Some complexity results for identifying codes in graphs

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Clermont-Ferrand

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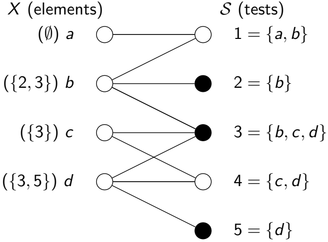
The test cover problem

Definition - TEST COVER (mentioned in Garey, Johnson, 1979)

INPUT: a set system (i.e. hypergraph) $(X, \mathcal{S})$
PROBLEM: find the minimum subset $\mathcal{T} \subseteq \mathcal{S}$ such that each element $x \in X$ belongs to a different set of sets in $\mathcal{T}$.



example: $\mathcal{T} = \{2, 3, 5\}$

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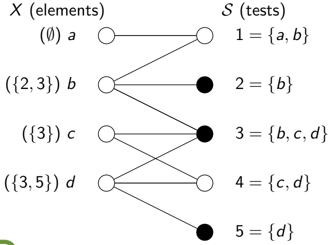

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Remark

Equivalently: for any pair x, y of elements of X, there is a set in $\mathcal{T}$ that contains **exactly** one of x, y, i.e. the symmetric difference of the sets of tests covering x, y is **nonempty**.

General bounds

Theorem (Folklore)

Given a set system $(X, \mathcal{S})$, a solution to TEST COVER has size at least $\log_2(|X|)$.

Proof: Must assign to each element of X, a distinct subset of $\mathcal{T}$.
Hence $|X| \leq 2^{|\mathcal{T}|}$. $\square$

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Hence $|X| \leq 2^{|\mathcal{T}|}$. □

Theorem (Bondy's theorem, 1972)

Given a set system $(X, \mathcal{S})$, a minimal solution to TEST COVER has size at most $|X| - 1$.

Proof: nice and short graph-theoretic argument. □

Complexity results

Theorem (Garey, Johnson, 1979)

TEST COVER is NP-complete.

Theorem (Charon, Cohen, Hudry, Lobstein, 2008)

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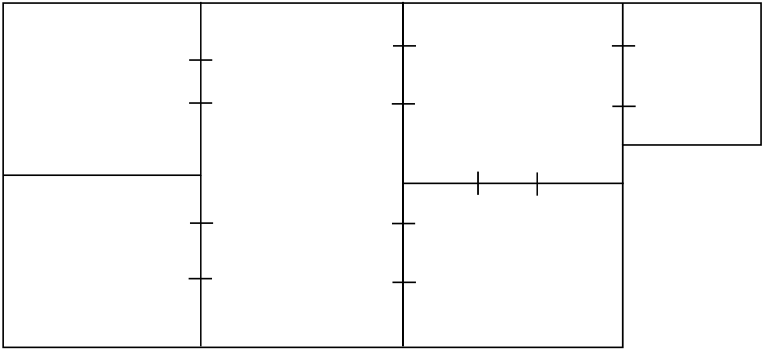
Theorem (De Bontridder, Haldorsson, Haldorsson, Hurkens, Lenstra, Ravi, Stougie, 2003)

MIN TEST COVER is $O(\log(|X|))$-approximable, but NP-hard to approximate within $o(log(|X|))$.

Proof: Reductions from and to MIN SET COVER. □

A special case: identifying the rooms of a building



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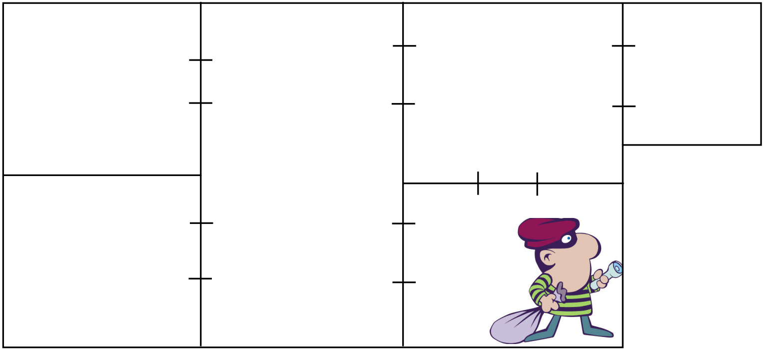

