Design Surface

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Locally, a surface **is** a 2D subset of 3D. **Is** means there is a 1 to 1 correspondence (homeomorphism) between each point of the surface and a point in 2D. Often, the surface is locally functional with respect to a plane, which constructs the homeomorphism implicitly. The smoothness of the surface is the differentiability of this implicit function. Implicitly C^n is G^n. An implicitly C^1 or G^1 surface is "smooth" and is characterized by a continuous unit normal.

When a smooth curve is in a surface, every derivative is in the surface. This **transfinite** property participates in the smoothness of the surface. When a smooth curve separates two surface patches which join G^1, the surface normal is the cross product of the curve tangent and an independent surface tangent. Factoring smoothness: Two surface patches, that share a curve and its perpendicular tangent direction, join smoothly.

Cubic edges provide an effective tool for surface design. Vertices are connected by cubic segments that have no other intersections or terminations. Bezier provides a convenient design handle for the cubics. Linear interpolation of the perpendicular tangent direction is shared if tangents describe unique vertex normals. Expanding this to quadratic interpolation to a shared mid cross tangent or BARB is motivated by the quadratic derivative of the cubic edge.

A triangle is a 3 cycle of segments.

Herron's 15 dof triangle: $H_{15} \equiv C_9 - H_6 (I - C_9)$ over a triangle $(x, y) = \sum_{i=1}^{3} b_i V_i$

$$C_9 = \sum_{i=1}^{3} b_i^2 \left((3 - 2 b_i) F_i + b_j T_{ij} + b_k T_{ik} \right) \qquad T_{ij} = \nabla F_i \bullet (V_j - V_i)$$

$$H_6 = \frac{\dfrac{9 b_1 b_2 b_3}{2} \left\{ \sum_{i=1}^{3} \dfrac{\left((2 b_k - b_j) X_{kj} + (2 b_j - b_k) X_{jk} \right)}{b_i} \right\}}{\sum_{i=1}^{3} \dfrac{1}{b_i}} \qquad X_{jk} = \left. \frac{\nabla (F - C_9) \cdot \nabla b_i}{\nabla b_i \cdot \nabla b_i} \right|_{(2 V_j + V_k)/3}$$

Vertices and tangents define C_9. The X_{jk} are determined from the interpolant to the perpendicular direction along the edge and the shape of any domain triangle.

$$\left.\frac{\partial C_9}{\partial b_1}\right|_{\frac{(2V_2+V_3)}{3}} = \frac{4\left(\frac{\partial b_2}{\partial b_1}F_2 + \frac{\partial b_3}{\partial b_1}F_3\right)}{3} + \frac{\left(4T_{21}+T_{31}-4T_{23}+\left(3\frac{\partial b_3}{\partial b_1}-1\right)T_{32}\right)}{9}$$

$$\frac{\partial F}{\partial b_i} \equiv \frac{\nabla F\cdot\nabla b_i}{\nabla b_i\cdot\nabla b_i} \qquad \frac{\partial b_j}{\partial b_i} = \frac{e_j\cdot e_i}{e_i\cdot e_i} \qquad e_i=V_k-V_j.$$

$\frac{\nabla b_n}{\nabla b_n\cdot\nabla b_n} = \sum_{i=1}^{3}\frac{\partial b_i}{\partial b_n}V_i$ is the n^{th} altitude vector.

$$\left.\frac{\partial C_9}{\partial b_1}\right|_{(2V_2+V_3)/3} = 4/3\left(\frac{\partial b_2}{\partial b_1}F_2+\frac{\partial b_3}{\partial b_1}F_3\right)+\left(4T_{21}+T_{31}-4T_{23}+\left(3\frac{\partial b_3}{\partial b_1}-1\right)T_{32}\right)/9$$

Let $N_{jk}=T_{jk}-T_{ji}\frac{T_{ji}\cdot T_{jk}}{T_{jk}\cdot T_{jk}}$ be the perpendicular to the i^{th} edge and let $n_{jk}=\frac{N_{jk}}{\|N_{jk}\|}$ be the unit direction perpendicular.

(If ∇F_j is orthonormal then $N_{jk}=\frac{\nabla F_j\nabla b_i}{\nabla b_i\cdot\nabla b_i}$.)

Linear interpolation: $n(s)=(1-s)n_{jk}+s\,n_{kj}$.

$$\left.\frac{\nabla F\nabla b_i}{\nabla b_i\cdot\nabla b_i}\right|_{((1-s)V_j+sV_k)} = \left((1-s)n_{jk}+s\,n_{kj}\right)\left((1-s)\|N_{jk}\|+s\|N_{jk}\|\right) \equiv X_i(s).$$

$X_i(1/3)=X_{jk}$ and $X_i(2/3)=X_{kj}$. Replacing $((1-s)n_{jk}+s\,n_{kj})$ with $((1-s)(1-2s)n_{jk}+4s(1-s)B_i+(2s-1)s\,n_{kj})$, a quadratic interpolant, uses the BARB, B_i, a mid-boundary cross-tangent. B_i is arbitrary but may be defaulted to $\frac{n_{jk}+n_{kj}}{2}$ for the linear interpolant or may be modified for perpendicularity or unit length. Sharing B_i between patches maintains smoothness. $\|B_i\|$ controls the fullness or tension of the surface.