Identifying codes in graphs of given maximum degree

(a probabilistic approach)

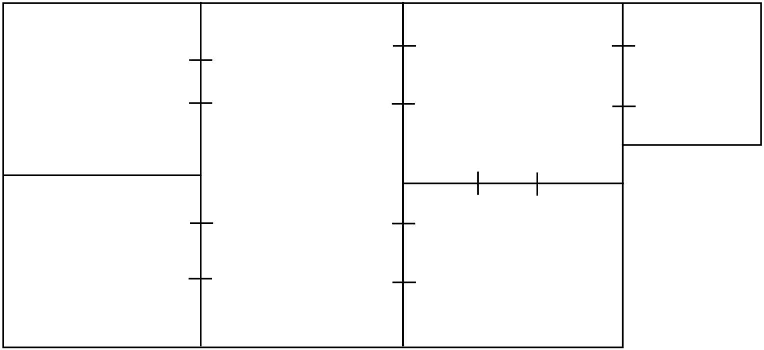
Florent Foucaud (LaBRI, Bordeaux, France)

JGA 2011 - November 18th, 2011

joint work with **Guillem Perarnau** (UPC, Barcelona, Spain)

ANR IDEA

Locating a burglar in a museum



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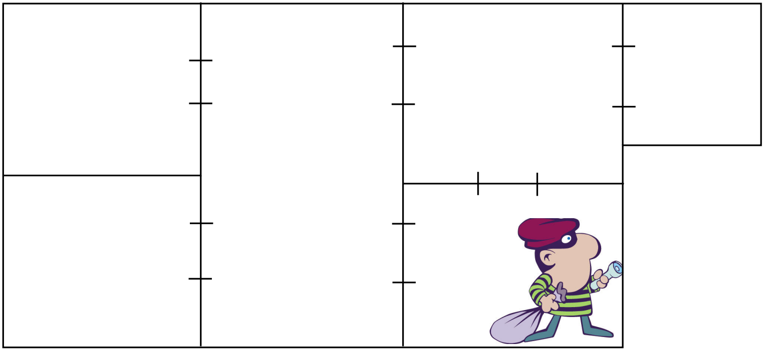

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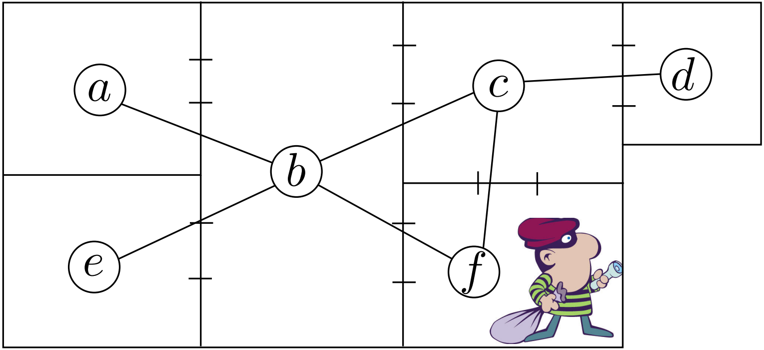

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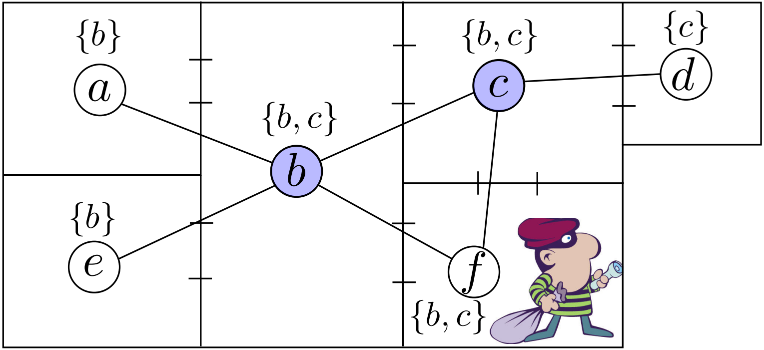

