Dead Physicists Society
Instituto de Física
Universidade de São Paulo

Mecânica Quântica Relativística

II Escola Jayme Tiomno de Física Teórica
Prof. Martín Dionisio Arteaga
22 a 31 de julho de 2019
Níckolas de Aguiar Alves

Mecânica Quântica Relativística

II Escola Jayme Tiomno de Física Teórica

Professor: Martín Dionisio Arteaga, DFMA-IFUSP
Notas por: Níckolas de Aguiar Alves

Nível: Graduação
Período: 22 a 31 de julho de 2019

São Paulo
22 a 31 de julho de 2019

II Escola Jayme Tiomno de Física Teórica

Mecânica Quântica Relativística - EJT2402

Martín Dionisio Arteaga Tupia

Nickolas Alves

2019

Mecânica Quântica Relativística

~~Good blood~~ Rain

Referências

1. J Bjorker & Dull
   Relativistic Quantum Mechanics
2. W Greiner
   RQM
3. P. Strange
   RQM

1905 - Relatividade Restrita

1915 - Relatividade Geral

Relatividade Restrita

↳ movimento de corpos macroscópicos com velocidades comparáveis às da luz

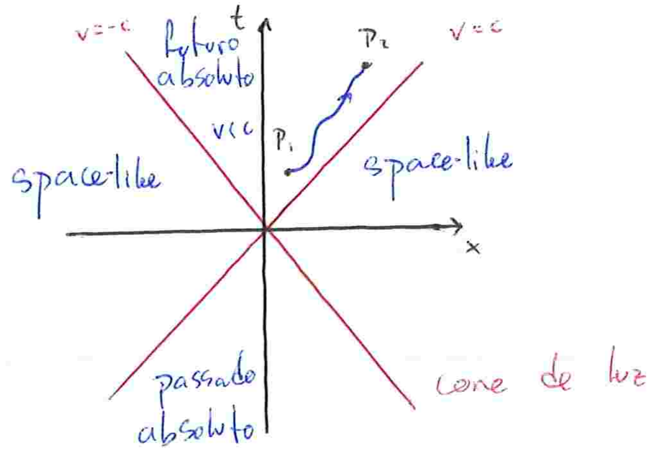
espaço de Minkowski

intervalo

invariante

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

$$= c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$



$P_1 \rightarrow (t_1, x_1, y_1, z_1)$

$P_2 \rightarrow (t_2, x_2, y_2, z_2)$

$$c^2 t_1^2 - x_1^2 - y_1^2 - z_1^2 = c^2 t_2^2 - x_2^2 - y_2^2 - z_2^2 = s^2$$

tome $x_2 = y_2 = z_2 = 0$

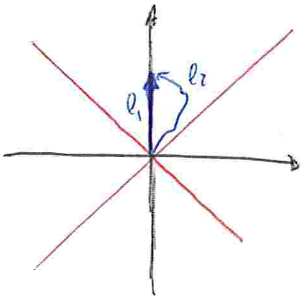
$$t_2^2 = \left(1 - \frac{(x_1^2 + y_1^2 + z_1^2)}{c^2 t_1^2}\right) t_1^2$$

$$= \left(1 - \frac{v^2}{c^2}\right) t_1^2$$

$$t_2 = \sqrt{1 - \frac{v^2}{c^2}}\, t_1$$

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$$

↳ fator de Lorentz



$l_1 > l_2 \rightarrow$ análogo ao tempo próprio

↳ os caminhos são máximos, não mínimos

$$\underbrace{S}_{\text{mínima ação}} = -K \underbrace{\int ds}_{\text{invariante}}$$

$K > 0$

$$S = -Kc \int \sqrt{1 - \frac{v^2}{c^2}}\, dt$$

$$\mathcal{L} = -Kc \sqrt{1 - \frac{v^2}{c^2}} \xrightarrow{K = mc} E = mc^2 + \frac{mv^2}{2} + \dots$$

$$\left.\begin{aligned} m &= \gamma m_0 \\ p &= \gamma m_0 v \\ E &= \gamma m_0 c^2 \end{aligned}\right\} E^2 = p^2c^2 + m_0^2 c^4$$

Equação de Schrödinger

$$\underbrace{-\frac{\hbar^2}{2m}\nabla^2 \Psi + V\Psi}_{\hat{H}\Psi} = \underbrace{E\Psi}_{i\hbar \frac{\partial \Psi}{\partial t}}$$

$\Psi(\vec{x}, t) = N e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

é uma possível solução

$$H = \frac{p^2}{2m} + V$$

$$\begin{cases} \hat{\vec{p}} = -i\hbar \nabla \\ \hat{H} = i\hbar \frac{\partial}{\partial t} \end{cases}$$

$$[\hat{x}, \hat{p}]\Psi = i\hbar \Psi$$

Equação de Continuidade

$$\psi^* \left(-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar \frac{\partial \psi}{\partial t} \right)$$

$$-\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi^*\psi = i\hbar\, \psi^* \frac{\partial \psi}{\partial t}$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}\right)\psi$$

$$-\frac{\hbar^2}{2m}(\nabla^2\psi^*) + V\psi^*\psi = -i\hbar\frac{\partial\psi^*}{\partial t}\psi$$

$$\underbrace{-\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + \frac{\hbar^2}{2m}\nabla^2\psi^*\psi}_{} = \underbrace{i\hbar\,\psi^*\frac{\partial\psi}{\partial t} + i\hbar\frac{\partial\psi^*}{\partial t}\psi}_{i\hbar\frac{\partial}{\partial t}(\psi^*\psi)}$$

$$-\frac{\hbar^2}{2m}\left(\nabla\cdot(\psi^*\nabla\psi) - \cancel{\nabla\psi^*\cdot\nabla\psi}\right)$$

$$+\frac{\hbar^2}{2m}\left(\nabla\cdot(\psi\nabla\psi^*) - \cancel{\nabla\psi^*\cdot\nabla\psi}\right)$$

$$= -\frac{\hbar^2}{2m}\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

$$-\frac{\hbar^2}{2m}\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = i\hbar\frac{\partial}{\partial t}(\psi^*\psi)$$

$$-\nabla\cdot\left[\left(\frac{i\hbar}{2m}\right)\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)\right] = \frac{\partial}{\partial t}(\psi^*\psi)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$$

$\rho = \psi^*\psi \longrightarrow$ densidade de probabilidade associada a uma partícula

$$\vec{J} = \left(\frac{i\hbar}{2m}\right)\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$$

Equação de Klein-Gordon

A equação de Schrödinger não é relativística

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t} \longrightarrow$$

ordem 2 no espaço, ordem 1 no tempo

↳ não estão em pé de igualdade

Para que seja relativista é preciso que

$$\nabla^2\psi \sim \frac{\partial^2\psi}{\partial t^2} \quad \text{ou} \quad \nabla\psi \sim \frac{\partial\psi}{\partial t}$$

↳ Klein-Gordon ↳ Dirac

Para Schrödinger, tínhamos

$$E = \frac{p^2}{2m} + V$$

$$\updownarrow \qquad \updownarrow$$

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{(-i\hbar\nabla)^2}{2m} + V\right)\psi$$

$$\hat{H} \to E \to i\hbar\frac{\partial}{\partial t}$$

$$\hat{\vec{p}} \to -i\hbar\nabla$$

Em Relatividade Restrita, a energia é expressa como

$$E^2 = p^2c^2 + m_0^2c^4 \longrightarrow$$ energia positiva ou negativa

$$E = \pm\sqrt{p^2c^2 + m_0^2c^4}$$

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2\psi = c^2(-i\hbar\nabla)^2\psi + m_0^2c^4\psi$$

