

Decision trees

Fraida Fund

Contents

In this lecture	2
Flexible decisions with cheap prediction?	2
Decision tree	2
Tree terminology	2
Note on notation	2
Stratification of feature space (1)	3
Stratification of feature space (2)	3
Stratification of feature space - illustration	3
Approximation using steps	4
Training a decision tree	5
Basic idea	5
Recursive binary splitting steps	6
Recursive binary splitting	6
Loss function for regression tree	6
Loss function for classification tree	7
Classification error rate	7
GINI index	7
Entropy	7
Comparison - measures of node impurity	7
Conditional entropy	8
Information gain	8
Example: should I play tennis? (1)	8
Example: should I play tennis? (2)	8
Example: should I play tennis? (3)	8
Example: should I play tennis? (4)	9
Example: should I play tennis? (5)	9
Variations	9
Interpretability	10
Feature importance	10
Bias and variance	10
Managing tree depth	10
Stopping criteria	10
Pruning	10
Weakest link pruning (1)	11
Weakest link pruning (2)	11
Cost complexity pruning	12
Runtime	12
Inference time	12
Training time (binary tree)	13
Summary - so far	14
The good and the bad (1)	14

Math prerequisites for this lecture: None

In this lecture

- Decision trees
 - Training decision trees
 - Bias and variance of decision trees

Flexible decisions with cheap prediction?

KNN was very flexible, but prediction is **slow**.

Next: flexible decisions, non-parametric approach, fast prediction

Idea: In KNN, we find the “neighborhood” of a test point and then give it the value of training points in that “neighborhood” - but it takes too long at inference time to define the “neighborhood”.

What if we define “neighborhoods” and their values in advance, at training time? Then at inference time, we only need to determine which “neighborhood” a test point belongs in.

However, we run into another **computationally hard** problem! To partition the feature space into *optimal* neighborhoods is too expensive. Instead, we will rely on some heuristics and get a non-optimal, but good enough, partition.

Decision tree

Tree terminology

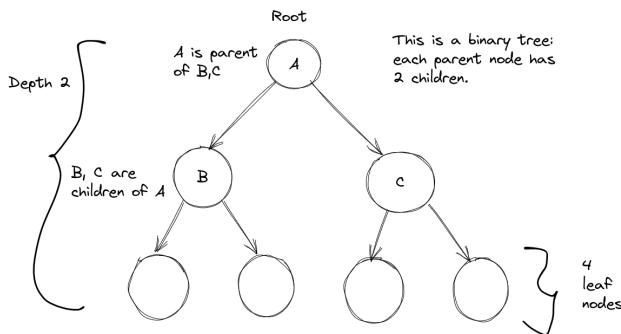


Figure 1: A binary tree.

- size of tree $|T|$ (number of leaf nodes)
 - depth (max length from root node to a leaf node)

Note on notation

Following notation of ISL, Chapter 8:

- X_j is feature j
 - x_i is sample i

Stratification of feature space (1)

Given set of possible predictors, X_1, \dots, X_p

- Training: Divide predictor space (set of possible values of X) into J non-overlapping regions: R_1, \dots, R_J , by splitting sequentially on one feature at a time.

