LARGE SUBGRAPHS WITHOUT SHORT CYCLES[*]

F. FOUCAUD[†], M. KRIVELEVICH[‡], AND G. PERARNAU[§]

Abstract. We study two extremal problems about subgraphs excluding a family $\mathcal{F}$ of graphs: (i) Among all graphs with m edges, what is the smallest size $f(m, \mathcal{F})$ of a largest $\mathcal{F}$-free subgraph? (ii) Among all graphs with minimum degree δ and maximum degree Δ, what is the smallest minimum degree $h(\delta, \Delta, \mathcal{F})$ of a spanning $\mathcal{F}$-free subgraph with largest minimum degree? These questions are easy to answer for families not containing any bipartite graph. We study the case where $\mathcal{F}$ is composed of all even cycles of length at most $2r$, $r \geq 2$. In this case, we give bounds on $f(m, \mathcal{F})$ and $h(\delta, \Delta, \mathcal{F})$ that are essentially asymptotically tight up to a logarithmic factor. In particular for every graph G, we show the existence of subgraphs with arbitrarily high girth and with either many edges or large minimum degree. These subgraphs are created using probabilistic embeddings of a graph into extremal graphs.

Key words. degrees, girth, forbidden subgraphs, spanning subgraphs, probabilistic methods

AMS subject classifications. 05C07, 05C35, 05C38, 05D40

DOI. 10.1137/140954416

1. Introduction. Let $G = (V, E)$ be a simple undirected graph with $|V| = n$ vertices. If F is a given graph, then we say that G is F-free if there is no subgraph of G isomorphic to F. The problem of determining the *Turán number* with respect to n and F, i.e., the largest size of an F-free graph on n vertices, has been extensively studied in the literature. This is the same as determining the size of a largest F-free spanning subgraph of K_n, the complete graph on n vertices. The latter quantity is denoted by $\mathrm{ex}(K_n, F)$. We can extend this notion in a natural way: for every graph G, let $\mathrm{ex}(G, F)$ denote the largest size of an F-free subgraph of G. Also, for a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$-free if G does not contain any graph from $\mathcal{F}$, and we denote by $\mathrm{ex}(G, \mathcal{F})$ the largest size of an $\mathcal{F}$-free subgraph of G.

In this paper, we provide lower bounds for $\mathrm{ex}(G, \mathcal{F})$ in terms of different graph parameters of G. If $\mathcal{F}$ does not contain any bipartite graph, it is easy to provide tight bounds for $\mathrm{ex}(G, \mathcal{F})$. Therefore, it is more interesting to study the behavior of $\mathrm{ex}(G, \mathcal{F})$ when $\mathcal{F}$ contains bipartite graphs. We mostly address the case of even cycles $\mathcal{F} = \{C_4, C_6, \ldots, C_{2r}\}$ with $r \geq 2$.

In the first part of the paper, we derive a lower bound for $\mathrm{ex}(G, \{C_4, C_6, \ldots, C_{2r}\})$ in terms of the size of G. In the second part of the paper, we study the largest minimum degree of a $\{C_4, C_6, \ldots, C_{2r}\}$-free spanning subgraph of G in terms of the maximum and minimum degrees of G. In both cases, for every graph G, we show the existence of subgraphs with either many edges or large minimum degree and

[†]Department of Mathematics, University of Johannesburg, Auckland Park 2006, South Africa; LAMSADE–CNRS UMR 7243, PSL, Université Paris-Dauphine, 75775 Paris, France (florent.foucaud@gmail.com).
[‡]School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel (krivelev@post.tau.ac.il). This author's research was supported in part by USA–Israel BSF grant 2010115 and by grant 912/12 from the Israel Science Foundation.
[§]School of Computer Science, McGill University, 845 Sherbrooke Street West, Montreal, QC, H3A 0G4, Canada (guillem.perarnaullobet@mcgill.ca).

arbitrarily high girth. As far as we know, this is the first study of these extremal problems.

Key definitions. Let F be a fixed graph. We define $f(m, F)$ as the smallest possible size of a largest F-free subgraph of a graph with m edges, that is,

$$f(m, F) := \min\{\mathrm{ex}(G, F) : |E(G)| = m\} .$$

More generally, for a family $\mathcal{F}$ of graphs, we define

$$f(m, \mathcal{F}) := \min\{\mathrm{ex}(G, \mathcal{F}) : |E(G)| = m\} .$$

Our second key definition is $h(\delta, \Delta, F)$, the smallest possible minimum degree of an F-free spanning subgraph H of a graph G with minimum degree δ and maximum degree Δ such that $\delta(H)$ is maximized. If we define $d(G, F) = \max\{\delta(H) : V(H) = V(G), H \subseteq G$ and $F \not\subset H\}$, then we can write

$$h(\delta, \Delta, F) := \min\{d(G, F) : \Delta(G) = \Delta \text{ and } \delta(G) = \delta\} .$$

As an illustration, consider the case $F = P_3$, the path on three vertices. Since every graph H with minimum degree at least 2 contains a copy of P_3, for any δ and Δ with $\Delta \geq \delta \geq 2$, we have $h(\delta, \Delta, P_3) = 1$.

Also, for a family $\mathcal{F}$ if we define $d(G, \mathcal{F}) = \max\{\delta(H) : V(H) = V(G), H \subseteq G$ and $\forall F \in \mathcal{F}, F \not\subset H\}$, then

$$h(\delta, \Delta, \mathcal{F}) := \min\{d(G, \mathcal{F}) : \Delta(G) = \Delta \text{ and } \delta(G) = \delta\} .$$

Thomassen's conjecture. A related problem for the girth and the average degree was stated by Thomassen [22]. Similarly to our definitions of $f(m, \mathcal{F})$ and $h(\delta, \Delta, \mathcal{F})$, we can define the smallest possible average degree of an $\mathcal{F}$-free subgraph F of a graph G with average degree $\overline{d}(G) \geq d$ such that $\overline{d}(F)$ is maximized. Formally, if we define $d'(G, \mathcal{F}) = \max\{\overline{d}(F) : F \subseteq G$ and $\forall F \in \mathcal{F}, F \not\subset F\}$, then

$$f'(d, \mathcal{F}) := \min\{d'(G, \mathcal{F}) : \overline{d}(G) \geq d\} .$$

A reformulation of Thomassen's conjecture is to say that for every fixed $g \geq 4$, $f'(d, \{C_3, \ldots, C_{g-1}\}) \to \infty$ when $d \to \infty$. If $g = 4$, it is easy to check that $f'(d, \{C_3\}) = d/2$. For $g = 6$, Kühn and Osthus [15] showed that $f'(d, \{C_3, C_4, C_5\}) = \Omega(\sqrt[3]{\log \log d})$, thus confirming the conjecture for this case.

