Homomorphism bounds for K_4-minor-free graphs

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joint work with

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Homomorphisms

Graph homomorphisms

Definition - Graph homomorphism of G to H

Mapping $h : V(G) \to V(H)$ which **preserves adjacency**:

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Notation: $G \to H$.

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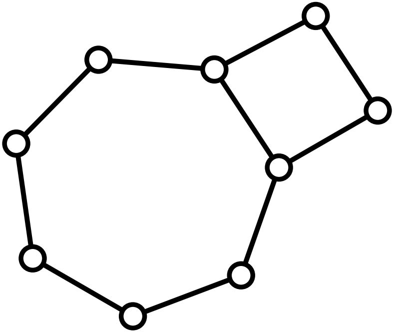
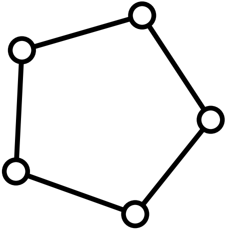


Target graph: $H = C_5$



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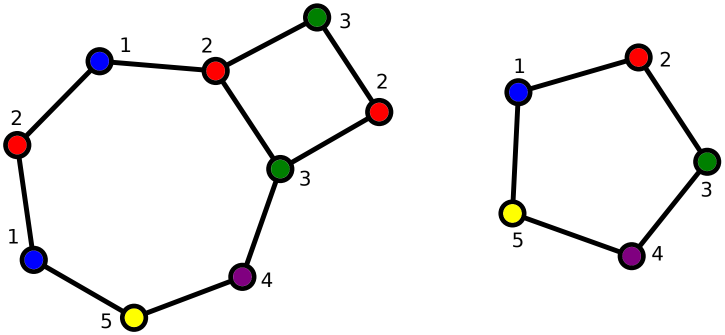

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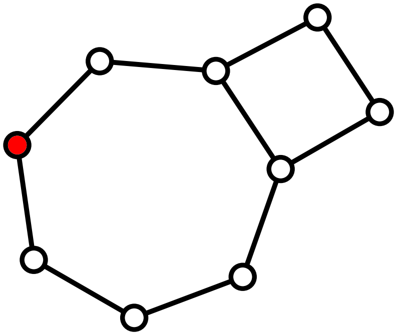

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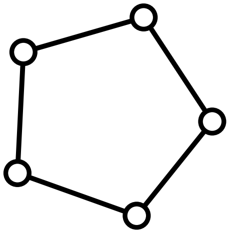

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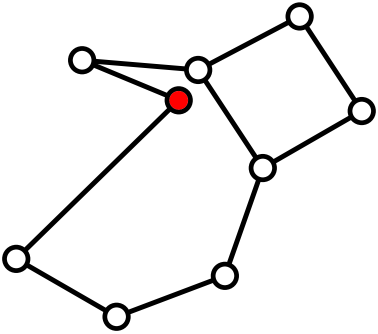

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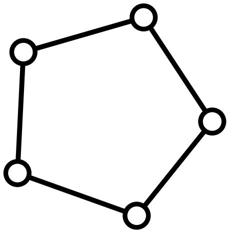

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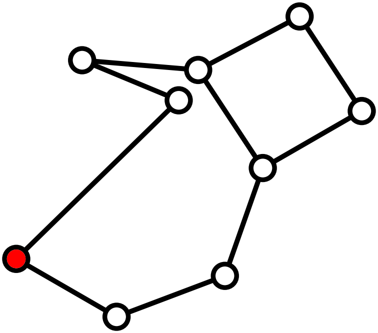

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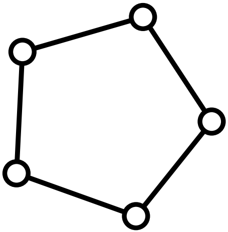

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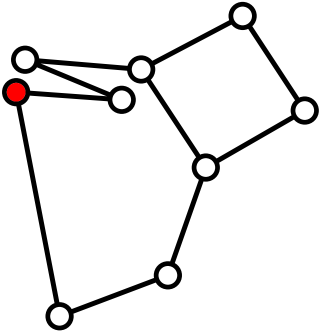

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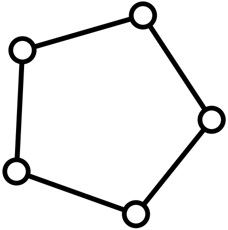

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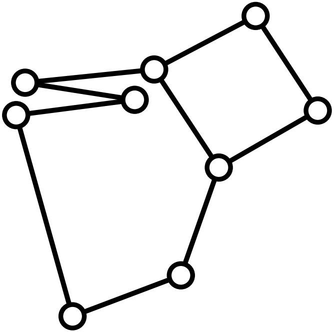

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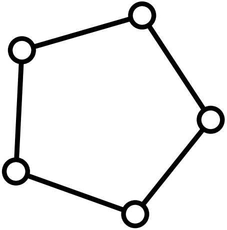

