The complexity of homomorphisms of signed graphs

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joint work (in progress) with:

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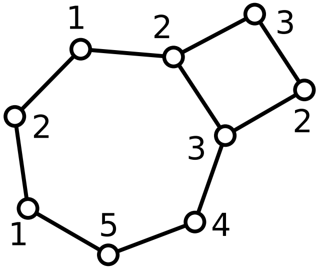
Pavol Hell (Simon Fraser U., Vancouver)
Reza Naserasr (U. Paris-Sud, Orsay)

JGA 2013

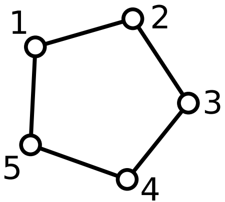
Graph homomorphisms

Definition - Graph homomorphism from G to H

Mapping from $V(G)$ to $V(H)$ which **preserves adjacency**.
If it exists, we note $G \to H$.



Target graph: $H = C_5$



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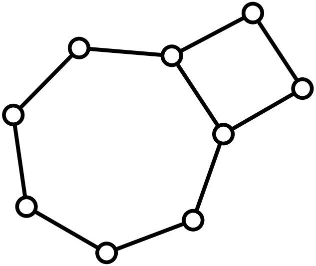

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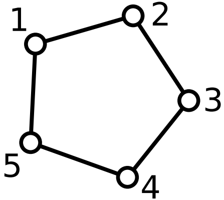

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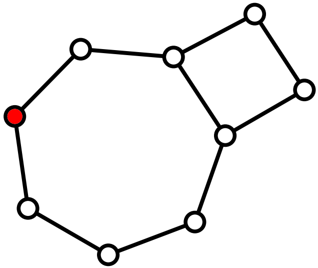

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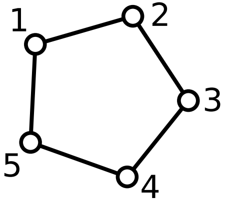

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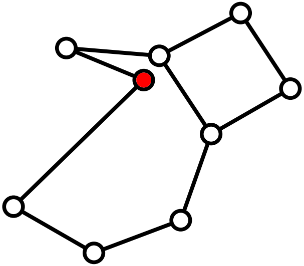

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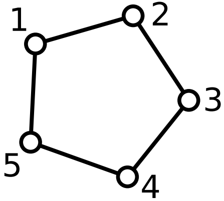

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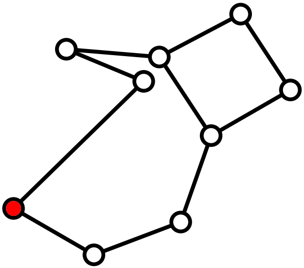

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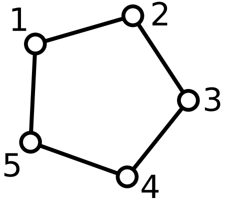

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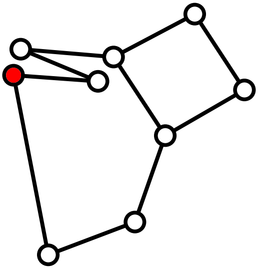

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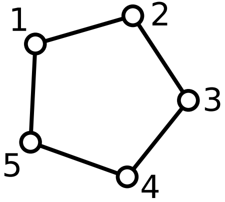

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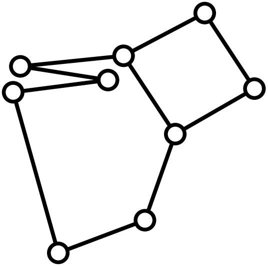

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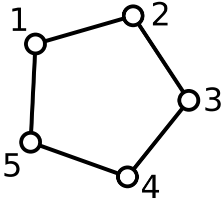

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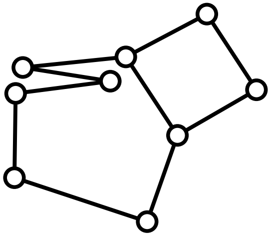

