

# Introduction to Statistical Learning Theory

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January 23, 2018

# Decision Theory: High Level View

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# What types of problems are we solving?

- In data science problems, we generally need to:
  - Make a decision
  - Take an action
  - Produce some output
- Have some evaluation criterion

# Actions

## Definition

An *action* is the generic term for what is produced by our system.

## Examples of Actions

- Produce a 0/1 classification [classical ML]
- Reject hypothesis that  $\theta = 0$  [classical Statistics]
- Written English text [image captioning, speech recognition, machine translation ]
- What's an action for predicting where a storm will be in 3 hours?
- What's an action for a self-driving car?

# Evaluation Criterion

*Decision theory* is about finding “optimal” actions, under various definitions of optimality.

## Examples of Evaluation Criteria

- Is classification correct?
- Does text transcription exactly match the spoken words?
  - Should we give partial credit? How?
- How far is the storm from the prediction location? [for point prediction]
- How likely is the storm's location under the prediction? [for density prediction]

# Real Life: Formalizing a Business Problem

- First two steps to formalizing a problem:
  - ① Define the *action space* (i.e. the set of possible actions)
  - ② Specify the evaluation criterion.
- Formalization may evolve gradually, as you understand the problem better

# Inputs

Most problems have an extra piece, going by various names:

- Inputs [ML]
- Covariates [Statistics]

## Examples of Inputs

- A picture
- A storm's historical location and other weather data
- A search query

# “Outcomes” or “Output” or “Label”

Inputs often paired with *outputs* or *outcomes* or *labels*

## Examples of outcomes/outputs/labels

- Whether or not the picture actually contains an animal
- The storm's location one hour after query
- Which, if any, of suggested the URLs were selected



# Typical Sequence of Events

Many problem domains can be formalized as follows:

- 1 Observe input  $x$ .
- 2 Take action  $a$ .
- 3 Observe outcome  $y$ .
- 4 Evaluate action in relation to the outcome (via a **loss function**  $\ell(a, y)$ )

## Note

- Outcome  $y$  is often **independent** of action  $a$
- But this is **not always the case**:
  - search result ranking
  - automated driving

# Formalization: The Spaces

The Three Spaces:

- Input space:  $\mathcal{X}$
- Action space:  $\mathcal{A}$
- Outcome space:  $\mathcal{Y}$

Concept check:

- What are the spaces for linear regression?
- What are the spaces for logistic regression?
- What are the spaces for a support vector machine?

# Some Formalization

## The Spaces

- $\mathcal{X}$ : input space
- $\mathcal{Y}$ : outcome space
- $\mathcal{A}$ : action space

## Prediction Function (or “decision function”)

A **prediction function** (or **decision function**) gets input  $x \in \mathcal{X}$  and produces an action  $a \in \mathcal{A}$  :

$$\begin{aligned} f: \mathcal{X} &\rightarrow \mathcal{A} \\ x &\mapsto f(x) \end{aligned}$$

## Loss Function

A **loss function** evaluates an action in the context of the outcome  $y$ .

$$\begin{aligned} \ell: \mathcal{A} \times \mathcal{Y} &\rightarrow \mathbf{R} \\ (a, y) &\mapsto \ell(a, y) \end{aligned}$$

# Evaluating a Prediction Function

- Loss function  $\ell$  evaluates a single action
- How to evaluate the prediction function as a whole?
- We will use the standard **statistical learning theory** framework.

# Statistical Learning Theory

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# A Simplifying Assumption

- Assume action has no effect on the output
  - includes all traditional prediction problems
  - what about stock market prediction?
  - what about stock market investing?
- What about fancier problems where this does not hold?
  - often can be reformulated or “reduced” to problems where it does hold
  - see literature on **reinforcement learning**

# Setup for Statistical Learning Theory

- Assume there is a **data generating distribution**  $P_{\mathcal{X} \times \mathcal{Y}}$ .
- All input/output pairs  $(x, y)$  are generated i.i.d. from  $P_{\mathcal{X} \times \mathcal{Y}}$ .
- Want prediction function  $f(x)$  that generally “does well on average”:

$\ell(f(x), y)$  is usually small, in some sense

- How can we formalize this?

# The Risk Functional

## Definition

The **risk** of a prediction function  $f : \mathcal{X} \rightarrow \mathcal{A}$  is

$$R(f) = \mathbb{E} \ell(f(x), y).$$

In words, it's the **expected loss** of  $f$  on a new example  $(x, y)$  drawn randomly from  $P_{\mathcal{X} \times \mathcal{Y}}$ .

## Risk function cannot be computed

Since we don't know  $P_{\mathcal{X} \times \mathcal{Y}}$ , we cannot compute the expectation.  
But we can estimate it...



# The Bayes Prediction Function

## Definition

A **Bayes prediction function**  $f^* : \mathcal{X} \rightarrow \mathcal{A}$  is a function that achieves the *minimal risk* among all possible functions:

$$f^* \in \arg \min_f R(f),$$

where the minimum is taken over all functions from  $\mathcal{X}$  to  $\mathcal{A}$ .

