LIKELY INTERSECTIONS IN POWERS OF THE MULTIPLICATIVE GROUP (PRELIMINARY VERSION)

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Abstract. We derive two finiteness properties as consequences of the *geometrical non-degeneracy* of an algebraic subvariety W of a power of the multiplicative group, concerning the intersections of W with cosets of an algebraic subgroup H of dimension greater than or equal to the codimension of W. The first one is that every coset of H intersects W, unless H is contained in one of finitely many proper algebraic subgroups depending only on W. The second one is that every coset of H by a torsion point intersects W, unless the coset is contained in one of finitely many proper algebraic subgroups, again depending only on W. We use methods from tropical geometry and equidistribution, as well as some very mild model theory.

Contents

1. Introduction

1.1. **Statement of the main results.** An algebraic subvariety W of the multiplicative group $\mathbb{G}_m^n$ is *geometrically non-degenerate* if for every algebraic subgroup J of $\mathbb{G}_m^n$, with associated algebraic quotient map $\pi_J : \mathbb{G}_m^n \twoheadrightarrow$

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$\mathbb{G}_m^n/J$, we have that $\dim \overline{\pi_J(W)} = \min\{\dim W, n-\dim J\}$. For example, it is easy to see that if W is a curve, then it is geometrically non-degenerate if and only if it is not contained in any coset of a proper algebraic subgroup of $\mathbb{G}_m^n$; and that if W is a hypersurface, then it is geometrically non-degenerate if and only if there is no infinite algebraic subgroup J of $\mathbb{G}_m^n$ such that $J \cdot W = W$.

Given equations defining W, the dimension condition needs to be checked for only finitely many algebraic subgroups J and these can be determined effectively; see [BMZ07, Theorem 1.4] and [Zan12, Sections 1.3.3 and 1.3.4]. One can also check that W is geometrically non-degenerate if and only if for every subtorus J of $\mathbb{G}_m^n$ of dimension $n - \dim W$ we have that $\dim \overline{\pi_J(W)} = \dim W$.

In this paper, we prove the following results on geometrically non-degenerate subvarieties of $\mathbb{G}_m^n$ defined over $\mathbb{C}$.

Theorem 1.1. *Let $n \in \mathbb{N}$ and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an irreducible geometrically non-degenerate subvariety.*

Then there exists a finite set $\mathcal{H} = \{H_1, \dots, H_N\}$ such that $H_i \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ is a subtorus for all $i = 1, \dots, N$ and such that for every subtorus $H \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim H + \dim W \geq n$ and for every $z \in (\mathbb{C}^\times)^n$, one of the following holds:

(i) $W(\mathbb{C}) \cap z \cdot H(\mathbb{C}) \neq \emptyset$ or
(ii) $H \subseteq H_i$ for some $i \in \{1, \dots, N\}$.

Theorem 1.2. *Let $n \in \mathbb{N}$ and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an irreducible geometrically non-degenerate subvariety.*

Then there exists a finite set $\mathcal{G} = \{G_1, \dots, G_N\}$ such that $G_i \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ is an algebraic subgroup for all $i = 1, \dots, N$ and such that for every subtorus $H \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim H + \dim W \geq n$ and for every torsion point $\zeta \in \mathbb{G}_{m,\mathbb{C}}^n(\mathbb{C})$, one of the following holds:

(i) $W(\mathbb{C}) \cap \zeta \cdot H(\mathbb{C}) \neq \emptyset$ or
(ii) $\zeta H \subseteq G_i$ for some $i \in \{1, \dots, N\}$.

The interest of these results comes from its connections with two areas of very active research which sit at the border of model theory and Diophantine geometry, namely *unlikely intersections* and *exponential algebraic-closedness.*

1.2. **Unlikely (and likely) intersections.** In the seminal paper [BMZ99], Bombieri, Masser, and Zannier proved that a curve in an algebraic torus (i.e. a power of the multiplicative group) contains at most finitely many points that satisfy two independent multiplicative relations provided that the curve is defined over the algebraic numbers and not contained in a coset of a proper subtorus. Intuitively, the intersection of a curve with an algebraic subgroup of codimension at least 2 should usually be empty for dimension reasons and the result of Bombieri, Masser, and Zannier states that, under the above-mentioned hypotheses on the curve, this holds true apart from at most finitely many exceptional points.

Since 1999, there has been a lot of research on generalizing the result of Bombieri, Masser, and Zannier in various ways: replacing the curve by a higher-dimensional subvariety, weakening the hypotheses on the curve, replacing the ambient algebraic torus by another ambient variety with a collection of "special" subvarieties similar to the collection of torsion cosets, i.e. of irreducible components of algebraic subgroups of the torus.

In particular, it follows from [Mau08] and [BMZ08] that the result of Bombieri, Masser, and Zannier continues to hold for a curve defined over $\mathbb{C}$ that is not contained in any proper algebraic subgroup of the torus and one can show that this last hypothesis is actually necessary.

Such results are collected under the umbrella term of "unlikely intersections" since they concern intersections that are expected to be empty for dimension reasons. Unlikely intersections have also been studied in abelian varieties and semiabelian varieties as well as in pure and mixed Shimura varieties. All of these ambient varieties possess a class $\mathcal{S}$ of special subvarieties S. It turns out that, in order to understand unlikely intersections, it is advantageous to look at the larger class of intersections of atypical dimension: for a fixed subvariety V and varying $S \in \mathcal{S}$, we call a component A of an intersection $V \cap S$ atypical if $\dim A > \dim V - \text{codim} S$. The intersection is unlikely if $\dim V - \text{codim} S < 0$.

General conjectures, due to Zilber [Zil02] (for semiabelian varieties), Pink [Pin05] (for mixed Shimura varieties), and Bombieri, Masser, and Zannier [BMZ07] (for algebraic tori), predict that unlikely or atypical intersections are rare in one way or another. Pink's and Zilber's conjectures are slightly different, but actually equivalent in many cases (see [BD21, Section 12]). In the case of algebraic tori, which is the relevant one for the purposes of this article, Bombieri, Masser, and Zannier have shown in the Appendix to [BMZ08] that all formulations of the conjecture are equivalent. The Zilberian formulation is equivalent to the statement that a given variety has at most finitely many atypical subvarieties that are maximal with this property with respect to inclusion. In other words, every atypical intersection and so in particular every unlikely intersection is explained by one of finitely many atypical intersections. There is an equivalent formulation that is more similar to our theorem (but not completely analogous; the proper algebraic subgroups contain the atypical intersections and not the algebraic subgroups with which one intersects, and indeed the latter would be impossible): every atypical subvariety is contained in one of finitely many proper algebraic subgroups of the algebraic torus.

Typically, the known results in the direction of these conjectures are confined to cases where either the dimension or the codimension is low (curves, hypersurfaces, subvarieties of codimension 2). However, more is known for geometrically non-degenerate subvarieties of algebraic tori and abelian varieties, see [Mau11, Théorème 1.1] and [HP16, Theorem 9.15(iv)].

More recently, people have started to also consider likely intersections, usually proving that the union of a certain class of likely intersections in a given subvariety is (analytically or Zariski) dense in that subvariety (of course, the intersection with the whole ambient variety is always likely, so this one will always be excluded from the class under consideration – the most natural option is to consider special subvarieties whose codimension is equal to the dimension of the fixed subvariety), see [G22; ES22] as well as the earlier [Hab10, Theorem 1.2] and [Zan12, Remark 3.4.5] for likely intersections in the square of the modular curve $Y(1)$.

For a subvariety V of an algebraic torus $\mathbb{G}_m^n$, Zariski density is easy to see: choose $\dim V$ monomials that remain algebraically independent when restricted to V and use that the image of the induced morphism from V to $\mathbb{G}_m^{\dim V}$ is constructible and Zariski dense. Note that analytic density fails already in the simplest example: the set of torsion points is not analytically dense in $\mathbb{G}_m^n$.

In this article, our goal is slightly different: we want to show that, under suitable hypotheses, almost every intersection that one expects to be non-empty for dimension reasons is actually non-empty. More concretely, for a geometrically non-degenerate subvariety V of an algebraic torus over $\mathbb{C}$, we show in Theorem 1.2 that, if $V \cap T = \emptyset$ for some torsion coset T of codimension at most $\dim V$, then T is contained in one of finitely many proper torsion cosets. We also show in Theorem 1.1 that, if $V \cap zH = \emptyset$ for some subtorus H of codimension at most $\dim V$ and some $\mathbb{C}$-point z, then H is contained in one of finitely many proper subtori. Here, one cannot expect zH to be contained in one of finitely many translates of proper subtori (unless V is a curve, see Corollary 1.4): take for example V to be the geometrically non-degenerate hypersurface that is defined by the equation $x + y + z = 1$. Then, for any $c \in \mathbb{C} \setminus \{0, 1\}$, the intersection of V with the coset $\{(1 - c, c)\} \times \mathbb{G}_m$ is empty, but the union of all these cosets is not contained in the union of finitely many cosets of proper subtori. Note that this does not contradict our statement about torsion cosets since c and $1 - c$ are simultaneously roots of unity if and only if c is a primitive sixth root of unity. The connection to unlikely intersections that becomes visible here will be explained in more detail later. This example also shows that Theorem 1.2 becomes hopelessly false in positive characteristic since then all the points $(1 - c, c)$ $(c \neq 0, 1)$ are torsion points.

The hypothesis of geometrical non-degeneracy is necessary for a result such as Theorem 1.2 to hold: if $\dim \overline{\pi_J(W)} < \dim W$ for some subtorus J of dimension $n - \dim W$, then $zJ \cap W = \emptyset$ for a Zariski dense set of torsion cosets zJ. Since geometrical non-degeneracy is not inherited by subvarieties, we can unfortunately not conclude that every unlikely non-intersection is explained by one of finitely many unlikely non-intersections. Furthermore, such a statement would just be plain false: for example, the hypersurface V_0 defined by the equation $x + 2y + z = 1$ in $\mathbb{G}_m^3$ has trivial stabilizer and is therefore geometrically non-degenerate, but has empty intersection with any torsion coset defined by a pair of equations $z = 1$ and $x = \zeta y$ for a root of unity ζ. But any algebraic subgroup that contains infinitely many of these torsion cosets will also contain the algebraic subgroup defined by the equation $z = 1$ and this algebraic subgroup has non-empty (but not geometrically non-degenerate) intersection with V_0.

Our result serves as motivation for the following more general conjecture:

Conjecture 1.3. *Let G be a semiabelian variety over an algebraically closed field F of characteristic 0 and let $V \subset G$ be a geometrically non-degenerate subvariety. Then there exists a finite set $\mathcal{S}$ of irreducible components of proper algebraic subgroups of G such that the following holds: suppose that*

$T \subset G$ is a special subvariety such that $\mathrm{codim}_G T \leq \dim V$ *and* $T \cap V = \emptyset$. *Then there exists* $T_0 \in \mathcal{S}$ *such that* $T \subset T_0$.

This can be regarded as a geometric analogue of the arithmetic results obtained in [BIR08; GI13]; the results in [BIR08] were the first author's original inspiration for this work. Several other recent results can be interpreted as density statements for certain classes of likely intersections in arithmetic schemes, e.g. [Cha18; ST20]

If $G = A$ is an abelian variety, Conjecture 1.3 trivially holds: if B is an abelian subvariety of A such that $\mathrm{codim}_A B \leq \dim V$, then the image of V under the quotient homomorphism $A \to A/B$ is Zariski dense and closed, so equal to A/B. The case where $G = \mathbb{G}_m^n$ is established in this article.

The statement of Conjecture 1.3 is simultaneously weaker and stronger than the density of some class of likely intersections: weaker since there might not exist enough algebraic subgroups to produce a dense union of a non-trivial class of likely intersections (consider for example a subvariety of a simple abelian variety of positive dimension and positive codimension), but stronger since mere density can still be achieved if most likely intersections are empty.

1.3. **Exponential-Algebraic Closedness.** In his work on the model theory of the complex exponential function [Zil05], Zilber conjectured a Nullstellensatz-type property for systems of exponential-polynomial equations over the complex numbers: he predicted that all such systems which are solvable in certain exponential fields extending $\mathbb{C}$ should already be solvable in $\mathbb{C}$. This conjecture, now known as the *Exponential-Algebraic Closedness Conjecture* (EAC Conejcture for short), is formulated geometrically: it states that all algebraic subvarieties of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ which satisfy the geometric conditions of *freeness* and *rotundity* should contain a point of the form $(z, \exp(z))$. An algebraic subvariety $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ is *free* if its projection $\mathbb{C}^n$ is not contained in a proper subspace of $\mathbb{C}^n$ and its projection to $(\mathbb{C}^\times)^n$ is not contained in a proper algebraic subgroup of $(\mathbb{C}^\times)^n$. The definition of a *rotund* algebraic subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ is given in Subsection 2.4; it is a condition stating that certain quotients of V satisfy appropriate dimension inequalities, for example $\dim V \geq n$.

The EAC conjecture has attracted considerable interest from people working in model theory and Diophantine geometry, and started a line of research which goes in a similar direction to that of likely intersection problems, aiming to prove that suitable conditions are sufficient to ensure the existence of the intersection between some algebraic varieties and the graphs of certain analytic functions, such as the complex exponential [BM17; AKM22; Gal23a; MM23], the exponential maps of abelian varieties [Gal23b], the modular j-invariant [EH21; Gal; AEM23] and the Γ function [EP23]. An important difference is that while likely intersection problems are usually seen as problems concerning the interaction between the arithmetic and the algebro-geometric structure of certain algebraic varieties, in EAC-type problems the analytic structure plays an important role.

In [Gal23a], the second author of the present paper established the EAC Conjecture for algebraic subvarieties of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ which split as a product of the form $L \times W$, where $L \leq \mathbb{C}^n$ is a linear subspace and $W \subseteq (\mathbb{C}^\times)^n$ is an algebraic subvariety. The proofs employed methods from *tropical geometry*, a relatively young branch of mathematics which has been described as a "combinatorial shadow" of algebraic geometry. The connection between EAC-type problems and this kind of methods had already been observed by Zilber in his early work on the conjecture [Zil02], but the development of tropical geometry which has taken place in the last few decades allowed to go well beyond Zilber's results. In this paper, we take this connection forward: we use tropical methods to prove results with a similar flavour to EAC-type questions, but which are purely algebraic.

Studying the intersection between an algebraic subvariety W of $(\mathbb{C}^\times)^n$ and the cosets of an algebraic subtorus J is the same as studying the intersection of the graph of the exponential with algebraic varieties of the form $L \times W$, where L is a translate of a linear subspace of $\mathbb{C}^n$ defined over $\mathbb{Q}$, such that $\exp(L)$ is a coset of J. Of course, such an algebraic variety will never satisfy the definition of freeness; hence the need to strengthen rotundity of $L \times W$ to geometrical non-degeneracy of W: it is a straightforward calculation that if W is geometrically non-degenerate then rotundity of $L \times W$ reduces to $\dim L + \dim W \geq n$, and therefore we have a sort of "uniform rotundity" as L varies in the Grassmannian of linear spaces of dimension $n - \dim L$. This uniformity will be exploited to prove our results, together with some equidistribution techniques.

If we see $\mathbb{C}$ as embedded in an algebraically closed valued field $\mathfrak{C}$, we are able to use tropical methods to study the behaviour of an algebraic variety $W \subseteq \mathbb{G}_m^n$ as the coordinates of its $\mathbb{C}$-points approach 0 or infinity, working with infinite and infinitesimal points rather than sequences on W. This approach is influenced by model theory: we will obtain the bigger field $\mathfrak{C}$ by taking an *elementary extension* of an appropriate structure. We remark that many of the statements and proofs could in theory be formulated in $\mathbb{C}$, for example working with a sequence of elements going to infinity instead of an infinite element; however, our approach makes things more readable.