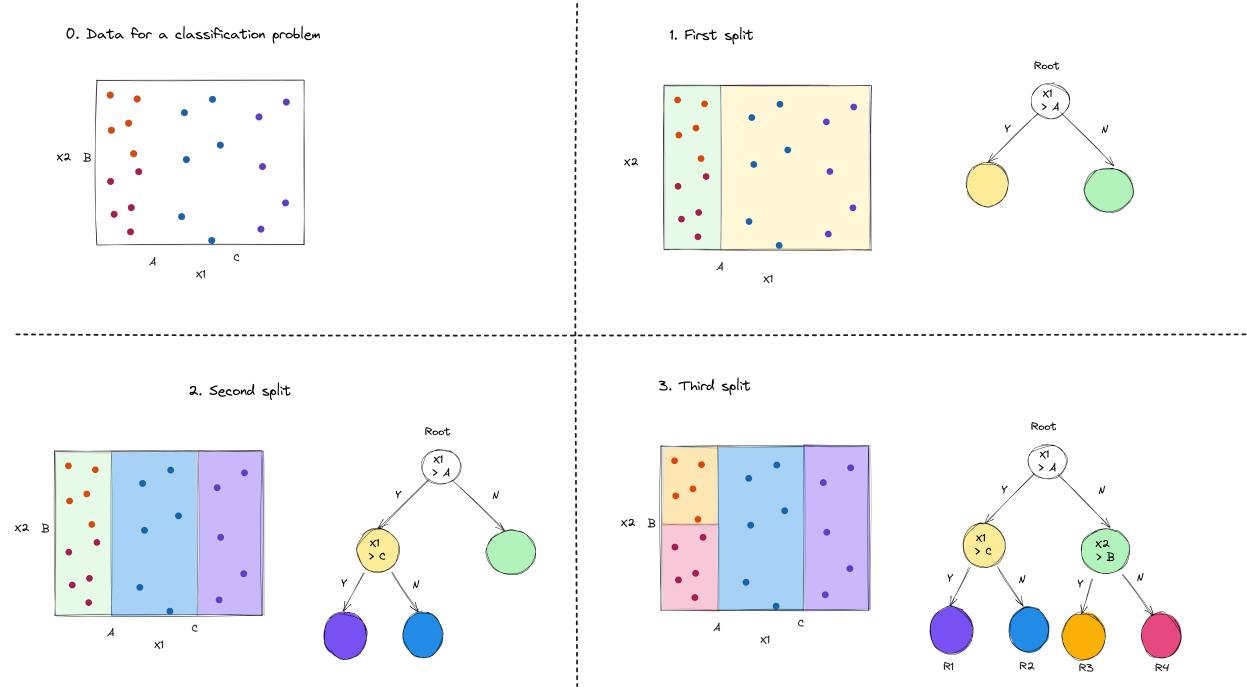


Figure 2: Dividing the feature space with a decision tree.

Stratification of feature space (2)

Prediction:

- At node that is not a leaf: test one feature X_i
- Branch from node depending on value of X_i
- At leaf node: for observation that falls in region R_j , predict \hat{y}_{R_j}
 - mean of labels of training points in R_j (regression)
 - mode of labels of training points in R_j (classification)

Stratification of feature space - illustration

The stratification on the top left cannot be produced by a decision tree using recursive binary splitting. The other three subfigures represent a single stratification. Note that the decision tree fits a piecewise step function!

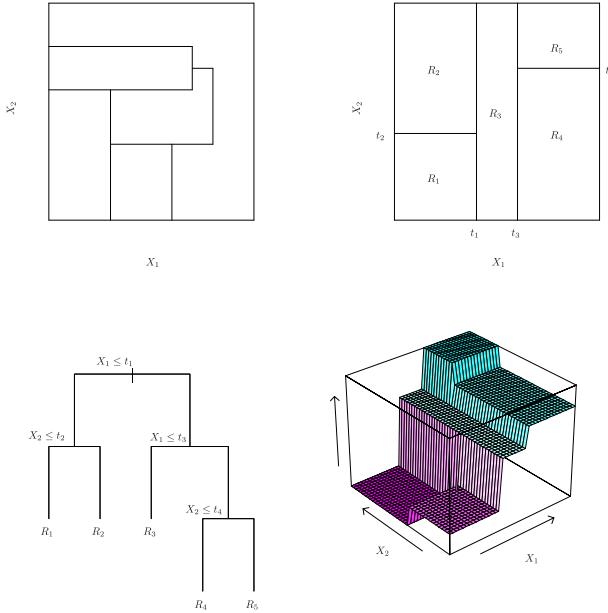


Figure 3: ISLR, Fig. 8.3.

