EXERCISE 1.13. Prove that Fib(n) is the closest integer to $\phi^n/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers to prove that Fib(n)= $(\phi^n - \psi^n)/\sqrt{5}$.

PROOF. First, it will be proven by induction on the variable n that Fib(n)= $(\phi^n - \psi^n)/\sqrt{5}$.

- $n = 0$ and $n = 1$. $\mathrm{Fib}(0) = 0$ and $\mathrm{Fib}(1) = 1$

$$\frac{\phi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 = \mathrm{Fib}(0)$$

$$\frac{\phi^1 - \psi^1}{\sqrt{5}} = \frac{1+\sqrt{5}-(1-\sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 = \mathrm{Fib}(1)$$

- Assuming it is true for $n-1$ and $n-1$, it is proven for n. Using the definition of the Fibonacci numbers, $\mathrm{Fib}(n) = \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2)$

$$\begin{aligned}
\mathrm{Fib}(n) &= \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \psi^{n-2}}{\sqrt{5}} = \frac{\phi^{n-1} + \phi^{n-2} - (\psi^{n-1} + \psi^{n-2})}{\sqrt{5}} = \\
&= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{n-1}}{2^{n-1}} + \frac{(1+\sqrt{5})^{n-2}}{2^{n-2}} - \frac{(1-\sqrt{5})^{n-1}}{2^{n-1}} - \frac{(1-\sqrt{5})^{n-2}}{2^{n-2}}\right] = \\
&= \frac{1}{2^n\sqrt{5}}\left[2(1+\sqrt{5})^{n-1} + 4(1+\sqrt{5})^{n-2} - 2(1-\sqrt{5})^{n-1} - 4(1-\sqrt{5})^{n-2}\right] = \\
&= \frac{1}{2^n\sqrt{5}}\left\{[2+(1+\sqrt{5})]\cdot 2(1+\sqrt{5})^{n-2} - [2+(1-\sqrt{5})]\cdot 2(1-\sqrt{5})^{n-2}\right\} = \\
&= \frac{1}{2^n\sqrt{5}}\left[(3+\sqrt{5})\cdot 2(1+\sqrt{5})^{n-2} - (3-\sqrt{5})\cdot 2(1-\sqrt{5})^{n-2}\right]
\end{aligned}$$

Now, using that

$$\frac{\phi^n - \psi^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^n}{2^n} - \frac{(1-\sqrt{5})^n}{2^n}\right] = \frac{1}{2^n\sqrt{5}}\left[(1+\sqrt{5})^n - (1-\sqrt{5})^n\right].$$

It is left to see if

$$(1+\sqrt{5})^n - (1-\sqrt{5})^n = \left[(3+\sqrt{5})\cdot 2(1+\sqrt{5})^{n-2} - (3-\sqrt{5})\cdot 2(1-\sqrt{5})^{n-2}\right].$$

Seeing that

$$(1+\sqrt{5})^n = (1+\sqrt{5})^{n-2}(1+\sqrt{5})^2 = (1+\sqrt{5})^{n-2}(6+2\sqrt{5}) = (1+\sqrt{5})^{n-2}\cdot 2(3+\sqrt{5})$$

and that

$$(1-\sqrt{5})^n = (1-\sqrt{5})^{n-2}(1-\sqrt{5})^2 = (1+\sqrt{5})^{n-2}(6-2\sqrt{5}) = (1+\sqrt{5})^{n-2}\cdot 2(3-\sqrt{5})$$

it is proven that Fib(n)= $(\phi^n - \psi^n)/\sqrt{5}$.

Now it is left to prove that Fib(n) is the closest integer to $\phi^n/\sqrt{5}$. Using that

$$\mathrm{Fib}(n) = \frac{(\phi^n - \psi^n)}{\sqrt{5}} = \frac{\phi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}}$$

can be written as

$$\frac{\phi^n}{\sqrt{5}} = \mathrm{Fib}(n) + \frac{\psi^n}{\sqrt{5}}$$

The objective is to prove that

$$-\frac{1}{2} < \mathrm{Fib}(n) - \frac{\phi^n}{\sqrt{5}} < \frac{1}{2}$$

Using the previous expression for $\phi^n/\sqrt{5}$, the objective is rewritten as

$$-\frac{1}{2} < -\frac{\psi^n}{\sqrt{5}} < \frac{1}{2}$$

As $\psi = (1-\sqrt{5})/2$, it follows that $-1 < \psi < 0$, then $-1 < \psi^n < 1$, so $-1 < -\psi^n < 1$.

$$-1 < \psi^n < 1 \Rightarrow -\frac{1}{\sqrt{5}} < -\frac{\psi^n}{\sqrt{5}} < \frac{1}{\sqrt{5}}$$

And as $\sqrt{5} > 2$ the objective expression is true as

$$\frac{1}{2} < -\frac{1}{\sqrt{5}} < -\frac{\psi^n}{\sqrt{5}} < \frac{1}{\sqrt{5}} < \frac{1}{2}$$
$$-\frac{1}{2} < \mathrm{Fib}(n) - \frac{\phi^n}{\sqrt{5}} < \frac{1}{2}$$

$\square$