A note on online and recursively arbitrarily vertex-partitionable balloons

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January 3, 2012

Abstract

In this note, we study arbitrarily vertex-partitionable balloons. In particular, we show that an online arbitrarily vertex-partitionable balloon has maximum degree at most 5. Moreover, we exhibit some infinite families of online and recursively vertex-partitionable balloons with four and five branches.

1 Preliminaries

In this note, the specific class of *balloons* is studied:

Definition 1 *We denote by $B(b_1,\dots,\dots,b_k)$ the graph of order $2+\sum_{1\le i\le k} b_i$ consisting of two vertices r_1, r_2, with k vertex-disjoint paths $P_1,\dots,P_k$ of order $b_1,\dots,b_k$ joining r_1 to r_2. The graph $B(b_1,\dots,\dots,b_k)$ is called a k-*balloon *with* roots r_1 *and* r_2 *and* branches $P_1,\dots,P_k$.

We will also need the following definition.

Definition 2 *We denote by $PB(b_1,\dots,\overline{b_i},\dots,\underline{b_j},\dots,b_k)$ the partial k-balloon with branches of sizes $b_1,\dots,b_k$ and roots r_1, r_2 where the branches denoted $\overline{b_i}$ are not adjacent to r_1 and the ones denoted $\underline{b_j}$ are not adjacent ro r_2.*

Let $n,\tau_1,\dots,\tau_k$ be positive integers such that $\sum_{1\le i\le k}\tau_i = n$. Then $\tau=(\tau_1,\dots,\tau_k)$ is called a *decomposition* of n.

We study the problem of partitioning the vertex set of graphs according to certain conditions. The following definitions have been introduced in [1], [4] and [3], respectively.

Definition 3 *Let G be a connected graph on n vertices.*

- *G is said to be* arbitrarily vertex-partitionable *(AP for short) if for every decomposition $\tau=(\tau_1,\dots,\tau_k)$ of n, there exists a partition $V_1\dots,V_k$ of $V(G)$ such that each part V_i has order τ_i and induces a connected subgraph of G.*

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- *G is said to be* online arbitrarily vertex-partitionable *(OLAP for short) if for every decomposition $\tau = (\tau_1, \ldots, \tau_k)$ of n, there exists a partition $V_1 \ldots, V_k$ of $V(G)$ such that each part V_i has order τ_i, induces a connected subgraph of G, and $V(G) \setminus V_i$ induces an OLAP subgraph of G.*

- *G is said to be* recursively arbitrarily vertex-partitionable *(RAP for short) if for every decomposition $\tau = (\tau_1, \ldots, \tau_k)$ of n, there exists a partition $V_1 \ldots, V_k$ of $V(G)$ such that each part V_i has order τ_i, and both V_i and $V(G) \setminus V_i$ induce a RAP subgraph of G.*

It is worth noting that every graph with a hamiltonian path is RAP, every RAP graph is OLAP, every OLAP graph is AP, and every AP graph contains a (quasi)-perfect matching.

Observation 4 ([3]) *Let $B = B(b_1, ..., b_k)$ be an AP balloon with $b_1 \leq ... \leq b_k$. If n is odd, B contains at most three branches of odd size. If n is even, B contains at most two branches of odd size.*

Lemma 5 ([2]) *Let $B(b_1, ..., b_k)$ be an AP balloon with $b_1 \leq ... \leq b_k$. Then for each $i \leq k$ we have $2b_i \geq \sum_{j<i} b_j$.*

2 OLAP balloons have maximum degree at most 5

It is shown in [3] that every RAP balloon has maximum degree at most 5. We show that this holds as well for OLAP balloons.

Theorem 6 *Let G be an OLAP k-balloon. Then $k \leq 5$.*

Proof The proof is by contradiction. Let G be a minimum counterexample to the theorem: G is an OLAP k-balloon ($k \geq 6$) with least possible order.

