EXERCISE 2.13. Show that under the assumption of small percentage tolerances there is a simple formula for the approximate percentage tolerance of the product of two intervals in terms of the tolerances of the factors. You may simplify the problem by assuming that all numbers are positive.

PROOF. Let's be $\{c_1, t_1\}$ and $\{c_2, t_2\}$ be two intervals defined by their centres and percentage tolerances. With the given definition of width, the intervals can be rewritten as $(c_1 - w_1, c_1 + w1) = (c1 - c1t1, c1 + c1t1)$ and $(c_2 - w_2, c_2 + w2) = (c2 - c2t2, c2 + c2t2)$.

Under the assumption that all endpoints are positive, it was seen in Exercise 2.11 that the endpoints of the obtained interval after multiplying two former ones are the product of the lower bounds and the product of the upper bounds respectively.

Besides, the percentage tolerance can be expressed as $t = \frac{w}{c} = \frac{\text{upper} - \text{bound} - \text{lower} - \text{bound}}{\text{upper} - \text{bound} + \text{lower} - \text{bound}}$. Using this, let's calculate the percentage tolerance of the product of the two intervals:

$$
\begin{aligned}
t_{12} = \frac{(c1 + c1t1)(c2 + c2t2) - (c1 - c1t1)(c2 - c2t2)}{(c1 + c1t1)(c2 + c2t2) + (c1 - c1t1)(c2 - c2t2)} &= \frac{c1c2(1 + t1)(1 + t2) - c1c2(1 - t1)(1 - t2)}{c1c2(1 + t1)(1 + t2) + c1c2(1 - t1)(1 - t2)} = \\
= \frac{(1 + t1)(1 + t2) - (1 - t1)(1 - t2)}{(1 + t1)(1 + t2) + (1 - t1)(1 - t2)} &= \frac{1 + t1 + t2 + t1t2 - 1 + t1 + t2 - t1t2}{1 + t1 + t2 + t1t2 + 1 - t1 - t2 + t1t2} = \\
&= \frac{2t1 + 2t2}{2 + 2t1t2} = \frac{t1 + t2}{1 + t1t2}
\end{aligned}
$$

Finally, if the percentage tolerances are small enough, the product $t1t2$ can be omitted, giving $t_{12} \approx t_1 + t_2$.

□