Cluster-size entropy in the Axelrod model of social influence: Small-world networks and mass media

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We study the Axelrod's cultural adaptation model using the concept of *cluster-size entropy* S_c, which gives information on the variability of the cultural cluster size present in the system. Using networks of different topologies, from regular to random, we find that the critical point of the well-known nonequilibrium monocultural-multicultural (order-disorder) transition of the Axelrod model is given by the maximum of the $S_c(q)$ distributions. The width of the cluster entropy distributions can be used to qualitatively determine whether the transition is first or second order. By scaling the cluster entropy distributions we were able to obtain a relationship between the critical cultural trait q_c and the number F of cultural features in two-dimensional regular networks. We also analyze the effect of the mass media (external field) on social systems within the Axelrod model in a square network. We find a partially ordered phase whose largest cultural cluster is not aligned with the external field, in contrast with a recent suggestion that this type of phase cannot be formed in regular networks. We draw a $q - B$ phase diagram for the Axelrod model in regular networks.

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I. INTRODUCTION

Recently, research in complex systems has paid particular attention to elucidating some of the mechanisms leading to interesting economic and social phenomena, such as opinion formation, self-organization, distribution of richness, formation of coalitions, land and air traffic, and the evolution of social structures [1–3]. Several theoretical approaches have been proposed to understand social systems. Axelrod [4] introduced a model to study, in particular, the dissemination of cultures among interacting individuals or agents in which (a) the more culturally similar the agents, the greater the chance of interaction between them, and (b) interaction increases similarity between individuals. Among the interesting results obtained with this model is a nonequilibrium transition from a monocultural state, where all agents share the same cultural features, to a multicultural state, where individuals mostly have their own features, as the cultural diversity increases [5].

The Axelrod model has been widely employed to analyze the effect of cultural drift caused by noise [6–8], repulsive interactions between individuals [9], and mass media [10–14] on social systems. The effect of the mass media, which is normally modeled as a uniform external field with values in the range [0, 1], has been of special interest since it was discovered that in finite square networks there is a critical field $B_c = 0.05$ above which the state of the system is always multicultural (disordered) [10,11]. It was also found that for values of the number of cultural traits $q > q_c$ the state is always multicultural, independently of the field value B [10]. Here, q_c is the critical value of the nonequilibrium multicultural-monocultural phase transition in the absence of an external field ($B = 0$). Later, it was shown that even for vanishing B the monocultural state is destabilized in very large systems [14]. More recently, it was found that for the Axelrod model in complex networks the system can order in a vector state different from the one imposed by the external field [12,13]. All of these are very counterintuitive findings, opposed to what is classically found in physics for spin systems that monotonously align with the external field. Interestingly, ordered phases that are not aligned with the external field are only encountered in fully connected, random, and scale-free networks. It was thus claimed that long-range interactions, absent in regular lattices, are fully required for the appearance of this result [12].

The nonequilibrium phase transition (hereafter we will omit the term "nonequilibrium") of the Axelrod model is characterized by an order parameter ϕ that is usually defined as the average size of the largest cultural cluster $C_{\max}$ in the system normalized by the total number of agents N; $\phi = C_{\max}/N$. In the monocultural (ordered) state $\phi \to 1$ and in the multicultural (disordered) state $\phi \to 0$. Even though this order parameter appropriately identified both the ordered and disordered phases emerging in the Axelrod model, it does not clearly define the critical region. Here, we show that the cluster-size entropy S_c, which is defined in terms of the probability that an occupied site of the lattice belongs to a cluster containing s sites, can be used as another powerful tool for the analysis of the phenomenology of the Axelrod model. The cluster-size entropy measures the number of clusters of different sizes and is related to the diversity of the system [15]. Theoretically, as a function of the probability of occupation, S_c should be zero in both the ordered and disordered phases, since the former is constituted by a single cluster of the size of the system and the latter is formed by a large number of small clusters of similar sizes. S_c should have a maximum at the transition where the diversity of cluster sizes is maximum. Thus, a peak develops as the phase transition takes place, which leads to a clear definition of the critical region.

