in M$ forms a tangent vector space T_pM of the same dimension as M, with coordinate basis $\frac{\partial}{\partial x^\mu}$. Any vector v can be expressed as $v = v^\mu \frac{\partial}{\partial x^\mu}$. □

$\boxed{\text{for proof, see R. M. Wald's "General Relativity"}}$

components basis

Example $T_p S^2$



the vectors don't live on the manifold, but on their own space. Furthermore, they live in different spaces for each point $p \in M$. How can we differentiate vectors with all this problems?

↳ we'll see in some time

the basis $\frac{\partial}{\partial x^\mu}$ depends on coordinate system. How do the components change when we change the coordinate system?

$$V = V^\mu(x) \frac{\partial}{\partial x^\mu} = \underbrace{V^\mu(x) \frac{\partial x'^\nu}{\partial x^\mu}}_{\substack{\text{new components} \\ V'^\nu(x')}} \underbrace{\frac{\partial}{\partial x'^\nu}}_{\text{basis in the new coordinate system}}$$

$$V'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu} V^\mu(x)$$

⟷ the vector doesn't change, but the components do

Definition: a tangent vectorfield is defined as

$\{V|_p \in T_pM, \forall p \in M;\ V(f) \text{ is smooth}\}$. ↗ assigns a vector $V|_p$ to each point $p \in M$

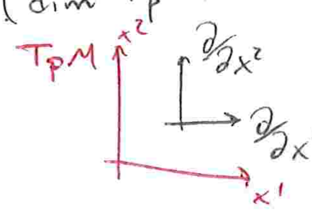
$(\forall f \in \mathcal{F}$

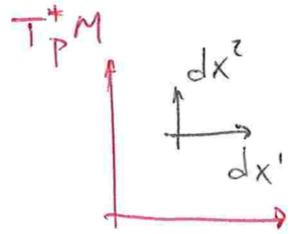
A tangent bundle: $TM = \bigcup_p T_pM$.

sort of ↙ ↳ local coordinates: (x^μ, v^ν)

Definition: a cotangent vector (or 1-form) ω at a point $p \in M$ is a map $\omega: T_pM \to \mathbb{R}$. The set of cotangent vectors at a point $p \in M$ is also a cotangent vector space T_p^*M with coordinate basis dx^μ ($\dim T_p^*M = \dim M$).

↳ dual to T_pM





The basis in T_p^*M is defined by the basis in T_pM through

$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu .$$

↳ Kronecker delta

Notice that we do not need a metric to define the cotangent vectors.

In terms of the basis dx^μ, we get

$$\omega = \underbrace{\omega_\mu}_{\text{~~basis~~ components}} \underbrace{dx^\mu}_{\text{~~components~~ basis}}$$

How do the cotangent vector's ~~coord~~ components change with coordinates?

$$\omega = \omega_\mu(x)\, dx^\mu = \underbrace{\omega_\mu(x) \frac{\partial x^\mu}{\partial x'^\nu}}_{\text{components}} \underbrace{dx'^\nu}_{\text{basis}}$$

$$\omega'_\nu(x') = \frac{\partial x^\mu}{\partial x'^\nu}\, \omega_\mu(x) \longrightarrow \text{transformation law}$$

Notice that $\dfrac{\partial x^\mu}{\partial x'^\nu} \dfrac{\partial x'^\nu}{\partial x^\pi} = \delta^\mu{}_\pi$ and thus tangent and cotangent vectors transform in "opposite" ways.

We may also define a cotangent bundle $T^*M = \bigcup_p T_p^*M$.

↳ tangent and cotangent bundles are examples of fiber bundles, though we don't need them now

↳ local coordinates (x^μ, ω_ν)

Note that $V \in T_pM$ uniquely defines $p \in M$ and $v \in T_pM$.

We might then define the canonical projection $\pi : TM \to M$ such that $\pi(V) = p$. For example,

$$\pi(x^1, \dots, x^n, v^1, \dots, v^n) = (x^1, \dots, x^n).$$

A tensor behaves like a product of vectors and forms.

Definition: a tensor of type (k, ℓ) and rank $k+\ell$ is a multilinear map

$$T : (T_p^* M)^k \times (T_p M)^\ell \to \mathbb{R}.$$

Its components transform as

once again, the tensor does not change, but the components do

$$T'^{\alpha\dots}{}_{\mu\dots}(x') = \underbrace{\frac{\partial x'^\alpha}{\partial x^\beta} \dots}_{k} \underbrace{\frac{\partial x^\nu}{\partial x'^\mu} \dots}_{\ell} T^{\beta\dots}{}_{\nu\dots}(x)$$

Example: metric g is a $(0,2)$ tensor: $g(V,W) \to \mathbb{R}$.

Tensor Algebra

i) Can add two tensors of the same type

ii) Tensor product: $T \otimes S$ is a $(t+s)$ rank tensor

iii) Contraction: $T^{\alpha\beta}{}_{\gamma}$

$$T^{\alpha\beta}{}_{\alpha} = V^\beta$$

free index

dummy indices

Examples:

tensor of ~~rank~~ type $(2,1)$

$\partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma$ is the basis

$$T = T^{\alpha\beta}{}_{\gamma}\, \partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma$$

The components are the tensor evaluated at the basis vectors

$$T^{\kappa\delta}{}_{\nu} = T(dx^{\kappa}, dx^{\delta}, \partial_{x^{\nu}}).$$

Indeed,

$$T(dx^{\kappa}, dx^{\delta}, \partial_{x^{\nu}}) = T^{\alpha\beta}{}_{r}\, dx^{\kappa}(\partial_{x^{\alpha}})\, dx^{\delta}(\partial_{x^{\beta}})\, dx^{r}(\partial_{x^{\nu}}),$$

$$= T^{\alpha\beta}{}_{r}\, \delta^{\kappa}_{\alpha}\, \delta^{\delta}_{\beta}\, \delta^{r}_{\nu} = T^{\kappa\delta}{}_{\nu}.$$

By linearity, we have

$$T(\omega, v, W) = T(\omega_\alpha dx^{\alpha}, v_\beta dx^{\beta}, W^{r}\partial_{x^{r}}),$$

$$= \omega_\alpha v_\beta W^{r}\, T(dx^{\alpha}, dx^{\beta}, \partial_{x^{r}}),$$

$$= T^{\alpha\beta}{}_{r}\, \omega_\alpha v_\beta W^{r},$$

which is a scalar function.

If S is a 1-form $S = S_\delta dx^{\delta}$, we have for $T \otimes S$

$$T \otimes S = \underbrace{T^{\alpha\beta}{}_{r} S_\delta}_{(T\otimes S)^{\alpha\beta}{}_{r\delta}}\, \partial_{x^{\alpha}} \otimes \partial_{x^{\beta}} \otimes dx^{r} \otimes dx^{\delta}.$$

Tensor field: map

$$T : (T^{*}M)^{k} \times (TM)^{\ell} \to \mathcal{F}$$

if we go over all the manifold, we get a function

Contraction

$$T_{\text{contr}} = T^{\alpha\beta}{}_{r}\, \partial x^{\alpha} \otimes \partial x^{\beta} \otimes dx^{r}$$

$$= T^{\alpha\beta}{}_{r}\, \partial x^{\beta}\, dx^{r}(\partial x^{\alpha}) = T^{\alpha\beta}{}_{r}\, \partial x^{\beta}\, \delta^{r}{}_{\alpha} = T^{\alpha\beta}{}_{\alpha}\, \partial x^{\beta}$$

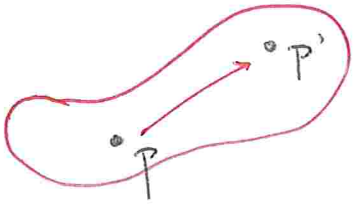
In order to formulate dynamics in manifolds, we must know how to differentiate tensors.

Lie Derivative

Problem when we want to differentiate

In real analysis: $\frac{df}{dt} = \lim_{s \to 0} \frac{f(t+s) - f(t)}{s}$

In the manifold:



$T_p M \neq T_{p'} M$

↳ we can't compare these vectors

Solution: move both vectors into the same tangent space → in general, can't be done and we need additional structure on the manifold

3 Standard Possibilities

i) Lie Derivative → requires the presence of a vector field U → guides how to move (and move vectors) on the manifold

ii) Exterior Derivative → doesn't need extra structure, but only works for special tensors (differential forms)

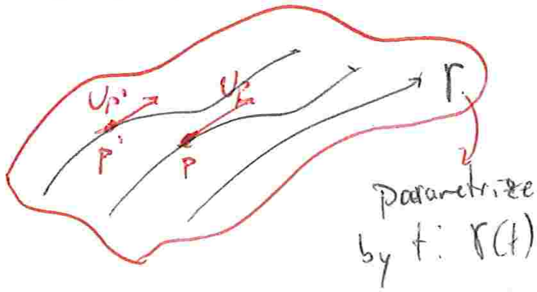
iii) Covariant Derivative

↓

requires a connection $\Gamma^{\alpha}_{\beta\gamma}$ → not a tensor

A vector field U defines its integral curves on M

↳ their tangent vectors coincide with $U_p \; \forall p \in M$



parametrize by t: $\gamma(t)$

Proof: coordinates x^μ

$$\frac{dx^\mu}{dt} = U^\mu(x) \longrightarrow \text{first-order ODE}$$

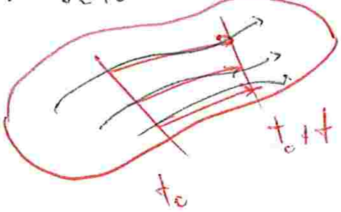
$\downarrow$

solution always exist

we want to find x^μ, which gives the curve's projection in $\mathbb{R}^n$

The integral lines define a map $\phi_t : M \to M$ by

$$\phi_t : \gamma(t_0) \mapsto \gamma(t_0 + t)$$



Def: diffeomorphism $\phi : M \to \tilde{M}$, bijective, ϕ is smooth, ϕ^{-1} is smooth

↳ with the inverse, one can map even the tensors from one manifold to another. Without inverse, things aren't as easy

a beautiful map

ϕ_t is really nice:

→ continuous in t

→ $\phi_0 = \mathrm{Id}$

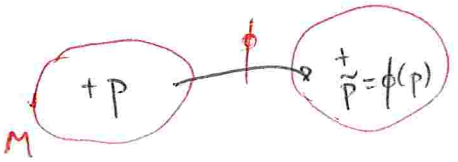
→ $\phi_{t+s} = \phi_t \circ \phi_s$

→ $\phi_{-t} = \phi_t^{-1}$

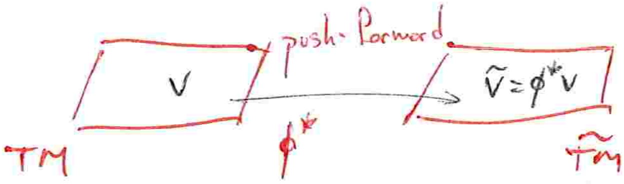
→ defines a 1-parametric (Lie) Group of diffeomorphisms