1.4. **Applications.** In Proposition 5.1, we use Theorem 1.2 to give a new proof of the Manin-Mumford conjecture for algebraic tori, proved by Laurent in [Lau84], by transforming an unlikely intersection happening into a likely intersection not happening. Given that we use equidistribution, which can also be used to prove the Manin-Mumford conjecture directly, this might not be too surprising. Nevertheless, the transformation of a problem about unlikely intersections into a problem about likely intersections seems not without interest.

If the subvariety W is a curve, then we will show that Theorem 1.1 can be strengthened as follows.

Corollary 1.4. *Let $n \in \mathbb{N}$ and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an irreducible curve that is not contained in any coset of a proper subtorus.*

Then there exist finitely many subtori $H_i \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ $(i = 1, \ldots, N)$ and finitely many points $z_1, \ldots, z_N$ of $\mathbb{G}_{m,\mathbb{C}}^n$ such that for every subtorus $H \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim H + \dim W \geq n$ and for every $z \in (\mathbb{C}^\times)^n$, one of the following holds:

(i) $W(\mathbb{C}) \cap z \cdot H(\mathbb{C}) \neq \emptyset$ *or*

(ii) $zH = z_iH_i$ *for some* $i \in \{1, \ldots, N\}$.

We will also apply Theorem 1.2 to prove that likely intersections are likely in a concrete probabilistic sense. Concretely, we will prove the following theorem.

Theorem 1.5. *Let* $n \in \mathbb{N}$ *and let* $V \subset \mathbb{G}_m^n$ *be an irreducible geometrically non-degenerate subvariety of dimension* $d > 0$*. For* $N \in \mathbb{N}$*, let* $\mathcal{S}(V, N) \subset \mathbb{Z}^{nd}$ *denote the set of vectors* $(a_{i,j})_{i=1,\ldots,d;\ j=1,\ldots,n}$ *such that*

$$\max_{i,j} |a_{i,j}| \leq N$$

and there exists some $x = (x_1, \ldots, x_n) \in V$ *with*

$$\prod_{j=1}^{n} x_j^{a_{i,j}} = 1 \quad (i = 1, \ldots, d).$$

Then

$$\lim_{N \to \infty} \frac{\#\mathcal{S}(V, N)}{(2N+1)^{nd}} = 1.$$

While the hypothesis of geometrical non-degeneracy is used in the proof of Theorem 1.5, it might well not be necessary for the theorem to hold.

Finally, we will show in Theorem 5.2 that Theorem 1.1 implies an analogous statement over any algebraically closed field K of characteristic 0. The same is true for Theorem 1.2 as well.

1.5. **Structure of the paper.** The paper is organised as follows. In Section 2 the necessary preliminaries from model theory and tropical geometry are introduced; this allows us to introduce the algebraically closed valued field $\mathfrak{C}$ in which many of our proofs will take place. Finally we discuss Zilber's notion of rotundity and show some characterisation for complex rotund varieties. In Section 2 we study the consequences of geometrical non-degeneracy, and in Section 3 we use an equidistribution argument to prove Theorems 1.1 and 1.2. Finally, in Section 5 we discuss some applications; in particular, we prove Corollary 1.4 and Theorem 1.5.

2. Preliminaries

2.1. **Model theory.** We will use very basic model theory in order to obtain some uniformity results, hence we use this subsection to fix some terminology. We will value conciseness over precision, and refer the reader to one of the many excellent textbooks in the area, for example [Hod97], for the details.

Recall that a *language* $\mathcal{L}$ consists of symbols for constants, functions, and relations, and that an *$\mathcal{L}$-formula* is an expression involving variables, the symbols in $\mathcal{L}$, and the usual logical symbols $\neg, \vee, \wedge, \exists, \forall$. A *structure* in a language $\mathcal{L}$ is a set S together with an assignment of the constants to some elements of S, of the functions to some functions $S^n \to S$, and of the relations to some subsets of S^m, for appropriate n and m. A formula *with parameters from* A for some subset $A \subseteq S$ is a formula $\varphi(x, a)$ where x, y are tuples of variables, $\varphi(x, y)$ is an $\mathcal{L}$-formula, and a is a tuple of elements from A (of the same length as y).

A subset $X \subseteq S^n$ is *definable* (over A) if there is some formula φ in n variables (with parameters from A) such that X consists of those points of S^n which satisfy φ.

An *elementary extension* S' of the structure S is an $\mathcal{L}$-structure such that $S \subseteq S'$ and every formula with parameters from S which is true in S is also true in S'. As an example, the complex field is not an elementary extension of the real field: the latter satisfies the sentence "$\forall x \neg(x^2 = -1)$" while the former does not. On the other hand, $\mathbb{C}$ is an elementary extension of the algebraic closure $\overline{\mathbb{Q}}$ of the rationals, as can be seen for example by applying the *Tarski-Vaught criterion* [Hod97, Theorem 2.5.1].

Given a structure S, an elementary extension S', and a definable subset X of S^n, we may look at the subset of $(S')^n$ defined by the same formula. This set will be denoted by $X(S')$: the geometrically-minded reader will notice the similarity with taking points of algebraic varieties in different fields containing the field of definition.

2.2. **Tropical geometry.** Here we recall the necessary preliminaries from tropical geometry. Everything in this subsection is known and can be found for example in [MS15], but we work out full proofs when we have not been able to find them in the literature.

Throughout we will talk about an algebraically closed valued field (K, val), with divisible (ordered) value group $(\Gamma, +, \leq)$, valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{m}$, and residue field $k = \mathcal{O}/\mathfrak{m}$. We will denote the residue map by $\mathrm{res} : \mathcal{O} \twoheadrightarrow k$. We abuse notation and write res also for the map on the polynomial rings $\mathrm{res} : \mathcal{O}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \twoheadrightarrow k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ In this subsection we will further assume that Γ is an Archimedean ordered abelian group, so that it can be embedded in $\mathbb{R}$; in most cases this assumption could be removed without causing any trouble, but we keep it in since we quote many results from [MS15] and it is always assumed there. In our applications, however, Γ is *not* going to be Archimedean; at the start of the next subsection we will explain why we do not need to worry about this.

We also fix a *splitting* of the valuation, that is, a group homomorphism $\varphi : \Gamma \to K^\times$ such that $\mathrm{val}(\varphi(\gamma)) = \gamma$ for all $\gamma \in \Gamma$. We are free to assume this exists by [MS15, Lemma 2.1.15]. With another abuse of notation, we will use val and φ also to denote the Cartesian powers of these maps, so the maps $\mathrm{val} : K^n \to \Gamma^n$ and $\varphi : \Gamma^n \to K^n$.

Given vectors $u \in \mathbb{Q}^n$ and $w \in \Gamma^n$, we write $\langle u, w\rangle$ for the element $u_1 w_1 + \dots + u_n w_n \in \Gamma$, with the action of $\mathbb{Q}$ on Γ defined in the obvious way. This is not to be confused with the operation $v \cdot v'$ on elements $v = (v_1, \dots, v_n), v' = (v_1', \dots, v_n') \in \mathbb{G}_m^n(K)$, which denotes the element $(v_1 \cdot v_1', \dots, v_n \cdot v_n')$. Finally, if $v \in \mathbb{G}_m^n(K)$ and $u \in \mathbb{Z}^n$, then $v^u = \Pi_{i=1}^n v_i^{u_i}$.

Definition 2.1. Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial and let $w \in \Gamma^n$. We write f as

$$f = \sum_{u \in S} c_u x^u$$

for some finite set $S \subseteq \mathbb{Z}^n$ (we assume $c_u \neq 0$ for all $u \in S$).

The *initial form* of f with respect to w is $\mathrm{in}_w(f) \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ defined as

$$\mathrm{in}_w(f) = \sum_{u \in S'} \mathrm{res}(c_u \varphi(-\mathrm{val}(c_u)))x^u$$

where

$$S' := \{u \in S \mid \mathrm{val}(c_u) + \langle u, w \rangle \leq \mathrm{val}(c_{u'}) + \langle u', w \rangle \, \forall u' \in S\}.$$

For example, if $\mathrm{val}(c_u) = 0$ for each $u \in S$, then $\mathrm{in}_0(f) = \mathrm{res}(f)$.

Given an ideal I and $w \in \Gamma^n$, we define the *initial ideal*

$$\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) \mid f \in I \rangle.$$

Lemma 2.2 ([MS15, Lemma 2.6.2(a)]). *If $g \in \mathrm{in}_w(I)$, then $g = \mathrm{in}_w(f)$ for some $f \in I$.*

Definition 2.3. Let W be an algebraic subvariety of $\mathbb{G}_m^n$ defined over K, and let I be the ideal of Laurent polynomials which vanish on W.

For $w \in \Gamma^n$, the *initial variety of W with respect to w* is the algebraic subvariety of $\mathbb{G}_m^n$ defined over k and consisting of the common zero locus of all polynomials in $\mathrm{in}_w(I)$. It is denoted by $\mathrm{in}_w(W)$.

Theorem 2.4 ([MS15, Theorem 3.2.3]). *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety defined over K and I the ideal of Laurent polynomials which vanish on W. Then the following sets coincide:*

(1) $\{\mathrm{val}(v) \in \Gamma^n \mid v \in W(K)\}$;
(2) $\{w \in \Gamma^n \mid \mathrm{in}_w(I) \neq \langle 1 \rangle\}$.

Proposition 2.5 ([MS15, Proposition 3.2.11]). *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety defined over K, $w \in \Gamma^n$. If $v \in \mathrm{in}_w(W)(k)$, then there is $v' \in W(K)$ such that $\mathrm{val}(v') = w$ and $\mathrm{res}(v' \cdot \varphi(-w)) = v$.*

Recall that a Γ-*rational polyhedron* is a subset τ of Γ^n consisting of the elements w which satisfy finitely many inequalities of the form $\langle u, w \rangle \leq \gamma$, for some $u \in \mathbb{Q}^n$ and $\gamma \in \Gamma$. A *face* of the polyhedron τ is a polyhedron τ' for which there exists some $\gamma \in \Gamma^n$ such that $\tau' = \{w \in \tau \mid \langle w, \gamma \rangle \leq \langle w', \gamma \rangle \, \forall w' \in \tau\}$. The *relative interior* of the polyhedron τ, denoted $\mathrm{relint}(\tau)$, is the set of points in τ which do not lie in any (proper) face of τ.

A Γ-*rational polyhedral complex* is a finite set Σ of Γ-rational polyhedra that is closed under taking faces, and such that for all $\tau_1, \tau_2 \in \Sigma$ we have that $\tau_1 \cap \tau_2$ is a face of both τ_1 and τ_2 (and hence an element of Σ). The *support* $\mathrm{Supp}(\Sigma)$ of the polyhedral complex Σ is the union $\bigcup_{\tau \in \Sigma} \tau \subseteq \Gamma^n$.

Theorem 2.6 ([MS15, Theorem 3.3.8]). *Let $W \subseteq \mathbb{G}_m^n$ be an irreducible algebraic subvariety defined over K. The set $\mathrm{val}(W(K))$ is the support of a Γ-rational polyhedral complex.*

By [MS15, Proposition 3.2.8], the polyhedral complex can be chosen so that for all $w_1, w_2 \in \mathrm{Supp}(\mathrm{Trop}(W))$, if w_1, w_2 lie in the relative interior of the same polyhedron $\tau \in \mathrm{Trop}(W)$ then $\mathrm{in}_{w_1}(I) = \mathrm{in}_{w_2}(I)$. We fix such a choice, so we are allowed to write W_τ instead of W_w and $\mathrm{in}_\tau(I)$ instead of $\mathrm{in}_w(I)$ when $w \in \mathrm{relint}(\tau)$.

Definition 2.7. Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety defined over K. The *tropicalization* of W, denoted by $\mathrm{Trop}(W)$, is the set of all polyhedra τ such that τ lies in the polyhedral complex associated by Theorem 2.6 to one of the irreducible components of W.

Hence, we have $\mathrm{val}(W(K)) = \mathrm{Supp}(\mathrm{Trop}(W))$.

Lemma 2.8. *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety defined over K, I the ideal of Laurent polynomials vanishing on W.*

Let $v \in W(K)$, $w = \mathrm{val}(v) \in \Gamma^n$, and let $W_\tau = \mathrm{in}_w(W)$, $\alpha = \varphi(w)$. Then

$$\mathrm{in}_w(I) = \{\mathrm{res}(g) \mid g \in \mathcal{O}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],\ g|_{\alpha^{-1}\cdot W} = 0\}$$

and

$$W_\tau(k) = \mathrm{res}((\alpha^{-1} \cdot W(K)) \cap (\mathcal{O}^\times)^n).$$

Proof. We prove first the equality of the two sets of polynomials.

($\supseteq$) Let g be a Laurent polynomial over $\mathcal{O}$ which vanishes on $\alpha^{-1} \cdot W$; then $g(\alpha^{-1} \cdot x) \in I$. If $\mathrm{res}(g) = 0$ then it is immediate that $\mathrm{res}(g) \in \mathrm{in}_w(I)$, so we assume $\mathrm{res}(g) \neq 0$, that is, g has at least one coefficient of valuation 0.

Write $g = \sum_{u \in S} c_u x^u$. Then $g(\alpha^{-1} \cdot x) = \sum_{u \in S} (c_u \alpha^{-u} x^u)$. Let

$$S' := \{u \in S \mid \mathrm{val}(c_u \alpha^{-u}) + \langle u, w\rangle \leq \mathrm{val}(c_{u'} \alpha^{-u'}) + \langle u', w\rangle\ \forall u' \in S\}.$$

Since $\mathrm{val}(\alpha) = w$, we have that $u \in S'$ if and only if $\mathrm{val}(c_u) \leq \mathrm{val}(c_{u'})$ for all $u' \in S$. By assumption on g, this happens if and only if $\mathrm{val}(c_u) = 0$. Hence

$$\begin{aligned}
\mathrm{in}_w(g(\alpha^{-1} \cdot x)) &= \sum_{u \in S'} \mathrm{res}\left(c_u \alpha^u \varphi(-\mathrm{val}(c_u \alpha^u)) x^u\right) \\
&= \sum_{u \in S'} \mathrm{res}\left(c_u \varphi(-\mathrm{val}(c_u)) x^u\right) \\
&= \mathrm{res}\left(\sum_{u \in S} c_u x^u\right) \\
&= \mathrm{res}(g)
\end{aligned}$$

hence $\mathrm{res}(g) \in \mathrm{in}_w(I)$.

($\subseteq$) Let $f \in I$, and write $f = \sum_{u \in S} c_u x^u$. Let

$$S' := \{u \in S \mid \mathrm{val}(c_u) + \langle u, w\rangle \leq \mathrm{val}(c_{u'}) + \langle u', w\rangle\ \forall u' \in S\}.$$

Take some $u_0 \in S'$, and let $\lambda := c_{u_0} \alpha^{u_0}$.