- The risk of a Bayes prediction function is called the **Bayes risk**.
- A Bayes prediction function is often called the “**target function**”, since it’s the best prediction function we can possibly produce.

## Example: Least Squares Regression

- Spaces:  $\mathcal{A} = \mathcal{Y} = \mathbf{R}$
- Square loss:

$$\ell(a, y) = (a - y)^2$$

- Risk:

$$\begin{aligned} R(f) &= \mathbb{E}[(f(x) - y)^2] \\ (\text{homework} \implies) &= \mathbb{E}[(f(x) - \mathbb{E}[y|x])^2] + \mathbb{E}[(y - \mathbb{E}[y|x])^2] \end{aligned}$$

- So Bayes prediction function is

$$f^*(x) = \mathbb{E}[y|x]$$

## Example 2: Multiclass Classification

- Spaces:  $\mathcal{A} = \mathcal{Y} = \{1, \dots, k\}$
- 0-1 loss:

$$\ell(a, y) = 1(a \neq y) := \begin{cases} 1 & \text{if } a \neq y \\ 0 & \text{otherwise.} \end{cases}$$

- Risk:

$$\begin{aligned} R(f) &= \mathbb{E}[1(f(x) \neq y)] = 0 \cdot \mathbb{P}(f(x) = y) + 1 \cdot \mathbb{P}(f(x) \neq y) \\ &= \mathbb{P}(f(x) \neq y), \end{aligned}$$

which is just the misclassification error rate.

- Bayes prediction function is just the assignment to the most likely class:

$$f^*(x) \in \arg \max_{1 \leq c \leq k} \mathbb{P}(y = c \mid x)$$

## But we can't compute the risk!

- Can't compute  $R(f) = \mathbb{E}\ell(f(x), y)$  because we **don't know**  $P_{\mathcal{X} \times \mathcal{Y}}$ .
- One thing we can do in ML/statistics/data science is

assume we have sample data.

Let  $\mathcal{D}_n = ((x_1, y_1), \dots, (x_n, y_n))$  be drawn i.i.d. from  $\mathcal{P}_{\mathcal{X} \times \mathcal{Y}}$ .

- Let's draw some inspiration from the Strong Law of Large Numbers:  
If  $z, z_1, \dots, z_n$  are i.i.d. with expected value  $\mathbb{E}z$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n z_i = \mathbb{E}z,$$

with probability 1.

# The Empirical Risk

Let  $\mathcal{D}_n = ((x_1, y_1), \dots, (x_n, y_n))$  be drawn i.i.d. from  $\mathcal{P}_{\mathcal{X} \times \mathcal{Y}}$ .

## Definition

The **empirical risk** of  $f : \mathcal{X} \rightarrow \mathcal{A}$  with respect to  $\mathcal{D}_n$  is

$$\hat{R}_n(f) = \frac{1}{n} \sum_{i=1}^n \ell(f(x_i), y_i).$$

- By the Strong Law of Large Numbers,

$$\lim_{n \rightarrow \infty} \hat{R}_n(f) = R(f),$$

almost surely.

- But we want to find the  $f$  that **minimizes**  $R(f)$  - will minimizing  $\hat{R}_n(f)$  be good enough?

# Empirical Risk Minimization

We want risk minimizer, is empirical risk minimizer close enough?

## Definition

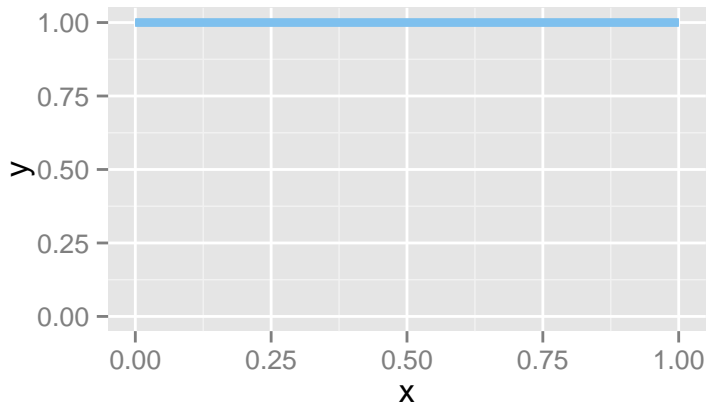
A function  $\hat{f}$  is an **empirical risk minimizer** if

$$\hat{f} \in \arg \min_f \hat{R}_n(f),$$

where the minimum is taken over all functions.

