Perspective Projection of an Ellipsoid

David Eberly
Geometric Tools, LLC
http://www.geometrictools.com/

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Let the eyepoint be $\mathbf{e}$ and let the view plane be $\mathbf{n} \cdot \mathbf{x} = \lambda$ where $\mathbf{n}$ is unit length and where $\mathbf{n} \cdot \mathbf{e} \neq \lambda$; that is, the eyepoint is not on the view plane. Moreover, let $\mathbf{n}$ be oriented away from the eyepoint in the sense that the point on the plane closest to the eyepoint is $\mathbf{e} + \ell\mathbf{n}$ with $\ell > 0$. An ellipsoid is defined by a quadratic equation $\mathbf{x}^{\mathrm{T}} A\mathbf{x} + \mathbf{b} \cdot \mathbf{x} + c = 0$ where A is positive definite (all positive eigenvalues). The problem is to compute the projection of the ellipsoid onto the view plane. NOTE: I am assuming that the ellipsoid is in front of the eyepoint; that is, the projection of the ellipsoid onto the ray $\mathbf{e} + t\mathbf{n}$ is an interval $[t_0, t_1]$ with $t_0 > 0$. In this case, the projection of the ellipsoid onto the view plane is an ellipse. If $t_0 \leq 0$, the projection is a hyperbola or parabola, and this document does not currently describe how to compute the standard form for these objects.

Consider rays of the form $\mathbf{x}(\tau) = \mathbf{e} + \tau\mathbf{d}$ for unit length direction vectors $\mathbf{d}$ and for $\tau \geq 0$. The intersection of a ray with the ellipsoid is determined by replacing $\mathbf{x}(\tau)$ in the equation for the ellipsoid and solving the resulting quadratic equation:

$$\begin{aligned} 0 &= \mathbf{x}(\tau)^{\mathrm{T}} A\mathbf{x}(\tau) + \mathbf{b} \cdot \mathbf{x}(\tau) + c \\ &= (\mathbf{e} + \tau\mathbf{d})^{\mathrm{T}} A(\mathbf{e} + \tau\mathbf{d}) + \mathbf{b} \cdot (\mathbf{e} + \tau\mathbf{d}) + c \\ &= (\mathbf{d}^{\mathrm{T}} A\mathbf{d})\tau^2 + (\mathbf{b} \cdot \mathbf{d} + 2\mathbf{d}^{\mathrm{T}} A\mathbf{e})\tau + (\mathbf{e}^{\mathrm{T}} A\mathbf{e} + \mathbf{b} \cdot \mathbf{e} + c) \\ &= \alpha\tau^2 + \beta\tau + \gamma \end{aligned}$$

If the quadratic has two distinct real roots, then the ray intersects the ellipsoid twice. The two projected points lie in the interior of the projected ellipse. If the quadratic has two non-real roots, then the ray misses the ellipsoid. The case of interest is when the ray has a repeated real root, in which case the ray is tangent to the ellipsoid. The projected point lies on the projected ellipse. For the quadratic to have a repeated real root, the discriminant $\beta^2 - 4\alpha\gamma$ must be zero:

$$\begin{aligned} 0 &= \beta^2 - 4\alpha\gamma \\ &= (\mathbf{b} \cdot \mathbf{d} + 2\mathbf{d}^{\mathrm{T}} A\mathbf{e})^2 - 4(\mathbf{d}^{\mathrm{T}} A\mathbf{d})(\mathbf{e}^{\mathrm{T}} A\mathbf{e} + \mathbf{b} \cdot \mathbf{e} + c) \\ &= \mathbf{d}^{\mathrm{T}} \left[(\mathbf{b} + 2A\mathbf{e})(\mathbf{b} + 2A\mathbf{e})^{\mathrm{T}} - 4(\mathbf{e}^{\mathrm{T}} A\mathbf{e} + \mathbf{b} \cdot \mathbf{e} + c)A\right] \mathbf{d} \\ &= \mathbf{d}^{\mathrm{T}} M\mathbf{d} \end{aligned}$$

The repeated root is $\tau = -\beta/(2\alpha)$ which does not play a role in computing the projected ellipse. If you want to know the contour of the ellipsoid whose projection is the ellipse, then you will need to do some more work using τ.

Now consider the intersection of rays with the view plane. The intersection of a ray with the plane is determined by replacing $\mathbf{x}(\tau)$ in the equation for the plane and solving the resulting linear equation:

$$\begin{aligned} \lambda &= \mathbf{n} \cdot \mathbf{x}(\tau) \\ &= \mathbf{n} \cdot (\mathbf{e} + \tau\mathbf{d}) \\ &= \mathbf{n} \cdot \mathbf{e} + \tau\mathbf{n} \cdot \mathbf{d} \end{aligned}$$

The solution is $\bar{\tau} = (\lambda - \mathbf{n} \cdot \mathbf{e})/(\mathbf{n} \cdot \mathbf{d})$. The point of intersection is

$$\mathbf{x} = \mathbf{e} + \bar{\tau}\mathbf{d}$$

Solving for the direction vector instead yields

$$\mathbf{d} = \frac{1}{\bar{\tau}} (\mathbf{x} - \mathbf{e})$$

Note that $\bar{\tau} \neq 0$ since $\mathbf{n} \cdot \mathbf{e} \neq \lambda$, so the reciprocal exists. Using those directions $\mathbf{d}$ which produce the contour of the ellipsoid, we have

$$\begin{aligned} 0 &= \mathbf{d}^{\mathrm{T}} M \mathbf{d} \\ &= \tfrac{1}{\bar{\tau}^2} (\mathbf{x} - \mathbf{e})^{\mathrm{T}} M (\mathbf{x} - \mathbf{e}) \end{aligned}$$

Consequently, the projected ellipse is defined by the equation

$$(\mathbf{x} - \mathbf{e})^{\mathrm{T}} M (\mathbf{x} - \mathbf{e}) = 0$$

where $\mathbf{x}$ are points in the view plane.