Approximation using steps

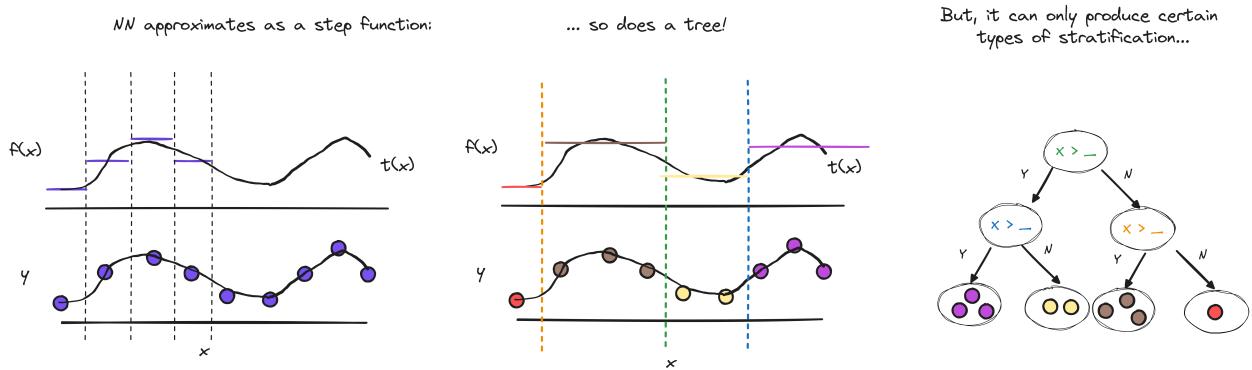


Figure 4: Like nearest neighbor, a tree represents an arbitrary function using steps.

Training a decision tree

Basic idea

- Goal: find the high-dimensional rectangles that minimize error
- Practical approach: recursive binary splitting (top-down, greedy approach)

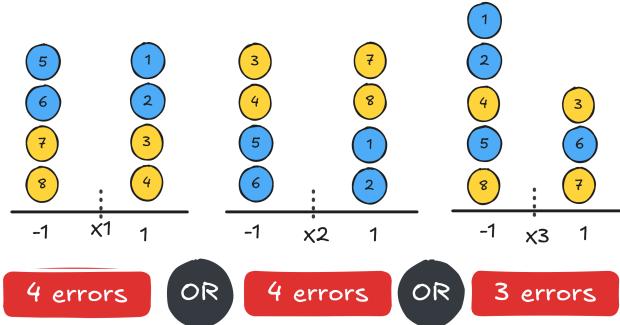
It is too computationally expensive to consider every possible partition. Instead, we'll use a greedy algorithm: at each step, make the best "local" decision at that step, without looking ahead and making a decision that might yield better results at future steps

For this example data:

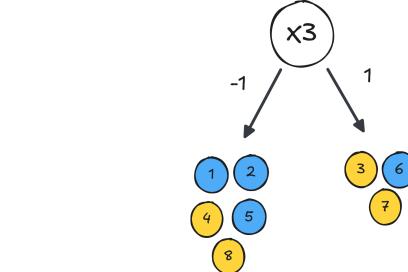
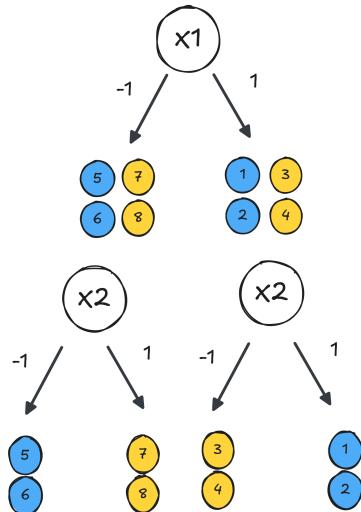
y	x_1	x_2	x_3
1	1	1	1
2	1	1	1
3	0	1	-1
4	0	1	1
5	1	-1	-1
6	1	-1	-1
7	0	-1	1
8	0	-1	1

But a greedy algorithm would not build that tree!