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = -\hbar^2c^2\nabla^2\psi + m_0^2c^4\psi$$

$$\hat{p}^\mu\hat{p}_\mu\psi = m_0^2c^2\psi$$

$$\hookrightarrow \hat{p}^\mu = i\hbar\left\{\frac{\partial}{c\partial t}, -\nabla\right\}$$

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\nabla^2\psi + \frac{m_0^2c^2}{\hbar^2}\psi$$

$$\left(\Box^2 + \frac{m_0^2c^2}{\hbar^2}\right)\psi = 0$$

$$\boxed{\left(\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi\right) + \frac{m_0^2c^2}{\hbar^2}\psi = 0}$$

Equação de Klein-Gordon

$\hookrightarrow$ spin 0

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \longrightarrow$$ tensor de Minkowski

$$\begin{cases} \hat{H} \to i\hbar\frac{\partial}{\partial t} \\ \hat{\vec{p}} \to -i\hbar\nabla \end{cases}$$

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$$

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

$$\partial^\mu\partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

$$= \Box^2$$

$\hookrightarrow$ D'Alembertiano

$$\hat{p}^\mu = \left(\frac{\hat{H}}{c}, \hat{\vec{p}}\right)$$

$$= \left(\frac{i\hbar}{c}\frac{\partial}{\partial t}, -i\hbar\nabla\right)$$

$$= i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) = i\hbar\partial^\mu$$

$$\left(\Box^2 + \frac{m_0^2 c^2}{\hbar^2}\right)\Psi(\vec{x},t) = 0$$

Como introduzir os potenciais eletromagnéticos?

acoplamento mínimo

$$\begin{cases} \vec{p} \to \vec{p} - e\vec{A} \\ E \to E - e\phi \end{cases}$$

quero manter as simetrias da teoria e respeitar superposição (para os campos)

↳ simetria de gauge

$$(E - e\phi)^2 = (\vec{p} - e\vec{A})^2 c^2 + m_0^2 c^4$$

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)^2 = \left(-i\hbar\nabla - e\vec{A}\right)^2 c^2 + m_0^2 c^4$$

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)^2 \Psi(\vec{x},t) = c^2\left(-i\hbar\nabla - e\vec{A}\right)^2 \Psi(\vec{x},t) + m_0^2 c^4 \Psi(\vec{x},t)$$

$$\nabla\cdot\vec{B} = 0 \Rightarrow \vec{B} = \nabla\times\vec{A} \rightarrow \vec{A} \to \vec{A} + \nabla\lambda$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\phi \rightarrow \phi \to \phi - \frac{\partial\lambda}{\partial t}$$

$$(\phi, \vec{A}) \to (\phi, -\vec{A}) - \frac{\partial\lambda}{\partial t} - \nabla\lambda$$

$$A_\mu \to A_\mu - \partial_\mu\lambda$$

Equação de Continuidade para Klein-Gordon

$$K_c \equiv \frac{m_0 c}{\hbar}$$

$$\begin{cases} \psi^*\left(\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + k_c^2\psi = 0\right) \\ \left(\frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2} - \nabla^2\psi^* + k_c^2\psi^* = 0\right)\psi \end{cases}$$

$$\begin{cases} \psi^*\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \psi^*\nabla^2\psi + k_c^2\psi^*\psi = 0 \\ \psi\frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2} - \psi\nabla^2\psi^* + k_c^2\psi^*\psi = 0 \end{cases}$$

$$\psi^*\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \psi\frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2} - \psi^*\nabla^2\psi + \psi\nabla^2\psi^* = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left[\psi^*\frac{1}{c}\frac{\partial\psi}{\partial t} - \psi\frac{1}{c}\frac{\partial\psi^*}{\partial t}\right] + \nabla\cdot\left[\psi\nabla\psi^* - \psi^*\nabla\psi\right] = 0$$

$$\rho = \frac{i\hbar}{2mc}\left[\psi^*\frac{1}{c}\frac{\partial\psi}{\partial t} - \psi\frac{1}{c}\frac{\partial\psi^*}{\partial t}\right]$$

a densidade de probabilidade relativística não é mais apenas positiva e/ou nula

$$\vec{J} = \frac{i\hbar}{2m}\left[\psi\nabla\psi^* - \psi^*\nabla\psi\right]$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)\left(c\rho, \vec{J}\right) = 0 \Rightarrow \partial_\mu J^\mu = 0$$

Ansatz: $\psi(\vec{x},t) = A\, e^{-i\omega t}$

normalização e parte espacial

$\omega = \frac{E}{\hbar}$

$$\psi^*(\vec{x},t) = A^* e^{i\omega t}$$

$$\frac{\partial\psi}{\partial t} = -i\omega A e^{-i\omega t} \qquad \frac{\partial\psi^*}{\partial t} = i\omega A^* e^{i\omega t}$$

$$\psi\frac{\partial\psi^*}{\partial t} = A e^{-i\omega t}\cdot i\omega A^* e^{i\omega t} = i\omega|A|^2$$

$$\psi^*\frac{\partial\psi}{\partial t} = A^* e^{i\omega t}(-i\omega A)e^{-i\omega t} = -i\omega|A|^2$$

$mc^2 \gg K$

$E = K + mc^2 \approx mc^2$

$$\rho = \frac{i\hbar}{2mc}\left(-i\omega|A|^2 - i\omega|A|^2\right)$$

$$\rho = \frac{E\,|A|^2}{mc^2} \Rightarrow \rho = \frac{E}{mc^2} \xrightarrow[\text{relativístico}]{\text{não}} \rho = \frac{mc^2}{mc^2} = 1$$

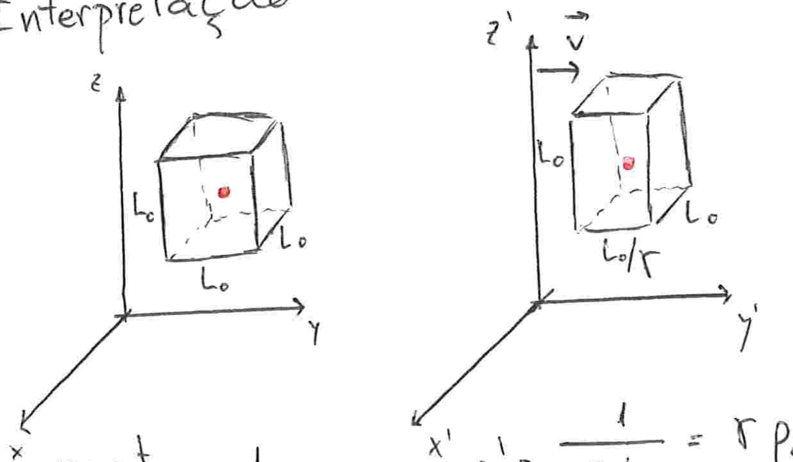
$|A|^2 = 1$

normalização

Mas $E = \gamma m c^2$, e portanto

$$\rho = \frac{\gamma m c^2}{m c^2} = \gamma$$

Interpretação



$$\rho_0 = \frac{\#\,\text{part}}{\text{Vol}} = \frac{1}{L_0^3} \qquad \rho' = \frac{1}{L_0^2 \frac{L_0}{\gamma}} = \gamma \rho_0$$

