1 Meta-theory of the translation

Translation of ambient contexts Γ.

$$\begin{array}{lcl}
\ulcorner . \urcorner & = & \cdot \\
\ulcorner \Gamma, \text{'}\mathtt{u} : [\Psi \vdash \mathbf{a}] \urcorner & = & \ulcorner \Gamma \urcorner,\ u : \ulcorner [\Psi \vdash \mathbf{a}] \urcorner
\end{array}$$

Translation of contextual types

$$\begin{array}{lcl}
\ulcorner [\Psi \vdash A] \urcorner & = & \mathtt{tm}[\ulcorner \Psi \urcorner, \ulcorner A \urcorner] \\
\ulcorner [\Psi \vdash \Phi] \urcorner & = & \mathtt{sub}[\ulcorner \Psi \urcorner, \ulcorner \Phi \urcorner]
\end{array}$$

Translation of the bound var. context Ψ

$$\begin{array}{lcl}
\ulcorner . \urcorner & = & \mathtt{nil}[] \\
\ulcorner \Psi, x : \mathbf{a} \urcorner & = & \mathtt{cons}[\ulcorner \Psi \urcorner, \mathbf{a}]
\end{array}$$

Translation of sim. substitutions

$$\begin{array}{lcl}
\ulcorner . \urcorner^{\Psi}_{\Psi} & = & \mathtt{Shift}[\ulcorner \Psi \urcorner, \ulcorner \Psi \urcorner](\mathtt{Id}[\ulcorner \Psi \urcorner, \ulcorner \Psi \urcorner]) \\
\ulcorner . \urcorner^{\Phi}_{\Psi, x:\mathbf{a}} & = & \mathtt{Shift}[\ulcorner \Psi, x : \mathbf{a} \urcorner, \ulcorner \Phi \urcorner] \\
 & & \quad (\mathtt{Suc}[\ulcorner \Psi, x : \mathbf{a} \urcorner, \ulcorner \Phi \urcorner] \ulcorner \cdot \urcorner^{\Phi}_{\Psi}) \\
\ulcorner \sigma, x/M \urcorner^{\Phi, x:A}_{\Psi} & = & \mathtt{Dot}[\ulcorner \Psi \urcorner, \ulcorner \Phi \urcorner, \mathbf{a}](\ulcorner \sigma \urcorner^{\Phi}_{\Psi}, \ulcorner M \urcorner_{\Phi \vdash \mathbf{a}})
\end{array}$$

Lemma 1 (Ambient Context)**.** *If* $\Gamma(u) = [\Psi \vdash \boldsymbol{a}]$ *then* $\ulcorner \Gamma \urcorner(u) = \boldsymbol{tm}[\ulcorner \Psi, \boldsymbol{a} \urcorner]$.

Proof. Induction on the structure of Γ. $\square$

Lemma 2 (Terms)**.**

1. *If* $\Gamma; \Psi \vdash M : A$ *then* $\cdot; \ulcorner \Gamma \urcorner \vdash \ulcorner M \urcorner_{\Psi \vdash A} : \ulcorner \Psi \vdash A \urcorner$.
2. *If* $\Gamma; \Psi \vdash \sigma : \Phi$ *then* $\cdot; \ulcorner \Gamma \urcorner \vdash \ulcorner \sigma \urcorner_{\Psi \vdash \Phi} : \ulcorner \Psi \vdash \Phi \urcorner$

Proof. Induction on the typing derivation.

Case $\mathcal{D} = \dfrac{\Psi(x) = \mathbf{a}}{\Gamma; \Psi \vdash x : \mathbf{a}}\ \mathtt{t\text{-}var}$

$\cdot; \ulcorner \Gamma \urcorner \vdash \mathtt{Var}[\ulcorner \Psi \urcorner,\ \mathbf{a}]\ k : \mathtt{tm}[\ulcorner \Psi \urcorner,\ \mathbf{a}]$ by the correctness of our translation function that computes the position k of x in Ψ.

$\cdot; \ulcorner \Gamma \urcorner \vdash \ulcorner x \urcorner_{\Psi \vdash \mathbf{a}} : \ulcorner \Psi \vdash \mathbf{a} \urcorner$ by definition

Case $\mathcal{D} = \dfrac{\Gamma; \Psi \vdash M : A \qquad \Gamma; \Phi \vdash \sigma : \Psi}{\Gamma; \Phi \vdash M[\sigma]^{\Phi}_{\Psi} : A}\ \mathtt{t\text{-}sub}$

$\cdot; \ulcorner \Gamma \urcorner \vdash \ulcorner M \urcorner_{\Psi \vdash A} : \ulcorner \Psi \vdash A \urcorner$ by I.H.

