Bornes pour la taille de codes identifiants dans les graphes de degré maximum Δ^*

Florent Foucaud † Ralf Klasing † Adrian Kosowski †‡ André Raspaud †

Abstract

Etant donné un graphe simple non orienté $G = (V, E)$, un code identifiant de G est un sous-ensemble $C \subseteq V$ tel que C est un ensemble dominant de G, et tout sommet de V est dominé par un sous-ensemble de sommets de C distinct. De par leurs applications à la détection d'erreurs dans des réseaux, les codes identifiants ont été étudiés intensément dans le contexte de la théorie des graphes et de la théorie des codes.

Pour un graphe G, soit $\gamma_{id}(G)$ la cardinalité minimum d'un code identifiant de G. Dans cet exposé, nous examinerons la relation entre le degré maximum d'un graphe et les bornes inférieure et supérieure pour la valeur de γ_{id}. Plus spécifiquement, nous rappellerons une borne inférieure pour γ_{id} dépendant de Δ déjà connue et caractériserons l'ensemble des graphes l'atteignant. Nous donnerons également des bornes supérieures pour γ_{id} dans le cas général et pour les graphes sans triangles. Nous donnons également une borne supérieure pour les graphes de maille au moins 5 faisant intervenir le degré minimum du graphe.

1 Introduction

Herein we study the question of the size of minimum identifying codes for graphs. We mainly focus on graphs of known maximum degree and present one result involving the minimum degree. Identifying codes, first studied in [5] are dominating sets with an additional property which allows us to tell apart any two vertices of the graph based on their neighbourhood within the identifying code. They have found numerous applications in networking and sensor systems, in particular, in fault-diagnosis of multiprocessor networks [5], and in the placement of networked fire detectors in complexes of rooms and corridors [7]. Studies of identifying codes are a natural extension of earlier works on the related concept of locating dominating sets, cf. e.g. [8]. For a given graph, the problem of finding a minimum identifying code is known to be NP-hard [3].

In all further considerations we assume that $G = (V, E)$ is a simple, connected, undirected graph. We denote by $n = |V|$ the order of the graph and by Δ the maximum vertex degree. For a vertex $v \in V$, the *ball* $B_r(v)$ is the set of all vertices of V which are at distance at most r from v. Let us also denote by $N[v] = B_1(v)$ the *closed neighbourhood* of v, and by $N(v) = N[v] \setminus v$, the *neighbourhood* of v. For an integer $k \geq 1$, a set $S \subseteq V$ is called a *k-independent set* if for all $x, y \in S$, $d(x, y) \geq k + 1$. A 1-independent set is simply called an *independent set*. We say that vertex $v \in V$ *dominates* vertex $u \in V$ if $u \in N[v]$. For subsets $C, U \subseteq V$, we say that C *dominates* U if each vertex of U is dominated by some vertex of C. Set $C \subseteq V$ is called a *dominating set* of G if C dominates V. A pair $\{u, v\}$ of vertices of V are *separated* by vertex $x \in V$ if x dominates exactly one of the vertices u and v. We call $C \subseteq V$ an *identifying code* of G if it is a dominating set of G, and for all pairs of vertices $u, v \in V$, u and v are separated by some vertex of C (the latter condition can be equivalently stated as $N[u] \cap C \neq N[v] \cap C$). A graph is called *triangle-free* if it contains no cycles of length 3.

A graph is *identifiable*, i.e., admits an identifying code, if and only if it does not contain a pair of vertices $u, v \in V$ such that $N[u] = N[v]$. An example of a graph which is not identifiable is the complete graph K_n. For an identifiable graph G we denote by $\gamma_{id}(G)$ the cardinality of the minimum identifying code of G.

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†LaBRI - Université Bordeaux 1 - CNRS, 351 cours de la Libération, 33405 Talence cedex, France.

‡Department of Algorithms and System Modeling, Gdańsk University of Technology, Narutowicza 11/12, 80952 Gdańsk, Poland.

For all identifiable graphs, the lower bound on this parameter is $\gamma_{id}(G) \geq \lceil \log_2(n+1) \rceil$ [5]. This bound is tight and the set of graphs reaching it has been characterized (see [6]). For graphs of maximum degree Δ, a lower bound is $\gamma_{id}(G) \geq \frac{2n}{\Delta+2}$ [5]. In Section 2, we show that this bound is tight and characterize the set of graphs reaching it.

The upper bound on parameter γ_{id} is $\gamma_{id}(G) \leq n-1$, and this bound is tight, in particular, for the star $K_{1,n-1}$ [2, 4]. When considering graphs of maximum degree Δ, we show that there exist examples of specific graphs such that $\gamma_{id}(G) = n - \frac{n}{\Delta}$ (e.g., the complete bipartite graph $K_{\Delta,\Delta}$).

