The complexity of homomorphisms of signed graphs

Florent Foucaud
(Postdoc at Universitat Politècnica de Catalunya, Barcelona)

joint work with Reza Naserasr
(CR CNRS at LRI, Orsay)

GT Graphes + Séminaire CPU-WP6

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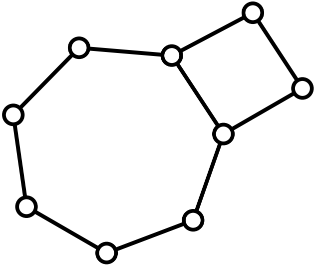

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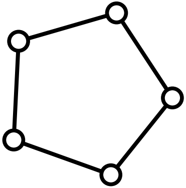

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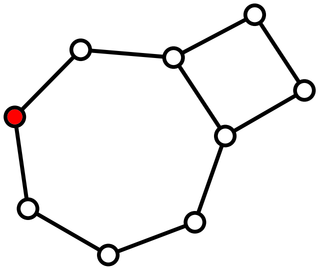

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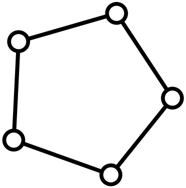

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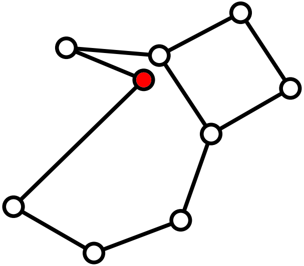

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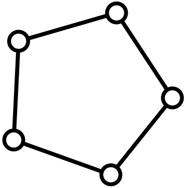

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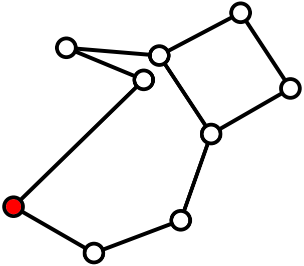

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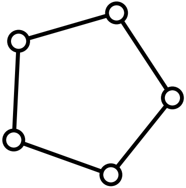

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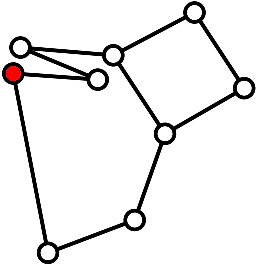

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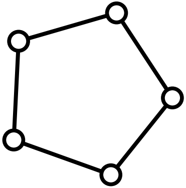

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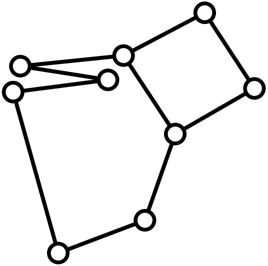

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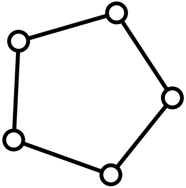

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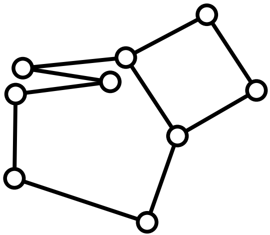

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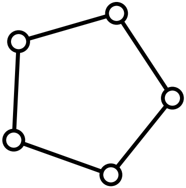

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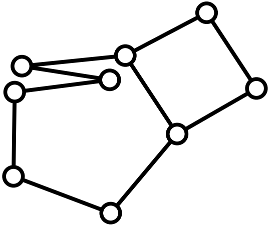

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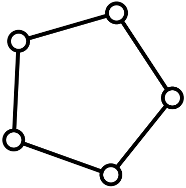

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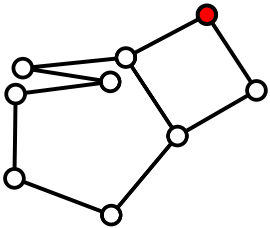

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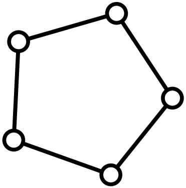

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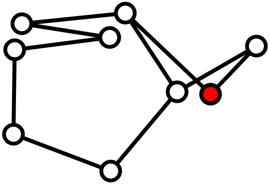

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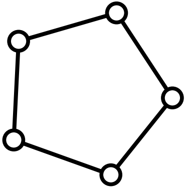

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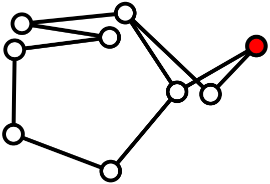

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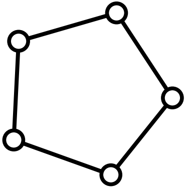

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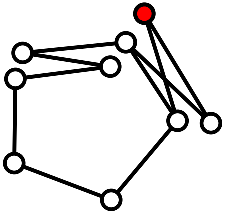

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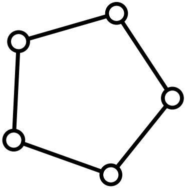

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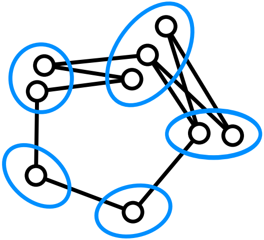

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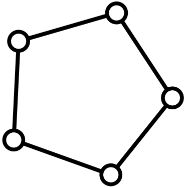

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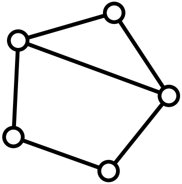

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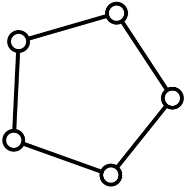

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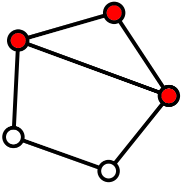

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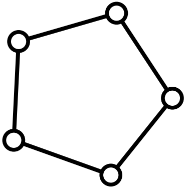

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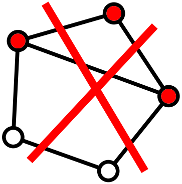

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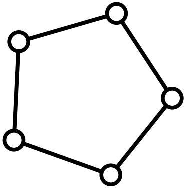

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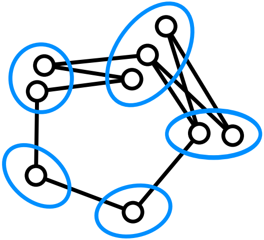

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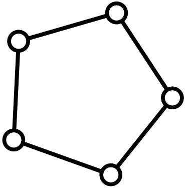

Observation

Equivalently: Sequence of **identifying non-adjacent vertices** of G to create subgraph of H.