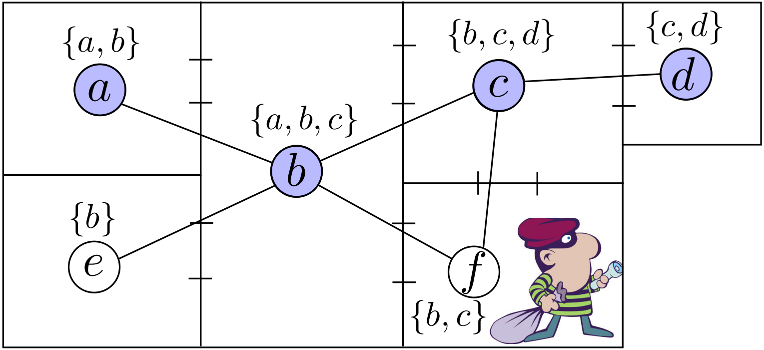
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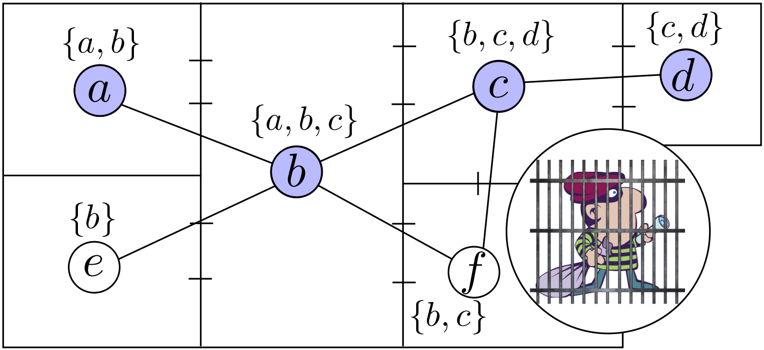
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Identifying codes: definition

Let $N[u]$ be the set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code of G (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of V such that:

- C is a **dominating set** in G: $\forall u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a **separating code** in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation - Identifying code number

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

Identifiable graphs

$N[u]$: set of vertices v s.t. $d(u, v) \leq 1$

Remark

Not all graphs have an identifying code!

Twins = pair u, v such that $N[u] = N[v]$.

A graph is **identifiable** iff it is **twin-free** (i.e. it has no twins).

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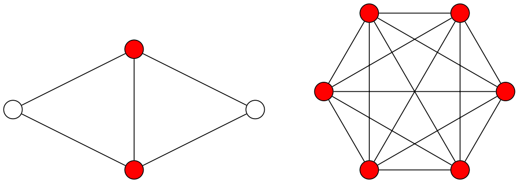
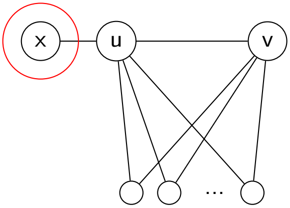


Forced vertices

u, v such that $N[v] \ominus N[u] = \{x\}$

Then $x \in C$, forced by uv.



Notation

Let $NF(G)$ be the proportion of **non forced vertices** of G

$$NF(G) = \frac{\#\text{non-forced vertices in G}}{\#\text{vertices in G}}$$

Note: if G regular, $NF(G) = 1$.

Previous results (1/2)

Theorem (Karpovsky, Chakrabarty, Levitin, 1998 + Gravier, Moncel, 2007)

Let G be an identifiable graph with at least one edge, then

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G) \leq n-1$$

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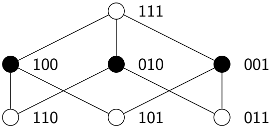

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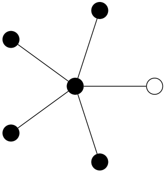

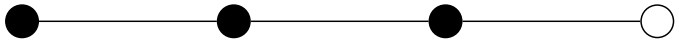

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Let G be an identifiable graph with maximum degree Δ, then

$$\frac{2n}{\Delta+2} \leq \gamma^{\text{ID}}(G)$$

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Conjecture (F., Klasing, Kosowski, Raspaud, 2009+)

Let G be a connected nontrivial identifiable graph of max. degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + O(1)$$

True for $\Delta = 2$ and $\Delta = n-1$.

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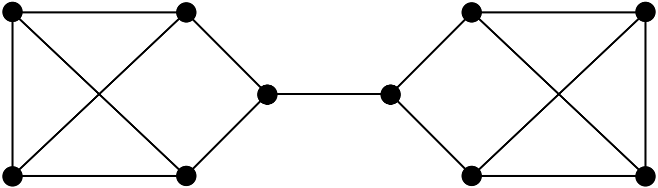

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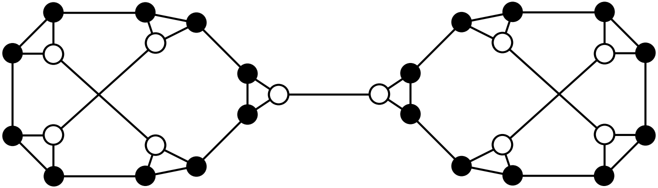

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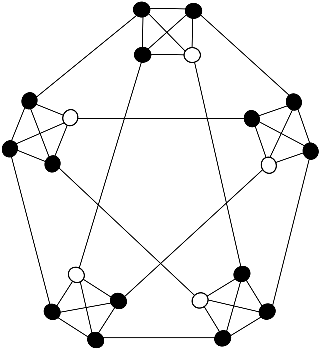

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Let G be a connected identifiable graph of maximum degree Δ. Then

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If G is Δ-regular, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$

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Question

Is it true that $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta)}$?