Observe that one can change the average degree for the minimum degree in the definition of $d'(G, F)$, since a graph with average degree δ has a subgraph with minimum degree at least $\delta/2$. It is interesting to notice that Thomassen's conjecture is true when restricted to graphs of large minimum degree δ with respect to the maximum degree Δ. In this case, a relatively straightforward application of the local lemma shows that for any family of graphs $\mathcal{F}$ there exists a spanning $\mathcal{F}$-free subgraph with minimum degree $\Omega(\delta \cdot \Delta^{-m(\mathcal{F})})$, where $m(\mathcal{F}) = \max_{F \in \mathcal{F}} \frac{|V(F)|-2}{|E(F)|-1}$.[1] One of the goals of this paper is to improve this lower bound when $\mathcal{F}$ is composed of cycles.

Families of nonbipartite graphs. It is rather easy to determine both functions $f(m, \mathcal{F})$ and $h(\delta, \Delta, \mathcal{F})$ asymptotically when $\chi(\mathcal{F}) > 2$ (there are no bipartite graphs in $\mathcal{F}$), as shown by the following proposition (a proof is provided in section 2.1).

PROPOSITION 1. *For every graph G and for every $k \geq 3$ there exists a $(k-1)$-partite subgraph H of G such that for every $v \in V(G)$,*

$$d_H(v) \geq \left(1 - \frac{1}{k-1}\right) d_G(v) .$$

---

[1] See Lemma 4 of [15] for a similar statement with $\mathcal{F} = \{C_4\}$.

Moreover, for every $\mathcal{F}$ with $\chi(\mathcal{F}) = k$, we have

$$f(m, \mathcal{F}) = \left(1 - \frac{1}{k-1} + o_m(1)\right) m$$

and

$$h(\delta, \Delta, \mathcal{F}) = \left(1 - \frac{1}{k-1} + o_\delta(1)\right) \delta .$$

Even cycles. It remains to study $f(m, \mathcal{F})$ and $h(\delta, \Delta, \mathcal{F})$ when $\mathcal{F}$ contains a bipartite graph. In this paper, we focus on the cases $\mathcal{F} = \mathcal{F}_r = \{C_3, C_4, \ldots, C_{2r+1}\}$ and $\mathcal{F} = \mathcal{F}_r^{\mathrm{even}} = \{C_4, C_6, \ldots, C_{2r}\}$ for some $r \geq 2$. We define

$$f(m, r) = f(m, \mathcal{F}_r^{\mathrm{even}})$$

and

$$h(\delta, \Delta, r) = h(\delta, \Delta, \mathcal{F}_r^{\mathrm{even}}) .$$

Using Proposition 1 one can easily show that every family $\mathcal{F}$ composed of $\mathcal{F}_r^{\mathrm{even}}$ and other graphs with chromatic number at least 3 satisfies

(1)
$$f(m, \mathcal{F}) = \Theta(f(m, r)),$$
$$h(\delta, \Delta, \mathcal{F}) = \Theta(h(\delta, \Delta, r)) .$$

This provides us a way to transfer the results obtained for $\mathcal{F}_r^{\mathrm{even}}$ to $\mathcal{F}_r$.

Denoting by $K_{s,t}$ the complete bipartite graph with parts of size s and t, our first result is the following theorem.

THEOREM 2. *For every $r \geq 2$ there exists $c = c(r) > 0$ such that for every large enough m,*

$$f(m, r) \geq \frac{c}{\log m} \min_{k \mid m} \mathrm{ex}(K_{k, m/k}, \mathcal{F}_r^{\mathrm{even}}) .$$

Observe that for every k dividing m, we have

$$f(m, r) \leq \mathrm{ex}(K_{k, m/k}, \mathcal{F}_r^{\mathrm{even}})$$

since $K_{k, m/k}$ has m edges. Thus, Theorem 2 is asymptotically tight up to a logarithmic factor. This theorem is proved in section 3.

The value of $\mathrm{ex}(K_{k, m/k}, \mathcal{F}_r^{\mathrm{even}})$ is not known for general k and r. Some results for small values of r can be found in [6, 12]. The results in [20] imply the following explicit upper bound:

$$f(m, r) = O(m^{(r+1)/2r}) .$$

The case $r = 2$ of the above problem appears to be more accessible. In particular, combining the upper bound of Kovári, Sós, and Turán [14] and Reiman [21] and the lower bound provided by Erdős and Rényi [7] we have $\mathrm{ex}(K_{n,n}, C_4) = \Theta(n^{3/2})$ (see Chapter 6.2 of [2] for a discussion). Here we derive the following corollary (proved in section 3).

COROLLARY 3. *There is a constant $c > 0$ such that for every large enough m,*

$$f(m, 2) \geq \frac{cm^{2/3}}{\log m} .$$

We remark that by applying a standard double counting argument of [14], one can show that $\mathrm{ex}(K_{m^{1/3},m^{2/3}}, C_4) = O(m^{2/3})$. Hence we obtain $f(m, 2) = \tilde{\Theta}(m^{2/3})$ (where the $\tilde{\Theta}$ notation neglects polylogarithmic terms). Then, by (1)

$$f(m, \{C_3, C_4, C_5\}) = \tilde{\Theta}(m^{2/3}) .$$

By using (1) with $\mathcal{F} = \mathcal{F}_r$, Theorem 2 provides a lower bound on $f(m, \mathcal{F}_r)$, which implies the existence of large subgraphs with high girth. We also provide a general lower bound for $h(\delta, \Delta, r)$, proved in section 4 (where $g(G)$ denotes the girth of graph G).

THEOREM 4. *Let G be a graph with minimum degree δ and (large enough) maximum degree Δ such that $\mathrm{ex}(K_\Delta, \mathcal{F}_r)\delta \geq \alpha\Delta^2 \log^4 \Delta$ for some large constant $\alpha > 0$. Then, for every $r \geq 2$ there exists a spanning subgraph H of G with $g(H) \geq 2r+2$ and $\delta(H) \geq \frac{c\cdot\mathrm{ex}(K_\Delta,\mathcal{F}_r)\delta}{\Delta^2 \log \Delta}$ for some small constant $c > 0$. In particular, under the above conditions on δ and Δ,*

$$h(\delta, \Delta, r) \geq h(\delta, \Delta, \mathcal{F}_r) \geq \frac{c \cdot \mathrm{ex}(K_\Delta, \mathcal{F}_r)\delta}{\Delta^2 \log \Delta} .$$

The existence of a graph with a given number of vertices, many edges, and large girth is one of the most interesting open problems in extremal graph theory (see section 4 in [11]). The best known result that holds for any value of $r \geq 2$ was given by Lazebnik, Ustimenko, and Woldar in [17]. They showed that $\mathrm{ex}(K_n, \mathcal{F}_r) = \Omega(n^{1+\frac{2}{3r-2}})$. Using that, we can obtain the following explicit corollary of Theorem 4.

