Edge identifying codes [1]

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Abstract

We study the edge identifying code problem, i.e. the identifying code problem in line graphs. If $\gamma^{\mathrm{ID}}(G)$ denotes the size of a minimum identifying code of a graph G, we show that the usual bound $\gamma^{\mathrm{ID}}(G) \geq \lceil \log_2(n+1) \rceil$, where n denotes the order of G, can be improved to $\Theta(\sqrt{n})$ in the class of line graphs. Moreover this bound is tight. We also prove that the upper bound $\gamma^{\mathrm{ID}}(\mathcal{L}(G)) \leq 2 \cdot |V(G)| - 4$ holds. This implies that a conjecture of R. Klasing, A. Kosowski, A. Raspaud and the first author holds for a subclass of line graphs. Finally, we show that the edge identifying code problem is NP-complete, even for the class of planar bipartite graphs of maximum degree 3 and arbitrarily large girth.

Keywords: Identifying codes, Dominating sets, Line graphs, NP-completeness

1 Introduction

An *identifying code* of a graph G is a subset C of vertices such that each vertex of G has a unique and non-empty neighbourhood within C. More formally, on the one hand, for each vertex $v \in V(G)$, $N[v] \cap C \neq \emptyset$ (where $N[v]$ denotes the closed neighbourhood of v), that is, C is a dominating set of G. On the other hand, for each pair u, v of distinct vertices of G, $N[u] \cap C \neq N[v] \cap C$. This concept was introduced in 1998 [9] and is a well-studied one (see e.g. [1,2,3,6,7,8,9,10]).

In this paper, we introduce the related concept of an *edge identifying code*, i.e. a set C_E of edges of a graph G, such that each edge e of G is uniquely determined by the set of edges belonging to C_E which are adjacent to e. More formally, for each edge $e \in E(G)$, $I[e] \cap C_E \neq \emptyset$ (where $I[e]$ denotes the set of edges adjacent to e together with e itself). Moreover, for each pair e, f of distinct edges of G, $I[e] \cap C_E \neq I[f] \cap C_E$.

Note that C_E is an edge identifying code of G if and only if C_E is an identifying code in $\mathcal{L}(G)$, the line graph of G. Hence, the study of edge identifying codes is the study of identifying codes in line graphs.

Not all graphs admit an identifying code or an edge identifying code. In fact a graph is *identifiable* if and only if it is *twin-free*, i.e., it has no pair of *twins*: vertices having the same closed neighbourhood. We say that a graph G is *edge-identifiable* if $\mathcal{L}(G)$ is identifiable. It is easy to observe that a graph is edge-identifiable if and only if it has no pair u, v of vertices such that $d(u) = d(v) = 1$ and they have a common neighbour, or $d(u) = d(v) = 2$, u and v are adjacent, and they have a common neighbour.

We define the *identifying code number* of an identifiable graph G, denoted $\gamma^{\mathrm{ID}}(G)$, to be the cardinality of a minimum-size identifying code of G. If G is edge-identifiable, the *edge identifying code number* of G, denoted $\gamma^{\mathrm{EID}}(G)$, is the cardinality of a minimum-size edge identifying code of G. If G is edge-identifiable, it follows that $\gamma^{\mathrm{EID}}(G) = \gamma^{\mathrm{ID}}(\mathcal{L}(G))$.

Determining the exact value of $\gamma^{\mathrm{ID}}(G)$ is an NP-complete problem, even for restricted classes of graphs (such as bipartite graphs of maximum degree 3 [2] or planar graphs of maximum degree 4 and arbitrarily large girth [1]).

In this paper, we present improved lower and upper bounds for the edge identifying code number of a graph, and show that the edge identifying code problem is NP-complete for a highly restricted class of graphs.

2   Lower bounds

In the case of vertex-identifying codes, given a graph G and an identifying code C of G, we have $|C| \geq \lceil \log(|V(G)| + 1) \rceil$ [9]. All graphs reaching this bound were described in [10], but they are far from being line graphs. We show that a stronger bound holds for line graphs. We first consider the case where the minimum edge identifying code induces a connected subgraph.

Theorem 2.1 *If an edge identifying code C_E of a nontrivial graph G induces a connected subgraph G' of G, then G has at most $\binom{|C_E|+2}{2} - 4$ edges.*

The main idea of the proof is that each edge $e = uv$ of G is uniquely

determined by the set of edges of G' incident to u together with the set of edges of G' incident to v. There are at most $|V(G')| + \binom{|V(G')|}{2} = \binom{|V(G')|+1}{2}$ such sets. Hence if G' is not a tree, $|V(G')| \leq |C_E|$ and we are done. If G' is a tree, we further show that at least four of these sets cannot be used, which proves the bound.

