Error modelling in difference imaging

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Aug 10th 2015, University of Washington

Weighted sum of random variables

The variance of a weighted sum is:

$$var(aX + bY) = a^2 var(X) + b^2 var(Y) + ab\ cov(X, Y), \tag{1}$$

where X and Y are random variables. More generally:

$$var\left(\sum_i a_i X_i\right) = \sum_{i,j} a_i a_j cov(X_i, X_j) \tag{2}$$

$$= \sum_i a_i^2 var(X) + \sum_{i \neq j} a_i a_j cov(X_i, X_j) \tag{3}$$

Sum over 2D kernel

Now consider a multidimensional kernel applied over an intensity image I:

$$var\left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} I_{\vec{y}}\right) = \sum_{\vec{y},\vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}} cov(I_{\vec{y}}, I_{\vec{z}}) \tag{4}$$

$$= \sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 var(I_{\vec{y}}) + \sum_{\vec{y}\neq\vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}} cov(I_{\vec{y}}, I_{\vec{z}}), \tag{5}$$

where now $\vec{x}$, $\vec{y}$ and $\vec{z}$ are positions in the image (e.g. pixels), $k_{\vec{y}}^{\vec{x}}$ is the kernel function centered in $\vec{x}$, evaluated at the position $\vec{y}$, and $I_{\vec{y}}$ is the image intensity at $\vec{y}$.

Approximations

Exact variance:

$$var\left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} I_{\vec{y}}\right) = \sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} var(I_{\vec{y}}) + \sum_{\vec{y}\neq\vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}} cov(I_{\vec{y}}, I_{\vec{z}}) \tag{6}$$

No covariance approximation:

$$var\left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} I_{\vec{y}}\right) \approx \sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 var(I_{\vec{y}}) \tag{7}$$

Local variance approximation:

$$var\left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} I_{\vec{x}}\right) \approx \left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}}\right)^2 var(I_{\vec{x}}) \tag{8}$$

Convolved variance approximation:

$$var\left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} I_{\vec{y}}\right) \approx \left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}}\right) \sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} var(I_{\vec{y}}) \tag{9}$$

New approximation

$$var\left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} I_{\vec{y}}\right) = \left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}}\right)^2 v\bar{a}r - \sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 v\bar{a}r - \sum_{\vec{y}\neq\vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}} v\bar{a}r + \sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 var(I_{\vec{y}}) + \sum_{\vec{y}\neq\vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}} cov(I_{\vec{y}}, I_{\vec{z}}), \tag{10}$$

Assuming

$$v\bar{a}r \equiv \frac{\sum_{\vec{y}\neq\vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}} cov(I_{\vec{y}}, I_{\vec{z}})}{\sum_{\vec{y}\neq\vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}}} \tag{11}$$

$$var\left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} I_{\vec{y}}\right) = \left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}}\right)^2 v\bar{a}r - \sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 v\bar{a}r + \sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 var(I_{\vec{y}}) \tag{12}$$

$$= \left(\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}}\right)^2 v\bar{a}r + \sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 \left[var(I_{\vec{y}}) - v\bar{a}r\right] \tag{13}$$

Optimal photometry

Given that $Flux(\vec{x}) k_{\vec{y}}^{\vec{x}} = I_{\vec{y}}$ for isolated stars, where now $k_{\vec{y}}^{\vec{x}}$ is the PSF centered in $\vec{x}$ evaluated at position $\vec{y}$, we can estimate $Flux(\vec{x})$ given $I_{\vec{y}}$ and $k_{\vec{y}}^{\vec{x}}$:

$$Flux(\vec{x}) = \frac{\sum_{\vec{y}} (I_{\vec{y}} / k_{\vec{y}}^{\vec{x}}) / var(I_{\vec{y}} / k_{\vec{y}}^{\vec{x}})}{\sum_{\vec{y}} 1 / var(I_{\vec{y}} / k_{\vec{y}}^{\vec{x}})} = \frac{\sum_{\vec{y}} k_{\vec{y}}^{\vec{x}} I_{\vec{y}} / var(I_{\vec{y}})}{\sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 / var(I_{\vec{y}})} \tag{14}$$

The we can write the variance of the flux as:

$$var[Flux(\vec{x})] = \frac{1}{\sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 / var(I_{\vec{y}})} \left\{ 1 + \frac{\sum_{\vec{y} \neq \vec{z}} \frac{k_{\vec{y}}^{\vec{x}}}{var(I_{\vec{y}})} \frac{k_{\vec{z}}^{\vec{x}}}{var(I_{\vec{z}})} cov(I_{\vec{y}}, I_{\vec{z}})}{\sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 / var(I_{\vec{y}})} \right\} \tag{15}$$