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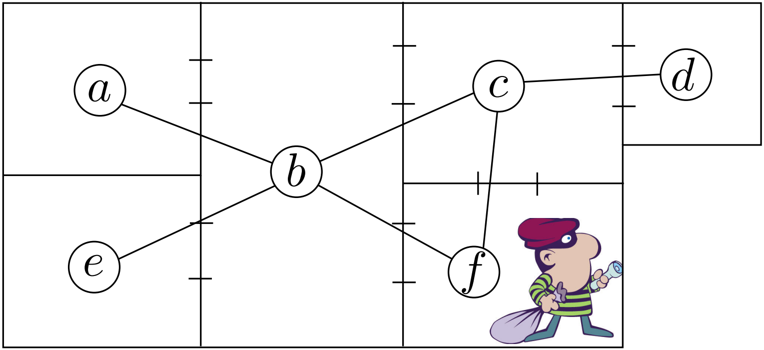

Graph $G = (V, E)$. V: vertices (rooms), $E \subseteq V \times V$: edges (doors)

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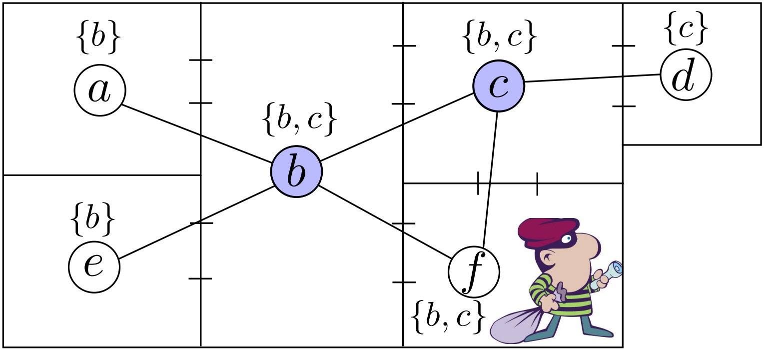

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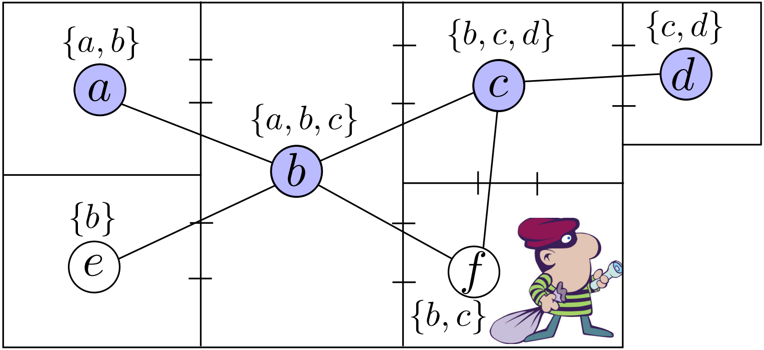

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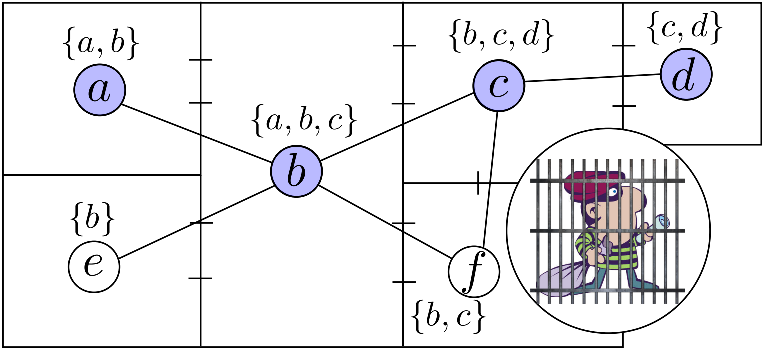