In statistical physics, cluster-size entropy has been used in the study of problems such as percolation [15–18] and complex systems [9,15,18–20]. In their studies on percolation, Tsang and co-workers [15,18] found that the cluster entropy shows a maximum at the percolation threshold, where a group of neighboring occupied sites forms a cluster that expands from one edge of the two-dimensional (2D) lattice to the opposite one causing an abrupt decrease in the cluster entropy of the system. In the context of the Axelrod model, cluster entropy measures the number of cultural groups of different sizes and was first used by Villegas-Febres and Olivares-Rivas [21] in an attempt to establish a connection with thermodynamics. Cluster entropy was also utilized within the Axelrod model to partially characterize the inclusion of repulsion among agents in a regular lattice [9].

To demonstrate the usefulness of the cluster entropy in the analysis of complex networks, here we employ this property to carry out a thorough study on the monocultural-multicultural phase transition of the Axelrod model. We analyze the effects of the topology of the network and the mass media on this phase transition. For the topology, we vary the probability p of random rewiring between sites from 0 (regular networks) to 1 (random networks). In addition to determining precisely the critical value q_c of the Axelrod model, we establish a mathematical expression that relates q_c with the important parameter F (number of cultural features) in finite 2D regular lattices and display in a much clearer form some other known properties. Considering the imposition of the mass media, we find that partially ordered states that are not aligned with the field can be formed in short-range-interaction regular networks, in contraposition to the claim by González-Avella *et al.* [12]. A $q-B$ phase diagram is proposed for the Axelrod model in regular lattices.

II. AXELROD MODEL

The original Axelrod model is defined on a square lattice of N sites (social agents). The state of the ith agent is defined by a set of F cultural features (e.g., religion, sports, politics, etc.) represented by a vector $C_i = (C_{i1}, C_{i2}, \dots, C_{iF})$. Each feature C_{ik} of the agent i is first randomly assigned with a uniform distribution of the integers in the interval $[0, q-1]$. The variable q defines the cultural traits allowed per feature and thus measures the cultural variability in the system. There are q^F possible cultural states.

The procedure to establish the dynamics of the system is as follows: (1) Choose randomly two nearest neighbor agents i and j, then (2) calculate the number of shared features (cultural overlap) between the agents $\ell_{ij} = \sum_k^F \delta_{C_{ik},C_{jk}}$. If $0 < \ell_{ij} < F$ then (3) pick up randomly a feature k such that $C_{ik} \neq C_{jk}$ and with probability ℓ_{ij}/F set $C_{ik} = C_{jk}$. These time steps are iterated and the dynamics stops when a frozen state is reached; i.e., either $\ell_{ij} = 0$ or $\ell_{ij} = F, \forall i, j$. A cluster is a set of connected agents with the same state. Monocultural or ordered phases are composed of a cluster of the size of the system where $\ell_{ij} = F$, $\forall i, j$. Multicultural or disordered phases consist of two or more clusters.

To study the effect of an external field in the original Axelrod model just described some modifications are needed. We define a uniform external field as a vector $M = (m_1, m_2, \dots, m_F)$, where $m_n \in [0, 1, \dots, q-1]$, with strength $B \in [0,1]$. This parameter B regulates the probability for the agent-field interactions. Each agent has a probability B of interacting with the field and a probability $(1-B)$ of interacting with one of its nearest neighbors.

In the dynamic described above the agent j is substituted by the field M and the whole sequence follows in the same way: (1) Choose randomly an agent i, then (2) with probability B agent i and the field M interact, (3) calculate the number of shared features (cultural overlap) between the agent and the field $\ell_{iM} = \sum_k^F \delta_{C_{ik},M_n}$. If $0 < \ell_{iM} < F$ then (4) pick up randomly a feature k such that $C_{ik} \neq m_k$ and with probability ℓ_{iM}/F set $C_{ik} = m_k$. If agent i and the field M do not interact, (5) choose randomly an agent j in the nearest neighborhood of agent i. If with probability $1-B$ agents i and j interact, (6) compute the cultural overlap $\ell_{ij} = \sum_k^F \delta_{C_{ik},C_{jk}}$. If $0 < \ell_{ij} < F$ (7) with probability ℓ_{ij}/F set $C_{ik} = C_{jk}$. The dynamics stops when a frozen state is reached.

To generate randomized lattices, required for the topology analysis, we used the Watts-Strogatz algorithm [22]. Starting with a regular network of N agents (with periodic boundary conditions), each link is visited and with probability p is removed and rewired at random (avoiding self-linked nodes). This random rewiring process produces networks with topologies that go from perfect regularity ($p = 0$) to full randomness ($p = 1$).