Technique developed, among others, by Erdős
used mainly in combinatorics (Ramsey theory, graph theory, ...)

1. Define a suitable probability space
2. Select some object from this space using randomness
3. Prove that with nonzero probability, certain "good" conditions hold
4. Conclusion: there always exists a "good" object

Notation

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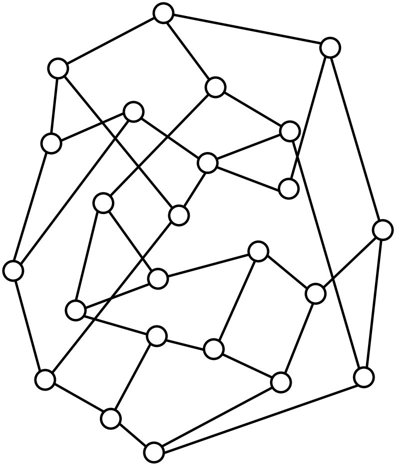
Theorem (F., Perarnau, 2011+)

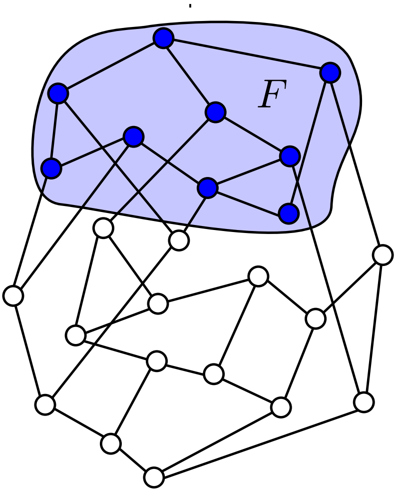
There exists an integer Δ_0 such that for each identifiable graph G on n vertices having maximum degree $\Delta \geq \Delta_0$ and no isolated vertices,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{85\Delta}$$

Proof - select a set at random...

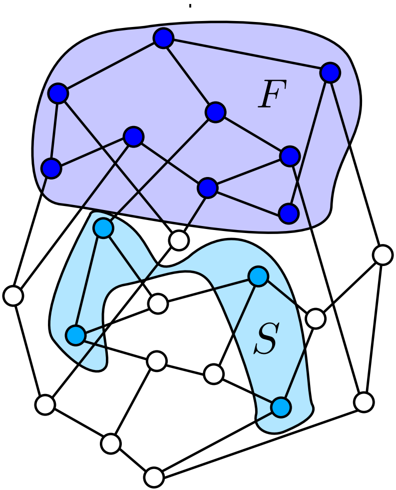


- F: forced vertices



- F: forced vertices

- Select a random set S from $V' = V \setminus F$: each vertex $v \in S$ with prob. p.



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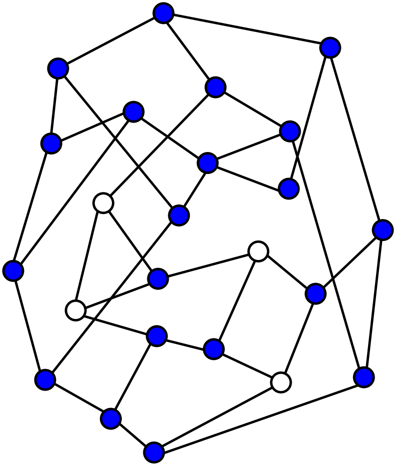

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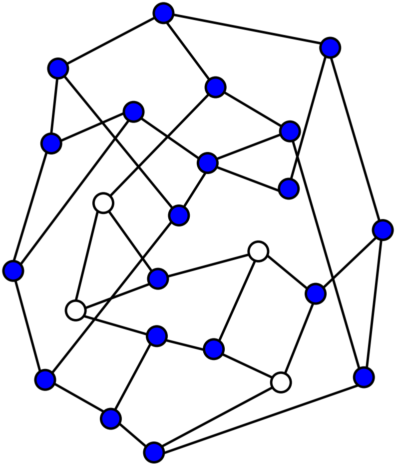
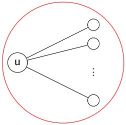


Proof - Using the (weighted) Lovász Local Lemma

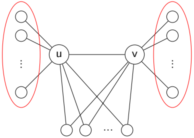
$\mathcal{E} = \{E_1, \ldots, E_M\}$: set of **"bad" events**, dependencies are "rare".

Then: with non-zero probability **none of the bad events occur**.

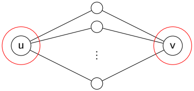
Moreover, this probability can be lower-bounded.