For each $\lambda < n$, it is possible to find a subset of vertices V_λ with $|V_\lambda| = \lambda$, $G[V_\lambda]$ is connected, and $G[V \setminus V_\lambda]$ is OLAP. Since by the minimality of G, $G' = G[V \setminus V_\lambda]$ cannot be a partial k-balloon (otherwise adding an edge to G' would provide an OLAP k-balloon with order strictly less than G), for each λ, one of the following conditions must be fulfilled:

1. $G[V \setminus V_\lambda]$ is a k'-balloon with $k' = k - 1$ (i.e. V_λ is a branch of G), or
2. $G[V \setminus V_\lambda]$ is a path (i.e. V_λ consists of a root of G together with $k - 2$ branches), or
3. $G[V \setminus V_\lambda]$ is an OLAP caterpillar (i.e. V_λ consists of a root of G together with $k-2$ branches but one vertex), or
4. $G[V \setminus V_\lambda]$ is the tripode $T_3(2, 4, 6)$ (i.e. V_λ consists of a root of G together with $k-2$ branches but two vertices of the same branch)

We claim that G has branches of sizes respectively 1, 2, 3, 4, 5 and 6. We show this using contradiction. Let $\lambda \in \{1, 2, 3, 4, 5, 6\}$ and suppose there is no branch of size λ in G.

- $\lambda \in \{1, 2, 3\}$: for any choice of V_λ, $G[V \setminus V_\lambda]$ is either a partial k-balloon or a tree with a vertex of degree 4 (since at most two branches can be completely included in V_λ), a contradiction.
- $\lambda = 4$: similarly as in the previous case, for any choice of V_λ, $G[V \setminus V_\lambda]$ is either a partial k-balloon or a tree with a vertex of degree 3. By Observation 4, $G[V \setminus V_\lambda]$ cannot be a caterpillar, thus it is $T_3(2, 4, 6)$ and $G = B_6(1, 1, 1, 2, 4, 6)$, a contradiction since G has a branch of size 4.

- $\underline{\lambda = 5}$: by the previous cases we know that G has branches of sizes 1, 2, 3 and 4. $G[V \setminus V_\lambda]$ cannot be a path since by Observation 4, five vertices cannot cover four branches and a root. If $G[V \setminus V_\lambda]$ is an OLAP caterpillar, then necessarily $k = 6$, the caterpillar is $Cat(4, 5)$ and G contains three branches of size 1: $G = B_6(1, 1, 1, 2, 3, 4)$. But then following Observation 4, G is not OLAP. Finally, if $G[V \setminus V_\lambda]$ is $T_3(2, 4, 6)$, V_λ must either totally cover the branch of size 3, or the branch of size 2 plus a vertex of the one of size 3. But then it is only possible to totally cover two branches, and the remaining part has a vertex of degree at least 4 and is not $T_3(2, 4, 6)$.

- $\underline{\lambda = 6}$: by the previous cases we know that G has branches of sizes 1, 2, 3, 4 and 5. Now suppose there is one additional branch b_i ($i \leq 6$) such that $b_i \leq 5$. By Lemma 5 there can be at most one such additional branch, and then $b_i = 1$. Thus $G = B_k(1, 1, 2, 3, 4, 5, ...)$. But now the only potential choice for V_λ would be the three smallest branches, a root and one additional vertex. If $k > 6$, $G[V \setminus V_\lambda]$ is a tree with a vertex of degree at least 4, which is not OLAP. Otherwise it is $T_3(2, 4, 5)$, which is not OLAP either. Thus, $G = B_k(1, 2, 3, 4, 5, b_6, ...)$ with $b_6 \geq 7$. But again for $\lambda = 6$ it is not possible to decompose it. Thus G is not OLAP, a contradiction.

So, $G = B_k(1, 2, 3, 4, 5, 6, ...)$. But using Lemma 5, G is not AP, a contradiction. □

3 Families of OLAP balloons

In this section, we show that there are infinitely many OLAP k-balloons, for each $k \leq 5$. The case where $k \leq 3$ is trivial since then any k-balloon has a hamiltonian path and is therefore RAP.

3.1 OLAP 4-balloons

Proposition 7 *$B(1, 1, n, m)$ is OLAP iff at most one of n and m is odd.*

Proof Easily partitionable for $\lambda = 1, 3$. For $\lambda = 2$, one can prove by induction that $B(1, 1, \overline{2k}, m)$ and $B(1, 1, n, \overline{2k})$ is OLAP. □

Proposition 8 *Let $a \leq b \leq c$ be such that $Cat(b + 1, a + c + 2)$ or $Cat(c + 1, a + b + 2)$ or $Cat(a + 1, b + c + 2)$ is OLAP. Then $B(1, a, b, c)$ is OLAP.*

Proof These caterpillars are spanning subgraphs of $B(1, a, b, c)$. □

Proposition 9 *$B(1, 2, 2, k)$, $k \geq 3$, is OLAP iff $k \equiv 0, 1 \bmod 3$.*

Proof Easily partitionable for $\lambda \neq 3$. If $k \equiv 0, 1 \bmod 3$, also OK for $\lambda = 3$ because of $Cat(3, k+1)$ which is OLAP.