ρ na presença de $V(\vec{r})$

$$E = E_0 + V(\vec{r})$$

$$(E - V(\vec{r}))^2 = E_0^2 = p^2c^2 + m^2c^4$$

$$(\hat{H} - \hat{V}(\vec{r}))^2 \psi = \hat{P}^2 c^2 \psi + m^2 c^4 \psi$$

$$\hat{H} \rightarrow i\hbar\frac{\partial}{\partial t} \qquad \hat{P} \rightarrow -i\hbar\nabla$$

$$\left(i\hbar\frac{\partial}{\partial t} - V(\vec{r})\right)^2 \psi = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi$$

$$\left(i\hbar\frac{\partial}{\partial t} - V(\vec{r})\right)\left(i\hbar\frac{\partial}{\partial t} - V(\vec{r})\right)\psi = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi$$

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} - 2i\hbar V(\vec{r})\frac{\partial \psi}{\partial t} + V^2(\vec{r})\psi = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi$$

$$V(\vec{r}) = \frac{e^2}{4\pi\epsilon_0}\Phi(\vec{r}) \Rightarrow -\hbar^2\frac{\partial^2\psi}{\partial t^2} - 2i\hbar\left(\frac{e^2}{4\pi\epsilon_0}\right)\Phi(\vec{r})\frac{\partial\psi}{\partial t} + \left(\frac{e^2}{4\pi\epsilon_0}\right)^2\Phi^2(\vec{r})\psi = -\hbar^2 c^2\nabla^2\psi + m^2c^4\psi$$

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - 2i\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)\Phi(\vec{r})\frac{1}{c}\frac{\partial\psi}{\partial t} + \left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2\Phi^2(\vec{r})\psi = -\nabla^2\psi + k_c^2\psi$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c}$$ → constante de estrutura fina

$$\psi^*\left(\frac{\partial^2\psi}{c^2\partial t^2} - \nabla^2\psi + 2i\alpha\,\Phi(\vec r)\frac{\partial\psi}{c\partial t} - \alpha^2\Phi^2(\vec r)\psi + k_c^2\psi = 0\right)$$

$$\left(\frac{\partial^2\psi^*}{c^2\partial t^2} - \nabla^2\psi^* - 2i\alpha\,\Phi(\vec r)\frac{\partial\psi^*}{c\partial t} - \alpha^2\Phi^2(\vec r)\psi^* + k_c^2\psi^* = 0\right)\psi$$

$$\psi^*\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \psi^*\nabla^2\psi + 2i\alpha\,\Phi(\vec r)\psi^*\frac{\partial\psi}{c\partial t} - \psi\frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2} + \psi\nabla^2\psi^* + 2i\alpha\,\Phi(\vec r)\psi\frac{\partial\psi^*}{c\partial t} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left[\psi^*\frac{\partial\psi}{c\partial t} - \psi\frac{\partial\psi^*}{c\partial t}\right] + \nabla\cdot\left[\psi\nabla\psi^* - \psi^*\nabla\psi\right] + 2i\alpha\,\Phi(\vec r)\frac{\partial|\psi|^2}{c\partial t} = 0$$

$$\frac{\partial}{\partial t}\underbrace{\left[\frac{i\hbar}{2mc}\left(\psi^*\frac{\partial\psi}{c\partial t} - \psi\frac{\partial\psi^*}{c\partial t} + 2i\alpha\,\Phi(\vec r)|\psi|^2\right)\right]}_{\rho} + \nabla\cdot\left[\underbrace{\frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)}_{\vec J}\right] = 0$$

Ansatz: $\psi = A e^{-i\omega t}$, $\psi^* = A^* e^{i\omega t}$

$$\rho = \frac{i\hbar}{2mc^2}\left(A^*e^{i\omega t}(-i\omega)Ae^{-i\omega t} - Ae^{-i\omega t}(i\omega)A^*e^{i\omega t} + 2c\alpha i\,\Phi(\vec r)|A|^2\right)$$

$$= \frac{i\hbar}{2mc^2}\left(-2i\omega|A|^2 + 2ci\frac{V(\vec r)}{\hbar c}|A|^2\right)$$ ~~|A|=1 por normalização~~

$$= \frac{i\hbar}{2mc^2}\left(-2i\omega + 2i\frac{V(\vec r)}{\hbar}\right)|A|^2$$

$$= \frac{\hbar\omega - V(\vec r)}{mc^2}|\psi|^2$$

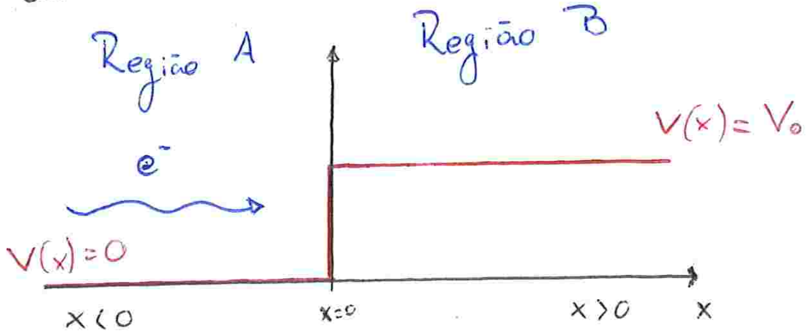
$$\rho_{V(\vec r)} = \left(\frac{E - V(\vec r)}{mc^2}\right)|\psi|^2$$

mesmo que as partículas tenham energia positiva, ainda temos densidades de probabilidade negativas

↓

Paradoxo de Klein

Paradoxo de Klein



uma onda eletrônica se propaga no eixo x e atinge um degrau de potencial

Região A

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

$$\psi_A(x,t) = \left[e^{ikx} + \underbrace{R}_{\text{coef. de reflexão}} e^{-ikx}\right] e^{-\frac{iEt}{\hbar}}$$

Região B

$$\left(i\hbar\frac{\partial}{\partial t} - V_0\right)^2\psi + c^2\hbar^2\frac{\partial^2\psi}{\partial x^2} = m^2c^4\psi$$

$$\psi_B(x,t) = \underbrace{T}_{\text{coef. de transmissão}} e^{-kx} e^{-\frac{iEt}{\hbar}}$$

Solução A + Eq. Partícula Livre: $\quad p = \hbar k$

$$E^2 = \hbar^2 k^2 c^2 + m^2 c^4$$

Solução B + Eq. com Potencial.