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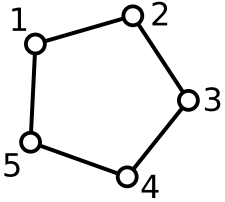

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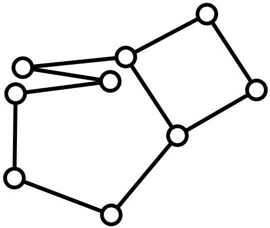

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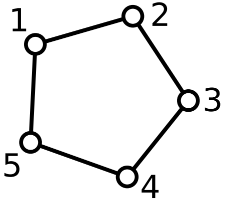

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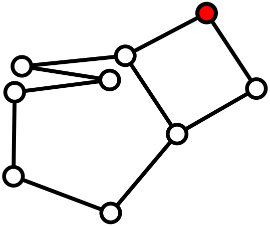

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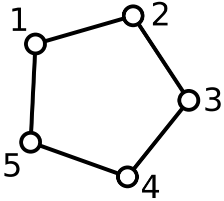

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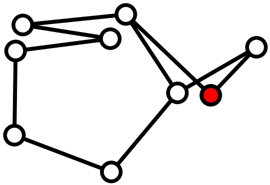

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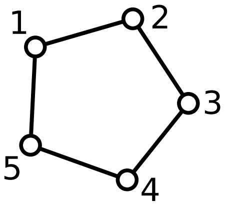

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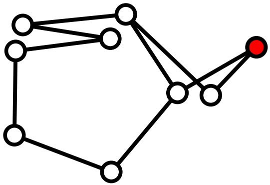

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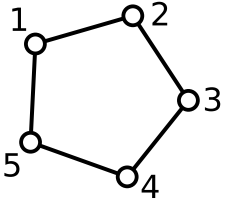

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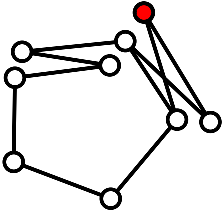

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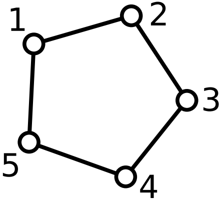

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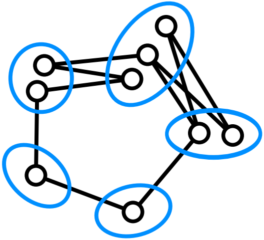

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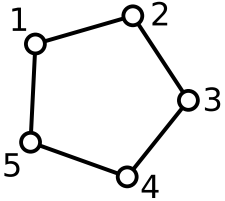

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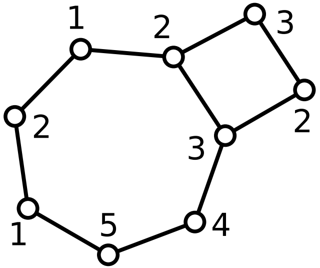

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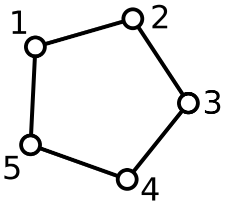

Remark: Homomorphisms generalize **proper vertex-colourings**

$$G \to K_k \iff G \text{ is } k\text{-colourable}$$

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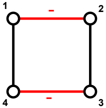
Σ: set of $-$ edges.



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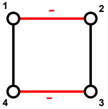

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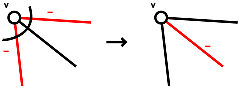
Re-signing operation at v: switch sign of each edge incident to v



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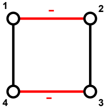

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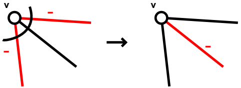

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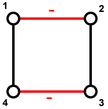

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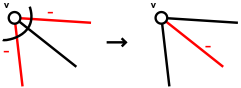

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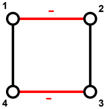

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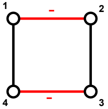

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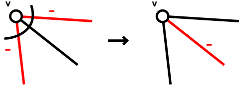

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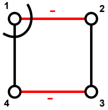

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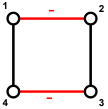