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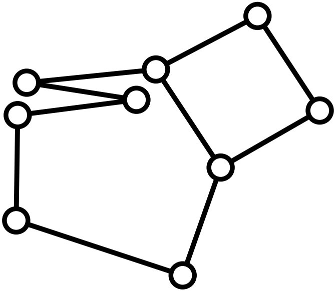

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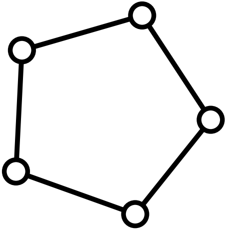

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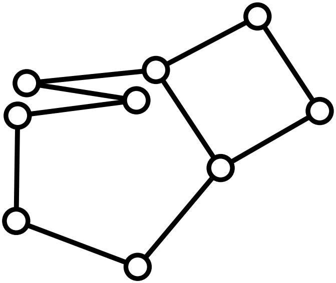

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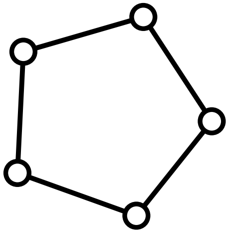

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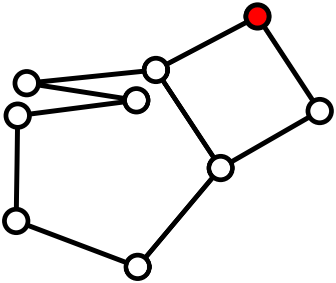

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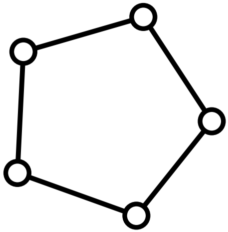

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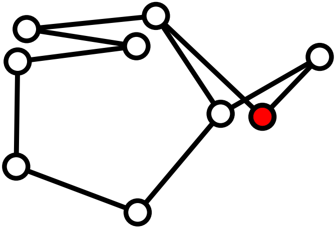

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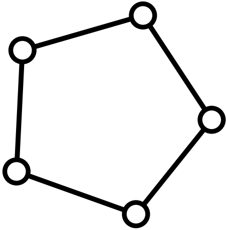

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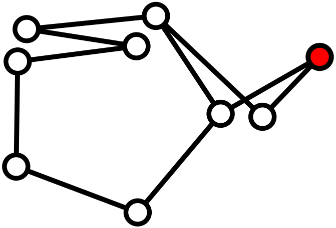

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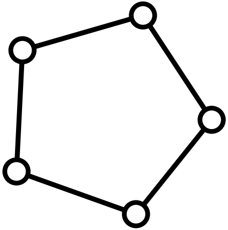

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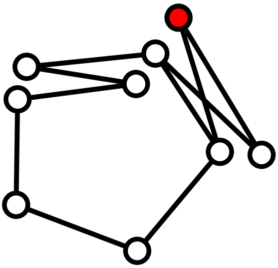

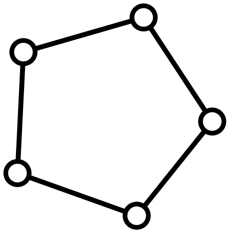

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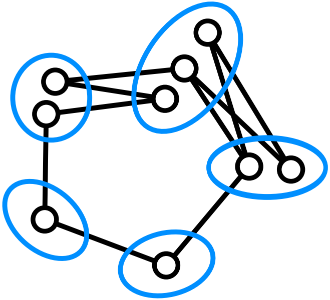

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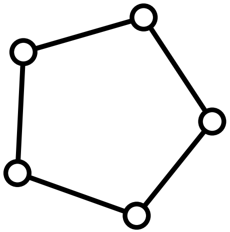

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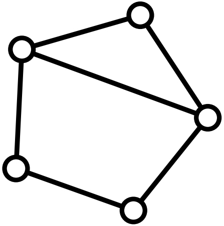

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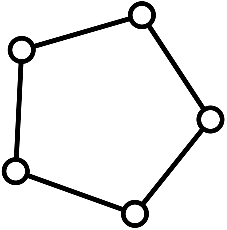

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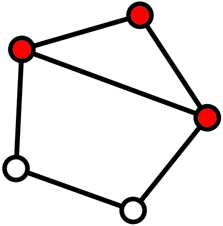

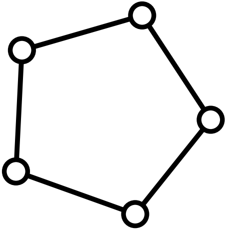

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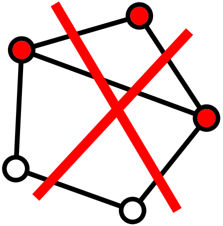

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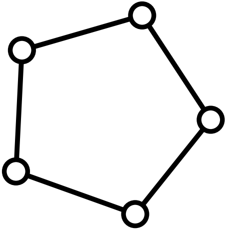

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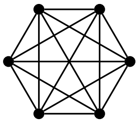