Now suppose $k \equiv 2 \bmod 3$. For $\lambda = 3$, for the balloon to be partitionable we should have either:

- $PB(1, 2, 2, \overline{k - 3})$ OLAP.
- $PB(1, 2, 2, \overline{a}, \underline{b})$ $(a + b = k - 3)$ OLAP

But in both cases these partial balloons are not $(3, \ldots, 3)$-partitionable. □

Proposition 10 *$B(1, 2, 3, k)$, $k \geq 3$, is OLAP.*

Proof Straightforward. $\square$

Proposition 11 *$PB(2,2,2,\overline{k})$, is OLAP iff $k \equiv 0 \bmod 3$.*

Proof By induction on k. $\square$

Observation 12 *$B(2,2,2,3)$, $B(2,2,3,4)$, $B(2,2,3,6)$, $B(2,2,3,18)$, $B(2,2,4,6)$, $B(2,3,4,6)$, are OLAP.*

3.2 OLAP 5-balloons

The proof of the following result is omitted.

Proposition 13 *Let G be a graph of Table 1. Then G is OLAP.*

$B(1,1,1,2,k)$ with $k \equiv 0,4 \bmod 6$ $B(1,1,1,4,k)$ with $k \in \{6,8,10,18\}$ $B(1,1,1,6,k)$ with $k \in \{8,10,12,14\}$ $B(1,1,1,8,10)$
$B(1,1,2,2,3)$ $B(1,1,2,2,6)$ $B(1,1,2,3,k)$ with $k \equiv 0 \bmod 2$ $B(1,1,2,4,6)$ $B(1,1,2,5,k)$ with $k \equiv 0,4 \bmod 6$ $B(1,1,3,4,6)$ $B(1,1,4,4,6)$
$B(1,2,2,3,4)$ $B(1,2,2,4,6)$ $B(1,2,3,4,5)$ $B(1,2,3,4,6)$ $B(1,2,3,4,7)$
$B(2,2,3,4,6)$

Table 1: Some OLAP 5-balloons

4 Families of RAP balloons

In this section, we show that there are infinitely many RAP k-balloons, for each $k \leq 5$.

4.1 RAP 4-balloons

Proposition 14 *$PB(1,1,1,\overline{k})$, $B(1,1,1,k)$, $PB(1,1,3,\overline{k})$ and $B(1,1,3,k)$ are RAP iff k is even.*

Proof If k is odd, they are not AP by Observation 4. On the other hand, $Cat(2,k+3)$ is a spanning subgraph of $PB(1,1,1,\overline{k})/B(1,1,1,k)$ and $Cat(2,k+5)$ is a spanning subgraph of $PB(1,1,3,\overline{k})/B(1,1,3,k)$. These caterpillars are RAP when k is even. $\square$

Proposition 15 *$PB(1,1,2,\overline{k})$ is RAP for any $k \geq 1$.*

Proof Observe that it is true for $k \leq 3$ since $Cat(3,5)$ and $Cat(4,5)$ are spanning subgraphs of $PB(1,1,2,\overline{1})$, $PB(1,1,2,\overline{2})$ and $PB(1,1,2,\overline{3})$, respectively.

Now, suppose the proposition is true for all $i \leq k-1$ and consider $PB(1,1,2,\overline{k})$. For $\lambda \leq 4$, $PB(1,1,2,\overline{k})$ can be partitioned into P_λ and $P_{n-\lambda}$. For $\lambda = 5$, it can be partitioned into $Cat(2,3)$ and P_{k+1}. For $\lambda = 6$, it can be partitioned into $B(1,1,2)$ and P_k. Finally, for $\lambda \geq 7$, it can be partitioned into $PB(1,1,2,\overline{\lambda-6})$ and $P_{k-\lambda+6}$, the first being RAP by induction hypothesis. $\square$

Proposition 16 *$PB(1,2,2,\overline{k})$ and $B(1,2,2,k)$ are RAP iff $k \equiv 1,2 \bmod 3$.*

Proof On the one hand, observe that $Cat(3,k+4)$ is a spanning subgraph of $PB(1,2,2,\overline{k})$ and $B(1,2,2,k)$, which is RAP when $k \equiv 1,2 \bmod 3$. On the other hand, if $k \equiv 0 \bmod 3$, they are not $(3,\ldots,3)$-partitionable. $\square$

Proposition 17 *$PB(1,2,3,\overline{k})$ is RAP for any $k \geq 3$.*

Proof For any $\lambda \leq \frac{n}{2}$, $PB(1,2,3,\overline{k})$ must be partitionable into two vertex-disjoint RAP subgraphs of orders λ and $n-\lambda$.

