Labs
**Machine Learning Course**
Spring 2025

**Westlake University**
Department of Artificial Intelligence, SOE
**Tao Lin & Kaicheng YU**
https://github.com/LINs-lab/course_machine_learning

# Problem Set Lab 01 (graded), Feb. 18, 2025 (Mathematical Foundation of Machine Learning)

**Goals.** The goals of this lab are to:

- Familiarize yourself with the mathematical foundations of the machine learning course.

**Submission instructions:**

- Please submit a PDF file to canvas.
- Deadline: 23.59 on Mar. 02, 2025

## Review of Linear Algebra

**Problem 1 (Idempotent Matrices and Rank Inequality):**

Given $A$ and $B$ are idempotent $n \times n$ matrices (i.e., $A^2 = A$ and $B^2 = B$), and they commute with each other ($AB = BA$):

1. Prove that $A + B - AB$ is also an idempotent matrix.
2. Further prove that

$$\operatorname{rank}(A + B - AB) \leq \operatorname{rank}(A) + \operatorname{rank}(B)\,. \tag{1}$$

**Problem 2 (Diagonalizability and Eigenvalues of a Linear Transformation):**

Let $V$ be a four-dimensional vector space, and let $T : V \to V$ be a linear transformation satisfying $T^3 - 2T^2 + T - 2I = 0$, where $I$ is the identity transformation on $V$. Prove that $T$ is diagonalizable and determine all its eigenvalues.

**Problem 3 (Existence of Real Matrix Roots for Positive Eigenvalue Matrices):**

If a real matrix $A$ has all eigenvalues as positive real numbers, then for any positive integer $m$, there exists a real matrix $B$ such that $B^m = A$.

**Problem 4 (Eigenvalue Equivalence under Commutator-like Condition):**

Let $A$ and $B$ be $n \times n$ square matrices satisfying

$$AB - BA = A - B\,. \tag{2}$$

Then, $A$ and $B$ have the same eigenvalues.

**Problem 5 (Matrix Determinant and Commutator):**

Let $A$ and $B$ be two $n \times n$ matrices satisfying the equation

$$AB - BA = A\,. \tag{3}$$

Prove that $\det(A) = 0$.

# Review of Probability Theory

**Problem 6 (Moment Bound for a Standard Normal Random Variable):**

For a standard normal random variable $X$, there exists a constant $C$ such that for all $p > 1$,

$$\left(\mathbb{E}\left[|X|^p\right]\right)^{1/p} \leq C\sqrt{p}\,. \tag{4}$$

**Problem 7 (Bounded Random Variable and Exponential Expectation):**

Let $X$ be a bounded random variable with $\mathbb{E}[X] = 0$ and $|X|_\infty \leq a$ for some $a > 0$. Prove that

$$\mathbb{E}\left[e^X\right] \leq \cosh(a)\,. \tag{5}$$

**Problem 8 (Almost Sure Convergence of Scaled Random Walks):**

Let $X$ be a random variable with distribution $P(X = 1) = P(X = -1) = \frac{1}{2}$. Define the partial sum $S_n = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) copies of $X$. For any $\alpha > \frac{1}{2}$, prove that

$$P\left(\lim_{n\to\infty} \frac{S_n}{n^\alpha} = 0\right) = 1\,. \tag{6}$$

**Problem 9 (Probability Bound for the Standardized Sum of Uniform Random Variables):**

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables uniformly distributed on the interval $(-1, 1)$. For any $r > 0$, prove that

$$P\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} < r\right) > 1 - \frac{1}{3r^2}\,. \tag{7}$$

**Problem 10 (Central Limit Theorem and Standardized Sum Convergence):**

Let $\{X_1, X_2, \ldots, X_n\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with finite mean $\mu = \mathbb{E}[X_i]$ and finite variance $\sigma^2 = \mathrm{Var}(X_i) > 0$. Define the standardized sum:

$$Z_n = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}\,. \tag{8}$$

Then, as $n \to \infty$, the distribution of $Z_n$ converges to the standard normal distribution $\mathcal{N}(0, 1)$. Formally, for all real numbers $z$:

$$\lim_{n\to\infty} P(Z_n \leq z) = \Phi(z)\,, \tag{9}$$

where $\Phi(z)$ is the cumulative distribution function (CDF) of the standard normal distribution.