$$\left(i\hbar\frac{\partial}{\partial t} - V_0\right)\left(i\hbar\frac{\partial}{\partial t} - V_0\right)\left(Te^{-kx}e^{-\frac{iEt}{\hbar}}\right) + c^2\hbar^2\frac{\partial^2}{\partial x^2}\left(Te^{-kx}e^{-\frac{iEt}{\hbar}}\right) = m^2c^4\,Te^{-kx}e^{-\frac{iEt}{\hbar}}$$

$$Te^{-kx}\left(i\hbar\frac{\partial}{\partial t} - V_0\right)\left(i\hbar\frac{\partial}{\partial t} - V_0\right)e^{-\frac{iEt}{\hbar}} + Te^{-\frac{iEt}{\hbar}}c^2\hbar^2\frac{\partial^2}{\partial x^2}e^{-kx} = Tm^2c^4e^{-kx}e^{-\frac{iEt}{\hbar}}$$

$$Te^{-kx}(E - V_0)^2 e^{-\frac{iEt}{\hbar}} + Te^{-\frac{iEt}{\hbar}}c^2\hbar^2k^2e^{-kx} = Tm^2c^4e^{-kx}e^{-\frac{iEt}{\hbar}}$$

$$c^2\hbar^2k^2 = m^2c^4 - (E - V_0)^2$$

Para $V_0 \to 0$

$$E^2 = -c^2\hbar^2 k^2 + m^2c^4$$

para a solução ser ondulatória, $Re[k] = 0$, então não há problema com o resultado anterior

$$c\hbar k = \sqrt{m^2c^4 - (E - V_0)^2}$$

↳ k pode ser real ou imaginário, a depender do valor de V_0

k imaginário

$$(E - V_0)^2 > m^2c^4$$

$$(E - V_0) > mc^2$$

$$\boxed{(E - V_0) < -mc^2}$$

~~Região IV~~

Região I

$$V_0 < E - mc^2$$

$$V_0 = E - m^2c^4$$

$$k = 0$$

k real

$$(E - V_0)^2 < m^2c^4$$

$$\boxed{-mc^2 < E - V_0} < m^2c^4$$

$$V_0 > E - m^2c^4$$

Região II

k cresce na região II

$E - V_0 > -mc^2$ Região III k real

$$V_0 < E + mc^2$$

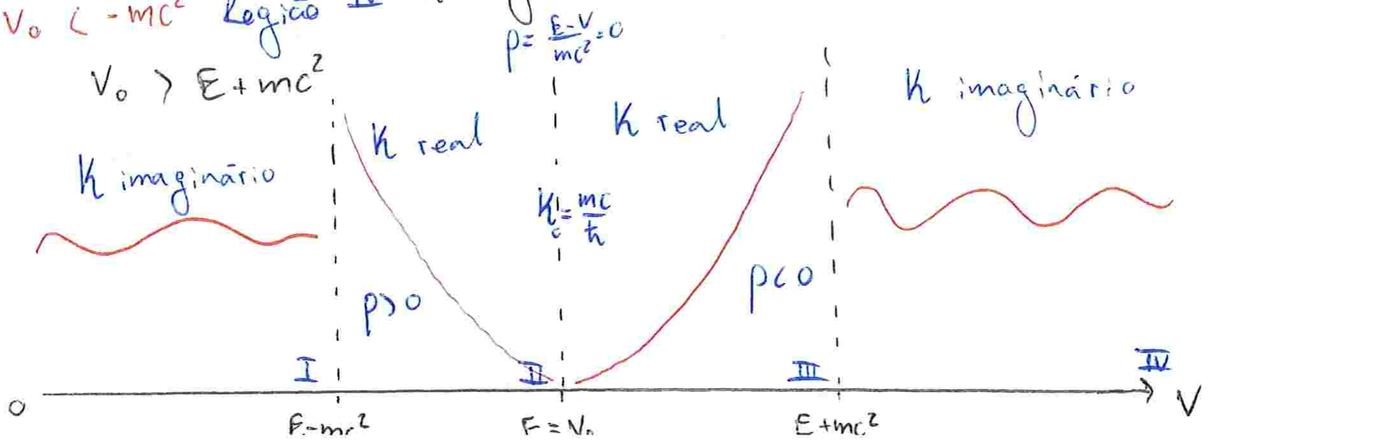
$V_0 = E + m^2c^4$

↳ $k = 0$

k decresce em III

$E - V_0 < -mc^2$ Região IV k imaginário

$$V_0 > E + mc^2$$



$E + mc^2 - (E - mc^2) = 2mc^2 \rightarrow$ não é arbitrário

Não é possível interpretar ψ como associada a uma única partícula! Criação e aniquilação de pares

RQM descreve partículas

QFT descreve campos

$V_0 \to +\infty$

$k \to i k'$

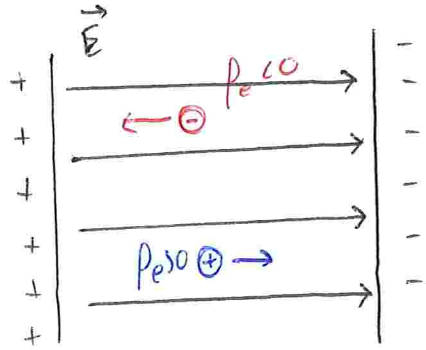
$$\psi_B(x,t) = T e^{-ik'x} e^{-\frac{iEt}{\hbar}}$$

$$J = \frac{i\hbar}{2m}\left[\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right]$$

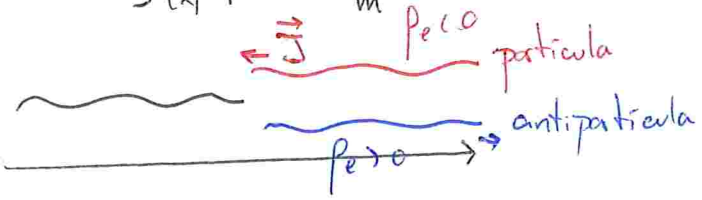
$$\psi_B^* = T^* e^{ik'x} e^{\frac{iEt}{\hbar}}$$

$$\frac{\partial \psi_B}{\partial x} = -ik' T e^{-ik'x} e^{-\frac{iEt}{\hbar}}$$

$$\frac{\partial \psi_B^*}{\partial x} = ik' T^* e^{ik'x} e^{\frac{iEt}{\hbar}}$$



$$J(x,t) = -\frac{\hbar k'}{m} |\psi|^2$$



interpretar como uma densidade de carga elétrica, por exemplo e pensar em antipartículas se propagando no sentido certo

Mas isso descreve apenas partículas de spin 0 (Equação de Klein-Gordon); podemos analisar outros casos? a posteriori

$s = 0$ bóson escalar

$s = 1/2$ elétrons, neutrinos

$s = 1$ fótons, $W^{\pm}$, Z, glúon

$s = 3/2$ gravitino

$s = 2$ gráviton

modelos SUSY

Observação: Gravidade em Mecânica Quântica

Relatividade Geral

Princípio de Equivalência: $m_{inercial} = m_{gravitacional}$

símbolos de Christoffel

$$\frac{d^2 x^k}{d\tau^2} + \Gamma^k_{ij} \frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0$$

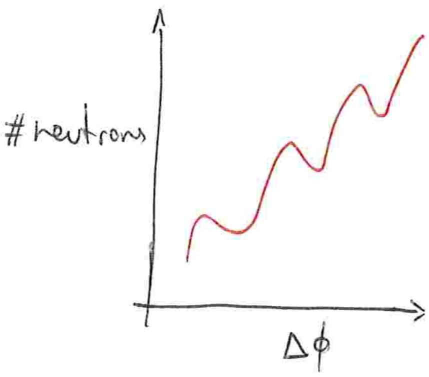
$$m_{in} \frac{d^2 x}{dt^2} = -m \nabla \Phi$$