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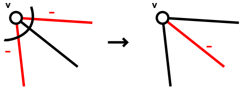

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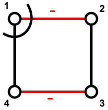

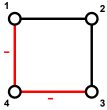

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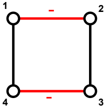

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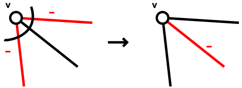

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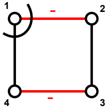

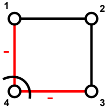

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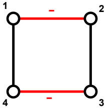

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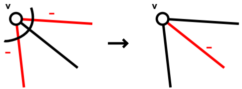

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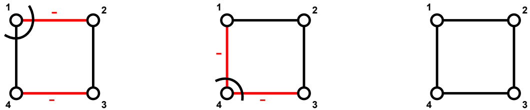

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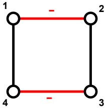

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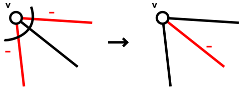

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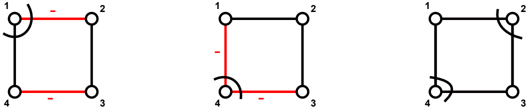

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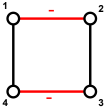

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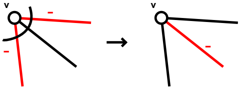

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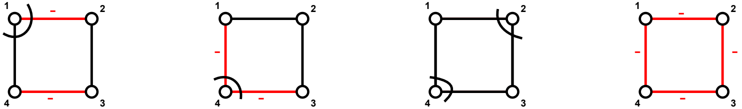

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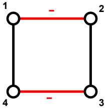

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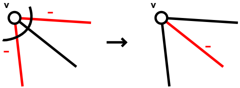

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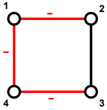

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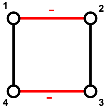

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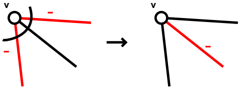

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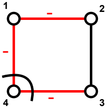

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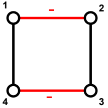

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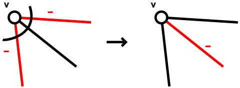

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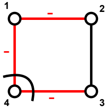

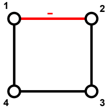

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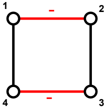

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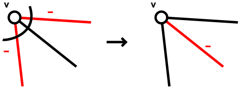

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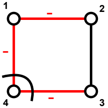

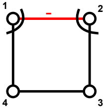

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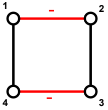

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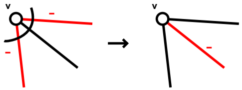

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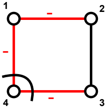

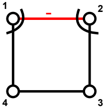

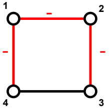

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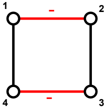

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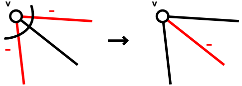

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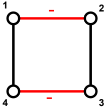
Signed graph: Graph G with an equivalence class $\mathcal{C}$ of signatures.

Notation: (G, Σ) with any $\Sigma \in \mathcal{C}$.

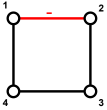
Signed graphs: (un)balanced cycles

Definition - Unbalanced cycle

Cycle with an odd number of negative edges.



balanced C_4



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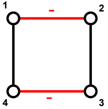

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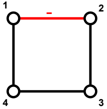

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Remark

Re-signing always preserves the balance of a cycle.

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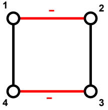

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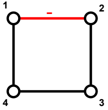

unbalanced C_4: UC_4

Remark

Re-signing always preserves the balance of a cycle.

Theorem (Zaslavsky, 1982)

Two signatures are equivalent if and only if they induce the same set of unbalanced cycles.

Why signed graphs?

Introduced by Harary (1953): notion of **balanced** signed graphs (each cycle is balanced)

$\rightarrow$ **Social psychology:** "like" and "dislike" relations in a social network. Balanced networks are socially stable. (Cartwright and Harary, 1956)

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Conjecture ("Odd Hadwiger" - Seymour; Gerards, 1993)

If $(G, E(G))$ has no $(K_k, E(K_k))$ as a minor, $\chi(G) \leq k-1$.