We conjecture that for $\Delta \geq 3$, the bound $\gamma_{id} \leq n - \frac{n}{\Delta}$ does in fact hold for all graphs. In this paper we show slightly weaker results. The first result holds for any graph G of maximum degree Δ: $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^4)}$ and $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^2)}$ if G is regular (Theorem 3.1). Our main result holds for triangle-free graphs of maximum degree Δ, namely, that $\gamma_{id} \leq n - \frac{n}{3\Delta+3}$ and $\gamma_{id} \leq n - \frac{n}{2\Delta+2}$ in regular triangle-free graphs (Theorem 3.2). Finally, we give a bound which is linear in n for graphs of girth at least 5 and minimum degree 2: in this case $\gamma_{id} \leq \frac{7n}{8} + 1$ (Theorem 3.3).

Due to space constraints, only proof ideas are presented in this extended abstract.

2 The lower bound

The following theorem has been stated in [5] for Δ-regular graphs, but it is also valid for graphs of maximum degree Δ:

Theorem 2.1 ([5]) *Let $G = (V, E)$ be an undirected, connected graph on n vertices and maximum degree Δ. Then $\gamma_{id}(G) \geq \frac{2n}{\Delta+2}$.*

We show that this bound is tight if $\frac{2n}{\Delta+2}$ is an integer. From the proof of this theorem (which is omitted here), it follows that in a graph reaching the lower bound, any optimal identifying code is an independent set and all non-code vertices have exactly two neighbours in the code, whereas code vertices have degree Δ. Given a value of Δ, we provide a method for constructing such graphs.

1. take any simple, Δ-regular graph D; its vertices will be the code vertices.

2. place an additional vertex on each edge of D: so, the new vertices are all connected to two different vertices of the regular graph D.

3. arbitrary edges can be added between the non-code vertices; this has no influence on the code.

Conversely, it can be shown that all graphs reaching the lower bound can be constructed using this method. An example with the Petersen graph as the graph D is given in Figure 2.1.

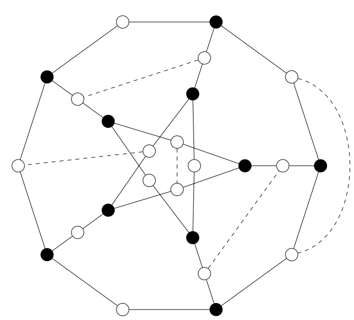

Figure 2.1: Example of an optimal graph of maximum degree 3

3 The upper bounds

First, note that for any Δ, there exists an infinite family of Δ-regular graphs with $\gamma_{id} = n - \frac{n}{\Delta}$. Indeed, consider any Δ-regular graph D and replace every vertex by a clique of Δ vertices. In the resulting graph

D', it can be shown that at least $\Delta - 1$ vertices must be in the identifying code for every clique, and that this is enough. So $\gamma_{id}(D') = n - \frac{n}{\Delta}$. Hence, an upper bound for γ_{id} cannot be smaller than this value.

We now give a first upper bound for parameter γ_{id} in graphs of maximum degree Δ.

Theorem 3.1 *Let $G = (V, E)$ be a connected, identifiable graph with n vertices, and maximum degree Δ. Then $\gamma_{id}(G) \leq n - \frac{n}{\Delta(\Delta^3+\Delta^2-\Delta+1)}$. If G is regular, then $\gamma_{id}(G) \leq n - \frac{n-1}{\Delta^2}$.*

Sketch of the proof We first construct a 4-independent set (resp. 2-independent set in regular graphs) S of G, using a greedy algorithm. The complementary set $V \setminus S$ is almost a valid identifying code of G; however some vertices may remain unseparated. We show that for every such pair of vertices, it is possible to put one neighbouring vertex out of the constructed code, and another one into it instead. Doing this, the resulting set is a valid identifying code having the same cardinality. □

Let us now consider triangle-free graphs. Note that the previously constructed family of graphs having $\gamma_{id} = n - \frac{n}{\Delta}$ is not triangle-free. However, it can easily be shown that the complete bipartite graph $K_{\Delta,\Delta}$ has $\gamma_{id}(K_{\Delta,\Delta}) = n - \frac{n}{\Delta}$. Moreover, for any value of Δ there exists an infinite family of graphs having maximum degree Δ and arbirrarily large order, and for which the value of γ_{id} is close to $n - \frac{n}{\Delta}$. Indeed, $\gamma_{id}(T) = \left\lceil n - \frac{n}{\Delta-1+\frac{1}{\Delta}} \right\rceil$ for any perfectly balanced $(\Delta-1)$-ary tree T [1]. For any value of Δ, we are also able to construct an infinite family of Δ-regular graphs of arbirarily large order for which γ_{id} has value $n - \frac{n}{\frac{2\Delta}{3}}$.