When the subgraph induced by C_E is not necessarily connected, we get:

Theorem 2.2 *Let G be an edge-identifiable graph with $\gamma^{EID}(G) = k$. Then $|E(G)| \leq \binom{\frac{4}{3}k}{2}$, $\binom{\frac{4}{3}(k-1)+1}{2}+1$, $\binom{\frac{4}{3}(k-2)+2}{2}+2$ for $k \equiv 0, 1, 2 \mod 3$ respectively.*

In order to prove this, we consider all graphs having maximum number of edges among the graphs having edge identifying code number k. We show that there exists such a graph, G, having an edge identifying code C_E with $|C_E| = k$ such that all but two connected components of $G[C_E]$ are isomorphic to P_4. The other two components can be paths of length at most 6 (with certain restrictions). The bound is tight and extremal examples are built inside the proof. This gives the following corollary:

Corollary 2.3 *Let G be an identifiable line graph. Then $\gamma^{ID}(G) > \frac{3\sqrt{2|V(G)|}}{4}$.*

3 Upper bounds

Several upper bounds for parameter γ^{ID} exist in the literature. It was shown in [8] that any non-trivial identifiable graph of order n admits an identifying code of size $n-1$. The class of graphs G with $\gamma^{ID}(G) = n-1$ was classified in [6]. It is easy to check that none but six of the graphs of this class are line graphs. Thus, if G is an identifiable line graph with $G \notin \{P_3, P_4, C_4, C_4 \bowtie K_1, P_4 \bowtie K_1, \mathcal{L}(K_4)\}$ (where $\bowtie$ denotes the join operation), then $\gamma^{ID}(G) \leq n - 2$. Since $\gamma^{EID}(K_{2,r}) = 2r - 2$, this bound is tight for an infinite family of graphs.

It was conjectured in [7] that for connected identifiable graphs of maximum degree Δ, the general upper bound of $n-1$ can be improved to $n - \frac{n}{\Delta} + O(1)$. For any Δ and arbitrarily large n, there exist Δ-regular line graphs G of order n having $\gamma^{ID}(G) = n - \frac{n}{\Delta}$. Indeed, consider the following construction. Given any Δ-regular loopless multigraph H, subdivide each edge of H once, and let H' be the resulting graph. Now, one can see that the graph $\mathcal{L}(H')$ is twin-free and $\gamma^{ID}(\mathcal{L}(H')) = n - \frac{n}{\Delta}$.

Theorem 3.1 *Let G be an edge-identifiable graph and let C_E be a minimum edge identifying code of G. Then $G' = G[C_E]$ is 2-degenerated.*

In order to prove this, we consider an edge uv belonging to C_E, and use the

local structure of the neighbourhood of uv to define an order $\{v_1, \ldots, v_{|V(G')|}\}$ over $V(G')$ such that each vertex v_i has degree at most 2 in $G[\{v_i, \ldots, v_{|V(G')|}\}]$. By further analysis of the proof we obtain the following result:

Corollary 3.2 *If G is an edge-identifiable graph on n vertices not isomorphic to K_4, then $\gamma^{EID}(G) \leq 2n - 4$.*

Considering the example of K_4 minus one edge, this bound is tight. Moreover, for $r \geq 3$ we have $\gamma^{\mathrm{EID}}(K_{2,r}) = 2r - 2 = 2n - 6$.

By Corollary 3.2 together with some easy calculations, we show that the conjecture of [7] holds for some subclass of line graphs:

Corollary 3.3 *If G is an edge-identifiable graph of average degree at least 5, then $\gamma^{EID}(G) \leq n - \frac{n}{\Delta(\mathcal{L}(G))}$, where $n = |V(\mathcal{L}(G))|$.*

4 Complexity

Given a graph G and an integer k, the EDGE IDCODE problem consists in deciding whether G has an edge identifying code of size at most k. The IDCODE problem asks whether G has an identifying code of size at most k.

The following restricted version of the PLANAR $(\leq 3,3)$-SAT problem was introduced and proved to be NP-complete in [5]. The instance is a collection $\mathcal{Q}$ of clauses over a set X of boolean variables, where each clause contains at most three distinct literals. Each variable appears in exactly two clauses in its non-negated form, and in exactly one clause in its negated form. Finally, the bipartite incidence graph $B(\mathcal{Q})$, defined over the clauses and the variables, is planar. The PLANAR $(\leq 3,3)$-SAT problem is about deciding if $\mathcal{Q}$ can be satisfied, *i.e.* whether there is a truth assignment of the variables of X such that each clause contains at least one true literal.