Extends the previous one; proved up to $k=5$.

Signed graph homomorphisms

Definition - Signed graph homomorphism from (G, Σ_G) to (H, Σ_H)

Homomorphism $f : G \to H$ such that there exists $\Sigma'_G \equiv \Sigma_G$ for which the signs are preserved with respect to Σ'_G, Σ_H.

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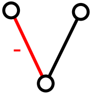

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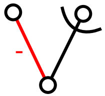

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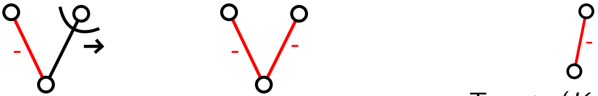

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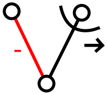

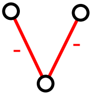

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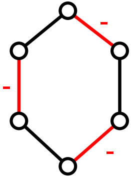

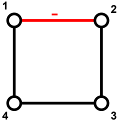

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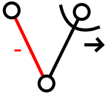

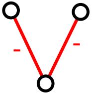

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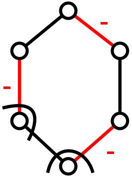

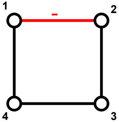

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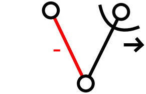

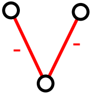

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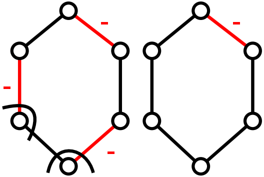

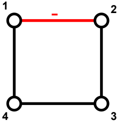

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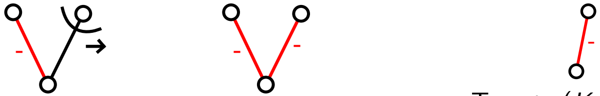

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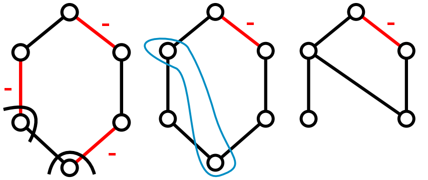

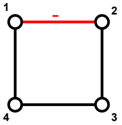

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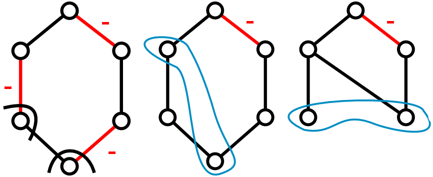

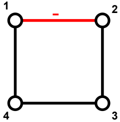

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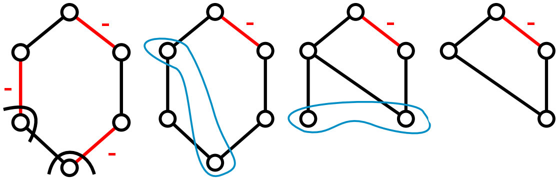

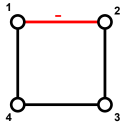

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Complexity: classical homomorphisms

Definition - H-COLOURING

INSTANCE: A graph G.
QUESTION: does $G \to H$?

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Theorem (Karp, 1972)

K_3-Colouring is NP-complete.

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INSTANCE: A graph G.
QUESTION: does $G \to H$?

Theorem (Hell, Nešetřil, 1990)

H-COLOURING is NP-complete for every non-bipartite graph H.
Polynomial (trivial) if H is bipartite or has a loop.

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Definition - H-COLOURING

INSTANCE: A graph G.
QUESTION: does $G \to H$?

Theorem (Hell, Nešetřil, 1990)

H-COLOURING is NP-complete for every non-bipartite graph H.
Polynomial (trivial) if H is bipartite or has a loop.

Conjecture (Feder-Vardi, 1998: Dichotomy conjecture)

For every **digraph** D, D-COLOURING is either NP-complete or polynomial-time solvable.

