On the Decidability of Termination for Polynomial Loops*

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Abstract. We consider the termination problem for triangular weakly non-linear loops (*twn*-loops) over a ring $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}$. The body of such a loop consists of a single assignment $\begin{bmatrix} x_1 \\ \cdots \\ x_d \end{bmatrix} \leftarrow \begin{bmatrix} c_1 \cdot x_1 + p_1 \\ \cdots \\ c_d \cdot x_d + p_d \end{bmatrix}$ where each x_i is a variable, $c_i \in \mathcal{S}$, and each p_i is a (possibly non-linear) polynomial over $\mathcal{S}$ and the variables $x_{i+1}, \ldots, x_d$.
We present a reduction from the question of termination to the existential fragment of the first-order theory of $\mathcal{S}$ and $\mathbb{R}$ ($\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$). For loops over $\mathbb{R}$, our reduction entails decidability of termination. For loops over $\mathbb{Z}$ or $\mathbb{Q}$, it proves semi-decidability of non-termination.
Furthermore, we show how to transform loops where the right-hand side of the assignment in the loop body consists of arbitrary polynomials into *twn*-loops. Then the original loop terminates iff the transformed loop terminates over a certain subset of $\mathbb{R}$, which can also be checked via our reduction to $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$. This transformation allows us to prove Co-NP-completeness for the termination problem over $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ for an important class of loops which can *always* be transformed into *twn*-loops.

1 Introduction

We consider loops of the form

$$\textbf{while } \varphi \textbf{ do } \vec{x} \leftarrow \vec{a}. \tag{1}$$

Here, $\vec{x}$ is a vector[3] of $d \geq 1$ pairwise different variables $x_1, \ldots, x_d$ that range over a ring $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}$, where $\leq$ denotes the subring relation. Moreover, $\vec{a} \in (\mathcal{S}[\vec{x}])^d$ where $\mathcal{S}[\vec{x}]$ is the set of all polynomials over $\vec{x}$ with coefficients from $\mathcal{S}$. The condition φ is a propositional formula over the atoms $\{p \triangleright 0 \mid p \in K_\mathcal{S}[\vec{x}], \triangleright \in \{\geq, >\}\}$,[4] where $K_\mathcal{S}$ is the quotient field of $\mathcal{S}$, i.e., the smallest field in which $\mathcal{S}$ can be embedded. So loops over $\mathcal{S} = \mathbb{Z}$ may contain polynomials from $\mathbb{Z}$'s quotient field $K_\mathcal{S} = \mathbb{Q}$

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[3] We use row- and column-vectors interchangeably to improve readability.

[4] In examples, we also use other relations like "=" and atoms of the form "$\alpha \triangleright \beta$" with $\beta \in K_\mathcal{S}[\vec{x}]$ as syntactic sugar. Negation is also syntactic sugar in our setting, as, e.g., $\neg(p > 0)$ is equivalent to $-p \geq 0$. So w.l.o.g. φ is built from atoms, $\wedge$, and $\vee$.

in its condition. As usual, we assume that all constants in (1) are algebraic numbers, as it is unclear how to represent transcendental numbers in programs.

We often represent a loop (1) by the tuple $(\varphi, \vec{a})$ of the *loop condition* φ and the *update* $\vec{a} = (a_1, \ldots, a_d)$. Unless stated otherwise, $(\varphi, \vec{a})$ is always a loop on $\mathcal{S}^d$ using the variables $\vec{x} = (x_1, \ldots, x_d)$ where $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}$ throughout the paper.

There exist several decidability results for the termination of linear[5] loops with only conjunctions in their loop conditions [3, 5, 12, 16, 22, 27]. In this paper, we present the first decidability results for termination of *non-linear* loops as well as for a novel class of linear loops whose conditions may also contain disjunctions. More precisely, we identify new sub-classes of loops of the form (1) where (non-)termination is (semi-)decidable. For the linear case, we also investigate the complexity of the termination problem.

After introducing preliminaries in Sect. 2, in Sect. 3 we show how to transform loops of the form (1) into a more restricted format, so-called *twn-loops*. While this transformation is incomplete in general (i.e., we cannot transform *arbitrary* loops into *twn*-loops), in Sect. 6 we show that it is complete for so-called *linear-update loops with real spectrum*.

Next, we present a reduction from termination of *twn*-loops to the existential fragment of the first-order theory of $\mathcal{S}$ and $\mathbb{R}$ in Sect. 4 and 5. It proceeds in two steps: We first compute closed forms for *twn*-loops in Sect. 4, which is a straightforward extension of our technique from [12] for linear updates. Then, we use these closed forms to reduce termination of *twn*-loops to the existential fragment of the respective first-order theory in Sect. 5. As an immediate consequence of Sect. 4 and 5, we obtain the novel results that termination of *twn*-loops over $\mathbb{R}$ is decidable and non-termination of *twn*-loops over $\mathbb{Z}$ and $\mathbb{Q}$ is semi-decidable. For those classes of loops where our transformation from Sect. 3 is complete (e.g., linear-update loops with real spectrum), we obtain analogous decidability results.

Finally, Sect. 6 analyzes the complexity of the transformation from Sect. 3 and the reduction from Sect. 4 and 5. This allows us to prove the new result that termination of *linear loops over $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ with real spectrum* is Co-NP-complete and that termination of *linear-update loops with real spectrum* is $\exists\mathbb{R}$-complete.

Sect. 7 discusses related work and the appendix contains all missing proofs.

2 Preliminaries

For any entity s, $s[x/t]$ is the entity that results from s by replacing all free occurrences of x by t. Similarly, if $\vec{x} = (x_1, \ldots, x_d)$ and $\vec{t} = (t_1, \ldots, t_d)$, then $s[\vec{x}/\vec{t}]$ results from s by replacing all free occurrences of x_i by t_i, for each $1 \leq i \leq d$.

Any vector of polynomials $\vec{a} \in (\mathcal{S}[\vec{x}])^d$ can also be regarded as a function $\vec{a} : (\mathcal{S}[\vec{x}])^d \to (\mathcal{S}[\vec{x}])^d$, where for any $\vec{p} \in (\mathcal{S}[\vec{x}])^d$, $\vec{a}(\vec{p}) = \vec{a}[\vec{x}/\vec{p}]$ results from *applying* the polynomials $\vec{a}$ to the polynomials $\vec{p}$. In a similar way, we can also apply

[5] In this paper "linear" refers to "linear polynomial arithmetic", which, e.g., also covers *affine* updates of the form $\vec{x} \leftarrow A \cdot \vec{x} + \vec{b}$.

a formula to polynomials $\vec{p} \in (\mathcal{S}[\vec{x}])^d$. To this end, we define $\psi(\vec{p}) = \psi[\vec{x}/\vec{p}]$ for first-order formulas ψ with free variables $\vec{x}$. As usual, function application associates to the left, i.e., $\vec{a}(\vec{b})(\vec{p})$ stands for $(\vec{a}(\vec{b}))(\vec{p})$. However, since applying polynomials only means that one instantiates variables, we obviously have $(\vec{a}(\vec{b}))(\vec{p}) = \vec{a}(\vec{b}(\vec{p}))$.

Def. 1 formalizes the intuitive notion of termination for a loop $(\varphi, \vec{a})$.

Definition 1 (Termination). *If*

$$\exists \vec{c} \in \mathcal{S}^d.\ \forall n \in \mathbb{N}.\ \varphi(\vec{a}^n(\vec{c})),$$

then $(\varphi, \vec{a})$ *is* non-terminating *and* $\vec{c}$ *is a* witness *for non-termination. Otherwise,* $(\varphi, \vec{a})$ terminates*.*

Here, $\vec{a}^n$ denotes the n-fold application of $\vec{a}$, i.e., $\vec{a}^0(\vec{c}) = \vec{c}$ and $\vec{a}^{n+1}(\vec{c}) = \vec{a}(\vec{a}^n(\vec{c}))$.

For any entity s, let $\mathcal{V}(s)$ be the set of all free variables that occur in s. Given an assignment $\vec{x} \leftarrow \vec{a}$, the relation $\succ_{\vec{a}} \in \mathcal{V}(\vec{a}) \times \mathcal{V}(\vec{a})$ is the transitive closure of $\{(x_i, x_j) \mid i, j \in \{1, \ldots, d\}, i \neq j, x_i \in \mathcal{V}(a_j)\}$. We call $(\varphi, \vec{a})$ *triangular* if $\succ_{\vec{a}}$ is well founded. So the restriction to triangular loops prohibits "cyclic dependencies" of variables (e.g., where the new values of x_1 and x_2 both depend on the old values of x_1 and x_2). For example, a loop whose body consists of the assignment $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \leftarrow \left[\begin{smallmatrix} x_1 + x_2^2 \\ x_2 - 1 \end{smallmatrix}\right]$ is triangular since $\succ = \{(x_1, x_2)\}$ is well founded, whereas a loop with the body $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \leftarrow \left[\begin{smallmatrix} x_1 + x_2^2 \\ x_1 - 1 \end{smallmatrix}\right]$ is not triangular. Triangularity allows us to compute *closed forms* for the n-fold application of the loop update $\vec{a}$, i.e., vectors $\vec{q}$ of d expressions over the variables $\vec{x}$ and n such that $\vec{q} = \vec{a}^n$, by handling one variable after the other.

Furthermore $(\varphi, \vec{a})$ is *weakly non-linear* if x_i does not occur in non-linear monomials in a_i, for all $1 \leq i \leq d$. So for example, a loop with the body $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \leftarrow \left[\begin{smallmatrix} x_1 + x_2^2 \\ x_2 - 1 \end{smallmatrix}\right]$ is weakly non-linear, whereas a loop with the body $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \leftarrow \left[\begin{smallmatrix} x_1 \cdot x_2 \\ x_2 - 1 \end{smallmatrix}\right]$ is not. Like triangularity, weak non-linearity is needed to ensure that we can always compute closed forms.

A *twn*-loop is triangular and weakly non-linear. If $(\varphi, \vec{a})$ is weakly non-linear and x_i's coefficient in a_i is non-negative for all $1 \leq i \leq d$, then $(\varphi, \vec{a})$ is *non-negative*. A *tnn*-loop is triangular and non-negative (and thus, also weakly non-linear).

If $\vec{a} = A \cdot \vec{x} + \vec{b}$ for some $A \in \mathcal{S}^{d \times d}$ without complex eigenvalues and some $\vec{b} \in \mathcal{S}^d$, then $(\varphi, \vec{a})$ is a *linear-update loop with real spectrum*. If moreover, φ only consists of linear inequations, then it is a *linear loop with real spectrum*.

For a ring $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}$, the *existential fragment of the first-order theory of* $\mathcal{S}$ is the set $\mathrm{Th}_\exists(\mathcal{S})$ of all formulas $\exists \vec{y} \in \mathcal{S}^k.\, \psi$, where ψ is a propositional formula over the atoms $\{p \triangleright 0 \mid p \in \mathcal{S}[\vec{y}, \vec{z}],\ \triangleright \in \{\geq, >\}\}$ and $k \in \mathbb{N}$. Here, $\vec{y}$ and $\vec{z}$ are pairwise disjoint vectors of variables (i.e., the variables $\vec{z}$ are *free*). Moreover, $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$ is the set of all formulas $\exists \vec{y}\,' \in \mathbb{R}^{k'}, \vec{y} \in \mathcal{S}^k.\, \psi$, with a propositional formula ψ over $\{p \triangleright 0 \mid p \in \mathcal{S}[\vec{y}\,', \vec{y}, \vec{z}],\ \triangleright \in \{\geq, >\}\}$ where $k', k \in \mathbb{N}$ and the variables $\vec{y}\,'$, $\vec{y}$, and $\vec{z}$ are pairwise disjoint. As usual, a formula is *closed* if it does not have any free variables.

For decision problems P and Q, we say that P is *reducible* to Q if there is a *reduction* from P to Q, i.e., a computable function $f : P \to Q$ that maps instances of P to instances of Q such that $f(x) \iff x$ for all $x \in P$.

3 Transformation to Triangular Weakly Non-Linear Form

In this section, we show how to handle loops over $\mathcal{S}$ that are not yet *twn*. To this end, we introduce a transformation of loops via *polynomial automorphisms* in Sect. 3.1 and show that our transformation (which also allows us to switch from $\mathcal{S}$ to a superring of $\mathcal{S}$) indeed preserves (non-)termination (Thm. 9). In Sect. 3.2, we use results from algebraic geometry to show that the question whether a loop can be transformed into *twn*-form is reducible to validity of $\mathrm{Th}_\exists(\mathcal{S})$-formulas (Thm. 18). Moreover, we show that it is decidable whether a *linear* automorphism can transform a loop into a special case of the *twn*-form (Thm. 20).

3.1 Transforming Loops

Clearly, the *polynomials* $x_1, \ldots, x_d$ are *generators* of the $\mathcal{S}$-algebra $\mathcal{S}[\vec{x}]$, i.e., every polynomial from $\mathcal{S}[\vec{x}]$ can be obtained from $x_1, \ldots, x_d$ and the operations of the algebra (i.e., addition and multiplication). So far, we have implicitly chosen a special "representation" of the loop based on the generators $x_1, \ldots, x_d$.

We now change this representation, i.e., we use a different set of d polynomials which are also generators of $\mathcal{S}[\vec{x}]$. Then the loop has to be modified accordingly in order to adapt it to this new representation. This modification does not affect the loop's termination behavior, but it may transform a non-*twn*-loop into *twn*-form.

The desired change of representation is described by $\mathcal{S}$*-automorphisms* of $\mathcal{S}[\vec{x}]$. As usual, an $\mathcal{S}$*-endomorphism* of $\mathcal{S}[\vec{x}]$ is a mapping $\eta : \mathcal{S}[\vec{x}] \to \mathcal{S}[\vec{x}]$ which is $\mathcal{S}$-linear and multiplicative.[6] We denote the ring of $\mathcal{S}$-endomorphisms of $\mathcal{S}[\vec{x}]$ by $\mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ (where the operations on this ring are pointwise addition and function composition $\circ$). The group of $\mathcal{S}$-automorphisms of $\mathcal{S}[\vec{x}]$ is $\mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$'s group of units, and we denote it by $\mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$. So an $\mathcal{S}$-automorphism of $\mathcal{S}[\vec{x}]$ is an $\eta \in \mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ that is *invertible*. Thus, there exists an $\eta^{-1} \in \mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ such that $\eta \circ \eta^{-1} = \eta^{-1} \circ \eta = id_{\mathcal{S}[\vec{x}]}$, where $id_{\mathcal{S}[\vec{x}]}$ is the identity function on $\mathcal{S}[\vec{x}]$.

Example 2 (Automorphism). Let $\eta \in \mathrm{End}_{\mathcal{S}}(\mathcal{S}[x_1, x_2])$ with $\eta(x_1) = x_2$ and $\eta(x_2) = x_1 - x_2^2$. Then $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[x_1, x_2])$, where $\eta^{-1}(x_1) = x_1^2 + x_2$ and $\eta^{-1}(x_2) = x_1$.