We may define an induced map $\phi^* :$ tensors on $M \to$ tensors on $\tilde{M}$ when we have a diffeomorphism

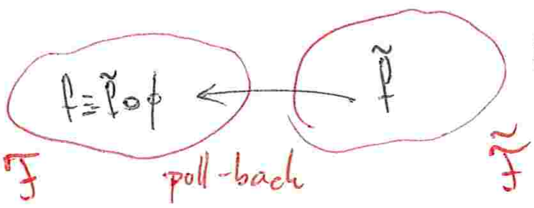
If we only had a smooth map (without the inverse) $\phi : M \to \tilde{M}$, we could define the induced maps



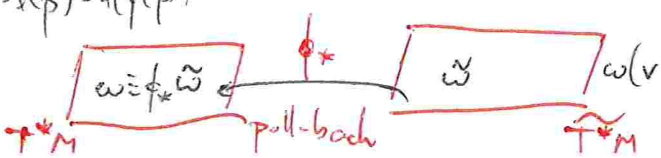
$\tilde{p} = \phi(p)$



$\tilde{v}(\tilde{f}) = v(f) = v(\tilde{f} \circ \phi)$



$f(p) = \tilde{f}(\tilde{p}) = \tilde{f}(\phi(p))$

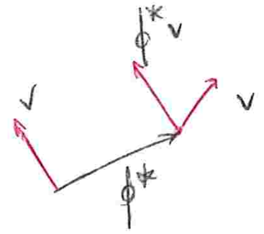


$\omega(v) = \tilde{\omega}(\tilde{v}) = \tilde{\omega}(\phi^* v)$

without the inverse, for a given type of object the map can only work one-way.

Definition: Let ϕ_t be a 1-parametric group of diffeomorphisms generated by a vector field U. Then the Lie derivative $\mathcal{L}_U$ is defined by

$$\mathcal{L}_U T|_p = \lim_{t \to 0} \frac{T|_p - \phi_t^* \{T|_p\}}{t}$$



↳ what was there minus what I carried there

Another viewpoint: we are $\overbrace{\text{changing the coordinate system}}^{\text{passive viewpoint}}$ instead of $\underbrace{\text{carrying the vectors}}_{\text{active viewpoint}}$ ↳ easier to calculate

Example:

$$\tilde{f}(t_0) = f(t_0 - t)$$

from passive view, we are changing coordinates and the value must be that at previous point

$$\mathcal{L}_U f = \lim_{t \to 0} \frac{f(t_0) - \tilde{f}(t_0)}{t}$$

$$= \frac{df}{dt} = \frac{dx^\mu}{dt} \frac{\partial f}{\partial x^\mu} = U^\mu \frac{\partial f}{\partial x^\mu}$$

Properties of the Lie derivative

i) $\mathcal{L}_U$ maps (k, l) tensors to (k, l) tensors

ii) $\mathcal{L}_U$ is linear and preserves contradiction

iii) $\mathcal{L}_U$ obeys the Leibnitz rule: $\mathcal{L}_U (T \otimes S) = (\mathcal{L}_U T) \otimes S + T \otimes (\mathcal{L}_U S)$

iv) $\mathcal{L}_U f = U(f)$, $\longrightarrow$ Lie bracket / commutator

$$\mathcal{L}_U V = [U,V] = UV - VU$$

$$\mathcal{L}_U T^{\alpha\cdots}{}_{\beta\cdots} = U^\gamma \frac{\partial}{\partial x^\gamma} T^{\alpha\cdots}{}_{\beta\cdots} - T^{\gamma\cdots}{}_{\beta\cdots} \frac{\partial U^\alpha}{\partial x^\gamma} + \cdots + T^{\alpha\cdots}{}_{\gamma\cdots} \frac{\partial U^\gamma}{\partial x^\beta}$$

negative sign if derivative wrt ~~for~~ higher index

positive sign if derivative wrt lower index

One way obtain the formula for a 1-form by using the formula for a vector and the Leibnitz rule, for example

Why is the Lie derivative interesting? Because it defines what we mean by a symmetry $\rightarrow \mathcal{L}_U T = 0$

If $\mathcal{L}_U T = 0$, U determines a special direction so that T doesn't change in that direction.

↳ e.g., for energy conservation, moving in the time direction keeps energy constant $\rightarrow$ when we have a symmetry $\rightarrow$ isometries

Symmetries of the metric: $\mathcal{L}_U g_{\alpha\beta} = 0$

Differential Forms $\rightarrow$ e.g. EM field tensor

Definition: a differential p-form ω is a totally antisymmetric tensor of type $(0,p)$. $\rightarrow$ antisymmetric under exchange of any two indices

$$\omega_{\alpha_1 \ldots \alpha_p} = \omega_{[\alpha_1, \ldots, \alpha_p]} = \frac{1}{p!} \sum_{\substack{\text{permutations} \\ \pi}} \operatorname{sign}(\pi)\, \omega_{\alpha_{\pi(1)} \ldots \alpha_{\pi(p)}}$$

$$\omega[\alpha,\beta] = \frac{1}{2}\left(\omega_{\alpha\beta} - \omega_{\beta\alpha}\right)$$

If we fix a point $x \in M$, the p-forms span a vector space denoted Λ^p_x.

Dimension of Λ^p_x: choosing p different indices out of n. Therefore,

$$\dim \Lambda^p_x = \binom{n}{p}$$

Definition: a exterior or wedge product $\wedge : \Lambda^p_x \times \Lambda^q_x \to \Lambda^{p+q}_x$:

$$(\omega \wedge \nu)_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} = \frac{(p+q)!}{p!\,q!}\, \underbrace{\omega_{[\alpha_1 \cdots \alpha_p} \nu_{\beta_1 \cdots \beta_q]}}_{\text{antisymmetrization of the tensor product}}$$

Notice that $\omega \wedge \nu = (-1)^{pq}\, \nu \wedge \omega$.

In coordinate basis, one can write any p-form ω as

$$\omega = \frac{1}{p!}\, \omega_{\alpha_1 \cdots \alpha_p}\, \underline{dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}}$$

once again, tensor product antisymmetrized

For any vector field V, we define an inner derivative $i_V : \Lambda^p \to \Lambda^{p-1}$

$$i_V \omega = V \lrcorner\, \omega = V \cdot \omega = \omega(V, \cdot, \cdot, \ldots, \cdot)$$

notation in math; notation in physics

still antisymmetric, for ω is antisymmetric in all indices

Properties of the inner-derivative

i) Linear and linear in V $\to$ $i_{fV+gW} = f\, i_V + g\, i_W$

ii) Leibnitz: $i_V(\omega \wedge \nu) = (i_V \omega) \wedge \nu + (-1)^p\, \omega \wedge (i_V \nu)$ $\to$ that's why we call it a derivative

$\hookrightarrow$ actually, graded Leibnitz rule

iii) $i_V i_W + i_W i_V = 0 \Rightarrow (i_V)^2 = 0$ $\rightarrow$ $\underbrace{V^\alpha V^\beta}_{\text{sym}} \underbrace{\omega_{\alpha\beta\ldots}}_{\text{antisym}} = 0$

Definition: the exterior derivative $d : \Lambda^p \to \Lambda^{p+1}$ is defined as follows

i) on a function f (0-form) we have
$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha.$$

ii) on a p-form ω we then have
$$d\omega = \frac{1}{p!} \underbrace{d\omega_{\alpha_1 \ldots \alpha_p}}_{\text{acting on components}} \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}.$$

In components,
$$(d\omega)_{\alpha_1 \ldots \alpha_p} = (p+1)\, \partial_{[\alpha_1} \omega_{\alpha_2 \ldots \alpha_{p+1}]}.$$

Note that $d^2 = 0$. $\rightarrow$ $\underbrace{\partial_{[\alpha} \partial_{\beta}}_{\text{partial derivatives are symmetric}} \omega_{\ldots]} = 0$

Definition: we say a p-form α is closed when $d\alpha = 0$. We say it is exact if $\alpha = d\beta$.

Notice that exact implies closed, for $d\alpha = d^2\beta = 0$. However, closed does not imply exact (only locally we may write $\alpha = d\beta$, not globally, in general).

$\rightarrow$ also known as Classical Mechanics !!

Cartan's Lemma:

For a p-form ω and a vector field V, we have the following identity:
$$\mathcal{L}_V \omega = V \lrcorner\, d\omega + d(V \lrcorner\, \omega).$$

In particular, $\mathcal{L}_V df = d \mathcal{L}_V f$.

$$\mathcal{L}_V df = V \cdot ddf + d(V \cdot df) = d\left(V^\mu \frac{\partial f}{\partial x^\mu}\right) = d\, \mathcal{L}_V f$$

Integration of Forms

Only possible to integrate a p-form on a p-dimensional manifold. Then we might always write ω as $\omega = f\,dx^1 \wedge \dots dx^p$. Then we define

$$\int_{O_p} \omega \equiv \underbrace{\int_{\psi(O_p)} f\,dx^1 \dots dx^p}$$

$\rightarrow$ p-dim Lebesgue integral

If we change the basis,

$$\omega = f'\,dx'^1 \wedge \dots \wedge dx'^p, \qquad f' = f \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right).$$

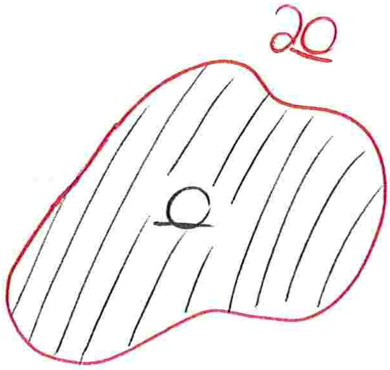
$\rightarrow$ Jacobian of transformation

$\downarrow$

the definition is basis-independent

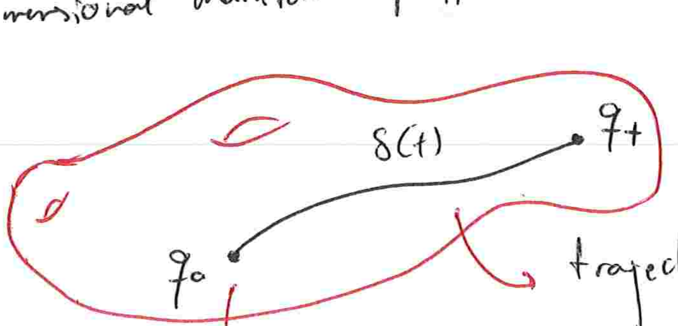
Stokes Theorem

$$\int_{\underline{O}} d\omega = \int_{\partial \underline{O}} \omega .$$



(Hints on) Geometric Formulation of Lagrangian Mechanics

A configuration space C with n degrees of freedom is a n-dimensional manifold equipped with local coordinates $(q^1, \dots, q^n)$:

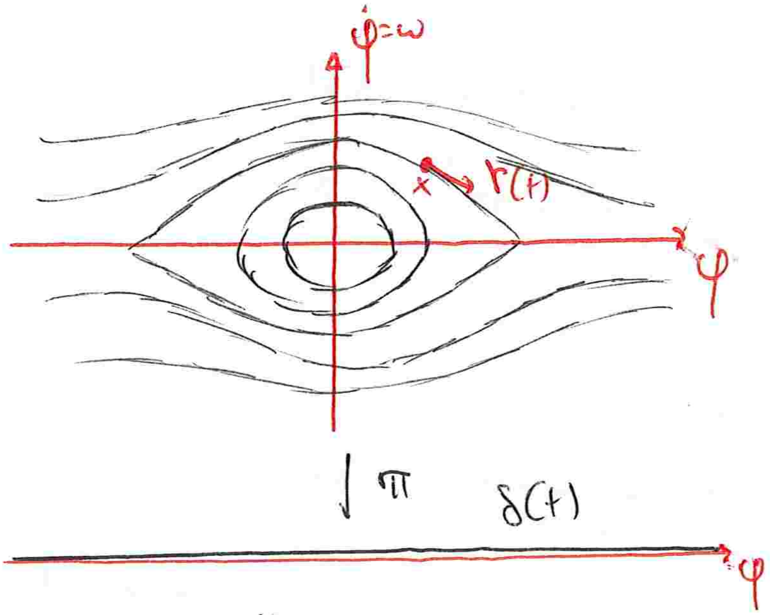


"Points" of $\delta(t)$ describe a "state of the system" at a given time

$\rightarrow$ trajectory $\simeq$ time evolution

$\downarrow$ conf. at $t=0$

A velocity phase space is a tangent bundle over the configuration space, TC. It has $\dim TC = 2n$ and generalised coordinates $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$.



Points of $\gamma(t)$ describe a physical state of the system at a given time.

A Lagrangian $L: TC \to \mathbb{R}$ determines the dynamics of the system. The dynamics is equally encoded in the dynamical vector field $X \in T(TC)$ which determines the system's integral curves $\gamma(t)$ on TC. Our aim is to find the corresponding canonical trajectory: $\delta(t) = \pi(\gamma(t))$, where π is the projection.

Procedure: $\underset{\text{problem}}{L} \to X \to \gamma(t) \to \underset{\text{solution}}{\delta(t)}$

The dynamical field X generates integral curves and hence

evolving in time in phase space

$$X = \frac{d}{dt} = \frac{dq^j}{dt}\frac{\partial}{\partial q^j} + \frac{d\dot{q}^j}{dt}\frac{\partial}{\partial \dot{q}^j} \quad + \quad \overset{\text{constraint}}{\frac{dq^j}{dt} = \dot{q}^j}$$

Thus, X is a special field of the form

$$X = \dot{q}^j \frac{\partial}{\partial q^j} + W^j(q^i, \dot{q}^i)\frac{\partial}{\partial \dot{q}^j}$$

how do we find this vector field?

Definition: we define the quantities: Lagrange 1-form: θ, Lagrange symplectic 2-form: ω, and Lagrange energy: E

$$\theta \equiv \frac{\partial L}{\partial \dot{q}^j} dq^j, \quad \omega \equiv d\theta, \quad E \equiv \frac{\partial L}{\partial \dot{q}^j}\dot{q}^j - L.$$

Theorem:
The physical state of the Lagrangian system described by the Lagrangian L is determined from integral curves of X, where X is given by

$$\mathcal{L}_X \theta = dL.$$

Euler-Lagrange Equations

unknown known

$\square$

Proof:
We want to show equivalence with the Euler-Lagrange Equations.

$$\mathcal{L}_X \theta = \left(\mathcal{L}_X \frac{\partial L}{\partial \dot{q}^i}\right) dq^i + \frac{\partial L}{\partial \dot{q}^i}\left(\mathcal{L}_X dq^i\right)$$

Cartan's Lemma

$$= \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right) dq^i + \frac{\partial L}{\partial \dot{q}^i}\left(d\mathcal{L}_X q^i\right)$$

Euler-Lagrange

$$= \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i = dL$$

$\blacksquare$

Corollary:
Equivalently, we may write

$$X \lrcorner\, \omega = -dE.$$

$\square$

Proof:
We know that

$$X = \dot{q}^i \frac{\partial}{\partial q^i} + W^i(q^i, \dot{q}^i)\frac{\partial}{\partial \dot{q}^i}, \quad \theta = \frac{\partial L}{\partial \dot{q}^i} dq^i.$$

Thus,

$$X \lrcorner\, \theta = \theta(X) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}.$$

$$X \lrcorner \omega = X \lrcorner d\theta \overset{\text{Cartan's}}{=} \mathcal{L}_X \theta - d(X \lrcorner \theta),$$
$$= dL - d(X \lrcorner \theta),$$
$$= -d(X \lrcorner \theta - L),$$
$$= -d\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right) = -dE.$$

Formal solution for X:

Let us define Ω as the inverse of ω: $\omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta^\gamma_\alpha$. The Corollary reads

$$X^\alpha \omega_{\alpha\beta} = (-dE)_\beta,$$
$$X^\alpha \omega_{\alpha\beta}\Omega^{\beta\gamma} = (-dE)_\beta \Omega^{\beta\gamma},$$
$$X^\gamma = (-dE)_\beta \Omega^{\beta\gamma},$$
$$X = -dE \cdot \Omega.$$

Conservation Laws and Noether's Theorem

Let the system admit a Lagrangian L, a dynamical vector field X and the associated diffeomorphism ϕ^X_t and an extra field Z with its associated diffeomorphism ϕ^Z_ϵ.

Noether's Theorem (Version IV):

Let $\mathcal{L}_Z L = 0$. Then $I = Z \lrcorner \theta$ is an integral of motion, i.e., $\mathcal{L}_X I = 0$. $\square$

Proof:

One has $\mathcal{L}_X I = \mathcal{L}_X (Z \lrcorner \theta) = (\mathcal{L}_X Z) \lrcorner \theta + Z \lrcorner (\mathcal{L}_X \theta)$

$$\mathcal{L}_X I = (\mathcal{L}_X Z) \lrcorner \theta + Z \lrcorner (\mathcal{L}_X \theta)$$

$$= [X, Z] \lrcorner \theta \quad \blacksquare$$

$$Z \lrcorner \mathcal{L}_X \theta = Z \lrcorner dL = \mathcal{L}_Z L = 0$$

"the fact that the last expression vanishes is left for you to prove during long Canadian winter evenings"

Note that this formulation naturally works on the velocity phase space. When we want to describe symmetries of the configuration space, the following elaborate construction is necessary. Let Z be a natural extension of Y that generates point transformations on C

$$Y: \quad q^i \to q^i_\epsilon = \phi_\epsilon(q^i), \quad \text{i.e.,} \quad Y^i = \left.\frac{dq^i_\epsilon}{d\epsilon}\right|_{\epsilon=0}$$

$$Z: \quad (q^j, \dot{q}^j) \to (q^j_\epsilon, \dot{q}^j_\epsilon) = (\phi_\epsilon(q^j), \phi^*_\epsilon(\dot{q}^j)).$$

That is,

$$Z = Y^i \frac{\partial}{\partial q_i} + \dot{Y}^i \frac{\partial}{\partial \dot{q}^i}, \quad \dot{Y}^i = \frac{dY^i(q^\ell)}{dt} = \frac{\partial Y^i}{\partial q^\ell}\dot{q}^\ell.$$

The fact that the restriction to "point transformations" is artificial and nowhere needed in the formulation of Noether's Theorem hints on the fact that one can have more general phase space symmetries. We shall return to this topic later in the course.

Hamiltonian Mechanics: Basic Theory

Hamilton's Canonical Equations

Canonical momentum:

$$p_j \equiv \frac{\partial L}{\partial \dot{q}^j}$$

Configuration Space

"plot" of the system

q^i, dim n for n d.o.f.

Phase Space

describes physical states

(q^i, p_i), dim $2n$ for n d.o.f.

The phase space here considered, in terms of momenta, is not the tangent bundle of the manifold, but the cotangent bundle.

↳ momentum is a form ↳ we'll see more details later

Hamiltonian → governs the dynamics on phase space
↳ Legendre transform of the Lagrangian

$$\mathcal{H}(q^i, p_i, t) = \left(p_i \dot{q}^i - L\right)\Big|_{\dot{q}^i = \dot{q}^i(q^i, p_i, t)}$$

→ the inversion of $p^i = \frac{\partial L}{\partial \dot{q}^i}$ may not always be possible, but we shall deal with this later

Theorem:
The system of n Euler-Lagrange equations is equivalent to the system of $2n$ first-order Hamilton's equations

$$\dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q^j}, \quad \dot{q}^j = \frac{\partial \mathcal{H}}{\partial p_j}$$

□

Proof:
i) Direct calculation: e.g. calculate $\frac{\partial \mathcal{H}}{\partial p_i}$ using the definition of $\mathcal{H}$ and the E-L equations
ii) Variational principle

$$0 = \delta S = \delta \int p_i \dot{q}^i - \mathcal{H} \, dt$$

$$= \int p_i \delta \dot{q}^i + \dot{q}^i \delta p_i - \frac{\partial \mathcal{H}}{\partial q^i} \delta q^i - \frac{\partial \mathcal{H}}{\partial p_i} \delta p_i \, dt$$

fixed endpoints

$$= \int \left(\dot{q}^i - \frac{\partial \mathcal{H}}{\partial p_i}\right) \delta p_i - \left(\dot{p}_i + \frac{\partial \mathcal{H}}{\partial q^i}\right) \delta q^i \, dt$$

And the statement follows. ■

Remark:

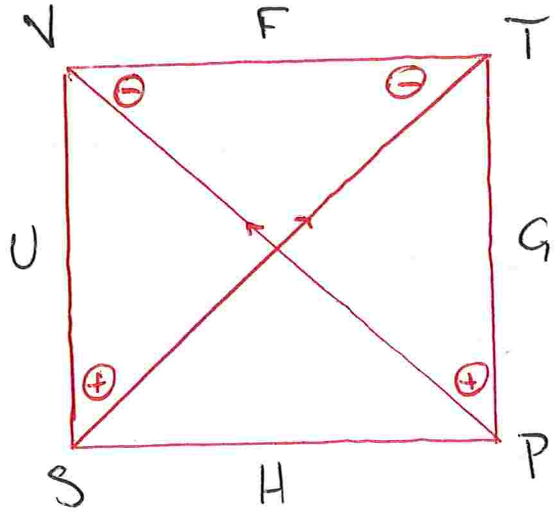
$$\frac{dH}{dt} = \frac{\partial H}{\partial q^i}\dot{q}^i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}$$

$$= + \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

Thus, if $H \neq H(t)$, H is conserved.