(Equivalent to dichotomy for CSP and MMSNP — tough conjecture!)

Complexity: questions for signed graphs

Definition - (H, Σ_H)-COLOURING

INSTANCE: A signed graph (G, Σ).
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Remark

- checking if $\Sigma \equiv \Sigma'$: polynomial
- $(G, \Sigma) \to (H, \emptyset)$ IFF $G \to H$ and $\Sigma \equiv \emptyset$.
- $(G, \Sigma) \to (H, E(H))$ IFF $G \to H$ and $\Sigma \equiv E(G)$.

$\to$ If $\Sigma_H \equiv \emptyset$ or $\Sigma_H \equiv E(H)$, (H, Σ_H)-COLOURING has same complexity as H-COLOURING.

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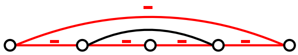
Polynomial cases:

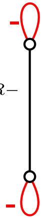
- H bipartite, $\Sigma_H \equiv \emptyset \equiv E(H)$
- H has one vertex with both $+$ loop and $-$ loop
- H has a loop and $\Sigma_H \equiv \emptyset$ or $\Sigma_H \equiv E(H)$
- H is bipartite and contains a multi-edge ($+$ and $-$)

Reduction from classical H-COLOURING

Theorem (Brewster, F., Hell, 2013+)

$$(G, E(G)) \to (H, \Sigma) \text{ IFF } G \to ((H, \Sigma) \times K_2^{R-})^U$$





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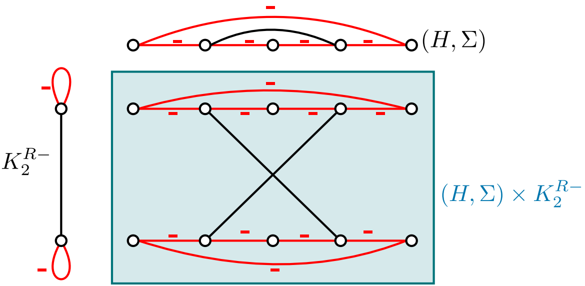

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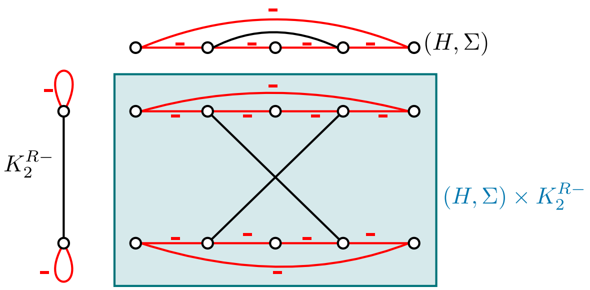

Corollary

If (H, Σ) has an **unbalanced** **odd** cycle, then (H, Σ)-COLOURING is NP-complete.
(symmetric result holds when (H, Σ) has *balanced* odd cycle)

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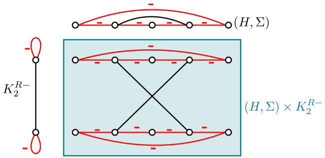

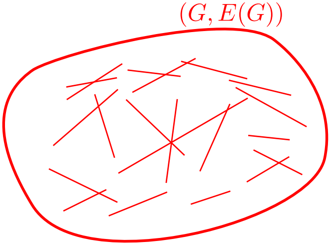

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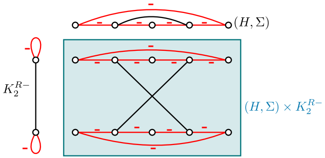

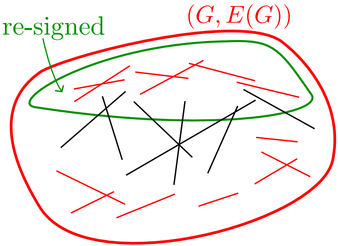

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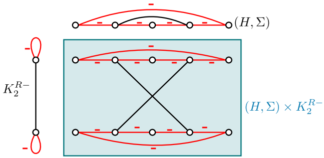