Then, for all $u \in S$,

$$\begin{aligned}
\mathrm{val}\left(\frac{c_u \alpha^u}{\lambda}\right) &= \mathrm{val}(c_u) + \mathrm{val}\left(\alpha^u\right) - \mathrm{val}(\lambda) \\
&= \mathrm{val}(c_u) + \langle u, w\rangle - \mathrm{val}(\lambda) \\
&= \mathrm{val}(c_u) + \langle u, w\rangle - \mathrm{val}(c_{u_0}) - \langle u_0, w\rangle \\
&\geq 0
\end{aligned}$$

with equality if and only if $u \in S'$, and hence the coefficients of the polynomial

$$\begin{aligned}\frac{f(\alpha \cdot x)}{\varphi(\mathrm{val}(\lambda))} &= \frac{\sum_{u\in S} c_u (\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))} \\ &= \sum_{u\in S} \frac{c_u \alpha^u}{\varphi(\mathrm{val}(\lambda))} x^u\end{aligned}$$

have non-negative valuation, strictly positive for all $u \in S \setminus S'$. Then

$$\begin{aligned}\mathrm{res}\left(\frac{f(\alpha \cdot x)}{\varphi(\mathrm{val}(\lambda))}\right) &= \mathrm{res}\left(\frac{\sum_{u\in S} c_u (\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))}\right) \\ &= \mathrm{res}\left(\frac{\sum_{u\in S'} c_u (\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))}\right) + \mathrm{res}\left(\frac{\sum_{u\in S\setminus S'} c_u (\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))}\right) \\ &= \mathrm{res}\left(\sum_{u\in S'} \frac{c_u (\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))}\right) + 0 \\ &= \mathrm{res}\left(\sum_{u\in S'} c_u \varphi(-\mathrm{val}(c_u)) x^u\right) \\ &= \mathrm{in}_w\,(f)\,.\end{aligned}$$

Hence if $f \in I$, then $\mathrm{in}_w(f) = \mathrm{res}\left(\frac{f(\alpha\cdot x)}{\varphi(\mathrm{val}(\lambda))}\right)$, and $\frac{f(\alpha\cdot x)}{\varphi(\mathrm{val}(\lambda))}$ is a polynomial over $\mathcal{O}$ which vanishes on $\alpha^{-1} \cdot W$.

We now prove the second equality.

($\subseteq$) Let $v \in W_\tau(k)$. Then by Proposition 2.5, there is $v' \in W(K)$ such that $\mathrm{val}(v') = w$ and $\mathrm{res}\left(\frac{v'}{\varphi(w)}\right) = \mathrm{res}(\alpha^{-1}\cdot v') = v$, so $v \in \mathrm{res}\left(\alpha^{-1} \cdot W(K) \cap (\mathcal{O}^\times)^n\right)$.

($\supseteq$) Let $v \in \alpha^{-1}\cdot W(K) \cap (\mathcal{O}^\times)^n$, $f \in \mathrm{in}_w(I)$. By the first part of the proof there is $g \in \mathcal{O}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that g vanishes on $\alpha^{-1}\cdot W$ (so in particular $g(v) = 0$) and $\mathrm{res}(g) = f$. Then, $f(\mathrm{res}(v)) = \mathrm{res}(g)(\mathrm{res}(v)) = \mathrm{res}(g(v)) = 0$. Thus $\mathrm{res}(v) \in W_\tau(k)$. □

Definition 2.9. Let $\tau \le \Gamma^n$ be a polyhedron. The *affine span* of τ, denoted $\mathrm{aff}(\tau)$, is the smallest affine subspace of Γ^n containing τ. We denote by $\mathrm{lin}(\tau)$ the linear subspace of Γ^n that is parallel to $\mathrm{aff}(\tau)$.

Definition 2.10. Let $\tau \le \Gamma^n$ be a polyhedron, $A \in M_{d\times n}(\mathbb{Z})$ a matrix with integer entries such that $\mathrm{aff}(\tau) = \{x \in \Gamma^n \mid Ax = b\}$ for some $b \in \Gamma^d$.

We denote by J_τ the identity component of the algebraic subgroup of $\mathbb{G}_m^n$ defined by $\{y \mid y^A = 1\}$.

Lemma 2.11. *Let $J \le \mathbb{G}_m^n$ be an algebraic subgroup defined by $J := \{x \in \mathbb{G}_m^n \mid x^A = 1\}$ for some matrix $A \in M_{d\times n}(\mathbb{Z})$.*

Then $\mathrm{Supp}(\mathrm{Trop}(J))$ *is the polyhedron τ defined by $\{x \in \Gamma^n \mid Ax = 0\}$.*

Proof. We use description (1) of the support of the tropicalization in Theorem 2.4, that is, $\mathrm{Supp}(\mathrm{Trop}(J)) = \mathrm{val}(J(K))$.

Let $v \in J(K)$; then, $v^A = 1$ implies that $A(\mathrm{val}(v)) = 0$, so $\mathrm{Supp}(\mathrm{Trop}(J)) \subseteq \tau$.

Conversely, if $w \in \tau$ then it satisfies $Aw = 0$, and thus $A\varphi(w) = 1$, so $\varphi(w) \in W(K)$. Since $\mathrm{val}(\varphi(w)) = w$, we get that $w \in \mathrm{Supp}(\mathrm{Trop}(J))$. □

In the next proof we will take initial forms of initial forms. To be able to do this, we see the residue field k as a valued field with the trivial valuation. Hence, for $f = \sum_{u \in S} c_u x^u \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $w \in \Gamma^n$ we have that $\mathrm{in}_w(f) := \sum_{u \in S'} c_u x^u$ where $S' := \{u \in S \mid \langle u, w \rangle \leq \langle u', w \rangle \, \forall u' \in S\}$.

Proposition 2.12. *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety, $\tau \in \mathrm{Trop}(W)$ a polyhedron. Then $J_\tau \leq \mathrm{Stab}(W_\tau)$.*

Proof. Let I be the ideal of Laurent polynomials vanishing on W and let $w \in \mathrm{relint}(\tau)$.

If $\dim J_\tau = 0$, then $J_\tau \leq \mathrm{Stab}(W_\tau)$ holds trivially.

Hence, we can assume that $\dim J_\tau > 0$. Then the set of points in J_τ which lie in a 1-dimensional algebraic subgroup of J_τ is Zariski-dense in J_τ; hence, it is sufficient to check that W_τ is stabilized by these points, as the stabilizer is a (Zariski closed) algebraic subgroup of $\mathbb{G}_m^n$.

Assume then that $z \in J_\tau(k)$ lies in some 1-dimensional algebraic subgroup H of J_τ. Let $z_1 \in H(K)$ be a point with valuation $v = (v_1, \dots, v_n) \neq 0$. By Lemma 2.11 $\mathrm{val}(J_\tau(K)) = \mathrm{lin}(\tau)$, so $v \in \mathrm{lin}(\tau)$; since $w \in \mathrm{relint}(\tau)$, we may then choose z_1 so that $w + v \in \tau$. Arguing as in [MS15, Lemma 3.3.6] we have that, possibly after replacing v by some other element of $\mathrm{Trop}(H)$ closer to 0, $\mathrm{in}_{w+v}(I) = \mathrm{in}_v(\mathrm{in}_w(I))$. Note that in that argument Γ is assumed to be an $\mathbb{R}$-vector space rather than a divisible ordered abelian group, and so the result is stated in a slightly different form; however, since our v varies in a 1-dimensional subgroup of Γ^n it makes sense to say that we can take it "sufficiently close to 0", say after fixing an isomorphism between Γ and $\mathrm{Supp}(\mathrm{Trop}(H))$, and the same proofs go through.

Let now $f \in \mathrm{in}_\tau(I)$. We apply [MS15, Lemma 2.6.2(2)] to deduce that $\mathrm{in}_w(I)$ is homogeneous with respect to the grading that weights the i-th variable by v_i $(i = 1, \dots, n)$. We deduce that the following equality of Laurent polynomials holds:

$$f(z \cdot x) = f(x).$$

Since $f \in \mathrm{in}_\tau(I)$ was arbitrary, it follows that $z \in \mathrm{Stab}(W_\tau)(k)$. Since z and H were arbitrary, we deduce that $J_\tau \leq \mathrm{Stab}(W_\tau)$.

□

Definition 2.13. Let Σ be a polyhedral complex, $\tau \in \Sigma$ a polyhedron. For every $\sigma \in \Sigma$ such that τ is a face of σ, consider the polyhedron $\bigcup_{x \in \sigma} x + \mathrm{lin}(\tau)$. These form a polyhedral complex Σ', invariant under translation by $\mathrm{lin}(\tau)$.

The *star* of τ in Σ, denoted $\mathrm{star}_\Sigma(\tau)$, is the polyhedral complex $\Sigma' - w$ for any $w \in \mathrm{relint}(\tau)$.

Proposition 2.14 ([MS15, Lemma 3.3.6])**.** *Let $\tau \in \mathrm{Trop}(W)$ be a polyhedron, $w \in \mathrm{relint}(\tau)$, W_τ the corresponding initial variety.*

Then $\mathrm{Trop}(W_\tau) = \mathrm{star}_{\mathrm{Trop}(W)}(\tau)$.

Remark 2.15. Every polyhedron $\sigma \in \mathrm{star}(\Sigma)(\tau)$ contains $\mathrm{lin}(\tau)$, and thus $\mathrm{Trop}(W_\tau)$ is invariant under translation by τ. This is the tropical counterpart to Proposition 2.12.

Finally, we recall the definition of *amoeba*, and the fact that tropicalizations are limits of amoebas.

Definition 2.16. Let $\mathrm{Log} : (\mathbb{C}^\times)^n \to \mathbb{R}^n$ denote the map

$$(z_1, \dots, z_n) \mapsto (\log |z_1|, \dots, \log |z_n|).$$

Given an algebraic subvariety $W \subseteq \mathbb{G}_m^n$ defined over $\mathbb{C}$, the *amoeba* $\mathcal{A}_W$ of W is the image of $W(\mathbb{C})$ under $-\mathrm{Log}$.

If $W \subseteq \mathbb{G}_m^n$ is defined over $\mathbb{C}$, we may see it as a subvariety of some algebraically closed valued field extension of $\mathbb{C}$, for example the Puiseux series field, with the trivial valuation induced on $\mathbb{C}$. The conditions defining the support of $\mathrm{Trop}(W)$, which in general have the form

$$\mathrm{val}(c_u) + \langle u, w\rangle \leq \mathrm{val}(c_{u'}) + \langle u', w\rangle$$

in this case then take the form $\langle u, w\rangle \leq \langle u', w\rangle$, because all coefficients of the polynomials defining W have valuation 0. In particular then the polyhedral complex $\mathrm{Trop}(W)$ will consist of polyhedra defined by equations of the form $Ax \leq 0$, for $A \in \mathbb{Z}^n$, and as such we may look at the set $\mathrm{Supp}(\mathrm{Trop}(W))(\mathbb{R})$ of points in $\mathbb{R}^n$ satisfying the given conditions. In tropical geometry this is known as the *constant coefficients case*, see [MS15, Corollary 3.3.5] for more details.

2.3. **Embedding $\mathbb{C}$ in a large valued field.** We use an approach based on nonstandard analysis: we embed $\mathbb{C}$ into a larger field, naturally endowed with a valuation. This field will contain infinite and infinitesimal elements, which intuitively correspond to the various order of growths of sequences in $\mathbb{C}$ going to infinity or 0; most of the statements and proofs in the next section could be formulated in the standard language, but that would make them far less readable.

Consider the model-theoretic structure $\mathbb{R}_{\exp,\sin}$, that is the field of real numbers in the language of ordered rings expanded by constants for all the elements of $\mathbb{R}$ and function symbols for exp and sin (here by sin we mean the **total** sine function, not the restricted version, so this structure is not o-minimal, and in fact not tame in any sense.) We may identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, and using exp and sin we may define the complex exponential function in this structure.

We denote by $\mathfrak{R}$ a proper elementary extension of $\mathbb{R}_{\exp,\sin}$, and by $\mathfrak{C} = \mathfrak{R} + i\mathfrak{R}$ its algebraic closure.

We may define an absolute value $|\cdot| : \mathfrak{C} \to \mathfrak{R}$ by setting $|x + iy| = \sqrt{x^2 + y^2}$. We use this to define the *Archimedean valuation* on $\mathfrak{C}$ as the quotient by the relation $z_1 \sim z_2$ if and only if there is $n \in \mathbb{N}$ such that $n|z_1| \geq |z_2|$ and $|z_2| \geq n|z_1|$. We denote by Γ the set $\mathfrak{C}^\times/\sim$, and by $\mathrm{val} : \mathfrak{C} \twoheadrightarrow \Gamma \cup \{\infty\}$ the valuation, where $\mathrm{val}(0) = \infty$. Endowing Γ with the ordering induced by the reverse ordering on the absolute values of elements of $\mathfrak{C}$, and with the sum induced by the product on $\mathfrak{C}$, makes Γ into an ordered abelian group and val into a valuation.

Slightly confusingly, even though this is called the Archimedean valuation, the resulting group Γ is not Archimedean: the first-order conditions which describe the growth of the real exponential function compared to polynomials force us to have that if $x \in \mathfrak{R}$ has valuation γ, then e^x has valuation greater than $n\gamma$ for every $n \in \mathbb{N}$. We note that while all the results from tropical geometry described in the previous subsection do assume that Γ is

Archimedean, we will only apply them to algebraic subvarieties $W \subseteq \mathbb{G}_m^n$ that are defined over $\mathbb{C}$. Hence they are also defined over some algebraically closed valued subfield $\mathfrak{C}_0$ of $\mathfrak{C}$ with Archimedean value group. Implicitly, when we apply the results from the previous subsection to such varieties, we are considering as subvarieties of $\mathbb{G}_m^n(\mathfrak{C}_0)$ and "lifting" the results to $\mathfrak{C}$ using *model completeness* of the theory of algebraically closed valued fields. This is sufficiently straightforward that we will never say it explicitly.

The field $\mathfrak{C}$ is then an algebraically closed valued field. By [MS15, Lemma 2.1.15] the valuation has a splitting, which as in the previous section we denote by φ (here also, in the reference $\Gamma \subseteq \mathbb{R}$ is assumed but only to deduce that Γ is torsion-free). By composing with the absolute value, we may and will assume that $\varphi(\Gamma) \subseteq \mathfrak{R}_{>0}$.

Since $\mathfrak{R}$ is an elementary extension of $\mathbb{R}$ in the language with exp and sin, it has an exponential function $\mathfrak{R} \to \mathfrak{R}_{>0}$, which may be extended to an exponential $\mathfrak{C} \to \mathfrak{C}$ by the usual formula $\exp(x+iy) = \exp(x)(\cos(y) + i\sin(y))$. For every $r \in \mathfrak{R}$ we have a well-defined power function on the positive elements defined by $x \mapsto x^r := \exp(r\log(x))$, where log here means the (well-defined) function $\mathfrak{R}_{>0} \to \mathfrak{R}$.

We can make Γ into an ordered $\mathfrak{R}$-vector space by setting $r\cdot\gamma = \mathrm{val}(\varphi(\gamma)^r)$ for $\gamma \in \Gamma$ and $r \in \mathfrak{R}$. This will allow us, given a vector subspace L of $\mathfrak{R}^n$ defined by some matrix A with entries in $\mathfrak{R}$, to look at the set of Γ-points of L, denoted by $L(\Gamma)$ and defined by the same matrix A. Note that $\dim_{\mathfrak{R}} \Gamma = 1$, because given $\gamma_1, \gamma_2 \in \Gamma$ we have that $\frac{\log(\varphi(\gamma_1))}{\log(\varphi(\gamma_2))} \cdot \gamma_2 = \mathrm{val}\left(\varphi(\gamma_2)^{\frac{\log(\varphi(\gamma_1))}{\log(\varphi(\gamma_2))}}\right) = \mathrm{val}(\varphi(\gamma_1)) = \gamma_1$. Obviously Γ is also an ordered $\mathbb{R}$-vector space, of infinite dimension.