A stronger bound than the one of Theorem 3.1 can be stated for triangle-free graphs in the following theorem:

Theorem 3.2 *Let $G = (V, E)$ be a connected, identifiable triangle-free graph of maximum degree Δ ($\Delta \geq 3$) with n vertices. Then $\gamma_{id}(G) \leq n - \frac{n}{3\Delta+3}$. If G is regular, then $\gamma_{id}(G) \leq n - \frac{n}{2\Delta+2}$.*

Sketch of the proof We first construct an independent set S of G having the property that two vertices of S do not share the same neighbourhood, using a greedy algorithm. The complementary set $V \setminus S$ is almost a valid identifying code of G; however some vertices may remain unseparated. These vertices have the property that they form pairs of code vertices having at least two neighbours each, all of these neighbours being out of the identifying code. We consider the subgraph H induced by these vertices and their neighbours, and show that it is possible to construct a partial identifying code for H which is not too big with respect to the previous solution. This solution is shown to be compatible with the solution for the rest of the graph, and its cardinality can be bounded to give the values of the theorem. □

Let us now consider graphs of girth 5 and of minimum degree $\delta \geq 2$. We are able to show an upper bound for γ_{id} being linear in n in this class of graphs.

Note that, as opposed to all previous results, this bound does not depend on the maximum degree of the graph. This shows that for this class of graphs (unlike in the previously considered graph classes, where for Δ close enough to n, we could exhibit graphs for which γ_{id} is close to $n - 1$), the maximum degree does not play such an important role for the value of parameter γ_{id}.

Note that since a Δ-regular graph has equal maximum and minimum degrees, this result holds for Δ-regular graphs if $\Delta \geq 2$.

Theorem 3.3 *Let $G = (V, E)$ be a connected, identifiable graph with minimum degree $\delta \geq 2$, having n vertices and girth $g \geq 5$. Then $\gamma_{id}(G) \leq \frac{7n}{8} + 1$.*

Sketch of the proof In this proof we construct a rooted Depth-First-Search (DFS) spanning tree T of G. We consider the levels of T and number them modulo 4 in order to partition the vertices into four classes. We take as an identifying code C, the union of three of these four classes such that $|C| \leq \frac{3n}{4}$. It can be shown that most of the vertices are correctly identified. Some specific cases must be looked at, especially sets of leaves in T which are not in C. Using the properties of DFS trees and the hypothesis on the girth and the minimum degree of G, the amount of extra vertices needed to form a correct identifying code of G can be bounded by $\frac{n}{8} + 1$, leading to the bound of the theorem. □

There exists a family of graphs of girth 5 and minimum degree 2 for which γ_{id} is close to $\frac{3n}{5}$: let $p \geq 2$ and $k \geq 2$ be two integers and consider a cycle on k vertices. To each vertex of this cycle, connect p cycles of length 5 by an edge. This graph $G_{p,k}$ has $n = (5p+1)k$ vertices and it can be shown that $\gamma_{id}(G_{p,k}) = \frac{3(n-k)}{5}$.

We summarize our results on upper bounds in Table 1. Each cell of the table corresponds to a class of graphs. Each entry is made out of a pair of two values. The first one corresponds to the value of γ_{id} for an infinite family of graphs having maximum degree Δ and belonging to the corresponding graph class. The second one represents the known upper bound for γ_{id} in the corresponding graph class.

$\Delta \geq 3$	maximum degree Δ	Δ-regular
arbitrary graphs	$\left\langle n - \frac{n}{\Delta},\ n - \frac{n}{\Theta(\Delta^4)} \right\rangle$	$\left\langle n - \frac{n}{\Delta},\ n - \frac{n-1}{\Delta^2} \right\rangle$
triangle-free graphs	$\left\langle n - \frac{n}{\Delta - 1 + \frac{1}{\Delta}},\ n - \frac{n}{3\Delta+3} \right\rangle$	$\left\langle n - \frac{n}{\frac{2\Delta}{3}},\ n - \frac{n}{2\Delta+2} \right\rangle$

	minimum degree $\delta \geq 2$
graphs of girth at least 5	$\left\langle \frac{3n}{5}, \frac{7n}{8} + 1 \right\rangle$

Table 1: Summary of the new upper bounds for identifying codes in graphs with n vertices

4 Conclusion

In this paper, we studied the relationship between the parameter γ_{id} and the maximum degree of a graph. We gave a characterization of all graphs of maximum degree Δ being optimal with respect to γ_{id}. We also gave a series of upper bounds for the value of γ_{id} in graphs of a given maximum degree. These bounds are not tight, but we exhibit some infinite families of graphs having values of γ_{id} close to these bounds. It would be interesting to close the gap between the upper bounds and the highest known values of γ_{id}. We conjecture that for all graphs of maximum degree Δ, $\gamma_{id} \leq n - \frac{n}{\Delta}$ is a tight upper bound for γ_{id}.

Finally, the value of Δ reveals itself to be less important in graphs of minimum degree 2 and girth at least 5, for which we got an upper bound for γ_{id} being linear in n. However, this bound is not tight either; hence, determining a tight upper bound for this class of graphs remains an open problem.

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