

1 Meta-theory of the translation

Translation of ambient contexts Γ .

$$\begin{aligned} \ulcorner \cdot \urcorner &= \cdot \\ \ulcorner \Gamma, u : [\Psi \vdash \mathbf{a}] \urcorner &= \ulcorner \Gamma \urcorner, u : \ulcorner [\Psi \vdash \mathbf{a}] \urcorner \end{aligned}$$

Translation of contextual types

$$\begin{aligned} \ulcorner [\Psi \vdash A] \urcorner &= \mathbf{tm}[\ulcorner \Psi \urcorner, \ulcorner A \urcorner] \\ \ulcorner [\Psi \vdash \Phi] \urcorner &= \mathbf{sub}[\ulcorner \Psi \urcorner, \ulcorner \Phi \urcorner] \end{aligned}$$

Translation of the bound var. context Ψ

$$\begin{aligned} \ulcorner \cdot \urcorner &= \mathbf{nil}[] \\ \ulcorner \Psi, x : \mathbf{a} \urcorner &= \mathbf{cons}[\ulcorner \Psi \urcorner, \mathbf{a}] \end{aligned}$$

Translation of sim. substitutions

$$\begin{aligned} \ulcorner \cdot \urcorner_{\Psi}^{\Psi} &= \mathbf{Shift}[\ulcorner \Psi \urcorner, \ulcorner \Psi \urcorner](\mathbf{Id}[\ulcorner \Psi \urcorner, \ulcorner \Psi \urcorner]) \\ \ulcorner \cdot \urcorner_{\Psi, x : \mathbf{a}}^{\Phi} &= \mathbf{Shift}[\ulcorner \Psi, x : \mathbf{a} \urcorner, \ulcorner \Phi \urcorner] \\ &\quad (\mathbf{Suc}[\ulcorner \Psi, x : \mathbf{a} \urcorner, \ulcorner \Phi \urcorner] \ulcorner \cdot \urcorner_{\Psi}^{\Phi}) \\ \ulcorner \sigma, x / M \urcorner_{\Psi}^{\Phi, x : A} &= \mathbf{Dot}[\ulcorner \Psi \urcorner, \ulcorner \Phi \urcorner, \mathbf{a}] (\ulcorner \sigma \urcorner_{\Psi}^{\Phi}, \ulcorner M \urcorner_{\Phi \vdash \mathbf{a}}^{\Psi}) \end{aligned}$$

Lemma 1 (Ambient Context). *If $\Gamma(u) = [\Psi \vdash \mathbf{a}]$ then $\ulcorner \Gamma \urcorner(u) = \mathbf{tm}[\ulcorner \Psi \urcorner, \mathbf{a}]$.*

Proof. Induction on the structure of Γ . □

Lemma 2 (Terms).

1. *If $\Gamma; \Psi \vdash M : A$ then $;\ulcorner \Gamma \urcorner \vdash \ulcorner M \urcorner_{\Psi \vdash A} : \ulcorner \Psi \vdash A \urcorner$.*
2. *If $\Gamma; \Psi \vdash \sigma : \Phi$ then $;\ulcorner \Gamma \urcorner \vdash \ulcorner \sigma \urcorner_{\Psi \vdash \Phi} : \ulcorner \Psi \vdash \Phi \urcorner$*

Proof. Induction on the typing derivation.

$$\mathbf{Case} \quad \mathcal{D} = \frac{\Psi(x) = \mathbf{a}}{\Gamma; \Psi \vdash x : \mathbf{a}} \mathbf{t-var}$$

$;\ulcorner \Gamma \urcorner \vdash \mathbf{Var}[\ulcorner \Psi \urcorner, \mathbf{a}] \ k : \mathbf{tm}[\ulcorner \Psi \urcorner, \mathbf{a}]$ by the correctness of our translation function that computes the position k of x in Ψ .

$;\ulcorner \Gamma \urcorner \vdash \ulcorner x \urcorner_{\Psi \vdash \mathbf{a}} : \ulcorner \Psi \vdash \mathbf{a} \urcorner$ by definition

$$\mathbf{Case} \quad \mathcal{D} = \frac{\Gamma; \Psi \vdash M : A \quad \Gamma; \Phi \vdash \sigma : \Psi}{\Gamma; \Phi \vdash M[\sigma]_{\Psi}^{\Phi} : A} \mathbf{t-sub}$$

$;\ulcorner \Gamma \urcorner \vdash \ulcorner M \urcorner_{\Psi \vdash A} : \ulcorner \Psi \vdash A \urcorner$ by I.H.