The valuation ring $\mathcal{O}$ is the convex hull of the complex numbers, and the residue field is isomorphic to $\mathbb{C}$. The maximal ideal $\mathfrak{m}$ consists of the elements whose absolute value is smaller than any positive real. Any element $z \in \mathcal{O}$ can be written uniquely as $\mathrm{res}(z) + \varepsilon$, where $\mathrm{res}(z) \in \mathbb{C}$ is the complex number closest to z and $\varepsilon \in \mathfrak{m}$. From this it is easy to see that for $z \in \mathcal{O}$ we have

$$\exp(z) = \exp(\mathrm{res}(z) + \varepsilon) = \exp(\mathrm{res}(z))\exp(\varepsilon) \in \exp(\mathrm{res}(z)) \cdot (1 + \mathfrak{m}) \subseteq \mathcal{O},$$

from which it follows that exp induces a function $\mathcal{O} \to \mathcal{O}^\times$ which satisfies $\mathrm{res}(\exp(z)) = \exp(\mathrm{res}(z))$.

2.4. **Rotundity.** Let J be an algebraic subgroup of $\mathbb{G}_m^n$. We denote by TJ its tangent bundle, which we identify with a subgroup of $\mathbb{G}_a^n \times \mathbb{G}_m^n$, and by $\pi_{TJ} : \mathbb{G}_a^n \times \mathbb{G}_m^n \twoheadrightarrow (\mathbb{G}_a^n \times \mathbb{G}_m^n)/TJ \cong \mathbb{G}_a^{n-\dim J} \times \mathbb{G}_m^{n-\dim J}$ the quotient mapping.

Definition 2.17. Let $V \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ an algebraic subvariety. Then V is *rotund* if for any algebraic subgroup $J \leq \mathbb{G}_m^n$,

$$\dim \pi_{TJ}(V) \geq n - \dim J.$$

The following proposition gives some characterizations of rotundity for subvarieties of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ which split as product $L \times W$ for some linear subspace $L \leq \mathbb{G}_a^n$, defined over $\mathbb{R}$, and some algebraic subvariety $W \subseteq \mathbb{G}_m^n$, defined over $\mathbb{C}$. If W is defined over $\mathbb{C}$, we may also see it as a subvariety of

$\mathbb{G}_m^n$ defined over the algebraically closed valued field $\mathfrak{C}$, so that we may take its tropicalization.

Proposition 2.18. *Let $L \leq \mathbb{G}_{a,\mathbb{C}}^n$ be a linear subspace defined over $\mathbb{R}$, and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an algebraic subvariety such that every irreducible component of W has dimension $n - \dim L$.*

The following are equivalent:

(1) *The subvariety $L \times W$ is rotund.*
(2) *There is a Zariski-open subset W° of W such that for all $w \in W^\circ$,*
$$\dim(\Gamma_{\exp} \cap (L \times (w^{-1} \cdot W^\circ))) = 0$$
where $\Gamma_{\exp}$ denotes the graph of the complex exponential function $\exp : \mathbb{C}^n \to (\mathbb{C}^\times)^n$, and $\dim$ is the complex analytic dimension.
(3) *There is a point $w \in W$ such that the map $\delta : (L \times W)(\mathbb{C}) \to (\mathbb{C}^\times)^n$ defined by $(l, w) \mapsto \frac{w}{\exp(l)}$ is open in a (Euclidean) neighbourhood of (l, w) for all $l \in L$.*
(4) *There is a Zariski-open subset $W^\circ \neq \varnothing$ of W such that the map $\delta : (L \times W^\circ)(\mathbb{C}) \to (\mathbb{C}^\times)^n$ defined by $(l, w) \mapsto \frac{w}{\exp(l)}$ is open.*
(5) *$\mathcal{A}_W + \mathrm{Re}(L) = \mathbb{R}^n$.*
(6) *$\bigcup_{\tau \in \mathrm{Trop}(W)} \tau(\Gamma) + \mathrm{Re}(L)(\Gamma) = \Gamma^n$.*
(7) *There is $\tau \in \mathrm{Trop}(W)$ such that $\dim(\tau + \mathrm{Re}(L)(\Gamma)) = n$.*
(8) *For all $z \in (\mathbb{C}^\times)^n$ there exist $z' \in z \cdot \mathbb{S}_1^n, \varepsilon > 0$ such that $B(z', \varepsilon) \subseteq \frac{W(\mathbb{C})}{\exp(L(\mathbb{C}))}$.*

Proof. For $(1 \Rightarrow 2)$ see [Kir19, Proposition 6.2 and Remark 6.3] and [Gal23a, Proposition 3.7]. For $(2 \Rightarrow 4)$, assuming W° as in (2) given, the same W° satisfies (4) by the Open Mapping Theorem. $(4 \Rightarrow 3)$ is obvious. $(3 \Rightarrow 1)$ is [Gal23a, Proposition 3.8].

We show $(1 \Rightarrow 5, 8)$.

Assume then that $L \times W$ is rotund, and let W° be the Zariski-open subset given by (3). Let $W' := W \setminus W^\circ$. Let $F : W \to \mathbb{C}$ be some non-zero algebraic function which vanishes on all points of W'.

Consider the algebraic subvariety W_1 of $(\mathbb{C}^\times)^{n+1}$ given by
$$\{(w_1, \ldots, w_{n+1}) \in (\mathbb{C}^\times)^{n+1} \mid (w_1, \ldots, w_n) \in (\mathbb{C}^\times)^n \wedge w_{n+1} = F(w_1, \ldots, w_n)\}.$$
Then $(L \times \mathbb{C}) \times W_1$ satisfies condition (4), with $W_1^\circ = W_1$. Hence we may apply [Gal23a, Lemma 6.15] which gives that $\mathcal{A}_{W_1} - (\mathrm{Re}(L) \times \mathbb{R} = \mathbb{R}^{n+1}$, from which it easily follows that $L \times W$ satisfies Condition (5). It also follows that $(L \times \mathbb{C}) \times W_1$ satisfies (8). But then $L \times W$ also does.

For $(8 \Rightarrow 3)$, note that, because of [Chi89, Proposition on p. 41 in Section 3.8], (8) implies that the map $\delta : L \times W \to (\mathbb{C}^\times)^n$ defined by $(l, w) \mapsto \frac{w}{\exp(l)}$ has a zero-dimensional fiber and this in turn implies that δ is open in a neighbourhood of some $(l_0, w_0) \in (L \times W)(\mathbb{C})$ by the Open Mapping Theorem [GR12, p. 107]. For any other $l_1 \in L(\mathbb{C})$, there is a neighbourhood U of (l_1, w_0) such that $\delta^* : L \times W \to (\mathbb{C}^\times)^n$ defined by $\delta^*(l, w) = \delta(l + l_0 - l_1, w) = \frac{w}{\exp(l + l_0 - l_1)} = \exp(l_1 - l_0)\delta(l, w)$ is open when restricted to U. Hence (3) holds.

$(6 \Rightarrow 7)$ is obvious. We now show $(5 \Rightarrow 6)$.

Let $x \in \mathbb{R}^n$. We first show that $x \in \operatorname{Supp}(\operatorname{Trop}(W))(\mathbb{R}) + \operatorname{Re}(L)(\mathbb{R})$. Since $0 \in \operatorname{Supp}(\operatorname{Trop}(W))$, if $x \in \operatorname{Re}(L)$ we are done, so we assume $x \notin \operatorname{Re}(L)$. Consider the notion of Kuratowski convergence induced on the set of subsets of $\mathbb{R}^n$ by the Euclidean topology on $\mathbb{R}^n$. The sequence of subspaces $\{jx + \operatorname{Re}(L)(\mathbb{R})\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$ diverges. By assumption, for every $j \in \mathbb{N}$ there is $a_j \in \mathcal{A}_W$ such that $jx + \operatorname{Re}(L) = a_j + \operatorname{Re}(L)$. If the sequence $\{a_j\}_{j\in\mathbb{N}}$ has a limit $a \in \mathbb{R}^n$ then $\lim_{j\in\mathbb{N}} jx + \operatorname{Re}(L) = a + \operatorname{Re}(L)$, so by the observation above we have that $\{a_j\}_{j\in\mathbb{N}}$ must be divergent.

Denote by $\mathbb{S}_{n-1}$ the unit $(n-1)$-sphere in $\mathbb{R}^n$. Let $b := \lim_{j\in\mathbb{N}} \frac{a_j}{|a_j|} \in \mathbb{S}_{n-1}$. Then $\mathbb{R}_{\geq 0}\cdot b = \mathbb{R}_{\geq 0}\cdot \lim_{j\in\mathbb{N}} \frac{a_j}{|a_j|}$ and by [MS15, Theorem 1.4.2] and divergence of the sequence $\{a_j\}_{j\in\mathbb{N}}$ we must have $\mathbb{R}_{\geq 0}\cdot b \subseteq \operatorname{Supp}(\operatorname{Trop}(W))(\mathbb{R})$. Writing $x = \frac{1}{j} jx$, for every j, we see

$$x \in \frac{1}{j}\left(|a_j|\frac{a_j}{|a_j|} + \operatorname{Re}(L)(\mathbb{R})\right) = \frac{|a_j|}{j}\frac{a_j}{|a_j|} + \operatorname{Re}(L)(\mathbb{R})$$

and thus

$$x \in \mathbb{R}_{\geq 0}\cdot \frac{a_j}{|a_j|} + \operatorname{Re}(L)(\mathbb{R}).$$

Since the sequence $\left\{\mathbb{R}_{\geq 0}\cdot \frac{a_j}{|a_j|}\right\}_{j\in\mathbb{N}}$ converges to the half-line $\mathbb{R}_{\geq 0}\cdot b$, we have that

$$\lim_{j\in\mathbb{N}}\left(\mathbb{R}_{\geq 0}\cdot \frac{a_j}{|a_j|} + \operatorname{Re}(L)\right) = \mathbb{R}_{\geq 0}\cdot b + \operatorname{Re}(L).$$

Hence $x \in \mathbb{R}_{\geq 0}\cdot b + \operatorname{Re}(L) \subseteq \operatorname{Supp}(\operatorname{Trop}(W))(\mathbb{R}) + \operatorname{Re}(L)(\mathbb{R})$.

Thus, the first-order sentence

$$\forall(x_1,\dots,x_n)\exists\gamma \in \operatorname{Supp}(\operatorname{Trop}(W)), l \in \operatorname{Re}(L)\,(\gamma + l = (x_1,\dots,x_n))$$

holds in the $\mathbb{R}$-vector space $\mathbb{R}$. Since $\mathbb{R}$ is 1-dimensional as an $\mathbb{R}$-vector space and Γ is infinite dimensional we have an embedding $\mathbb{R} \hookrightarrow \Gamma$ of ordered vector spaces; by quantifier elimination for ordered vector spaces over $\mathbb{R}$ [Dri98, Corollary I.7.6] and the Tarski-Vaught criterion [Hod97, Theorem 2.5.1] the embedding is in fact elementary, so the above first-order sentence is also true in Γ.

Finally, let us prove that $(7 \Rightarrow 1)$ (more precisely, we prove the contrapositive $(\neg 1 \Rightarrow \neg 7)$).

Claim: If W has an initial variety W_τ such that $L \times W_\tau$ is rotund, then $L \times W$ is rotund.

Proof of Claim: Assume that $L \times W_\tau$ is rotund for some $\tau \in \operatorname{Trop}(W)$, and let $x \in \operatorname{relint}(\tau)$ and $\alpha := \varphi(x)$. Let W_τ° be the Zariski-open dense subset of W_τ obtained by applying (4) to $L \times W_\tau$. By Lemma 2.8, $W_\tau(\mathbb{C}) = \operatorname{res}((\alpha^{-1}\cdot W(\mathfrak{C})) \cap (\mathcal{O}^\times)^n)$. We now consider the reduct of $\mathbb{R}$ and $\mathfrak{R}$ to the language of ordered rings expanded by constants for the elements of $\mathbb{R}$ and function symbols for exp and the **restricted** sine function. Then $\mathfrak{R}$ is still an elementary extension of $\mathbb{R}$ in this language, by the Tarski-Vaught criterion, and the structures are o-minimal. Moreover, the extension $\mathfrak{R}$ is a *tame* extension of $\mathbb{R}$ in the sense of [DL95; Dri97].

Let $U_L \subseteq L$ be a bounded open semialgebraic subset of L, of standard radius. Note that using the restricted sine function, the complex exponential restricted to U_L is definable in the o-minimal structure we are working in, and therefore so is the set $\frac{\alpha^{-1}\cdot W}{\exp(U_L)}$. Now we have that

$$\operatorname{res}\left(\frac{\alpha^{-1}\cdot W}{\exp(U_L)}(\mathfrak{C})\cap(\mathcal{O}^\times)^n\right)$$

has non-empty interior, since $W_\tau^\circ \subseteq \operatorname{res}(\alpha^{-1}\cdot W) = W_\tau$. Then by [Dri97, Lemma 1.9] we have that $\frac{\alpha^{-1}\cdot W}{\exp(U_L)}(\mathfrak{C})$ has non-empty interior. Multiplying this by α, we get that $\frac{W}{\exp(U_L)}(\mathfrak{C})$ has non-empty interior; since $\frac{W}{\exp(U_L)}$ is defined over $\mathbb{C}$, and having non-empty interior is a first-order condition, $\frac{W}{\exp(L)}(\mathbb{C})$ also has non-empty interior. Hence, $L\times W$ is rotund by Condition (3), again using [Chi89, Proposition on p. 41 in Section 3.8] and the Open Mapping Theorem. This completes the proof of the above claim.

Let then $L\times W$ be a non-rotund variety. We show that

$$\dim\left(\bigcup_{\tau\in\operatorname{Trop}(W)}\tau(\Gamma)+\operatorname{Re}(L)(\Gamma)\right)\neq n$$

by induction on $\dim W$.

If $\dim W=0$, then $\operatorname{Trop}(W)=\{\{0\}\}$ and $L=\mathbb{G}_{a,\mathbb{C}}^n$, so $\cup_{\tau\in\operatorname{Trop}(W)}\tau(\Gamma)+\operatorname{Re}(L)(\Gamma)=\operatorname{Re}(L)(\Gamma)=\Gamma^n$.

If $\dim W=d>0$, let $\tau\in\operatorname{Trop}(W)$. Since $\dim W>0$, we have $\dim L<n$ and so we can assume that $\dim\tau>0$. By assumption and the above claim, $L\times W_\tau$ is not rotund. Let J_τ be the algebraic subgroup that W_τ is invariant by, and LJ_τ its Lie algebra. Then $\pi_{TJ_\tau}(L\times W_\tau)$ is also not rotund, and $\dim(\pi_{J_\tau}(W_\tau))<\dim W$, so we can apply the induction hypothesis and get that

$$\dim\left(\bigcup_{\tau'\in\operatorname{Trop}(\pi_{J_\tau}(W_\tau))}\tau'(\Gamma)+\operatorname{Re}(\pi_{LJ_\tau}(L))(\Gamma)\right)\neq n-\dim J_\tau$$

and hence

$$\dim\left(\bigcup_{\tau'\in\operatorname{Trop}(W),\tau\subseteq\tau'}\tau'(\Gamma)+\operatorname{Re}(L)(\Gamma)\right)\neq n$$

where we are using that by [MS15, Corollary 3.2.13] the map induced on Γ^n by π_{J_τ} is π_{LJ_τ}. Varying τ, we obtain the result. □

3. Geometrical non-degeneracy

Definition 3.1. Let $W\subseteq\mathbb{G}_m^n$ be an irreducible algebraic subvariety. Then W is *geometrically non-degenerate* if for any algebraic subgroup $J\leq\mathbb{G}_m^n$,

$$\dim\pi_J(W)=\min\{\dim W, n-\dim J\}.$$

Notation 3.2. For a field k, we denote by $G_k(d,n)$ the Grassmannian of linear subspaces $L\leq\mathbb{G}_a^n$ defined over k of dimension d.