As $\mathcal{S}[\vec{x}]$ is free on the generators $\vec{x}$, an endomorphism $\eta \in \mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ is uniquely determined by the images of the variables, i.e., by $\eta(x_1), \ldots, \eta(x_d)$. Hence we have a one-to-one correspondence between elements of $(\mathcal{S}[\vec{x}])^d$ and $\mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$. In particular, every tuple $\vec{a} = (a_1, \ldots, a_d) \in (\mathcal{S}[\vec{x}])^d$ corresponds to the unique endomorphism $\widetilde{a} \in \mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ with $\widetilde{a}(x_i) = a_i$ for all $1 \le i \le d$. So for any $p \in \mathcal{S}[\vec{x}]$ we have $\widetilde{a}(p) = p(\vec{a})$. Thus, the update of a loop induces an endomorphism which operates on polynomials.

Example 3 (Updates as Endomorphisms). Consider the loop

$$\textbf{while } x_2^3 + x_1 - x_2^2 > 0 \textbf{ do } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

[6] So we have $\eta(r \cdot p + s \cdot q) = r \cdot \eta(p) + s \cdot \eta(q)$, $\eta(1) = 1$, and $\eta(p \cdot q) = \eta(p) \cdot \eta(q)$ for all $r, s \in \mathcal{S}$ and all $p, q \in \mathcal{S}[\vec{x}]$.

where $a_1 = \Big(\big(-x_2^2+x_1\big)^2 + x_2\Big)^2 - 2\cdot x_2^2 + 2\cdot x_1$ and $a_2 = \big(-x_2^2+x_1\big)^2 + x_2$, i.e., $\varphi = (x_2^3 + x_1 - x_2^2 > 0)$ and $\vec{a} = (a_1, a_2)$. Then $\vec{a}$ induces the endomorphism $\widetilde{a}$ with $\widetilde{a}(x_1) = a_1$ and $\widetilde{a}(x_2) = a_2$. So we have $\widetilde{a}(2\cdot x_1 + x_2^3) = (2\cdot x_1 + x_2^3)(\vec{a}) = 2\cdot a_1 + a_2^3$.

For tuples of numbers (e.g., $\vec{c} = (5,2)$), the endomorphism $\widetilde{c}$ is $\widetilde{c}(x_1) = 5$ and $\widetilde{c}(x_2) = 2$. Thus, we have $\widetilde{c}(x_2^3 + x_1 - x_2^2) = (x_2^3 + x_1 - x_2^2)(5,2) = 2^3 + 5 - 2^2 = 9$.

We extend the application of endomorphisms $\eta : \mathcal{S}[\vec{x}] \to \mathcal{S}[\vec{x}]$ to vectors of polynomials $\vec{a} = (a_1, \ldots, a_d)$ by defining $\eta(\vec{a}) = (\eta(a_1), \ldots, \eta(a_d))$ and to formulas $\varphi \in \mathrm{Th}_\exists(\mathcal{S})$ by defining $\eta(\varphi) = \varphi(\eta(\vec{x}))$, i.e., $\eta(\varphi)$ results from φ by applying η to all polynomials that occur in φ. This allows us to transform $(\varphi, \vec{a})$ into a new loop $Tr_\eta(\varphi, \vec{a})$ using any automorphism $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$.

Definition 4 **(*Tr*).** *Let $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$. We define $Tr_\eta(\varphi, \vec{a}) = (\varphi', \vec{a}')$ where*

$$\varphi' = \eta^{-1}(\varphi) \qquad \textit{and} \qquad \vec{a}' = (\eta^{-1} \circ \widetilde{a} \circ \eta)(\vec{x}).$$[7]

Example 5 (Transforming Loops). We transform the loop $(\varphi, \vec{a})$ from Ex. 3 with the automorphism η from Ex. 2. We obtain $Tr_\eta(\varphi, \vec{a}) = (\varphi', \vec{a}')$ where

$$\begin{aligned}
\varphi' &= \eta^{-1}(\varphi) = \Big(\big(\eta^{-1}(x_2)\big)^3 + \eta^{-1}(x_1) - \big(\eta^{-1}(x_2)\big)^2 > 0\Big) \\
&= (x_1^3 + x_1^2 + x_2 - x_1^2 > 0) = (x_1^3 + x_2 > 0) \qquad\qquad \text{and} \\
\vec{a}' &= \big((\eta^{-1} \circ \widetilde{a} \circ \eta)(x_1), (\eta^{-1} \circ \widetilde{a} \circ \eta)(x_2)\big) = \big(\eta^{-1}(\widetilde{a}(x_2)), \eta^{-1}(\widetilde{a}(x_1 - x_2^2))\big) \\
&= \qquad \big(\eta^{-1}(a_2), \eta^{-1}(a_1 - a_2^2)\big) \qquad = \qquad (x_1 + x_2^2, 2\cdot x_2).
\end{aligned}$$

So the resulting transformed loop is:

$$\textbf{while } x_1^3 + x_2 > 0 \textbf{ do } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 + x_2^2 \\ 2\cdot x_2 \end{bmatrix}$$

Note that while the original loop $(\varphi, \vec{a})$ is neither triangular nor weakly non-linear, the resulting transformed loop is *twn*. Also note that we used a *non-linear* automorphism with $\eta(x_2) = x_1 - x_2^2$ for the transformation.

While the above example shows that our transformation can indeed transform non-*twn*-loops into *twn*-loops, it remains to prove that this transformation preserves (non-)termination. Then we can use our techniques for termination analysis of *twn*-loops for *twn-transformable*-loops as well, i.e., for all loops $(\varphi, \vec{a})$ where $Tr_\eta(\varphi, \vec{a})$ is *twn* for some automorphism η. (The question how to find such automorphisms will be addressed in Sect. 3.2.)

To this end, we first prove that our transformation is "compatible" with the operation $\circ$ of the group $\mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$, i.e., that it is an *action*.[8]

Lemma 6 **(*Tr* is an Action).** *The transformation from Def. 4 is a* right action *of $\mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ on loops, i.e.,*

- $Tr_{id_{\mathcal{S}[\vec{x}]}}(\varphi, \vec{a}) = (\varphi, \vec{a})$ *and*
- $Tr_{\eta_1 \circ \eta_2}(\varphi, \vec{a}) = Tr_{\eta_2}(Tr_{\eta_1}(\varphi, \vec{a}))$ *for all $\eta_1, \eta_2 \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$.*

[7] In other words, we have $\vec{a}' = (\eta(\vec{x}))\,(\vec{a})\,(\eta^{-1}(\vec{x}))$, since $(\eta^{-1} \circ \widetilde{a} \circ \eta)(\vec{x}) = \eta^{-1}(\eta(\vec{x})[\vec{x}/\vec{a}]) = \eta(\vec{x})[\vec{x}/\vec{a}][\vec{x}/\eta^{-1}(\vec{x})] = (\eta(\vec{x}))(\vec{a})(\eta^{-1}(\vec{x}))$.

[8] We consider right actions.

Now we show that a witness for non-termination of $(\varphi, \vec{a})$ is transformed by $\eta(\vec{x})$ into a witness for non-termination of $Tr_\eta(\varphi, \vec{a})$.

Lemma 7 (*Tr* Preserves Witnesses). *If $\vec{c} \in \mathcal{S}^d$ witnesses non-termination of $(\varphi, \vec{a})$, then $\widehat{\eta}(\vec{c})$ witnesses non-termination of $Tr_\eta(\varphi, \vec{a})$. Here, $\widehat{\eta} : \mathcal{S}^d \to \mathcal{S}^d$ maps $\vec{c}$ to $\widehat{\eta}(\vec{c}) = \widetilde{c}(\eta(\vec{x})) = (\eta(\vec{x}))\,(\vec{c})$.*

Example 8 (Transforming Witnesses). For the tuple $\vec{c} = (5, 2)$ from Ex. 3 and the automorphism η from Ex. 2 with $\eta(x_1) = x_2$ and $\eta(x_2) = x_1 - x_2^2$, we obtain

$$\widehat{\eta}(\vec{c}) = (\eta(x_1), \eta(x_2))\,(\vec{c}) = (2, 5 - 2^2) = (2, 1).$$

As $\vec{c} = (5, 2)$ witnesses non-termination of Ex. 3, $\widehat{\eta}(\vec{c}) = (2, 1)$ witnesses non-termination of $Tr_\eta(\varphi, \vec{a})$ due to Lemma 7.

Finally, we can prove that transforming a loop preserves (non-)termination.

Theorem 9 (*Tr* Preserves Termination). *If $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$, then $(\varphi, \vec{a})$ terminates iff $Tr_\eta(\varphi, \vec{a})$ terminates. Furthermore, $\widehat{\eta}$ is a bijection between the respective sets of witnesses for non-termination.*

Up to now we only transformed a loop $(\varphi, \vec{a})$ on $\mathcal{S}_1^d$ using elements of $\mathrm{Aut}_{\mathcal{S}_1}(\mathcal{S}_1[\vec{x}])$. However, we can also transform it into the loop $Tr_\eta(\varphi, \vec{a})$ on $\mathcal{S}_2^d$ if $\mathcal{S}_1 \leq \mathcal{S}_2$ and $\eta \in \mathrm{Aut}_{\mathcal{S}_2}(\mathcal{S}_2[\vec{x}])$. Nevertheless, our goal remains to prove termination on $\mathcal{S}_1^d$ instead of $\mathcal{S}_2^d$, which is not equivalent in general. Thus, in Sect. 5 we will show how to analyze termination of loops on certain subsets F of $\mathcal{S}_2^d$. This allows us to analyze termination of $(\varphi, \vec{a})$ on $\mathcal{S}_1^d$ by checking termination of $Tr_\eta(\varphi, \vec{a})$ on the subset $\widehat{\eta}(\mathcal{S}_1^d) \subseteq \mathcal{S}_2^d$ instead.

By our definition of loops over a ring $\mathcal{S}$, we have $\vec{a}(\vec{c}) \in \mathcal{S}^d$ for all $\vec{c} \in \mathcal{S}^d$, i.e., $\mathcal{S}^d$ is *$\vec{a}$-invariant*. This property is preserved by our transformation.

Definition 10 ($\vec{a}$-Invariance). *Let $(\varphi, \vec{a})$ be a loop on $\mathcal{S}^d$ and let $F \subseteq \mathcal{S}^d$. We call F* $\vec{a}$-invariant *or* update-invariant *if for all $\vec{c} \in F$ we have $\vec{a}(\vec{c}) \in F$.*

Lemma 11 (*Tr* Preserves Invariance). *Let $\mathbb{Z} \leq \mathcal{S}_1 \leq \mathcal{S}_2 \leq \mathbb{R}$ be rings, let $(\varphi, \vec{a})$ be a loop on $\mathcal{S}_1^d$, let $F \subseteq \mathcal{S}_1^d$ be an $\vec{a}$-invariant set, and let $\eta \in \mathrm{Aut}_{\mathcal{S}_2}(\mathcal{S}_2[\vec{x}])$. Furthermore, let $Tr_\eta(\varphi, \vec{a}) = (\varphi', \vec{a}')$. Then $\widehat{\eta}(F)$ is $\vec{a}'$-invariant.*

Recall that our goal is to reduce termination to validity of a $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$-formula. Clearly, *termination on* F cannot be encoded with such a formula if F cannot be defined via $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$. Thus, we require that F is $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$*-definable*.

Definition 12 ($\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$-Definability). *A set $F \subseteq \mathbb{R}^d$ is* $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$-definable *if there is a formula $\psi \in \mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$ with free variables $\vec{x}$ such that for all $\vec{c} \in \mathbb{R}^d$*

$$\vec{c} \in F \quad \textit{iff} \quad \psi(\vec{c}) \textit{ is valid.}$$

An example for a $\mathrm{Th}_\exists(\mathbb{Z}, \mathbb{R})$-definable set is $\{(a, 0, a) \mid a \in \mathbb{Z}\}$, which is characterized by the formula $\exists a \in \mathbb{Z}.\ x_1 = a \wedge x_2 = 0 \wedge x_3 = a$.

To analyze termination of $(\varphi, \vec{a})$ on $\mathcal{S}_1^d$, by Thm. 9 we can analyze termination of $Tr_\eta(\varphi, \vec{a})$ on $\widehat{\eta}(\mathcal{S}_1^d) \subseteq \mathcal{S}_2^d$ instead. While $\mathcal{S}_1^d$ is clearly $\mathrm{Th}_\exists(\mathcal{S}_1, \mathbb{R})$-definable, the following lemma shows that $\widehat{\eta}(\mathcal{S}_1^d)$ is $\mathrm{Th}_\exists(\mathcal{S}_1, \mathbb{R})$-definable, too. More precisely, $\mathrm{Th}_\exists(\mathcal{S}_1, \mathbb{R})$-definability is preserved by polynomial endomorphisms.

Lemma 13 (*Tr* Preserves Definability). *Let $\mathbb{Z} \le \mathcal{S}_1 \le \mathcal{S}_2 \le \mathbb{R}$ be rings and let $\eta \in \mathrm{End}_{\mathcal{S}_2}(\mathcal{S}_2[\vec{x}])$. If $F \subseteq \mathbb{R}^d$ is $\mathrm{Th}_\exists(\mathcal{S}_1, \mathbb{R})$-definable then so is $\widehat{\eta}(F)$.*

Example 14. The set $\mathbb{Z}^2$ is $\mathrm{Th}_\exists(\mathbb{Z}, \mathbb{R})$-definable, as we have $(x_1, x_2) \in \mathbb{Z}^2$ iff

$$\exists a, b \in \mathbb{Z}.\ x_1 = a \wedge x_2 = b.$$

Let $\eta \in \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[\vec{x}])$ with $\eta(x_1) = \frac{1}{2} \cdot x_1^2 + x_2^2$ and $\eta(x_2) = x_2^2$. Then $\widehat{\eta}(\mathbb{Z}^2)$ is also $\mathrm{Th}_\exists(\mathbb{Z}, \mathbb{R})$-definable, because for $x_1, x_2 \in \mathbb{R}$, we have $(x_1, x_2) \in \eta(\mathbb{Z}^2)$ iff

$$\exists y_1, y_2 \in \mathbb{R}, a, b \in \mathbb{Z}.\ y_1 = a \wedge y_2 = b \ \wedge\ x_1 = \tfrac{1}{2} \cdot y_1^2 + y_2^2 \wedge x_2 = y_2^2.$$

We recapitulate our most important results on *Tr* in the following corollary.