$\therefore \ulcorner \Gamma \urcorner \vdash \ulcorner \sigma \urcorner_{\Phi \vdash \Psi} : \ulcorner \Phi \vdash \Psi \urcorner$ by I.H.

$e = \text{apply_sub } \ulcorner M \urcorner_{\Psi \vdash A} \ulcorner \sigma \urcorner_{\Phi \vdash \Psi}$ and
 $\therefore \ulcorner \Gamma \urcorner \vdash e : \ulcorner \Phi \vdash A \urcorner$ by property of `apply_sub`

Case $\mathcal{D} = \frac{\Gamma(u) = [\Psi \vdash \mathbf{a}]}{\Gamma; \Psi \vdash 'u : \mathbf{a}}$ **t-qvar**

$\ulcorner \Gamma \urcorner(u) = \text{tm}[\ulcorner \Psi, \mathbf{a} \urcorner]$ by Lemma 1

$\therefore \ulcorner \Gamma \urcorner \vdash \ulcorner 'u \urcorner : \ulcorner \Psi \vdash \mathbf{a} \urcorner$ by rule `g-var` and definition

Case $\mathcal{D} = \frac{\Gamma; \Psi, x : \mathbf{a} \vdash M : A}{\Gamma; \Psi \vdash \lambda x. M : \mathbf{a} \rightarrow A}$ **t-lam**

$\therefore \ulcorner \Gamma \urcorner \vdash \ulcorner M \urcorner_{\Psi, \mathbf{a} \vdash A} : \ulcorner \Psi, \mathbf{a} \vdash A \urcorner$ by I.H.

$\ulcorner \Psi \vdash \mathbf{a} \rightarrow A \urcorner = \text{tm}[\ulcorner \Psi \urcorner, \ulcorner \mathbf{a} \rightarrow A \urcorner] = \text{tm}[\ulcorner \Psi \urcorner, \text{arr}[\mathbf{a}, \ulcorner A \urcorner]]$ by definition

$\therefore \ulcorner \Gamma \urcorner \vdash \text{Lam } [\text{cons } [\ulcorner \Psi \urcorner, \mathbf{a}, \ulcorner A \urcorner]] \ulcorner M \urcorner_{\Gamma, \mathbf{a} \vdash A} : \ulcorner \Psi \vdash \mathbf{a} \rightarrow A \urcorner$ by using `g-con`

Similar for the other cases. □

Lemma 3 (Pat.). *If $\vdash pat : \tau \downarrow \Gamma$ then $\cdot \vdash \ulcorner pat \urcorner_{\Psi \vdash A}^\Gamma : \ulcorner \tau \urcorner \downarrow \Gamma$.*

Proof. By induction on the type derivation for patterns. □

Lemma 4 (Ctx. Pat.). *If $\Psi \vdash R : A \downarrow \Gamma$ then $\cdot \vdash \ulcorner R \urcorner_{\Psi \vdash A}^\Gamma : \ulcorner \Psi \vdash A \urcorner \downarrow \Gamma$.*

Proof. By induction on the typing derivation. The interesting case is the one where R is a pattern variable.

Case: $\mathcal{D} = \frac{}{\Psi \vdash 'u : \mathbf{a} \downarrow u : [\Psi \vdash \mathbf{a}]}$ **tp-mvar**

$\cdot \vdash u : \text{tm}[\ulcorner \Psi \urcorner, \mathbf{a}] \downarrow \therefore u : \text{tm}[\ulcorner \Psi \urcorner, \mathbf{a}]$ by `gp-var`

$\cdot \vdash \ulcorner 'u \urcorner_{\mathbf{a}}^{u : [\Psi \vdash \mathbf{a}]} : \ulcorner \Psi \vdash \mathbf{a} \urcorner \downarrow \therefore \ulcorner u : [\Psi \vdash \mathbf{a}] \urcorner$ by definition

The other cases are similar. □

Thm. 1 (Main).