At the root, it would find that a split on x_3 is best:



A split on x_1 and then x_2 make a depth 2 tree with 0 error:



and then at the next level, a split on x_1 or x_2 would still result in 3 errors total:

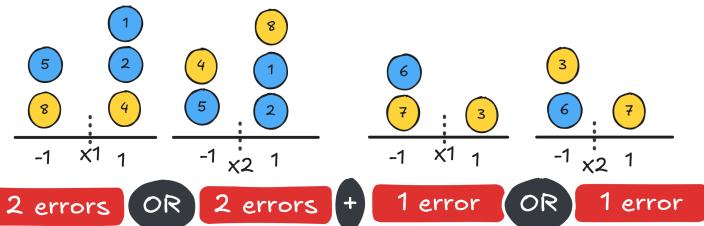


Figure 5: Example showing how a greedy algorithm might not find the best tree.

Recursive binary splitting steps

Start at root of the tree, considering all training samples.

1. At the current node,
2. Find feature X_j and *cutpoint s* that minimizes some *local loss function* (?)
3. Split training samples at that node into two leaf nodes
4. Stop when no training error (?)
5. Otherwise, repeat at leaf nodes

Recursive binary splitting

For any feature j and *cutpoint s*, define the regions

$$R_1(j, s) = \{X | X_j < s\}, \quad R_2(j, s) = \{X | X_j \geq s\}$$

where $\{X | X_j < s\}$ is the region of predictor space in which X_j takes on a value less than s .

Loss function for regression tree

For regression: look for feature j and cutpoint s that leads to the greatest possible reduction in squared error, where the “new” squared error is (\hat{y}_{R_j} is the prediction for the samples in R_j):

$$\sum_{i: x_i \in R_1(j, s)} (y_i - \hat{y}_{R_1})^2 + \sum_{i: x_i \in R_2(j, s)} (y_i - \hat{y}_{R_2})^2$$

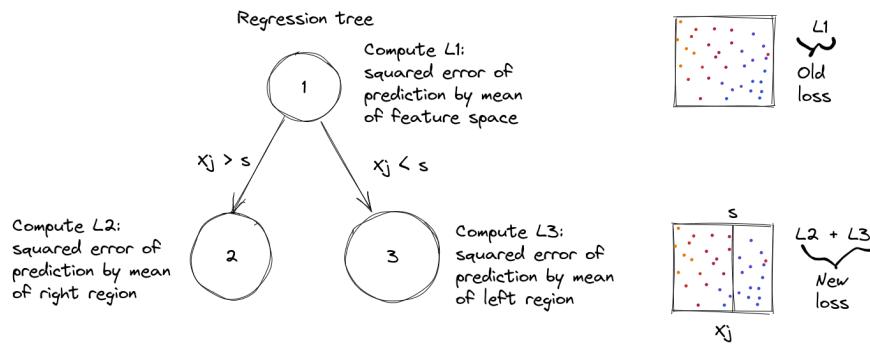


Figure 6: Training a regression tree.

Note: we are *not* computing the average loss of each region, we want the loss to be weighted by the number of samples it applies to!

Loss function for classification tree

For classification, find a split that minimizes some measure of node *impurity*:

- A node whose samples all belong to the same class - most *pure*
- A node whose samples are evenly distributed among all classes - highly *impure*

Classification error rate

For classification: one possible way is to split on 0-1 loss or *misclassification rate*:

$$\sum_{x_i \in R_m} 1(y_i \neq \hat{y}_{R_m})$$

Not used often ([this is a good explanation re: why](#)), but used for *pruning*.