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner\sigma\urcorner_{\Phi\vdash\Psi} : \ulcorner\Phi \vdash \Psi\urcorner$ by I.H.

$e = \texttt{apply\_sub}\ \ulcorner M\urcorner_{\Psi\vdash A}\ \ulcorner\sigma\urcorner_{\Phi\vdash\Psi}$ and
$\cdot; \ulcorner\Gamma\urcorner \vdash e : \ulcorner\Phi \vdash A\urcorner$ by property of `apply_sub`

Case $\mathcal{D} = \dfrac{\Gamma(\mathtt{u}) = [\Psi \vdash \mathbf{a}]}{\Gamma; \Psi \vdash \mathtt{'u} : \mathbf{a}}\ \texttt{t-qvar}$

$\ulcorner\Gamma\urcorner(\mathtt{u}) = \texttt{tm}[\ulcorner\Psi, \mathbf{a}\urcorner]$ by Lemma 1

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner\mathtt{'u}\urcorner : \ulcorner\Psi \vdash \mathbf{a}\urcorner$ by rule `g-var` and definition

Case $\mathcal{D} = \dfrac{\Gamma; \Psi, x : \mathbf{a} \vdash M : A}{\Gamma; \Psi \vdash \lambda x.M : \mathbf{a} \to A}\ \texttt{t-lam}$

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner M\urcorner_{\Psi,\mathbf{a}\vdash A} : \ulcorner\Psi, \mathbf{a} \vdash A\urcorner$ by I.H.

$\ulcorner\Psi \vdash \mathbf{a} \to A\urcorner = \texttt{tm}[\ulcorner\Psi\urcorner, \ulcorner\mathbf{a} \to A\urcorner] = \texttt{tm}[\ulcorner\Psi\urcorner, \texttt{arr}[\mathbf{a}, \ulcorner A\urcorner]]$ by definition

$\cdot; \ulcorner\Gamma\urcorner \vdash \texttt{Lam}\ [\texttt{cons}\ [\ulcorner\Psi\urcorner, \mathbf{a}, \ulcorner A\urcorner\ ]\,]\ \ulcorner M\urcorner_{\Gamma,\mathbf{a}\vdash A} : \ulcorner\Psi \vdash \mathbf{a} \to A\urcorner$ by using `g-con`

Similar for the other cases.

□

Lemma 3 (Pat.)**.** *If* $\vdash pat : \tau \downarrow \Gamma$ *then* $\cdot \vdash \ulcorner pat\urcorner^{\Gamma}_{\Psi\vdash A} : \ulcorner\tau\urcorner \downarrow \Gamma$.

Proof. By induction on the type derivation for patterns. □

Lemma 4 (Ctx. Pat.)**.** *If* $\Psi \vdash R : A \downarrow \Gamma$ *then* $\cdot \vdash \ulcorner R\urcorner^{\Gamma}_{\Psi\vdash A} : \ulcorner\Psi \vdash A\urcorner \downarrow \Gamma$.

Proof. By induction on the typing derivation. The interesting case is the one where R is a pattern variable.

Case: $\mathcal{D} = \dfrac{}{\Psi \vdash \mathtt{'u} : \mathbf{a} \downarrow u : [\Psi \vdash \mathbf{a}]}\ \texttt{tp-mvar}$

$\cdot \vdash u : \texttt{tm}[\ulcorner\Psi\urcorner, \mathbf{a}] \downarrow \cdot; u : \texttt{tm}[\ulcorner\Psi\urcorner, \mathbf{a}]$ by `gp-var`

$\cdot \vdash \ulcorner\mathtt{'u}\urcorner^{u:[\Psi\vdash\mathbf{a}]}_{\mathbf{a}} : \ulcorner\Psi \vdash \mathbf{a}\urcorner \downarrow \cdot; \ulcorner u : [\Psi \vdash \mathbf{a}]\urcorner$ by definition

The other cases are similar.

□

Thm. 1 (Main)**.**

1. *If* $\Gamma \vdash e \Leftarrow \tau$ *then* $\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner e\urcorner_{\Gamma\vdash\tau} : \ulcorner\tau\urcorner$.
2. *If* $\Gamma \vdash i \Rightarrow \tau$ *then* $\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner i\urcorner_{\Gamma\vdash\tau} : \ulcorner\tau\urcorner$.