Corollary 15 (Properties of *Tr*). *Let $\mathbb{Z} \le \mathcal{S}_1 \le \mathcal{S}_2 \le \mathbb{R}$ be rings, let $(\varphi, \vec{a})$ be a loop on $\mathcal{S}_1^d$, let $\eta \in \mathrm{Aut}_{\mathcal{S}_2}(\mathcal{S}_2[\vec{x}])$, let $Tr_\eta(\varphi, \vec{a}) = (\varphi', \vec{a'})$, and let $F \subseteq \mathcal{S}_1^d$ be $\vec{a}$-invariant and $\mathrm{Th}_\exists(\mathcal{S}_1, \mathbb{R})$-definable. Then*

1. *$\widehat{\eta}(F) \subseteq \mathcal{S}_2^d$ is $\vec{a}'$-invariant and $\mathrm{Th}_\exists(\mathcal{S}_1, \mathbb{R})$-definable,*
2. *$(\varphi, \vec{a})$ terminates on F iff $(\varphi', \vec{a}')$ terminates on $\widehat{\eta}(F)$, and*
3. *$\vec{c} \in F$ witnesses non-termination of $(\varphi, \vec{a})$ iff $\widehat{\eta}(\vec{c}) \in \widehat{\eta}(F)$ witnesses non-termination of $(\varphi', \vec{a}')$.*

3.2 Finding Automorphisms to Transform Loops into *twn*-Form

The goal of the transformation from Sect. 3 is to transform $(\varphi, \vec{a})$ into *twn*-form, such that termination of the resulting loop $Tr_\eta(\varphi, \vec{a})$ can be analyzed by the technique which will be presented in Sect. 4 and 5. Hence, the remaining challenge is to find a suitable automorphism $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ such that $Tr_\eta(\varphi, \vec{a})$ is *twn*. In this section, we will present two techniques to find such automorphisms.

Note that the question whether a loop is *twn*-transformable is closely related to the question whether a polynomial endomorphism can be conjugated into a so-called “de Jonquiéres”-automorphism, a difficult question from algebraic geometry (cf. [9]). So future advances in this field may help to improve the results of Sect. 3.2.

The first technique shows that the search for a suitable automorphism *of bounded degree* can be reduced to $\mathrm{Th}_\exists(\mathcal{S})$. The degree of $\eta \in \mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ is

$$\deg(\eta) = \max_{1 \le i \le d} \deg(\eta(x_i)).$$

For any automorphism, there is an upper bound on the degree of its inverse.

Theorem 16 (Degree of Inverse [9, Cor. 2.3.4]). *Let $\mathcal{S}$ be a reduced ring.*[9] *Then for every $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$, we have $\deg(\eta^{-1}) \le (\deg(\eta))^{d-1}$.*

By Thm. 16, checking if an endomorphism is indeed an automorphism can be reduced to $\mathrm{Th}_\exists(\mathcal{S})$. To do so, one encodes the existence of suitable coefficients of the polynomials $\eta^{-1}(x_1), \ldots, \eta^{-1}(x_d)$, which all have at most degree $(\deg(\eta))^{d-1}$.

[9] A *reduced* ring is a ring $\mathcal{S}$ without non-trivial nilpotent elements, i.e., where for all $r \in \mathcal{S}$, $r^n = 0$ for some $n \in \mathbb{N}$ implies $r = 0$. Note that all subrings of $\mathbb{R}$ are reduced.

Lemma 17 (Checking Automorphisms in $\mathrm{Th}_\exists(\mathcal{S})$). *Let $\mathcal{S}$ be a reduced ring and let $\eta \in \mathrm{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$. Then the question whether $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ holds is reducible to $\mathrm{Th}_\exists(\mathcal{S})$.*

Based on Lemma 17, we now present our first technique to find an automorphism η that transforms a loop into *twn*-form. Thm. 18 states that the existence of such an automorphism *with bounded degree* can be reduced to $\mathrm{Th}_\exists(\mathcal{S})$.

Theorem 18 (*Tr* with Automorphisms of Bounded Degree). *For any $\delta \geq 0$, the question whether there exists an $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ with $\deg(\eta) \leq \delta$ such that $Tr_\eta(\varphi, \vec{a})$ is twn is reducible to $\mathrm{Th}_\exists(\mathcal{S})$.*

Thus, if $\mathrm{Th}_\exists(\mathcal{S})$ is decidable (which is the case for $\mathcal{S} = \mathbb{R}$), then it is *semi-decidable* whether there exists an $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ such that $Tr_\eta(\varphi, \vec{a})$ is *twn*, and it is *decidable* if the degree of η is bounded a priori. Recall that even if $\mathrm{Th}_\exists(\mathcal{S})$ is undecidable, one can still consider the loop $(\varphi, \vec{a})$ on $(\mathcal{S}')^d$ where $\mathcal{S}' > \mathcal{S}$ and $\mathrm{Th}_\exists(\mathcal{S}')$ is decidable. If there is an $\eta' \in \mathrm{Aut}_{\mathcal{S}'}(\mathcal{S}'[\vec{x}])$ such that $Tr_{\eta'}(\varphi, \vec{a})$ is *twn*, then one can use our technique from Sect. 4 and 5 to analyze termination of $Tr_{\eta'}(\varphi, \vec{a})$ on $\widehat{\eta}'(\mathcal{S}^d)$ in order to prove termination of $(\varphi, \vec{a})$ on $\mathcal{S}^d$.

Our second technique to find automorphisms for a transformation into *twn*-form is restricted to *linear* automorphisms $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ where $\mathcal{S}$ is a *field* (instead of a ring). In this case, it is decidable whether a loop can be transformed into a *twn*-loop $(\varphi', \vec{a}')$ where the monomial for x_i has the coefficient 1 in each a'_i. The decision procedure checks whether a certain Jacobian matrix is *strongly nilpotent*, i.e., it is not based on a reduction to $\mathrm{Th}_\exists(\mathcal{S})$. Hence, it can also be used if $\mathrm{Th}_\exists(\mathcal{S})$ is undecidable.

Definition 19 (Strong Nilpotence). *Let $J \in (\mathcal{S}[\vec{x}])^{d \times d}$ be a matrix of polynomials. For all $1 \leq i \leq d$, let $\vec{y}^{(i)}$ be a vector of fresh variables. J is* strongly nilpotent *if $\prod_{i=1}^{d} J[\vec{x}/\vec{y}^{(i)}] = 0^{d \times d}$, where $0^{d \times d} \in (\mathcal{S}[\vec{x}])^{d \times d}$ is the zero matrix.*

Our second technique is formulated in the following theorem which follows from an existing result in linear algebra [10, Thm. 1.6.].

Theorem 20 (*Tr* with Linear Automorphisms, cf. [10, Thm. 1.6.]). *For a loop $(\varphi, \vec{a})$ over a field $\mathcal{S}$, the Jacobian matrix $(\frac{\partial(a_i - x_i)}{\partial x_j})_{1 \leq i,j \leq d} \in (\mathcal{S}[\vec{x}])^{d \times d}$ is strongly nilpotent iff there exists a linear automorphism $\eta \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ with*

$$Tr_\eta(\varphi, \vec{a}) = (\varphi', (x_1 + p_1, \ldots, x_d + p_d))$$

and $p_i \in \mathcal{S}[x_{i+1}, \ldots, x_d]$. Thus, $Tr_\eta(\varphi, \vec{a})$ is twn.

Since strong nilpotence of the Jacobian matrix is clearly decidable, Thm. 20 gives rise to a decision procedure even if $\mathrm{Th}_\exists(\mathcal{S})$ is undecidable.

Example 21 (Finding Automorphisms). The following loop on $\mathbb{Z}^d$ shows how our results enlarge the class of loops where termination is reducible to $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$.

$$\textbf{while } 4 \cdot {x_2}^2 + x_1 + x_2 + x_3 > 0 \textbf{ do } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leftarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \text{with} \tag{2}$$

$$a_1 = x_1 + 8 \cdot x_1 \cdot {x_2}^2 + 16 \cdot {x_2}^3 + 16 \cdot {x_2}^2 \cdot x_3$$
$$a_2 = x_2 - {x_1}^2 - 4 \cdot x_1 \cdot x_2 - 4 \cdot x_1 \cdot x_3 - 4 \cdot {x_2}^2 - 8 \cdot x_2 \cdot x_3 - 4 \cdot {x_3}^2$$
$$a_3 = x_3 - 4 \cdot x_1 \cdot {x_2}^2 - 8 \cdot {x_2}^3 - 8 \cdot {x_2}^2 \cdot x_3 + {x_1}^2 + 4 \cdot x_1 \cdot x_2 + 4 \cdot x_1 \cdot x_3 + 4 \cdot {x_2}^2 + 8 \cdot x_2 \cdot x_3 + 4 \cdot {x_3}^2$$

It is clearly *not* in *twn*-form. Since $\mathcal{S}_1 = \mathbb{Z}$ is not a field, we consider the loop on $\mathcal{S}_2^3 = \mathbb{R}^3$ to use Thm. 20. The Jacobian matrix J of $(a_1 - x_1, a_2 - x_2, a_3 - x_3)$ is

$$\begin{bmatrix} 8 \cdot x_2^2 & 16 \cdot x_1 \cdot x_2 + 48 \cdot x_2^2 + 32 \cdot x_2 \cdot x_3 & 16 \cdot x_2^2 \\ -2 \cdot x_1 - 4 \cdot x_2 - 4 \cdot x_3 & -4 \cdot x_1 - 8 \cdot x_2 - 8 \cdot x_3 & -4 \cdot x_1 - 8 \cdot x_2 - 8 \cdot x_3 \\ -4 \cdot x_2^2 + 2 \cdot x_1 + 4 \cdot x_2 + 4 \cdot x_3 & -8 \cdot x_1 \cdot x_2 - 24 \cdot x_2^2 - 16 \cdot x_2 \cdot x_3 + 4 \cdot x_1 + 8 \cdot x_2 + 8 \cdot x_3 & -8 \cdot x_2^2 + 4 \cdot x_1 + 8 \cdot x_2 + 8 \cdot x_3 \end{bmatrix}$$

One easily checks that J is strongly nilpotent.[10] By Thm. 20 this means that the loop can be transformed into *twn*-form by a linear automorphism. Indeed, consider the linear automorphism $\eta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}[\vec{x}])$ induced by the matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$, i.e.,

$$x_1 \mapsto x_1 + x_2 + x_3, \quad x_2 \mapsto 2 \cdot x_2, \quad x_3 \mapsto x_1 + 2 \cdot x_2 + 2 \cdot x_3$$

with its inverse η^{-1}

$$x_1 \mapsto 2 \cdot x_1 - x_3, \quad x_2 \mapsto \tfrac{1}{2} \cdot x_2, \quad x_3 \mapsto -x_1 - \tfrac{1}{2} \cdot x_2 + x_3.$$

If we transform our loop with η, we obtain the following *twn*-loop on $\mathcal{S}_2^d = \mathbb{R}^d$:

$$\textbf{while } x_1 + x_2^2 > 0 \textbf{ do } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 + x_2^2 \cdot x_3 \\ x_2 - 2 \cdot x_3^2 \\ x_3 \end{bmatrix} \tag{3}$$

By Thm. 9, (3) terminates on $\mathbb{R}^3$ iff (2) terminates on $\mathbb{R}^3$. However, we are not interested in termination of (2) on $\mathbb{R}^3$, but in termination on $\mathbb{Z}^3$. Note that $\widehat{\eta}$ maps $\mathbb{Z}^3$ to the set of all $\mathbb{Z}$-linear combinations of columns of M, i.e.,

$$\widehat{\eta}(\mathbb{Z}^3) = \left\{ a \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

By Cor. 15, (3) terminates on $\widehat{\eta}(\mathbb{Z}^3)$ iff (2) terminates on $\mathbb{Z}^3$. Moreover, $\widehat{\eta}(\mathbb{Z}^3)$ is $\mathrm{Th}_\exists(\mathbb{Z}, \mathbb{R})$-definable: We have $(x_1, x_2, x_3) \in \widehat{\eta}(\mathbb{Z}^3)$ iff

$$\exists a, b, c \in \mathbb{Z}.\ x_1 = a \cdot 1 + b \cdot 1 + c \cdot 1 \land x_2 = b \cdot 2 \land x_3 = a \cdot 1 + b \cdot 2 + c \cdot 2.$$

In the following sections, we will see how to analyze termination of loops like (3) on sets that can be characterized by such formulas.

When comparing our two techniques from Thm. 18 and 20, one notices that whenever Thm. 20 is applicable, a suitable linear automorphism can also be found by considering the loop on $\mathbb{R}^d$ and using Thm. 18 for some fixed degree $\delta \geq 1$. So our first technique subsumes our second one. However, for Thm. 18 one has to check validity of a possibly *non-linear* formula over the reals, where the degree of the occurring polynomials depends on δ and on the degree of the polynomials in the update $\vec{a}$ of the loop. So even when searching for a linear automorphism, one may obtain a non-linear formula if the loop is non-linear. On the other hand, Thm. 20 only requires linear algebra. Thus, Thm. 18 is *always* applicable, whereas Thm. 20 is *easier* to apply.

[10] We used the computer algebra system Maple 17.

Note that the proofs of Thm. 18 and 20 are constructive. Thus, we can not only check the existence of a suitable automorphism, but we can also compute it whenever its existence can be proven.

4 Computing Closed Forms

Now we show how to reduce the termination problem of a *twn*-loop on $\mathcal{S}^d$ to validity of a formula from $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$. The first step towards our reduction is to show that for *twn*-loops $(\varphi, \vec{a})$ there is a closed form for the n-fold application of $\vec{a}$ which can be represented as a vector of *poly-exponential expressions*. This is a straightforward generalization of our analogous results from [12] to loops with non-linear arithmetic. Therefore, we only present the terminology and the main results needed in the rest of this work and refer to App. A for details.

As in [12], the reason that we restrict ourselves to *tnn*-loops (instead of *twn*-loops) is that each *twn*-loop can be transformed into a *tnn*-loop via *chaining*, which preserves (witnesses for) non-termination.

Definition 22 (Chaining). Chaining *a loop* $(\varphi, \vec{a})$ *yields* $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$.

Clearly, $(\varphi, \vec{a})$ terminates iff $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$ terminates. Moreover, if $(\varphi, \vec{a})$ is a *twn*-loop then $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$ is a *tnn*-loop, i.e., the coefficient of each x_i in $\vec{a}(\vec{a})$ is non-negative. Thus, analogous to [12], we obtain the following theorem.

Theorem 23 (Reducing Termination to *tnn*-Loops). *Termination of twn-loops is reducible to termination of tnn-loops.*

Poly-exponential expressions are sums of arithmetic terms over the variables $\vec{x}$ and an additional designated variable n, where it is always clear which addend determines the asymptotic growth of the whole expression when increasing n. This is crucial for our reducibility proof in Sect. 5. In the following, for any set $X \subseteq \mathbb{R}$, any $k \in X$, and $\triangleright \in \{\geq, >\}$, let $X_{\triangleright k} = \{x \in X \mid x \triangleright k\}$.

