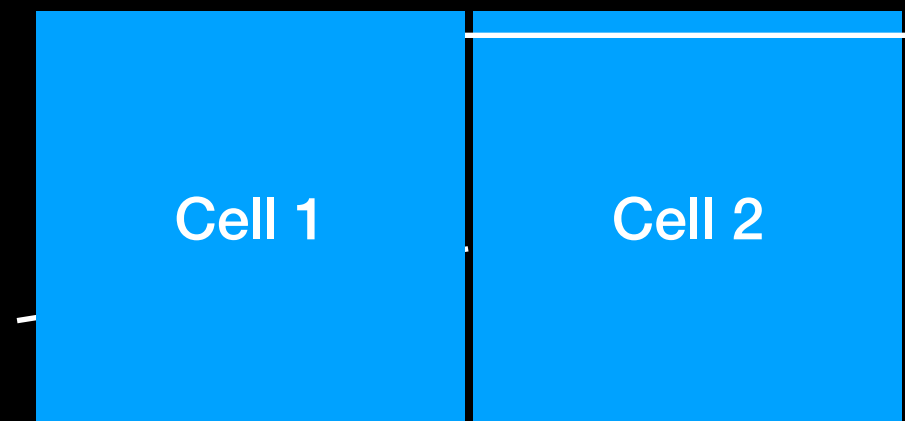


Shock-capturing numerical methods

Overview

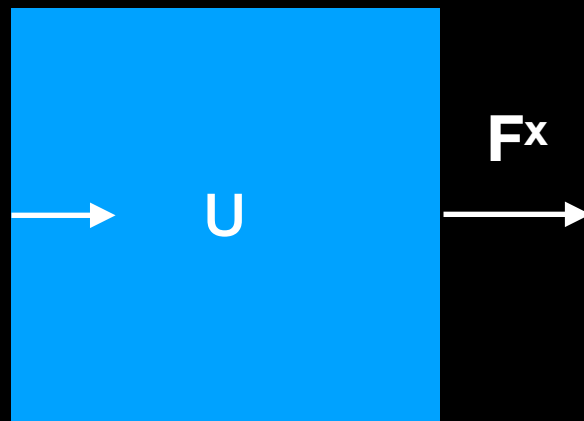
- Many numerical methods are unstable / lead to large oscillations in solution when shocks are present
- Shock capturing methods are built to take into account the presence of a discontinuity in the solution in between grid cells



- Solution at cell interface is assumed to be different on the “left” and “right” side of the face
- These two states have to satisfy appropriate “jump” condition (e.g. Rankine-Hugoniot)

Methods

$$\partial_t U + \partial_i F^i(U) = S(U)$$



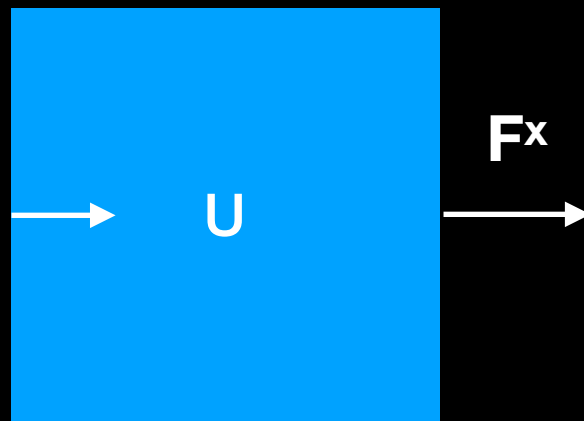
$$\partial_t U = - \frac{F(x + \Delta x/2) - F(x - \Delta x/2)}{\Delta x}$$

- “Conservative” scheme: fluxes are estimated on cell faces, and we conserve (up to source terms) :

$$U_{\text{tot}} = \int dV U$$

Methods

$$\partial_t U + \partial_i F^i(U) = S(U)$$



$$\partial_t U = - \frac{F(x + \Delta x/2) - F(x - \Delta x/2)}{\Delta x}$$

- Fluxes on faces are computed from the “right” and “left” states $F_{L,R}$, so that if $F_L \neq F_R$, the flux satisfies the jump conditions = solving the Riemann problem

$$F^i = f(F_R^i, F_L^i)$$

Algorithm

- To compute the time derivative $d_t U$:
 - Compute “Left” and “Right” value of the **fluxes at cell faces**
 - Compute F from $F_{R,L}$ using a **Riemann solver**
 - Compute the divergence $d_i F_i$ from the fluxes at cell faces
 - Compute the source terms at cell centers
- To evolve, we also need to **discretize in time** the equations

Reconstruction

- Simplest method (not very accurate):

$$F_L(x + \Delta x/2) = F(x) \quad F_R(x + \Delta x/2) = F(x + \Delta x)$$

- Slightly better : minmod reconstruction

$$F_L(x + \Delta x/2) = F(x) + s(x)/2$$

$$F_R(x - \Delta x/2) = F(x) - s(x)/2$$

$$s(x) = \text{minmod}[F(x - \Delta x), F(x), F(x + \Delta x)]$$

$$\text{minmod}[a, b, c] = \begin{cases} 0 & (c - b)(b - a) \leq 0 \\ \text{sign}[b - a] \times \min[(b - a), (c - b)] & \text{otherwise} \end{cases}$$

- Alternative: **reconstruct \mathbf{U} on faces, then compute $\mathbf{F}(\mathbf{U})$**
[use this for our test problem]

Riemann solver

- Simplest: Local Lax Friedrich method

$$F = \frac{F_L + F_R}{2} - \frac{c}{2}(U_R - U_L)$$

with c the largest *characteristic speed* (in absolute value)
of the sytems of equations

- Note that for this, we need $U_{R,L}$ and $F_{R,L}$
- The second term adds *dissipation* at shocks: if $U_R > U_L$, F becomes more negative (i.e. the flux of U goes from right to left)

Time stepping

- A simple 2nd order method to take a time step dt is the 2nd order Runge-Kutta algorithm :

(1) Take a half-step to estimate $U(t+dt/2)$

$$U\left(t + \frac{dt}{2}\right) = U(t) + \frac{dt}{2} \partial_t U(t)$$

(2) Take a full step, with the time derivative computed using $U(t+dt/2)$

$$U(t + dt) = U(t) + dt \partial_t U\left(t + \frac{dt}{2}\right)$$

Example

- Try to implement the following equation in 1D (use e.g. 100 grid cells in $[-1,1]$ with periodic boundaries):

$$\partial_t y + \partial_x \left(\frac{y^2}{2} \right) = 0$$

$$y(x, t = 0) = \exp(-16x^2)$$

- The (only) characteristic speed of this equation is $c=y$
- You may also try this problem *without* using a Riemann solver, to see the importance of using shock-capturing methods