Now for conversion to an equation involving two variables. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors such that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{n}$ form an orthonormal system (all vectors unit length and are mutually orthogonal). Let $\boldsymbol{\theta}$ be the projection of the eyepoint onto the view plane. This point is obtained by intersecting the ray $\mathbf{e} + \tau \mathbf{n}$ with the plane $\mathbf{n} \cdot \mathbf{x} = \lambda$. The solution is $\tau = \lambda - \mathbf{n} \cdot \mathbf{e}$ and

$$\boldsymbol{\theta} = \mathbf{e} + (\lambda - \mathbf{n} \cdot \mathbf{e}) \mathbf{n}$$

Any point on the plane is given by

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} + \boldsymbol{\theta}$$

Replacing this in the ellipse equation yields

$$\begin{aligned} 0 &= (\mathbf{x} - \mathbf{e})^{\mathrm{T}} M (\mathbf{x} - \mathbf{e}) \\ &= (s\mathbf{u} + t\mathbf{v} + \boldsymbol{\theta} - \mathbf{e})^{\mathrm{T}} M (s\mathbf{u} + t\mathbf{v} + \boldsymbol{\theta} - \mathbf{e}) \\ &= k_0 s^2 + 2k_1 st + k_2 t^2 + k_3 s + k_4 t + k_5 \end{aligned}$$

where

$$\begin{aligned} k_0 &= \mathbf{u}^{\mathrm{T}} M \mathbf{u} \\ k_1 &= \mathbf{u}^{\mathrm{T}} M \mathbf{v} \\ k_2 &= \mathbf{v}^{\mathrm{T}} M \mathbf{v} \\ k_3 &= 2(\lambda - \mathbf{n} \cdot \mathbf{e}) \mathbf{u}^{\mathrm{T}} M \mathbf{n} \\ k_4 &= 2(\lambda - \mathbf{n} \cdot \mathbf{e}) \mathbf{v}^{\mathrm{T}} M \mathbf{n} \\ k_5 &= (\lambda - \mathbf{n} \cdot \mathbf{e})^2 \mathbf{n}^{\mathrm{T}} M \mathbf{n} \end{aligned}$$

Finally, this equation can be converted into the standard one for an ellipsoid via rotation of axes and completing the square. The equation is

$$\boldsymbol{\sigma}^{\mathrm{T}} P \boldsymbol{\sigma} + \mathbf{q} \cdot \boldsymbol{\sigma} + r = 0 \tag{1}$$

where $\boldsymbol{\sigma} = (s, t)$, $\mathbf{q} = (k_3, k_4)$, $r = k_5$, and

$$P = \begin{bmatrix} k_0 & k_1 \\ k_1 & k_2 \end{bmatrix}$$

Using eigenmethods, the matrix P can be factored as $P = R^{\mathrm{T}} D R$ where R is a rotation matrix and D is a diagonal matrix. Either D has all positive diagonal entries or all negative diagonal entries. If they are negative, multiply equation (1) by -1. That is, replace P by $-P$, $\mathbf{q}$ by $-\mathbf{q}$, and r by $-r$. This does not change the solution set to the equation. After the replacement, the new P is positive definite so D has all positive diagonal entries. Define $\mathbf{w} = R\boldsymbol{\sigma}$ and $\boldsymbol{\beta} = R\mathbf{q}$. The quadratic equation becomes

$$\begin{aligned} 0 &= \mathbf{w}^{\mathrm{T}} D \mathbf{w} + \boldsymbol{\beta} \cdot \mathbf{w} + r \\ &= d_1 w_1^2 + d_2 w_2^2 + \beta_1 w_1 + \beta_2 w_2 + r \\ &= d_1 \left(w_1 + \tfrac{\beta_1}{2d_1} \right)^2 + d_2 \left(w_2 + \tfrac{\beta_2}{2d_2} \right)^2 + r - \tfrac{\beta_1^2}{4d_1} - \tfrac{\beta_2^2}{4d_2} \end{aligned}$$

Define

$$\phi = \frac{\frac{\beta_1^2}{4d_1} + \frac{\beta_2^2}{4d_2} - r}{d_1 d_2}$$

In standard form we have

$$\frac{\left(w_1 + \frac{\beta_1}{2d_1} \right)^2}{d_2 \phi} + \frac{\left(w_2 + \frac{\beta_2}{2d_2} \right)^2}{d_1 \phi} = 1$$

The ellipse center is $(-\beta_1/(2d_1), -\beta_2/(2d_2))$ and the axis lengths are $\sqrt{d_2\phi}$ and $\sqrt{d_1\phi}$.