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Identifying codes, a special case of test covers

G: undirected graph
$N[u]$: set of vertices v s.t. $d(u, v) \leq 1$

Definition - Identifying code (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of $V(G)$ such that:

- C is a **dominating set** in G: $\forall u \in V(G)$, $N[u] \cap C \neq \emptyset$, and
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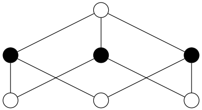

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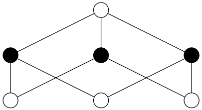

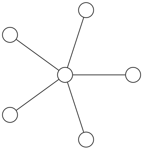

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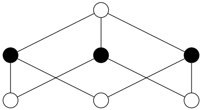

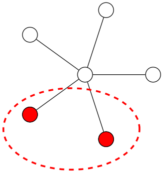

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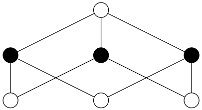

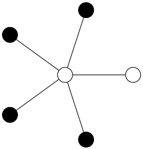

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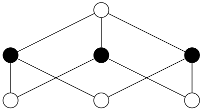

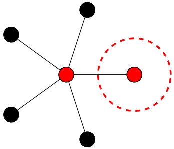

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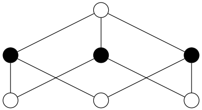

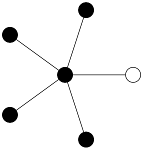

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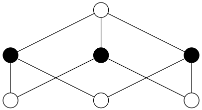

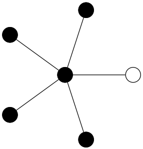

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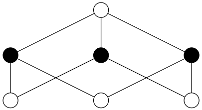

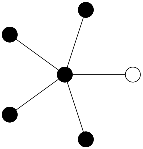

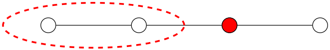

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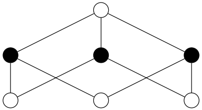

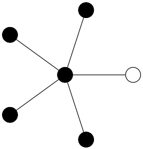

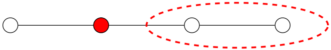

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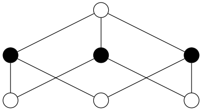

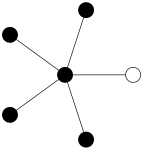

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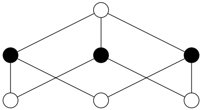
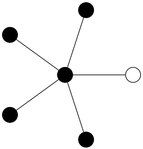




Computational problems

Definition - IDCODE

INPUT: graph G, integer k
QUESTION: does G have an identifying code of size at most k?

Theorem (Cohen, Honkala, Lobstein, Zémor, 1999)

IDCODE is NP-complete (reduction from 3SAT).

NP-completeness also holds for planar subcubic graphs, planar bipartite unit disk graphs, line graphs, etc.

Computational problems

Definition - MIN IDCODE

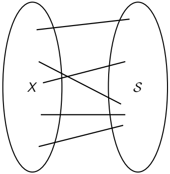
INPUT: graph G
PROBLEM: find a minimum-size identifying code of G

Theorem (Berger-Wolf, Laifenfeld, Trachtenberg, 2006)

MIN IDCODE is approximable within $O(\log(n))$, but NP-hard to approximate within $o(\log(n))$ (reduction from MIN SET COVER).

New and non-approximability reductions

Reduction: MIN TEST COVER to MIN IDCODE for bipartite graphs.



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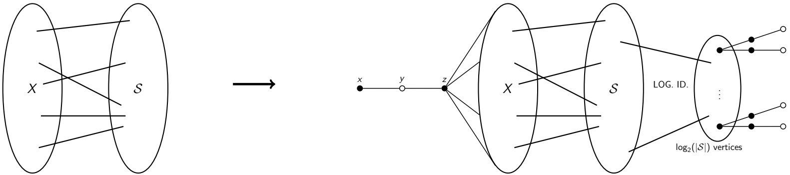

Theorem (F.)