Eq. de Schrödinger

$$-\frac{\hbar^2}{2m}\nabla^2\psi + m\Phi\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\tilde{\psi}(x,t) = e^{-\frac{m\Phi t}{\hbar}}\psi(x,t)$$

$$\phi_1 - \phi_2 \approx \cos\left(\frac{m\Phi t}{\hbar}\right)\dots$$



~~$V(x) = m_{grav}$~~

$$V(x) = m_{grav}\Phi(x)$$

$$p = \frac{E - V(x)}{mc^2}$$

$$= \frac{\gamma mc^2 - m\Phi}{mc^2}$$

$$= \gamma - \frac{\Phi}{c^2} \begin{cases} p \geq 0 \\ p < 0 \end{cases}$$

$p \geq 0$

$$\gamma \geq \frac{\Phi}{c^2}$$

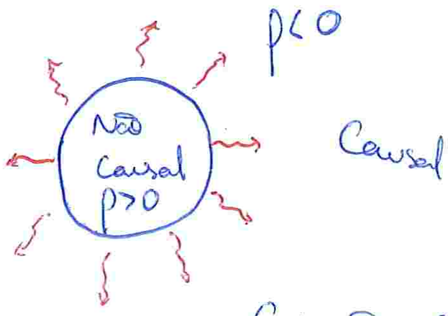
$$\sqrt{1 - \frac{v^2}{c^2}} = \gamma^{-1} \leq \left(\frac{\Phi}{c^2}\right)^{-1}$$

$$1-\frac{v^2}{c^2} \leq \left(\frac{\Phi}{c^2}\right)^{-2}$$

$\rho > 0$ p/ $v > c$
$\rho < 0$ p/ $v < c$

$$c^2 - v^2 \leq c^2 \frac{c^4}{\Phi^2}$$

$$c^2 \leq v^2 + \frac{c^6}{\Phi^2}$$



Criação e aniquilação na fronteira de um buraco negro

↓

possível interpretação

$\rho < 0$

$$\gamma < \frac{\Phi}{c^2}$$

$$\sqrt{1-\frac{v^2}{c^2}} = \gamma^{-1} > \left(\frac{\Phi}{c^2}\right)^{-1} \Rightarrow 1-\frac{v^2}{c^2} > \frac{c^4}{\Phi^2}$$

$$c^2 - v^2 > \frac{c^6}{\Phi^2} \Rightarrow c^2 > v^2 + \frac{c^6}{\Phi^2}$$

De volta ao Paradoxo de Klein

$$\psi_A = \left(e^{ikx} + R e^{-ikx}\right) e^{-\frac{iEt}{\hbar}}$$

$$\psi_B = T e^{-\eta x} e^{-\frac{iEt}{\hbar}}$$

Impondo que ψ é contínua (em $x=0$) obteremos

$$1 + R = T$$

Continuidade de ψ'

$$\frac{\partial \psi_A}{\partial x} = \frac{\partial \psi_B}{\partial x}$$

$$\alpha(1-R) = kT$$

EJ12402

$$R = \frac{ik + k}{ik - k} \qquad T = \frac{2ik}{ik-k}$$

$$R = \frac{k + k^2}{k \cdot k^2} > 1$$

$$T = \frac{2k}{k - k^2} > 1$$

Spin 1/2: Equação de Dirac

$$E^2 = p^2c^2 + m^2c^4 \begin{cases} E > 0 \\ E < 0 \end{cases}$$

$\cancel{\hat{H}^2\psi \quad p^2 \text{ escalar} \quad \alpha_i}$

$$\hat{H}^2\psi = p^2c^2\psi + m^2c^4\psi$$

quero que valha
quero ainda linearidade em $\hat{H}$

α_i e β não podem ser números, ou não valeriam invariâncias rotacionais

matrizes?

ψ não é um escalar, mas um vetor no fim do processo

Ansatz: $\hat{H} = c\,\hat{\alpha}_i \hat{p}^i + \hat{\beta} mc^2$

$$\hat{\alpha}_i \hat{p}_i = \hat{\alpha}_x \hat{p}_x + \hat{\alpha}_y \hat{p}_y + \hat{\alpha}_z \hat{p}_z$$

$$i\hbar \frac{\partial \psi}{\partial t} = c\hat{\alpha}_i \left(-i\hbar \frac{\partial}{\partial x^i}\right)\psi + mc^2\hat{\beta}\psi$$

$$\frac{i\hbar}{c}\frac{\partial \psi}{\partial t} = -i\hbar\,\hat{\alpha}_i \frac{\partial}{\partial x^i}\psi + mc\,\hat{\beta}\psi$$

$$\frac{\hat{H}}{c} = \hat{\alpha}_j p_j + \hat{\beta} mc \Rightarrow \frac{i\hbar}{c}\frac{\partial}{\partial t} = -i\hbar\,\hat{\alpha}_j \frac{\partial}{\partial x^i} + \hat{\beta} mc$$

$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}\right)\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}\right)\psi = \left(-i\hbar\hat{\alpha}_j\frac{\partial}{\partial x^j} + \hat{\beta}mc\right)\left(-i\hbar\hat{\alpha}_i\frac{\partial}{\partial x^i} + \hat{\beta}mc\right)\psi$$

$$-\frac{\hbar^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\hbar^2\left(\frac{\hat{\alpha}_j\hat{\alpha}_i + \hat{\alpha}_i\hat{\alpha}_j}{2}\right)\frac{\partial^2\psi}{\partial x^i\partial x^j} - i\hbar mc\left(\hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i\right)\frac{\partial\psi}{\partial x^i} + \hat{\beta}^2 m^2c^2\psi$$

$$\sum_{i,j}\hat{\alpha}_i\hat{\alpha}_j = \sum_{i<j}\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i = \frac{1}{2}\sum_{i,j}\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i$$

Para satisfazer $E^2 = p^2c^2 + m^2c^4$, é preciso que cada componente de ψ satisfaça Klein-Gordon

$$\left(\Box^2 + \frac{m^2c^2}{\hbar^2}\right)\psi_\sigma = 0$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi_\sigma = 0$$

$$-\frac{\hbar^2}{c^2}\frac{\partial^2\psi_\sigma}{\partial t^2} = -\hbar^2\frac{\partial^2\psi_\sigma}{\partial x^i\partial x^i} + m^2c^2\psi_\sigma$$

Por comparação com a equação anterior, vemos que

$$\left.\begin{aligned}\hat{\alpha}_j\hat{\alpha}_i + \hat{\alpha}_i\hat{\alpha}_j &= 2\delta_{ij}\mathbb{1}\\ \hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i &= 0\\ \hat{\beta}^2 &= \mathbb{1}\end{aligned}\right\}\text{álgebra de Clifford}$$

$$\left.\begin{aligned}\hat{\alpha}_i^\dagger &= \hat{\alpha}_i\\ \hat{\beta}^\dagger &= \hat{\beta}\end{aligned}\right\}\text{necessário para } \hat{H} \text{ ser hermitiano}$$

