Bounds on the size of identifying codes for graphs of maximum degree Δ

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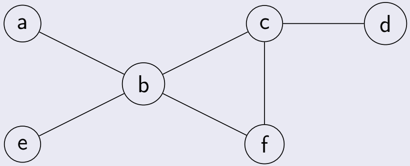
September 2009

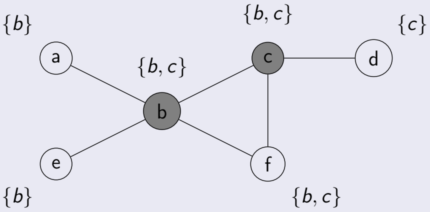
Locating a fire in a building

simple, undirected graph : models a building



simple detectors : able to detect a fire in a neighbouring room

goal : locate an eventual fire



		b	c	
a		•		
b		•	•	
c		•	•	
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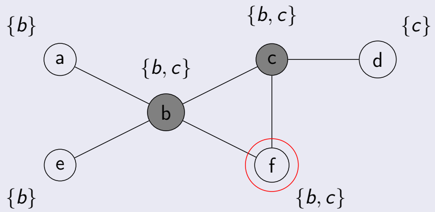

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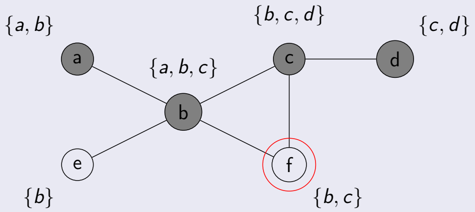
the *identifying sets* of all vertices must be distinct



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Identifying codes : definition

Definition : identifying code of a graph $G = (V, E)$ (Karpovsky et al. 1998 [2])

subset C of V such that :

- C is a dominating set in G, and
- for all distinct u, v of V, u and v have distinct *identifying sets* : $N[u] \cap C \neq N[v] \cap C$

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Note : close to *locating-dominating sets* (Slater, Rall 84 [4])

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Notation

$\gamma_{id}(G)$: minimum cardinality of an identifying code in a graph G

Identifiable graphs

Remark : not all graphs admit an identifying code

u and v are *twin* vertices if $N[u] = N[v]$.
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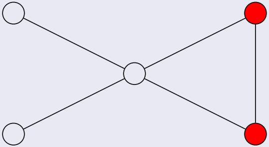

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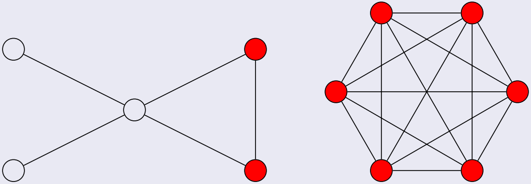

Lower bound and maximum degree

Thm (Karpovski et al. 98 [2])

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Thm (Karpovski et al. 98 [2])

Let G be an identifiable graph with n vertices and maximum degree Δ. Then $\gamma_{id}(G) \geq \frac{2n}{\Delta+2}$.

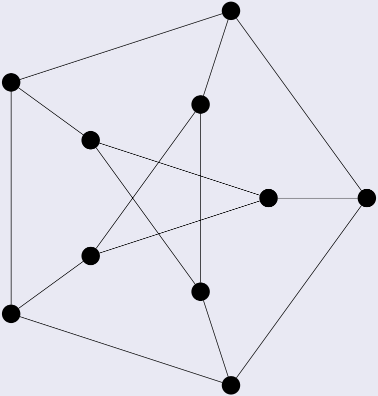
Graphs reaching the lower bound

Characterization

- n vertices
- independent set C of size $\frac{2n}{\Delta+2}$ (id. code)
- every vertex of C has exactly Δ neighbours
- $\frac{\Delta n}{\Delta+2}$ vertices connected to exactly 2 code vertices each

Graphs reaching the lower bound - example

Example : D=Petersen graph, $\Delta = 3$, $n = 10$



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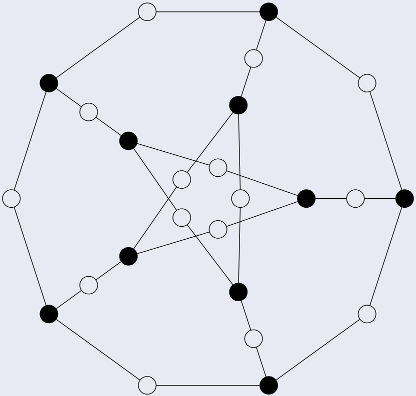

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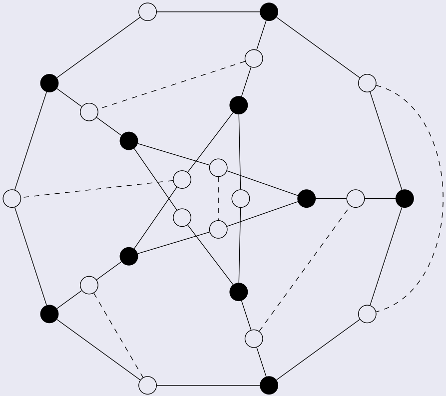

A general upper bound

Thm (Gravier, Moncel 07 [1])

Let G be an identifiable connected graph with $n \geq 3$ vertices.
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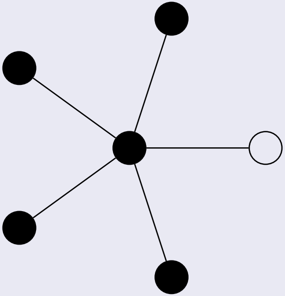
For all $n \geq 3$, there exist identifiable graphs with n vertices with $\gamma_{id}(G) = n - 1$.