Remark 3.3. We will see the Grassmannian $G_{\mathbb{R}}(d,n)$ as a definable set in $\mathbb{R}_{\exp,\sin}$ by identifying each linear subspace L with the matrix that defines it, seen as an element in $\mathbb{R}^{dn}$. Hence, $G_{\mathfrak{R}}(d,n)$ is the corresponding set in $\mathfrak{R}$.

It is straighforward to verify that if W is defined over $\mathbb{C}$, geometrically non-degenerate, and of codimension d, then $L \times W$ is rotund for all $L \in G_{\mathbb{R}}(d,n)$. Our goal in this section is to show that under these assumptions Condition 8 in Proposition 2.18 holds uniformly in z and L: there is an $\varepsilon > 0$ such that for all $z \in (\mathbb{C}^\times)^n$ and $L \in G_{\mathbb{R}}(d,n)$, there is $z' \in z \cdot \mathbb{S}_1^n(\mathbb{C})$ such that $B(z',\varepsilon) \subseteq \frac{W(\mathbb{C})}{\exp(L(\mathbb{C}))}$.

Lemma 3.4. *Let $L \leq \mathbb{G}_{a,\mathfrak{C}}^n$ be a linear subspace and let*

$$I = \left\{ (a_1,\dots,a_n) \in \mathcal{O}^n \mid \sum_{i=1}^n a_i x_i = 0 \forall (x_1,\dots,x_n) \in L(\mathfrak{C}) \right\}$$

and

$$\mathrm{res}(I) = \{ b \in \mathbb{C}^n \mid b = \mathrm{res}(a), a \in I \}.$$

Set $L(\mathcal{O}) = L(\mathfrak{C}) \cap \mathcal{O}^n$. Then

$$\mathrm{res}(L(\mathcal{O})) = \left\{ (v_1,\dots,v_n) \in \mathbb{C}^n \mid \sum_{i=1}^n b_i v_i = 0 \forall b = (b_1,\dots,b_n) \in \mathrm{res}(I) \right\}.$$

Furthermore, $\dim \mathrm{res}(L(\mathcal{O})) = \dim L$.

Proof. We induct on $\dim L$, the case $L = \{0\}$ being trivial.

If $L \neq \{0\}$, then L contains some non-zero vector $v = (v_1,\dots,v_n)$. After rescaling and permuting the coordinates, we can assume without loss of generality that $v \in \{1\} \times \mathcal{O}^{n-1}$. There exists an invertible matrix $A \in \mathrm{GL}_n(\mathcal{O})$ such that $Av = (1,0,\dots,0)^t$.

Set $L' = A \cdot L$, then $L' = \mathfrak{C} \times L''$ for some linear subspace L'' of $\mathfrak{C}^{n-1}$ and $L'(\mathcal{O}) = A \cdot L(\mathcal{O})$. Furthermore, if

$$I' = \left\{ (a_1,\dots,a_n) \in \mathcal{O}^n \mid \sum_{i=1}^n a_i x_i = 0 \forall (x_1,\dots,x_n) \in L'(\mathfrak{C}) \right\},$$

then

$$I' = \{ a \cdot A^{-1} \mid a \in I \}.$$

Furthermore, $I' \subseteq \{0\} \times \mathcal{O}^{n-1}$ and in fact $I' = \{0\} \times I''$, where

$$I'' = \left\{ (a_1,\dots,a_{n-1}) \in \mathcal{O}^{n-1} \mid \sum_{i=1}^{n-1} a_i x_i = 0 \forall (x_1,\dots,x_{n-1}) \in L''(\mathfrak{C}) \right\}.$$

By induction,

$$\mathrm{res}(L(\mathcal{O})) = \mathrm{res}(A)^{-1} \cdot \mathrm{res}(L'(\mathcal{O})) = \mathrm{res}(A)^{-1} \cdot (\mathbb{C} \times \mathrm{res}(L''(\mathcal{O})))$$

is the linear subspace defined by the linear equations with coefficient vectors in the set

$$\{(0,b) \cdot \mathrm{res}(A) \mid b \in \mathrm{res}(I'')\} = \{\mathrm{res}(a \cdot A) \mid a \in I'\} = \mathrm{res}(I).$$

Furthermore, $\dim \mathrm{res}(L(\mathcal{O})) = 1 + \dim \mathrm{res}(L''(\mathcal{O})) = 1 + \dim L'' = \dim L' = \dim L$. $\square$

If $L \leq \mathbb{G}_{a,\mathfrak{C}}^n$ is a linear subspace, we can therefore denote by $\mathrm{res}(L) \leq \mathbb{G}_{a,\mathbb{C}}$ the linear subspace defined by the residues of the equations defining L whose coefficients all lie in $\mathcal{O}$.

We remark that Lemma 3.4 stays true, by the same proof, if we replace $\mathfrak{C}, \mathbb{C}$ and $\mathcal{O}$ by $\mathfrak{R}, \mathbb{R}$ and $\mathcal{O} \cap \mathbb{R}$ respectively.

Lemma 3.5. *Let $L \leq \mathbb{G}_{a,\mathfrak{C}}^n$ be a linear subspace of dimension d defined over $\mathfrak{R}$. Then* $\mathrm{val}(\exp(L(\mathfrak{C}))) = \mathrm{Re}(L)(\Gamma)$.

Proof. We fix a $(n-d) \times n$-matrix A of maximal rank with coefficients in $\mathfrak{R}$ such that L is the kernel of A. After dividing each row by its entry of minimal valuation, we can and will assume that A has coefficients in $\mathcal{O} \cap \mathfrak{R}$.

We start by proving "$\subseteq$". Let $\ell \in L(\mathfrak{C})$. We want to show that

$$A \cdot \mathrm{val}(\exp(\ell)) = 0.$$

By definition

$$\mathrm{val}(\exp(\ell)) = \mathrm{val}(|\exp(\ell)|) = \mathrm{val}(\exp(\mathrm{Re}(\ell))).$$

We know that $A \cdot \mathrm{Re}(\ell) = 0$. By definition

$$\exp(\mathrm{Re}(\ell))^A = \exp(A \cdot \mathrm{Re}(\ell)) = \exp(0) = 1.$$

It remains to be shown that

$$\mathrm{val}(\exp(\mathrm{Re}(\ell))^A) = A \cdot \mathrm{val}(\exp(\mathrm{Re}(\ell))).$$

But by definition

$$A \cdot \mathrm{val}(\exp(\mathrm{Re}(\ell))) = \mathrm{val}(\varphi(\mathrm{val}(\exp(\mathrm{Re}(\ell))))^A)$$

and

$$\mathrm{val}(\varphi(\mathrm{val}(\exp(\mathrm{Re}(\ell)))) \cdot \exp(\mathrm{Re}(\ell))^{-1}) = 0.$$

Since the coefficients of A are in $\mathcal{O} \cap \mathfrak{R}$, it follows that

$$\mathrm{val}(\varphi(\mathrm{val}(\exp(\mathrm{Re}(\ell))))^A \cdot \exp(\mathrm{Re}(\ell))^{-A}) = 0$$

or equivalently

$$A \cdot \mathrm{val}(\exp(\mathrm{Re}(\ell))) - \mathrm{val}(\exp(\mathrm{Re}(\ell))^A) = 0.$$

We next prove "$\supseteq$". We take $\gamma \in \mathrm{Re}(L)(\Gamma)$, so $A \cdot \gamma = 0$. We set $x = \varphi(\gamma)$. Then

$$\mathrm{val}(x^A) = A \cdot \gamma = 0,$$

so $x^A \in (\mathcal{O}^\times)^{n-d}$. It follows that there exists $y \in (\mathcal{O}^\times)^n$ such that $x \cdot y \in \exp(L(\mathfrak{C}))$ and so $\gamma = \mathrm{val}(x) = \mathrm{val}(x \cdot y) \in \mathrm{val}(\exp(L(\mathfrak{C})))$. □

Lemma 3.6. *Let $L \leq \mathbb{G}_{a,\mathbb{C}}^n$ be a linear subspace of dimension d defined over $\mathbb{R}$ and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an algebraic subvariety of codimension d. Let $\tau \in \mathrm{Trop}(W)$ such that $L \times W_\tau$ is rotund. Then, for every $\tau' \in \mathrm{Trop}(W)$ such that $\tau' \subseteq \tau$, the variety $L \times W_{\tau'}$ is rotund.*

Proof. Every polyhedron in $\mathrm{Trop}(W_\tau)$ contains τ. By Proposition 2.12, W_τ is invariant under translation by J_τ, which implies that each of its irreducible components is.

By rotundity of $L \times W_\tau$,

$$\begin{aligned} n - \dim J_\tau &\leq \dim \pi_{TJ_\tau}(L \times W_\tau) \\ &\leq \dim L + \dim W - \dim J_\tau \\ &= n - \dim J_\tau. \end{aligned}$$

As rotundity is preserved by taking quotients, $\pi_{TJ_\tau}(L \times W_\tau)$ is rotund; by the above it has dimension $n - \dim J_\tau$.

Then there is a polyhedron $\sigma_0 \in \mathrm{Trop}(\pi_{J_\tau}(W_\tau))$ such that

$$\dim \sigma_0 + \mathrm{Re}(\pi_{LJ_\tau}(L))(\Gamma) = n - \dim J_\tau$$

and then by [MS15, Corollary 3.2.13] there is $\sigma \supseteq \tau$ in $\mathrm{Trop}(W_\tau)$ such that $\dim(\sigma + \mathrm{Re}(L)(\Gamma)) = n$.

Now let $\tau' \subseteq \tau$ be in $\mathrm{Trop}(W)$. By Proposition 2.14, $\mathrm{Trop}(W_{\tau'}) = \mathrm{star}_{\mathrm{Trop}(W)}(\tau')$ and $\mathrm{Trop}(W_\tau) = \mathrm{star}_{\mathrm{Trop}(W)}(\tau)$. Since $\tau' \subseteq \tau \subseteq \sigma$, $\mathrm{Trop}(W_{\tau'})$ contains then a face σ' that has affine span parallel to σ. Then $\sigma' + \mathrm{Re}(L)$ has dimension n, so $L \times W_{\tau'}$ satisfies Proposition 2.18(7) and is rotund. □

Lemma 3.7. *Let $L_0 \leq \mathbb{G}_{a,\mathbb{C}}^n$ be a linear subspace of dimension d defined over $\mathbb{R}$ and $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ an algebraic subvariety such that $\dim L_0 + \dim W = n$ and $L_0 \times W$ is rotund.*

Let $S := \{\tau \in \mathrm{Trop}(W) \mid L_0 \times W_\tau$ is rotund$\}$ and let $T := \bigcup_{\tau \in S} \tau$. There is a neighbourhood U of L_0 in the Grassmannian $G_\mathbb{R}(d, n)$ such that for all $L \in U$, $T(\mathbb{R}) + L(\mathbb{R}) = \mathbb{R}^n$.

Proof. It follows from Propositions 2.14 and 2.18(7) that for $L \in G_\mathbb{R}(d, n)$ and $\tau \in \mathrm{Trop}(W)$, $L \times W_\tau$ being rotund is equivalent to τ being contained in some $\tau' \in \mathrm{Trop}(W)$ such that $\dim L(\mathbb{R}) + \tau'(\mathbb{R}) = n$.

Therefore, we can find a neighbourhood $U \subseteq G_\mathbb{R}(d, n)$ of L_0 in the Grassmannian such that for all $L \in U$ and all $\tau \in S$, we have that $L \times W_\tau$ is rotund.

It follows from Proposition 2.18 that $\mathbb{R}^n \backslash (T(\mathbb{R}) + L_0(\mathbb{R}))$ is contained in a finite union of affine subspaces of dimension $< n$. At the same time, this set must be open because $T(\mathbb{R}) + L_0(\mathbb{R})$ is a union of finitely many closed polyhedra. We deduce that

$$T(\mathbb{R}) + L_0(\mathbb{R}) = \mathbb{R}^n.$$

We now prove the statement by induction on $\dim W$.

If $\dim W = 1$, then every $\tau \in \mathrm{Trop}(W)$ (and hence every $\tau \in S$) is either the singleton $\{0\}$ or a non-negative half-line spanned by some element in $\mathbb{R}^n$, and $T(\mathbb{R})$ is connected. Moreover $d = n - 1$, so $L_0(\mathbb{R})$ is a hyperplane in $\mathbb{R}^n$, say defined by an equation $\sum_{i=1}^n a_i x_i = 0$ with $a_1, \ldots, a_n \in \mathbb{R}$. It follows from $T(\mathbb{R}) + L_0(\mathbb{R}) = \mathbb{R}^n$ that there exist $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ in $T(\mathbb{R})$ such that

$$\sum_{i=1}^n a_i y_i < 0 < \sum_{i=1}^n a_i z_i.$$

These inequalities stay true if we replace $(a_1, \ldots, a_n)$ with a vector $(b_1, \ldots, b_n)$ that is sufficiently close to $(a_1, \ldots, a_n)$, and that is enough to deduce that $T(\mathbb{R}) + L(\mathbb{R}) = \mathbb{R}^n$ for all L in some neighbourhood of L_0.

Assume now $\dim W > 1$, and fix $L \in U$. Then $d < n-1$, from which it follows that $\mathbb{R}^n \setminus L(\mathbb{R})$ is connected. Let $x \in T(\mathbb{R}) + L(\mathbb{R})$, $x \notin L(\mathbb{R})$. Then there is a non-zero polyhedron $\tau_0 \in S$ such that $x \in y + L(\mathbb{R})$ for some $y \in \operatorname{relint}(\tau_0)(\mathbb{R})$.

By the definition of S and our choice of U, $L \times W_{\tau_0}$ is rotund. Let

$$S_0 := \{\tau \in \operatorname{Trop}(W_{\tau_0}) \mid L_0 \times (W_{\tau_0})_\tau \text{ is rotund}\}$$

and let $T_0 := \bigcup_{\tau \in S_0} \tau$, $Q := \operatorname{lin}(\tau_0)$ and $T_1 := \bigcup_{\tau \in S,\ \tau_0 \subseteq \tau} \tau$. Note that, locally around y, T_1 and T coincide since $y \in \operatorname{relint}(\tau_0)(\mathbb{R})$.

