Design Surface

Gary Herron & Frank Little

Locally, a surface **is** a 2D subset of 3D. **Is** means there is a 1 to 1 correspondence (homeomorphism) between each point of the surface and a point in 2D. Often, the surface is locally functional with respect to a plane, which constructs the homeomorphism implicitly. The smoothness of the surface is the differentiability of this implicit function. Implicitly C^n is G^n. An implicitly C^1 or G^1 surface is "smooth" and is characterized by a continuous unit normal.

Every derivative of a smooth curve in a surface is in the surface. This **transfinite** property participates in the smoothness of the surface. When a smooth curve separates two surface patches which join G^1, the surface normal is the cross product of the curve tangent and an independent surface tangent. Factoring smoothness: "Two surface patches, that share a curve and its perpendicular tangent direction, join smoothly."

The cubic triangulation provides an effective tool for surface design. Vertices are connected by cubic segments that have no other intersections or terminations. A triangle is a 3 cycle of segments. Interpolation of the perpendicular direction between vertex values allows a triangular interpolant to surface the cubic triangulation. Where two triangles share an edge and the vertex normals are perpendicular to curve tangents, this surface is smooth. Bezier provides a convient design handle on the cubic triangulation. This is augmented by BARBS, tangent vectors drawn across the midpoint of cubics, in the case quadratic interpolation is used. Linear interpolation is matched when the midpoint value is taken as the average of the endpoint values.

Herron's 15 dof triangle: $H_{15} = C_9 + H_6(I - C_9)$ over a triangle $(x, y) = \sum_{i=1}^{3} b_i V_i$

$$C_9 = \sum_{i=1}^{3} b_i^2\left(\left(3 - 2b_i\right)F_i + b_j T_{ij} + b_k T_{ik}\right) \qquad T_{ij} = \nabla F_i \bullet \left(V_j - V_i\right)$$

$$H_6 = \frac{\dfrac{9 b_1 b_2 b_3}{2}\left\{\sum_{i=1}^{3} \dfrac{\left(\left(2b_k - b_j\right)X_{kj} + \left(2b_j - b_k\right)X_{jk}\right)}{b_i}\right\}}{\sum_{i=1}^{3} \dfrac{1}{b_i}} \qquad X_{jk} = \left.\frac{\nabla\left(F - C_9\right)\cdot\nabla b_i}{\nabla b_i \cdot \nabla b_i}\right|_{\left(2V_j + V_k\right)/3}$$

$$H_{crap}=\frac{\frac{9b_1b_2b_3}{2}\left\{\sum_{i=1}^{3}\frac{\left(X_{kj}+X_{jk}\right)}{2}\right\}}{\sum_{i=1}^{3}\frac{1}{b_i}}$$ was originally added to H_6.

The vertices and tangents from the Cubic Triangulation define C_9. The X_{jk} are determined from the interpolant to the perpendicular direction along the edge and any domain triangle.

$$\left.\frac{\partial C_9}{\partial b_1}\right|_{\frac{(2V_2+V_3)}{3}}=\frac{4\left(\frac{\partial b_2}{\partial b_1}F_2+\frac{\partial b_3}{\partial b_1}F_3\right)}{3}+\frac{\left(4T_{21}+T_{31}-4T_{23}+\left(3\frac{\partial b_3}{\partial b_1}-1\right)T_{32}\right)}{9}$$

$$\frac{\partial F}{\partial b_i}\equiv\frac{\nabla F\cdot\nabla b_i}{\nabla b_i\cdot\nabla b_i}\qquad\frac{\partial b_j}{\partial b_i}=\frac{e_j\cdot e_i}{e_i\cdot e_i}$$ $e_i=V_k-V_j$.

$$\frac{\nabla b_n}{\nabla b_n\cdot\nabla b_n}=\sum_{i=1}^{3}\frac{\partial b_i}{\partial b_n}V_i$$ is the n^{th} altitude vector.

$$\left.\frac{\partial C_9}{\partial b_1}\right|_{(2V_2+V_3)/3}=4/3\left(\frac{\partial b_2}{\partial b_1}F_2+\frac{\partial b_3}{\partial b_1}F_3\right)+\left(4T_{21}+T_{31}-4T_{23}+\left(3\frac{\partial b_3}{\partial b_1}-1\right)T_{32}\right)/9$$

Let $N_{jk}=T_{jk}-T_{ji}\frac{T_{ji}\cdot T_{jk}}{T_{jk}\cdot T_{jk}}$ be the perpendicular to the i^{th} edge and let $n_{jk}=\frac{N_{jk}}{\|N_{jk}\|}$ be the unit direction perpendicular.

(If ∇F_j is orthonormal then $N_{jk}=\frac{\nabla F_j\nabla b_i}{\nabla b_i\cdot\nabla b_i}$.)

Linear interpolation: $n(s)=(1-s)n_{jk}+s\,n_{kj}$.

$$\left.\frac{\nabla F\,\nabla b_i}{\nabla b_i\cdot\nabla b_i}\right|_{((1-s)V_j+sV_k)}=\left((1-s)n_{jk}+s\,n_{kj}\right)\left((1-s)\|N_{jk}\|+s\|N_{jk}\|\right)\equiv X_i(s).$$

Evaluation at 1/3 and 2/3 completes X_{jk} and X_{kj}. Replacing $\left((1-s)n_{jk}+s\,n_{kj}\right)$ with $\left((1-s)(1-2s)n_{jk}+4s(1-s)B_i+(2s-1)s\,n_{kj}\right)$, the quadratic interpolant, defines the Barb, B_i, a mid-boundary vector. B_i is arbitrary but may be defaulted to $\frac{n_{jk}+n_{kj}}{2}$ for the linear interpolant or may be modified for unit length or perpendicularity. Sharing B_i between patches maintains smoothness. $\|B_i\|$ controls the fullness or tension of the surface.