III. CLUSTER ENTROPY

The cluster-size entropy is defined as [15]

$$S_c(P) = -\sum_s W_s(P) \ln\, W_s(P)\,, \tag{1}$$

where P is the probability of occupation (probability $1/q$ of taking a particular value of the cultural trait) and $W_s(P)$ is the probability that an agent belongs to a cluster of size s.

IV. NETWORK TOPOLOGY IN THE AXELROD MODEL

Here we vary the disorder parameter p of the network from 0 to 1 to see the effect on the monocultural-multicultural phase transition of the Axelrod model. Figure 1 displays the cluster-size entropy as a function of the probability of occupation $P = 1/q$ and, for comparison, the order parameter ϕ against the cultural trait q for $F = 2$ and $F = 5$ in networks of $N = 40 \times 40$ agents. There are various relevant issues in this figure that need consideration. The maximum of the cluster entropy curves is located around the corresponding onset trait q_c of the monocultural-multicultural phase transitions as studied with ϕ. The peak of a cluster-size entropy distribution corresponds to the state of maximum size disorder that occurs at the transition. Thus, the average size of the largest cluster becomes finite (transition onset) when the maximum size disorder occurs. The size entropy brings out the fact that as the system moves from a disordered phase, in which most agents each occupy a single cluster, to an ordered phase, in which all agents occupy just one cluster of the system size, the system goes through a critical region where agents become agglomerated in clusters of different sizes. Then, the maximum of the cluster-size entropy defines the critical value q_c.

Another feature to observe in Fig. 1 is that the critical value $1/q_c$ (q_c) becomes smaller (larger) as p goes from 0 (regular networks) to 1 (fully random networks), in complete agreement with the results obtained with the order parameter ϕ by Klemm *et al.* [23]. We believe that the collapse of the curves corresponding to $p > 0.5$ is due to finite-size effects

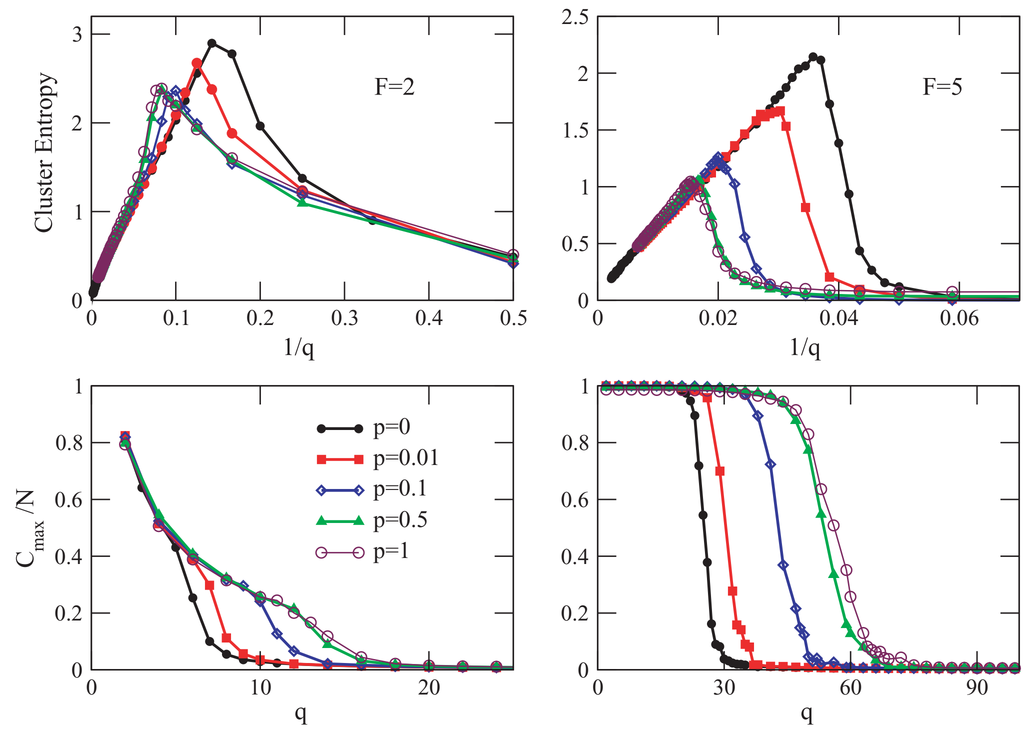

FIG. 1. (Color online) Cluster-size entropy as a function of the probability of occupation $1/q$ and order parameter versus q for $F = 2$ and $F = 5$ in networks of 40×40 with disorder parameter p equal to 0, 0.01, 0.1, 0.5, and 1. The peak of the cluster entropy distributions occurs at the onset of the phase transitions as characterized by the order parameter ϕ (see text).

and not to the lack of any further dynamics that indeed seems to develop up to $p = 1$ in networks of large size [23].