Changing from the Lagrangian formalism to the Hamiltonian formalism is a Legendre transformation which changes from a scalar function in TM to a scalar function in T^*M

Magic Square in Thermodynamics



→ corners are

traditional quantities

→ sides are td potentials corresponding to vertices

we can read

→ the thermodynamic potentials, which are total differentials expressed in terms of their variables

$$\overset{\oplus}{} \qquad \overset{\ominus}{}$$

$$dU = + TdS - PdV$$

if V and S are held constant, $dU = 0$

$$dG = + VdP - SdT$$

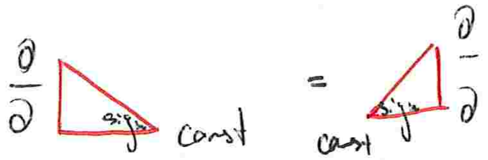
$$\underset{\oplus}{} \qquad \underset{\ominus}{}$$

→ Legendre transformation

If you go $S \to T$, you can read the sign at T, for example

$$F = U - TS, \qquad T = \frac{\partial U}{\partial S}$$

→ Maxwell's relations



$$+\left(\frac{\partial V}{\partial S}\right)_P = +\left(\frac{\partial T}{\partial P}\right)_S$$

$$\left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial S}{\partial P}\right)_T$$

proof: $dH = TdS + VdP$

$$\left(\frac{\partial T}{\partial P}\right)_S = \frac{\partial^2 H}{\partial S \partial P} = \left(\frac{\partial V}{\partial S}\right)_P$$

Poisson Brackets

Definition: let f, g be two phase space functions. Their canonical Poisson bracket is a new function on phase space, defined by

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

Properties

i) Fundamental brackets are given by

$$\{q^i, p_j\} = \delta^i_j, \quad \{q^i, q^j\} = 0, \quad \{p_i, p_j\} = 0.$$

↳ play an important role in the transition to QM by applying

$$\{\cdot, \cdot\} \Leftrightarrow \frac{1}{i\hbar}[\cdot, \cdot]$$

↳ $\{\cdot, \cdot\} \Rightarrow \frac{1}{i\hbar}[\cdot, \cdot]$ is heuristic and in general doesn't work

ii) antisymmetry: $\{f,g\} = -\{g,f\}$,

iii) linearity: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$, $\forall \alpha, \beta \in \mathbb{R}$,

iv) Leibnitz rule: $\{fg,h\} = f\{g,h\} + \{f,h\}g$,

v) Jacobi identity:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

We may derive this properties from the definition or obtain our definition from these properties.

These properties also mean the Poisson bracket determines a Lie algebra.

Using these properties, one can drive Mechanics. Relaxing them, Feynman managed to get Maxwell's Electrodynamics. → see tutorial

In terms of the Poisson brackets, the time evolution is written as

$$\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

→ open $\frac{df}{dt}$ in chain rule
eliminate $\dot{q}, \dot{p}$ with Hamilton's eqs
use definition of $\{f,\mathcal{H}\}$

Finally, if an integral of motion I has $\frac{\partial I}{\partial t} = 0$, it obeys $\{I,\mathcal{H}\} = 0$.

→ integrals of motion Poisson-commute with $\mathcal{H}$

Having two integrals of motion, I_1 and I_2,

$$I_3 = \{I_1, I_2\}$$

is also an integral of motion.

→ but sometimes it is a trivial one

→ more on tutorial

Canonical Transformations

↳ the strongest point of Hamiltonian Mechanics is the possibility to exploit the freedom in its description encoded in the so called canonical transformations

Definition

A canonical transformation is a transformation of phase space coordinates

$$Q^i = Q^i(q^i, p_i, t), \quad P_i = P_i(q^i, p_i, t)$$

that preserves the form of Hamilton's equations, i.e., one can find a new Hamiltonian $H' = H'(Q^i, P_i, t)$ such that

$$\frac{\partial H'}{\partial P_i} = \dot{Q}^i, \quad \frac{\partial H'}{\partial Q^i} = -\dot{P}_i$$

still describes the same dynamics.

Note: not every transformation is canonical

Example:

$H = \frac{p^2}{2m}$, transformation: $Q = \sqrt{q}$, $P = p$.

Guessing $H' = \frac{P^2}{2m}$ we have

$$\frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} = 2Q\dot{Q} \neq \dot{Q}$$

later we shall see that it would have to be so

How do we find canonical transformations?

↳ they are generated by generating functions of the type

$F = F(\text{old}, \text{new})$ → old $\in (q, p)$, new $\in (Q, P)$

↓

four possibilities

not all equivalent for simple systems due to degeneracies and extra symmetries, but if there are no degeneracies they are related by Legendre transformations

Example: consider $F(q, Q, t)$. Then we have

freedom

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt}\right) dt = \int \left[\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right)\right] dt$$

I want

$$= \int P\dot{Q} - H'(P, Q, t)\, dt$$

$$\int P\dot{Q} - H'\, dt = \int \dot{q}\underbrace{\left(p - \frac{\partial F}{\partial q}\right)}_{0} \underbrace{- \frac{\partial F}{\partial Q}}_{P}\dot{Q} - \underbrace{\left(H + \frac{\partial F}{\partial t}\right)}_{H'} dt$$

for the equality to hold

$$P = -\frac{\partial F}{\partial Q}$$

$$p = \frac{\partial F}{\partial q}$$

$$H' = H + \frac{\partial F}{\partial t}$$

the transformation generated by F in this way is guaranteed to be canonical

For a transformation to be generated by F, notice we have the integrability condition

$$\frac{\partial^2 F}{\partial Q\, \partial q} = \frac{\partial^2 F}{\partial q\, \partial Q},$$

$$\frac{\partial p}{\partial Q} = -\frac{\partial P}{\partial q}.$$

Remarks

i) Poisson brackets are invariant wrt canonical transformations, i.e.,

$\{f, g\}_{q,p} = \{f, g\}_{Q,P}$.

Hint on proof:

Usually, $\{f, \mathcal{H}\} = \frac{df}{dt}$. If we treat g as a Hamiltonian function generating "time flow t", it is intuitively obvious that

$$"\frac{df}{dt}" = \{f, g\}$$

does not depend on coordinate systems.

ii) (Q, P) are new canonical coordinates and hence

$$\{Q^i, P_j\} = \delta^i_j, \quad \{Q^i, Q^j\} = 0, \quad \{P_i, P_j\} = 0.$$

Since Poisson brackets are canonical invariants, such results must hold when calculated in the (q, p) coordinates.

iii) Time evolution corresponds to a change of coordinates. Or, more precisely, time evolution is a canonical transformation generated by $F = -S$, where S is the so called Hamilton's function

Definition:

A Hamilton's function is an "action understood as a function of coordinates and time" $\longrightarrow$ action evaluated on-shell

In order to obtain this function we

i) solve the Euler-Lagrange equations for q, expressing it in terms of the boundary data $q = (t_1, q_1, t_2, q_2, t)$

ii) once the solution is known, we plug it back to L and integrate L over t to get

$$S(t_1, t_2, q_1, q_2) = \int_{t_1}^{t_2} L(t_1, q_1, t_2, q_2, t)\, dt.$$

Since $L\,dt = p_i\,dq^i - H\,dt$, it can be shown that the total differential of S reads

see Landau & Lifshitz

$$dS = (p_i\,dq^i - H\,dt)_{final} - (p_i\,dq^i - H\,dt)_{initial}.$$

Identifying $t = t_2$ and $q = q_2$, the last equality implies two important relations that lead to the Hamilton-Jacobi equation (derived in a different fashion later on)

$$\frac{\partial S}{\partial q^i} = p_i, \quad \frac{\partial S}{\partial t} = -H.$$

Also, upon $q_0 = q_1$ and $t_0 = t_1$, we have $\frac{\partial S}{\partial q_0^i} = -p_{0i}$, $\frac{\partial S}{\partial t_0} = H_0$.

Finally, notice we may write time evolution as a coordinate transformation

$$q(t) = q(q_0, p_0, t), \qquad p(t) = p(q_0, p_0, t).$$

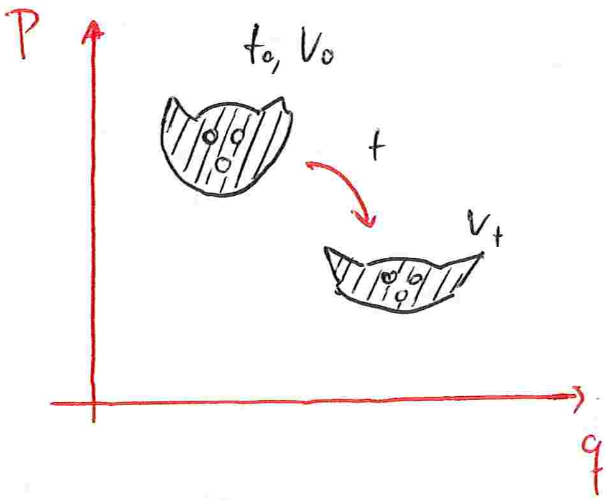
Integrating dS, we find

$$\int [p\,dq - H\,dt] = \int \left[p_0\,dq_0 - H_0\,dt - \frac{d(-S)}{dt}\,dt\right].$$

Hence, the transformation is canonical and generated by the generating function $-S = -S(q_0, q, t)$.

vi) Liouville's Theorem: important for statistical Physics

Volume of the phase space remains invariant under canonical transformations. In particular, time-evolution preserves the phase space volume. $\square$



In one dimension (1 dof):

$$V_t = \int dQ\, dP = \int |J|\, dq\, dp$$

$$\{J\} = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \{Q, P\} = 1$$

$$\therefore V_t = \int dq\, dp = V_0 \qquad \blacksquare$$

Hamilton - Jacobi Theory

Idea: find special generating function S such that $H' = 0$ (and thus every momentum and every coordinate is constant).

$$\dot{Q}^i = \frac{\partial H'}{\partial P_i} = 0 \;, \qquad \dot{P}_j = \frac{\partial H'}{\partial Q^j} = 0,$$

$$\Downarrow$$

$$Q^i = \alpha^i \;, \qquad P_j = -\beta_j .$$

in the new coordinates, the dynamics is trivial

We have though

$$P_i = -\frac{\partial S}{\partial Q^i} \iff \beta_j = \frac{\partial S(q^i, \alpha^i, t)}{\partial \alpha^i}$$

$\rightarrow$ implicit solution

Inverting the last expression we find

$$q^i = q^i(t, \alpha^j, \beta_j)$$

and we are done.