COROLLARY 5. *For every Δ and δ such that $\mathrm{ex}(K_\Delta, \mathcal{F}_r)\delta \geq \alpha\Delta^2 \log^4 \Delta$ for some large constant $\alpha > 0$, we have*

$$h(\delta, \Delta, r) \geq h(\delta, \Delta, \mathcal{F}_r) = \Omega\left(\frac{\delta}{\Delta^{1-\frac{2}{3r-2}} \log \Delta}\right) .$$

The upper bound for the Turán number of even cycles $\mathrm{ex}(K_d, C_{2r}) = O(d^{1+1/r})$, shown by Bondy and Simonovits [3], implies that

$$(2) \qquad\qquad h(d, d, r) \leq \frac{2}{d}\mathrm{ex}(K_{d+1}, C_{2r}) = O(d^{\frac{1}{r}}) .$$

Since the above cited upper bound for $\mathrm{ex}(K_{d+1}, C_{2r})$ is conjectured to be of the right order, Corollary 5 is probably not tight.

For the particular case $r = 2$ we can derive a better bound. Erdős, Rényi, and Sós [8] and Brown [4] showed that for every prime p there exists a C_4-free p-regular graph with $p^2 - 1$ vertices. By the density of primes, for every n there exists a graph of order at most $2n$ satisfying the former properties. Thus, we can obtain the following corollary of Theorem 4.

COROLLARY 6. *For every Δ and δ such that $\delta \geq \alpha\sqrt{\Delta} \log^4 \Delta$ for some large constant $\alpha > 0$, we have*

$$h(\delta, \Delta, 2) \geq h(\delta, \Delta, \{C_3, C_4, C_5\}) = \Omega\left(\frac{\delta}{\sqrt{\Delta} \log \Delta}\right) .$$

This corollary is tight up to a logarithmic factor, as will be shown in Proposition 15 (section 4). The condition on δ and Δ is also tight up to a logarithmic factor since, if $\Delta \geq \delta^2$, any spanning C_4-free subgraph H of $K_{\delta,\Delta}$ satisfies $\delta(H) \leq 1$. Similar results can be derived for $r = 3, 5$, since there exist graphs with girth at least 8 and 12, respectively, and large minimum degree [18, 16].

Outline of the paper. We begin with some preliminary considerations in section 2. We then prove Theorem 2 and Corollary 3 in section 3. Section 4 is devoted to the proof of Theorem 4 and Proposition 15. We conclude with some remarks and open questions in section 5.

2. Preliminaries. For every vertex $v \in V$ let $N_G(v)$ denote the set of neighbors of v in G and $d_G(v) = |N_G(v)|$. If the graph G is clear from the context we will denote the above quantities by $N(v)$ and $d(v)$, respectively. We use $\log(x)$ to denote the natural logarithm of x.

2.1. Proof of Proposition 1. We start by proving Proposition 1. We will use the well-known Erdős–Stone–Simonovits theorem [10, 9]: for every graph F with $\chi(F) = k$,

$$(3) \qquad \qquad \operatorname{ex}(K_n, F) = \left(1 - \frac{1}{k-1} + o(1)\right)\binom{n}{2}.$$

Proof of Proposition 1. Let G be a graph with m edges and consider a $(k-1)$-partition $\mathcal{P} = \{P_1, P_2, \ldots, P_{k-1}\}$ of $V(G)$ that maximizes the number of edges in the subgraph $G_{\mathcal{P}} = G \setminus (G[P_1] \cup \cdots \cup G[P_{k-1}])$. Then, we claim that for every $v \in V(G)$, $d_{G_{\mathcal{P}}}(v) \geq (1 - \frac{1}{k-1})d_G(v)$.

For the sake of contradiction, suppose that there is a vertex $v \in P_i$, $i \in [k-1]$, with degree in $G_{\mathcal{P}}$ less than $(1 - \frac{1}{k-1})d_G(v)$. Then, there are more than $\frac{d_G(v)}{k-1}$ neighbors of v in P_i and there is a part P_j, $j \neq i$, with at most $\frac{d_G(v)}{k-1}$ neighbors of v. Moving v from P_i to P_j increases the number of edges in $G_{\mathcal{P}}$ by at least one, which gives a contradiction by the choice of $\mathcal{P}$.

Clearly, $G_{\mathcal{P}}$ does not contain any copy of F with $\chi(F) \geq k$, and thus it does not contain any graph in $\mathcal{F}$. Observe also that $G_{\mathcal{P}}$ has at least $(1 - \frac{1}{k-1})m$ edges and minimum degree $\delta(G_{\mathcal{P}}) \geq (1 - \frac{1}{k-1})\delta(G)$.

Notice that $f(m, \mathcal{F}) \leq f(m, F)$ and $h(\delta, \Delta, \mathcal{F}) \leq h(\delta, \Delta, F)$ for every $F \in \mathcal{F}$. Let $F \in \mathcal{F}$ be such that $\chi(F) = k$. The upper bound for $f(m, F)$ follows from (3) by choosing n for which $\binom{n-1}{2} < m \leq \binom{n}{2}$, and then by taking G to be any subgraph of K_n with exactly m edges. For the upper bound on $h(\delta, \Delta, F)$ (assuming $\delta \geq 2$) take Δ disjoint copies of $K_{\delta+1}$, add a new vertex v, and connect it to one vertex from each of the cliques $K_{\delta+1}$. If a subgraph H of the so obtained graph G is F-free, then the subgraph of H spanned by the vertex set of each of the cliques $K_{\delta+1}$ is F-free as well, thus implying by (3) that $\delta(H) \leq (1 - \frac{1}{k-1} + o(1))\delta$. $\quad\square$

2.2. Useful definitions. The following definitions will be useful in our proofs.

DEFINITION 7. *For every graph G, every $\mathcal{G}$ with $V(\mathcal{G}) = [\ell]$, and every vertex labeling $\chi : V(G) \to [\ell]$ we define the spanning subgraph $H'_{(\chi, \mathcal{G})} \subseteq G$ as the subgraph with vertex set $V(G)$ where an edge $e = uv$ is present if and only if $uv \in E(G)$ and $\chi(u)\chi(v) \in E(\mathcal{G})$.*

DEFINITION 8. *For every graph G, every $\mathcal{G}$ with $V(\mathcal{G}) = [\ell]$, and every vertex labeling $\chi : V(G) \to [\ell]$ we define the spanning subgraph $H^*_{(\chi, \mathcal{G})} \subseteq G$ as the subgraph with vertex set $V(G)$ such that an edge $e = uv$ is present in H^* if all the following properties are satisfied:*
   (i) *$uv \in E(G)$ and $\chi(u)\chi(v) \in E(\mathcal{G})$, that is, $e \in E(H')$;*
   (ii) *for every $w \neq v$, $w \in N_G(u)$, we have $\chi(w) \neq \chi(v)$; and*
   (iii) *for every $w \neq u$, $w \in N_G(v)$, we have $\chi(w) \neq \chi(u)$.*

These two definitions are crucial for our proofs. We will use them to randomly embed a (large) graph G into a fixed extremal graph $\mathcal{G}$ satisfying certain conditions,

therefore creating a subgraph of G. The way in which we construct the subgraph will allow us to prove that it preserves some of the properties of $\mathcal{G}$.