Next, we point out that the height of the cluster entropy curves in Fig. 1 reveals a characteristic of the Axelrod model that was impossible to uncover by means of the order parameter ϕ. As p increases, the maximum of the size entropies becomes smaller, suggesting less cluster diversity at the transition. Thus, it appears that in random networks, due to the presence of long-range links, the formation of the system-size cluster defining the ordered state of the Axelrod model is more efficient (explores a smaller region of the cluster-size space) than in regular networks.

The most highlighted findings of the present analysis on the effect of topology in the Axelrod model are observed in the scaled data of Figs. 2–4 for network randomness $p = 0$,0.1, and 1, respectively. These figures display the normalized cluster entropy S_c/S_c^{max} versus the normalized probability of occupation q_c/q for different values of the cultural-feature number F. Here, S_c^{max} is the value of the cluster entropy at the peak maximum. The insets show the corresponding regular data. All three plots show that, independently of the network randomness, the overall dynamics of the Axelrod model for $F = 2$ is unambiguously different from that for any value $F > 2$, and that the dynamics is the same for all values $F > 2$. The plots indicate that the Axelrod model for $F = 2$ and $F > 2$ must belong to different "universality classes" in the terminology of statistical physics. This agrees with previous works that claimed that the transition is first-order type for $F > 2$ and second-order type for $F = 2$ [5,6].

The width of the cluster-size entropy distribution may give a clue of the order of the transition. The more narrow distributions of the cluster entropy for $F > 2$ (Figs. 2–4) compared with the distributions for $F = 2$ indicate that the system crosses over from disorder (a large number of clusters exist) to order (just one cluster size is present) in a sudden manner, which is typical of first-order-like transitions. Broad distributions, instead, suggest that the system moves from one regime to the other in a smooth manner. A finite number of

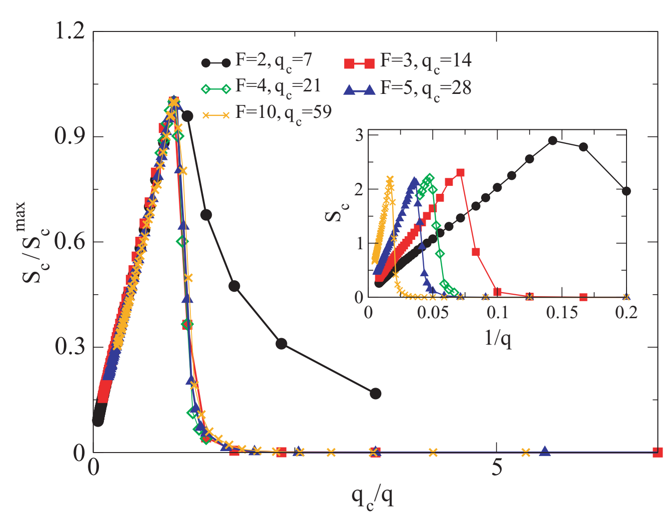

FIG. 2. (Color online) Normalized cluster-size entropy S_c/S_c^{max} against the normalized probability of occupation q_c/q for a network of disorder $p = 0$ and size 40×40 for different values of F. The data suggest that the Axelrod model for $F = 2$ is in a different universality class from that for $F > 2$ (see text). The inset shows the regular data.