Note that EDGE IDCODE is expressable in monadic second order logic using quantification over edges. Hence it follows from Courcelle's theorem (see [4]) that it is linear time solvable for any class of graphs having bounded tree-width (e.g. trees, series-parallel graphs or outerplanar graphs). However:

Theorem 4.1 *EDGE IDCODE is NP-complete even when restricted to planar bipartite graphs of maximum degree 3 and arbitrarily large girth.*

The proof is a reduction from PLANAR $(\leq 3,3)$-SAT to EDGE IDCODE. Let $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ be an instance of PLANAR $(\leq 3,3)$-SAT over the set of boolean variables $X = \{x_1, \ldots, x_n\}$. Given two integers $\lambda \geq 2$ and $\mu \geq 1$, we build the graph $G_{\mathcal{Q}}(\lambda, \mu)$ as follows. For each variable x_j and each

clause Q_i we build the subgraphs $G_{x_j}(\mu)$ and $G_{Q_i}(\lambda)$ respectively, as shown in Figures 1, 2 and 3. For each clause $Q_i = \{l_{i_1}, l_{i_2}, l_{i_3}\}$, we identify the vertex representing l_{i_k} in clause gadget $G_{Q_i}(\lambda)$ with one of the vertices representing l_{i_k} in the corresponding variable gadget. The identification of the vertices can be done such that $G_{\mathcal{Q}}(\lambda, \mu)$ is planar. The whole construction can be done in polynomial time. Moreover $G_{\mathcal{Q}}(\lambda, \mu)$ is bipartite, has maximum degree 3, and girth $\min\{4\mu, 8(\lambda+1)\}$. Now, it can be shown that $\mathcal{Q}$ is satisfiable if and only if $G_{\mathcal{Q}}(\lambda, \mu)$ has an identifying code of size at most $k = (21\lambda+4)m+(17\mu-12)n$.
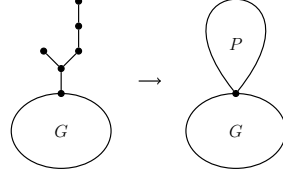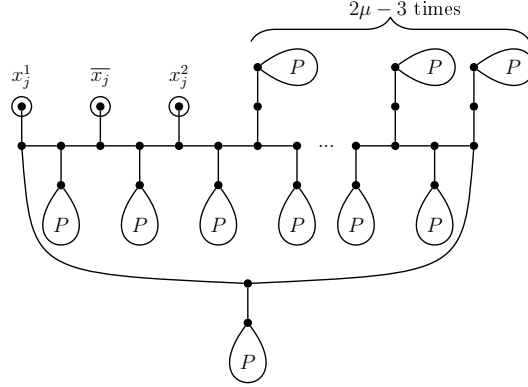

Figure 1. Generic P-gadget

Figure 2. Variable gadget $G_{x_j}(\mu)$

$G_{\mathcal{Q}}(\lambda, \mu)$ is bipartite. Hence $\mathcal{L}(G_{\mathcal{Q}}(\lambda, \mu))$ is perfect [11]. Moreover $\mathcal{L}(G_{\mathcal{Q}}(\lambda, \mu))$ is planar, has maximum degree 4 and clique number 3. Hence we get:

Corollary 4.2 *IDCODE is NP-complete even when restricted to perfect 3-colorable planar line graphs of maximum degree 4.*

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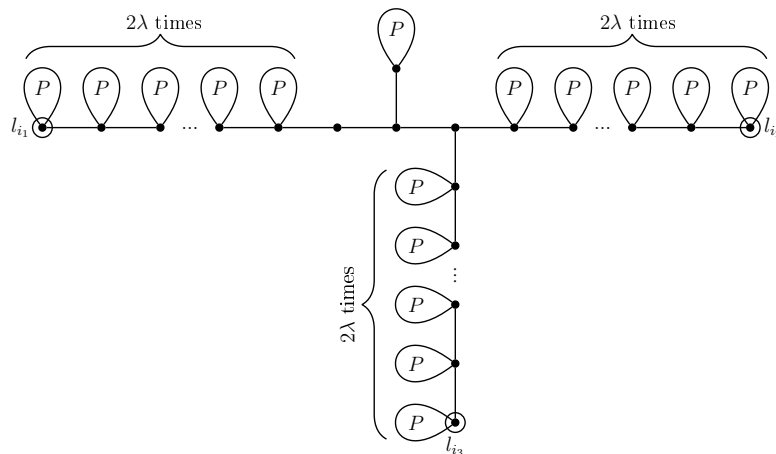
Figure 3. Clause gadget $G_{Q_i}(\lambda)$

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