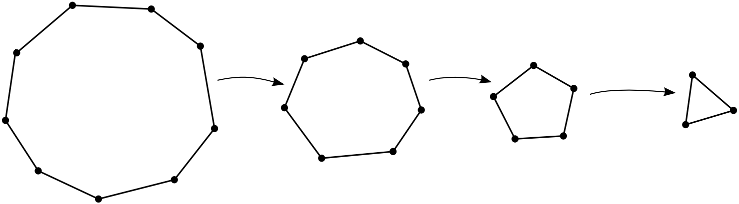
Complete graph K_6

Remark: Homomorphisms generalize proper colourings

$$G \to K_k \text{ if and only if } \chi(G) \le k$$

Proposition

$C_{2k+1} \to C_{2\ell+1}$ if and only if $\ell \leq k$



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$G \to H$ if and only if $core(G) \to core(H)$

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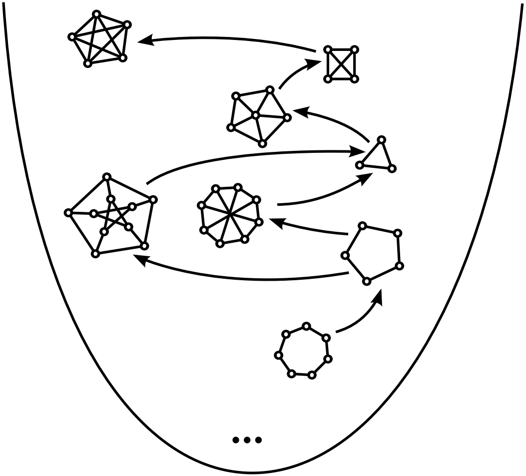
Definition - Homomorphism quasi-order

Defined by $G \preceq H$ iff $G \to H$ (if restricted to cores: partial order).

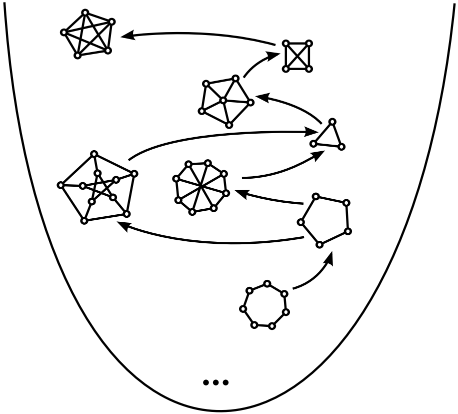


- reflexive
- transitive
- antisymmetric (cores)

Bounds

Definition - Bound in the order

Graph B is a **bound** for graph class $\mathscr{C}$ if for each $G \in \mathscr{C}$, $G \to B$.



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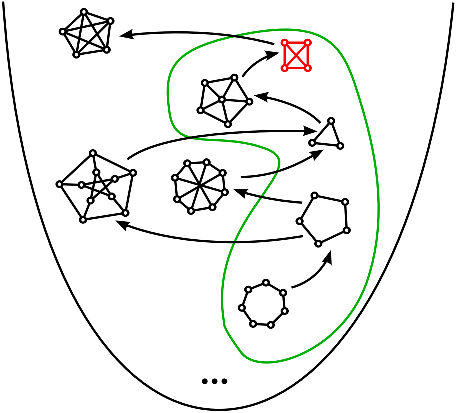

K_4: bound for planar graphs (4CT)

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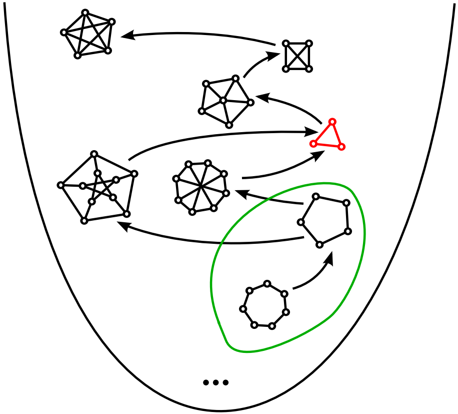

K_3: bound for planar triangle-free graphs (Grötzsch's theorem)

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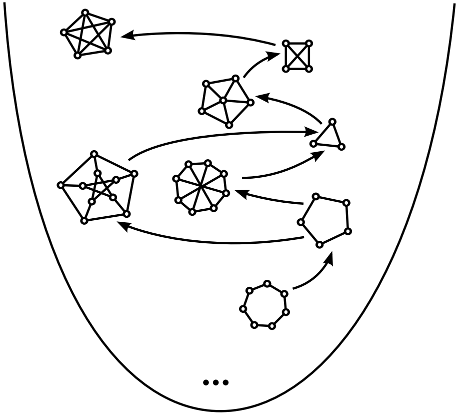

Question

Given graph class $\mathscr{C}$, is there a bound for $\mathscr{C}$ having specific properties?

Definition

$\mathscr{F}$: finite set of graphs. *Forb*($\mathscr{F}$): all graphs G s.t. for any $F \in \mathscr{F}$, $F \not\to G$.

