1 Meta-theory of the translation

Thm. 1 (Main)**.** *If* $\Gamma \vdash e \Leftarrow \tau$ *then* $\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner e \urcorner_{\Gamma\vdash\tau} : \ulcorner\tau\urcorner$

Proof. By mutual induction on the type derivation:

Case: $\dfrac{\Gamma; \Psi \vdash M : \mathbf{a}}{\Gamma \vdash [\hat{\Psi} \vdash M] \Leftarrow [\Psi \vdash \mathbf{a}]}$ `t-ctx-obj`

we have: $\ulcorner\Gamma\urcorner \vdash \ulcorner M \urcorner_{\Psi\vdash\mathbf{a}} : \mathbf{a}$ from lemma 2.
and that $\ulcorner \Psi \vdash \mathbf{a} \urcorner = \ulcorner \mathbf{a} \urcorner = \mathbf{a}$.
and that $\ulcorner \Psi \vdash \mathbf{M} \urcorner_{\Psi\vdash\mathbf{a}} = \ulcorner M \urcorner_{\Psi\vdash\mathbf{a}}$ by definition.
we have $\ulcorner\Gamma\urcorner \vdash \ulcorner \hat{\Psi} \vdash M] \urcorner_{\Psi\vdash\mathbf{a}} : \mathbf{a}$

Case: $\dfrac{\Gamma \vdash i \Rightarrow [\Psi \vdash \mathbf{a}] \quad \forall b \in \overrightarrow{b}\ .\ \Gamma \vdash b \Leftarrow [\Psi \vdash \mathbf{a}] \to \tau}{\Gamma \vdash \texttt{cmatch}\, i\, \texttt{with}\ \overrightarrow{b} \Leftarrow \tau}$ `t-cm`

WTS: $\ulcorner\Gamma\urcorner \vdash \ulcorner \texttt{cmatch}\, i\, \texttt{with}\ \overrightarrow{b} \urcorner_{\Gamma\vdash\tau} \Leftarrow \ulcorner\tau\urcorner$

we know each $b_1 \in \overrightarrow{b}$ is of the form $[\Psi \vdash R] \mapsto e$
and translates to $\ulcorner R \urcorner^{\Gamma'}_{\Psi\vdash\mathbf{a}}$
by lemma 3 is of type $\cdot \vdash \ulcorner R \urcorner^{\Gamma'}_{\Psi\vdash\mathbf{a}} : \mathbf{a} \downarrow \Gamma'$
$\cdot; \ulcorner \Gamma, \Gamma' \urcorner \vdash \ulcorner e \urcorner_{\Gamma,\Gamma'\vdash\tau} : \ulcorner\tau\urcorner$ by I.H. on the branch
and $\cdot; \Gamma, \Gamma' \vdash \ulcorner R \urcorner^{\Gamma'}_{\Psi\vdash\mathbf{a}} \mapsto \ulcorner e \urcorner_{\Gamma,\Gamma'} : \mathbf{a} \to \ulcorner\tau\urcorner$ by `g-branch` for each branch(1)
we also know that $\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner i \urcorner_{\Gamma\vdash\mathbf{a}} : \mathbf{a}$ bi I.H (2).
then the rest follows from `g-match` on (1) and (2).

Case: $\dfrac{\Gamma \vdash i \Rightarrow \tau' \quad \tau = \tau'}{\Gamma \vdash i \Leftarrow \tau}$ `t-emb` follows from the mutually recursive theorem

The other cases are similar. ∎

Thm. 2. *and If* $\Gamma \vdash i \Rightarrow \tau$ *then* $\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner i \urcorner_{\Gamma\vdash\tau} : \ulcorner\tau\urcorner$.

Proof. By induction on the type derivation:

Case: $\dfrac{\Gamma(x) = \tau}{\Gamma \vdash x \Rightarrow \tau}$ `t-var`
Trivial using `g-var`.

Case: $\dfrac{\Gamma \vdash i \Rightarrow \tau' \to \tau \quad \Gamma \vdash e \Leftarrow \tau'}{\Gamma \vdash i\, e \Rightarrow \tau}$ `t-app`

$\ulcorner\Gamma\urcorner \vdash \ulcorner i \urcorner_{\gamma\vdash\tau'\to\tau} : \ulcorner\tau'\urcorner \to \ulcorner\tau\urcorner$ by I.H.
$\ulcorner\Gamma\urcorner \vdash \ulcorner e \urcorner_{\gamma\vdash\tau'} : \ulcorner\tau'\urcorner$ by I.H.
The goal follows from `g-app`

The other cases are similar. □

Lemma 1 (Pat.)**.** *If* $\vdash pat : \tau \downarrow \Gamma$ *then* $\cdot \vdash \ulcorner pat \urcorner^{\Gamma}_{\Psi\vdash A} : \ulcorner\tau\urcorner \downarrow \ulcorner\Gamma\urcorner$.

Proof. By induction on the type derivation for patterns. □

Lemma 2 (Terms)**.** *If* $\Gamma; \Psi \vdash M : A$ *then* $\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner M \urcorner_{\Psi\vdash A} : \ulcorner A \urcorner$.

Proof. By induction on the type derivation:
Case: $\dfrac{\Psi(x) = \mathbf{a}}{\Psi \vdash x : \mathbf{a}}$ `t-var`
WTS: $\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner x \urcorner_{\Psi\vdash\mathbf{a}}$
follow by a straightforward lemma on variables and de Bruijn index.

Case: $\dfrac{\Psi, x : \mathbf{a} \vdash M : A}{\Psi \vdash \lambda x.M : \mathbf{a} \to A}$ `t-lam`

WTS: $\cdot; \ulcorner\Gamma\urcorner \vdash \mathtt{Lam}[\mathtt{cons}[\ulcorner\Psi\urcorner, \mathbf{a}, \ulcorner A \urcorner]] \ulcorner M \urcorner_{\Gamma,\mathbf{a}\vdash A}$
we know: $\cdot; \ulcorner\Gamma\urcorner \vdash \ulcorner M \urcorner_{\Psi,\mathbf{a}\vdash A} : \ulcorner A \urcorner$ by I.H.
the goal follow by application of `g-con`.

Similar for the other cases. □

Lemma 3 (Ctx. Pat.)**.** *If* $\Psi \vdash R : A \downarrow \Gamma$ *then* $\cdot \vdash \ulcorner R \urcorner^{\Gamma}_{\Psi\vdash A} : \ulcorner A \urcorner \downarrow \cdot; \ulcorner\Gamma\urcorner$.

Proof. By induction on the typing derivation.
The interesting cases are the ones that bind variables.

Case: $\dfrac{}{\Psi \vdash \mathtt{'u} : \mathbf{a} \downarrow u : [\Psi \vdash \mathbf{a}]}$ `tp-mvar`

WTS: $\cdot \vdash \ulcorner \mathtt{'u} \urcorner^{u:[\Psi\vdash\mathbf{a}]}_{\mathbf{a}} : \ulcorner \Psi \vdash \mathbf{a} \urcorner \downarrow \cdot; \ulcorner u : [\Psi \vdash \mathbf{a}] \urcorner$
which translates to : $\cdot \vdash u : \mathtt{tm}[\ulcorner\Psi\urcorner, \mathbf{a}] \downarrow \cdot; u : \mathtt{tm}[\ulcorner\Psi\urcorner, \mathbf{a}]$
Which bind the variable u with the right type and follows from `gp-var`.

The other terms are similar to Lemma 2. □