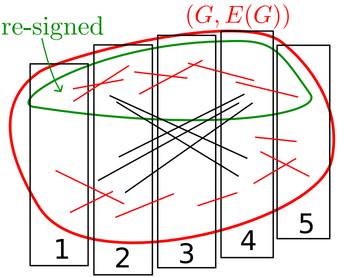

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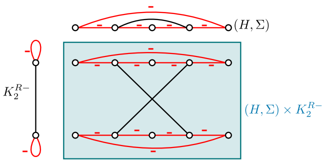

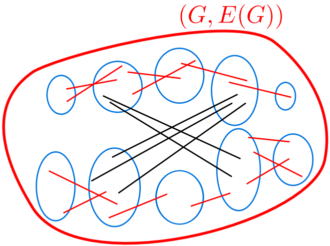

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Reduction from NAE-3SAT

Definition - UC_{2k}-Colouring

INSTANCE: A (bipartite) signed graph (G, Σ).
QUESTION: does $(G, \Sigma) \to UC_{2k}$?

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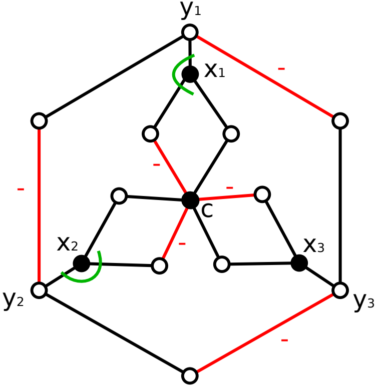
Definition - MONOTONE NOT-ALL-EQUAL-3SAT

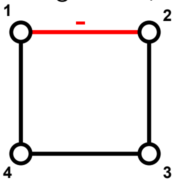
INSTANCE: A set of clauses of 3 Boolean variables from set X.
QUESTION: Is there a truth assignment $X \to \{0, 1\}$ s.t. each clause has variables with different values?

NAE-3SAT $\leq_R$ UC_4-Colouring: clause gadget

Gadget for clause $\{x_1, x_2, x_3\}$



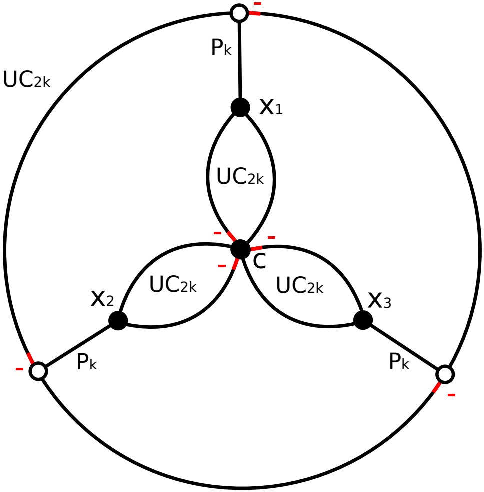
Target: UC_4



Construction of $G(F)$: one clause gadget per clause of F.
All vertices with same labels (c or x_i) identified with each other.

Main idea: In a mapping, re-signing at $x_i \iff x_i$=TRUE

NAE-3SAT $\leq_R$ UC_{2k}-COLOURING: clause gadget



(where P_k has length $k-1$)

CSPs, signed CSPs

Constraint Satisfaction Problem (CSP) for relational system $T = (X_T, V_T)$: domain X_T, set V relations $R_1, \dots, R_k$ of arity $a_1, \dots, a_k$ with $R_i \subseteq X^{a_i}$ (vocabulary).

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Examples:

• (Di)graph homomorphism to D: $X_T = V(D)$, V_T is one binary (non-)symmetric relation.

• 3SAT: $X_T = \{0, 1\}$, V_T: one ternary relation with all triples except 000.

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Signed CSP: $+$ and $-$ tuples, re-signing allowed.

Proposition

Dichotomy for CSP $\iff$ Dichotomy for signed CSP

Perspectives

- Prove dichotomy for (H, Σ)-Colouring.
  $\rightarrow$ remaining cases: H bipartite or has both kinds of loops
- Study signed CSPs