Examples:

- *Forb*($\{K_\ell\}$): graphs with **clique number** at most $\ell - 1$
- *Forb*($\{C_{2k-1}\}$): graphs of **odd-girth** at least $2k+1$

(odd-girth: length of a smallest odd cycle)

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Minor of G: graph obtained by sequence of edge-contractions and deletions.

Classic minor-closed graph classes:
trees, planar graphs, bounded genus, classed defined by forbidden minor...

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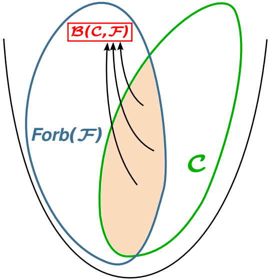
Theorem (Nešetřil and Ossona de Mendez, 2008)

For any minor-closed class $\mathscr{C}$ of graphs:
$\mathscr{C} \cap Forb(\mathscr{F})$ is bounded by a finite graph $B(\mathscr{C},\mathscr{F})$ from $Forb(\mathscr{F})$.



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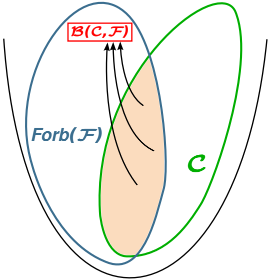

Proved using machinery of classes of bounded expansion

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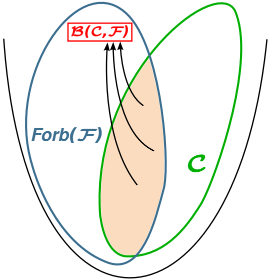

Example 1. $\mathscr{C}$: planar graphs
$\mathscr{F} = \{C_{2k-1}\}$

$\longrightarrow$ all planar graphs of odd-girth at least $2k+1$ map to some graph $B_{n,k}$ of odd-girth $2k+1$.

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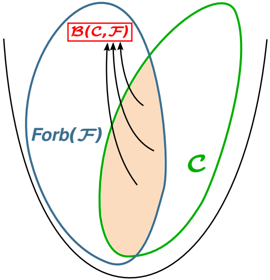

Example 2. $\mathscr{C}$: K_n-minor-free graphs
$\mathscr{F} = \{K_n\}$

$\longrightarrow$ all K_n-minor-free graphs map to some graph B_n of clique number $n-1$.

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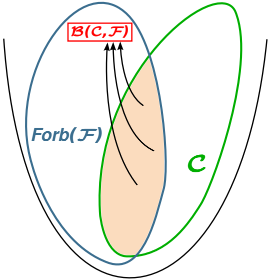

Note: there could be no bound in $\mathscr{C} \cap Forb(\mathscr{F})$ itself! (e.g. planar triangle-free graphs)

Definition

$\mathscr{F}$: finite set of graphs. $Forb(\mathscr{F})$: all graphs G s.t. for any $F \in \mathscr{F}$, $F \not\to G$.

Theorem (Nešetřil and Ossona de Mendez, 2008)

For any minor-closed class $\mathscr{C}$ of graphs:
$\mathscr{C} \cap Forb(\mathscr{F})$ is bounded by a finite graph $B(\mathscr{C}, \mathscr{F})$ from $Forb(\mathscr{F})$.

Question

What is a bound of smallest order?

Example: $\mathscr{C}$: K_n-minor-free graphs, $\mathscr{F} = \{K_n\}$

$\longrightarrow$Hadwiger's conjecture states that smallest B_n is K_{n-1}.

Projective cubes and planar graphs

Conjecture (Naserasr, 2007)

The class of planar graphs of odd-girth at least $2k+1$ is bounded by the projective cube $PC(2k)$.

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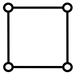
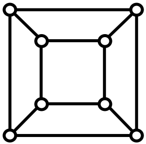


$H(2)$



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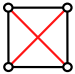
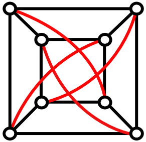


$PC(2) = K_4$



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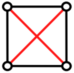

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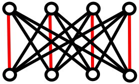

$PC(3) = K_{4,4}$

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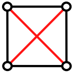

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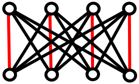

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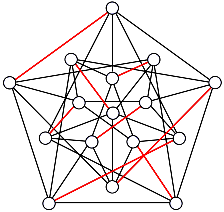

$PC(4)$: Clebsch graph
a.k.a Greenwood-Gleason

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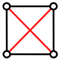

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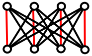

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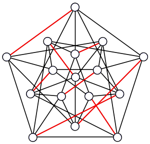

$PC(4)$: Clebsch graph

Remark

$PC(d)$ is distance-transitive: for any two pairs $\{x, y\}$, $\{u, v\}$ with $d(x, y) = d(u, v)$, there is an automorphism with $x \to u$ and $y \to v$

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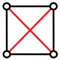

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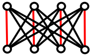

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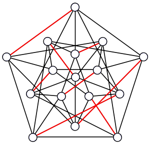

$PC(4)$: Clebsch graph

Remark

$d = 2k+1$ odd: $PC(2k+1)$ bipartite
$d = 2k$ even: $PC(2k)$ has odd-girth $2k+1$

Conjecture (Naserasr, 2007)

The class of planar graphs of odd-girth at least $2k+1$ is bounded by the projective cube $PC(2k)$.