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Remark: Homomorphisms generalize proper colourings

$$G \to K_k \iff \chi(G) \leq k$$

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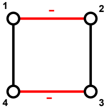

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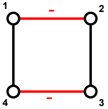

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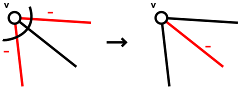
Re-signing operation at v: switch sign of each edge incident to v



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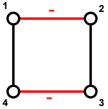

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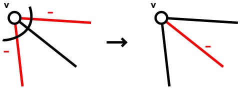

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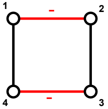

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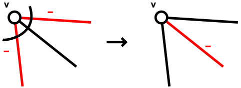

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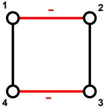

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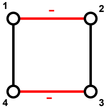

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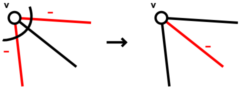

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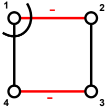

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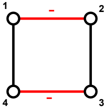

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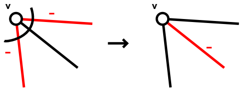

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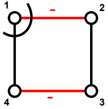

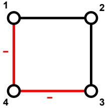

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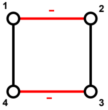

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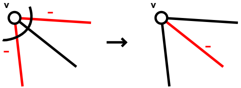

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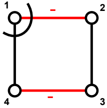

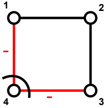

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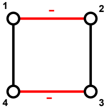

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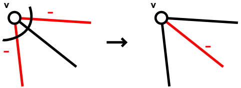

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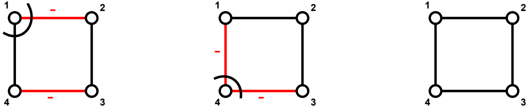

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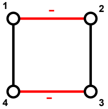

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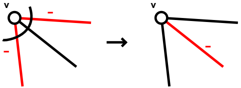

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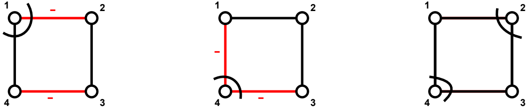

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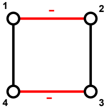

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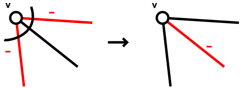

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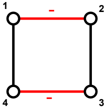

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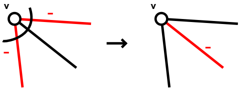

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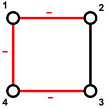

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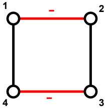

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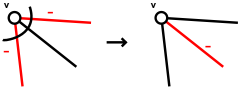

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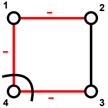

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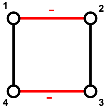

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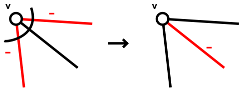

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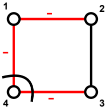

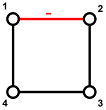

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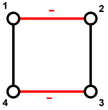

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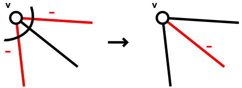

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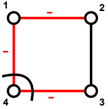

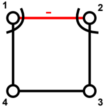

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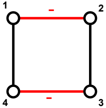

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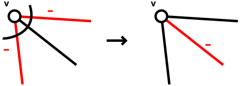

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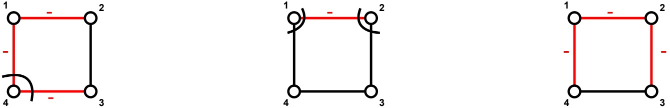

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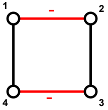

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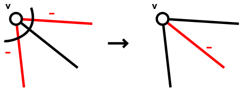

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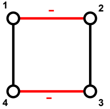
Signed graph: Graph G with an equivalence class $\mathcal{C}$ of signatures.

Notation: (G, Σ) with any $\Sigma \in \mathcal{C}$.

Signed graphs: (un)balanced cycles

Definition - Unbalanced cycle

Cycle with an odd number of negative edges.



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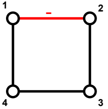

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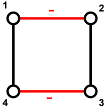

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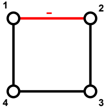

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Remark

Re-signing always preserves the balance of a cycle.

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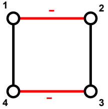

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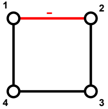

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Remark

Re-signing always preserves the balance of a cycle.

Theorem (Zaslavsky)

Two signatures are equivalent if and only if they induce the same set of unbalanced cycles.

Why signed graphs?

Introduced by Harary (1953): notion of **balanced** signed graphs (each cycle is balanced)

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Graph theory - minors and colourings: extension of Hadwiger's conjecture

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Conjecture (Hadwiger, 1943)

If G has no K_k as a minor, $\chi(G) \leq k-1$.

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- proof for $k = 5$: Wagner, 1937 with Appel and Haken, 4CT, 1976
- $\chi(G) = O(k\sqrt{\log k})$ (Kostochka, 1984; Thomason, 1984)
- proof for $k = 6$: Robertson, Seymour and Thomas, 1993

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If $(G, E(G))$ has no $(K_k, E(K_k))$ as a signed minor, $\chi(G) \leq k - 1$.

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• $k = 3$: $(K_3, E(K_3))$-minor-free graphs $(G, E(G))$ are exactly those where G is bipartite (to compare with K_3-minor free graphs)

• proof for $k = 4$: Catlin, 1979

Signed graph homomorphisms

Definition - Signed graph homomorphism from (G, Σ_G) to (H, Σ_H)

Homomorphism $f : G \rightarrow H$ s.t. it exists $\Sigma'_G \equiv \Sigma_G$:
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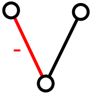

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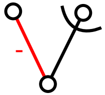

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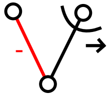

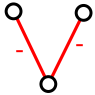

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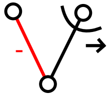

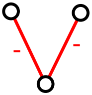

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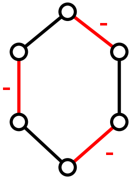

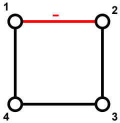

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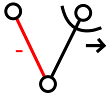

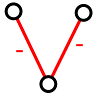

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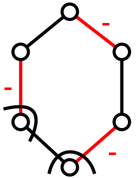