The concept of frugal coloring was introduced by Hind, Molloy, and Reed in [13]. We say that a proper coloring $\chi : V(G) \to [\ell]$ is t-frugal if for every vertex v and every color $c \in [\ell]$,

$$|N_G(v) \cap \chi^{-1}(c)| \leq t \; ,$$

that is, there are at most t vertices of the same color in the neighborhood of each vertex. For instance, a 1-frugal coloring of G is equivalent to a proper coloring of G^2.

2.3. Probabilistic tools. Here we state some (standard) lemmas we will use in the proofs.

LEMMA 9 (Chernoff inequality for binomial distributions [1]). *Let $X \sim Bin(N, p)$ be a Binomial random variable. Then $\forall 0 < \varepsilon < 1$,*
  1. $\Pr(X \leq (1 - \varepsilon)Np) < \exp\left(-\frac{\varepsilon^2}{2}Np\right)$,
  2. $\Pr(X \geq (1 + \varepsilon)Np) < \exp\left(-\frac{\varepsilon^2}{3}Np\right)$.

Let $L : S^T \to \mathbb{R}$ be a functional. We say that L satisfies the Lipschitz condition if for every g and g' differing in just one coordinate from the product space S^T, we have

$$|L(g) - L(g')| \leq 1 \; .$$

LEMMA 10 (Azuma inequality, Theorem 7.4.2 in [1]). *Let L satisfy the Lipschitz condition relative to a gradation of length l (i.e., $|T| = l$). Then $\forall \lambda > 0$*
  1. $\Pr(L \leq \mathbb{E}(L) - \lambda\sqrt{l}) < e^{-\frac{\lambda^2}{2}}$,
  2. $\Pr(L \geq \mathbb{E}(L) + \lambda\sqrt{l}) < e^{-\frac{\lambda^2}{2}}$.

LEMMA 11 (weighted Lovász local lemma [19]). *Let $\mathcal{A} = \{A_1, \ldots, A_N\}$ be a set of events and let H be a dependency graph for $\mathcal{A}$.*

If there exist weights $w_1, \ldots, w_N \geq 1$ and a real $p \leq \frac{1}{4}$ such that for each $i \in [N]$,
  1. $\Pr(A_i) \leq p^{w_i}$ *and*
  2. $\sum_{j : ij \in E(H)} (2p)^{w_j} \leq \frac{w_i}{2}$,

then

$$\Pr\left(\bigcap_{i=1}^{N} \overline{A_i}\right) > 0 \; .$$

3. Subgraphs with large girth and many edges. This section is devoted to the proofs of Theorem 2 and Corollary 3.

Proof of Theorem 2. Let G be a graph with m edges. Define V_1 to the set of vertices of G with degree at least $2\sqrt{m}$, and $V_2 = V \setminus V_1$. Observe that

$$|V_1| \leq 2m/2\sqrt{m} = \sqrt{m}$$

and thus V_1 spans at most $m/2$ edges. Recall that $\mathcal{F}_r^{\text{even}} = \{C_4, C_6, \ldots, C_{2r}\}$ is the family of all even cycles of length at most $2r$. In order to find an $\mathcal{F}_r^{\text{even}}$-free subgraph with many edges, we will remove all the edges inside V_1 and only look at the edges between V_1 and V_2, and the edges inside V_2. We will split the proof into two cases in terms of the number of edges between V_1 and V_2.

Case 1: $e(V_1, V_2) \geq m/4$. Consider the following partition of V_1 into the sets $U_1, \ldots, U_s$, where U_p contains all vertices of V_1 whose degree into V_2 is between 2^p and 2^{p+1}.

Notice that $s \leq \log_2 m$. Hence, there is a subset U_q incident to at least $m/4 \log_2 m$ edges leading to V_2. Let $|U_q| = k$; then $k = \Omega(m/(2^q \log m))$ and $k = O(m/2^q)$ since all the degrees in U_q are between 2^q and 2^{q+1}. Let $u_1, \ldots, u_k$ be the vertices of U_q.

Fix an $\mathcal{F}_r^{\text{even}}$-free bipartite graph $\mathcal{G}$ with parts $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_{m/k}\}$, and $\text{ex}(K_{k,m/k}, \mathcal{F}_r^{\text{even}})$ edges, and let $\chi : U_q \cup V_2 \to A \cup B$ be the $(A \cup B)$-coloring defined by $\chi(u_i) = a_i$ for any $u_i \in U_q$ and by $\chi(v) = b$ for any $v \in V_2$, where $b \in B$ is chosen independently and uniformly at random.

We say that $b \in B$ is *good* for $u_i \in U_q$ if
1. $(a_i, b) \in E(\mathcal{G})$ and
2. exactly one neighbor of u_i in G is colored b by χ.

Let G_1 be the subgraph of G with vertex set $U_q \cup V_2$ that only contains the edges $e = (u_i, v) \in E(G)$, where $u_i \in U_q$, $v \in V_2$, and $\chi(v) = b$ is good for u_i. We claim that G_1 is an $\mathcal{F}_r^{\text{even}}$-free graph. Assume that there is a cycle of length l of even length in G_1 with $l \leq 2r$ and let $w_1, \ldots, w_l$ be its vertices. Observe that there should be at least one repeated color in the vertices of the cycle, otherwise $\mathcal{G}$ would contain a C_l. Let $x, y \in [l]$ ($x < y$) such that $\chi(w_x) = \chi(w_y)$ and for every $x < z < y$, $\chi(w_x) \neq \chi(w_z)$. Besides, since χ is 1-frugal in G_1 (in the sense that for each vertex of G_1, all its neighbors are colored differently by χ), $y - x \geq 3$. Thus, there exists a cycle of length at least 3 and at most $2r$ in $\mathcal{G}$, a contradiction since $\mathcal{G}$ is bipartite and, since it is $\mathcal{F}_r^{\text{even}}$-free, it has no even cycles of length at most $2r$.

Let us compute the expected size of G_1. Fix $u_i \in U_q$. Then, the probability that a given edge $e = (u_i, v) \in E(G)$ exists in G_1 is

$$\Pr(e \in E(G_1)) \geq \frac{d_{\mathcal{G}}(a_i)}{m/k} \left(1 - \frac{1}{m/k}\right)^{2^{q+1}} = \Omega\left(\frac{d_{\mathcal{G}}(a_i)k}{m} e^{-\frac{2^q k}{m}}\right) = \Omega\left(\frac{d_{\mathcal{G}}(a_i)k}{m}\right)$$

since $2^q k = O(m)$.