Theorem (Naserasr, Sen, Sun, 2014)

If true, the conjecture is optimal: there is a planar graph of odd-girth $2k+1$ whose smallest image of odd-girth $2k+1$ has order 2^{2k}.

Proof idea: construct planar $(2k-1)$-walk-power clique of odd-girth $2k+1$

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Conjecture (Seymour, 1981)

Every planar r-graph is r-edge-colourable.

(r-graph: r-regular multigraph without odd ($< r$)-cut)
$\longrightarrow$ Proved up to $r = 8$.

Theorem (Naserasr, 2007)

Planar graphs of odd-girth at least $2k+1$ are bounded by $PC(2k)$ if and only if every planar $(2k+1)$-graph is $(2k+1)$-edge-colourable.

Conjecture (Naserasr, 2007)

The class of planar graphs (also, K_5-minor-free graphs) of odd-girth at least $2k+1$ is bounded by the projective cube $PC(2k)$.

odd girth

forbidden minor	3	5	7	9	
K_4	K_3				
K_5	K_4=PC(2)	PC(4)	PC(6)	PC(8)?	...
K_6	K_5				
K_7	K_6?				
	...				

Naserasr's conjecture

Hadwiger's conjecture

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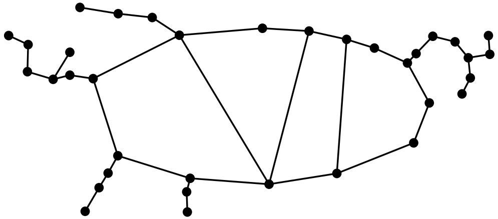
Naserasr's conjecture

Hadwiger's conjecture

Outerplanar graphs

Outerplanar graph: Planar graphs with all vertices on the outer face

$\longrightarrow$ Exactly the class of $\{K_4, K_{2,3}\}$-minor-free graphs.



Theorem (Gerards, 1988)

The class of outerplanar graphs of odd-girth at least $2k+1$ is bounded by the cycle C_{2k+1}.

K_4-minor-free graphs

Question

What is an optimal bound of odd-girth $2k+1$ for K_4-minor-free graphs of odd-girth at least $2k+1$?

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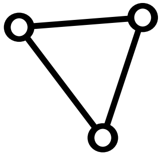
Proposition

A graph is K_4-minor free if and only if it is a partial 2-tree.



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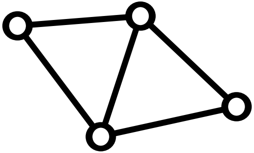

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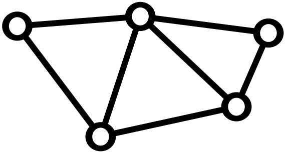

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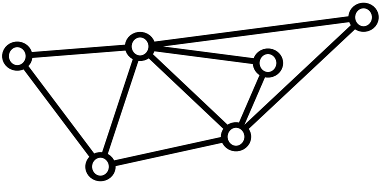

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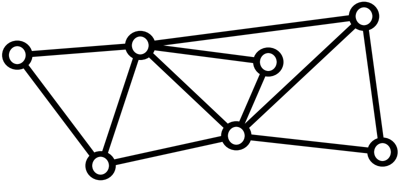

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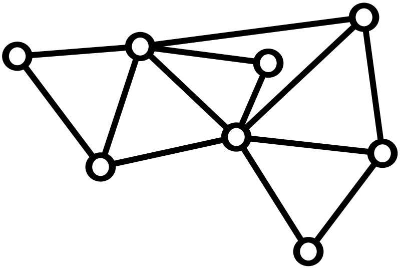

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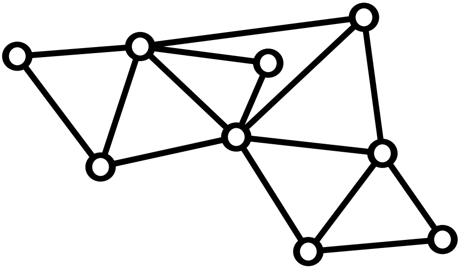

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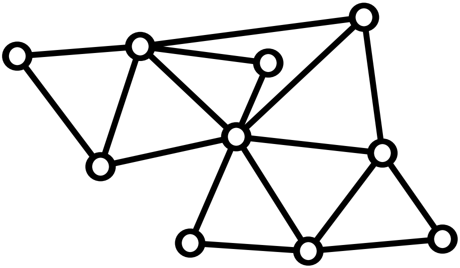

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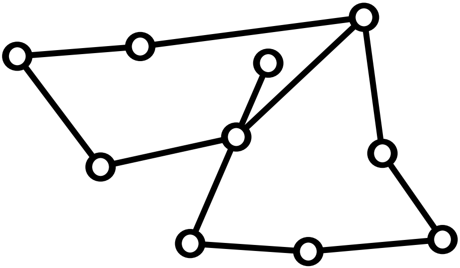

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K_4-minor-free graphs are 2-degenerate $\Longrightarrow$ 3-colourable.