- $(X, \mathcal{S})$ has a test cover of size k if and only if $G(X, \mathcal{S})$ has an identifying code of size $k + 3\lceil \log_2(|\mathcal{S}| + 1) \rceil + 2$. Constructive.
- If MIN IDCODE has an α-approximation algorithm, then MIN TEST COVER has a 4α-approximation algorithm.

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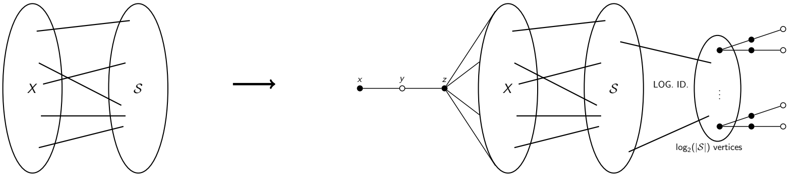

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Proof: Build approximate id. code C with $|C| \leq \alpha OPT_{ID}$

Build test cover T: $|T| \leq \alpha OPT_{ID} - 3\log_2(|\mathcal{S}|) - 2$

$$\begin{aligned} &\leq \alpha(OPT_{TC} + 3\log_2(|\mathcal{S}|) + 2) - 3\log_2(|\mathcal{S}|) - 2 \\ &\leq \alpha OPT_{TC} + (\alpha - 1)3\log_2(|\mathcal{S}|) \\ &\leq 4\alpha OPT_{TC} \end{aligned}$$

□

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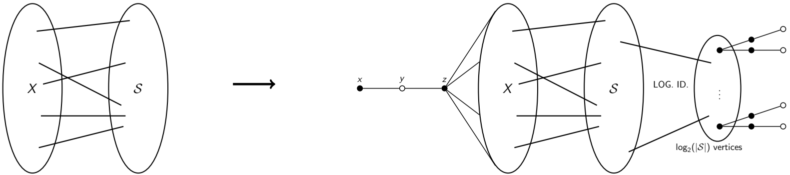

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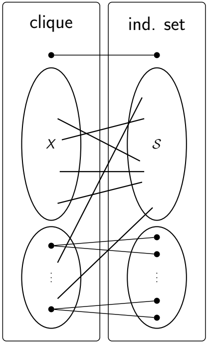
- If MIN IDCODE has an α-approximation algorithm, then MIN TEST COVER has a 4α-approximation algorithm.

Corollary

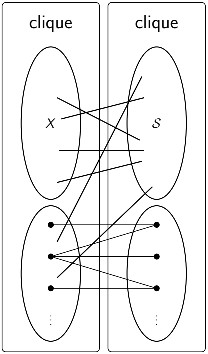
It is NP-hard to approximate MIN IDCODE within $o(\log(n))$, even for bipartite graphs.

New non-approximability reductions

Similar reductions for split graphs and co-bipartite graphs.



split graphs



co-bipartite graphs

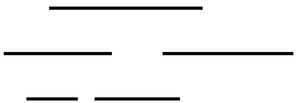
Theorem (F.)

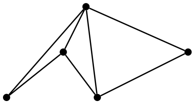
It is NP-hard to approximate MIN IDCODE within $o(\log(n))$, even for split graphs and even for co-bipartite graphs.

Interval graphs

Definition - Interval graph

Intersection graph of intervals of the real line.





IDCODE for interval graphs

Theorem (F., Kosowski, Mertzios, Naserasr, Parreau, Valicov)

IDCODE is NP-complete for interval graphs. Reduction from 3-DIMENSIONAL MATCHING.

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Definition - 3-DIMENSIONAL MATCHING

INPUT: three sets A, B, C with $|A| = |B| = |C|$, and a set T of triples of $A \times B \times C$.
QUESTION: is there a perfect 3-dimensional matching $M \subseteq T$, i.e. each element of $A \cup B \cup C$ belongs to eaxctly one triple of M?