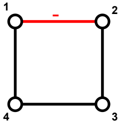

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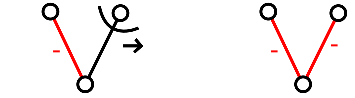

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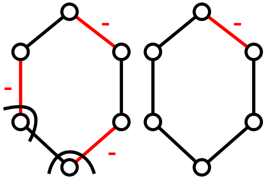

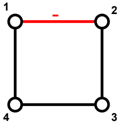

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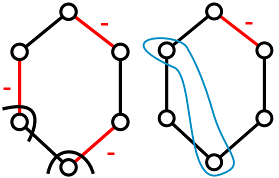

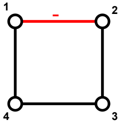

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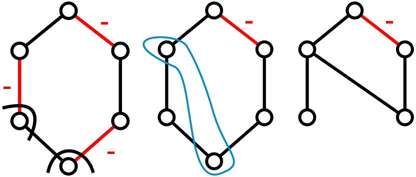

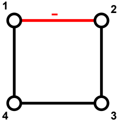

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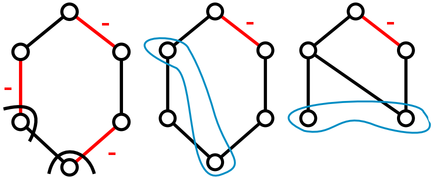

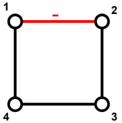

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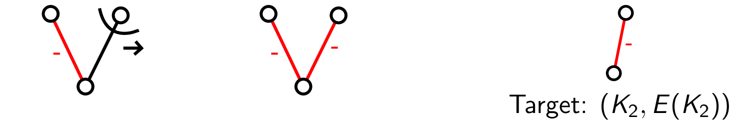

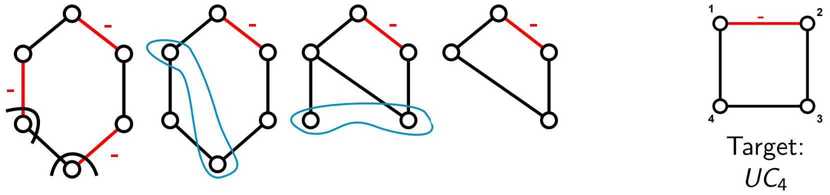

Remarks, concepts, known results (1)

Definition - Signed chromatic number

$\chi(G,\Sigma)$: smallest integer k s.t. $(G,\Sigma) \to (H,\Sigma_H)$ with $|V(H)| = k$

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Remark

$$G \to H \Leftrightarrow (G, \emptyset) \to (H, \emptyset) \Leftrightarrow (G, E(G)) \to (H, E(H)).$$

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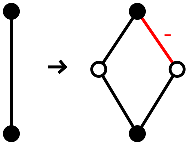
Theorem (Naserasr, Rollova and Sopena, 2012+)

(G,Σ) is an S-clique $\Leftrightarrow$ between each pair of vertices, there is an edge or an UC_4.

Remarks, concepts, known results (2)

Construction: $S(G)$



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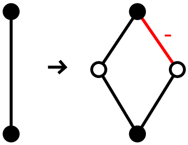

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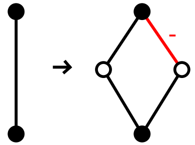

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Theorem (Naserasr, Rollova and Sopena, 2012+)

$G \rightarrow K_k \Leftrightarrow S(G) \rightarrow (K_{k,k}, M)$ with M a perfect matching.

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INSTANCE: A graph G.
QUESTION: does $G \to H$?

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Theorem (MacGillivray, Siggers, 2009 + Hell, Naserasr, Tardiff)

C_{2k+1}-COLOURING is NP-complete for every $k \geq 1$, even when the input graph is planar.

Complexity: questions for signed graphs

Definition - (H, Σ_H)-COLOURING

INSTANCE: A signed graph (G, Σ).
QUESTION: does $(G, \Sigma) \to (H, \Sigma_H)$?

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If $(H, \Sigma_H) = (H, \emptyset)$ or $(H, \Sigma_H) = (H, E(H))$, same complexity as H-Colouring.

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What can we say about the complexity of (H, Σ_H)-Colouring?
First interesting cases: S-cliques, bipartite unbalancd cycles,...

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Question

True that (H, Σ_H)-Colouring is NP-complete unless $\chi(H, \Sigma_H) \leq 2$?

Reduction from NAE-3SAT

Definition - UC_{2k}-COLOURING

INSTANCE: A (bipartite) signed graph (G, Σ).
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UC_{2k}-Colouring is NP-complete for every $k \geq 2$.

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UC_{2k}-Colouring is NP-complete for every $k \geq 2$.

Definition - Monotone Not-All-Equal-3SAT

INSTANCE: A set of clauses of 3 Boolean variables from set X.
QUESTION: Is there a truth assignment $X \to \{0, 1\}$ s.t. each clause has variables with different values?

Theorem (Schaefer, 1978; Moret, 1998)

Monotone Not-All-Equal-3SAT is NP-complete.

Polynomial-time reduction: principle

Polynomial-time reduction from SAT to H-Colouring

Procedure to build for each SAT-formula F a graph $G(F)$ s.t.

- it takes time polynomial in $|F|$
- F is satisfiable $\Leftrightarrow G(F) \to H$

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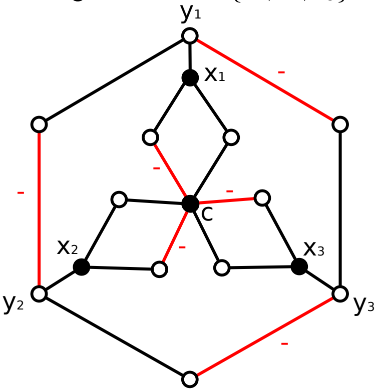
Consequence

If H-$\textsc{Colouring}$ is solvable in polynomial time:

1. given F, construct $G(F)$
2. solve H-$\textsc{Colouring}$ in time $poly(|G(F)|) = poly(|F|)$
3. Hence SAT is solvable in polynomial time: **Contradiction**

Gadget for clause $\{x_1, x_2, x_3\}$



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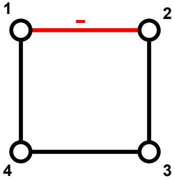
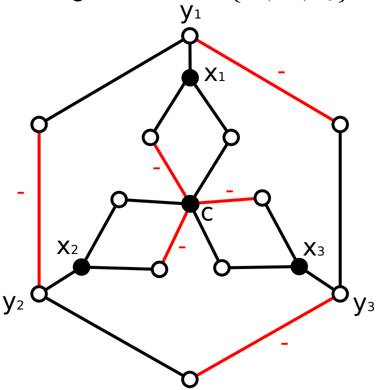


Construction of $G(F)$: one clause gadget per clause of F.
All vertices with same labels (c or x_i) identified with each other.