Como $\hat{\alpha}_i$ e $\hat{\beta}$ são hermitianas com $\hat{\alpha}_i^2 = \hat{\beta}^2 = \mathbb{1}$, todos seus autovalores são ± 1. Na base que diagonaliza $\hat{\alpha}_i$,

$$\hat{\alpha}_i = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix}$$

$$\hat{\alpha}_i^2 = \begin{pmatrix} A_1^2 & & & 0 \\ & A_2^2 & & \\ & & \ddots & \\ 0 & & & A_n^2 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix} = \mathbb{1}$$

$$\therefore A_j^2 = 1 \Rightarrow A_j = \pm 1$$

Como $\hat{\alpha}_i \hat{\beta} + \hat{\beta}\hat{\alpha}_i = 0$, e $\hat{\beta}^2 = \mathbb{1}$, $\hat{\alpha}_i = -\hat{\beta}\hat{\alpha}_i\hat{\beta}$. Como $\mathrm{Tr}(\hat{A}\hat{B}) = \mathrm{Tr}(\hat{B}\hat{A})$, temos

$$\mathrm{Tr}(\hat{\alpha}_i) = \mathrm{Tr}(\hat{\beta}^2\hat{\alpha}_i) = \mathrm{Tr}(\hat{\beta}\hat{\alpha}_i\hat{\beta}),$$
$$= -\mathrm{Tr}(\hat{\alpha}_i),$$
$$2\,\mathrm{Tr}(\hat{\alpha}_i) = 0,$$
$$\mathrm{Tr}(\hat{\alpha}_i) = 0.$$

idem para $\hat{\beta}$

como o traço é a soma dos autovalores, $\hat{\alpha}_i$ e $\hat{\beta}$ tem tantos autovalores positivos quanto negativos e é preciso que a dimensão seja par.

a dimensão não pode ser $N=2$ → só há três matrizes de Pauli

próxima menor dimensão: $N=4$

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

uma das possíveis representações explícitas das matrizes de Dirac

$$\hat{\alpha}_1 = \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix} \quad \hat{\alpha}_2 = \begin{pmatrix} 0&0&0&-i\\ 0&0&i&0\\ 0&-i&0&0\\ i&0&0&0 \end{pmatrix} \quad \hat{\alpha}_3 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&-1&0&0 \end{pmatrix} \quad \hat{\beta} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}$$

Resta construirmos a 4-corrente de probabilidade

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c\, \hat{\alpha}_i \frac{\partial \psi}{\partial x^i} + mc^2 \hat{\beta}\psi$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = i\hbar c \frac{\partial \psi^*}{\partial x^i}\hat{\alpha}_i + mc^2 \psi^* \hat{\beta} \quad \rightarrow (AB)^\dagger = B^\dagger A^\dagger$$

$$i\hbar\, \psi^* \frac{\partial \psi}{\partial t} = -i\hbar c\, \psi^* \hat{\alpha}_i \frac{\partial \psi}{\partial x^i} + mc^2 \psi^* \hat{\beta}\psi$$

$$-i\hbar \frac{\partial \psi^*}{\partial t}\psi = i\hbar c \frac{\partial \psi^*}{\partial x^i}\hat{\alpha}_i \psi + mc^2 \psi^* \hat{\beta}\psi$$

$$i\hbar\left(\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi\right) = -i\hbar c\left(\psi^* \hat{\alpha}_i \frac{\partial \psi}{\partial x^i} + \frac{\partial \psi^*}{\partial x^i}\hat{\alpha}_i \psi\right) + mc^2\left(\psi^* \hat{\beta}\psi - \psi^* \hat{\beta}\psi\right)$$

$$i\hbar \frac{\partial}{\partial t}(\psi^*\psi) = -i\hbar c \frac{\partial}{\partial x^i}\left(\psi^* \hat{\alpha}_i \psi\right)$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\partial}{\partial x^i}\left(\psi^* \hat{\alpha}_i \psi\right) = 0$$

$$\rho \equiv \psi^*\psi, \quad \vec{J} = c\, \psi^* \vec{\hat{\alpha}}\, \psi$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

Equação de Dirac usando matrizes γ → forma covariante

$$\hat{H} = \hat{\alpha}^i c \hat{p}_i + \hat{\beta} mc^2$$

$$\hat{H}\psi = \left(\hat{\alpha}^i_i c \hat{p}_i + \hat{\beta} mc^2\right)\psi$$

$\hat{H} \rightarrow i\hbar \frac{\partial}{\partial t}$

$\vec{p} \rightarrow -i\hbar \nabla$

$\hat{p}_i \rightarrow -i\hbar \frac{\partial}{\partial x^i}$

$$i\hbar \frac{\partial \psi}{\partial t} = \left(\hat{\alpha}^i c\left(-i\hbar \frac{\partial}{\partial x^i}\right) + \hat{\beta} mc^2\right)\psi$$

$$i\hbar \frac{\partial \psi}{\partial t} + i\hat{\alpha}^i c\hbar \frac{\partial \psi}{\partial x^i} - \hat{\beta} mc^2 \psi = 0$$

Observação

Note que $\psi_{KG} \neq \psi_D$

$$i\hbar\left[\frac{\partial}{\partial t} + c\,\hat{\alpha}^i\frac{\partial}{\partial x^i}\right]\psi - \hat{\beta}mc^2\psi = 0$$

$$i\hbar\hat{\beta}\left[\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}^i\frac{\partial}{\partial x^i}\right]\psi - \hat{\beta}^2 mc\psi = 0$$

$$i\hbar\underbrace{\left[\hat{\beta}\frac{1}{c}\frac{\partial}{\partial t} + (\hat{\beta}\hat{\alpha}^i)\frac{\partial}{\partial x^i}\right]}_{(\hat{\beta},\ \hat{\beta}\hat{\alpha}^i)\left(\frac{1}{c}\frac{\partial}{\partial t},\ \frac{\partial}{\partial x^i}\right) = \gamma^\mu\partial_\mu}\psi - mc\psi = 0$$

$\gamma^0 = \hat{\beta}$

$\gamma^1 = \hat{\beta}\hat{\alpha}^1$

$\gamma^2 = \hat{\beta}\hat{\alpha}^2$

$\gamma^3 = \hat{\beta}\hat{\alpha}^3$

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

notação slash de Feynman $\gamma^\mu A_\mu = \not{A}$

$$\boxed{(i\hbar\not{\partial} - mc)\psi = 0}$$

Equação de Dirac

$(i\not{\partial} - m)\psi = 0$ em unidades naturais

Equação de Continuidade → desta vez pelo formalismo covariante

$$(i\hbar\not{\partial} - mc)\psi = 0 \Rightarrow i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

$$\psi^\dagger(i\hbar\not{\partial}^\dagger - mc) = 0 \Rightarrow -i\hbar\psi^\dagger\partial_\mu\gamma^{\mu\dagger} - \psi^\dagger mc = 0$$

$$\not{\partial}^\dagger = (\gamma^\mu\partial_\mu)^\dagger = \partial_\mu\gamma^{\mu\dagger}$$

$\gamma^{\mu\dagger}$

$$\begin{cases}\hat{\alpha}_i^\dagger = \hat{\alpha}_i \\ \hat{\beta}^\dagger = \hat{\beta}\end{cases} \rightarrow \gamma^0 = \hat{\beta},\quad \gamma^{0\dagger} = \gamma^0$$

$$\gamma^i = \hat{\beta}\hat{\alpha}^i$$

$$\hat{\alpha}^{i\dagger}\hat{\beta}^\dagger = \hat{\alpha}^i\hat{\beta}^\dagger \rightarrow \hat{\alpha}^{i\dagger} = \hat{\alpha}^i$$

$$= \hat{\alpha}^i\hat{\beta} \rightarrow \hat{\beta}^\dagger = \hat{\beta}$$

$$\gamma^{i\dagger} = \mathbb{1}\,\hat{\alpha}^i\hat{\beta}$$
$$= \hat{\beta}\hat{\beta}\hat{\alpha}^i\hat{\beta}$$
$$= \gamma^0\gamma^i\gamma^0$$
$$\therefore\ \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

$$\gamma^{0\dagger} = \gamma^0\gamma^0\gamma^0$$
$$= \mathbb{1}\,\gamma^0$$
$$= \gamma^0$$