How can we find the generating function S?

There are still two equations for the transformation due to a generating function which we have not used:

$$p_i = \frac{\partial S}{\partial q^i}, \qquad H' = 0 = H + \frac{\partial S}{\partial t}.$$

Therefore,

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) = 0.$$

Hamilton-Jacobi Equation

partial differential equation

When solving the HJE, we are not interested in a general solution (depending on an arbitrary function), but rather in a complete integral $S(q^i, \alpha^i, t)$ (a solution that depends on n constants α^i)

Trick for solving: additive separation of variables

$$S(q, t) = S_1(q^1) + S_2(q^2) + \dots + S_n(q^n) + S_t(t)$$

If $H \neq H(t)$, then

$$S(q^i, t) = S_0(q^i) - Et.$$

If $H \neq H(q_c)$, then

$$S(q^i, t) = \alpha_c q^c + S_0(q^1, \dots, q^{c-1}, q^{c+1}, \dots, q^n, t).$$

Remark: separation of variables is coordinate-dependent.

↳ may be possible in a system, while impossible in another system of coords.

Example: free fall in homogeneous gravitational field

$$H = \frac{p^2}{2m} + V(x), \qquad V = -mgx.$$

The HJE reads

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V + \frac{\partial S}{\partial t} = 0.$$

$H \neq H(t)$, so we try $S = S_0(x) - Et$ and get

$$\frac{1}{2m}\left(\frac{dS_0}{dx}\right)^2 + V - E = 0 \Rightarrow S_0 = \int \sqrt{2m(E-V)}\, dx.$$

Performing the integral for $V(x) = -mgx$ we get

$$S = \frac{1}{3gm^2}\left[2m(E+mgx)\right]^{3/2} - Et.$$

Now we set

$$\beta = \frac{\partial S}{\partial E} = \frac{1}{mg}\sqrt{2m(E-V)} - t.$$

Inverting this expression we finally find

$$x = \frac{g}{2}(t+\beta)^2 - \frac{E}{mg}.$$

the meaning of β is to set the initial time in this case, while E is the total energy

our new heavy machinery does yield the solution we expected

Connection to Quantum Mechanics

The Schrödinger Equation reads

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + V(x), \qquad \hat{p} = -i\hbar\nabla.$$

Let us use the geometric optics (WKB) approximation:

$$\psi = \psi_0 \, e^{\frac{i}{\hbar} S(x,t)} \longrightarrow \text{ansatz}$$

So we get,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi}_{\left(\frac{(\nabla S)^2}{2m} - \frac{i\hbar}{2m}\nabla^2 S + V\right)\psi} = \hat{H}\psi$$

If we take the limit $\hbar \to 0$, we get

$$\frac{(\nabla S)^2}{2m} + V + \frac{\partial S}{\partial t} = 0. \longrightarrow \text{we recovered the Hamilton-Jacobi Equation}$$

An Overview of

Integrable Systems $\longrightarrow$ can be solved analitically

Definition: a dynamical system with n degrees of freedom ($2n$-dimensional phase-space) is completely (Liouville) integrable if it possess n independent conserved quantities $F_i(q,p) = f_i$, $\{H, F_i\} = 0$ that are in involution, i.e., $\{F_i, F_j\} = 0$, $\forall\, i,j$.

Liouville's Theorem:

The solution of equations of motion of a completely integrable system can be obtained by "quadrature", i.e., by a finite number of algebraic operations and integrations. $\square$

Proof Sketch:

We want to find a canonical transformation $(q^i, p_i) \to (F^i, \psi_i)$, where F^i are the conserved quantities. If we succeed, then

$$\dot{F}^i = \{H, F^i\} = 0$$

$$\dot{\psi}_i = \{H, \psi_i\} = \frac{\partial H}{\partial F^i} = \Omega_i(F_j)$$

constant in time

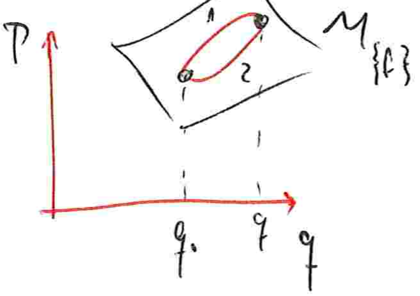
it holds that $H = H(F_i)$, as we shall see after the sketch

$$\therefore F^i(t) = F^i(0), \quad \psi_i(t) = \psi_i(0) + \Omega_i t$$

To construct this, let us find the corresponding generating function S. We have $M_{\{f\}} = \{F^i(p,q) = f_i\}$. In principle we may invert this to get $p_i = p_i(f,q)$ on $M_{\{f\}}$ and define

hypersurface in phase space

$$S(q, F) = \int_{q_0}^{q} p_i \, dq^i$$



If such an integral exists, then

$$\psi_i = -\frac{\partial S}{\partial F^i}$$

gives the desired canonical transformation.

notice that $p_i = \frac{\partial S}{\partial q^i}$ is automatically satisfied

One finally needs to show that S is well-defined, i.e., the integral does not depend on integration path. One can show that this is exactly equivalent to the requirement that the F^i are in involution: $\{F^i, F^j\} = 0$.

So we have obtained the solution of the EoM by making one integral and some algebraic operations (needed to express $p = p(q, F)$). ∎

Remark

i. Note that to integrate $2n$ ODE's we must know $2n$ integrals of motion, but for a given canonical system it is sufficient to have only n. This is possible because each integral of motion can be "used twice".

Analogous statement in QFT:

In any gauge theory we have

$$\#(\text{true dof}) = \#(\text{apparent dof}) - 2\cdot(\text{dof of gauge function}).$$

Examples

a) Electromagnetism is described by the potentials $(\varphi, \vec{A})$, i.e., 4 apparent dof. The following gauge transformation

$$\varphi \to \varphi + \frac{\partial \Lambda}{\partial t}, \quad \vec{A} \to \vec{A} + \nabla \Lambda$$

there might be a sign mistake in this expression

Λ is a gauge function $\downarrow$ scalar function

leaves observable fields $\vec{E}$ and $\vec{B}$ invariant. The gauge function is just one scalar function, and thus ammounts to 1 dof. Therefore,

$$\#(\text{true dof}) = 4 - 2\cdot 1 = 2$$

two polarizations of photon

b) Gravity is described by a 4x4 symmetric matrix (metric $g_{\mu\nu}$), which has 10 apparent dof. Gauge transformations

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

ξ_μ is a 4-vector $\downarrow$ 4 dof

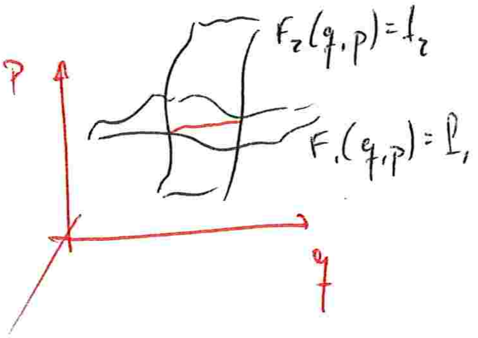
leave field strength invariant. Thus,

$$\#(\text{true dof}) = 10 - 2\cdot 4 = 2$$

two polarization of the graviton

ii. Independence

Each integral defines a hypersurface in phase space, dynamical trajectories must remain in this surface



Independent: hypersurfaces for F_i and f_j are "never tangent"

(global ≈ foliate space time)

$M_{\{f\}}$ given by $\{F_i = f_i\}$ has dimension n

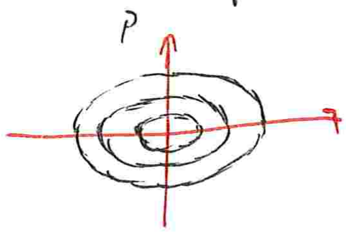
iii. One cannot have more than n independent integrals of motion that are in involution. Otherwise, the Poisson bracket would be degenerate. This implies that $H = H(F_i)$.

iv. Under a suitable global hypothesis, $M_{\{f\}}$ is an n-dimensional tori T_n

Example: harmonic oscillator

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$

except (0,0) → stationary point



The phase space is fibred into ellipses → T_1

$$H = E$$

Introduce $p = \rho\cos\theta$, $q = \frac{\rho}{\omega}\sin\theta$

$$\text{motion} = \left\{\rho = \sqrt{2E},\ \theta = \omega t + \theta_0\right\}$$

Generalization: $H = \frac{1}{2}\sum_i \underbrace{\left(p_i^2 + \omega_i q_i^2\right)}_{F_i}$ n conserved quantities in involution

$M_f \equiv \{F_i = f_i\}$ is T_n → all integrable systems look like this

v. One can show that whenever the HJE completely separates the motion is completely integrable

vi. For small perturbations of integrable systems, the (Liouville) tori exist almost everywhere. This is a content of the Kolmogorov-Arnold-Moser (KAM) Theorem and is connected to the field of deterministic chaos.

Constraints → for details, see Dirac's "Principles of QM"

↳ not all q^i, p_i are independent, i.e., at any time we have a constraint

$$\phi(q^i, p_i) = 0$$

Any phase space function f can be understood as a "Hamiltonian" and generates a corresponding "flow" (in fictitious time t_f)

$$\frac{\partial q^i}{\partial t_f} = \{q^i, f\}, \quad \frac{\partial p_i}{\partial t_f} = \{p_i, f\}.$$

In particular, if we have a conserved quantity I, then

$$0 = \{I, H\} = -\{H, I\} = 0.$$

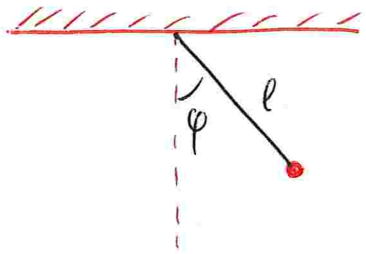
The Hamiltonian is unaffected by the flow generated by I.