Let $\tau \in S_0$. Then $L_0 \times (W_{\tau_0})_\tau$ is rotund. Moreover $\tau \in \operatorname{Trop}(W_{\tau_0}) = \operatorname{star}_{\operatorname{Trop}(W)}(\tau_0)$, so there is some $\tau' \in \operatorname{Trop}(W)$ such that $\tau_0 \subseteq \tau'$ and $\tau = \tau' + Q$. Now there is some $\sigma \in \operatorname{Trop}(W_{\tau_0})$ such that $\tau \subseteq \sigma$ and $\dim(L_0(\mathbb{R}) + \sigma(\mathbb{R})) = n$. Hence there is $\sigma' \in \operatorname{Trop}(W)$ such that $\sigma = \sigma' + Q$ and $\tau' + Q \subseteq \sigma' + Q$. If $\tau' \not\subseteq \sigma'$, then $\operatorname{lin}(\tau' \cap \sigma')$ is a proper subspace of $\operatorname{lin}(\tau')$ that contains Q. On the other hand, $\tau' \subseteq \sigma' + Q = \sigma$ and so for every $x \in \tau'$, there exists $q \in Q$ such that $x \in q + \sigma'$. But then there exists $y \in \tau_0$ such that $q + y \in \tau_0$ and so $x + y \in \sigma' + \tau_0 = \sigma'$. At the same time, $x + y \in \tau' + \tau_0 = \tau'$ and so $x + y \in \tau' \cap \sigma'$. This implies that $\operatorname{lin}(\tau' \cap \sigma') = \operatorname{lin}(\sigma')$, a contradiction. We deduce that $\tau' \subseteq \sigma'$, and hence $\tau' \in S$ because $\dim(L_0(\mathbb{R}) + \sigma'(\mathbb{R})) = n$, applying Lemma 3.6. Conversely, if $\tau \in S$ and $\tau_0 \subseteq \tau$, then there is $\sigma \in S \subseteq \operatorname{Trop}(W)$ such that $\tau_0 \subseteq \tau \subseteq \sigma$ and $\dim(L_0(\mathbb{R}) + \sigma(\mathbb{R})) = n$. Therefore, $\tau + Q \subseteq \sigma + Q \in \operatorname{Trop}(W_{\tau_0})$ and $L_0 \times (W_{\tau_0})_{\tau+Q}$ is rotund by Lemma 3.6. Therefore $\tau + Q \in S_0$. Hence, we have that $S_0 = \{\tau + Q \mid \tau \in S, \tau_0 \subseteq \tau\}$, and thus $T_0(\mathbb{R}) = T_1(\mathbb{R}) + Q(\mathbb{R})$.

At the same time, since $y \in \operatorname{relint}(\tau_0(\mathbb{R}))$ and so $y \in \tau(\mathbb{R})$ for every $\tau \in S$ such that $\tau_0 \subseteq \tau$, we have that there exists some $\varepsilon_{Q,y} > 0$ such that

$$T_1(\mathbb{R}) + (B(0, \varepsilon_{Q,y}) \cap Q) \subseteq T_1(\mathbb{R}) - y \subseteq T_0(\mathbb{R}).$$

In particular,

$$\mathbb{R}_{>0} \cdot (T_1(\mathbb{R}) - y) = T_0(\mathbb{R}).$$

For $z \in \mathbb{R}^n$ of norm 1, let $\lambda(z) \in [0, \infty]$ denote the supremum of the set of λ such that $\lambda z \in T_1(\mathbb{R}) - y$. Note that $\lambda(z) > 0$ if and only if $\mathbb{R}_{>0} \cdot z \subseteq T_0(\mathbb{R})$ if and only if $\mathbb{R}_{>0} \cdot z \cap T_0(\mathbb{R}) \neq \emptyset$. If $\lambda(z) < \infty$, then there exists $\tau_z \in S$ such that $\tau_0 \subseteq \tau_z$ and $y + \lambda(z) z \in \tau_z(\mathbb{R})$. We take τ_z minimal with these properties with respect to inclusion. But $y + \lambda(z) z$ cannot lie in $\operatorname{relint}(\tau_z)(\mathbb{R})$ (otherwise, $\lambda(z)$ could not be the supremum), so it must lie in some face $\tau_z' \neq \tau_z$ of τ_z. But also $\tau_z' \in S$ by Lemma 3.6 and so we conclude that $\tau_0 \not\subseteq \tau_z'$. This implies that $\lambda(z) \geq \varepsilon' > 0$, where ε' is chosen such that

$$B(y, \varepsilon') \subseteq \mathbb{R}^n \setminus \bigcup_{\tau \in \operatorname{Trop}(W),\ \tau_0 \not\subseteq \tau} \tau(\mathbb{R}).$$

We deduce that

$$B(y, \varepsilon') \cap T_0(\mathbb{R}) = B(y, \varepsilon') \cap T_1(\mathbb{R}) = B(y, \varepsilon') \cap T(\mathbb{R}).$$

Let π_Q and $\pi_{\exp(Q)}$ denote the usual projections associated to Q. Then $\pi_Q(L) \times \pi_{\exp(Q)}(W_{\tau_0})$ is rotund by rotundity of $L \times W_{\tau_0}$, and it has dimension $n - \dim Q$ by rotundity of $L \times W_{\tau_0}$ and because by Proposition 2.12 $\dim \pi_{\exp(Q)}(W_{\tau_0}) = \dim W_{\tau_0} - \dim Q$. Since $y \in \operatorname{relint}(\tau_0)$, $\pi_Q(y) = 0$. By the induction hypothesis, 0 then lies in the interior of $\pi_Q(T_0)(\mathbb{R}) + \pi_Q(L)(\mathbb{R})$. From this, it follows that y lies in the interior of

$$\pi_Q^{-1}(\pi_Q(T_0)(\mathbb{R}) + \pi_Q(L)(\mathbb{R})) = T_0(\mathbb{R}) + L(\mathbb{R}).$$

For each maximal $\tau \in S_0$, $\mathrm{lin}(\tau) \cap L = \{0\}$ and the addition map $\varphi_\tau : \mathrm{lin}(\tau)(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}^n$ is a homeomorphism. We can choose $\varepsilon'' > 0$ such that for every such τ and for every $z \in B(y, \varepsilon'') \cap (\tau(\mathbb{R}) + L(\mathbb{R}))$, we have that $\varphi_\tau^{-1}(z) \in B(y, \varepsilon') \times L(\mathbb{R})$. Since

$$B(y, \varepsilon'') \subseteq \bigcup_{\tau \in S_0 \text{ maximal}} \varphi_\tau(\tau(\mathbb{R}) \times L(\mathbb{R})),$$

it follows that

$$B(y, \varepsilon'') \subseteq (B(y, \varepsilon') \cap T_0(\mathbb{R})) + L(\mathbb{R}) = (B(y, \varepsilon') \cap T(\mathbb{R})) + L(\mathbb{R}).$$

Hence, y lies in the interior of $T(\mathbb{R}) + L(\mathbb{R})$, and therefore so does x.

Hence we have showed that $(T(\mathbb{R}) + L(\mathbb{R})) \setminus L(\mathbb{R})$ is open in $\mathbb{R}^n \setminus L(\mathbb{R})$, and it is also relatively closed in this set because $T(\mathbb{R}) + L(\mathbb{R})$ is closed as it is the union of finitely many closed polyhedra. Therefore

$$(T(\mathbb{R}) + L(\mathbb{R})) \setminus L(\mathbb{R}) = \mathbb{R}^n \setminus L(\mathbb{R})$$

by connectedness of the latter. Since $0 \in T$, we can then conclude that $T(\mathbb{R}) + L(\mathbb{R}) = \mathbb{R}^n$. □

Proposition 3.8. *Let $L \leq \mathbb{G}_{a,\mathfrak{C}}^n$ be a linear subspace of dimension d defined over $\mathfrak{R}$ and $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ an algebraic subvariety such that W is geometrically non-degenerate and* $\dim L + \dim W = n$.

Then there is a finite set of polyhedra $S = \{\tau_1, \ldots, \tau_k\} \subseteq \mathrm{Trop}(W)$ such that the following hold:

(1) *For all $z \in (\mathfrak{C}^\times)^n$ there is $z' \in z \cdot \exp(L(\mathfrak{C})) \cdot (\mathcal{O}^\times)^n \cap W(\mathfrak{C})$ such that $\mathrm{val}(z') \in \tau$ for some $\tau \in S$.*
(2) *$\mathrm{res}(L) \times W_\tau$ is rotund for all $\tau \in S$.*

Proof. Since W is geometrically non-degenerate, $\mathrm{res}(L) \times W$ is rotund. Let $S = \{\tau \in \mathrm{Trop}(W) \mid \mathrm{res}(L) \times W_\tau \text{ is rotund}\}$, and let $T = \bigcup_{\tau \in S} \tau$. By Lemma 3.7, there exists a neighbourhood $U \subseteq G_{\mathbb{R}}(d, n)$ of $\mathrm{res}(L)$ such that for all $L' \in U(\mathbb{R})$, we have that $T(\mathbb{R}) + L'(\mathbb{R}) = \mathbb{R}^n$.

Fixing a covering of $G_{\mathbb{R}}(d, n)$ by affine spaces $\mathbb{R}^{(n-d)d}$, we may treat $G_{\mathbb{R}}(d, n)$ locally around $\mathrm{res}(L)$ as a metric space. Therefore there is some positive $\varepsilon \in \mathbb{R}$ such that the first-order sentence

$$\forall L' \in B(\mathrm{res}(L), \varepsilon) \left(T(\mathbb{R}) + L'(\mathbb{R}) = \mathbb{R}^n\right)$$

is true in the structure $\mathbb{R}$; it only has parameters in $\mathbb{R}$, so it is also true in $\mathfrak{R}$. Since $L \in B(\mathrm{res}(L), \varepsilon)$ for every positive $\varepsilon \in \mathbb{R}$, by Lemma 3.4, we have that $\mathrm{Trop}(W)(\mathfrak{R}) + L'(\mathfrak{R}) = \mathfrak{R}^n$; using the $\mathfrak{R}$-vector space isomorphism between $\mathfrak{R}$ and Γ it follows that this holds in Γ, too.

So for every $z \in \mathfrak{C}^n$ there exists $\tau \in S$ such that $(\mathrm{val}(z) + L(\Gamma)) \cap \tau \neq \varnothing$. Let α be a point in this intersection. By Lemma 3.5, there is $z_0 \in z \cdot \exp(L(\mathfrak{C}))$ such that $\mathrm{val}(z_0) = \alpha \in \tau$.

Since $\alpha \in \tau$, by Theorem 2.4, there exists $z' \in W(\mathfrak{C})$ with $\mathrm{val}(z') = \alpha$. It follows that $\mathrm{val}\left(\frac{z_0}{z'}\right) = \mathrm{val}\left(\frac{z'}{z_0}\right) = 0$ and so $\frac{z'}{z_0} \in (\mathcal{O}^\times)^n$. Therefore

$$z' = \frac{z'}{z_0} \cdot z_0 \in z \cdot \exp(L(\mathfrak{C})) \cdot (\mathcal{O}^\times)^n$$

and we are done. □

Lemma 3.9. *Let $L \leq \mathbb{G}_{a,\mathfrak{C}}^n$ be a linear subspace defined over $\mathfrak{R}$, and let $W \subseteq \mathbb{G}_{m,\mathfrak{C}}^n$ be an algebraic subvariety such that every irreducible component of W has dimension $d := n - \dim L$. Let $\delta : (L \times W)(\mathfrak{C}) \to (\mathfrak{C}^\times)^n$ be the map defined by $(l, w) \mapsto \frac{w}{\exp(l)}$.*

Let U_L be the intersection of L with a ball around 0. Let $w \in W(\mathfrak{C})$ satisfy

$$(w \cdot \exp(\partial U_L(\mathfrak{C}))) \cap W(\mathfrak{C}) = \varnothing.$$

Then w lies in the interior of $\frac{W}{\exp(U_L)}(\mathfrak{C})$.

In particular if there is some open subset U_W of $W(\mathfrak{C})$ such that every $w \in U_W$ satisfies the assumption, then $\frac{U_W}{\exp(U_L)}(\mathfrak{C})$ is an open set.

Proof. Let $\mathcal{W}$ be a parametric family, definable over $\mathbb{C}$, of equidimensional algebraic subvarieties of dimension d such that the given W belongs to $\mathcal{W}$.

We consider the following first-order sentence ϑ, which has parameters in $\mathbb{C}$ as $\mathcal{W}$ is defined over $\mathbb{C}$. We use L and W as variables in it, hopefully without causing confusion with the L and W given in the statement of the Lemma:

$$\begin{aligned}&\forall L \in G(d,n) \forall W \in \mathcal{W} \forall w \in W \\ &\qquad \Bigg(\exists \varepsilon_0 > 0 \, (w \cdot \exp(B(0,\varepsilon_0) \cap L) \cap W = \{w\}) \to \\ &\qquad\qquad \forall \varepsilon_1, \varepsilon_2 > 0 \exists \varepsilon_3 > 0 \left(B(w, \varepsilon_3) \subseteq \frac{W \cap B(w,\varepsilon_1)}{\exp(L \cap B(0,\varepsilon_2))} \right) \Bigg).\end{aligned}$$

This is true in the structure $\mathbb{R}$ by the Open Mapping Theorem: the condition in the first two lines asserts that the point $(0, w)$ is isolated in the fibre $\delta^{-1}(w)$ (and so every point w' sufficiently close to $(0, w)$ is isolated in the fibre $\delta^{-1}(\delta(w'))$ by [Chi89, Proposition 4 on p. 36]), and the conclusion asserts that w lies in the interior of the image of every restriction of δ to neighbourhoods of $(0, w)$. This is implied by the Open Mapping Theorem because the map δ is complex analytic between spaces of the same dimension and hence if its restriction to some neighbourhood of a point in $\delta^{-1}(w)$ has discrete fibres, then it is open. Hence, as it is first-order, it is true in $\mathfrak{R}$ as well.

For the "in particular", assume U_W is as in the statement and let $z \in \frac{U_W}{\exp(U_L)}(\mathfrak{C})$. Then there is $(\ell, w) \in (U_L \times U_W)(\mathfrak{C})$ such that $z = \frac{w}{\exp(\ell)}$. Choose $U_L' \subseteq U_L$ equal to the intersection of L with another ball around 0 such that $\ell + U_L' \subseteq U_L$. Since the first-order sentence ϑ in the proof of the first part holds in $\mathfrak{R}$, the point z is in the interior of $\frac{U_W \cdot \exp(-\ell)}{\exp(U_L')}(\mathfrak{C})$ and therefore in the interior of $\frac{U_W}{\exp(U_L)}(\mathfrak{C})$. □

Theorem 3.10. *Let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an algebraic subvariety of codimension d that is geometrically non-degenerate.*

There is $\varepsilon > 0$ such that for every linear subspace $L \in G_{\mathbb{R}}(d,n)$ and every $z \in (\mathbb{C}^\times)^n$ there is $s \in \mathbb{S}_1^n(\mathbb{C})$ such that $B(s,\varepsilon) \subseteq \frac{z^{-1}\cdot W(\mathbb{C})}{\exp(L(\mathbb{C}))}$.

Proof. Let $L \in G_{\mathfrak{R}}(d,n)$. We will show that for all $z \in (\mathfrak{C}^\times)^n$, there are $s \in \mathbb{S}_1^n(\mathfrak{C})$ and $\varepsilon \in \mathbb{R}_{>0}$ such that $B(s,\varepsilon) \subseteq \frac{z^{-1}\cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))}$.

Let $z \in (\mathfrak{C}^\times)^n$. Then by Proposition 3.8, there is $z' \in z\cdot\exp(L(\mathfrak{C}))\cdot(\mathcal{O}^\times)^n\cap W(\mathfrak{C})$ such that $\alpha := \mathrm{val}(z') \in \tau$ for some $\tau \in \mathrm{Trop}(W)$ and $\mathrm{res}(L) \times W_\tau$ is rotund. There exists $\tau' \in \mathrm{Trop}(W)$ such that $\tau' \subseteq \tau$ and $\alpha \in \mathrm{relint}(\tau')$. By Lemma 3.6, also $\mathrm{res}(L) \times W_{\tau'}$ is rotund and so, after replacing τ by τ', we can assume without loss of generality that $\mathrm{val}(z') \in \mathrm{relint}(\tau)$.