GINI index

The GINI index is:

$$\sum_{k=1}^K \hat{p}_{mk}(1 - \hat{p}_{mk})$$

where \hat{p}_{mk} is the proportion of training samples in R_m belonging to class k .

You can see that this is small when all values of \hat{p}_{mk} are around 0 or 1.

Entropy

Entropy as a measure of impurity on subset of samples:

$$-\sum_{k=1}^K \hat{p}_{mk} \log_2 \hat{p}_{mk}$$

where \hat{p}_{mk} is the proportion of training samples in R_m belonging to class k .

Comparison - measures of node impurity

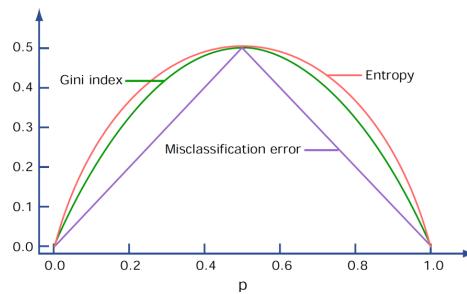


Figure 7: Measures of node “impurity”.

Conditional entropy

- Splitting on feature X creates subsets S_1 and S_2 with different entropies
- Conditional entropy:

$$\text{Entropy}(S|X) = \sum_v \frac{|S_v|}{|S|} \text{Entropy}(S_v)$$

Information gain

- Choose feature to split so as to maximize information gain, the expected reduction in entropy due to splitting on X :

$$\text{Gain}(S, X) := \text{Entropy}(S) - \text{Entropy}(S|X)$$

Example: should I play tennis? (1)

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Figure 8: Via Tom Mitchell.

Example: should I play tennis? (2)

For top node: $S = \{9+, 5-\}, |S| = 14$

$$\text{Entropy}(S) = -\frac{9}{14} \log_2 \frac{9}{14} - \frac{5}{14} \log_2 \frac{5}{14} = 0.94$$

Example: should I play tennis? (3)

If we split on Wind:

Considering the Weak branch:

- $S_{\text{weak}} = \{6+, 2-\}, |S_{\text{weak}}| = 8$
- $\text{Entropy}(S_{\text{weak}}) = -\frac{6}{8} \log_2 \left(\frac{6}{8}\right) - \frac{2}{8} \log_2 \left(\frac{2}{8}\right) = 0.81$

Considering the Strong branch:

- $S_{\text{strong}} = \{3+, 3-\}, |S_{\text{strong}}| = 6$
- $\text{Entropy}(S_{\text{strong}}) = 1$

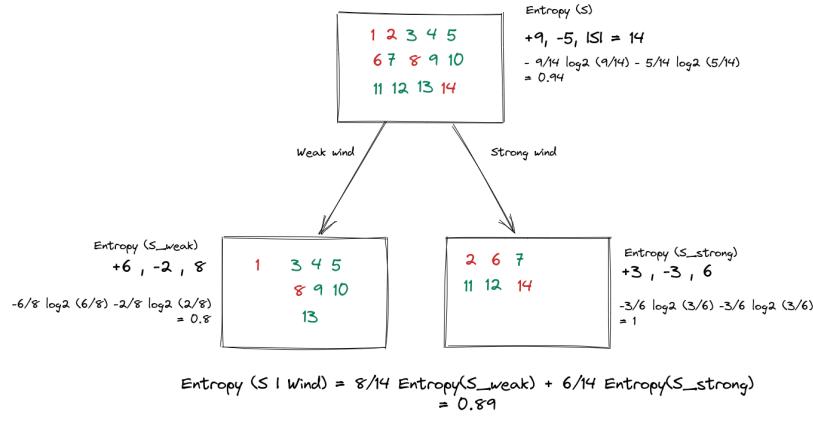


Figure 9: Considering the split on Wind.