Upper bound - example

Example : the star $K_{1,n-1}$



Upper bound - example

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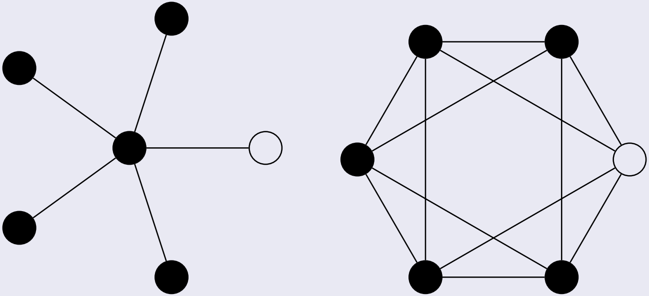

Upper bound and maximum degree

Remark

All these graphs have a high maximum degree $\Delta(G)$: $n-1$ or $n-2$.

Result - general case

Thm (F., Klasing, Kosowski and Raspaud 09)

Let G be a connected identifiable graph of maximum degree Δ.
Then $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^4)}$.
If G is regular, $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^2)}$.

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Sketch of the proof

- Greedily construct a 4-independant (resp. 2-independent) set S : distance between two vertices is at least 5 (resp. 3)
- take $C = V \backslash S$ as a code
- C must be modified locally

Connected cliques

- Take any Δ-regular graph H with m vertices
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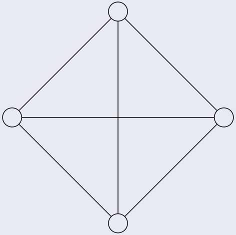

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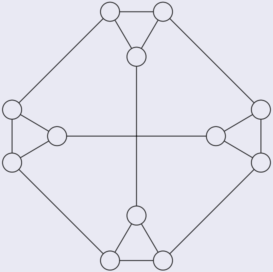

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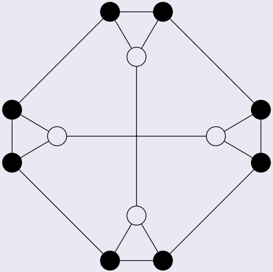

For every clique, at least $\Delta - 1$ vertices in the code

$\Rightarrow \gamma_{id}(G) \geq m \cdot (\Delta - 1) = n - \frac{n}{\Delta}$

Large codes in triangle-free graphs

Proposition

Let $K_{m,m}$ be the complete bipartite graph with $n = 2m$ vertices.
$id(K_{m,m}) = 2m - 2 = n - \frac{n}{\Delta}$.

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Thm (Bertrand et al. 05)

Let T_k^h be the k-ary tree with h levels and n vertices.

$$id(T_k^h) = \left\lceil \frac{k^2 n}{k^2 + k + 1} \right\rceil = n - \frac{n}{\Delta - 1 + \frac{1}{\Delta}}.$$

Triangle-free graphs - Result

Thm (F., Klasing, Kosowski and Raspaud 09)

Let G be a connected triangle-free identifiable graph G with $n \geq 3$ vertices and maximum degree Δ.
Then $\gamma_{id}(G) \leq n - \frac{n}{3\Delta+3}$.
If G is regular, $\gamma_{id}(G) \leq n - \frac{n}{2\Delta+2}$.

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Sketch of the proof

- Greedily construct an independent set S with special properties : $|S| \geq \frac{n}{\Delta+1}$
- Take $C = V \backslash S$ as a code
- Some vertices may not be identified correctly
- $\rightarrow$ locally modify C. It is possible to add not too much vertices to C

Graphs of girth at least 5

Thm (F., Klasing, Kosowski and Raspaud 09)

Let G be an identifiable graph with n vertices, of minimum degree $\delta \geq 2$ and girth $g \geq 5$.
Then $\gamma_{id}(G) \leq \frac{7n}{8} + 1$.

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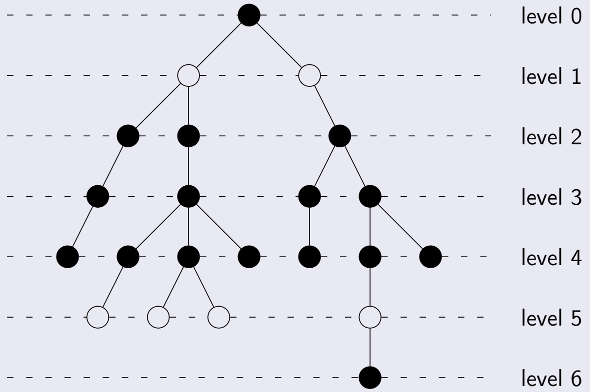
Sketch of the proof

- Construct a DFS spanning tree T of G
- Partition the vertices into 4 classes V_0, V_1, V_2, V_3 depending on their level in T
- Take $C = V \setminus V_i$ as a code, $|V_i| \geq \frac{n}{4}$: $|V_i| \leq \frac{3n}{4}$
- C must be modified locally; the size of C might increase

Graphs of girth at least 5



Summary

	arbitrary graphs	Δ-regular graphs
arbitrary graphs	$\left\langle n-\frac{n}{\Delta},\ n-\frac{n}{\Theta(\Delta^4)}\right\rangle$	$\left\langle n-\frac{n}{\Delta},\ n-\frac{n-1}{\Delta^2}\right\rangle$
triangle-free graphs	$\left\langle n-\frac{n}{\Delta-1+\frac{1}{\Delta}},\ n-\frac{n}{3\Delta+3}\right\rangle$	$\left\langle n-\frac{n}{\frac{2\Delta}{3}},\ n-\frac{n}{2\Delta+2}\right\rangle$

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	minimum degree $\delta \geq 2$
graphs of girth at least 5	$\left\langle \frac{3n}{5},\ \frac{7n}{8}+1\right\rangle$

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