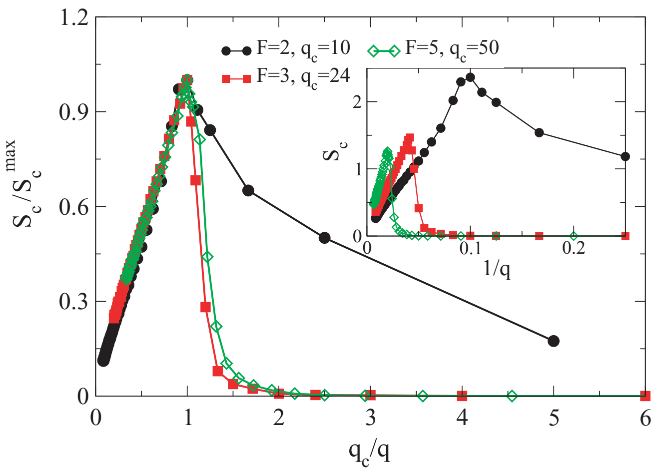

FIG. 3. (Color online) Normalized cluster-size entropy $S_c/S_c^{\max}$ against the normalized probability of occupation q_c/q for a network of disorder $p = 0.1$ and size 40×40 for different values of F. The inset shows the regular data.

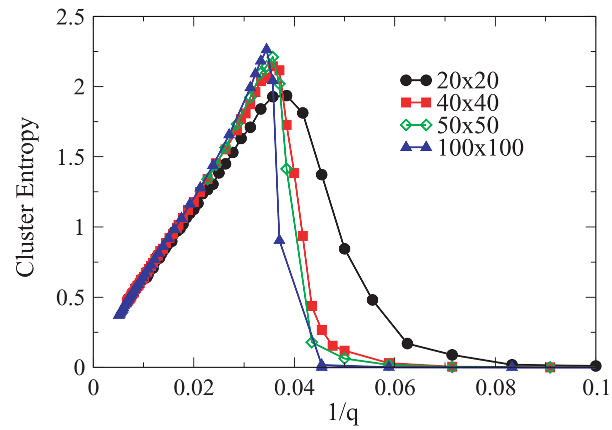

FIG. 5. (Color online) Cluster-size entropy versus probability of occupation in regular networks ($p = 0$) of different sizes.

clusters of all sizes starts to form in the disordered phase and as the system crosses over to the ordered side, those clusters of all sizes agglomerate in just one of the sizes of the system. This is the behavior expected in second-order-like transitions.

Notably, the normalized cluster entropies are independent of F for $(q_c/q) < 1$, indicating that there is a unique Axelrod dynamics for values $q > q_c$. This conclusion is robust against finite-size effects, as is demonstrated by the data shown in Fig. 5.

It is widely known that the monocultural-multicultural phase transition occurs at a critical value q_c that increases as F augments. In previous works that considered q_c as the cultural trait at which the system is completely ordered ($\phi = 1$), it was found that $q_c \approx F$ in one-dimensional lattices [23,24]. Using our definition of q_c as the maximum of the cluster-size entropy (or equivalently the onset of the transition when ϕ becomes finite), here we provide an equation that relates q_c and F in two-dimensional regular lattices ($p = 0$) of finite sizes:

$$q_c \approx 7(F - 1). \tag{2}$$

This expression (1) defines the value of the cultural trait q above which the system will always be in a multicultural state and (2) states that for any value of F the system will always be monocultural for $q < 7$. Equation (2) appears to hold for other values of the network randomness p with a prefactor that becomes larger as p increases.

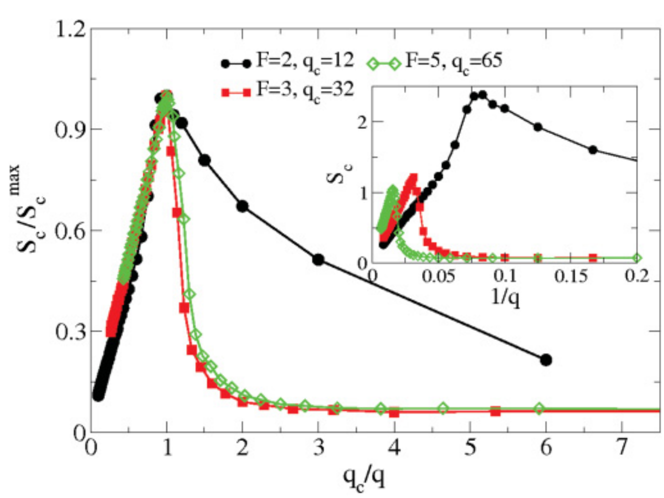

FIG. 4. (Color online) Normalized cluster-size entropy $S_c/S_c^{\max}$ against the normalized probability of occupation q_c/q for a network of disorder $p = 1$ and size 40×40 for different values of F. The inset shows the regular data.