2D Markov process correlation inversely proportional to distance

2D images are approximated by 2D Markov processes where the correlation is inversely proportional to the distance. For such a process the following relation holds:

$$cov(I_{\vec{x}}, I_{\vec{y}}) = C \exp\Big\{-\frac{||\vec{x}-\vec{y}||}{\lambda}\Big\} = var_0 \exp\Big\{-\frac{||\vec{x}-\vec{y}||}{\lambda}\Big\}, \quad (16)$$

where var_0 is the variance when $\vec{x} = \vec{y}$ and λ is some length scale, e.g. the FWHM. Then

$$v\bar{a}r \equiv \frac{\sum_{\vec{y}\neq\vec{z}} k^{\vec{x}}_{\vec{y}} k^{\vec{x}}_{\vec{z}} cov(I_{\vec{y}}, I_{\vec{z}})}{\sum_{\vec{y}\neq\vec{z}} k^{\vec{x}}_{\vec{y}} k^{\vec{x}}_{\vec{z}}} = \frac{\sum_{\vec{y}\neq\vec{z}} k^{\vec{x}}_{\vec{y}} k^{\vec{x}}_{\vec{z}} var_0 \exp\big\{-\frac{||\vec{x}-\vec{y}||}{\lambda}\big\}}{\sum_{\vec{y}\neq\vec{z}} k^{\vec{x}}_{\vec{y}} k^{\vec{x}}_{\vec{z}}} \quad (17)$$

Convolution with correlation inversely prop. to distance

Combining equations 11 and 16 we have:

$$\bar{var} \approx \frac{\sum_{\vec{y} \neq \vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}} var_0 \exp\left\{-\frac{||\vec{y}-\vec{z}||}{\lambda}\right\}}{\sum_{\vec{y} \neq \vec{z}} k_{\vec{y}}^{\vec{x}} k_{\vec{z}}^{\vec{x}}} \tag{18}$$

Photometry with correlation inversely prop. to distance

Combining equations 13 and 16 we have:

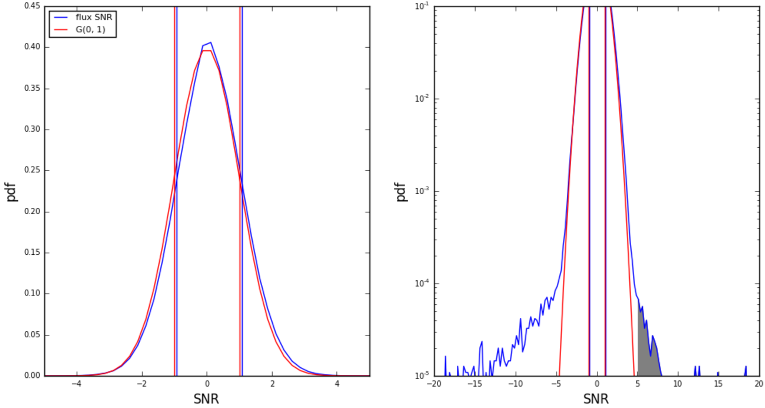
$$var[Flux(\vec{x})] \approx$$

$$\frac{1}{\sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 / var(I_{\vec{y}})} \left\{ 1 + \frac{\sum_{\vec{y} \neq \vec{z}} \frac{k_{\vec{y}}^{\vec{x}}}{var(I_{\vec{y}})} \frac{k_{\vec{z}}^{\vec{x}}}{var(I_{\vec{z}})} var_0 \exp\left\{ -\frac{||\vec{y}-\vec{z}||}{\lambda} \right\}}{\sum_{\vec{y}} (k_{\vec{y}}^{\vec{x}})^2 / var(I_{\vec{y}})} \right\} \quad (19)$$

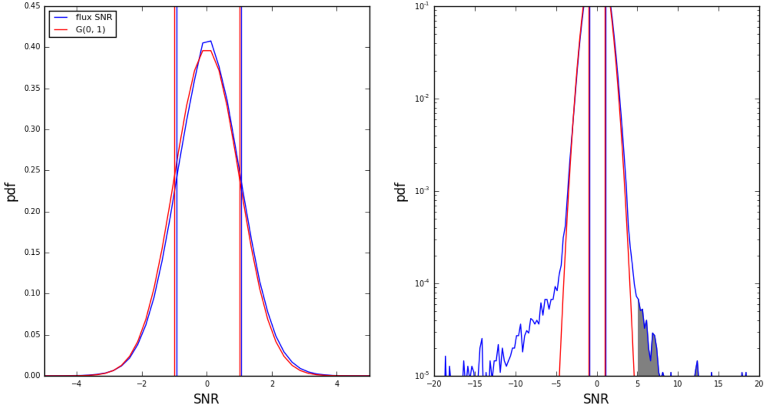
Test with real data

- Measure the distribution of $SNR(\vec{x}) \equiv \frac{Flux(\vec{x})}{\sqrt{var[Flux(\vec{x})]}}$, if errors are properly modeled we should see a standard normal distribution.
- Assuming correlation inversely proportional to distance, var_0 and λ in order to reproduce standard normal distribution ($var_0 \approx var_{\vec{x}}$ and $\lambda \approx FWHM$?).