Proof. By mutual induction on the type derivations.

Case $\mathcal{D} = \dfrac{\Gamma; \Psi \vdash M : \mathbf{a}}{\Gamma \vdash [\hat{\Psi} \vdash M] \Leftarrow [\Psi \vdash \mathbf{a}]}$ `t-ctx-obj`

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner M\urcorner_{\Psi\vdash\mathbf{a}} : \ulcorner\Psi \vdash \mathbf{a}\urcorner$ from Lemma 2.

$\ulcorner\Psi \vdash \mathbf{a}\urcorner = \mathtt{tm}[\ulcorner\Psi\urcorner, \ulcorner A\urcorner]$ and $\ulcorner\Psi \vdash \mathbf{M}\urcorner_{\Psi\vdash\mathbf{a}} = \ulcorner M\urcorner_{\Psi\vdash\mathbf{a}}$

$\ulcorner\Gamma\urcorner \vdash \ulcorner\hat{\Psi} \vdash M]\urcorner_{\Psi\vdash\mathbf{a}} : \ulcorner[\Psi \vdash \mathbf{a}]\urcorner$ by definition

Case $\mathcal{D} = \dfrac{\Gamma \vdash i \Rightarrow [\Psi \vdash \mathbf{a}] \qquad \forall b \in \overrightarrow{b}\ .\ \Gamma \vdash b \Leftarrow [\Psi \vdash \mathbf{a}] \to \tau}{\Gamma \vdash \mathtt{cmatch}\, i\, \mathtt{with}\ \overrightarrow{b} \Leftarrow \tau}$ `t-cm`

We note that each $b_i \in \overrightarrow{b}$ is of the form $[\Psi \vdash R] \mapsto e$.

$\Psi \vdash R : A \downarrow \Gamma$

$\Gamma, \Gamma' \vdash e \Leftarrow \tau$ by typing inversion

$\cdot \vdash \ulcorner R\urcorner^{\Gamma'}_{\Psi\vdash\mathbf{a}} : \mathbf{a} \downarrow \Gamma'$ by Lemma 4

$\cdot; \ulcorner\Gamma, \Gamma'\urcorner \vdash \ulcorner e\urcorner_{\Gamma,\Gamma'\vdash\tau} : \ulcorner\tau\urcorner$ by I.H. (1).

$\cdot; \Gamma, \Gamma' \vdash \ulcorner R\urcorner^{\Gamma'}_{\Psi\vdash\mathbf{a}} \mapsto \ulcorner e\urcorner_{\Gamma,\Gamma'} : \mathbf{a} \to \ulcorner\tau\urcorner$ by `g-branch`

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner i\urcorner_{\Gamma\vdash\mathbf{a}} : \mathbf{a}$ by I.H (2).

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner\mathtt{cmatch}\, i\, \mathtt{with}\ \overrightarrow{b}\urcorner_{\Gamma\vdash\tau} : \ulcorner\tau\urcorner$ by `g-match`

Case $\mathcal{D} = \dfrac{\Gamma \vdash i \Rightarrow \tau' \qquad \tau = \tau'}{\Gamma \vdash i \Leftarrow \tau}$ `t-emb`

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner i\urcorner_{\Gamma\vdash\tau} : \ulcorner\tau\urcorner$ by I.H.(2)

The other cases for part 1) are similar.

Case $\mathcal{D} = \dfrac{\Gamma(x) = \tau}{\Gamma \vdash x \Rightarrow \tau}$ `t-var`

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner x\urcorner_{\Gamma\vdash\tau} : \ulcorner\tau\urcorner$ trivial using `g-var`.

Case $\mathcal{D} = \dfrac{\Gamma \vdash i \Rightarrow \tau' \to \tau \qquad \Gamma \vdash e \Leftarrow \tau'}{\Gamma \vdash i\, e \Rightarrow \tau}$ `t-app`

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner i\urcorner_{\Gamma\vdash\tau'\to\tau} : \ulcorner\tau' \to \tau\urcorner$ by I.H.

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner i\urcorner_{\Gamma\vdash\tau'\to\tau} : \ulcorner\tau'\urcorner \to \ulcorner\tau\urcorner$ by definition

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner e\urcorner_{\Gamma\vdash\tau'} : \ulcorner\tau'\urcorner$ by I.H.

$\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner i\, e\urcorner_{\Gamma\vdash\tau} : \ulcorner\tau\urcorner$ by `g-app`

The other cases for part 2) are similar.

$\square$