Definition 24 (Poly-Exponential Expressions). *Let $\mathcal{C}$ be the set of all finite conjunctions over the literals $n = c, n \neq c$ where n is a designated variable and $c \in \mathbb{N}$. Literals of the form $n = c$ are called* positive*. Furthermore, given a formula ψ over n, let $[\![\psi]\!] : \mathbb{N} \to \{0, 1\}$ be the characteristic function of ψ, i.e., we have $[\![\psi]\!]\,(c) = 1$ if $\psi[n/c]$ holds and $[\![\psi]\!]\,(c) = 0$, otherwise. The set of all* poly-exponential expressions *over $\mathcal{S}$ with the variables $\vec{x}$ is*

$$\mathbb{PE}[\vec{x}] = \left\{ \sum\nolimits_{j=1}^{\ell} [\![\psi_j]\!] \cdot \alpha_j \cdot n^{a_j} \cdot b_j^n \;\middle|\; \ell, a_j \in \mathbb{N},\ \psi_j \in \mathcal{C},\ \alpha_j \in K_{\mathcal{S}}[\vec{x}],\ b_j \in \mathcal{S}_{>0} \right\}.$$

As n ranges over $\mathbb{N}$, we use $[\![n > c]\!]$ as syntactic sugar for $[\![\bigwedge_{i=0}^{c} n \neq i]\!]$. So an example for a poly-exponential expression over $\mathcal{S} = \mathbb{Z}$ (where $K_{\mathcal{S}} = \mathbb{Q}$) is

$$[\![n > 2]\!] \cdot (\tfrac{1}{2} \cdot {x_1}^2 + \tfrac{3}{4} \cdot x_2 - 1) \cdot n^3 \cdot 3^n \;+\; [\![n = 2]\!] \cdot (x_1 - x_2).$$

As in [12, Sect. 3], we can compute a closed form from $(\mathbb{PE}[\vec{x}])^d$ for every *tnn*-loop.

Theorem 25 (Closed Forms for *tnn*-Loops). *Let $(\varphi, \vec{a})$ be a tnn-loop. Then one can compute a $\vec{q} \in (\mathbb{PE}[\vec{x}])^d$ such that $\vec{q} = \vec{a}^n$.*

By the restriction to triangular loops, the update of x_i only depends on the previous values of $x_i, x_{i+1}, \ldots, x_d$. Thus, we can compute closed forms for one variable after the other and when computing the closed form for x_i's value after n loop iterations, we already know the closed forms for $x_{i+1}, \ldots, x_d$.

The restriction to weakly non-linear loops ensures that we can always find *poly-exponential* closed forms. In particular, it prevents super-exponential growth of variables, which cannot be captured by expressions from $\mathbb{PE}[\vec{x}]$. For example, consider the loop **while** $\ldots$ **do** $x_1 \leftarrow x_1^2$ which is not weakly non-linear. Here, the value of x_1 after n iterations is $x_1^{(2^n)}$. The same effect can be achieved with mixed monomials, e.g., with the loop **while** $x_1 = x_2 \wedge \ldots$ **do** $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 \cdot x_2 \\ x_1 \cdot x_2 \end{bmatrix}$ which is also not weakly non-linear.

Example 26 (Closed Forms). Reconsider the loop (3) on $\mathcal{S}^3 = \mathbb{Z}^3$ from Ex. 21.

$$\textbf{while } {x_1}^2 + x_2 > 0 \textbf{ do } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 + x_2^2 \cdot x_3 \\ x_2 - 2 \cdot x_3^2 \\ x_3 \end{bmatrix}$$

This loop is *tnn* as $\succ_{(3)} = \{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}$ is well founded. Moreover, every variable occurs with a non-negative coefficient in its update. A closed form for the update after $n \in \mathbb{N}$ loop iterations is:

$$\vec{q} = \begin{bmatrix} \frac{4}{3} \cdot {x_3}^5 \cdot n^3 + \left(-2 \cdot {x_3}^5 - 2 \cdot x_2 \cdot {x_3}^3\right) \cdot n^2 + \left({x_2}^2 \cdot x_3 + \frac{2}{3} \cdot {x_3}^5 + 2 \cdot x_2 \cdot {x_3}^3\right) \cdot n + x_1 \\ -2 \cdot {x_3}^2 \cdot n + x_2 \\ x_3 \end{bmatrix}$$

We can clearly see that the update of (3) consists of integer polynomials whereas the closed form involves coefficients from $K_{\mathcal{S}} = \mathbb{Q}$ (e.g., $\frac{4}{3}$).

5 Reducing Termination of *tnn*-Loops to $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$

Now we reduce termination of *tnn*-loops to $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$. While the idea of our reduction is similar to [12, Sect. 4], our technique extends [12] in several ways:

- In [12], we only considered linear loops while we handle arbitrary loops now.
- In [12], we only considered loops over $\mathbb{Z}$. In contrast, we now analyze termination of $(\varphi, \vec{a})$ on an $\vec{a}$-invariant $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$-definable subset of $\mathbb{R}^d$.
- In [12], all atoms could be normalized to the form $\alpha > 0$. As we now also handle numbers from $\mathbb{Q}$ or $\mathbb{R}$, we have to deal with atoms $\alpha \geq 0$, too.
- In [12], we only considered loop conditions that consist of conjunctions of inequalities. In contrast, we now allow arbitrary propositional formulas.
- In [12], we exploited that every addend $[\![\psi]\!] \cdot \alpha \cdot n^a \cdot b^n$ of a poly-exponential expression was weakly monotonically increasing for large enough n, as we required $b \geq 1$. In contrast, we now only have $b > 0$ and thus we also need to deal with addends that are weakly monotonically *decreasing* for large enough n (e.g., the closed form of $x_1 \leftarrow \frac{1}{2} \cdot x_1$ after n iterations is $x_1^{(n)} = \frac{1}{2}^n \cdot x_1^{(0)}$).

Thus, the proofs for this section differ substantially from the ones in [12]. For reasons of space, here we only present the major steps of our reduction. For more details on the individual proof steps, we refer to App. B.

In the following, let $(\varphi, \vec{a})$ be *tnn*, let $F \subseteq \mathbb{R}^d$ be $\vec{a}$-invariant and $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$-definable by the formula ψ_F, and let $\vec{q} \in (\mathbb{PE}[\vec{x}])^d$ be the closed form of $\vec{a}^n$.

We now show how to encode termination of $(\varphi, \vec{a})$ on F into a $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$-formula. More precisely, we show that there is a function with the following specification that is computable in polynomial time:

$$\begin{aligned} \text{Input}: &\quad (\varphi, \vec{a}),\ \vec{q}, \text{ and } \psi_F \text{ as above} \\ \text{Result}: &\quad \text{a closed formula } \chi \in \mathrm{Th}_\exists(\mathcal{S}, \mathbb{R}) \text{ such that} \\ &\quad \chi \text{ is valid iff } (\varphi, \vec{a}) \text{ does not terminate on } F \end{aligned} \tag{4}$$

We rely on the concept of *eventual non-termination* [5, 12, 22], where the idea is to disregard the loop condition during a finite prefix of the run, i.e.,

$$\begin{aligned} &\vec{c} \in F \text{ witnesses } \textit{eventual non-termination} \text{ of } (\varphi, \vec{a}) \text{ on } F \\ \text{iff} \quad &\exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \varphi(\vec{a}^n(\vec{c})). \end{aligned}$$

Clearly, $(\varphi, \vec{a})$ is non-terminating iff it is eventually non-terminating (cf. [22]). The formula χ in (4) will be constructed in such a way that it encodes the existence of a witness for eventual non-termination of $(\varphi, \vec{a})$.

By the definition of the closed form $\vec{q}$, we immediately obtain that $(\varphi, \vec{a})$ is eventually non-terminating on F iff

$$\exists \vec{x} \in F,\ n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \varphi(\vec{q}). \tag{5}$$

Example 27 (Eventual Non-Termination). We continue Ex. 21 and 26. The loop (3) is eventually non-terminating on

$$F = \widehat{\eta}(\mathbb{Z}^3) = \left\{ a \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}$$

iff there is a corresponding witness $\vec{c} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, i.e., iff

$$\exists x_1, x_2, x_3 \in F,\ n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ p > 0, \qquad \text{where} \tag{6}$$

$$\begin{aligned} p = \left(\tfrac{4}{3} \cdot x_3^5\right) \cdot n^3 \quad &+ \left(-2 \cdot x_3^5 - 2 \cdot x_2 \cdot x_3^3 + 4 \cdot x_3^4\right) \cdot n^2 \\ + \left(x_2^2 \cdot x_3 + \tfrac{2}{3} \cdot x_3^5 + 2 \cdot x_2 \cdot x_3^3 - 4 \cdot x_2 \cdot x_3^2\right) \cdot n &+ \left(x_1 + x_2^2\right). \end{aligned}$$

Let $\vec{q}_{norm}$ be like $\vec{q}$, but all factors $[\![\psi]\!]$ where ψ contains a positive literal are replaced by 0 and all other factors $[\![\psi]\!]$ are replaced by 1. The reason is that for large enough n, positive literals will become false and negative literals will become true. Thus, it follows that (5) is equivalent to

$$\exists \vec{x} \in F,\ n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \varphi(\vec{q}_{norm}). \tag{7}$$

In this way, we obtain *normalized* poly-exponential expressions.

Definition 28 (Normalized $\mathbb{PE}$s). *We call $p \in \mathbb{PE}[\vec{x}]$* normalized *if it is in*

$$\mathbb{NPE}[\vec{x}] = \left\{ \sum\nolimits_{j=1}^{\ell} \alpha_j \cdot n^{a_j} \cdot b_j^n \;\middle|\; \ell, a_j \in \mathbb{N},\ \alpha_j \in K_{\mathcal{S}}[\vec{x}],\ b_j \in \mathcal{S}_{>0} \right\}.$$

W.l.o.g., we always assume $(b_i, a_i) \neq (b_j, a_j)$ if $i \neq j$. We define $\mathbb{NPE} = \mathbb{NPE}[\varnothing]$.

As φ is a propositional formula over $K_{\mathcal{S}}[\vec{x}]$-inequations, $\varphi(\vec{q}_{norm})$ is a propositional formula over $\mathbb{NPE}[\vec{x}]$-inequations. By (7), we need to check if there is an $\vec{x} \in F$ such that $\varphi(\vec{q}_{norm})$ is valid for large enough n. To do so, we generalize [12, Lemma 24]. As usual, $g : \mathbb{N} \to \mathbb{R}$ dominates $f : \mathbb{N} \to \mathbb{R}$ asymptotically ($f \in o(g)$) if for all $m > 0$ there is an $n_0 \in \mathbb{N}$ such that $|f(n)| < m \cdot |g(n)|$ for all $n \in \mathbb{N}_{>n_0}$.

Lemma 29 (Asymptotic Growth). *Let $b_1, b_2 \in \mathbb{R}_{>0}$ and $a_1, a_2 \in \mathbb{N}$. If $(b_2, a_2) >_{lex} (b_1, a_1)$, then $n^{a_1} \cdot b_1^n \in o(n^{a_2} \cdot b_2^n)$. Here, $>_{lex}$ is the lexicographic order, i.e., $(b_2, a_2) >_{lex} (b_1, a_1)$ iff $b_2 > b_1$ or $b_2 = b_1 \wedge a_2 > a_1$.*

In the following, let $p \geq 0$ or $p > 0$ occur in $\varphi(\vec{q}_{norm})$. Then we can order the coefficients of p according to the asymptotic growth of their addends w.r.t. n.

Definition 30 (Ordering Coefficients). Marked coefficients *are of the form $\alpha^{(b,a)}$ where $\alpha \in K_{\mathcal{S}}[\vec{x}], b \in \mathcal{S}_{>0}$, and $a \in \mathbb{N}$. We define* unmark $\left(\alpha^{(b,a)}\right) = \alpha$ *and $\alpha_2^{(b_2,a_2)} \succ_{coef} \alpha_1^{(b_1,a_1)}$ if $(b_2, a_2) >_{lex} (b_1, a_1)$. Let $p = \sum_{j=1}^{\ell} \alpha_j \cdot n^{a_j} \cdot b_j^n \in \mathbb{NPE}[\vec{x}]$, where $\alpha_j \neq 0$ for all $1 \leq j \leq \ell$. Then the marked coefficients of p are*

$$\operatorname{coefs}(p) = \begin{cases} \{0^{(1,0)}\} & \text{if } \ell = 0 \\ \{\alpha_j^{(b_j,a_j)} \mid 0 \leq j \leq \ell\} & \text{otherwise.} \end{cases}$$

Example 31 (Marked Coefficients). Continuing Ex. 27, the marked coefficients of p are $\operatorname{coefs}(p) = \{\alpha_1^{(1,3)}, \alpha_2^{(1,2)}, \alpha_3^{(1,1)}, \alpha_4^{(1,0)}\}$ with

$$\alpha_1 = \tfrac{4}{3} \cdot x_3^5 \qquad \alpha_2 = -2 \cdot x_3^5 - 2 \cdot x_2 \cdot x_3^3 + 4 \cdot x_3^4$$
$$\alpha_3 = x_2^2 \cdot x_3 + \tfrac{2}{3} \cdot x_3^5 + 2 \cdot x_2 \cdot x_3^3 - 4 \cdot x_2 \cdot x_3^2 \qquad \alpha_4 = x_2^2 + x_1$$

Note that $p(\vec{c}) \in \mathbb{NPE}$ for any $\vec{c} \in \mathbb{R}^d$, i.e., the only variable in $p(\vec{c})$ is n. Now the $\succ_{coef}$-maximal addend determines the asymptotic growth of $p(\vec{c})$:

$$o(p(\vec{c})) = o(k \cdot n^a \cdot b^n) \qquad \text{where } k^{(b,a)} = \max\nolimits_{\succ_{coef}} (\operatorname{coefs}(p(\vec{c}))). \tag{8}$$

Note that (8) would be incorrect for the case $k = 0$ if we replaced $o(p(\vec{c})) = o(k \cdot n^a \cdot b^n)$ with $o(p(\vec{c})) = o(n^a \cdot b^n)$ as $o(0) = \varnothing \neq o(1)$. Obviously, (8) implies

$$\exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \operatorname{sign}(p(\vec{c})) = \operatorname{sign}(k) \tag{9}$$

where $\operatorname{sign}(0) = 0$, $\operatorname{sign}(k) = 1$ if $k > 0$, and $\operatorname{sign}(k) = -1$ if $k < 0$. This already allows us to reduce eventual non-termination to $\mathrm{Th}_{\exists}(\mathcal{S}, \mathbb{R})$ if φ is an atom.

Lemma 32 (Reduction for Atoms). *Let $\varphi(\vec{q}_{norm})$ be $p \triangleright 0$ with $\triangleright \in \{\geq, >\}$. Then one can reduce validity of*

$$\exists \vec{x} \in F,\ n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ p \triangleright 0 \tag{10}$$

to validity of the closed formula $\exists \vec{x} \in \mathbb{R}^d.\ \psi_F \wedge \operatorname{red}(p \triangleright 0) \in \mathrm{Th}_{\exists}(\mathcal{S}, \mathbb{R})$ in polynomial time.[11]

[11] More precisely, the reduction of Lemma 32 and of the following Thm. 34 takes polynomially many steps in the size of the input of the function in (4).