Thus, the expected degree of u_i in G_1 is of order

$$\mathbb{E}(d_{G_1}(u_i)) = \Omega\left(2^q \frac{d_{\mathcal{G}}(a_i)k}{m}\right) .$$

Recall that $\sum_{a_i \in A} d_{\mathcal{G}}(a_i) = \text{ex}(K_{k,m/k}, \mathcal{F}_r^{\text{even}})$; hence the expected number of edges of G_1 is of order $2^q \text{ex}(K_{k,m/k}, \mathcal{F}_r^{\text{even}}) k/m$. Since $2^q k \geq cm/\log m$ for some small constant $c > 0$, the expected number of edges is of order at least

$$\min_{k|m} 2^q \frac{\text{ex}(K_{k,m/k}, \mathcal{F}_r^{\text{even}})k}{m} \geq \frac{c}{\log m} \min_{k|m} \text{ex}(K_{k,m/k}, \mathcal{F}_r^{\text{even}}) .$$

Case 2: $e(V_1, V_2) < m/4$. Then $e(V_2) > m/4$, and all the degrees in $G[V_2]$ are less than $2\sqrt{m}$. For the sake of convenience we will assume that $2\sqrt{m}$ is an integer.

We will find a large $\mathcal{F}_r^{\text{even}}$-free graph inside V_2. For this, let $\mathcal{G}$ be an $\mathcal{F}_r^{\text{even}}$-free graph on $2\sqrt{m}$ vertices with the largest possible number of edges. Assume $V(\mathcal{G}) = [2\sqrt{m}]$ and let $\chi : V_2 \to V(\mathcal{G})$ be a random labeling of the vertices of V_2. Observe that using Proposition 1 we can select a bipartite subgraph $\mathcal{G}'$ of $\mathcal{G}$ with at least $\text{ex}(K_{2\sqrt{m}}, \mathcal{F}_r^{\text{even}})/2$ edges (in particular, $\mathcal{G}'$ is $\mathcal{F}_r$-free).

Consider the graph $H^* = H^*_{(\chi, \mathcal{G}')}$ from Definition 8 applied to the induced subgraph $G[V_2]$. Since $\mathcal{G}'$ is $\mathcal{F}_r$-free, H^* is also $\mathcal{F}_r$-free. For each edge $e = uv$ of G

spanned by V_2, its probability for belonging to H^* is at least

$$\frac{1}{2\sqrt{m}} \sum_{i \in V(\mathcal{G}')} \frac{d_{\mathcal{G}'}(i)}{2\sqrt{m}} \left(1 - \frac{1}{2\sqrt{m} - 2}\right)^{d_G(u) + d_G(v) - 2} = \Omega\left(\frac{\mathrm{ex}(K_{2\sqrt{m}}, \mathcal{F}_r^{\mathrm{even}})}{m}\right)$$

since $d_G(u), d_G(v) \leq 2\sqrt{m}$. To justify the above estimate, first choose a label i for u, then require a label j of v to be one of the neighbors of i in $\mathcal{G}'$, and finally for each of the neighbors of u and v in G choose a label different from i and j.

Thus, we expect

$$\Omega\left(\mathrm{ex}(K_{2\sqrt{m}}, \mathcal{F}_r^{\mathrm{even}})\right) = \Omega\left(\mathrm{ex}(K_{\sqrt{m},\sqrt{m}}, \mathcal{F}_r^{\mathrm{even}})\right) = \omega\left(\frac{\mathrm{ex}(K_{\sqrt{m},\sqrt{m}}, \mathcal{F}_r^{\mathrm{even}})}{\log m}\right)$$

edges in the $\mathcal{F}_r^{\mathrm{even}}$-free subgraph H^* of G.   □

We now prove Corollary 3.

Proof of Corollary 3. By Theorem 2 we have

$$f(m, 2) \geq \frac{c}{\log m} \min_{k|m} \mathrm{ex}(K_{k,m/k}, C_4)$$

for some small constant $c > 0$. By the symmetry of $K_{s,t}$ and $K_{t,s}$, we may assume without loss of generality that the minimum is attained when $k \leq \sqrt{m}$.

We provide two constructions. First, the star of size m/k, which is a C_4-free graph, shows that $\mathrm{ex}(K_{k,m/k}, C_4) \geq m/k$.

On the other hand, one can construct a C_4-free graph by selecting a subgraph of a larger C_4-free graph. Let G_1 be a largest C_4-free subgraph of $K_{m/k,m/k}$. We construct a C_4-free bipartite graph G_2 by keeping the k vertices with highest degrees in one of the parts of G_1. Then G_2 is a subgraph of $K_{k,m/k}$ and has at least $\frac{k^2}{m}\mathrm{ex}(K_{m/k,m/k}, C_4) = \Omega(\sqrt{mk})$ edges.

Hence, for every k,

$$\mathrm{ex}(K_{k,m/k}, C_4) \geq \max\{m/k, \Omega(\sqrt{mk})\} = \Omega\left(m^{2/3}\right) .   □$$

4. Subgraphs with large girth and large minimum degree. We devote this section to the proof of Theorem 4. Before proving the theorem, let us state an important observation and an auxiliary lemma.

In this proof we will use a graph $\mathcal{G}$ that satisfies the following: it has $|V(\mathcal{G})| = \ell \in (k, 2k)$ (where $k = 2e^4\Delta$) vertices, girth at least $2r + 2$, and minimum degree at least $q = \mathrm{ex}(K_k, \mathcal{F}_r)/2k$. The existence of such $\mathcal{G}$ is provided by the following observation.

OBSERVATION 12. *Let $\mathcal{G}'$ be an $\mathcal{F}_r$-free graph on $2k$ vertices with $\mathrm{ex}(K_{2k}, \mathcal{F}_r)$ edges. Then we claim that $\mathcal{G}'$ has a subgraph $\mathcal{G}$ satisfying the desired properties. Assume it is not the case, namely, all its subgraphs of size at least k have minimum degree smaller than q. It is also clear that $\mathrm{ex}(K_{2k}, \mathcal{F}_r) \geq 2\mathrm{ex}(K_k, \mathcal{F}_r)$. By iteratively removing vertices of degree smaller than q, we can obtain an $\mathcal{F}_r$-free graph on k vertices with at least $\mathrm{ex}(K_{2k}, \mathcal{F}_r) - qk = \mathrm{ex}(K_{2k}, \mathcal{F}_r) - \mathrm{ex}(K_k, \mathcal{F}_r)/2 \geq 3 \cdot \mathrm{ex}(K_k, \mathcal{F}_r)/2$ edges, a contradiction.*

In a vertex-colored graph, a cycle is called *rainbow* if all its vertices have distinct colors. A path is called *maximal inner-rainbow* if its endpoints have the same color i, but all other vertices of the path are colored with distinct colors (other than i). We will use the following lemma.