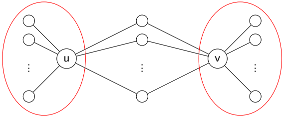
Event A_u



Event $B_{u,v}$

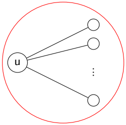


Event $C_{u,v}$



Event $D_{u,v}$

Set the bad events...



$p^{\Delta+1} \leq Pr(A_u) \leq p^2$

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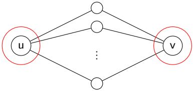
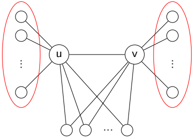


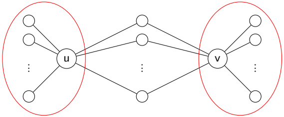
Event $C_{u,v}$

$Pr(C_{u,v}) = p^2$



Event $B_{u,v}$

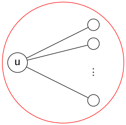
$p^{2\Delta-2} \leq Pr(B_{u,v}) \leq p^2$



Event $D_{u,v}$

$p^{2\Delta} \leq Pr(D_{u,v}) \leq p^4$

Set the bad events...



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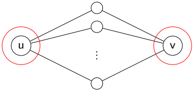
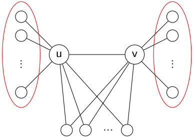


Event $C_{u,v}$

$Pr(C_{u,v}) = p^2$



Event $B_{u,v}$

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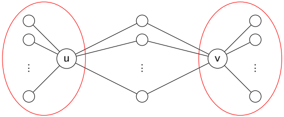

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$p^{2\Delta} \leq Pr(D_{u,v}) \leq p^4$

Taking $p = \frac{1}{k\Delta} \Longrightarrow$ **LLL can be applied**

Proof - the set can be small...

By the LLL we know that

There exists some set S with $\mathbb{E}(|S|) = \frac{n \cdot NF(G)}{k \cdot \Delta}$
such that no bad event occurs

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And we also know: if $S = \emptyset$, $\mathcal{C} = V \setminus S = V$ is a trivial code!

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And we also know: if $S = \emptyset$, $\mathcal{C} = V \setminus S = V$ is a trivial code!

But by the LLL we also know that the probability to have a **good set** S is:

$$\Pr\left(\bigcap_{i=1}^{m} \overline{E_i}\right) > \exp\left\{-\frac{9}{k^2\Delta}n\right\}$$

We have a set of n independent Bernouilli random variables.

Using the **Chernoff bound**, probability that S is **too small**:

$$\Pr\left(\mathbb{E}(|S|) - |S| > \frac{n \cdot NF(G)}{c\Delta}\right) \leq \exp\left\{\frac{kNF(G)}{2c^2\Delta}n\right\}$$

We want: $\Pr(S \text{ good}) - \Pr(S \text{ too small}) > 0$

Proof (regular case) - size of the code

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We get that there exists S such that $V \setminus S$ is an **identifying code**, and

$$|\mathcal{C}| = |V \setminus S| \leq n - \frac{n \cdot NF(G)^2}{85\Delta}$$

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Theorem (F., Perarnau, 2011+)

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Proposition

$$\frac{1}{\Delta+1} \leq NF(G) \leq 1$$

Proof:

Lemma Bertrand, Hudry, 2005

Let G be an identifiable graph having no isolated vertices. Let x be a vertex of G. There exists a non forced vertex y in $N[x]$.

$\Rightarrow$ The set S of non-forced vertices forms a dominating set. Hence $|S| \geq \frac{n}{\Delta+1}$.

Bounding the number of forced vertices

Proposition

Let G be a graph of clique number at most k. There exists a function ρ such that:

$$\frac{1}{\rho(k)} \leq NF(G) \leq 1$$

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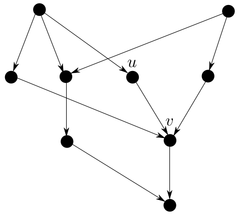
- Define graph $\overrightarrow{H}(G)$
- Max. degree of $\overrightarrow{H}(G)$: $2k-3$
- Longest directed chain of $\overrightarrow{H}(G)$: $k-1$
- Each component has a non-forced vertex
- $\Rightarrow \rho(k) \leq \sum_{i=0}^{k-2}(2k-3)^i$



$u \to v$ if $N[v] = N[u] \cup \{x\}$

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Corollary

- In general, $NF(G) \geq \frac{1}{\Delta+1}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$
- If G is Δ-regular, $NF(G) = 1$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{85\Delta} = n - \frac{n}{\Theta(\Delta)}$
- If G has clique number bounded by k, $NF(G) \geq \frac{1}{\rho(k)}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{85 \cdot (\rho(k))^2 \cdot \Delta} = n - \frac{n}{\Theta(\Delta)}$