Gadget for clause $\{x_1, x_2, x_3\}$



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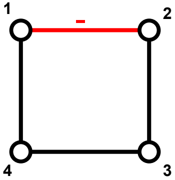

We want: Formula NAE-satisified $\Leftrightarrow$ $G(F)$ maps.

Main idea: In a mapping, re-signing at $x_i \iff x_i$=TRUE

NAE-3SAT $\leq_R$ UC_4-Colouring: clause gadget

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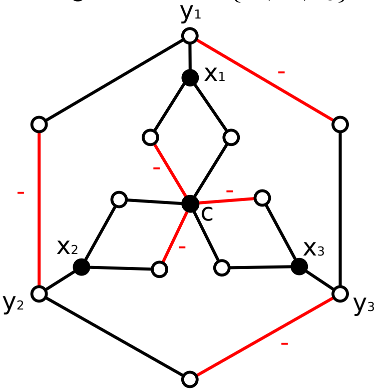

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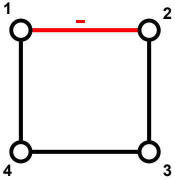

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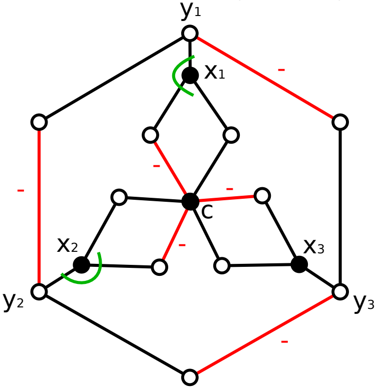

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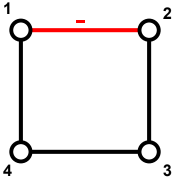

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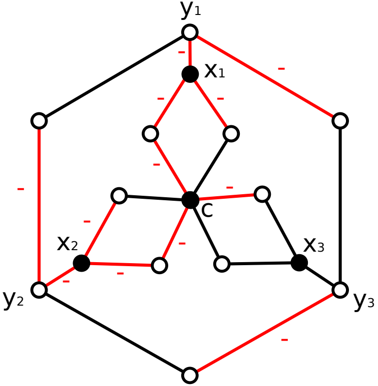

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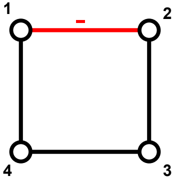

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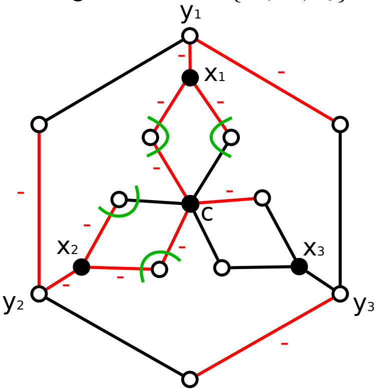

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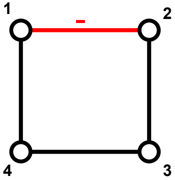

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NAE-3SAT $\leq_R$ UC_4-Colouring: clause gadget

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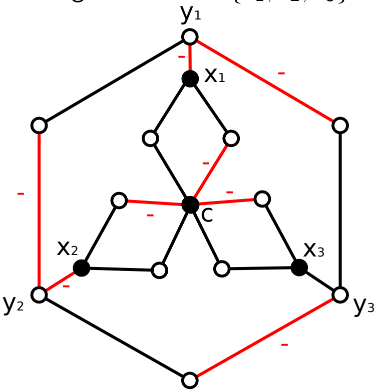

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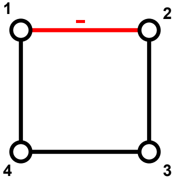

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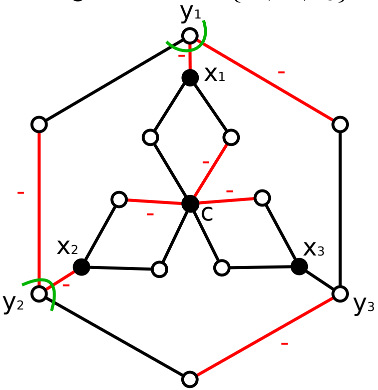

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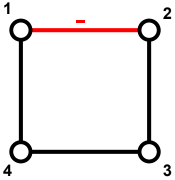

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NAE-3SAT $\leq_R$ UC_4-COLOURING: clause gadget

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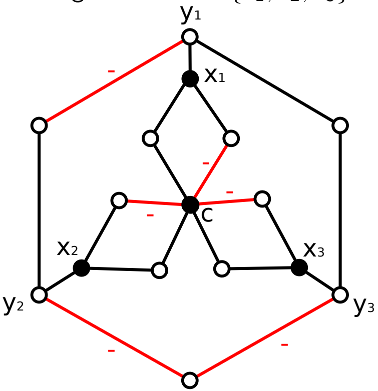

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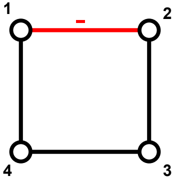

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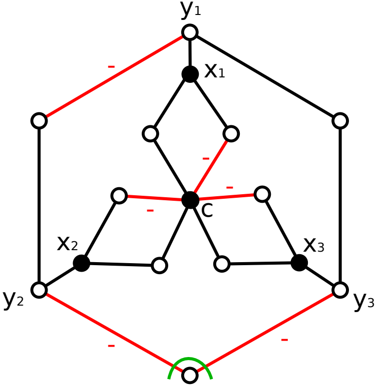

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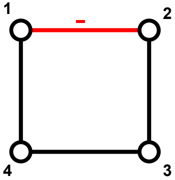

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NAE-3SAT $\leq_R$ UC_4-Colouring: clause gadget