LEMMA 13. *Let G be a graph with maximum degree Δ and minimum degree δ that admits a t-frugal coloring χ without rainbow cycles of length at most $2r + 1$ and*

maximal inner-rainbow paths of length l for every $3 \leq l \leq 2r$. If $\delta > 129t^3 \log \Delta$ and Δ is large enough, then there exists a subgraph $H \subseteq G$ such that

    1. $\forall v \in V(G)$, $d_H(v) \geq \frac{d_G(v)}{4t}$, *and*

    2. $g(H) \geq 2r + 2$.

Proof. Let the color classes of χ be $S_1, S_2, \ldots$. Assign to each edge $e \in E(G)$ a random variable $f(e)$ uniformly distributed in $(0, 1)$. We construct the following subgraph H: for every pair of color classes (S_i, S_j) of χ, an edge e between S_i and S_j is retained in H if $f(e)$ is less than $f(e')$ for every e' between S_i and S_j incident with e. Observe that, by construction, χ is a 1-frugal coloring of H, that is, the vertices of any pair of color classes of χ induce a matching.

We claim that deterministically (i.e., with probability 1) $g(H) \geq 2r + 2$. For the sake of contradiction, suppose that $g(H) < 2r + 2$ and let $C = (u_1, \ldots, u_l)$ be a cycle in H with $l < 2r + 2$. Since χ does not induce any rainbow cycle of length at most l in G, there exist at least two vertices of C with the same color. Let $a, b \in [l]$ be such that $a < b$, $\chi(u_a) = \chi(u_b)$ and for every $a < j < b$, $\chi(u_a) \neq \chi(u_j)$. Then $3 \leq b - a \leq 2r$ since χ is a 1-frugal coloring in H and C is not rainbow. Then $u_a, \ldots, u_b$ is a maximal inner-rainbow path with a forbidden length, a contradiction.

Now, it remains to show that with positive probability the obtained subgraph H has the desired minimum degree.

Observe first that a given edge $e = (u, v)$, with $u \in S_i$ and $v \in S_j$, is preserved in H with probability

$$\frac{1}{d_G(u, S_j) + d_G(v, S_i) - 1} \geq \frac{1}{2t - 1},$$

where $d_G(w, S) = |N_G(w) \cap S|$.

For every $v \in V$ consider the random variable $L(v)$ equal to the degree of v in H. We have $\mathbb{E}(L(v)) \geq d_G(v)/2t$. By applying Azuma's inequality (Lemma 10) we will now show that with probability exponentially close to 1, $L(v)$ is large enough.

First of all, observe that $L(v)$ depends only on the edges that connect a neighbor of v to a vertex of color $\chi(v)$. Let T_v be the set of these edges:

$$T_v = \bigcup_{u \in N_G(v)} \{vu\} \cup \{uw : w \in N_G(u) \text{ and } \chi(v) = \chi(w)\} .$$

Since χ is t-frugal, we have $|T_v| \leq td_G(v)$. Let $S = (0, 1)$. Then $L(v)$ depends only on a vector in S^{T_v}.

If two functions $f, f' : S^{T_v} \to \mathbb{R}$ differ only on one edge of T_v, $|L(v)(f(T_v)) - L(v)(f'(T_v))| \leq 1$. Thus $L = L(v)$ satisfies the 1-Lipschitz condition. Let A_v be the event that $L(v) \leq d_G(v)/4t$. Since the martingale length is at most $td_G(v)$, by setting $\lambda = \frac{\sqrt{d_G(v)}}{4t^{3/2}}$ in Lemma 10,

$$\Pr(A_v) = \Pr\left(L(v) \leq \frac{d_G(v)}{4t}\right) = \Pr\left(L(v) \leq \mathbb{E}(L(v)) - \frac{d_G(v)}{4t}\right)$$

$$< e^{-\frac{d_G(v)}{32t^3}} < e^{-\frac{\delta}{32t^3}} .$$

Since A_v is influenced only by the edges in T_v, it is thus independent of each A_u unless u is at distance at most 4 from v, and there are at most $\Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 + \Delta(\Delta - 1)^3 \leq \Delta^4$ such events. Notice that

$$2\Pr(A_v)\Delta^4 < 2e^{-\frac{\delta}{32t^3}}\Delta^4 < \frac{1}{2}$$

since $\delta > 129t^3 \log \Delta$. Thus, by the Lovász local lemma (Lemma 11) with $p = \max_{v \in V} \Pr(A_v)$ and $w_i = 1$ we have

$$\Pr(\cap_{v \in V} \overline{A_v}) > 0 \ ,$$

and thus there is a way to assign values to $f(e)$ so that $\delta(H) \geq d_G(v)/4t$. □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. The idea in this proof is to randomly color the vertices from G with ℓ colors, where $\ell = |V(\mathcal{G})|$ for the graph $\mathcal{G}$ introduced above. We then consider the subgraph $H' = H'_{(\chi,\mathcal{G})}$ from Definition 7 induced by the coloring and the graph $\mathcal{G}$. We will show that with positive probability, such a coloring is t-frugal in H' and that H' contains neither rainbow cycles of length at most $2r$ nor maximal inner-rainbow paths of length $3 \leq l \leq 2r$. The value of t will be set later in the proof. Then, we will use Lemma 13 to obtain the desired subgraph.

Let χ be a uniformly random coloring of $V(G)$ with ℓ colors. Consider the $\mathcal{F}_r$-free graph $\mathcal{G}$ of order $\ell \geq 2e^4\Delta$ that satisfies $g(\mathcal{G}) \geq 2r + 2$ and $\delta(\mathcal{G}) \geq q = \frac{\mathrm{ex}(K_{2e^4\Delta}, \mathcal{F}_r)}{4e^4\Delta}$ provided by Observation 12. Then construct the spanning subgraph $H' = H'_{(\chi,\mathcal{G})}$ of G.

Since $g(\mathcal{G}) \geq 2r + 2$, χ does not induce any rainbow cycle of length at most $2r+1$ in H', nor any maximal inner-rainbow path of length at least 3 and at most $2r$. Moreover, χ is a proper coloring of H' since $\mathcal{G}$ has no loops.

We will use the Lovász local lemma to show that there is a positive probability that the random ℓ-coloring of G satisfies the following properties:

1. for every $v \in V(G)$, $d_{H'}(v) = \Omega(\frac{qd_G(v)}{\ell})$ and
2. χ is t-frugal in H'.

For this purpose we will define the following events:

1. Type A: for each $v \in V(G)$, A_v is the event $d_{H'}(v) \leq \frac{qd_G(v)}{2\ell}$,
2. Type B: for each vertex $v \in V(G)$ and each set $X = \{x_1, \ldots, x_{t+1}\} \subseteq N_G(v)$, $B_{v,X}$ is the event $\chi(x_i) = \chi(x_j)$ for every distinct $i, j \in [t+1]$.