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Theorem (Karp, 1972)

3-DIMENSIONAL MATCHING is NP-complete.

IDCODE for interval graphs - a P_5 as covering gadget

Main idea of the reduction: an interval can separate several pairs of intervals that are **far away** from each other (without affecting what lies in between).

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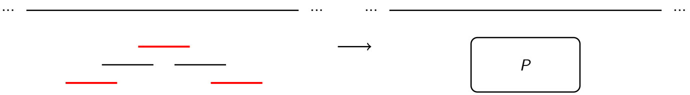
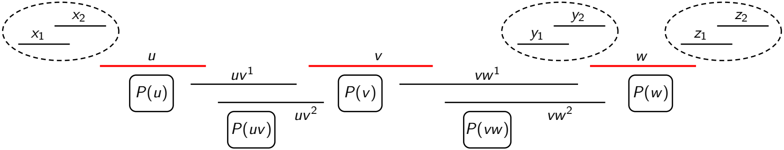
A small gadget to ensure that intervals are covered and separated:



IDCODE for interval graphs - transmitter gadget

Idea ("Transmitter gadget"): in order to separate $\{uv^1, uv^2\}$ and $\{vw^1, vw^2\}$, either:

1. take only v into the id. code, or
2. take both u, w — and separate the three pairs $\{x_1, x_2\}$, $\{y_1, y_2\}$, $\{z_1, z_2\}$ "for free".



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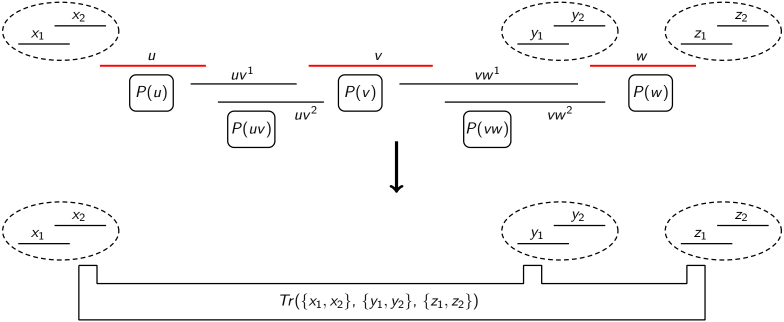
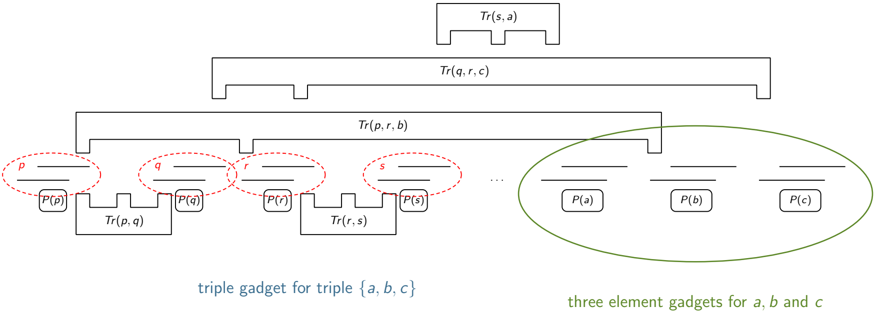


IDCODE for interval graphs - the reduction



Idea: to separate pairs p, q, r, s we need the identifying vertices of some transmitter gadgets. Either take:

1. $Tr(p, q)$ and $Tr(r, s)$, but separate NO element pair a, b, c, or
2. $Tr(p, r, b)$, $Tr(q, r, c)$ and $Tr(s, a)$ and separate all three element pairs "for free"

IDCODE for unit interval graphs

Definition - Unit interval graph

Intersection graph of intervals of the real line all having unit length. Equivalent to *proper* interval graphs (no interval properly contains another).

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Our reduction creates interval graphs that are far from proper/unit.

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Our reduction creates interval graphs that are far from proper/unit.

Question

What is the complexity of IDCODE for *unit* interval graphs?