To see how to construct $\mathrm{red}(p \triangleright 0)$, note that by (9), we have $p(\vec{c}) > 0$ for large enough values of n iff the coefficient of the asymptotically fastest-growing addend $\alpha(\vec{c}) \cdot n^a \cdot b^n$ of p that does not vanish (i.e., where $\alpha(\vec{c}) \neq 0$) is *positive*. Similarly, we have $p(\vec{c}) < 0$ for large enough n iff $\alpha(\vec{c}) < 0$. If *all* addends of p vanish when instantiating $\vec{x}$ with $\vec{c}$, then $p(\vec{c}) = 0$. In other words, (10) holds iff there is a $\vec{c} \in F$ such that $\mathrm{unmark}\left(\max_{\succ_{coef}}(\mathrm{coefs}(p(\vec{c})))\right) \triangleright 0$. To express this in $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$, let $\alpha_1, \ldots, \alpha_\ell$ be the coefficients of p, ordered according to the asymptotic growth of the respective addends where α_1 belongs to the fastest-growing addend. Then

$$\begin{aligned} &\mathrm{red}(p > 0) \;\text{ is }\; \bigvee\nolimits_{j=1}^{\ell} \left(\alpha_j > 0 \wedge \bigwedge\nolimits_{i=1}^{j-1} \alpha_i = 0\right) \\ \text{and} \qquad &\mathrm{red}(p \geq 0) \;\text{ is }\; \mathrm{red}(p > 0) \vee \bigwedge\nolimits_{i=1}^{\ell} \alpha_i = 0. \end{aligned}$$

Hence, (10) is equivalent to $\exists \vec{x} \in \mathbb{R}^d.\ \psi_F \wedge \mathrm{red}(p \triangleright 0)$. If $\mathcal{S} \neq K_\mathcal{S}$, then $\mathrm{red}(p \triangleright 0)$ can be transformed into a $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$-formula by multiplying with all denominators.

Example 33 (Reducing Eventual Non-Termination to $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$*).* We finish Ex. 31 resp. 21, where $\mathrm{unmark}\left(\max_{\succ_{coef}}(\mathrm{coefs}(p))\right) = \frac{4}{3} \cdot x_3^5$. Moreover, ψ_F is

$$\exists a, b, c \in \mathbb{Z}.\ x_1 = a + b + c \wedge x_2 = b \cdot 2 \wedge x_3 = a + b \cdot 2 + c \cdot 2.$$

Thus, (6) is valid iff $\exists x_1, x_2, x_3 \in \mathbb{R}.\ \psi_F \wedge \mathrm{red}(p > 0)$ is valid where

$$\begin{aligned} \mathrm{red}(p > 0) = \quad & \alpha_1 > 0 \qquad\qquad\qquad\qquad \vee (\alpha_2 > 0 \wedge \alpha_1 = 0) \\ & \vee (\alpha_3 > 0 \wedge \alpha_1 = \alpha_2 = 0) \vee (\alpha_4 > 0 \wedge \alpha_1 = \alpha_2 = \alpha_3 = 0). \end{aligned}$$

Then $[x_1/1, x_2/0, x_3/1]$ satisfies $\psi_F \wedge \alpha_1 > 0$ as $\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right]$ is an element of F (cf. Ex. 27) and we have $\left(\frac{4}{3} \cdot x_3^5\right)[x_1/1, x_2/0, x_3/1] > 0$. Thus $\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right]$ witnesses eventual non-termination of (3). Hence, the original loop (2) is non-terminating on $\mathbb{Z}^3$ by Cor. 15 resp. Thm. 9.

Now we lift our reduction to propositional formulas.

Theorem 34 (Reduction for Formulas). *Let* $\xi = \varphi(\vec{q}_{norm})$ *be a propositional formula over* $\{p \triangleright 0 \mid p \in \mathbb{NPE}[\vec{x}], \triangleright \in \{\geq, >\}\}$. *Then one can reduce validity of*

$$\exists \vec{x} \in F, n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \xi \tag{11}$$

to validity of a formula $\exists \vec{x} \in \mathbb{R}^d.\ \psi_F \wedge \mathrm{red}(\xi) \in \mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$ *in polynomial time.*

Here, $\mathrm{red}(\xi)$ results from replacing each atom $p \triangleright 0$ in ξ by $\mathrm{red}(p \triangleright 0)$.

So Thm. 34 shows that the function (4) is computable (in polynomial time). This allows us to prove the main theorem of this section.

Theorem 35 (Reducing Termination). *Termination of tnn-loops (and thus also of twn-loops) over* F *is reducible to* $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$.

However, in general this reduction is not computable in polynomial time. The reason is that closed forms $\vec{q}$ cannot be computed in polynomial time if the update contains *non-linear* terms. For example, consider the following *tnn*-loop:

$$\textbf{while } \text{true } \textbf{do } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \\ x_d \end{bmatrix} \leftarrow \begin{bmatrix} 2 \cdot x_1 \\ x_1^2 \\ \vdots \\ x_{d-2}^2 \\ x_{d-1}^2 \end{bmatrix} \tag{12}$$

Let $x_i^{(n)}$ be the value of x_i after n loop iterations, where $x_i^{(0)} = x_i$. Then we have $x_i^{(n)} = 2^{2^{i-1} \cdot (n-i+1)} \cdot x_1^{2^{i-1}}$ for all $n \geq d$. As $\log 2^{2^{d-1}}$ is exponential in d, the closed form $q_d \in \mathbb{PE}[\vec{x}]$ for $x_d^{(n)}$ contains constants whose logarithm is exponential in the size of φ and $\vec{a}$. Thus, q_d cannot be computed in polynomial time.

Thm. 35 immediately leads to the following corollary.

Corollary 36 ((Semi-)Decidability of (Non-)Termination over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$). *Consider the class of twn-(transformable)-loops on* $\mathrm{Th}_\exists(\mathcal{S})$*-definable and update-invariant sets* $F \subseteq \mathcal{S}^d$*.*

(a) For this class, termination is decidable if $\mathcal{S} = \mathbb{R}$*.*
(b) For this class, non-termination is semi-decidable if $\mathcal{S} = \mathbb{Z}$ *or* $\mathcal{S} = \mathbb{Q}$*.*

For $\mathcal{S} = \mathbb{R}$ it is also semi-decidable if a loop is *twn-transformable*, cf. Thm. 18.

Our technique does not yield witnesses for non-termination, but the formula constructed by Thm. 34 describes *all* witnesses for *eventual* non-termination.

Lemma 37 (Witnessing Non-Termination). *Let* $\xi = \varphi(\vec{q}_{norm})$*. Then* $\vec{c} \in \mathbb{R}^d$ *witnesses eventual non-termination of* $(\varphi, \vec{a})$ *on* F *iff* $\psi_F(\vec{c}) \wedge (\mathrm{red}(\xi))\,(\vec{c})$*.*

If $(\varphi, \vec{a})$ results from the original loop by first transforming it into *twn*-form (cf. Sect. 3) and by subsequently chaining it in order to obtain a loop in *tnn*-form (cf. Sect. 4), then our approach can also be used to obtain witnesses for eventual non-termination of the original loop. In other words, one can compute a witness for the original loop from the witness for the transformed loop as in Cor. 15, since chaining clearly preserves witnesses for eventual non-termination. Alg. 1 summarizes our technique to check termination of *twn*-transformable-loops.

Algorithm 1: Checking Termination

Input: a *twn*-transformable-loop $(\varphi, \vec{a})$ and $\psi_F \in \mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$
Result: $\top$ resp. $\bot$ if (non-)termination of $(\varphi, \vec{a})$ on F is proven, ? otherwise
$(\varphi, \vec{a}) \leftarrow Tr_\eta(\varphi, \vec{a})$, $\psi_F \leftarrow \psi_{\widehat{\eta}(F)}$, such that $(\varphi, \vec{a})$ becomes *twn*
$(\varphi, \vec{a}) \leftarrow (\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$, such that $(\varphi, \vec{a})$ becomes *tnn*
$\vec{q} \leftarrow$ closed form of $\vec{a}^n$
if (un)satisfiability of $\psi_F \wedge \mathrm{red}(\varphi(\vec{q}_{norm}))$ cannot be proven **then return** ?
if $\psi_F \wedge \mathrm{red}(\varphi(\vec{q}_{norm}))$ is satisfiable **then return** $\bot$ **else return** $\top$

6 Complexity Analysis in the Linear Case

We now analyze the complexity of our technique for *linear* loops where the guard is a propositional formula over linear inequations and the update is of the form

$\vec{x} \leftarrow A \cdot \vec{x} + \vec{b}$ with $A \in \mathcal{S}^{d \times d}$ and $\vec{b} \in \mathcal{S}^d$. More precisely, we show that termination of linear loops with real spectrum is Co-NP-complete if $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.

To do so, w.l.o.g. we assume $\vec{b} = \vec{0}$. The reason is that termination of

$$\textbf{while } \varphi \textbf{ do } \vec{x} \leftarrow A \cdot \vec{x} + \vec{b} \qquad \text{and} \tag{13}$$

$$\textbf{while } \varphi \wedge x_{\vec{b}} = 1 \textbf{ do } \begin{bmatrix} \vec{x} \\ x_{\vec{b}} \end{bmatrix} \leftarrow \begin{bmatrix} A & \vec{b} \\ \vec{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} \vec{x} \\ x_{\vec{b}} \end{bmatrix} \tag{14}$$

is equivalent, where $x_{\vec{b}}$ is a fresh variable. Moreover, $\vec{c}$ witnesses (eventual) non-termination for (13) iff $\begin{bmatrix} \vec{c} \\ 1 \end{bmatrix}$ witnesses (eventual) non-termination for (14). Note that the only eigenvalue of $\begin{bmatrix} A & \vec{b} \\ \vec{0}^T & 1 \end{bmatrix}$ whose multiplicity increases in comparison to A is 1. Thus, to decide termination of linear loops with real spectrum, it suffices to decide termination of linear loops of the following form where A has only real eigenvalues.

$$\textbf{while } \varphi \textbf{ do } \vec{x} \leftarrow A \cdot \vec{x}$$

Such loops can *always* be transformed into *twn*-form using our transformation Tr_η from Sect. 3. To compute the required automorphism η, we compute the Jordan normal form Q of A together with the corresponding transformation matrix T, i.e., T is an invertible real matrix such that $A = T^{-1} \cdot Q \cdot T$. Then Q is a block diagonal matrix where each block has the form $\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{bmatrix}$. The value $\lambda \in \mathbb{R}$ is the corresponding eigenvalue of A. Note that λ is a real *algebraic* number as it is a root of the characteristic polynomial of A.[12]

Now we define $\eta \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}[\vec{x}])$ by $\eta(\vec{x}) = T \cdot \vec{x}$. Then $\eta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}[\vec{x}])$ has the inverse $\eta^{-1}(\vec{x}) = T^{-1} \cdot \vec{x}$. Thus, $Tr_\eta(\varphi, A \cdot \vec{x})$ is a linear *twn*-loop with the update

$$(\eta(\vec{x}))\,(A \cdot \vec{x})\,(\eta^{-1}(\vec{x})) = T \cdot A \cdot T^{-1} \cdot \vec{x} = Q \cdot \vec{x}.$$

The Jordan normal form Q as well as the matrix T and its inverse T^{-1} can be computed in polynomial time [25]. Using Q we can then decide whether all eigenvalues are real numbers by just checking its diagonal entries, which can also be done in polynomial time. Thus, we obtain the following lemma.

Lemma 38 (Poly-Time Transformation to *twn*-Form). *Let $(\varphi, A \cdot \vec{x})$ be a linear loop on $\mathcal{S}^d$.*

(a) It is decidable in polynomial time whether A has only real eigenvalues.
(b) If A has only real eigenvalues, then we can compute $\eta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}[\vec{x}])$ such that $Tr_\eta(\varphi, A \cdot \vec{x})$ is a linear twn-loop in polynomial time.

As a consequence, the transformation from Sect. 3 is complete for linear loops with real spectrum. Completeness for linear-*update* loops with real spectrum follows analogously. Note that the first part of Lemma 38 yields an efficient check whether a given linear loop has real spectrum, i.e., whether it belongs to the class of loops considered in the current section.

[12] Recall that we restricted ourselves to algebraic constants in the loop, cf. Sect. 1.

To analyze termination of a loop, our technique of Sect. 4 computes a closed form for the n-fold application of the loop body. In other words, to analyze a linear loop with update $Q \cdot \vec{x}$ we need to compute a vector $\vec{q} \in (\mathbb{PE}[\vec{x}])^d$ with $\vec{q} = Q^n \cdot \vec{x}$. If Q is in Jordan normal form, this can be achieved by computing the n^{th} power of each Jordan block. This gives rise to an alternative technique to Sect. 4, which eases our Co-NP-completeness proof and does not require a further transformation into *tnn*-form. If the size of a Jordan block is ν, then

$$\begin{bmatrix} \lambda & 1 & 0 \cdots & 0 \\ 0 & \lambda & 1 \cdots & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 \cdots & 0 & \lambda & 1 \\ 0 \cdots & 0 & 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n \cdot \lambda^{n-1} & \binom{n}{2} \cdot \lambda^{n-2} & \cdots & \binom{n}{\nu-1} \cdot \lambda^{n-(\nu-1)} \\ 0 & \lambda^n & n \cdot \lambda^{n-1} & \cdots & \binom{n}{\nu-2} \cdot \lambda^{n-(\nu-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \cdot \lambda^{n-1} \\ 0 & 0 & 0 & \cdots & \lambda^n \end{bmatrix}$$

where $\binom{n}{k} = 0$ if $n < k$. Note that $\binom{n}{k} = \frac{\prod_{j=0}^{k-1}(n-j)}{k!}$ is a polynomial of degree k in the variable n. Thus, we obtain a poly-exponential expression for each component of $\vec{q}$ by elementary arithmetic conversions. As each eigenvalue λ is a real algebraic number, it follows that $\vec{q}$ can be computed in polynomial time [21]. Note that the resulting poly-exponential expressions are clearly linear in $\vec{x}$.

According to our approach in Sect. 5, we now proceed as in Alg. 1 and compute $\mathrm{red}(\varphi(\vec{q}_{norm})) \in \mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$. The construction of this formula can be done in polynomial time due to Thm. 34. Thus, we obtain the following lemma.

Lemma 39 (Poly-Time Reduction to $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$). *Let $(\varphi, A \cdot \vec{x})$ be a linear loop on $\mathcal{S}^d$ with real spectrum. Then we can compute a $\psi \in \mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$ with only linear atoms in polynomial time such that ψ is valid iff the loop is non-terminating.*

As ψ is *linear* and does not contain universal quantifiers, invalidity of ψ is in Co-NP if $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. The reason is that validity of such formulas is in NP (see, e.g., [23]). Consequently, we obtain the main theorem of this section.