Proposition 2.18(2) associates to each irreducible component of W_τ such that $\mathrm{res}(L) \times W_\tau$ is rotund a Zariski-open subset; let W_τ° be the union of such sets. Note that the same set satisfies condition (4) of that proposition, as stated in its proof. This is then a Zariski-open subset of W_τ such that $\mathrm{res}(L)\times\overline{W_\tau \setminus W_\tau^\circ}^{\mathrm{Zar}}$ is not rotund. Since $z' \in z\cdot\exp(L(\mathfrak{C}))\cdot(\mathcal{O}^\times)^n$, there exists $\ell \in L(\mathfrak{C})$ such that $y = z' \cdot z^{-1} \cdot \exp(-\ell) \in (\mathcal{O}^\times)^n$ and so $\mathrm{res}(y) \in (\mathbb{C}^\times)^n$. We also deduce that

$$\frac{z^{-1}\cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))} = \frac{z'^{-1}\cdot y\cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))}$$

By Proposition 2.18(8), there are $s_0 \in \mathbb{S}_1^n(\mathbb{C})$ and $\varepsilon_0 \in \mathbb{R}_{>0}$ such that

$$B(s_0,\varepsilon_0) \subseteq \frac{\mathrm{res}(\varphi(\alpha)z'^{-1}y)\cdot W_\tau(\mathbb{C})}{\exp(\mathrm{res}(L)(\mathbb{C}))} = \mathrm{res}\left(\frac{\left(W(\mathfrak{C})\cdot z'^{-1}\cdot y\right)\cap(\mathcal{O}^\times)^n}{\exp(L(\mathcal{O}))}\right)$$

(the equality holds by Lemmas 2.8 and 3.4). More precisely, by our assumption that $\mathrm{res}(L) \times \overline{W_\tau \setminus W_\tau^\circ}^{\mathrm{Zar}}$ is not rotund, we have that by [Chi89, Proposition on p.41] the set

$$\frac{\mathrm{res}(\varphi(\alpha)z'^{-1}y)\cdot(W_\tau \setminus W_\tau^\circ)(\mathbb{C})}{\exp(\mathrm{res}(L)(\mathbb{C}))}$$

is contained in a countable union of complex analytic subsets of $(\mathbb{C}^\times)^n$ of positive codimension. No open subset of $\mathbb{S}_1^n(\mathbb{C})$ is contained in a complex analytic subset of positive codimension (see [Gal23a, Corollary 6.11] for a proof), so by the Baire category theorem it is also not contained in a countable union of such sets. We therefore may assume, after maybe changing s_0 and decreasing ε_0, that

$$B(s_0,\varepsilon_0) \subseteq \frac{\mathrm{res}(\varphi(\alpha)z'^{-1}y)\cdot W_\tau^\circ(\mathbb{C})}{\exp(\mathrm{res}(L)(\mathbb{C}))}.$$

To ease notation, we now replace W by $s_0^{-1}\cdot W$, so that we may assume $s_0 = 1$. So there is $w_0 \in \mathrm{res}(\varphi(\alpha)z'^{-1}y)\cdot W_\tau^\circ(\mathbb{C})$ such that $1 \in \frac{w_0}{\exp(\mathrm{res}(L))(\mathbb{C})}$ - that is, such that $w_0 = \exp(\ell_0)$ for some $\ell_0 \in \mathrm{res}(L)(\mathbb{C})$.

As in the proof of ($7 \Rightarrow 1$) in Proposition 2.18, we now consider $\mathbb{R}$ and $\mathfrak{R}$ as structures in the language with the restricted sine function; the cost is that we can only work with the restricted exponential function, but we can use techniques from o-minimal geometry.

By definition of W_τ°, there are positive reals r_1, r_2 for which the sets $U_1 = B(\ell_0, r_1) \cap (\mathrm{res}(L)(\mathbb{C}))$ and $U_{W_\tau} = B(w_0, r_2) \cap (\mathrm{res}(\varphi(\alpha) z'^{-1} y) \cdot W_\tau(\mathbb{C}))$ are connected and such that the set

$$\left\{ (\ell, w) \in \overline{U_1 \times U_{W_\tau}} \mid \frac{w}{\exp(\ell)} = c \right\}$$

is finite (possibly empty) for all $c \in (\mathbb{C}^\times)^n$, and it has precisely one element for $c = 1$. We have that $\frac{w_0}{\exp(\partial U_1)} \cap \mathrm{res}(\varphi(\alpha) z'^{-1} y) \cdot W_\tau(\mathbb{C}) = \varnothing$, that $\frac{w_0}{\exp(\partial U_1)}$ is compact and that $\mathrm{res}(\varphi(\alpha) z'^{-1} y) \cdot W_\tau(\mathbb{C})$ is closed, so that every point in $\frac{w_0}{\exp(\partial U_1)}$ is at distance at least R from $\mathrm{res}(\varphi(\alpha) z'^{-1} y) \cdot W_\tau(\mathbb{C})$ for some real positive R. By taking r_2 sufficiently small we can make sure that every point in $\frac{U_{W_\tau}}{\exp(\partial U_1)}$ is at distance at most $\frac{R}{2}$ from $\frac{w_0}{\exp(\partial U_1)}$, and thus

$$\frac{U_{W_\tau}}{\exp(\partial U_1)} \cap \mathrm{res}(\varphi(\alpha) z'^{-1} y) \cdot W_\tau(\mathbb{C}) = \varnothing.$$

Let $\ell_1 \in L(\mathfrak{C}) \cap (\mathcal{O}^\times)^n$ such that $\mathrm{res}(\ell_1) = \ell_0$ Let $U_2 := B(\ell_1, r_1) \cap L(\mathfrak{C})$; let $w_1 \in (z'^{-1} y) \cdot W(\mathfrak{C})$ satisfy $\mathrm{res}(w_1) = w_0$, and let $U_W := B(w_1, r_2) \cap (z'^{-1} y) \cdot W(\mathfrak{C})$. Then U_2 and U_W are open semialgebraic subsets of L and $W \cdot z'^{-1} y(\mathfrak{C})$ respectively, and we have $\mathrm{res}(U_W) = U_{W_\tau}$ and $\mathrm{res}(U_2) = U_1$. It follows that every point in U_W satisfies the assumptions of Lemma 3.9, and therefore, by the "in particular" part of that Lemma, $\frac{U_W}{\exp(U_2)}$ is an open set.

Let $U := U_1 \times U_{W_\tau}$. Since $\partial \overline{U}$ is compact, there is some $\varepsilon_1 \in \mathbb{R}_{>0}$ such that for each $(\ell, w) \in \partial \overline{U}$ we have that $\frac{w}{\exp(\ell)}$ is at distance greater that ε_1 from 1. Moreover, $\partial(U_2 \times U_W)$ is a closed and bounded set definable in an o-minimal structure, so by [Dri98, Corollary 6.1.13] its image under the function $(\ell, w) \mapsto \frac{w}{\exp(\ell)}$ is closed. Assume now $z \in B(1, \varepsilon_1) \setminus \frac{U_W}{\exp(U_2)}$. By the choice of ε_1, there is no $(\ell, w) \in \partial(U_2 \times U_W)$ such that $\frac{w}{\exp(\ell)} = z$. By [Dri98, Corollary 6.1.13] applied to the image of $\overline{U_W \times U_2}$ under the same function as before, z then has a neighbourhood which is disjoint from $\frac{U_W}{\exp(U_2)}$. It follows that $\frac{U_W}{\exp(U_2)} \cap B(1, \varepsilon_1)$ is relatively closed in $B(1, \varepsilon_1)$. It is also non-empty since $1 \in \mathrm{res}\left(\frac{U_W}{\exp(U_2)}\right)$ and $\varepsilon_1 \in \mathbb{R}$, and definable in an o-minimal structure since the exponential is only applied on a bounded domain. Thus, since $B(1, \varepsilon_1)$ is *definably connected* (see [Dri98, Section 1.3]), we have that $B(1, \varepsilon_1)(\mathfrak{C}) \subseteq \frac{(z'^{-1} y) \cdot W}{\exp(U_2)}(\mathfrak{C}) \subseteq \frac{z^{-1} \cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))}$.

We now consider again $\mathbb{R}$ and $\mathfrak{R}$ in the language with the total sine function. We have proved that in $\mathfrak{R}$ the following first-order sentence holds

$$\exists \varepsilon > 0 \left(\forall z \in \mathbb{G}_m^n \forall L \in G(d, n) \left(\exists s \in \mathbb{S}_1^n \left(B(s, \varepsilon) \subseteq \frac{z^{-1} \cdot W}{\exp(L)} \right) \right) \right)$$

because any infinitesimal element is a witness for it. Hence, this sentence must also hold true in $\mathbb{R}_{\exp,\sin}$, completing the proof. $\square$

Remark 3.11. If $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ is not geometrically non-degenerate, then there is $L \leq \mathbb{G}_a^n$ defined over $\mathbb{Q}$ such that $\frac{W}{\exp(L)}$ has empty interior, so there is no such ε. Therefore the converse of Theorem 3.10 holds as well, giving a characterization of geometrically non-degenerate subvarieties of $\mathbb{G}_{m,\mathbb{C}}^n$ in a similar spirit as the characterization of rotundity in Proposition 2.18.

4. An equidistribution argument

In this section, we will apply the following well-known equidistribution theorem for Galois orbits of torsion points in algebraic tori. We denote the set of roots of unity in $\mathbb{C}^\times$ by μ_∞, and the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ by $\overline{\mathbb{Q}}$. For an element $z = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n$ and a set S of field automorphisms of $\mathbb{C}$, we define

$$S \cdot z = \{\sigma(z) \mid \sigma \in S\},$$

where $\sigma(z)$ is defined to be

$$(\sigma(z_1), \ldots, \sigma(z_n))$$

for a field automorphism σ of $\mathbb{C}$.

Theorem 4.1. *Let $K \subset \mathbb{C}$ be a subfield that is finitely generated over $\mathbb{Q}$, let $n \in \mathbb{N}$, and let $B \subset (\mathbb{C}^\times)^n$ be some open Euclidean ball such that $B \cap \mathbb{S}_1^n(\mathbb{C}) \neq \emptyset$.*

Let $(\zeta_j)_{j\in\mathbb{N}}$ be a sequence in μ_∞^n such that for every algebraic subgroup $G \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$, the set of $j \in \mathbb{N}$ such that $\zeta_j \in G(\mathbb{C})$ is finite.

Then there exists $N \in \mathbb{N}$ such that for all $j \in \mathbb{N}$:

$$j \geq N \implies \mathrm{Aut}(\mathbb{C}/K) \cdot \zeta_j \cap B \cap \mathbb{S}_1^n(\mathbb{C}) \neq \emptyset.$$

Proof. Set $L = \overline{\mathbb{Q}} \cap K$. Then L is a finite extension of $\mathbb{Q}$ by [Isa09, Theorem 24.9] and the homomorphism $\mathrm{Aut}(\mathbb{C}/K) \to \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ is surjective since $\mathrm{Gal}(\overline{\mathbb{Q}}K/K) \to \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ is surjective by [Lan02, Chapter VI, Theorem 1.12] and $\mathrm{Aut}(\mathbb{C}/K) \to \mathrm{Gal}(\overline{\mathbb{Q}}K/K)$ is surjective as well. Hence, we can assume without loss of generality that K is a finite extension of $\mathbb{Q}$.

The theorem then follows from Bilu's equidistribution theorem; see [Bil97], where the theorem is formulated over $\mathbb{Q}$, and see [Küh22] for a proof of a much more general statement over an arbitrary number field. □

Lemma 4.2. *Let $K \subset \mathbb{C}$ be a subfield that is finitely generated over $\mathbb{Q}$, let $n \in \mathbb{N}$, and let $\varepsilon > 0$.*

Let $\mathscr{N} : \mu_\infty^n \to \mathbb{Z}_{>0}$ be defined by

$$\mathscr{N}(x) = \min\{\|\underline{N}\|_1; \underline{N} \in \mathbb{Z}^n \backslash \{0\} \text{ such that } x^{\underline{N}} = 1\}.$$

There exists $N = N(n, K, \varepsilon) \in \mathbb{Z}_{>0}$ such that for every $\xi \in \mu_\infty^n$ with $\mathscr{N}(\xi) > N$, the set $\mathrm{Aut}(\mathbb{C}/K) \cdot \xi$ intersects every open Euclidean ball of radius ε centered at a point of $\mathbb{S}_1^n(\mathbb{C})$.

Proof. We argue by contradiction and assume that the lemma is false. Hence, there is a sequence $(\xi_j)_{j\in\mathbb{N}}$ in μ_∞^n and a sequence of open Euclidean balls B_j ($j \in \mathbb{N}$) of radius ε, each centered at some point of $\mathbb{S}_1^n(\mathbb{C})$, such that $\lim_{j\to\infty} \mathscr{N}(\xi_j) = \infty$ and

$$\mathrm{Aut}(\mathbb{C}/K) \cdot \xi_j \cap B_j = \emptyset$$

for all $j \in \mathbb{N}$.

Since $\mathbb{S}_1^n(\mathbb{C})$ is compact, we can find finitely many open Euclidean balls $\widetilde{B}_1, \ldots, \widetilde{B}_M$ in $(\mathbb{C}^\times)^n$ centered at points of $\mathbb{S}_1^n(\mathbb{C})$ and of radius $\varepsilon/2$ such that

$$\mathbb{S}_1^n(\mathbb{C}) \subseteq \widetilde{B}_1 \cup \cdots \cup \widetilde{B}_M.$$

For each $j \in \mathbb{N}$, there exists an $i(j) \in \{1, \ldots, M\}$ such that $\widetilde{B}_{i(j)}$ contains the center of B_j. It follows that $\widetilde{B}_{i(j)} \subseteq B_j$.

We obtain that *a fortiori*

$$\operatorname{Aut}(\mathbb{C}/K) \cdot \xi_j \cap \widetilde{B}_{i(j)} = \emptyset$$

for all $j \in \mathbb{N}$. After passing to a subsequence of $(\xi_j)_{j\in\mathbb{N}}$, we can assume that $i(j) = i_0$ for some $i_0 \in \{1, \ldots, M\}$ and all $j \in \mathbb{N}$.

It follows that

$$\operatorname{Aut}(\mathbb{C}/K) \cdot \xi_j \cap \widetilde{B}_{i_0} = \emptyset$$

for all $j \in \mathbb{N}$. At the same time, we deduce from $\lim_{j\to\infty} \mathscr{N}(\xi_j) = \infty$ that for every algebraic subgroup $G \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$, the set of $j \in \mathbb{N}$ such that $\xi_j \in G(\mathbb{C})$ is finite. We have found a contradiction with Theorem 4.1, which finishes the proof. □

Lemma 4.3. *Let $n \in \mathbb{N}$, let $K \subset \mathbb{C}$ be a subfield that is finitely generated over $\mathbb{Q}$, and let $\varepsilon > 0$. There exists a finite set $\mathcal{G} = \{G_1, \ldots, G_N\}$, depending only on n, K, and ε, such that $G_i \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ is an algebraic subgroup for all $i = 1, \ldots, N$ and such that for every subtorus $J \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ and every torsion point $\zeta \in \mathbb{G}_{m,\mathbb{C}}^n(\mathbb{C})$, one of the following holds:*

- *(i) for every Euclidean ball of radius ε centered at a point of $\mathbb{S}_1^n(\mathbb{C})$, the intersection of $\mathbb{S}_1^n(\mathbb{C}) \cap \sigma(\zeta) \cdot J(\mathbb{C})$ and the ball is not empty for some $\sigma \in \operatorname{Aut}(\mathbb{C}/K)$ or*
- *(ii) $\zeta \cdot J \subseteq G_i$ for some $i \in \{1, \ldots, N\}$.*

Proof. Let $N = N(n, K, \varepsilon)$ be the positive integer provided by Lemma 4.2.