V. EXTERNAL FIELD OR MASS MEDIA IN THE AXELROD MODEL

González-Avella *et al.* [12] found that in the presence of an external field the Axelrod model displays a rich $q - B$ phase diagram in fully connected, random, and scale-free 2D networks for $F = 10$. A second ordered phase was found that is not aligned with the external field and that in complex networks does not cover the whole system. González-Avella *et al.* argue that this ordered phase is caused by the long-range interactions characteristic of complex networks, and that such a phase does not exist in regular (short-range interaction) networks.

Figure 6 shows the cluster entropy against $1/q$ and the order parameter versus q in a 40×40 regular network for several values of the field strength B and for $F = 3, 5$, and 10. One of the effects of the external field is to move the critical value $1/q_c$ toward higher occupation probabilities, as occurs when the number of cultural features F is reduced (see Figs. 1 and 2). Interestingly, the limiting $(1/q_c) = 0.128$ as $B \to 1$ is very close to the limiting $(1/q_c) = 0.142$ as $F \to 2$.

More relevant, a second peak is observed in the cluster entropies of Fig. 6 for $0.2 < B < 0.8$ that indicates the occurrence of a second phase transition. These cluster entropies are constituted by two overlapping entropy distributions whose peaks correspond to the onset of phase transitions. The second peaks in S_c in the upper panels of Fig. 6 correspond to reentrant behaviors in the order parameter ϕ in the lower panels. That is, as the value of q is lowered the system moves from a multicultural to an ordered-like phase, then returns to the multicultural phase, and finally goes to a monocultural phase. The ordered-like phase makes the phase diagram

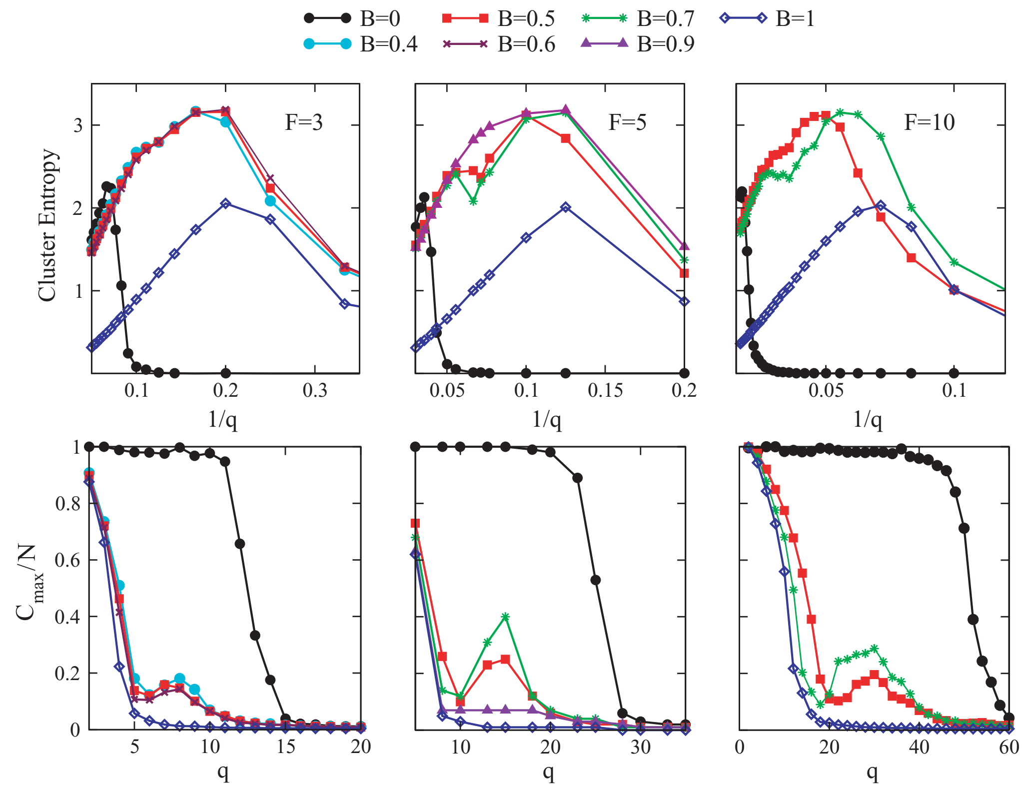

FIG. 6. (Color online) Cluster entropy S_c against $1/q$ and order parameter ϕ as a function of q for $F = 3, 5,$ and 10 in a 40×40 regular network for various field strengths B. For all three values of F a second transition develops in the field range $0.2 < B < 0.8$.

of the Axelrod model in regular networks more complex than previously thought [12]. The appearance of the second transition for $F = 3, 5,$ and 10 indicates that its existence does not depend on F.