Theorem 40 (Co-NP Completeness). *Termination of linear loops $(\varphi, A \cdot \vec{x} + \vec{b})$ with real spectrum over $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ is Co-NP-complete.*

For Co-NP-hardness, let ξ be a propositional formula over the variables $\vec{x}$. Then

$$\textbf{while } \xi[x_1/(x_1 > 0), \ldots, x_d/(x_d > 0)] \textbf{ do } \vec{x} \leftarrow \vec{x}$$

terminates (over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$) iff ξ is unsatisfiable. So Co-NP-hardness of termination follows from Co-NP-hardness of unsatisfiability of propositional formulas.

Moreover, as the presented technique to compute closed forms via Jordan normal forms is independent of the loop condition φ, the following corollary immediately follows due to the $\exists\mathbb{R}$-completeness of $\mathrm{Th}_\exists(\mathbb{R})$.

Corollary 41 ($\exists\mathbb{R}$ Completeness). *Termination of linear-update loops with real spectrum (and possibly non-linear loop conditions) on $\mathbb{R}^d$ is $\exists\mathbb{R}$-complete.*

7 Related Work and Conclusion

We presented a reduction from termination of *twn*-loops on a $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$-definable update-invariant set $F \subseteq \mathbb{R}^d$ to $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$. For these loops, this implies decidability of termination over $\mathcal{S} = \mathbb{R}$ and semi-decidability of non-termination over

$\mathcal{S} = \mathbb{Z}$ and $\mathcal{S} = \mathbb{Q}$. The restriction to *twn*-loops excludes super-exponential growth and "cyclic" dependencies between variables. Moreover, we introduced a transformation based on polynomial automorphisms, which allows us to transform non-*twn*-loops into *twn*-form and generalizes our results to a wider class of loops. We also showed that checking *twn*-transformability over $\mathbb{R}$ is semi-decidable. Furthermore, we used our results to prove that termination of linear loops with real spectrum over $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ is Co-NP-complete.

Related Work: There is a large body of work on *automated* termination analysis (e.g., [2, 4, 6, 7, 11, 13, 14, 15, 17, 18, 19, 20, 24]). In contrast, we are interested in *decidability* of termination for certain classes of loops as well as in the complexity of the corresponding decision problems. The most closely related works are concerned with decidability of termination for *conjunctive linear loops* where only linear arithmetic and conjunctive loop conditions are permitted. For such loops, termination is known to be decidable if the variables range over the real [27] or the rational numbers [5]. For the integer case, decidability has been conjectured 15 years ago [27] and after several partial solutions [3, 5, 12, 22], this conjecture has been confirmed very recently [16].

In contrast to [3, 5, 16, 22, 27], our proof from [12] hardly relies on the absence of non-linear arithmetic. This allows us to generalize the approach from [12] to loops with non-linear polynomial arithmetic over any ring $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}$ in the current work, whereas all techniques mentioned above are specialized to either $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ and restricted to linear arithmetic. Note that the technique presented in this work is a proper generalization of [12], i.e., it processes loops that belong to the fragment considered in [12] in precisely the same way. Moreover, in the current paper we allow disjunctions in the loop condition φ, whereas all other approaches mentioned above are restricted to conjunctions of linear constraints.

Regarding complexity, [22] proves that termination of conjunctive linear loops over $\mathbb{Z}$ with assignment $\vec{x} \leftarrow A \cdot \vec{x} + \vec{b}$ is in EXPSPACE if A is diagonalizable resp. in P if $|\vec{x}| \leq 4$. The paper [16] does not investigate the complexity of the presented technique. Moreover, [22] states that the techniques from [5, 27] run in polynomial time. Thus, termination of conjunctive linear loops over $\mathbb{Q}$ or $\mathbb{R}$ is in P.

Our Co-NP-completeness result for linear loops with real spectrum is orthogonal to those results as, in contrast to [5, 16, 22, 27], we allow disjunctions in the loop condition. Moreover, Co-NP-completeness also holds for termination over $\mathbb{Z}$, whereas [5, 27] only consider termination over $\mathbb{Q}$ resp. $\mathbb{R}$.

Future Work: There are several directions for future work. First of all, we want to adapt our technique such that it allows us to synthesize witnesses for non-termination (instead of just *eventual* non-termination). Furthermore, for (sub-classes of) *twn*-loops we will investigate whether their (asymptotic) *runtime complexity* is computable as well. Finally, we plan to analyze the complexity of the termination problem for classes of loops beyond those considered in Sect. 6.

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Appendix

A Details on Computing Closed Forms

This section is devoted to the details of Sect. 4, i.e., we will show how to compute closed forms for the n-fold application of the loop update $\vec{a}$ by generalizing our technique from [12, Sect. 3].

The crux of the proof that poly-exponential expressions can represent closed forms is to show that certain sums over products of exponential and poly-exponential expressions can be represented by poly-exponential expressions, cf. Lemma 43. To construct these expressions, we use a variant of [1, Lemma 3.5]. It expresses a polynomial q via the difference of another polynomial r at the positions n and $n-1$, using an additional factor c which can be chosen freely. Moreover, its proof directly yields an algorithm to compute r, cf. [12, Alg. 1].

Lemma 42 (Expressing Polynomials by Differences [1]). *If $q \in K_{\mathcal{S}}[n]$, and $c \in K_{\mathcal{S}}$, then there is an $r \in K_{\mathcal{S}}[n]$ such that $q = r - c \cdot r[n/n-1]$ for all $n \in \mathbb{N}$.*

The proof of Lemma 42 is identical to the proof of [12, Lemma 10].

The following result generalizes [12, Lemma 12] and can be proven in the same way using Lemma 42.

Lemma 43 (Closure of $\mathbb{PE}$ under Sums of Products and Exponentials). *If $m \in \mathbb{N}$ and $p \in \mathbb{PE}[\vec{x}]$, then one can compute a $q \in \mathbb{PE}[\vec{x}]$ which is equivalent to $\sum_{i=1}^{n} m^{n-i} \cdot p[n/i-1]$.*

Recall that our goal is to compute closed forms for loops. As a first step, instead of the n-fold update function $h(n, \vec{x}) = \vec{a}^n$, we consider a recursive update function for a single variable $x \in \vec{x}$:

$$g(0, \vec{x}) = x \quad \text{and} \quad g(n, \vec{x}) = m \cdot g(n-1, \vec{x}) + p[n/n-1] \quad \text{for all } n > 0$$

Here, $m \in \mathcal{S}_{\geq 0}$ and $p \in \mathbb{PE}[\vec{x}]$. Using Lemma 43, it is easy to show that g can be represented by a poly-exponential expression.

Lemma 44 (Closed Form for Single Variables [12, Lemma 14]). *If $x \in \vec{x}$, $m \in \mathcal{S}_{\geq 0}$, and $p \in \mathbb{PE}[\vec{x}]$, then one can compute a $q \in \mathbb{PE}[\vec{x}]$ which satisfies*

$$q[n/0] = x \quad \textit{and} \quad q = (m \cdot q + p)[n/n-1] \quad \textit{for all } n > 0. \tag{15}$$

The restriction to triangular loops now allows us to generalize Lemma 44 to vectors of variables. The reason is that due to triangularity, the update of each program variable x only depends on the previous values of those variables y with $x \succeq_{\vec{a}} y$. So when regarding x, we can assume that we already know the closed forms for all variables that are smaller w.r.t. $\succ_{\vec{a}}$. This allows us to find closed forms for one variable after the other by applying Lemma 44 repeatedly. In other words, it allows us to find a vector $\vec{q}$ of poly-exponential expressions that satisfies

$$\vec{q}[n/0] = \vec{x} \quad \text{and} \quad \vec{q} = \vec{a}[\vec{x}/\vec{q}][n/n-1] \quad \text{for all } n > 0.$$

This claim follows from Lemma 45 that generalizes [12, Lemma 16]. Here, we extend the notion of *tnn*-loops to assignments with vectors of poly-exponential expressions (instead of just vectors of polynomials). For $\vec{a} = (a_1, \ldots, a_d) \in (\mathbb{PE}[\vec{x}])^d$ we say that an assignment $\vec{x} \leftarrow \vec{a}$ is *tnn* iff it is triangular and for all $1 \leq i \leq d$ there exist $m_i \in \mathcal{S}_{\geq 0}$ and $p_i \in \mathbb{PE}[\vec{x} \setminus \{x_i\}]$ such that $a_i = m_i \cdot x_i + p_i$. In the following, for a vector $\vec{v} = (v_1, \ldots, v_d)$, let $\vec{v}_{1,\ldots,d-1}$ denote the vector $(v_1, \ldots, v_{d-1})$.

Lemma 45 (Closed Forms for Vectors of Variables). *Consider a tnn-assignment $\vec{x} \leftarrow \vec{a}$ where $\vec{a} \in (\mathbb{PE}[\vec{x}])^d$. Then one can compute a $\vec{q} \in (\mathbb{PE}[\vec{x}])^d$ such that:*

$$\vec{q}[n/0] = \vec{x} \quad \textit{and} \tag{16}$$

$$\vec{q} = \vec{a}[\vec{x}/\vec{q}][n/n-1] \quad \textit{for all } n > 0 \tag{17}$$

Proof. Let $\vec{q} = (q_1, \ldots, q_d)$. We use induction on d. W.l.o.g., assume that x_d is minimal w.r.t. $\succ_{\vec{a}}$. For each $d \geq 1$ and $n > 0$ we have:

$$\begin{array}{lll}
 & \vec{q} \ = \vec{a}[\vec{x}/\vec{q}][n/n-1] & \\
\iff & q_d = a_d[\vec{x}/\vec{q}][n/n-1] \quad \wedge & \\
 & q_i \ = a_i[\vec{x}/\vec{q}][n/n-1] \quad \text{for all } 1 \leq i < d & \\
\iff & q_d = a_d[x_d/q_d][n/n-1] \quad \wedge & \text{as } \vec{x}_d \text{ is minimal} \\
 & q_i \ = a_i[\vec{x}/\vec{q}][n/n-1] \quad \text{for all } 1 \leq i < d & \\
\iff & q_d = (m \cdot q_d + p)\,[n/n-1] \ \wedge \text{where } m \in \mathcal{S}_{\geq 0}, p \in \mathbb{PE}[\vec{x}], & \text{as } \vec{x} \leftarrow \vec{a} \text{ is } \textit{tnn} \\
 & q_i \ = a_i[\vec{x}/\vec{q}][n/n-1] \quad \text{for all } 1 \leq i < d &
\end{array}$$

By Lemma 44, we can compute a $q_d \in \mathbb{PE}[\vec{x}]$ that satisfies

$$q_d[n/0] = x_d \quad \text{and} \quad q_d = (m \cdot q_d + p)\,[n/n-1] \quad \text{for all } n > 0.$$

In the induction base ($d = 1$), there is no i with $1 \leq i < d$. In the induction step ($d > 1$), it remains to show that we can compute $\vec{q}_{1,\ldots,d-1}$ such that

$$q_i[n/0] = x_i \quad \text{and} \quad q_i = a_i[\vec{x}/\vec{q}][n/n-1]$$

for all $n > 0$ and all $1 \leq i < d$, which is equivalent to

$$\vec{q}_{1,\ldots,d-1}[n/0] = \vec{x}_{1,\ldots,d-1} \quad \text{and}$$
$$\vec{q}_{1,\ldots,d-1} = \vec{a}_{1,\ldots,d-1}[x_d/q_d][\vec{x}_{1,\ldots,d-1}/\vec{q}_{1,\ldots,d-1}][n/n-1]$$

for all $n > 0$. As $\vec{a}_{1,\ldots,d-1}[x_d/q_d] \in (\mathbb{PE}[\vec{x}])^{d-1}$, the claim follows from the induction hypothesis. $\square$

The proof of Lemma 45 is by induction on d, which is reflected by the following algorithm to compute a solution for (16) and (17).

Algorithm 2: closed_form

Input: a *tnn*-assignment $\vec{x} \leftarrow \vec{a}$. W.l.o.g. let x_d be minimal w.r.t. $\succ_{\vec{a}}$.
Result: $\vec{q} \in (\mathbb{PE}[\vec{x}])^d$ which satisfies (16) & (17) for the given $\vec{a}$
let $a_d = m \cdot x_d + p$
let $q_d \in \mathbb{PE}[\vec{x}]$ satisfy (15) for x_d, m, and p (cf. Lemma 44)
if $d > 1$ **then** $\vec{q}_{1,\dots,d-1} \leftarrow$ closed_form($\vec{x}_{1,\dots,d-1} \leftarrow \vec{a}_{1,\dots,d-1}[x_d/q_d]$)
return $\vec{q}$

We now arrive at the main theorem of this section which generalizes [12, Thm. 17].

Theorem 25 (Closed Forms for *tnn*-Loops). *Let $(\varphi, \vec{a})$ be a tnn-loop. Then one can compute a $\vec{q} \in (\mathbb{PE}[\vec{x}])^d$ such that $\vec{q} = \vec{a}^n$.*

Proof. Consider a *tnn*-loop $(\varphi, \vec{a})$. By Lemma 45, we can compute a $\vec{q} \in (\mathbb{PE}[\vec{x}])^d$ that satisfies

$$\vec{q}[n/0] = \vec{x} \quad \text{and} \quad \vec{q} = \vec{a}[\vec{x}/\vec{q}][n/n-1] \quad \text{for all } n > 0.$$

We prove $\vec{a}^c = \vec{q}[n/c]$ by induction on $c \in \mathbb{N}$. If $c = 0$, we get

$$\vec{a}^0 = \vec{x} = \vec{q}[n/0] = \vec{q}[n/c].$$

If $c > 0$, we get:

$$\begin{aligned}
\vec{a}^c &= \vec{a}(\vec{a}^{c-1}) \\
&= \vec{a}(\vec{q}[n/c-1]) \quad \text{by the induction hypothesis} \\
&= \vec{a}[\vec{x}/\vec{q}][n/c-1] \text{ as } \vec{a} \in (\mathcal{S}[\vec{x}])^d \text{ does not contain } n \\
&= \vec{q}[n/c]
\end{aligned}$$

□

So invoking Alg. 2 on $\vec{x} \leftarrow \vec{a}$ yields the closed form of a *tnn*-loop $(\varphi, \vec{a})$.