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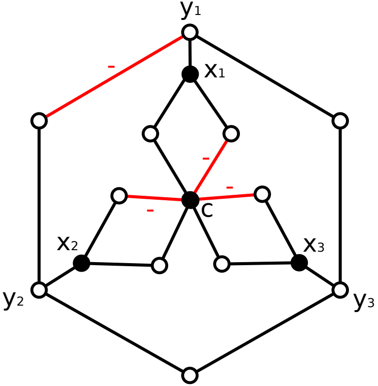

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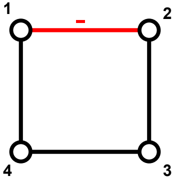

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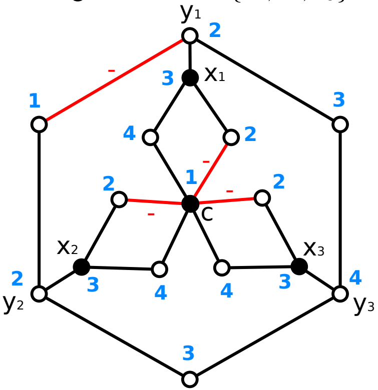

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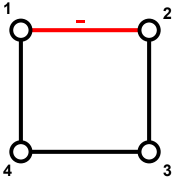

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Remark: all x_i's must map to 4

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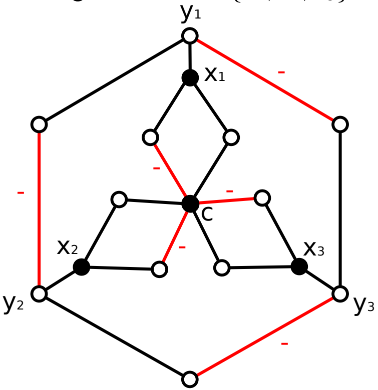

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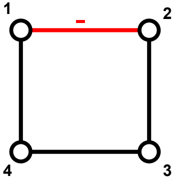

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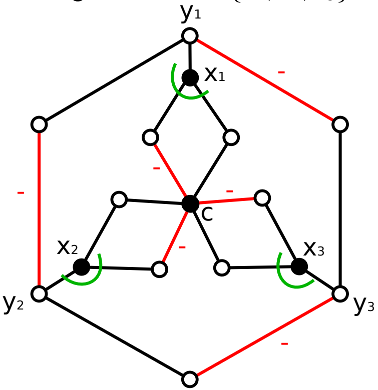

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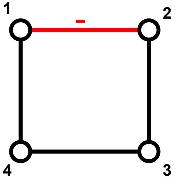

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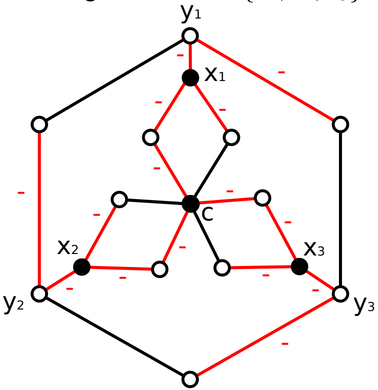

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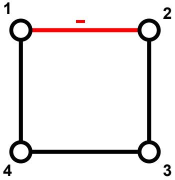

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NAE-3SAT $\leq_R$ UC_4-Colouring: clause gadget

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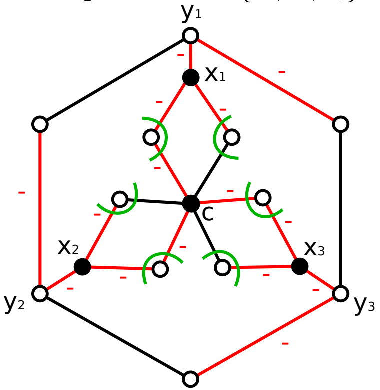

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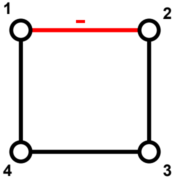

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NAE-3SAT $\leq_R$ UC_4-COLOURING: clause gadget

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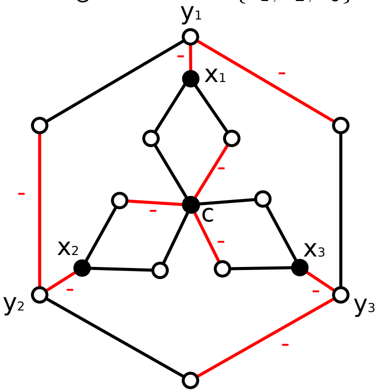

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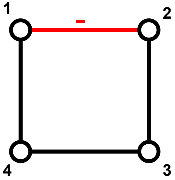

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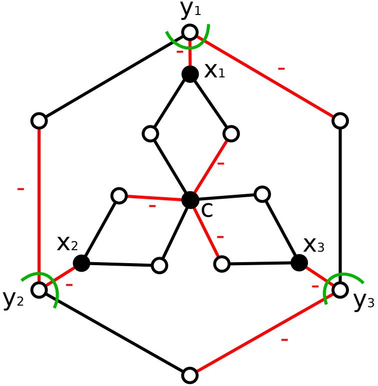

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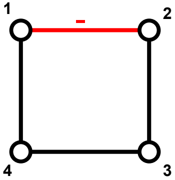


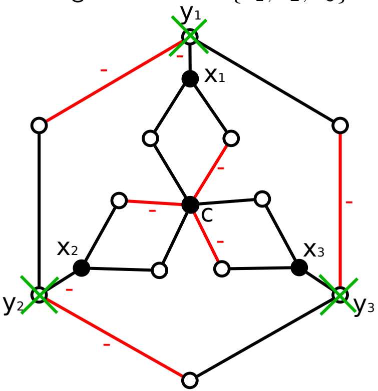
2. (⇐) Assume $G(F) \to UC_4$.

Set x_i to TRUE ⇔ x_i has been re-signed in the mapping.

Key observation: re-signing x_1, x_2, x_3 or none is impossible.

NAE-3SAT $\leq_R$ UC_4-Colouring: clause gadget

Gadget for clause $\{x_1, x_2, x_3\}$



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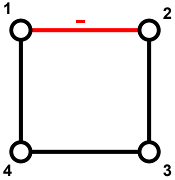

2. (⇐) Assume $G(F) \to UC_4$.

Set x_i to TRUE $\Leftrightarrow$ x_i has been re-signed in the mapping.