Example: should I play tennis? (4)

$$\text{Entropy}(S) = -\frac{9}{14} \log_2 \frac{9}{14} - \frac{5}{14} \log_2 \frac{5}{14} = 0.94$$

$$\text{Entropy}(S \mid \text{Wind}) = \frac{8}{14} \text{Entropy}(S_{\text{weak}}) + \frac{6}{14} \text{Entropy}(S_{\text{strong}}) = 0.89$$

$$\text{Gain}(S, \text{Wind}) = 0.94 - 0.89 = 0.05$$

Example: should I play tennis? (5)

- $\text{Gain}(S, \text{Outlook}) = 0.246$
- $\text{Gain}(S, \text{Humidity}) = 0.151$
- $\text{Gain}(S, \text{Wind}) = 0.048$
- $\text{Gain}(S, \text{Temperature}) = 0.029$

→ Split on Outlook!

Variations

In this example, the data had only categorical variables, and no missing values.

What if we had a continuous (not categorical) variable? We would need to also decide how to partition the continuous feature into a discrete set of intervals.

There are a few well-known algorithms for fitting decision trees - CART, ID3, C4.5 - that have different capabilities with respect to continuous features, features with missing values, and what measure of node impurity is used.

e.g. C4.5 introduces the idea that if a sample has a missing value for a feature,

- when training, compute information gain using only samples where the feature is defined
- when using, we decide which branch to follow based on which is more probable

Interpretability

Feature importance

- For each feature X_j , find all nodes where the feature was used as the split variable
- Add up information gain due to split (or for GINI index, difference in loss weighted by number of samples.)
- This sum reflects feature importance

This feature importance can be used for feature selection or feature weighting!

It tends to do reasonable things both with (1) features that are only useful in combination and (2) features that are highly correlated.

Bias and variance

Managing tree depth

- If tree is too deep - likely to overfit (high variance)
- If tree is not deep enough - likely to have high bias

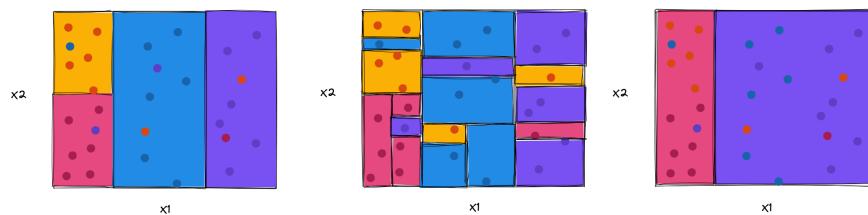


Figure 10: The depth/size of the tree (number of regions) controls the complexity of the regression line or decision boundaries, and the bias variance tradeoff.

Stopping criteria

If we build tree until there is zero error on training set, we have “memorized” training data.

Other stopping criteria:

- Max depth
- Max size (number of leaf nodes)
- Min number of samples to split
- Min number of samples in leaf node
- Min decrease in loss function due to split

(Can select depth, etc. by CV)

Pruning

- Alternative to stopping criteria: build entire tree, then *prune*
- With greedy algorithm - a very good split may descend from a less-good split

Weakest link pruning (1)

Prune a large tree from leaves to root:

- Start with full tree T_0
- Merge two adjacent leaf nodes into their parent to obtain T_1 by minimizing:

$$\frac{Err(T_1) - Err(T_0)}{|T_0| - |T_1|}$$

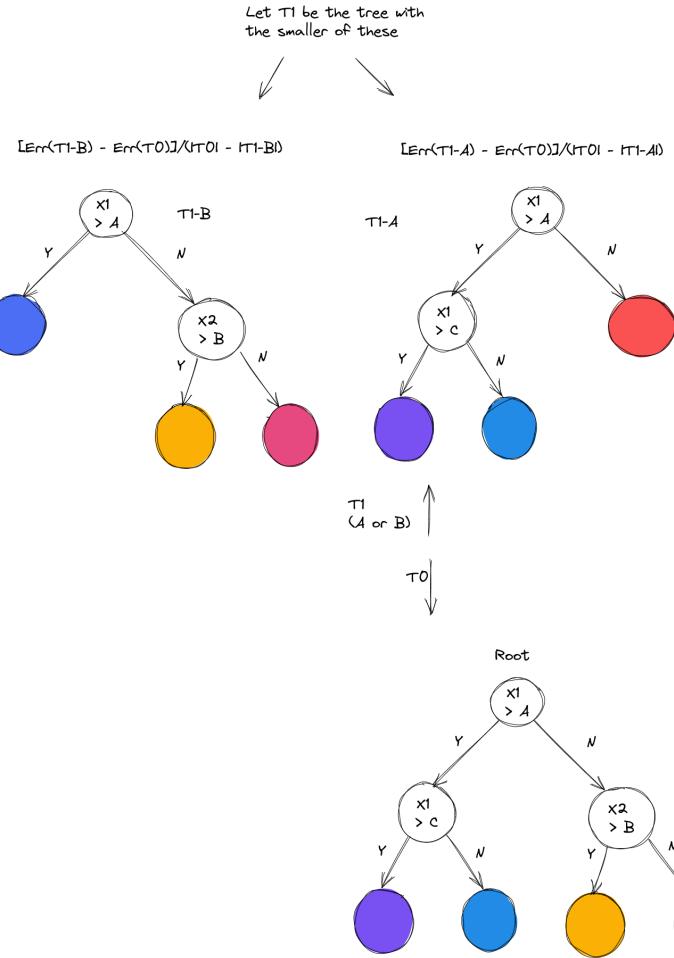


Figure 11: Weakest link pruning.

We usually prune classification trees using classification error rate as loss function, even if tree was built using GINI or entropy.

Weakest link pruning (2)

- Iterate to produce a sequence of trees T_0, T_1, \dots, T_m where T_m is a tree of minimum size.
- Select optimal tree by CV

Candidate trees: T_0, T_1, T_2, T_3
 Use CV to select the candidate tree that has best validation accuracy.

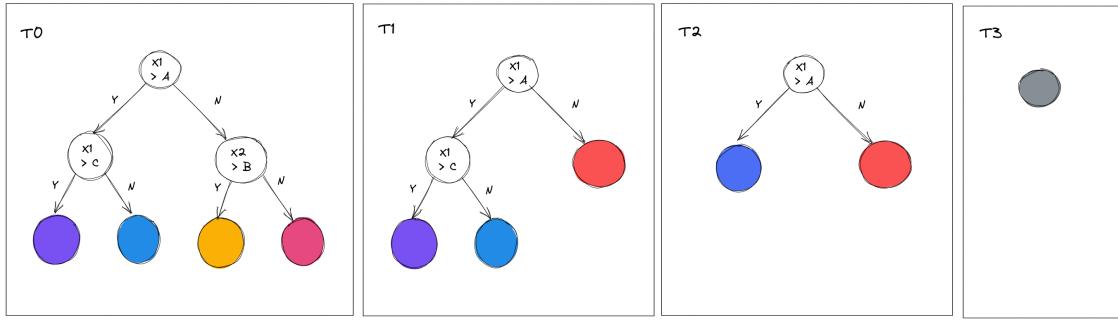


Figure 12: Selecting tree from the set of candidate trees.

Cost complexity pruning

Equivalent to: Minimize

$$\sum_{m=1}^{|T|} \sum_{x_i \in R_m} (y_i - \hat{y}_{R_m})^2 + \alpha |T|$$

Choose α by CV, 1-SE rule ($\uparrow \alpha, \downarrow |T|$).

Runtime

Inference time

$O(d)$ where d is depth of tree

- Worst case: $O(n)$
- Well-balanced: $O(\log n)$

(In a well-balanced tree, we halve the number of samples in each split.)