B Proofs

B.1 Proof of Lemma 6

Proof. Let $(\varphi, \vec{a})$ be a loop. Since $id_{\mathcal{S}[\vec{x}]}^{-1} = id_{\mathcal{S}[\vec{x}]}$, we obtain $Tr_{id_{\mathcal{S}[\vec{x}]}}(\varphi, \vec{a}) = (\varphi', \vec{a}')$ with

$$\begin{aligned}
\varphi' &= id_{\mathcal{S}[\vec{x}]}^{-1}(\varphi) = \varphi \\
\vec{a}' &= (id_{\mathcal{S}[\vec{x}]}^{-1} \circ \widetilde{a} \circ id_{\mathcal{S}[\vec{x}]})(\vec{x}) = \widetilde{a}(\vec{x}) = \vec{a}
\end{aligned}$$

Now we take $\eta_1, \eta_2 \in \mathrm{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$. Note that $(\eta_1 \circ \eta_2)^{-1} = \eta_2^{-1} \circ \eta_1^{-1}$. Let $Tr_{\eta_1 \circ \eta_2}(\varphi, \vec{a}) = (\varphi', \vec{a}')$, $Tr_{\eta_1}(\varphi, \vec{a}) = (\varphi'', \vec{a}'')$, and $Tr_{\eta_2}(\varphi'', \vec{a}'') = (\varphi''', \vec{a}''')$. We have

$$\begin{aligned}
\varphi' &= (\eta_2^{-1} \circ \eta_1^{-1})\,(\varphi) \\
\varphi'' &= \eta_1^{-1}\,(\varphi) \\
\varphi''' &= \eta_2^{-1}\,(\varphi'') \\
&= \eta_2^{-1}\left(\eta_1^{-1}\,(\varphi)\right) \\
&= (\eta_2^{-1} \circ \eta_1^{-1})\,(\varphi) \\
&= \varphi'
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\vec{a}' &= \left(\eta_2^{-1} \circ \eta_1^{-1} \circ \widetilde{a} \circ \eta_1 \circ \eta_2\right)(\vec{x}) \\
&= (\eta_2(\vec{x}))\;(\eta_1(\vec{x}))\;(\vec{a})\;(\eta_1^{-1}(\vec{x}))\;(\eta_2^{-1}(\vec{x})) \\
\vec{a}'' &= \left(\eta_1^{-1} \circ \widetilde{a} \circ \eta_1\right)(\vec{x}) \\
&= (\eta_1(\vec{x}))\;(\vec{a})\;(\eta_1^{-1}(\vec{x})) \\
\vec{a}''' &= \left(\eta_2^{-1} \circ \widetilde{a''} \circ \eta_2\right)(\vec{x}) \\
&= (\eta_2(\vec{x}))\;(\eta_1(\vec{x}))\;(\vec{a})\;(\eta_1^{-1}(\vec{x}))\;(\eta_2^{-1}(\vec{x})) \\
&= \vec{a}'
\end{aligned}$$

□

B.2 Proof of Lemma 7

Proof. Let $\vec{c}$ be a witness for non-termination of $(\varphi, \vec{a})$, i.e., $\varphi(\vec{a}^n(\vec{c}))$ holds for all $n \in \mathbb{N}$. Let $Tr_\eta(\varphi, \vec{a}) = (\varphi', \vec{a}')$. To prove the lemma, we show that

$$\varphi'(\vec{a}'^n((\eta(\vec{x}))\,(\vec{c}))) = \varphi(\vec{a}^n(\vec{c}))$$

for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
&\varphi'(\vec{a'}^n((\eta(\vec{x}))\,(\vec{c}))) \\
&= \eta^{-1}\,(\varphi)\;\;(\vec{a'}^n((\eta(\vec{x}))\,(\vec{c}))) \\
&= \varphi\,[\vec{x}/\eta^{-1}(\vec{x})]\;\underbrace{[\vec{x}/\vec{a'}]}_{n \text{ times}}[\vec{x}/\eta(\vec{x})]\,[\vec{x}/\vec{c}] \\
&= \varphi\,[\vec{x}/\eta^{-1}(\vec{x})]\,[\vec{x}/\eta(\vec{x})]\;\;\underbrace{[\vec{x}/\vec{a}]}_{n \text{ times}}\;[\vec{x}/\eta^{-1}(\vec{x})]\,[\vec{x}/\eta(\vec{x})]\,[\vec{x}/\vec{c}] \\
&= \varphi\;\;\underbrace{[\vec{x}/\vec{a}]}_{n \text{ times}}\;[\vec{x}/\vec{c}] \\
&= \varphi(\vec{a}^n(\vec{c}))
\end{aligned}$$

□

B.3 Proof of Thm. 9

Proof. The second statement implies the first. In Lemma 7 we have seen that if $\vec{c}$ is a witness for non-termination of $(\varphi, \vec{a})$, then $\widehat{\eta}(\vec{c})$ witnesses non-termination of $Tr_\eta(\varphi, \vec{a})$. Now let $\vec{u}$ be a witness for non-termination of $Tr_\eta(\varphi, \vec{a})$. Then by Lemma 7, $\widehat{\eta^{-1}}(\vec{u})$ witnesses non-termination of $Tr_{\eta^{-1}}(Tr_\eta(\varphi, \vec{a})) \overset{\text{Lemma 6}}{=} Tr_{\eta \circ \eta^{-1}}(\varphi, \vec{a}) = (\varphi, \vec{a})$. Hence, $\widehat{\eta}$ maps witnesses for non-termination of $(\varphi, \vec{a})$ to witnesses for non-termination of $Tr_\eta(\varphi, \vec{a})$ and $\widehat{\eta^{-1}}$ maps witnesses for non-termination of $Tr_\eta(\varphi, \vec{a})$ to witnesses for non-termination of $(\varphi, \vec{a})$. These two mappings are inverse to each other: For $\vec{u} \in \mathcal{S}^d$ we have

$$\begin{aligned}
&\widehat{\eta}(\widehat{\eta^{-1}}(\vec{u})) \\
={}& \widehat{\eta}((\eta^{-1}(\vec{x}))(\vec{u})) && \text{by definition of } \widehat{\eta^{-1}} \\
={}& (\eta(\vec{x}))((\eta^{-1}(\vec{x}))(\vec{u})) && \text{by definition of } \widehat{\eta} \\
={}& \eta(\vec{x})[\vec{x}/\eta^{-1}(\vec{x})][\vec{x}/\vec{u}] \\
={}& \vec{u}
\end{aligned}$$

$$\begin{aligned}
&\widehat{\eta^{-1}}(\widehat{\eta}(\vec{u})) \\
={}& \widehat{\eta^{-1}}((\eta(\vec{x}))(\vec{u})) && \text{by definition of } \widehat{\eta} \\
={}& (\eta^{-1}(\vec{x}))((\eta(\vec{x}))(\vec{u})) && \text{by definition of } \widehat{\eta^{-1}} \\
={}& \eta^{-1}(\vec{x})[\vec{x}/\eta(\vec{x})][\vec{x}/\vec{u}] \\
={}& \vec{u}.
\end{aligned}$$

Hence, $\widehat{\eta}$ is indeed a bijection with inverse mapping $\widehat{\eta^{-1}}$. □

B.4 Proof of Lemma 11

Proof. Let $\vec{c}\,' \in \widehat{\eta}(F)$. Then $\vec{c}\,' = \widehat{\eta}(\vec{c})$ for some $\vec{c} \in F$. As F is $\vec{a}$-invariant, we have $\vec{a}(\vec{c}) \in F$. We obtain

$$\begin{aligned}
\vec{a}\,'(\vec{c}\,') &= (\eta(\vec{x}))\,(\vec{a})\,(\eta^{-1}(x))\,(\vec{c}\,') \\
&= (\eta(\vec{x}))\,(\vec{a})\,(\eta^{-1}(x))\,(\eta(\vec{x}))\,(\vec{c}) \\
&= (\eta(\vec{x}))\,(\vec{a})\,(\vec{c}) \\
&= \widehat{\eta}(\vec{a}(\vec{c})) \in \widehat{\eta}(F).
\end{aligned}$$

□

B.5 Proof of Lemma 13

Proof. Let F be characterized by $\psi_F \in \mathrm{Th}_\exists(\mathcal{S}_1, \mathbb{R})$. Consider the following formula $\psi \in \mathrm{Th}_\exists(\mathcal{S}_1, \mathbb{R})$:

$$\exists \vec{y} \in \mathbb{R}^d.\ \psi_F(\vec{y}) \wedge \vec{x} = (\eta(\vec{x}))\,(\vec{y})$$

Then $\psi(\vec{c})$ holds for a point $\vec{c} \in \mathbb{R}^d$ iff $\vec{c} = \widehat{\eta}(\vec{u})$ for some $\vec{u} \in \mathbb{R}^d$ where $\psi_F(\vec{u})$ holds, i.e., where $\vec{u} \in F$. □

B.6 Proof of Lemma 17

Proof. Let $\delta = \deg(\eta)$. Note that for any $e \in \mathbb{N}$, there is only a finite number of monomials over $\vec{x}$ of degree e. (More precisely, the number of monomials of exactly degree e is $\binom{d+e-1}{e}$.) Hence, for any $1 \le i \le d$ we can construct the following term that stands for $\eta^{-1}(x_i)$:

$$\sum_{m \text{ is a monomial of (at most) degree } \delta^{d-1}} a_{i,m} \cdot m$$

Here, the monomials m contain the variables $\vec{x}$ and the $a_{i,m}$ are variables that stand for the unknown coefficients of the polynomial $\eta^{-1}(x_i)$.

Hence, for any $1 \le i \le d$ we now build a formula $\varphi_{r,i}$ which stands for the requirement "$\eta(\eta^{-1}(x_i)) = x_i$" (i.e., that η^{-1} is a right inverse of η):

$$\varphi_{r,i}: \qquad \sum_{m \text{ is a monomial of (at most) degree } \delta^{d-1}} a_{i,m} \cdot \eta(m) = x_i$$

Similarly, for any $1 \le i \le d$ we construct a formula $\varphi_{l,i}$ which stands for the requirement "$\eta^{-1}(\eta(x_i)) = x_i$" (i.e., that η^{-1} is a left inverse of η):

$$\varphi_{l,i}: \qquad \eta(x_i) \left(\sum_{m \text{ is a monomial of (at most) degree } \delta^{d-1}} a_{i,m} \cdot m \right) = x_i$$

Thus, the formula

$$\forall \vec{x} \in \mathcal{S}^d. \quad \bigwedge_{i=1}^{d} \varphi_{r,i} \wedge \bigwedge_{i=1}^{d} \varphi_{l,i} \tag{18}$$

is valid iff η has an inverse of degree at most δ^{d-1}. By Thm. 16, this is equivalent to the question whether η has an inverse, i.e., whether η is an automorphism. Unfortunately, (18) $\notin \mathrm{Th}_{\exists}(\mathcal{S})$. However, $\bigwedge_{i=1}^{d} \varphi_{r,i} \wedge \bigwedge_{i=1}^{d} \varphi_{l,i}$ has to hold for all $\vec{x} \in \mathcal{S}^d$. So, we can reduce this formula to a system of equations: one simply has to check whether there is an instantiation of the unknown coefficients $a_{i,m}$ such that all monomials in $\varphi_{r,i}$ and $\varphi_{l,i}$ except x_i get the coefficient 0 and the monomial x_i gets the coefficient 1. When building the conjunction of these equations and existentially quantifying the unknown coefficients $a_{i,m}$, one indeed obtains a formula $\mathrm{Th}_{\exists}(\mathcal{S})$. □

B.7 Proof of Thm. 18

Proof. For every $1 \le i \le d$, let

$$\eta(x_i) = \sum_{m \text{ is a monomial of (at most) degree } \delta} b_{i,m} \cdot m,$$

where the $b_{i,m}$ are variables that stand for unknown coefficients. By Lemma 17 there is a $\mathrm{Th}_{\exists}(\mathcal{S})$-formula that contains both $b_{i,m}$ and the variables $a_{i,m}$ (for the coefficients of η^{-1}) which expresses that η is an automorphism.

Furthermore, using these coefficients we can construct a formula from $\mathrm{Th}_{\exists}(\mathcal{S})$ which expresses that the update $\vec{a}' = (a'_1, \ldots, a'_d) = (\eta^{-1} \circ \widetilde{a} \circ \eta)(\vec{x})$ is a *twn*-loop:

Note that we have $\deg(\vec{a}') = \deg((\eta^{-1} \circ \widetilde{a} \circ \eta)(\vec{x})) \leq \deg(\eta^{-1}) \cdot \deg(\widetilde{a}) \cdot \deg(\eta) \leq \delta^{d-1} \cdot \deg(\widetilde{a}) \cdot \delta$. So there is a bound on the degree of the polynomials occurring in the transformed loop $Tr_\eta(\varphi, \vec{a})$. Hence, for every $1 \leq i \leq d$, let

$$a_i' = \sum_{m \text{ is a monomial of (at most) degree } \delta^{d-1} \cdot \deg(\widetilde{a}) \cdot \delta} c_{i,m} \cdot m,$$

where the variables $c_{i,m}$ stand again for unknown coefficients. Now we can build a $\mathrm{Th}_\exists(\mathcal{S})$-formula which is valid iff $\vec{a}'$ is in *twn*-form by requiring that certain coefficients $c_{i,m}$ are zero. Moreover, we can construct a $\mathrm{Th}_\exists(\mathcal{S})$-formula which is valid iff $\vec{a}' = (\eta^{-1} \circ \widetilde{a} \circ \eta)(\vec{x})$. This proves the theorem. $\square$

B.8 Proof of Thm. 23

Proof. We first prove:

$$\text{If } (\varphi, \vec{a}) \text{ is } twn, \text{ then } (\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a})) \text{ is } tnn. \tag{19}$$

Due to weak non-linearity, we have $a_i = m_i \cdot x_i + \alpha_i$ with $x_i \notin \mathcal{V}(\alpha_i)$ for all $1 \leq i \leq d$. Then

$$a_i(\vec{a}) = m_i \cdot (m_i \cdot x_i + \alpha_i) + \alpha_i(\vec{a}) = m_i^2 \cdot x_i + m_i \cdot \alpha_i + \alpha_i(\vec{a}).$$

Assume that $x_i \in \mathcal{V}(\alpha_i(\vec{a}))$. Since $x_i \notin \mathcal{V}(\alpha_i)$ by weak non-linearity, there must be an $x_j \in \mathcal{V}(\alpha_i)$ with $x_j \neq x_i$ and $x_i \in \mathcal{V}(a_j)$. By definition, $x_j \neq x_i$ and $x_i \in \mathcal{V}(a_j)$ implies $x_j \succ_{\vec{a}} x_i$. But $x_j \in \mathcal{V}(\alpha_i)$ also implies $x_i \succ_{\vec{a}} x_j$, which would violate well-foundedness of $\succ_{\vec{a}}$, i.e., it would contradict the triangularity of $(\varphi, \vec{a})$.

Hence, m_i^2 is the coefficient of x_i in $a_i(\vec{a})$. Since $m_i^2 \geq 0$, this proves that $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$ is non-negative.