# Empirical Risk Minimization

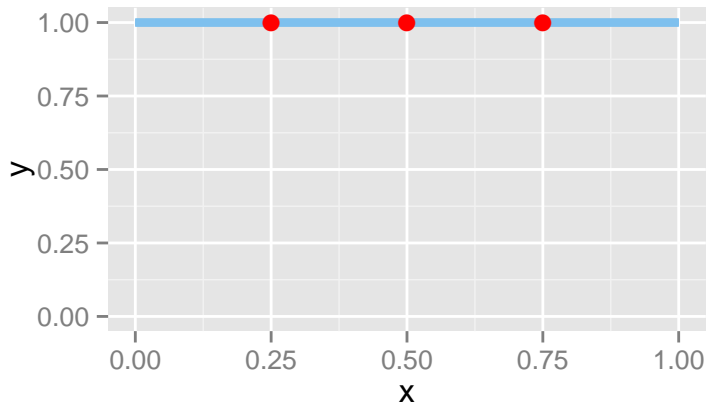
$P_{\mathcal{X}} = \text{Uniform}[0, 1]$ ,  $Y \equiv 1$  (i.e.  $Y$  is always 1).



$\mathcal{P}_{\mathcal{X} \times \mathcal{Y}}$ .

# Empirical Risk Minimization

$P_{\mathcal{X}} = \text{Uniform}[0, 1]$ ,  $Y \equiv 1$  (i.e.  $Y$  is always 1).

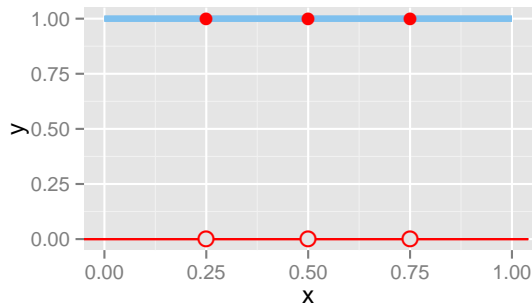


A sample of size 3 from  $\mathcal{P}_{\mathcal{X} \times \mathcal{Y}}$ .



# Empirical Risk Minimization

$P_{\mathcal{X}} = \text{Uniform}[0, 1]$ ,  $Y \equiv 1$  (i.e.  $Y$  is always 1).

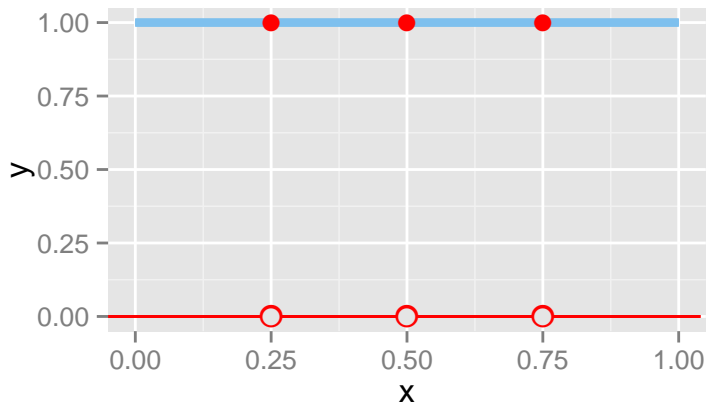


A proposed prediction function:

$$\hat{f}(x) = 1(x \in \{0.25, 0.5, 0.75\}) = \begin{cases} 1 & \text{if } x \in \{0.25, .5, .75\} \\ 0 & \text{otherwise} \end{cases}$$

# Empirical Risk Minimization

$P_{\mathcal{X}} = \text{Uniform}[0, 1]$ ,  $Y \equiv 1$  (i.e.  $Y$  is always 1).



Under square loss or 0/1 loss:  $\hat{f}$  has Empirical Risk = 0 and Risk = 1.

# Empirical Risk Minimization

- ERM led to a function  $f$  that just memorized the data.
- How to spread information or “**generalize**” from training inputs to new inputs?
- Need to smooth things out somehow...
  - A lot of modeling is about spreading and extrapolating information from one part of the input space  $\mathcal{X}$  into unobserved parts of the space.
- One approach: “Constrained ERM”
  - Instead of minimizing empirical risk over all prediction functions,
  - constrain to a particular subset, called a **hypothesis space**.

# Hypothesis Spaces

## Definition

A **hypothesis space**  $\mathcal{F}$  is a set of functions mapping  $\mathcal{X} \rightarrow \mathcal{A}$ .

- It is the collection of prediction functions we are choosing from.

## Want Hypothesis Space that...

- Includes only those functions that have desired “regularity”
  - e.g. smoothness, simplicity
- Easy to work with

Example hypothesis spaces?

# Constrained Empirical Risk Minimization

- Hypothesis space  $\mathcal{F}$ , a set of [prediction] functions mapping  $\mathcal{X} \rightarrow \mathcal{A}$
- **Empirical risk minimizer (ERM)** in  $\mathcal{F}$  is

$$\hat{f}_n \in \arg \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \ell(f(x_i), y_i).$$

- **Risk minimizer** in  $\mathcal{F}$  is  $f_{\mathcal{F}}^* \in \mathcal{F}$ , where

$$f_{\mathcal{F}}^* \in \arg \min_{f \in \mathcal{F}} \mathbb{E} \ell(f(x), y).$$