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$
$$-i\hbar\psi^\dagger\partial_\mu(\gamma^0\gamma^\mu\gamma^0) - \psi^\dagger mc = 0$$
$$-i\hbar\underbrace{\psi^\dagger\gamma^0}_{\bar{\psi}=\psi^\dagger\gamma^0}\gamma^\mu\partial_\mu\gamma^0 - \psi^\dagger mc = 0$$

$$-i\hbar\bar{\psi}\gamma^\mu\partial_\mu\gamma^0\gamma^0 - \psi^\dagger\gamma^0 mc = 0$$
$$-i\hbar\bar{\psi}\gamma^\mu\partial_\mu - \bar{\psi}mc = 0$$

$$i\hbar\bar{\psi}\gamma^\mu\overrightarrow{\partial_\mu}\psi - mc\bar{\psi}\psi = 0$$
$$-i\hbar\bar{\psi}\gamma^\mu\overleftarrow{\partial_\mu}\psi - mc\bar{\psi}\psi = 0$$

$\overrightarrow{\partial_\mu}$ age para a direita

$\overleftarrow{\partial_\mu}$ age para a esquerda

Lei conjugado

$$i\hbar\bar{\psi}\gamma^\mu(\partial_\mu\psi) + i\hbar(\partial_\mu\bar{\psi})\gamma^\mu\psi = 0$$
$$\bar{\psi}\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\gamma^\mu\psi = 0$$
$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$
$$\partial_\mu J^\mu = 0 \rightarrow$$ equação de continuidade

$$c\rho = J^0$$
$$\rho = \frac{1}{c}\bar{\psi}\gamma^0\psi$$
$$= \frac{1}{c}\psi^\dagger\gamma^0\gamma^0\psi$$
$$= \frac{1}{c}|\psi|^2$$

Observação

$\hat{\alpha}_i^2 = \mathbb{1}$ $\quad \hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i = 2\delta_{ij}$

$\hat{\beta}^2 = \mathbb{1}$ $\quad \hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i = 0$

$$\Downarrow\quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\underbrace{\eta^{\mu\nu}}_{\text{métrica de Minkowski}}$$

Clifford $\quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

EJT2402

Ansatz: Onda Plana

$$\psi(\vec{x},t) = u(p)\, e^{-iEt/\hbar + i\vec{p}\cdot\vec{x}/\hbar}$$

↳ amplitude do spinor

$$\gamma^0 = \hat{\beta} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\gamma^i = \hat{\beta}\hat{\alpha}^i = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\left[i\hbar \gamma^0 \partial_0 + i\hbar \gamma^i \partial_i - mc \right]\psi = 0$$

$$\gamma^0 \partial_0 = \begin{pmatrix} \mathbb{1}\partial_0 & 0 \\ 0 & -\mathbb{1}\partial_0 \end{pmatrix}$$

$$\gamma^i \partial_i = \begin{pmatrix} 0 & \sigma^i \partial_i \\ -\sigma^i \partial_i & 0 \end{pmatrix}$$

Seguindo os cálculos, obtém-se eventualmente

$$\begin{pmatrix} mc^2 \cdot \mathbb{1} & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \cdot \mathbb{1} \end{pmatrix}\psi = E\psi$$

observação

$$(\vec{\sigma}\cdot\vec{p})^2 = (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})$$

$$= \sigma^i p^i \sigma^j p^j$$

$$= \sigma^i \sigma^j p^i p^j$$

$$= \left(\frac{\sigma^i\sigma^j + \sigma^j\sigma^i}{2}\right) p^i p^j = \frac{\{\sigma^i, \sigma^j\}}{2} p^i p^j = \frac{2\delta^{ij}}{2} p^i p^j = p^i p^i = |\vec{p}|^2$$

$$(\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2$$

$$\therefore \begin{cases} (\vec{\sigma}\cdot\vec{p})\,\chi_+ = +\,|\vec{p}|\,\chi_+ \\ \underbrace{(\vec{\sigma}\cdot\vec{p})}_{2\times 2}\,\underbrace{\chi_-}_{2\times 1} = -\,|\vec{p}|\,\chi_- \end{cases}$$

$$u_+(p) = \begin{pmatrix} A\,\chi_+ \\ B\,\chi_+ \end{pmatrix} \Rightarrow |u|^2 = 1 = |A|^2 + |B|^2$$

$$\begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix} \begin{pmatrix} A\chi_+ \\ B\chi_+ \end{pmatrix} = E \begin{pmatrix} A\chi_+ \\ B\chi_+ \end{pmatrix}$$

$\mathbb{1}$ implicito

$$Amc^2\chi_+ + cB(\vec{\sigma}\cdot\vec{p})\chi_+ = EA\chi_+$$

$$\underbrace{\left[A(mc^2 - E) + B(c|\vec{p}|)\right]}_{0}\chi_+ = 0$$

Sistema

$$\begin{cases} A(mc^2 - E) + B(c|\vec{p}|) = 0 \\ |A|^2 + |B|^2 = 1 \end{cases} \Rightarrow \quad A = \sqrt{\frac{E + mc^2}{2mc^2}} \qquad B = \sqrt{\frac{E - mc^2}{2mc^2}}$$

$$\psi = Ae^{-kx} + Be^{+kx}$$

$$\begin{cases} u_+(p) = \begin{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}}\,\chi_+(p) \\ \sqrt{\frac{E-mc^2}{2mc^2}}\,\chi_+(p) \end{pmatrix} \\ u_-(p) = \begin{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}}\,\chi_-(p) \\ -\sqrt{\frac{E-mc^2}{2mc^2}}\,\chi_-(p) \end{pmatrix} \end{cases}$$

$E,\ \vec{p}$

$\#\text{DoF} = 2$

$2s+1 = 2$

$s = \frac{1}{2}$

Também podemos resolver a equação para energia negativa

$$\psi = V(p)\, e^{-(i(-E)t - i(-\vec{p})\cdot\vec{x})}$$

também posso interpretar que $E>0$, mas o tempo corre no sentido negativo

↓

interpretação de antipartículas voltando no tempo

$$\begin{pmatrix} -mc^2 & c(\vec{\sigma}\cdot\vec{p}) \\ c(\vec{\sigma}\cdot\vec{p}) & mc^2 \end{pmatrix}\psi = E\psi$$

$$V_+(p) = \begin{pmatrix} C\chi_+ \\ D\chi_- \end{pmatrix}, \quad V_-(p) = \begin{pmatrix} E\chi_- \\ G\chi_- \end{pmatrix}$$

Conservação do Momento Angular

$$\frac{d\vec{L}}{dt} = 0, \quad \frac{d\hat{\vec{L}}}{dt} = 0\ ?$$

Em Mecânica Quântica,

$$[\hat{H}, \hat{\theta}] = i\hbar\frac{\partial\hat{\theta}}{\partial t}$$

$\hat{\theta}$ é conservado se

$$[\hat{H}, \hat{\theta}] = 0$$

O que obtemos com o momento angular e a Hamiltoniana de Dirac?