Observe that the probability that an edge is retained in H' is at least q/ℓ. Thus, $\mathbb{E}(d_{H'}(v)) \geq qd_G(v)/\ell$. Using Part 1 of Lemma 9 with $\varepsilon = 1/2$, one can check that

$$\Pr(A_v) = \Pr\left(d_{H'}(v) \leq \frac{qd_G(v)}{2\ell}\right) \leq \Pr\left(d_{H'}(v) \leq \frac{1}{2}\mathbb{E}(d_{H'}(v))\right)$$

$$\leq e^{-\Omega(\mathbb{E}(d_{H'}(v)))} \leq e^{-\frac{c'qd_G(v)}{\ell}}$$

for some small enough constant $c' > 0$. We also have

$$\Pr(B_{v,X}) = \ell^{-t} \ .$$

Here, it is convenient to define the auxiliary event D_e as the event $e \in E(H')$. Observe that any event A_v or $B_{v,X}$ can be expressed in terms of the events D_e.

CLAIM 14. *Let $v \in V(G)$ and let $F \subset E(G)$ be a set of edges not incident to v. Then, for each $i \in [\ell]$,*

$$\Pr(\chi(v) = i \mid \cup_{f \in F} D_f) = \frac{1}{\ell} \ .$$

To prove the claim, observe that all the unveiled information is about nonincident edges and thus no information about the color of v has been provided. While

TABLE 1
Table of dependencies.

	Type A	Type B
Type A	$2\Delta^2$	$2\Delta^2\binom{\Delta}{t}$
Type B	$(t+1)\Delta$	$(t+1)\Delta\binom{\Delta}{t}$

information on the existence of the edges in F may affect the degree of v, it cannot affect its color.

By the previous claim, the color given to a vertex v depends only on the events that intersect this vertex. Thus, it depends on at most Δ events of Type A, precisely the events A_w where $vw \in E(G)$, and on at most $\Delta\binom{\Delta}{t}$ events of Type B, precisely the events $B_{w,X}$ where $vw \in E(G)$ and $v \in X$. Moreover, the existence of an edge $e = uv$ in H' depends only on the colors of u and v.

Since an event of Type A depends on the existence of at most Δ edges and an event of Type B depends on the colors given to a set of $t+1$ vertices, we have at most the number of dependencies given in Table 1.

Then, by applying the weighted version of the local lemma (Lemma 11) with $p = \ell^{-1}$, $w_i = \frac{c'q\delta}{\ell \log \ell}$ if the weight corresponds to an event of Type A, and $w_i = t$ if it corresponds to an event of Type B, we have that a subgraph H' avoiding all events exists if

$$2\Delta^2 e^{-\frac{c'q\delta}{\ell}\left(1-\frac{\log 2}{\log \ell}\right)} + 2\Delta^2\binom{\Delta}{t}(2p)^t \leq \frac{c'q\delta}{2\ell \log \ell} \text{ and}$$

$$(t+1)\Delta e^{-\frac{c'q\delta}{\ell}\left(1-\frac{\log 2}{\log \ell}\right)} + (t+1)\Delta\binom{\Delta}{t}(2p)^t \leq \frac{t}{2} \ .$$

Recall that by the hypothesis of the theorem, $\mathrm{ex}(K_\Delta, \mathcal{F}_r)\delta \geq \alpha\Delta^2 \log^4 \Delta$, which in particular implies $c'q\delta \geq \frac{c'\alpha}{4e^4}\Delta \log^4 \Delta \geq 4\ell \log \Delta$, if Δ is large enough. Set $t = \log \Delta$. Then $p^{-1} = \ell \geq 2e^4\Delta = 2\Delta^{1+\frac{4}{t}}$ and the previous inequalities are satisfied if Δ is large enough.

Therefore, there is a subgraph H' such that for every $v \in V$, $d_{H'}(v) \geq q\delta/2\ell$ and H' admits a $\log \Delta$-frugal coloring with ℓ colors, no rainbow cycle of length at most $2r+1$, and no maximal inner-rainbow path of length at least 3 and at most $2r$.

Let us check that the minimum degree of H' is large enough to apply Lemma 13. By using the facts that $q = \frac{\mathrm{ex}(K_k, \mathcal{F}_r)}{2k}$, where $k = 2e^4\Delta(G)$, that $\ell \in (k, 2k)$ and that $\mathrm{ex}(K_{\Delta(G)}, \mathcal{F}_r)\delta(G) \geq \alpha(\Delta(G))^2 \log^4 \Delta(G)$, we have

$$\delta(H') \geq \frac{q\delta(G)}{2\ell} \geq \frac{\mathrm{ex}(K_{2e^4\Delta(G)}, \mathcal{F}_r)\delta(G)}{8e^4\Delta(G)\ell} \geq \frac{\mathrm{ex}(K_{\Delta(G)}, \mathcal{F}_r)\delta(G)}{32e^8(\Delta(G))^2}$$

$$\geq \frac{\alpha \log^4 \Delta(G)}{32e^8} > 129t^3 \log \Delta(G) \geq 129t^3 \log \Delta(H') \ ,$$

provided that α is large enough.

Then Lemma 13 provides a spanning subgraph H of G with $g(H) \geq 2r+2$ and for every $v \in V$,

$$d_H(v) \geq \frac{q\delta/2\ell}{4t} \geq \frac{c \cdot \mathrm{ex}(K_\Delta, \mathcal{F}_r)\delta}{\Delta^2 \log \Delta} \ ,$$

for some small constant $c > 0$. ☐

The following proposition shows that Corollary 6 is tight up to a logarithmic factor.

PROPOSITION 15. *For every δ, Δ satisfying $\Delta \leq \delta^2 \leq \Delta^2$, there exists a graph G with minimum degree δ and maximum degree Δ such that for every spanning C_4-free subgraph H of G,*

$$\delta(H) = O\left(\frac{\delta}{\sqrt{\Delta}}\right) .$$

Proof. Let G be the complete bipartite graph with parts A, B of sizes Δ and δ, respectively. For the sake of contradiction, suppose that there exists a C_4-free subgraph H such that $\delta(H) > 2\delta/\sqrt{\Delta}$. We call a pair of edges incident to a common vertex v a *cherry* of v. We will get a contradiction by double counting the number of cherries of vertices of A. On the one hand, since $\delta(H) > 2\delta/\sqrt{\Delta}$, there are at least $\Delta \cdot \binom{2\delta/\sqrt{\Delta}}{2}$ cherries of vertices of A.

On the other hand, since H is C_4-free, each pair of vertices in B has at most one common neighbor in A having a cherry; thus there are at most $\binom{\delta}{2}$ cherries of vertices of A, and we have

$$2\delta^2 - \delta\sqrt{\Delta} = \Delta\binom{\frac{2\delta}{\sqrt{\Delta}}}{2} \leq \binom{\delta}{2} = \frac{\delta^2 - \delta}{2} ,$$

providing a contradiction.   □

5. Remarks and open questions.

1. There are still logarithmic gaps between the lower bounds (Theorem 2 and Corollary 6) and the upper bounds for f and h. We conjecture that the upper bounds are asymptotically tight.