We can do some kind of transformation on the phase space without affecting the dynamics (Hamiltonian): symmetry

Constraints are special conserved quantities and as such they define a symmetry

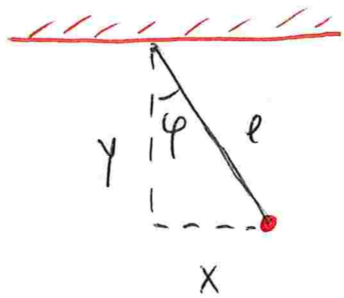
Standard example:

"Clever description"



vs.

Constraint System



- $n=1$ dof
- phase space (φ, p_φ)
- no constraint

eliminate constraint

$x = \ell \cos\varphi$

$y = \ell \sin\varphi$

- $n=2$ dof
- phase space (x, y, p_x, p_y)
- constraint

$$\phi = x^2 + y^2 - \ell^2 = 0$$

Advantages:

- extra symmetry
- can calculate "constraint forces"
  ↳ e.g. string tension

Remark

Although the description with constraints is more complicated, the extra symmetry can be powerful and worth the complication

To deal with constraints, one uses the method of Lagrange multipliers. In the presence of m constraints ϕ_j, we define the total Hamiltonian:

$$H_T = H + \sum_{j=1}^{m} \lambda_j \phi_j$$

Hamiltonian without constraints

Lagrange multipliers $\lambda_j = \lambda_j(t)$

unknowns

Two possibilities:

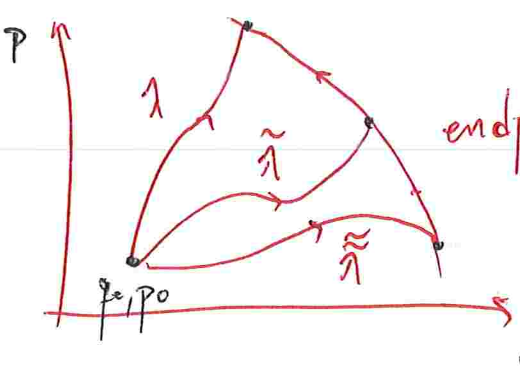
→ we enlarge not only the Hamiltonian, but also the phase space and regard the λ_j as new coordinates to get a system with $(n+m)$ dof. Varying the action with respect to the λ_j as well and defining $L_T = L - \lambda_j \phi_j$ as the total Lagrangian, we get the Euler-Lagrange equations for L_T and the on-shell constraints:

$$\frac{\partial L_T}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot{q}^i}\right) = 0, \quad \phi_j = 0.$$

We can use these to solve for q^i as well as λ_j. If we proceed to the Hamiltonian picture, we see that $p_{\lambda_j} = 0$ and recover H_T as defined before.

→ keep $\lambda_j(t)$ as arbitrary functions. We then have n E-L equations which depend on $\lambda_j(t)$. In this situation the solutions to the EoM depend on λ_j and are therefore not unique. However, the end configurations are related by a symmetry of the system.

The underlying physics is unique, but the description we chose is not.



endpoints related by symmetry

Connected by the flow generated by the symmetry

More details on tutorial

A typical origin of constraints is the Legendre transformation from the Lagrangian to the Hamiltonian description. The constraints emerge when the expression $p_i = \frac{\partial L}{\partial \dot{q}^i}$ can't be inverted to solve for $\dot{q}^i$.

This happens when $\det A = 0$, where $A_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$. One can then write

$$\dot{q}^i = \dot{q}^i(q^j, p_j, \lambda^k)$$

where λ^k are arbitrary functions and we have that many of them as there are zero eigenvalues of A.

Toy Example:

Let us consider the motion of a free particle from the perspective of an observer with a broken clock that doesn't keep time steady (sometimes faster, sometimes slower).

Starting from the Lagrangian written in terms of the lousy time and performing the standard Legendre transformation we find that the corresponding Hamiltonian vanishes

gravity is another example

$$H = 0.$$

This is a characteristic of a totally constraint system. At the same time we find a constraint

details on tutorial

$$\phi = p_T + \frac{p_x^2}{2m}.$$

So there is no time-evolution and the only flow is wrt the constraint (total Hamiltonian is non-trivial and determines this flow). Despite the lack of true time evolution, relational predictions are still possible for this type of system.

Interestingly, this is a prototype of what happens for gauge theories and, in particular, gravity.

Geometric Formulation of Hamiltonian Mechanics

two options:
- symplectic geometry
- contact geometry

contact allows for $H = H(t)$ and is more general

Symplectic Geometry $\to H \neq H(t)$

works for autonomous systems

Definition: Let M^{2n} be an even-dimensional manifold. A symplectic structure on M is a closed non-degenerate 2-form ω, i.e.,

i. $d\omega = 0$;

ii. $\forall X \neq 0,\ \exists Y;\ \omega(X, Y) \neq 0.$ $\to$ thinking of it as a matrix ω has no zero eigenvalues

The pair (M^{2n}, ω) is called a symplectic manifold.

Remark:
the very existence of a non-degenerate 2-form implies that $\dim M = 2n$.

Define now Ω as an "inverse" of ω:

$$\Omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}.$$

Since ω is non-degenerate, this is unique. So ω defines an isomorphism between TM and T^*M as follows

$$\omega : TM \to T^*M; \quad X \to \omega(X, \cdot) = X \lrcorner\, \omega \quad \text{or} \quad X^{\alpha} \to X^{\alpha}\omega_{\alpha\beta} = X_{\beta}$$

$$\Omega : T^*M \to TM; \quad \eta \to \Omega \cdot \eta \quad \text{or} \quad \eta_{\alpha} \to \Omega^{\beta\alpha}\eta_{\alpha} = \eta^{\beta}$$

One may then rise and lower indices, similar to when we have a metric. However, ω and Ω are antisymmetric (while the metric is symmetric) and therefore contracting the first index is different from contracting the second by a minus sign.

Given a function f on M, ω defines a Hamiltonian vector field

$$X_f = \underline{\Omega} \cdot df \iff X_f \lrcorner \omega = -df.$$

Indeed, in components we have

$$X_f^\alpha = \underline{\Omega}^{\alpha\beta} \partial_\beta f$$

$$X_f^\alpha \omega_{\alpha\gamma} = \underbrace{\underline{\Omega}^{\alpha\beta} \omega_{\alpha\gamma}}_{-\delta^\beta_\gamma} \partial_\beta f = -\partial_\gamma f$$

$$\underline{\Omega}^{\alpha\beta} \omega_{\alpha\gamma} = -\underline{\Omega}^{\beta\alpha} \omega_{\alpha\gamma} = -\delta^\beta_\gamma$$

Theorem:

A Hamiltonian vector field preserves the symplectic structure:

$$\mathcal{L}_{X_f} \omega = 0. \qquad \square$$

Proof:

From Cartan's Lemma, we have

$$\mathcal{L}_{X_f} \omega = X_f \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{X_f \lrcorner \omega}_{-df})$$

$$= -d^2 f = 0 \qquad \blacksquare$$

Corollary (Liouville's Theorem):

A Hamiltonian vector field preserves the volume element

$$\epsilon = \omega^{\wedge n} = \frac{1}{n!} \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ times}}$$

(and hence also volume) on M. $\square$

Proof:

We have

$$\mathcal{L}_{X_f}\epsilon = \frac{1}{n!}\mathcal{L}_{X_f}(\omega\wedge\dots\wedge\omega)$$ Leibnitz

$$\propto \omega^{\wedge(n-1)}\wedge\mathcal{L}_{X_f}\omega = 0$$

Poisson Bracket

this M will be the phase space when we get to Ham. Mech.

If we let f, g be scalar ~~phase space~~ functions on M, we define the Poisson bracket through

$$\{f,g\} \equiv df \cdot \underline{\Omega} \cdot dg = -\omega(X_f, X_g)$$

satisfies properties i) - v) of PB from before

↓

Jacobi ⟺ $d\omega = 0$

We can also get

sometimes used as definition of $\{f,g\}$

$$X_{\{f,g\}} = [X_f, X_g].$$

In order to obtain the fundamental brackets, we must introduce q and p into this picture.

Darboux Theorem:

dim $M = 2n$
first n coords: q
remaining coords: p

Since ω is antisymmetric, closed and non-degenerate, there exists a coordinate system on M $x^\alpha = (q^i, p_i)$ such that

$$\underline{\Omega}^{\alpha\beta} = \begin{array}{cc} & \begin{array}{cc} q^i & p_i \end{array} \\ \begin{array}{c} q^i \\ p_i \end{array} & \left(\begin{array}{cc|c|c|c} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & & & \\ \hline 0 & & \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} & 0 & 0 \\ \hline 0 & & 0 & \ddots & 0 \\ \hline 0 & & 0 & 0 & \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \end{array}\right) \end{array}, \quad \omega_{\alpha\beta} = \begin{array}{cc} & \begin{array}{cc} q^i & p_i \end{array} \\ \begin{array}{c} q^i \\ p_i \end{array} & \begin{pmatrix} \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} & & & \\ & \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} & & \\ & & \ddots & \\ & & & \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \end{pmatrix} \end{array}$$

Therefore,

$$\omega = dp_1 \wedge dq^1 + dp_2 \wedge dq^2 + \dots + dp_n \wedge dq^n,$$
$$= dp_i \wedge dq^i. \qquad \square$$

"canonical form"

Consequence:

$$\epsilon = \omega \wedge \dots \wedge \omega = dp_1 \wedge dq^1 \wedge \dots \wedge dp_n \wedge dq^n.$$

$$\int f = \int f \epsilon = \int f \, dp_1 dq^1 \cdots dp_n dq^n.$$

In coordinates, we have for $\{f, g\}$:

$$\{f, g\} = \partial_\alpha f \, \Omega^{\alpha\beta} \, \partial_\beta g$$

Darboux

$$= \Omega^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta}$$
$$= \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}$$

we recovered the fundamental bracket, but in adequate coordinates

we'll deal with canonical transformations in a while

canonical or Darboux coordinates

Notice that

$$\{f, g\} = df \cdot \Omega \cdot dg$$
$$= df \cdot X_g = X_g(f) = \frac{df}{dt_g}$$
$$= -\{g, f\} = -X_f(g) = -\frac{dg}{dt_f}$$

$$\therefore \quad X_f = \frac{d}{dt_f} = \{\cdot, f\} = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}$$

Darboux coordinates

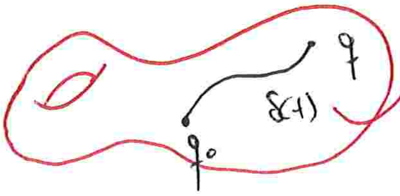
Hamiltonian Mechanics

Basic Theory

A configuration space C for a system with n dof is a manifold of dimension n equipped with local coordinates (q^i)

"photo" of the system

where the motion happens



trajectory $\simeq$ time evolution

A phase space is a cotangent bundle T^*C over the configuration space. It has $\dim T^*C = 2n$ with local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$.