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K_4-minor-free graphs: almost equivalent to series-parallel graphs.

Circular chromatic number

Definition - $\frac{p}{q}$-colouring of G

Mapping $c : V(G) \to \{1,\dots,p\}$ s.t. $xy \in E(G) \Rightarrow q \leq |c(x) - c(y)| \leq p - q$.

Circular chromatic number: $\chi_c(G) = \inf\{\frac{p}{q} \mid G \text{ is } \frac{p}{q}\text{-colourable}\}$

Remark

- Equivalently, homomorphism to circular clique $K(p/q)$
- $\frac{2k+1}{k}$-colouring $\iff$ homomorphism to C_{2k+1}
- Refinement of chromatic number: $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$

Theorem (Hell & Zhu, 2000 + Pan & Zhu, 2002)

If G K_4-minor-free and triangle-free, $\chi_c(G) \leq \frac{8}{3}$.

If moreover G has odd-girth at least 7, $\chi_c(G) \leq \frac{5}{2}$.

General bounds for K_4-minor-free graphs

Conjecture (Naserasr, 2007)

The class of planar graphs of odd-girth at least $2k+1$ is bounded by the projective cube $PC(2k)$.

Theorem (Beaudou, F., Naserasr)

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Corollary

Every K_4-minor-free $(2k+1)$-graph is $(2k+1)$-edge-colourable.

$\longrightarrow$ A more general result already proved by Seymour (1990)

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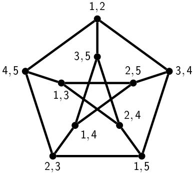
The class of planar graphs of odd-girth at least $2k+1$ is bounded by the projective cube $PC(2k)$.

Theorem (Beaudou, F., Naserasr)

The Kneser graph ("odd graph") $Kn(2k+1,k) \subset PC(2k)$ is a bound for K_4-minor-free graphs of odd-girth at least $2k+1$.

Kneser graph $Kn(a,b)$:
vertices are b-subsets of $\{1,\ldots,a\}$
adjacent if and only if disjoint.
Example: $Kn(5,2) =$ Petersen graph.



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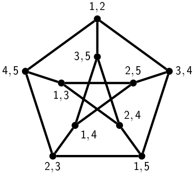

Corollary

K_4-minor-free graphs of odd-girth at least $2k+1$ have fractional chromatic number at most $2+\frac{1}{k}$.

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Theorem (Beaudou, F., Naserasr)

The $2k \times 2k$ projective toroidal grid $PTG_{2k,2k} \subset PC(2k)$ is a bound for K_4-minor-free graphs of odd-girth at least $2k+1$.

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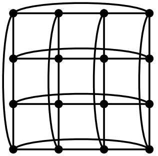

$k = 2$: $PTG_{4,4}$

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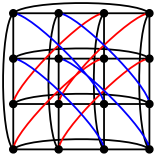

$k = 2$: $PTG_{4,4} = PC(4)$
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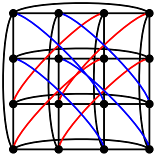

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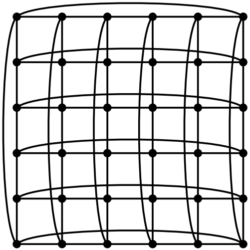

$k=3$: $PTG_{6,6}$

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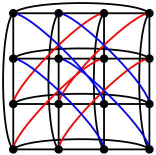

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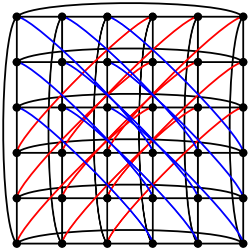

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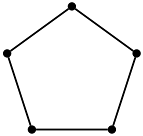

$k=2$: $M_1(C_5)$

Conjecture (Naserasr, 2007)

The class of planar graphs of odd-girth at least $2k+1$ is bounded by the projective cube $PC(2k)$.