Let J be a subtorus of $\mathbb{G}_{m,\mathbb{C}}^n$ and let $\zeta \in \mathbb{G}_{m,\mathbb{C}}^n(\mathbb{C})$ be a torsion point. Since $\zeta \cdot J(\mathbb{C}) \cap \mu_\infty^n$ is Zariski dense in ζJ, it follows that either ζJ itself is contained in an algebraic subgroup defined by an equation $x^{\underline{N}} = 1$ with $\underline{N} \in \mathbb{Z}^n$ such that $0 < \|\underline{N}\|_1 \leq N$ or there is a point $\xi \in \zeta \cdot J(\mathbb{C}) \cap \mu_\infty^n$ with $\mathscr{N}(\xi) > N$. In the second case, it follows from Lemma 4.2 that the first alternative in the conclusion of Lemma 4.3 holds (note that the set $J(\mathbb{C})$ is invariant under any field automorphism of $\mathbb{C}$). We are now done by setting

$$\begin{aligned}\mathcal{G} = \{G;\ & G \text{ algebraic subgroup defined by an equation} \\ & x^{\underline{N}} = 1 \text{ with } \underline{N} \in \mathbb{Z}^n \text{ such that } 0 < \|\underline{N}\|_1 \leq N\}.\end{aligned}$$

□

Proof of Theorem 1.1. By Theorem 3.10, there exists $\varepsilon > 0$ such that for every $z \in (\mathbb{C}^\times)^n$ and every subtorus $J \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim J + \dim W \geq n$ (in Theorem 3.10, the dimensions add up to n, but this is no restriction since one can replace J by a suitable subtorus), the image of the map

$$\psi_{W,J,z} : W(\mathbb{C}) \times J(\mathbb{C}) \to (\mathbb{C}^\times)^n, \quad (w, y) \mapsto wyz$$

contains a Euclidean ball of radius ε centered at a point of $\mathbb{S}_1^n(\mathbb{C})$.

Let $x \in (\mathbb{C}^\times)^n$ be an arbitrary point. Note that

$$x \in \psi_{W,J,1}(W(\mathbb{C}) \times J(\mathbb{C})) \Leftrightarrow 1 \in \psi_{W,J,x^{-1}}(W(\mathbb{C}) \times J(\mathbb{C})).$$

We know that the image of $\psi_{W,J,x^{-1}}$ contains a Euclidean ball of radius ε centered at a point of $\mathbb{S}_1^n(\mathbb{C})$. Since

$$J(\mathbb{C}) \cdot \psi_{W,J,x^{-1}}(W(\mathbb{C}) \times J(\mathbb{C})) = \psi_{W,J,x^{-1}}(W(\mathbb{C}) \times J(\mathbb{C})),$$

the theorem follows from Lemma 4.3 applied with $\zeta = (1, \ldots, 1)$. $\square$

Proof of Theorem 1.2. By Theorem 3.10, there exists $\varepsilon > 0$ such that for every $z \in (\mathbb{C}^\times)^n$ and every subtorus $J \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim J + \dim W \geq n$ (again, we may replace J by a suitable subtorus of dimension $\operatorname{codim} W$), the image of the map

$$\psi_{W,J,z} : W(\mathbb{C}) \times J(\mathbb{C}) \to (\mathbb{C}^\times)^n, \quad (w, y) \mapsto wyz$$

contains a Euclidean ball of radius ε centered at a point of $\mathbb{S}_1^n(\mathbb{C})$.

Let now J be a fixed subtorus of $\mathbb{G}_{m,\mathbb{C}}^n$ such that $\dim J + \dim W \geq n$ and let $\zeta \in \mathbb{G}_{m,\mathbb{C}}^n(\mathbb{C})$ be a torsion point. We deduce that the image of $\psi_{W,J,1}$ contains a Euclidean ball of radius ε centered at a point of $\mathbb{S}_1^n(\mathbb{C})$. We have

$$W(\mathbb{C}) \cap \zeta \cdot J(\mathbb{C}) \neq \emptyset \Leftrightarrow 1 \in \psi_{W,J,\zeta^{-1}}(W(\mathbb{C}) \times J(\mathbb{C})).$$

The subvariety W of $\mathbb{G}_{m,\mathbb{C}}^n$ is defined by equations with coefficients in some field $K \subseteq \mathbb{C}$ that is finitely generated over $\mathbb{Q}$. Since

$$z^{-1} \cdot J(\mathbb{C}) \cdot \psi_{W,J,1}(W(\mathbb{C}) \times J(\mathbb{C})) = \psi_{W,J,z^{-1}}(W(\mathbb{C}) \times J(\mathbb{C}))$$

for all $z \in (\mathbb{C}^\times)^n$, it follows from Lemma 4.3 that either $\zeta \cdot J$ is contained in one of finitely many algebraic subgroups $G \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ or $W(\mathbb{C}) \cap \sigma(\zeta) \cdot J(\mathbb{C}) \neq \emptyset$ for some $\sigma \in \operatorname{Aut}(\mathbb{C}/K)$. We can assume without loss of generality that the second alternative holds.

But since W is defined over K, the set $W(\mathbb{C})$ is invariant under σ^{-1} (and the same holds for $J(\mathbb{C})$). It follows that $W(\mathbb{C}) \cap \zeta \cdot J(\mathbb{C}) \neq \emptyset$ and we are done. $\square$

5. Applications

We start this section by showing how our results on likely intersections allow us to give a new proof of the Manin-Mumford conjecture for algebraic tori.

Proposition 5.1. *Let V be a subvariety of $\mathbb{G}_m^n$, defined over $\mathbb{C}$. Then the Zariski closure of the set of torsion points that lie on V is a finite union of torsion cosets.*

Proof. Any subvariety of $\mathbb{G}_m^n$ is a finite union of irreducible subvarieties of $\mathbb{G}_m^n$, so we can assume that V is irreducible. Any irreducible subvariety of $\mathbb{G}_m^n$ is an intersection of finitely many irreducible hypersurfaces in $\mathbb{G}_m^n$. Since the intersection of two torsion cosets is a finite union of torsion cosets, we can assume that V is an irreducible hypersurface.

We then induct on n, the case $n = 1$ being trivial.

If the stabilizer of V is positive-dimensional, then we can find a surjective homomorphism $\varphi : \mathbb{G}_m^n \to \mathbb{G}_m^k$ for some $k \in \{1, \ldots, n-1\}$ such that $V' = \varphi(V)$ is closed in $\mathbb{G}_m^k$ and $V = \varphi^{-1}(V')$. We can then apply the inductive hypothesis for V' and are done. Hence, we can assume that the stabilizer of V is finite.

The hypersurface V is defined by some polynomial equation

$$f(x_1, \ldots, x_n) = 0.$$

We now consider the hypersurface V' in $\mathbb{G}_m^{n+1}$ that is defined by the equation

$$f(x_1, \ldots, x_n) = x_{n+1}.$$

Clearly, V' is irreducible. We deote by $\pi : \mathbb{G}_m^{n+1} \to \mathbb{G}_m^n$ the projection to the first n coordinates, then $\pi(V')$ is the complement of V. It follows that π maps the stabilizer of V' into the stabilizer of V, which is finite. Furthermore, if $(z_1, \ldots, z_{n+1})$ belongs to the stabilizer of V', then

$$z_{n+1} = f(z_1 x_1, \ldots, z_n x_n) f(x_1, \ldots, x_n)^{-1},$$

where $(x_1, \ldots, x_n)$ is an arbitrary point at which f does not vanish. Hence, the restriction of π to the stabilizer of V' is injective. We deduce that the stabilizer of V' is finite and so V' is geometrically non-degenerate.

Suppose now that $(\zeta_1, \ldots, \zeta_n) \in V(\mathbb{C})$ is a torsion point. It follows that the intersection of V' with the torsion coset

$$\{(\zeta_1, \ldots, \zeta_n)\} \times \mathbb{G}_m$$

is empty. Theorem 1.2 now yields a finite set of proper torsion cosets in $\mathbb{G}_m^{n+1}$ such that $\{(\zeta_1, \ldots, \zeta_n)\} \times \mathbb{G}_m$ is contained in one of them or, equivalently, a finite set of proper torsion cosets in $\mathbb{G}_m^n$ such that $(\zeta_1, \ldots, \zeta_n)$ belongs to one of them.

Let ζH be such a torsion coset, where ζ is a torsion point and H is a proper subtorus of $\mathbb{G}_m^n$. If $\zeta H \subset V$, we add ζH to our (finite) list of torsion cosets contained in V and containing all torsion points that lie on V and we are done. Otherwise, $\zeta H \cap V$ is either empty, in which case there is nothing to prove, or a union of irreducible hypersurfaces inside ζH. Recalling that H is isomorphic as an algebraic group to $\mathbb{G}_m^k$ for some $k \in \{1, \ldots, n-1\}$ and using that translation by $\zeta^{\pm 1}$ maps torsion cosets onto torsion cosets, we are then again done by induction. □

We next prove Theorem 1.5.

Proof of Theorem 1.5. For a vector $a = (a_{i,j})_{i=1,\ldots,d;\ j=1,\ldots,n} \in \mathbb{Z}^{nd}$, we set

$$H_a = \left\{ (x_1, \ldots, x_n) \in \mathbb{G}_m^n;\ \prod_{j=1}^n x_j^{a_{i,j}} = 1\ (i = 1, \ldots, d) \right\},$$

which is an algebraic subgroup of $\mathbb{G}_m^n$.

We apply Theorem 1.2 to an intersection of the form $V \cap H_a$. We find that such an intersection is non-empty unless one of finitely many non-zero vectors $b_1, \ldots, b_k \in \mathbb{Z}^n$ is contained in the vector subspace of $\mathbb{Q}^n$ that is generated by $(a_{i,1}, \ldots, a_{i,n})$ $(i = 1, \ldots, d)$.

Therefore, it is enough to show that, for a given $b \in \mathbb{Z}^n \backslash \{0\}$, we have that

$$\lim_{N \to \infty} \frac{\# \mathcal{T}(n, d, b, N)}{(2N+1)^{nd}} = 0,$$

where $\mathcal{T}(n, d, b, N)$ is the set of tuples of vectors $(a_1, \ldots, a_d) \in \mathbb{Z}^{nd}$ such that their entries are bounded in absolute value by N and the set $\{a_1, \ldots, a_d, b\}$ is linearly dependent.

Since $b = (b_1, \dots, b_n) \neq 0$, there exists some j_0 such that $b_{j_0} \neq 0$. Given $(a_1, \dots, a_d) \in \mathcal{T}(n, d, b, N)$, where $a_i = (a_{i,1}, \dots, a_{i,n})$ $(i = 1, \dots, d)$, we can then find $k \in \{1, \dots, d\}$ and pairs of indices $(i_1, j_1), \dots, (i_k, j_k)$ such that

$$\#\{j_0, \dots, j_k\} = \#\{i_1, \dots, i_k\} + 1 = k + 1$$

and

$$\det \begin{pmatrix} a_{i_1,j_0} & a_{i_1,j_1} & \dots & a_{i_1,j_k} \\ \vdots & & & \vdots \\ a_{i_k,j_0} & a_{i_k,j_1} & \dots & a_{i_k,j_k} \\ b_{j_0} & b_{j_1} & \dots & b_{j_k} \end{pmatrix} = 0,$$

but

$$\det \begin{pmatrix} a_{i_1,j_0} & a_{i_1,j_1} & \dots & a_{i_1,j_{k-1}} \\ \vdots & & & \vdots \\ a_{i_{k-1},j_0} & a_{i_{k-1},j_1} & \dots & a_{i_{k-1},j_{k-1}} \\ b_{j_0} & b_{j_1} & \dots & b_{j_{k-1}} \end{pmatrix} \neq 0.$$

It follows from this that a_{i_k,j_k} is uniquely determined by the other $a_{i,j}$ and so the number of possibilities for $(a_1, \dots, a_d)$ is at most $(2N + 1)^{nd-1}$. This suffices to conclude. $\square$

We continue with the proof of Corollary 1.4.

Proof of Corollary 1.4. Note first that, if $H \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ is a subtorus such that $\dim H + \dim W \geq n$ and $W(\mathbb{C}) \cap z \cdot H(\mathbb{C}) = \emptyset$ for some $z \in (\mathbb{C}^\times)^n$, then $\dim H = n-1$. By Theorem 1.1, we then have that H is one of finitely many subtori. It remains to prove that there are only finitely many possibilities for the coset zH. But the restriction of the quotient homomorphism $\pi_H : \mathbb{G}_m^n \to \mathbb{G}_m^n/H$ to W is not constant and hence dominant since its domain and its codomain are both curves. It follows that $\pi_H(W)$ is constructible and dense in $\mathbb{G}_m^n/H$ and therefore $(\mathbb{G}_m^n/H) \setminus \pi_H(W)$ is finite. It follows that $zH = \pi_H^{-1}(y)$ for one of finitely many y and we are done. $\square$

Finally, we remark that it is possible to use Theorem 1.1 to deduce a version of it over arbitrary algebraically closed fields of characteristic 0. The second author thanks Vahagn Aslanyan and Vincenzo Mantova for pointing this out.

Theorem 5.2. *Let K be an algebraically closed field of characteristic* 0*, $W \subseteq \mathbb{G}_m^n$ an irreducible geometrically non-degenerate algebraic subvariety defined over K.*

Then there exists a finite set $\mathcal{H} = \{H_1, \dots, H_N\}$ of subtori of $\mathbb{G}_m^n$ such that for every subtorus $H \subseteq \mathbb{G}_m^n$ with $\dim H + \dim W \geq n$ and every $z \in (K^\times)^n$, one of the following holds:

(i) $W(K) \cap z \cdot H(K) \neq \varnothing$ or
(ii) $H \subseteq H_i$ for some $i \in \{1, \dots, N\}$.

Proof. Let K_0 be a countable algebraically closed subfield of K such that W is defined over K_0. Let $\iota : K_0 \hookrightarrow \mathbb{C}$ be an embedding, so that we may see W as defined over $\mathbb{C}$. We may then apply Theorem 1.1 and obtain a finite set $\mathcal{H} = \{H_1, \dots, H_N\}$ of subtori. Let $H \subseteq \mathbb{G}_m^n$ be a subtorus with $\dim H + \dim W \geq n$, let $z \in (K^\times)^n$, and assume $H \not\subseteq H_i$ for every $i =$

$1, \dots, N$. Extend the embedding ι to an embedding ι' of $K_0(z)$ into $\mathbb{C}$. By Theorem 1.1, $W(\mathbb{C}) \cap \iota'(z) \cdot H(\mathbb{C}) \neq \varnothing$. We may then conclude using the fact that K is algebraically closed and that ι' is an embedding. $\square$

A similar argument allows us to generalise also Theorem 1.2 to arbitrary algebraically closed fields of characteristic 0.

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