Note that $x_i \succ_{\vec{a}(\vec{a})} x_j$ implies $x_j \in \mathcal{V}(\alpha_i)$ (in this case we also have $x_i \succ_{\vec{a}} x_j$) or it implies that there is an $x_k \in \mathcal{V}(\alpha_i)$ with $x_j \in \mathcal{V}(a_k)$ (in this case we have $x_i \succ_{\vec{a}} x_k$ and $x_k \succeq_{\vec{a}} x_j$). So in both cases, $x_i \succ_{\vec{a}(\vec{a})} x_j$ implies $x_i \succ_{\vec{a}} x_j$. Thus, we obtain $\succ_{\vec{a}(\vec{a})} \subseteq \succ_{\vec{a}}$. As $\succ_{\vec{a}}$ is well founded, this means that $\succ_{\vec{a}(\vec{a})}$ is well founded, too. Hence, $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$ is triangular.

Now we prove that $(\varphi, \vec{a})$ terminates iff $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$ terminates. Then the claim immediately follows due to (19), as chaining is clearly computable.

$$\begin{aligned}
& (\varphi, \vec{a}) \text{ does not terminate} \\
\iff & \exists \vec{c} \in \mathcal{S}^d.\ \forall n \in \mathbb{N}.\ \varphi(\vec{a}^n(\vec{c})) \qquad \text{by Def. 1} \\
\iff & \exists \vec{c} \in \mathcal{S}^d.\ \forall n \in \mathbb{N}.\ \varphi(\vec{a}^{2 \cdot n}(\vec{c})) \wedge \varphi(\vec{a}^{2 \cdot n+1}(\vec{c})) \\
\iff & \exists \vec{c} \in \mathcal{S}^d.\ \forall n \in \mathbb{N}.\ \varphi(\vec{a}^{2 \cdot n}(\vec{c})) \wedge \varphi(\vec{a})(\vec{a}^{2 \cdot n}(\vec{c})) \\
\iff & \exists \vec{c} \in \mathcal{S}^d.\ \forall n \in \mathbb{N}.\ (\varphi \wedge \varphi(\vec{a}))\,(\vec{a}(\vec{a}))^n \\
\iff & (\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a})) \text{ does not terminate}
\end{aligned}$$

$\square$

B.9 Proof of Lemma 29

Proof. Recall that for $f, g : \mathbb{N} \to \mathbb{R}$, $f(n) \in o(g(n))$ means

$$\forall m > 0.\ \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ |f(n)| < m \cdot |g(n)|.$$

First consider the case $b_2 > b_1$. We have $b_2^n = b_1^n \cdot \left(\frac{b_2}{b_1}\right)^n$ where $\frac{b_2}{b_1} > 1$. As we clearly have $n^{a_1} \in o\left(\left(\frac{b_2}{b_1}\right)^n\right)$, we obtain $n^{a_1} \cdot b_1^n \in o\left(\left(\frac{b_2}{b_1}\right)^n \cdot b_1^n\right) = o(b_2^n) \subseteq o(n^{a_2} \cdot b_2^n)$, i.e., $n^{a_1} \cdot b_1^n \in o(n^{a_2} \cdot b_2^n)$.

Now consider the case $b_2 = b_1$ and $a_2 > a_1$. Then $n^{a_1} \cdot b_1^n \in o(n^{a_2} \cdot b_2^n)$ trivially holds. □

B.10 Proof of Equation (8)

Proof. If $p(\vec{c}) = 0$, then $k = 0$ by Def. 30 and hence $o(p(\vec{c})) = o(k \cdot n^a \cdot b^n) = o(0)$. Otherwise, $p(\vec{c})$ has the form

$$k \cdot n^a \cdot b^n + \sum_{i=1}^{\ell} k_i \cdot n^{a_i} \cdot b_i^n$$

for $k \neq 0$ and $\ell \geq 0$. We have $k_i^{(b_i,a_i)} \in \operatorname{coefs}(p(\vec{c}))$ and hence $(b, a) >_{lex} (b_i, a_i)$ for all $1 \leq i \leq \ell$. Thus, Lemma 29 implies $n^{a_i} \cdot b_i^n \in o(n^a \cdot b^n)$ and hence we get

$$o(p(\vec{c})) = o\left(k \cdot n^a \cdot b^n + \sum_{i=1}^{\ell} k_i \cdot n^{a_i} \cdot b_i^n\right) = o(n^a \cdot b^n) = o(k \cdot n^a \cdot b^n).$$

□

B.11 Proof of Equation (9)

Proof. If $k = 0$, then the claim is trivial, so assume $k \neq 0$, i.e., $p(\vec{c}) = k \cdot b^n \cdot n^a + p'$ for some $p' \in \mathbb{NPE}$. By Lemma 29 we have

$$\begin{aligned}
& p' \in o(k \cdot b^n \cdot n^a) \\
\iff\ & \forall m > 0.\ \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ |p'| < m \cdot |k \cdot b^n \cdot n^a| \\
\implies\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ |p'| < |k \cdot b^n \cdot n^a|.
\end{aligned}$$

Assume $k > 0$. Then

$$\begin{aligned}
& \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ |p'| < |k \cdot b^n \cdot n^a| \\
\implies\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ -p' < |k \cdot b^n \cdot n^a| \\
\iff\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ -p' < k \cdot b^n \cdot n^a \\
\iff\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ 0 < k \cdot b^n \cdot n^a + p' \\
\iff\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ 0 < p(\vec{c}) \\
\iff\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \operatorname{sign}(p(\vec{c})) = \operatorname{sign}(k).
\end{aligned}$$

If $k < 0$, then

$$
\begin{aligned}
& \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ |p'| < |k \cdot b^n \cdot n^a| \\
\implies\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ p' < |k \cdot b^n \cdot n^a| \\
\iff\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ p' < -k \cdot b^n \cdot n^a \\
\iff\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ k \cdot b^n \cdot n^a + p' < 0 \\
\iff\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ p(\vec{c}) < 0 \\
\iff\ & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \operatorname{sign}(p(\vec{c})) = \operatorname{sign}(k).
\end{aligned}
$$

□

B.12 Proof of Lemma 32

Proof. By the definition of $\vec{q}_{norm}$, we have $p \in \mathbb{NPE}[\vec{x}]$ and thus $p(\vec{c}) \in \mathbb{NPE}$ for any $\vec{c} \in \mathbb{R}^d$. Hence,

$\exists n_0 \in \mathbb{N}. \forall n \in \mathbb{N}_{>n_0}. p(\vec{c}) \triangleright 0$ iff $\operatorname{unmark}\left(\max_{\succ_{coef}}(\operatorname{coefs}(p(\vec{c})))\right) \triangleright 0$ (by (9)).

Let $\operatorname{coefs}(p) = \{\alpha_1^{(b_1,a_1)}, \ldots, \alpha_\ell^{(b_\ell,a_\ell)}\}$ where $\alpha_i^{(b_i,a_i)} \succ_{coef} \alpha_j^{(b_j,a_j)}$ for all $1 \le i < j \le \ell$. If $p(\vec{c}) = 0$, then $\alpha_1(\vec{c}) = \ldots = \alpha_\ell(\vec{c}) = 0$ and thus $\operatorname{coefs}(p(\vec{c})) = \{0^{(1,0)}\}$ and $\operatorname{unmark}\left(\max_{\succ_{coef}}(\operatorname{coefs}(p(\vec{c})))\right) = 0$. Otherwise, there is a $1 \le j \le \ell$ with $\operatorname{unmark}\left(\max_{\succ_{coef}}(\operatorname{coefs}(p(\vec{c})))\right) = \alpha_j(\vec{c}) \neq 0$ and $\alpha_i(\vec{c}) = 0$ for all $1 \le i \le j-1$. Thus,

$$\operatorname{unmark}\left(\max_{\succ_{coef}}(\operatorname{coefs}(p(\vec{c})))\right) \triangleright 0 \quad \text{iff} \quad (\operatorname{red}(p \triangleright 0))(\vec{c}) \text{ holds.}$$

Hence, (10) is equivalent to

$$\exists \vec{x} \in \mathbb{R}^d.\ \psi_F \land \operatorname{red}(p \triangleright 0). \tag{20}$$

As $\vec{q}$ can clearly be transformed into $\vec{q}_{norm}$ in polynomial time, p can be obtained from the inputs φ and $\vec{q}$ in (4) in polynomial time. Thus, the size of p is polynomial in the size of the input and hence we can compute and sort $\operatorname{coefs}(p)$ in polynomial time. Furthermore, $\operatorname{red}(p \triangleright 0)$ is a disjunction of at most $\ell + 1$ subformulas, where each subformula consists of at most ℓ (in-)equations over $\operatorname{coefs}(p)$. As ℓ is bounded by the size of p, $\operatorname{red}(p \triangleright 0)$ can be computed in polynomial time. Since ψ_F is part of the input of (4), it follows that (20) can be computed in polynomial time. □

B.13 Proof of Thm. 34

Proof. We have to prove

$$(11) \iff \exists \vec{x} \in \mathbb{R}^d.\ \psi_F \land \operatorname{red}(\xi), \tag{21}$$

where $\operatorname{red}(\xi)$ results from replacing each atom $p \triangleright 0$ in ξ by $\operatorname{red}(p \triangleright 0)$. Since each $\operatorname{red}(p \triangleright 0)$ can be computed in polynomial time due to Lemma 32, the computation of the formula "$\exists \vec{x} \in \mathbb{R}^d.\ \psi_F \land \operatorname{red}(\xi)$" clearly works in polynomial time.

To prove (21), we introduce the notion of a *fundamental* set. Let $p_1 \triangleright_1 0, \ldots, p_k \triangleright_k 0$ denote the atoms in ξ. We call a subset $I \subseteq \{1, \ldots, k\}$ *fundamental* if $\bigwedge_{i \in I} p_i \triangleright_i 0 \implies \xi$. Recall that w.l.o.g., we can assume that ξ does not contain any Boolean connectives except $\wedge$ and $\vee$. Thus, whenever $\xi \neq$ false, the formula ξ must have fundamental sets. Clearly, we have

$$\exists \vec{x} \in \mathbb{R}^d.\ \psi_F \wedge \mathrm{red}(\xi) \iff \exists \text{ fundamental set } I.\ \psi_F \wedge \bigwedge_{i \in I} \mathrm{red}(p_i \triangleright_i 0) \text{ is valid.}$$

Thus, to prove (21), it suffices to show the following:

$$(11) \iff \exists \text{ fundamental set } I.\ \psi_F \wedge \bigwedge_{i \in I} \mathrm{red}(p_i \triangleright_i 0) \text{ is valid.} \tag{22}$$

For the " $\Longleftarrow$ "-direction of (22), assume that there is such a fundamental set, i.e.,

$$\psi_F \wedge \bigwedge_{i \in I} \mathrm{red}(p_i \triangleright_i 0)$$

is valid. Then as in the proof of Lemma 32, we obtain that for each $i \in I$, there is an $n_i \in \mathbb{N}$ such that

$$\exists \vec{x} \in F.\ \forall n \in \mathbb{N}_{>n_i}.\ p_i \triangleright_i 0.$$

As I is finite, $n_{\max} = \max\{n_i \mid i \in I\}$ exists. Hence, we get

$$\exists \vec{x} \in F.\ \forall n \in \mathbb{N}_{>n_{max}}.\ \bigwedge_{i \in I} p_i \triangleright_i 0.$$

Since I is fundamental, this implies (11).

For the " $\Longrightarrow$ "-direction, assume (11). Then there is a $\vec{c} \in F$ and an $n_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}_{>n_0}$, there is a fundamental set I_n such that $\bigwedge_{i \in I_n} p_i(\vec{c}) \triangleright_i 0$ holds. As there are only finitely many fundamental sets, there is some fundamental set I that occurs infinitely often in $(I_n)_{n \in \mathbb{N}_{>n_0}}$. Hence we get

$$\exists n_0 \in \mathbb{N}.\ \exists^\infty n \in \mathbb{N}_{>n_0}.\ \bigwedge_{i \in I} p_i(\vec{c}) \triangleright_i 0. \tag{23}$$

By definition of poly-exponential expressions, each $p_i(\vec{c})$ is weakly monotonic in n for large enough n. Thus, for large enough n, the $p_i(\vec{c})$ with $i \in I$ must be weakly monotonically *increasing*. Thus, (23) implies

$$\exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \bigwedge_{i \in I} p_i(\vec{c}) \triangleright_i 0.$$

As $\vec{c} \in F$, this implies that there is a fundamental set I such that $\psi_F \wedge \bigwedge_{i \in I} \mathrm{red}(p_i \triangleright_i 0)$ holds. $\square$

B.14 Proof of Thm. 35

Proof. By Thm. 23, termination of *twn*-loops is reducible to termination of *tnn*-loops. Given a *tnn*-loop $(\varphi, \vec{a})$, we obtain $\vec{q}_{norm} \in (\mathbb{NPE}[\vec{x}])^d$ such that $(\varphi, \vec{a})$ is (eventually) non-terminating iff (7) holds, where φ is a propositional formula over the atoms $\{\alpha \geq 0, \alpha > 0 \mid \alpha \in K_{\mathcal{S}}[\vec{x}]\}$. Hence, $\varphi(\vec{q}_{norm})$ is a propositional formula over the atoms $\{p \triangleright 0 \mid p \in \mathbb{NPE}[\vec{x}], \triangleright \in \{\geq, >\}\}$. Thus, by Thm. 34, validity of (7) resp. (11) is reducible to $\mathrm{Th}_\exists(\mathcal{S}, \mathbb{R})$. $\square$

B.15 Proof of Cor. 36

Proof. By Thm. 23, termination of *twn*-loops is reducible to termination of *tnn*-loops. By Thm. 35, termination of *tnn*-loops is reducible to invalidity of a formula $\chi \in \mathrm{Th}_{\exists}(\mathcal{S}, \mathbb{R})$. If $\mathcal{S} = \mathbb{R}$, then validity of χ is decidable, and if $\mathcal{S} = \mathbb{Z}$ or $\mathcal{S} = \mathbb{Q}$, then validity of ξ semi-decidable (cf. [8, 26]). But χ is valid iff the loop is non-terminating. Hence, non-termination is decidable for $\mathcal{S} = \mathbb{R}$ and semi-decidable if $\mathcal{S} = \mathbb{Z}$ or $\mathcal{S} = \mathbb{Q}$. The claim follows since *deciding* non-termination is equivalent to deciding termination. $\square$

B.16 Proof of Lemma 37

Proof. Let ξ contain the atoms $p_i \triangleright_i 0$ for $1 \leq i \leq k$. We have:

$$\begin{aligned}
& \vec{c} \text{ witnesses eventual non-termination of } (\varphi, \vec{a}) \\
\iff & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ (\varphi(\vec{q}_{norm}))\,(\vec{c}) && \text{(by (7))} \\
\iff & \exists n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ \xi(\vec{c}) \\
\iff & \psi_F(\vec{c}) \wedge \mathrm{red}(\xi)(\vec{c}) && \text{(as in the proof of Thm. 34)}
\end{aligned}$$

$\square$