Cartan's 1-form $\theta \in T^*(T^*C)$, given by

$$\theta = p_j \, dq^j$$

the phase space always admits a symplectic structure

defines a natural symplectic structure on T^*C.

$$\omega = d\theta = dp_j \wedge dq^j!$$

clearly closed and non-degenerate

The dynamics is defined by specifying a Hamiltonian $H: T^*C \to \mathbb{R}$

$$H = H(p_i, q^i).$$

$H(q,p,t)$ won't work with symplectic geometry we must refer to contact geometry in these situations

Systems with $H \neq H(t)$ are said to be autonomous.

H defines a dynamical Hamiltonian vector field, X_H.

$$X_H = \frac{d}{dt} = \Omega \cdot dH = \{\cdot, H\} \longrightarrow X_H \lrcorner \omega = -dH$$

we drop the H sub index

the field generates its integral curves $\gamma(t) = (q^i(t), p_i(t))$:

$$X_H = \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i}$$

However,

$$X_H = \{\cdot, H\} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

By comparison we have

$$\dot{q}^j = \frac{\partial H}{\partial p_i}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^i}$$

Hamilton's Equations

in non-canonical coords, the equations would be more complicated, but a non-canonical transf. would solve it

analogy to Newton's law in non-inertial frames of reference (when Coriolis and etc show up).

these equations determine $\gamma(t)$, and we may then obtain the trajectory $\delta(t)$ through canonical projection

$$\delta(t) = \pi(\gamma(t)).$$

Since X_H is a Hamiltonian vector field, it holds that

$$\mathcal{L}_{X_H}\omega = 0$$

and therefore volume is preserved by time translations.

in phase space

~~Canoni~~ Note that the Legendre transform $L \leftrightarrow H$ is a map $TC \leftrightarrow T^*C$ that identifies $\dot{q}^j \leftrightarrow p_j$ according to

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad \dot{q}^j = \frac{\partial H}{\partial p_j}, \quad H = \dot{q}^j p_j - L \Big|_{\dot{q}^j = \dot{q}^j(q^i, p_i)}$$

$$\begin{tikzcd}
& \mathbb{R} & \\
(q^i, \dot{q}^i) = z \in TC \arrow[ur, "L"] \arrow[rr, leftrightarrow, "\text{Legendre}"] \arrow[dr, "\pi"'] & & T^*C \ni z^* = (q^i, p_i) \arrow[ul, "H"'] \arrow[dl, "\bar{\pi}"] \\
& C \ni q^i &
\end{tikzcd}$$

Any function $f: TC \to \mathbb{R}$ can be used to identify TC with T^*C as follows: it defines a map $\Lambda_f : TC \to T^*C$ mapping $(x,v) \mapsto (x,p)$ by

$$p(v) = \frac{d}{dt} f(x, w + tv) \Big|_{t=0}$$

i.e.

$$p = \frac{\partial f}{\partial \dot{q}^i} dq^i$$

Canonical Transformations

A transformation

$$Q^j = Q^j(q^i, p_i), \quad P_j = P_j(q^i, p_i)$$

is canonical if it preserves the form of ω:

$$dp_j \wedge dq^j = dP_j \wedge dQ^j.$$

For example, let us take $n=1$.

$$\omega = dP \wedge dQ = \left(\frac{\partial P}{\partial q} dq + \frac{\partial P}{\partial p} dp \right) \wedge \left(\frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp \right) = \{Q, P\}_{q,p} \, dp \wedge dq.$$

We get a condition that $\{Q, P\} = 1$.

Having two Cartan 1-forms,

$$\theta = p_i \, dq^i, \quad \tilde{\theta} = P_i \, dQ^i,$$

$$d\theta = \omega = d\tilde{\theta}.$$

since θ and $\tilde{\theta}$ are 1-forms F is a function

$$\therefore \; d(\theta - \tilde{\theta}) = 0 \Rightarrow \theta - \tilde{\theta} = dF \text{ locally}.$$

Thus, there is a generating function $F = F(q,p)$ such that

$$p_i \, dq^i - P_i \, dQ^i = dF.$$

locally, every canonical transformation has a generating function

For example, assume that in a neighborhood of some (q_0, p_0) we can take (Q, q) as new independent coordinates $\left[\det \frac{\partial(Q,q)}{\partial(q,p)} \neq 0\right]$, i.e., we can write

$$F_1(q^i, Q^i) = F(q^i, p_i(Q^i, q^i)).$$

If so, then one has

$$p_i \, dq^i - P_i \, dQ^i = \frac{\partial F_1}{\partial q^i} dq^i + \frac{\partial F_1}{\partial Q^i} dQ^i \Rightarrow \quad p_i = \frac{\partial F_1}{\partial q^i}, \quad P_i = -\frac{\partial F_1}{\partial Q^i}.$$

Similarly, one could take $F_2(q^i, P_i) = F + P_i Q^i$ and other possibilities.

Since H is an scalar function, it is unaffected by the change of coordinates:

$$H = H'.$$

gets more complicated when $H = H(t)$

Symmetries

Noether's Theorem (Version V):

Let $Y \in T(T^*C)$ such that $\mathcal{L}_Y \omega = 0$ and $\mathcal{L}_Y H = 0$. Then there is a quantity I such that

$$\frac{dI}{dt} = \mathcal{L}_{X_H} I = 0. \qquad \square$$

Proof:

$$0 = \mathcal{L}_Y \omega = Y \lrcorner \cancel{d\omega}^{\,0} + d(\underbrace{Y \lrcorner \omega}_{-dI})$$

locally

$$Y \lrcorner \omega = -dI$$

$$\Downarrow$$

$$Y = Y_I$$

$$0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\},$$
$$= -\{I, H\} = -\frac{dI}{dt} = 0.$$

Notice that this is on the level of the phase space.

Y is a symmetry of the phase space. What happens if we consider its canonical projection $\pi^* Y$?

→ first possibility: vector field on C
symmetry of the configuration space
"isometry"

→ second possibility: not well defined on C
still carries information about momenta
"dynamical or hidden symmetry"

No symmetry seen on the manifold, but if you let some particles move around, you notice they move in some nice way

the symmetry is not seen on C, it is hidden in T^*C

Example of a Hidden Symmetry: Kepler Problem

$$\vec{F} = -\frac{k}{r^2}\hat{r}$$

Since the force is spherically symmetric and does not depend on time, we find the isometries:

stationary: conservation of energy E,

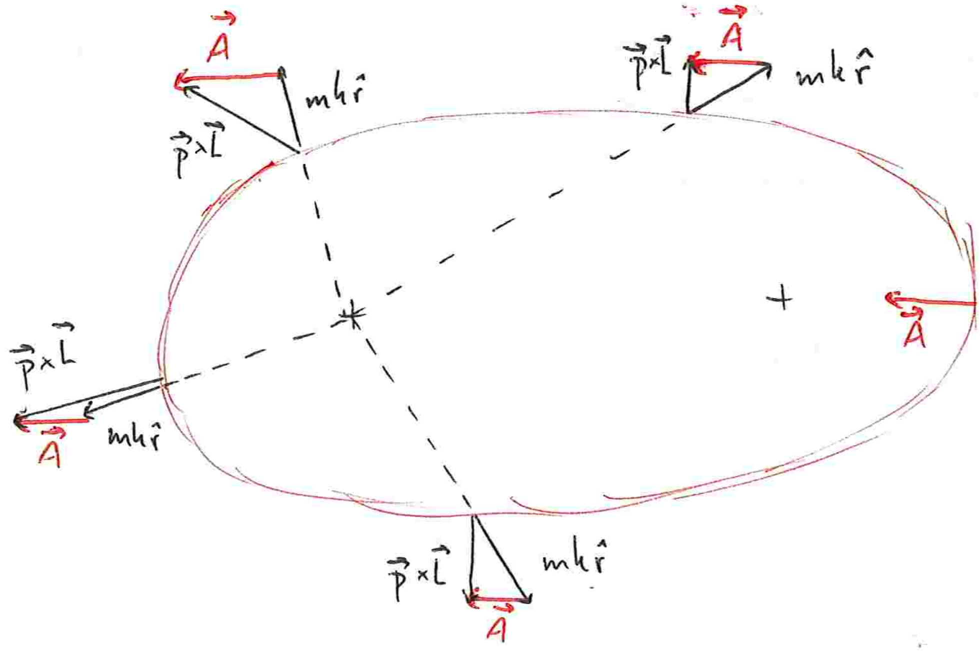
spherical symmetry: conservation of angular momentum $\vec{L}$.

In this case, we also have a hidden symmetry given by the (Laplace-)Runge-Lenz vector

i.e., the Kepler problem

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}.$$

For a geometric derivation of $\vec{A}$, see Feyman Lectures on Physics.



Dynanical symmetry: no associated cyclic coordinate can be found

One can show that this is related to a hidden $SO(4)$ symmetry (free particle on a 3-sphere). Notice we have the two following constraints:

$$\vec{A}\cdot\vec{L} = 0, \qquad A^2 = m^2k^2 + 2mEL^2.$$

Thus, the set $\{E, \vec{L}, \vec{A}\}$ gives $7-2=5$ independent conserved quantity. Since #dof $=3$, we need 3 independent integrals of motion to have complete integrabity. We have $2n-1$. If we have more than n integrals, the system is said to be superintegrable. If we have $2n-1$, it is said to be maximally superintegrable.

the last one is always a shift in time

At most n integrals are in involution, since the opposite would imply the Poisson bracket is degenerate (and it is not).

Superintegrability allows to "integrate the equations in a 'fancy' way", for example.

job is easier

These integrals of motion can be used as extra Hamiltonians in terms of Nambu Mechanics.

what if a system had to be described through not only one Hamiltonian but 2, 3, ..., n instead?

↳ works very well at theoretical level, but there are no known examples of systems that must be described by more than one Hamiltonian.

Overview of Nambu Mechanics

Hamiltonian Mechanics → why should this always work?

- $\dim M = 2n$
- ω is a 2-form
- $\{f, g\}$ is a binary operator
- dynamics described by a single Hamiltonian

↳ e.g. (q^i, p_i, r^i)

Nambu in 1973 postulated the existence of a Nambu tensor η: nondegenerate p-form such that $D\eta = 0$ → this D is more complicated

The Poisson bracket becomes → inverse of η

$$\{f_1, \dots, f_p\} = \tilde{\eta}(df_1, \dots, df_p)$$

→ Example: 3D space with $\eta = \epsilon$

$$\{f, g, h\} = \epsilon^{ijk}\partial_i f \partial_j g \partial_k h$$

↳ to know the dynamics of f_1, we need $p-1$ other functions

↳ $p-1$ Hamiltonians yielding

$$\frac{df}{dt} = \{f, H_1, \dots, H_{p-1}\}$$

Applications:
- → superintegrable systems
- → brane dynamics in String Theory

Contact Geometry $\longrightarrow$ can deal with non-autonomous systems

Definition:
A vector X for which $\omega(X, Y) = 0 \ \forall \ Y$ is called a null vector of the 2-form ω.