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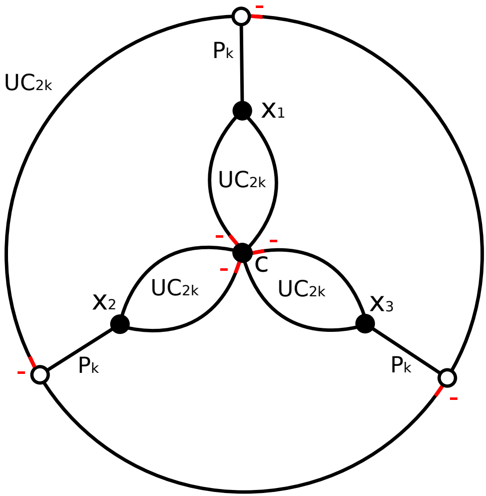
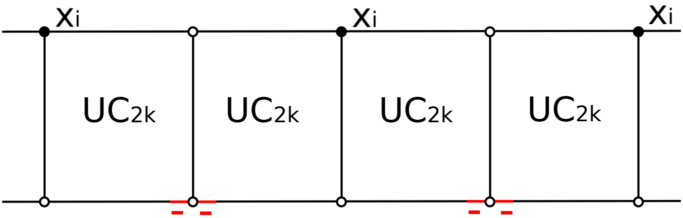
Mapping is impossible!



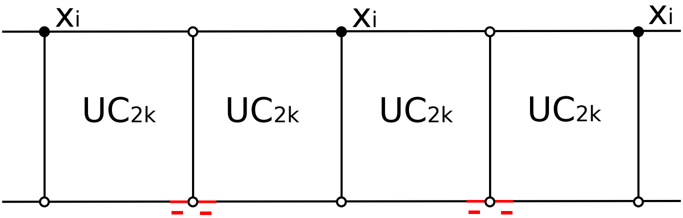
(where P_k has length $k-1$)

Bounding the maximum degree



One replicating gadget for each x_i and for c.

Bounding the maximum degree



One replicating gadget for each x_i and for c.

Corollary

UC_{2k}-Colouring is NP-complete even for graphs of max. degree 6.

What about planar graphs?

Remark

$\mathrm{NAE\text{-}3SAT}$ is polynomial-time solvable when the bipartite incidence graph of the formula is planar (Moret, 1988).

What about planar graphs?

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NAE-3SAT is polynomial-time solvable when the bipartite incidence graph of the formula is planar (Moret, 1988).

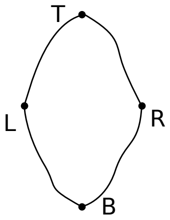
Question

Is UC_{2k}-Colouring NP-complete for planar graphs?

Planar instances: crossover gadgets

Definition - Crossover gadget for H

Planar (signed) graph C_H with vertices L, T, R, B on outer face and:
- $C_H \to H$
- in each mapping f to H, $f(L) = f(R)$ and $f(T) = f(B)$
- for any pair $x, y \in V(H)$ (possibly $x = y$), there is a mapping f with $f(L) = x$ and $f(T) = y$



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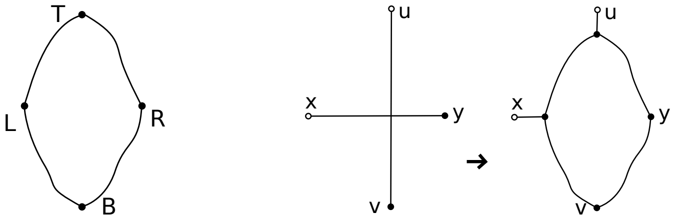

Proposition

If H has a crossover gadget, then H-Colouring $\leq_R$ H-Colouring for planar instances

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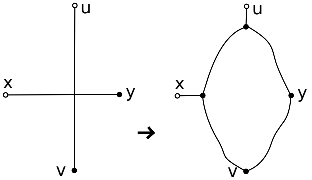

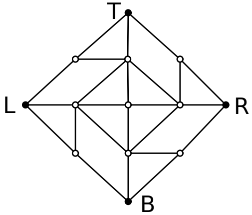

Example: crossover for K_3
(Garey and Johnson, 1979)

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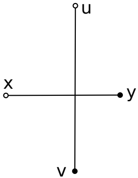

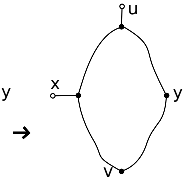

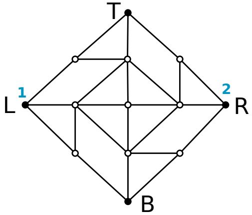

$f(L) \neq f(R)$?

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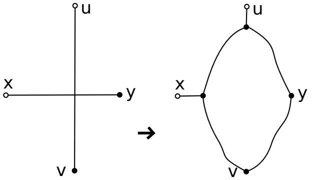

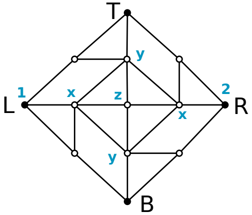

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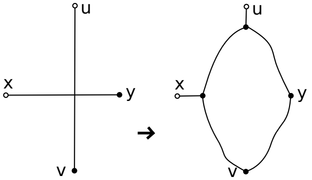

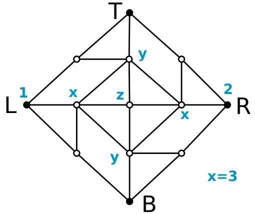

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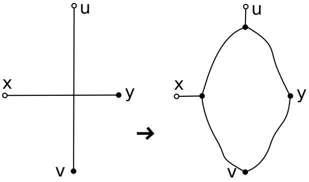

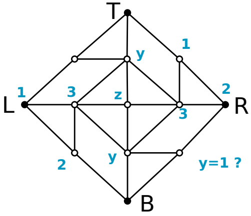

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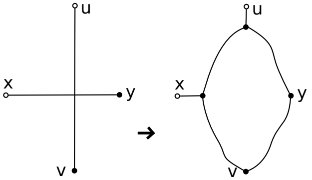

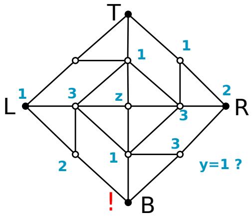

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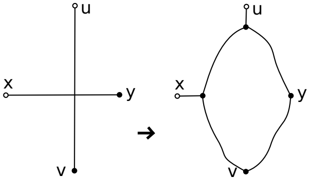

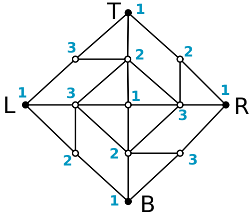

$f(L) = f(T) = 1$

Example: crossover for K_3
(Garey and Johnson, 1979)

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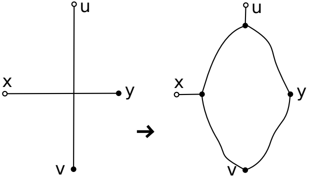

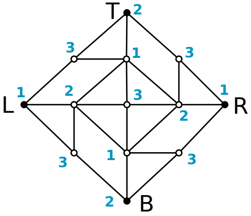

$f(L) = 1, f(T) = 2$

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Theorem (Hell, Naserasr, Tardiff, 2006)

There is a crossover gadget for any odd cycle C_{2k+1}.