- $\lambda = 1$: $P_1 + P_{n-1}$
- $\lambda = 2$: $P_2 + P_{n-2}$
- $\lambda = 3$: $P_3 + P_{n-3}$
- $\lambda = 4$: $P_4 + P_{k+4}$
- $\lambda = 5$: $P_5 + P_{k+3}$
- $\lambda = 6$: $P_6 + P_{k+2}$
- $\lambda = 7$: $Cat(3,4) + P_{k+1}$
- $\lambda = 8$: $B(1,2,3) + P_k$

Note that by the previous cases, the proposition is true for $n \leq 16$ (i.e. $k \leq 8$).

Now, let $\lambda \geq 9$, suppose it is true for any $i \leq k-1$, and consider $PB(1,1,2,3,\overline{k})$. Now one can partition the graph into $P_{n-\lambda}$ and $PB(1,2,3,\overline{\lambda-8})$, which is RAP by induction hypothesis. $\square$

4.2 RAP 5-balloons

Proposition 18 *$PB(1,1,2,3,\overline{2k})$ is RAP for any $k \geq 0$.*

Proof For any $\lambda \leq \frac{n}{2}$, $PB(1,1,2,3,\overline{2k})$ must be partitionable into two vertex-disjoint RAP subgraphs of orders λ and $n-\lambda$.

- $\lambda = 1$: $P_1 + B(1,2,3,\overline{2k})$ (RAP by Proposition 17)
- $\lambda = 2$: $P_2 + B(1,1,3,\overline{2k})$ (RAP by Proposition 14)
- $\lambda = 3$: $P_3 + B(1,1,2,\overline{2k})$ (RAP by Proposition 15)
- $\lambda = 4$: $P_4 + Cat(4,2k+1)$
- $\lambda = 5$: $Cat(2,3) + P_{2k+4}$

- $\lambda = 6$: $P_6 + Cat(2, 2k+1)$
- $\lambda = 7$: $Cat(3, 4) + P_{2k+2}$

Note that by the previous cases, the proposition is true for $n \leq 14$ (i.e. $k \leq 2$).
Now, let $\lambda \geq 8$, suppose it is true for any $i \leq k-1$, and consider $PB(1, 1, 2, 3, \overline{2k})$.

- λ even:
  - if $\lambda \leq 2k$, one can partition the graph into P_λ and $PB(1, 1, 2, 3, \overline{2k-\lambda})$ by placing P_λ at the end of the last branch, and use the induction hypothesis.
  - if $\lambda > 2k$, then $n - \lambda \leq 7$ (since both $2k$ and λ are even), thus one can use the cases where $\lambda \leq 7$.
- λ odd. Then partition the graph into $PB(1, 1, 2, 3, \overline{\lambda - 9})$ (note that it is RAP by induction hypothesis since $\lambda - 9$ is even and positive) and $P_{2k-\lambda+9}$.

□

This proves the existence of an infinite family of RAP 5-balloons:

Corollary 19 *$B(1, 1, 2, 3, 2k)$ is RAP for any $k \geq 1$.*

In [3], it is shown that $B(1, 1, 1, 2, 4)$ and $B(1, 1, 2, 2, 3)$ are RAP.

Proposition 20 *Other RAP 5-balloons are $B(1, 2, 2, 3, 4)$ and $B(2, 2, 3, 4, 6)$.*

References

[1] O. Baudon, D. Barth and J. Puech. Decomposable trees: a polynomial algorithm for tripodes. *Discrete applied mathematics*, 119(3):205–216, 2002.

[2] O. Baudon, F. Foucaud, J. Przybyło and M. Woźniak. Structure of k-connected Arbitrarily Partitionable Graphs. Manuscript, 2011.

[3] O. Baudon, F. Gilbert and M. Woźniak. Recursively arbitrarily vertex-decomposable graphs. Submitted, 2010.

[4] M. Horňák, Z. Tuza and M. Woźniak. On-line arbitrarily vertex decomposable trees. *Discrete Applied Mathematics* 155(11):1420–1429, 2007.