$\longrightarrow X \lrcorner \omega = 0$

$\omega \cdot X = 0$

↓

eigenvector of ω with eigenvalue 0

Definition:
A 2-form ω is non-singular when the vector space generated by its null vectors has minimal possible dimension, i.e.,

$$\left.\begin{aligned} &\dim = 0, \quad \text{for even-dimensional manifold } M^{2n}, \\ &\dim = 1, \quad \text{for odd-dimensional manifold } M^{2n+1}. \end{aligned}\right\} \text{holds}$$

Definition: When ω is closed and non-singular, the pair (M^{2n+1}, ω) is called ~~the contact geometry~~ a contact manifold.

→ for odd dimensions, we allow a single zero eigenvalue

Remarks

i. This is an "odd-dimensional version of symplectic geometry"

ii. Since ω is closed, it is locally generated by the contact 1-form θ: $\omega = d\theta$. ω also determines a unique vortex direction X such that $X \lrcorner \omega = 0$. If X is normalized $(\theta(X) = 1)$ we call it the Reeb vector.

iii. Branch of mathematics: special Riemannian manifolds

→ Kähler manifolds: $(M^{2n}$, Kähler 2-form $\omega)$ → String Theory

→ Sasaki manifolds: $(M^{2n+1}$, Reeb vector $X)$ → String Theory

particular cases of symplectic and contact geometries

String Theory Mathematics → compactification from 11 to 4 dimensions

iv. Note that

$$\mathcal{L}_X \omega = 0.$$

preserves form of ω as we move along X, which shall represent time evolution

⇓

time evolution is a canonical transformation

Indeed,

$$\mathcal{L}_X \omega = X \lrcorner \underbrace{\cancel{d\omega}^{\,0}}_{\text{closed}} + \underbrace{\cancel{d(X \lrcorner \omega)}^{\,0}}_{\text{hypothesis}} = 0.$$

v. Darboux theorem

Since ω is closed and nonsingular, there is a coordinate system $x^\alpha = (q^i, p_i, t)$ such that

$$\omega_{\alpha\beta} = \begin{array}{c|ccccccc} & q^1 & p_1 & & & q^n & p_n & t \\ \hline q^1 & 0 & -1 & & & & & \\ p_1 & 1 & 0 & & & & & \\ & & & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & & 0 & \\ & & & & \ddots & & & \\ q^n & & & & & 0 & -1 & \\ p_n & & & 0 & & 1 & 0 & \\ t & & & & & & & 0 \end{array}$$

ω and X take the following form

$$\omega = dp_i \wedge dq^i, \quad X = \frac{\partial}{\partial t}$$

vi. One can define a Lagrange bracket by requiring

$$[X_f, X_g] = X_{\{f,g\}}$$

Hamiltonian Mechanics Revisited

Definition: we define the extended phase space through

$$M^{2n+1} = T^*C \times \mathbb{R}$$

with local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n, t)$.

Given a Hamiltonian function $H(p_i, q^i, t)$, the extended phase space admits a contact 1-form θ given by

$$\theta = p_i dq^i - H dt,$$

$$\omega = d\theta = dp_i \wedge dq^i - dH \wedge dt.$$

we know from even dimension that $dp_i \wedge dq^i$ is non-singular, and thus ω is non-singular for the odd-dimension

only $dH \wedge dt$ could give a zero

Hence, (M, ω) is a contact manifold.

the corresponding vortex field is given by the solution of $X \lrcorner \omega = 0$, which is

$$X = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}.$$

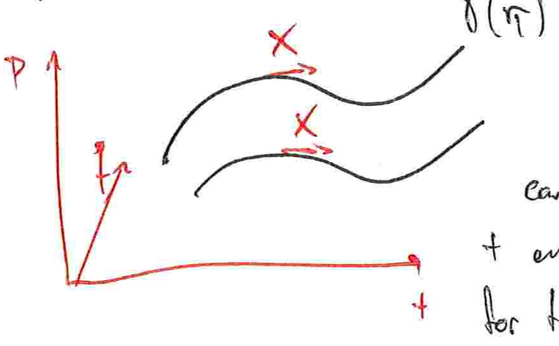
Therefore, the integral curve $\gamma(\pi) = (q^i(\pi), p_i(\pi), t(\pi))$ are determined from

$$X \equiv \frac{d}{d\pi} = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} + \dot{t}\frac{\partial}{\partial t}.$$

After identification $\pi = t$, we get

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}.$$

Hamilton's Equations



can't pick t as parameter, for t is a coordinate now

Thus, X determines time evolution for any function f

$$X(f) = \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} = \mathcal{L}_X f$$

stands for the standard Poisson bracket for notational reasons

In particular, integrals of motion satisfy $X(I) = 0$.

Definition:

A transformation

$$Q^i = Q^i(q^i, p_i, t), \quad P_j^k = P_j^k(q^i, p_i, t)$$

is said to be canonical if it preserves the form of ω, i.e., there exists a new Hamiltonian H' such that

$$\omega = dp_i \wedge dq^i - dH \wedge dt = dP_j \wedge dQ^i - dH' \wedge dt.$$

Note that a diffeomorphism ϕ_X generated by X gives a canonical transformation, for $\mathcal{L}_X \omega = 0$ and thus $\phi_X^* \omega = \omega$.

If there are two different contact forms θ and $\tilde{\theta}$ with

$$\theta = p_i dq^i - H dt, \quad \tilde{\theta} = P_i dQ^i - H' dt$$

$$d\theta = \omega = d\tilde{\theta},$$

then $d(\theta - \tilde{\theta}) = 0$ and we have locally that $\theta - \tilde{\theta} = dF$, where F is a generating function. Choosing, e.g., $F(q^i, Q^i, t)$, we have

$$p_i dq^i - H dt - P_i dQ^i + H' dt = \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial Q^i} dQ^i + \frac{\partial F}{\partial t} dt.$$

Therefore,

$$p_i = \frac{\partial F}{\partial q^i}, \quad P_i = -\frac{\partial F}{\partial Q^i}, \quad H' - H = \frac{\partial F}{\partial t}.$$

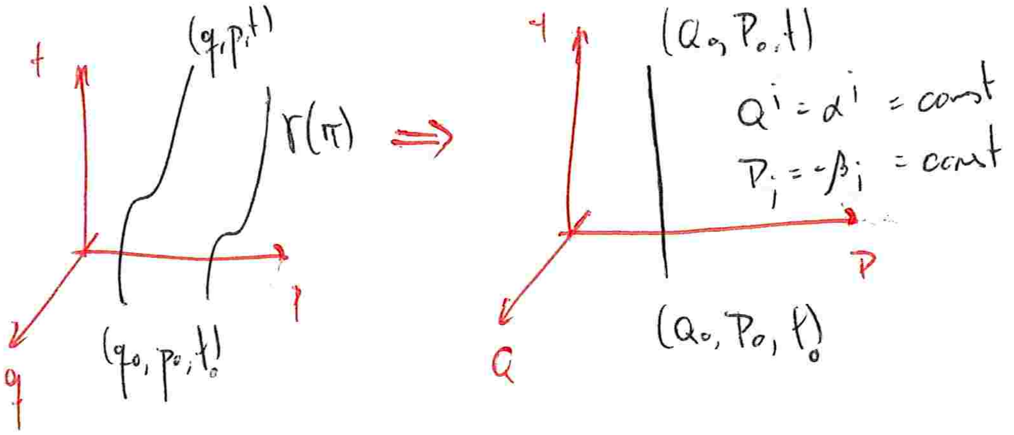
Geometrical Picture of the Hamilton-Jacobi theory

Idea: make a canonical transformation such that the new coordinates (Q^i, P_i, t) are Darboux canonical coordinates for ω.

always possible due to Darboux's Theorem

In Darboux coordinates,

$$\omega = dP_i \wedge dQ^i, \quad H' = 0, \quad X = \frac{\partial}{\partial t}.$$



We have seen previously that such a canonical transformation is generated by $F = S$ where

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) = 0.$$

Summary

→ Classical Mechanics finds a natural description in terms of symplectic and contact geometries

time independent

time dependent

	Hamiltonian: symplectic (2n)	Hamiltonian: contact (2n+1)	Lagrangian: symplectic (2n)
manifold M	phase space T^*C	extended phase space $T^*C \times \mathbb{R}$	velocity phase space TC
dynamics	Hamiltonian $H(q,p)$	Hamiltonian $H(q,p,t)$	Lagrangian $L(q,\dot{q})$
1-form	Cartan $\theta = p_i\, dq^i$	contact $\theta_H = p_i dq^i - H dt$	Lagrange $\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i$
2-form	symplectic non-degenerate, closed $\omega = d\theta$	non-singular, closed $\omega_H = d\theta_H$	Lagrange symplectic 2-form $\omega_L = d\theta_L$
EoM for dynamical field X	X_H $X_H \lrcorner d\omega = -dH$	$\tilde{X}_H$ $\tilde{X}_H \lrcorner \omega_H = 0$	$\tilde{X}_L$ $\mathcal{L}_{X_L} \theta_L = dL$ or $X_L \lrcorner \omega_L = -dE$

$$E = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L$$

Lagrange energy

→ similar framework works in thermodynamics. Namely,

$$dE = T dS - P dV$$

defines a contact 1-form

→ there is a generalization of Hamiltonian mechanics in terms of Nambu mechanics

→ differential geometry has many applications in theoretical physics

- General Relativity
- Mechanics
- Thermodynamics

↓

arXiv: 0711.4319

- Gauge theories → gauge potentials as connections
- String theory
- Mathematical Physics → Clifford algebra, special manifolds, Grassman algebra

Immediate future:

Quantum Mechanics:

$$\{q^i, p_i\} \to [\hat{q}^i, \hat{p}_i]$$

Quantum Field Theory:

$$q(t) \to \phi(\vec{x}, t)$$

$$L(q, \dot{q}, t) \to \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \to \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$$

General Relativity:

Classical Mechanics: central object is a symplectic 2-form ω which is non-degenerate, antisymmetric and has $d\omega = 0$

General Relativity: central object is a metric g that describes the gravitational field. g is non-degenerate, symmetric and has $\nabla g = 0$.