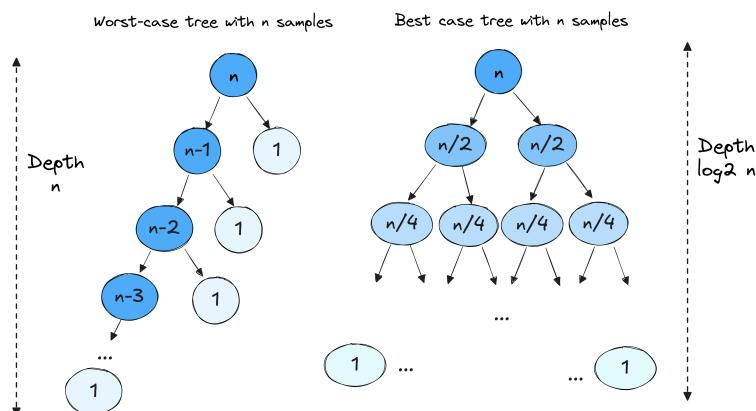


Figure 13: Tree depth in worst-case and well-balanced trees.

Training time (binary tree)

At a node with n_i samples,

- consider up to $n_i - 1$ cutpoints
- for p features $\rightarrow O(pn_i)$ impurity computations per node

What is the cost of computing impurity? If we sort at the beginning (cost: $O(pn \log n)$ for sorting), we can do it much faster than the naive way!

Naive way:

- count samples of each class to the left vs right of the cut: $O(n_i)$
- do this for $n_i - 1$ split points

With sorted features, using linear scan:

- start with empty $<$ side, $>$ side
- move one (sorted) sample from the right to the left of the cut: $O(1)$

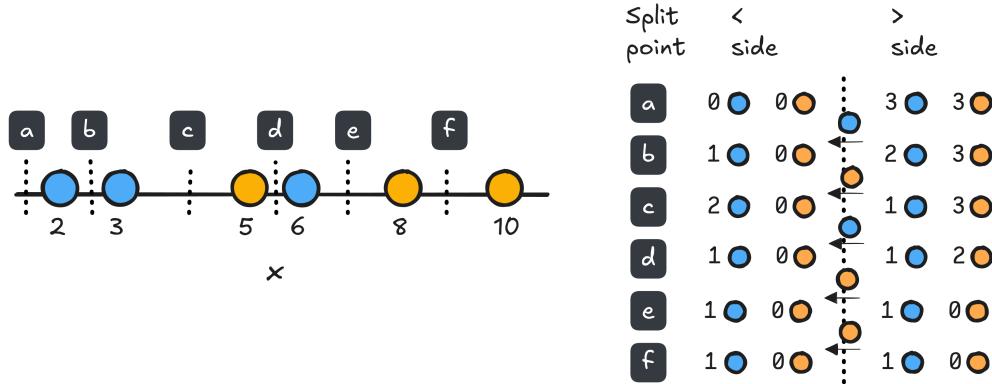


Figure 14: Efficient computation of p_{mk} with linear scan.

Then, going from node \rightarrow tree:

Balanced case: At each depth level of the tree, we handle all n samples, so the total cost is:

$O(pn)$ cost per depth level $\times \log n$ number of levels

\rightarrow total cost of recursive splitting for balanced case: $O(pn \log n)$.

Worst case (tree depth n):

- cost at level i is $pn_i = p(n - i)$
- sum across levels $p(1 + 2 + \dots + (n - 2) + (n - 1) + n)$
- using the rule for sum of integers up to n : $p \frac{n(n-1)}{2}$

\rightarrow total cost of recursive splitting for worst case: $O(pn^2)$

Summary - so far

The good and the bad (1)

Good:

- Flexible with much faster inference time than KNN
- Easy to interpret, close to human decision-making
- Can derive feature importance
- Easily handles mixed types, different ranges

The good and the bad (2)

Bad:

- Need greedy heuristic to train
- Deep trees have large variance
- Non-robust: Small change in data can cause large change in estimated tree