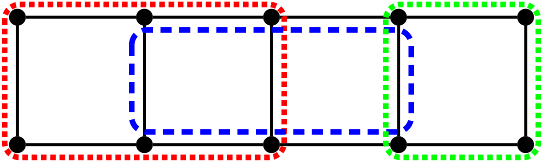
IDCODE for unit interval graphs

Definition - Ladder graph L_m

L_m is the grid graph $P_2 \square P_m$.

Definition - Cycle cover

Set $\mathcal{S}$ of cycles of graph G s.t. $\bigcup_{S \in \mathcal{S}} E(S) = E(G)$.



IDCODE for unit interval graphs

Definition - LADDER CYCLE COVER

INPUT: An integer m and an integer k, and a set $\mathcal{S}$ of cycles of L_m.
QUESTION: Is there a set $\mathcal{S}' \subseteq \mathcal{S}$ of size k which is a cycle cover of L_m?

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IDCODE for unit interval graphs of order n can be reduced to LADDER CYCLE COVER for L_{n+1} and an input of n cycles.

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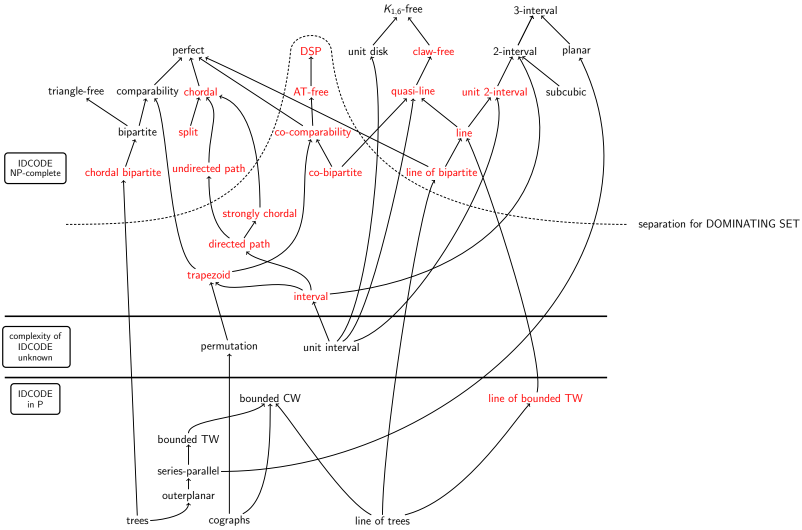
Complexity of IDCODE for various graph classes



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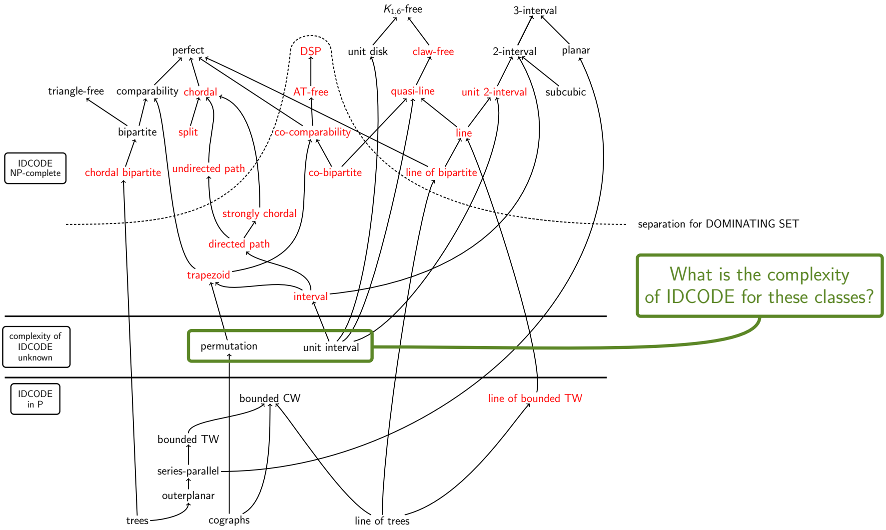

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