$$\hat{H}_D = c\hat{\vec{\alpha}}\cdot\vec{p} + \hat{\beta}mc^2$$

$$L^k_i = \epsilon_{ikl}x^k\hat{p}^l$$

$$\vec{L} = \vec{r}\times\vec{p}$$

$$[\hat{H}_D, L_i] = [c\hat{\alpha}^j\hat{p}^j + \hat{\beta}mc^2, \epsilon_{ikl}x^k\hat{p}^l]$$

$$= [c\hat{\alpha}^j\hat{p}^j, \epsilon_{ikl}x^k\hat{p}^l] + \cancel{[\hat{\beta}mc^2, \epsilon_{ikl}x^k\hat{p}^l]}$$

$$\hat{\beta} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$[\hat{H}_D, L_i] = c\hat{\alpha}^j \epsilon_{ikl} [\hat{p}^j, x^k \hat{p}^l]$$

$$[\hat{p}^j, x^k \hat{p}^l] = \left[-i\hbar \frac{\partial}{\partial x^j}, -x^k i\hbar \frac{\partial}{\partial x^l}\right]$$

$$= (-i\hbar)^2 \left(\left[\frac{\partial}{\partial x^j}, x^k\right] \frac{\partial}{\partial x^l} + x^k \left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l}\right] \right)$$

$$= (-i\hbar)^2 \delta^{kj} \frac{\partial}{\partial x^l}$$

$$= -i\hbar\, \delta^{kj} \hat{p}^l$$

$$[\hat{H}_D, L_i] = c\hat{\alpha}^j \epsilon_{ikl} (-i\hbar) \delta^{kj} \hat{p}^l$$

$$= -i\hbar c\, \epsilon_{ijk} \hat{\alpha}^j \hat{p}^k$$

$$= -i\hbar c\, \hat{\vec{\alpha}} \times \hat{\vec{p}}_i \longrightarrow \vec{L} \text{ não é conservado!}$$

Consideramos

$$\hat{\Sigma}^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$[\hat{H}_D, \Sigma^i] = 2ic\, \epsilon_{ijk} \hat{\alpha}^j \hat{p}^k$$

$$= 2ic\, \hat{\vec{\alpha}} \times \hat{\vec{p}}_i$$

posso compor $\vec{L}$ e $\vec{\Sigma}$ para obter um objeto conservado

$$\hat{\vec{J}} = \hat{\vec{L}} + \frac{\hbar}{2} \vec{\Sigma}$$

$$[\hat{H}_D, \vec{J}] = [\hat{H}_D, \vec{L}] + \frac{\hbar}{2} [\hat{H}_D, \vec{\Sigma}]$$

$$= -i\hbar c\, \hat{\vec{\alpha}} \times \hat{\vec{p}} + \frac{\hbar}{2} (2ic\, \hat{\vec{\alpha}} \times \hat{\vec{p}}) = 0$$

$$\hat{\Sigma}\psi = \lambda\psi \Rightarrow \lambda = \pm 1$$

$$\vec{J}\cdot\vec{P} = \underbrace{\vec{L}\cdot\vec{P}}_{\substack{0 \\ \vec{L}=\vec{r}\times\vec{p}}} + \underbrace{\frac{\hbar}{2}\vec{\Sigma}\cdot\vec{P}}_{\vec{S}=\frac{\hbar}{2}\vec{\Sigma}} \Rightarrow \begin{cases} (\vec{J}\cdot\vec{P})u_+ = +\frac{\hbar}{2}|\vec{p}|u_+ \\ (\vec{J}\cdot\vec{P})u_- = -\frac{\hbar}{2}|\vec{p}|u_- \end{cases}$$

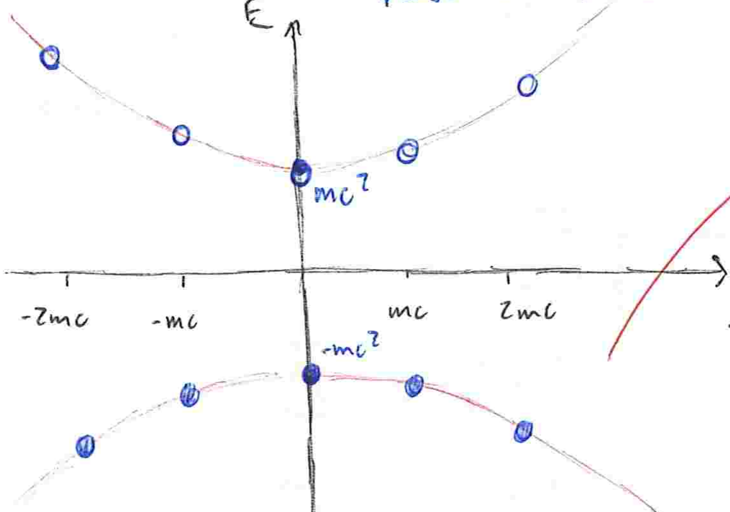
$$\left(\frac{\vec{J}\cdot\vec{P}}{|\vec{p}|}\right) = \hat{h} \rightarrow \text{operador de helicidade}$$

$$\begin{cases} \hat{h}\,u_+ = +\frac{\hbar}{2}u_+ \\ \hat{h}\,u_- = -\frac{\hbar}{2}u_- \end{cases}$$

Duas soluções de energia positiva (spin up e spin down) e duas soluções de $\underbrace{\text{energia negativa}}$ (spin up e spin down)

continuam aparecendo mesmo na Equação de Dirac

Dirac: Antipartículas → interpretação distinta da atual de Teoria Quântica de Campos, mas resultados semelhantes



Dirac: os estados de energia negativa já estão preenchidos

vácuo

mar de Dirac

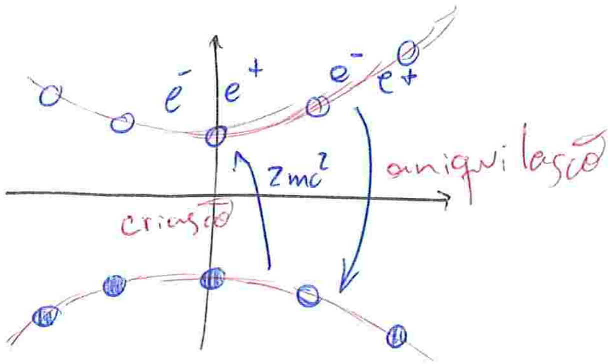
Podemos excitar as partículas ocupando estados de energia negativa?

$\vec{p}$ → momento e^- com $E < 0$

$-\sqrt{|\vec{p}|^2 c^2 + m^2 c^4}$ → energia e^-

$$\begin{array}{ccc} \vec{p} & \longrightarrow & -\vec{p} \\ -\sqrt{|\vec{p}|^2 c^2 + m^2 c^4} & \longrightarrow & +\sqrt{|\vec{p}|^2 c^2 + m^2 c^4} \\ e^- & \longrightarrow & e^+ \end{array}$$

mar de Dirac $\xrightarrow[\text{excitação}]{}$ pósitron



Problema resolvido! Ou não?

↳ ψ não descreve mais apenas uma partícula

↓

solução:

Teoria Quântica de Campos

EJT2402