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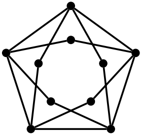

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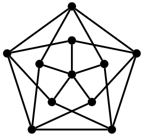

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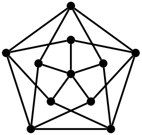

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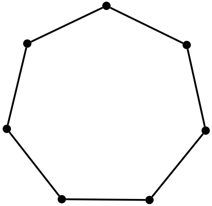

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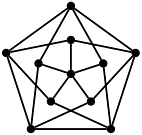

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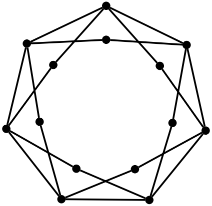

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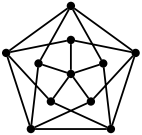

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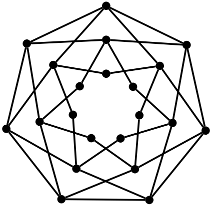

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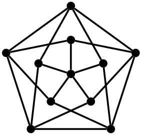

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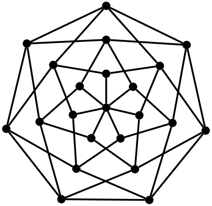

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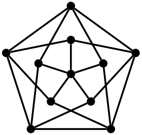

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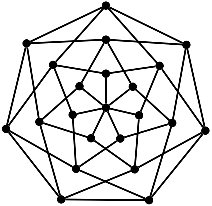

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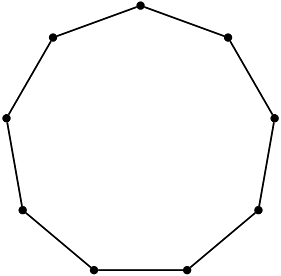

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K_4-minor-free graphs and projective cubes

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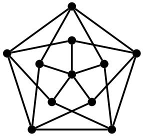

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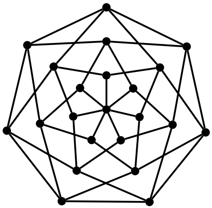

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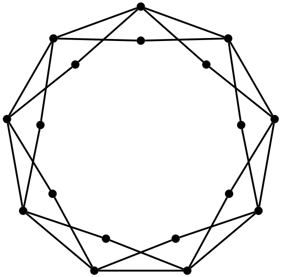

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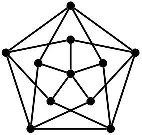

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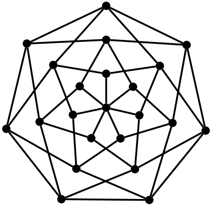

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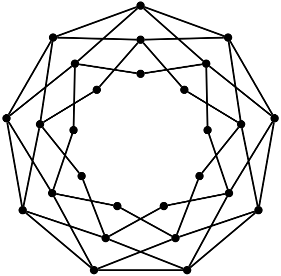

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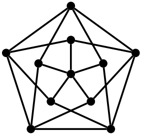

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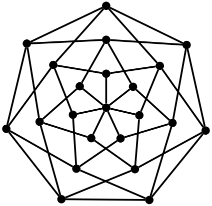

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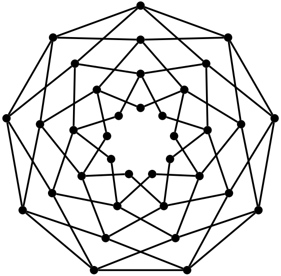

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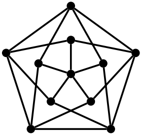

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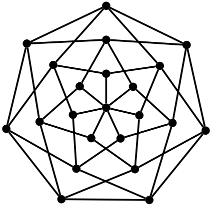

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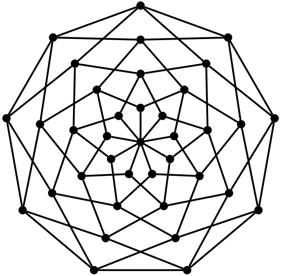

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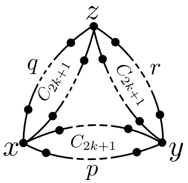
Let $1 \leq p, q, r \leq k$.

Graph $T_{2k+1}(p, q, r)$:



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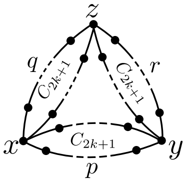

Definition

• Let $G \subseteq \widetilde{G}$. Partial distance (weighted) graph $(\widetilde{G}, d_G)$ of G: weighted extension of G (weights are distances in G).

• $(\widetilde{G}, d_G)$ is k-good if:

– For every $1 \leq p \leq k$, $\widetilde{G}$ has an edge of weight p

– For each edge uv of weight p and every q, r s.t. $T_{2k+1}(p,q,r)$ has odd-girth at least $2k+1$, there is $w \in V(G)$ with uw, vw in $E(\widetilde{G})$ and $d_G(uw) = q$, $d_G(vw) = r$.

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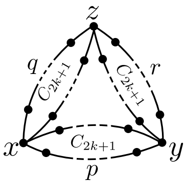

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Theorem (Beaudou, F., Naserasr)

B: graph with odd-girth $2k+1$.

- If B has a k-good $(\widetilde{B}, d_B)$, then B bounds the K_4-minor-free graphs of odd-girth at least $2k+1$.
- If B is a minimal such bound, then B has a k-good $(\widetilde{B}, d_B)$.