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Remark

For many H, it is open whether H-COLOURING is NP-hard for planar instances.

Crossover gadget for UC_{2k}

Theorem (F., Naserasr, 2012+)

For every $k \geq 2$, UC_{2k} has a crossover gadget (of bounded max. degree). Hence, UC_{2k}-Colouring is NP-complete even for planar graphs (of bounded max. degree).

Crossover gadget for UC_4

Definition - Crossover gadget for **bipartite** H

Planar (signed) graph C_H with vertices L, T, R, B on outer face and:

- $C_H \to H$
- in each mapping f to H, $f(L) = f(R)$ and $f(T) = f(B)$
- for any pair $x, y \in V(H)$ **in different parts**, there is a mapping f with $f(L) = x$ and $f(T) = y$.

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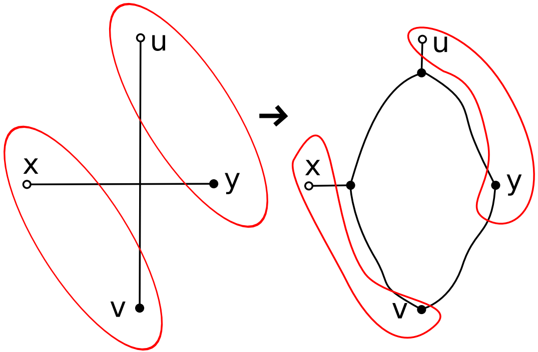

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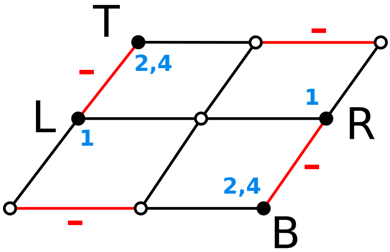
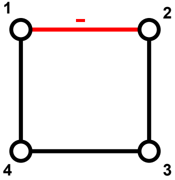
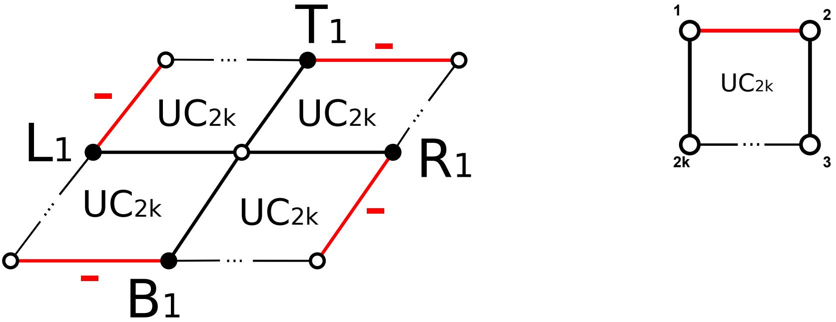




Crossover gadget for UC_{2k}

Crossover gadget for UC_{2k}: step 1



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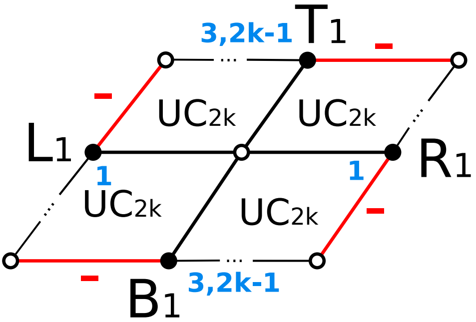

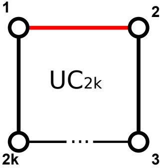

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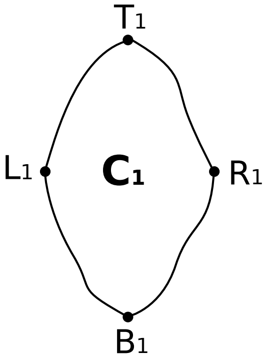

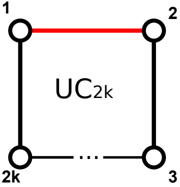

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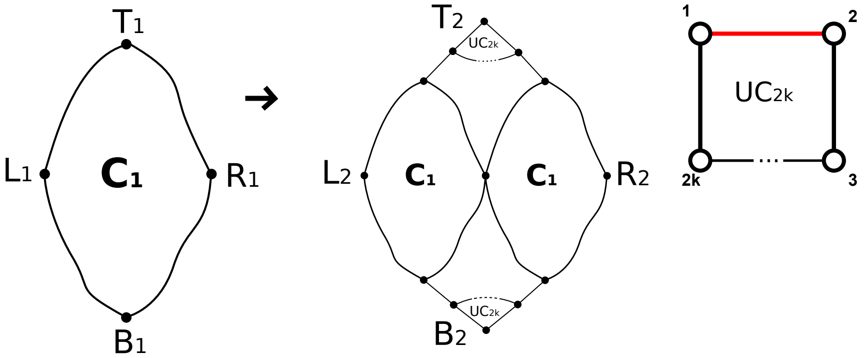