A further analysis reveals that the system in the extra ordered phase, here called the "crossing phase," is mainly formed by a large cultural cluster whose state is not aligned with the external field. This crossing phase, whose order parameter $\phi < 1$ (meaning that the largest cluster does not have the size of the system), is closely related to the "orthogonal" ordered phase reported by González-Avella *et al.* [12] of the Axelrod model in 50×50 random networks. Contrasting with the conclusions by González-Avella *et al.*, our results indicate that long-range interactions are *not* fully required in the Axelrod model to form an ordered state whose orientation is not parallel to the external field.

In Fig. 7 we draw a phase diagram of the Axelrod model in regular networks by using the peaks of the size entropy distributions for $F = 5$. The highest peak at q_c of each S_c distribution in Fig. 6 agrees with the onset of the monocultural-multicultural transition in the corresponding ϕ curve. The second peak at q^* in the entropy distributions is assigned to the multicultural-crossing transition. In the monocultural phase (colored red, dark gray) the state ($\phi \sim 1$) is aligned with the external field for $q < q_c$. In the crossing phase (colored yellow, light gray) the ordered state ($\phi < 1$) is not parallel to the external field for $q_c < q < q^*$ and $0.2 < B < 0.8$. In this region there is no overlap between the ordered state and the field. The multicultural phase (colored blue, gray) is completely disordered. In particular, the state in regular networks for $F = 5$ is always monocultural for $q < 8$ and multicultural for $q > 28$ independently of the external field, in agreement with Eq. (2).

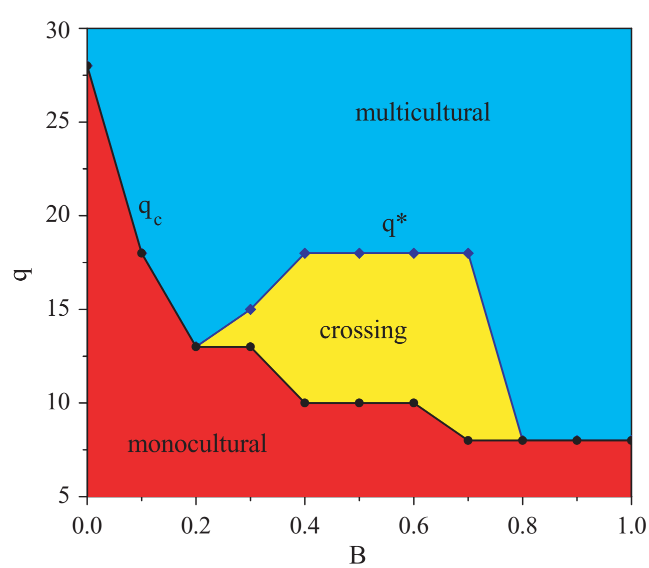

FIG. 7. (Color online) $q - B$ phase diagram of the Axelrod model obtained for $F = 5$ in a 40×40 regular network. Each point represents an average of 50–100 realizations.

VI. SUMMARY

Thus far, the cluster-size entropy has hardly been y used in the analysis of phase transitions. Here, we showed that the cluster-size entropy S_c is a valuable tool that can be utilized as a complement of the order parameter ϕ. Using the cluster entropy, we were able to both reproduce most of the results previously known for the Axelrod model in square networks and find additional relevant results. We showed by a simple analysis that the Axelrod model for $F = 2$ and $F > 2$ belongs to a different "universality" class. For two-dimensional regular lattices an expression was determined that relates q_c and F and that defines the asymptotic values of the trait q for the presence of multicultural and monocultural phases in the system.

We found a partially ordered phase for the Axelrod model in regular lattices, in which the vector state of the largest cultural cluster is not aligned with the external field or mass media. This phase is similar to one previously reported for fully connected, scale-free, and random networks, and leads to a cultural trait-field $(q - B)$ phase diagram for the Axelrod model in regular networks. The results suggest that long-range interactions are not completely necessary for the existence of a partially ordered state that is not oriented along the applied field.

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