1. *If $\Gamma \vdash e \Leftarrow \tau$ then $\therefore \ulcorner \Gamma \urcorner \vdash \ulcorner e \urcorner_{\Gamma \vdash \tau} : \ulcorner \tau \urcorner$.*

2. *If $\Gamma \vdash i \Rightarrow \tau$ then $\therefore \ulcorner \Gamma \urcorner \vdash \ulcorner i \urcorner_{\Gamma \vdash \tau} : \ulcorner \tau \urcorner$.*

Proof. By mutual induction on the type derivations.

$$\text{Case } \mathcal{D} = \frac{\Gamma; \Psi \vdash M : \mathbf{a}}{\Gamma \vdash [\hat{\Psi} \vdash M] \Leftarrow [\Psi \vdash \mathbf{a}]} \text{t-ctx-obj}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil M \rceil_{\Psi \vdash \mathbf{a}} : \lceil \Psi \vdash \mathbf{a} \rceil \quad \text{from Lemma 2.}$$

$$\lceil \Psi \vdash \mathbf{a} \rceil = \text{tm}[\lceil \Psi \rceil, \lceil A \rceil] \text{ and } \lceil \Psi \vdash \mathbf{M} \rceil_{\Psi \vdash \mathbf{a}} = \lceil M \rceil_{\Psi \vdash \mathbf{a}}$$

$$\lceil \Gamma \rceil \vdash \lceil \hat{\Psi} \vdash M \rceil_{\Psi \vdash \mathbf{a}} : \lceil [\Psi \vdash \mathbf{a}] \rceil \quad \text{by definition}$$

$$\text{Case } \mathcal{D} = \frac{\Gamma \vdash i \Rightarrow [\Psi \vdash \mathbf{a}] \quad \forall b \in \vec{b}. \Gamma \vdash b \Leftarrow [\Psi \vdash \mathbf{a}] \rightarrow \tau}{\Gamma \vdash \text{cmatch } i \text{ with } \vec{b} \Leftarrow \tau} \text{t-cm}$$

We note that each $b_i \in \vec{b}$ is of the form $[\Psi \vdash R] \mapsto e$.

$$\Psi \vdash R : A \downarrow \Gamma$$

$$\Gamma, \Gamma' \vdash e \Leftarrow \tau \quad \text{by typing inversion}$$

$$\cdot \vdash \lceil R \rceil_{\Psi \vdash \mathbf{a}}^{\Gamma'} : \mathbf{a} \downarrow \Gamma' \quad \text{by Lemma 4}$$

$$\cdot; \lceil \Gamma, \Gamma' \rceil \vdash \lceil e \rceil_{\Gamma, \Gamma' \vdash \tau} : \lceil \tau \rceil \quad \text{by I.H. (1).}$$

$$\cdot; \Gamma, \Gamma' \vdash \lceil R \rceil_{\Psi \vdash \mathbf{a}}^{\Gamma'} \mapsto \lceil e \rceil_{\Gamma, \Gamma'} : \mathbf{a} \rightarrow \lceil \tau \rceil \quad \text{by g-branch}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil i \rceil_{\Gamma \vdash \mathbf{a}} : \mathbf{a} \quad \text{by I.H. (2).}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil \text{cmatch } i \text{ with } \vec{b} \rceil_{\Gamma \vdash \tau} : \lceil \tau \rceil \quad \text{by g-match}$$

$$\text{Case } \mathcal{D} = \frac{\Gamma \vdash i \Rightarrow \tau' \quad \tau = \tau'}{\Gamma \vdash i \Leftarrow \tau} \text{t-emb}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil i \rceil_{\Gamma \vdash \tau} : \lceil \tau \rceil \quad \text{by I.H. (2)}$$

The other cases for part 1) are similar.

$$\text{Case } \mathcal{D} = \frac{\Gamma(x) = \tau}{\Gamma \vdash x \Rightarrow \tau} \text{t-var}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil x \rceil_{\Gamma \vdash \tau} : \lceil \tau \rceil \quad \text{trivial using g-var.}$$

$$\text{Case } \mathcal{D} = \frac{\Gamma \vdash i \Rightarrow \tau' \rightarrow \tau \quad \Gamma \vdash e \Leftarrow \tau'}{\Gamma \vdash ie \Rightarrow \tau} \text{t-app}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil i \rceil_{\Gamma \vdash \tau' \rightarrow \tau} : \lceil \tau' \rightarrow \tau \rceil \quad \text{by I.H.}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil i \rceil_{\Gamma \vdash \tau' \rightarrow \tau} : \lceil \tau' \rceil \rightarrow \lceil \tau \rceil \quad \text{by definition}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil e \rceil_{\Gamma \vdash \tau'} : \lceil \tau' \rceil \quad \text{by I.H.}$$

$$\cdot; \lceil \Gamma \rceil \vdash \lceil ie \rceil_{\Gamma \vdash \tau} : \lceil \tau \rceil \quad \text{by g-app}$$

The other cases for part 2) are similar.

□