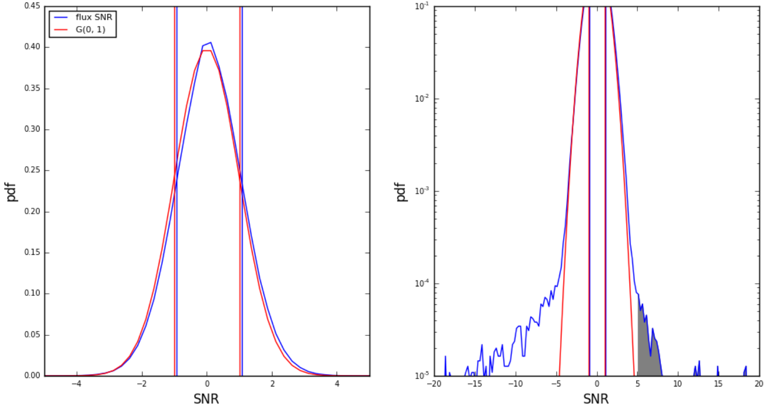
No covariance approximation, small kernel



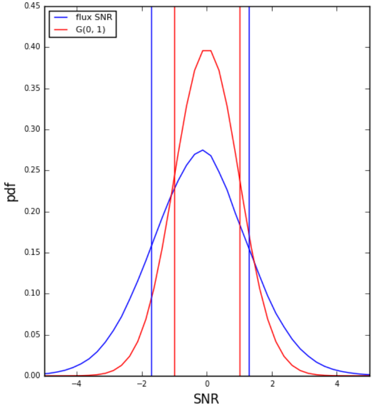
Local variance approximation, small kernel

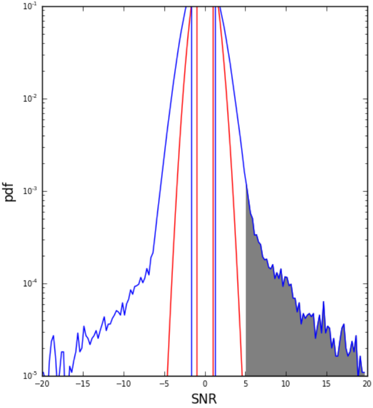


Convolved variance approximation, small kernel



No covariance approximation, large kernel





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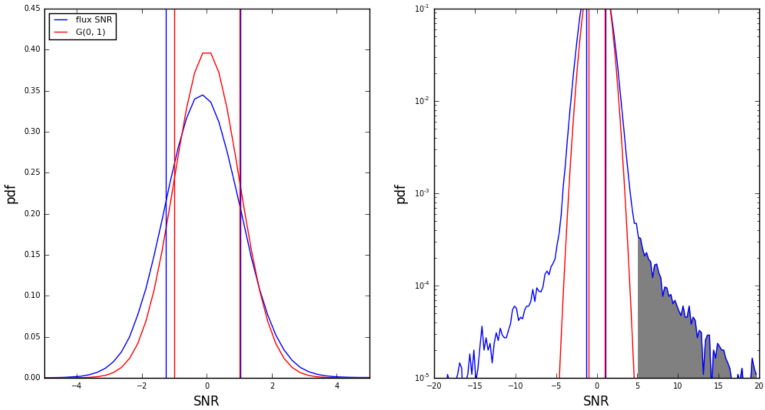
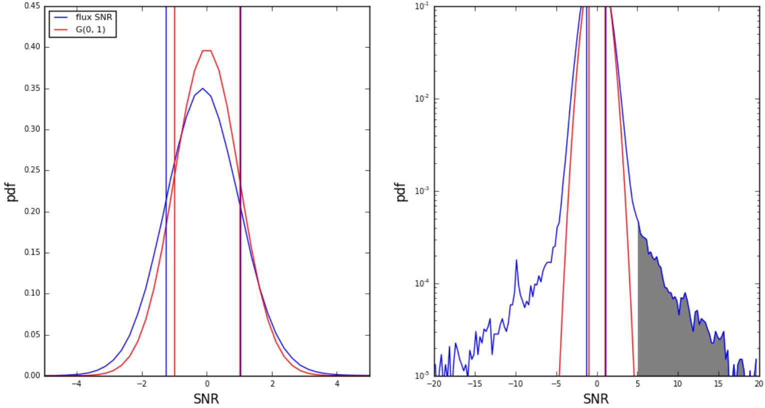


Convolved variance approximation, large kernel



Conclusions

- Exact expression for the variance of the convolution and photometry of an image difference were obtained, which depend on the full covariance matrix
- We expect non–diagonal terms of the covariance matrix to increase the variance of the convolved image and the photometry.
- The larger the convolution kernel, the larger the effect of the covariance matrix during convolution
- The larger the PSF and the smaller the variance in every pixel, the larger the effect of the covariance matrix during photometry
- An analytic approximation for the covariance matrix is proposed assuming that the correlation between pixels is inversely proportional to distance. The approximation has two free–parameters: a scaling factor and a length scale. We plan to test whether these parameters are related to the local variance and the FWHM of the PSF.