2. In order to give a more explicit result in Theorem 2 it is interesting to determine the value $k^* = k^*(m, \mathcal{F}_r^{\text{even}})$ that minimizes $\text{ex}(K_{k,m/k}, \mathcal{F}_r^{\text{even}})$. It is clear that for every $r \geq 2$, $k^* = \Omega(m^{1/3})$ and $k^* = O(m^{2/3})$. Indeed, any extremal bipartite $\mathcal{F}$-free graph has at least as many edges as the size of the largest stable set which is $\Omega(m^{2/3})$ in both previous cases.

   In the proof of Corollary 3, we showed that $k^*(m, C_4) = \Theta(m^{1/3})$.

   Observe that

$$\lim_{r \to \infty} k^*(m, \mathcal{F}_r) = m^{1/2}.$$

   When r tends to infinity, $\mathcal{F}_r$ is composed of all cycles of length up to $2r+1$, and thus, the extremal graph becomes more and more tree-like. The minimum over k of the extremal function $ex(K_{k,m/k}, \mathcal{F}_\infty)$ (i.e., when forbidding all cycles) is obtained when both stable sets are approximately of the same size. Thus, we get that for every $r \geq 2$,

$$f(m, r) = \Theta(f(m, \mathcal{F}_r)) = \Omega\left(\frac{m^{1/2}}{\log m}\right) .$$

   However, this is meaningless since it is clear that any graph G with m edges has a spanning forest with at least $\Omega(m^{1/2})$ edges: such a G contains a star with $\sqrt{m}$ edges, or a matching of size $\Omega(m^{1/2})$.

3. Recently, Conlon, Fox, and Sudakov showed in [5] that for every $r \geq 2$, $f(m, K_{r,r}) = \Theta(m^{r/r+1})$. For $r = 2$, this result improves Corollary 3 by a logarithmic factor. It is worth mentioning that they were able to determine $f(m, K_{r,r})$ even though the Turán numbers for $K_{r,r}$ are not known.

4. Regarding the function $h(\delta, \Delta, \mathcal{F})$, we conjecture that the following holds.

CONJECTURE 16. *For every δ, Δ and every family $\mathcal{F}$, we have*

$$h(\delta, \Delta, \mathcal{F}) = \Omega\left(\frac{ex(K_\Delta, \mathcal{F})\delta}{\Delta^2}\right) .$$

Proposition 1 shows that this conjecture is true for $\mathcal{F}$ not containing any bipartite graph. The bipartite case remains wide open. Our results imply that, for any $r \geq 2$, Conjecture 16 is true up to a logarithmic factor when $\mathcal{F}$ contains only the bipartite graphs in $\mathcal{F}_r^{\text{even}}$—if the maximum degree is not too large with respect to the minimum degree. In particular, it would be interesting to set the conjecture for other important families of bipartite graphs, such as complete bipartite graphs.

REFERENCES

[1] N. ALON AND J. H. SPENCER, *The Probabilistic Method*, 3rd ed., Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley & Sons, New York, 2008.

[2] B. BOLLOBÁS, *Extremal Graph Theory*, Academic Press, New York, 1978.

[3] J. A. BONDY AND M. SIMONOVITS, *Cycles of even length in graphs*, J. Combin. Theory Ser. B, 16 (1974), pp. 97–105.

[4] W. G. BROWN, *On graphs that do not contain a Thomsen graph*, Canad. Math. Bull., 9 (1966), pp. 1–2.

[5] D. CONLON, J. FOX, AND B. SUDAKOV, *Large Subgraphs Without Complete Bipartite Graphs*, preprint, arXiv:1401.6711, 2014.

[6] D. DE CAEN AND L. A. SZÉKELY, *The maximum size of 4- and 6-cycle free bipartite graphs on m, n vertices*, in Sets, Graphs and Numbers, Colloq. Math. Soc. János Bolyai 60, North-Holland, Amsterdam, 1992, pp. 135–142.

[7] P. ERDŐS AND A. RÉNYI, *On a problem in the theory of graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl., 7 (1962), pp. 623–641.

[8] P. ERDŐS, A. RÉNYI, AND V. T. SÓS, *On a problem of graph theory*, Studia Sci. Math. Hungar., 1 (1966), pp. 215–235.

[9] P. ERDŐS AND M. SIMONOVITS, *A limit theorem in graph theory*, Studia Sci. Math. Hungar., 1 (1966), pp. 51–57.

[10] P. ERDŐS AND A. H. STONE, *On the structure of linear graphs*, Bull. Amer. Math. Soc., 52 (1946), pp. 1087–1091.

[11] Z. FÜREDI AND M. SIMONOVITS, *The History of Degenerate (Bipartite) Extremal Graph Problems*, arXiv:1306.5167, 2013.

[12] E. GYŐRI, *C_6-free bipartite graphs and product representation of squares*, Discrete Math., 165 (1997), pp. 371–375.

[13] H. HIND, M. MOLLOY, AND B. REED, *Colouring a graph frugally*, Combinatorica, 17 (1997), pp. 469–482.

[14] T. KOVÁRI, V. SÓS, AND P. TURÁN, *On a problem of K. Zarankiewicz*, Colloq. Math., 3 (1954), pp. 50–57.

[15] D. KÜHN AND D. OSTHUS, *Every graph of sufficiently large average degree contains a C_4-free subgraph of large average degree*, Combinatorica, 24 (2004), pp. 155–162.

[16] T. LAM AND J. VERSTRAËTE, *A note on graphs without short even cycles*, Electron. J. Combin., 12 (2005), p. N5.

[17] F. LAZEBNIK, V. A. USTIMENKO, AND A. J. WOLDAR, *A new series of dense graphs of high girth*, Bull. Amer. Math. Soc., 32 (1995), pp. 73–79.

[18] F. LAZEBNIK, V. A. USTIMENKO, AND A. J. WOLDAR, *Polarities and* $2k$*-cycle-free graphs*, Discrete Math., 197 (1999), pp. 503–513.

[19] M. MOLLOY AND B. REED, *Graph Colouring and the Probabilistic Method*, Springer, New York, 2002.

[20] A. NAOR AND J. VERSTRAËTE, *A note on bipartite graphs without* $2k$*-cycles*, Combin. Probab. Comput., 14 (2005), pp. 845–849.

[21] I. REIMAN, *Über ein problem von K. Zarankiewicz*, Acta Math. Hungar., 9 (1958), pp. 269–273.

[22] C. THOMASSEN, *Girth in graphs*, J. Combin. Theory Ser. B, 35 (1983), pp. 129–141.