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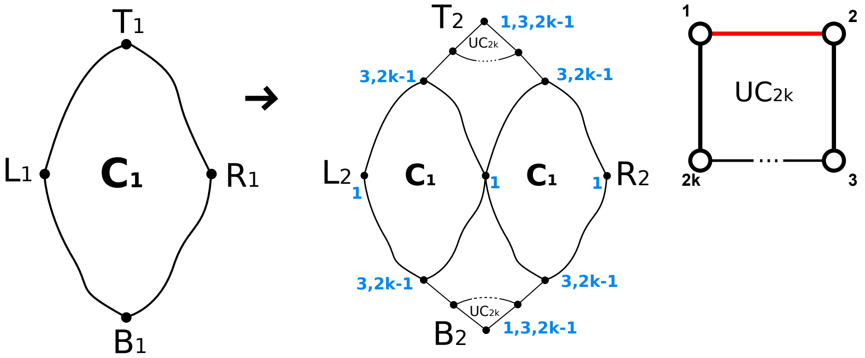
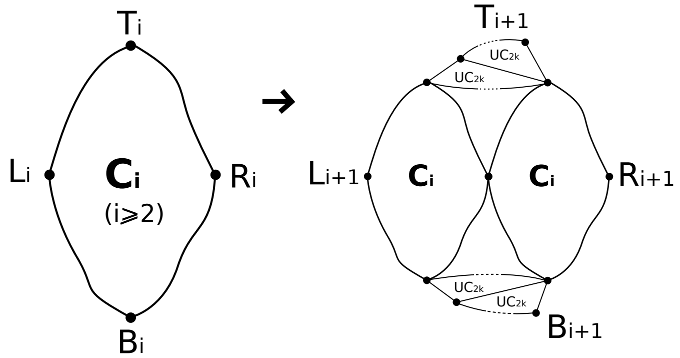
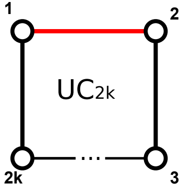


Crossover gadget for UC_{2k}: step i





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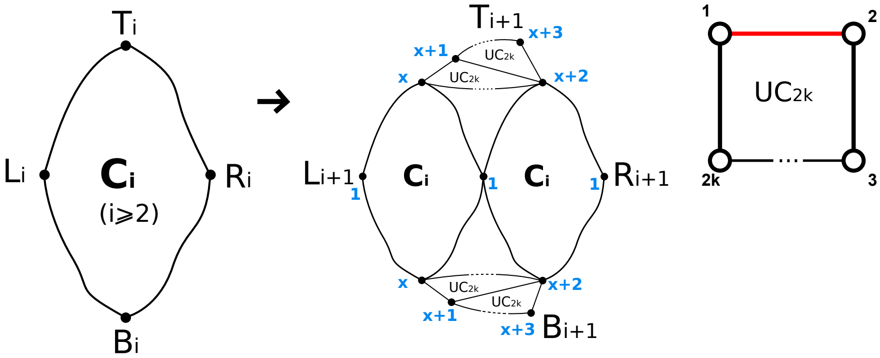

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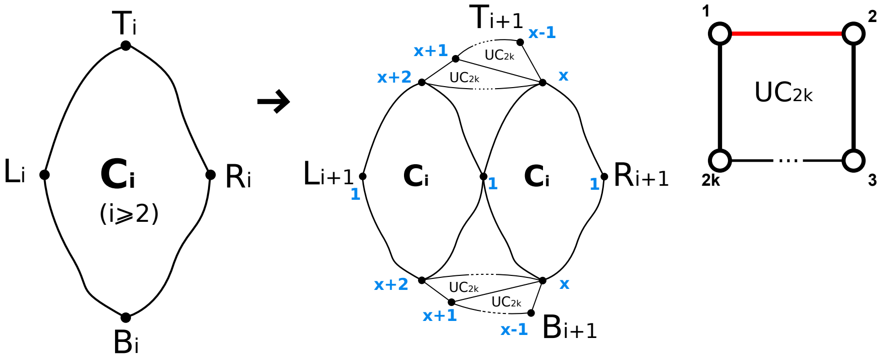

At step $i = 2j$:
$$f(T_i) \in S_{2i} = \{2\ell + 1 \mid 0 \le \ell \le i\} \cup \{2k - (2\ell + 1) \mid 0 \le \ell \le i\}$$

At step $i + 1 = 2j + 1$: $f(T_{i+1}) \in S_{2i} \pm 1$

Planar graphs of large unbalanced girth

Conjecture (Jaeger, 1988 - Stockmeyer, Zhang, 2000)

If G is a planar graph with odd-girth at least $4k+1$, then $G \to C_{2k+1}$.

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If (G, Σ) is a planar bipartite signed graph with unbalanced-girth at least $4k-2$, then $(G, \Sigma) \to UC_{2k}$.

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Theorem (Charpentier, Naserasr, Sopena, 2012+)

If (G, Σ) is a planar bipartite signed graph with unbalanced-girth at least 8, then $(G, \Sigma) \to UC_4$. (conjecture: true for ub-girth 6)

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Theorem (F., Naserasr, 2012+)

Either every planar bipartite signed graph with unbalanced-girth at least 6 maps to UC_4, or UC_4-COLOURING is NP-complete for planar graphs with unbalanced-girth at least 6.

Planar graphs of large unbalanced girth

Proof: inspired by Esperet, Montassier, Ochem, Pinlou, 2013.

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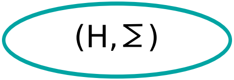
1. Assume conjecture is false: (H, Σ) is a planar signed graph with unbalanced-girth at least 6 not mapping to UC_4 of **smallest order**.



Planar graphs of large unbalanced girth

Proof: inspired by Esperet, Montassier, Ochem, Pinlou, 2013.

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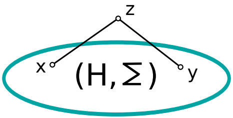

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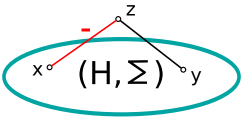

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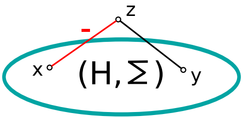

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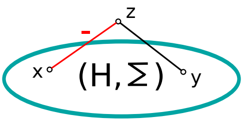

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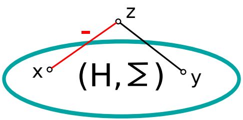

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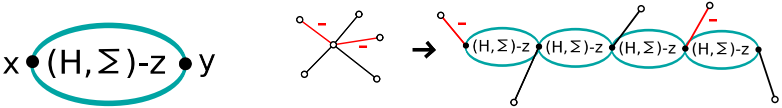
6. UC_4-Colouring $\leq_R$ UC_4-Colouring:



Perspectives

- Prove dichotomy for (H, Σ_H)-Colouring. $\rightarrow$ H bipartite
- Extension of Feder-Vardi's dichotomy conjecture to signed CSPs via signed digraphs?