Corollary

Given a graph B of odd-girth $2k+1$, one can test in time polynomial in B whether B bounds all K_4-minor-free graphs of odd-girth at least $2k+1$.

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Question

Given a graph B of odd-girth $2k+1$, is there a finite time algorithm to decide whether B bounds all planar graphs of odd-girth at least $2k+1$?

Theorem (Beaudou, F., Naserasr)

The complete distance graphs of $PC(2k)$, $Kn(2k+1,k)$ and $PTG_{2k,2k}$ have the k-good property.

$PC(2k)$ has order 2^{2k}

$Kn(2k+1,k)$ has order $\binom{2k+1}{k} < 2^{2k}/2$

$PTG(2k,2k)$ has order $4k^2$

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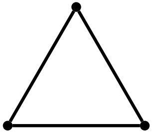
Question

Are these bounds optimal?

Bounds for small odd-girth

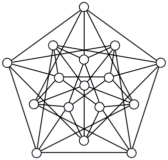
Proposition

K_4-minor-free graphs are 3-colourable: optimal bound is K_3

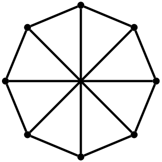


Odd-girth 5

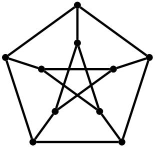
Odd-girth 5 (i.e. triangle-free): $PC(4)$, $K(8/3)$, $Kn(5,2)$, $M_1(C_5)$ are bounds.



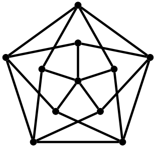
Clebsch graph $PC(4)$



Wagner graph $K(8/3)$



Petersen graph $Kn(5,2)$



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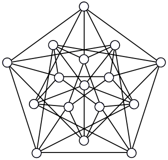
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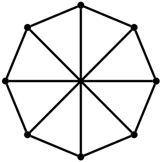
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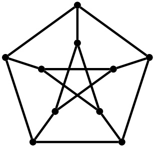
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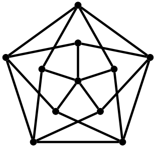
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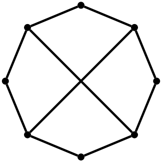
Theorem (Beaudou, F., Naserasr)

C_8^{++} is the smallest triangle-free bound for K_4-minor-free triangle-free graphs. It is unique.



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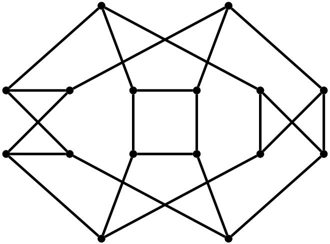
Theorem (Beaudou, F., Naserasr)

The graph below (order 16) is a bound for K_4-minor-free graphs of odd-girth at least 7.



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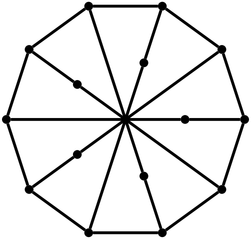

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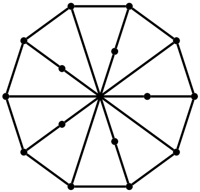
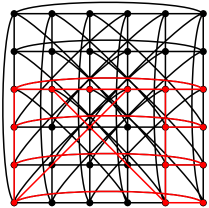
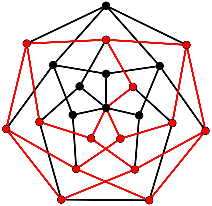
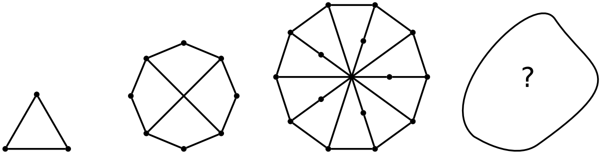
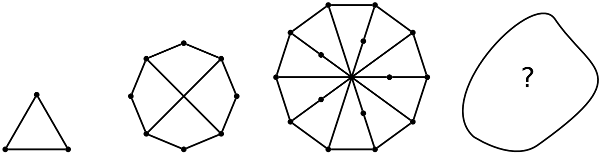








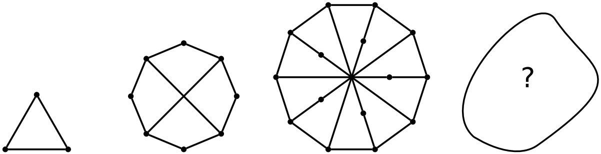
odd-girth 3 odd-girth 5 odd-girth 7 odd-girth 9



odd-girth 3 odd-girth 5 odd-girth 7 odd-girth 9

Question

Is the optimal bound of odd-girth 9 a common subgraph of $K(9,4)$, $M_3(C_9)$ and $TPG(8,8)$?



odd-girth 3 odd-girth 5 odd-girth 7 odd-girth 9

Question

Is the optimal bound of odd-girth 9 a common subgraph of $K(9,4)$, $M_3(C_9)$ and $TPG(8,8)$?

THE END