# LIKELY INTERSECTIONS IN POWERS OF THE MULTIPLICATIVE GROUP (PRELIMINARY VERSION)

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Abstract. TODO.

## Contents



## 1. Introduction

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**Theorem 1.1.** *Let $n \in \mathbb{N}$ and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an irreducible geometrically non-degenerate subvariety.*

*Then there exists a finite set $\mathcal{H} = \{H_1, \ldots, H_N\}$ such that $H_i \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ is a subtorus for all $i = 1, \ldots, N$ and such that for every subtorus $H \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim H + \dim W \geq n$ and for every $z \in (\mathbb{C}^\times)^n$, one of the following holds:*

*(i) $W(\mathbb{C}) \cap z \cdot H(\mathbb{C}) \neq \emptyset$ or*
*(ii) $H \subseteq H_i$ for some $i \in \{1, \ldots, N\}$.*

**Theorem 1.2.** *Let $n \in \mathbb{N}$ and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an irreducible geometrically non-degenerate subvariety.*

*Then there exists a finite set $\mathcal{G} = \{G_1, \ldots, G_N\}$ such that $G_i \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ is an algebraic subgroup for all $i = 1, \ldots, N$ and such that for every subtorus $H \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim H + \dim W \geq n$ and for every torsion point $\zeta \in \mathbb{G}_{m,\mathbb{C}}^n(\mathbb{C})$, one of the following holds:*

*(i) $W(\mathbb{C}) \cap \zeta \cdot H(\mathbb{C}) \neq \emptyset$ or*
*(ii) $\zeta H \subseteq G_i$ for some $i \in \{1, \ldots, N\}$.*

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*Date*: July 1, 2024.
2020 *Mathematics Subject Classification.* TODO.
*Key words and phrases.* TODO..

## 2. Preliminaries

2.1. **Model theory.** We will use very basic model theory in order to obtain some uniformity results, hence we use this subsection to fix some terminology. We will value conciseness over precision, and refer the reader to one of the many excellent textbooks in the area, for example [Hod97], for the details.

Recall that a *language* $\mathcal{L}$ consists of symbols for constants, functions, and relations, and that an *$\mathcal{L}$-formula* is an expression involving variables, the symbols in $\mathcal{L}$, and the usual logical symbols $\neg, \vee, \wedge, \exists, \forall$. A *structure* in a language $\mathcal{L}$ is a set $S$ together with an assignment of the constants to some elements of $S$, of the functions to some functions $S^n \to S$, and of the relations to some subsets of $S^m$, for appropriate $n$ and $m$. A formula *with parameters from* $A$ for some subset $A \subseteq S$ is a formula $\varphi(x, a)$ where $x, y$ are tuples of variables, $\varphi(x, y)$ is an $\mathcal{L}$-formula, and $a$ is a tuple of elements from $A$ (of the same length as $y$).

A subset $X \subseteq S^n$ is *definable* (over $A$) if there is some formula $\varphi$ in $n$ variables (with parameters from $A$) such that $X$ consists of those points of $S$ which satisfy $\varphi$.

An *elementary extension* $S'$ of the structure $S$ is an $\mathcal{L}$-structure such that every formula with parameters from $S$ which is true in $S$ is also true in $S'$. As an example, the complex field is not an elementary extension of the real field: the latter satisfies the sentence "$\forall x \neg(x^2 = -1)$" while the former does not. On the other hand, $\mathbb{C}$ is an elementary extension of the algebraic closure $\overline{\mathbb{Q}}$ of the rationals, as can be seen for example applying the *Tarski-Vaught criterion* [Hod97, Theorem 2.5.1].

Given a structure $S$, an elementary extension $S'$, and a definable subset $X$ of $S^n$, we may look at the subset of $(S')^n$ defined by the same formula. This set will be denoted by $X(S')$: the geometrically-minded reader will notice the similarity with taking points of algebraic varieties in different fields containing the field of definition. In other words, a set definable over some structure $S$ may be seen as a functor from the category of elementary extensions of $S$ with embeddings to the category of sets.

2.2. **Tropical geometry.** Here we recall the necessary preliminaries from tropical geometry. Everything in this subsection is known and can be found for example in [MS15], although we give some proofs which we have not been able to find in the literature.

Throughout we will talk about an algebraically closed valued field $(K, \mathrm{val})$, with divisible (ordered) value group $(\Gamma, +, \leq)$, valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{m}$, and residue field $k = \mathcal{O}/\mathfrak{m}$. We will denote the residue map by $\mathrm{res} : \mathcal{O} \twoheadrightarrow k$. We abuse notation and write res also for the map on the polynomial rings $\mathrm{res} : \mathcal{O}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \twoheadrightarrow k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

We also fix a *splitting* of the valuation, that is, a group homomorphism $\varphi : \Gamma \to K^\times$ such that $\mathrm{val}(\varphi(\gamma)) = \gamma$ for all $\gamma \in \Gamma$. We are free to assume this exists by [MS15, Lemma 2.1.15] (there they assume $\Gamma \subseteq \mathbb{R}$, but they only use that $\Gamma$ is torsion-free). With another abuse of notation, we will use val and $\varphi$ also to denote the Cartesian powers of these maps, so the maps $\mathrm{val} : K^n \to \Gamma^n$ and $\varphi : \Gamma^n \to K^n$.

Given vectors $u \in \mathbb{Q}^n$ and $w \in \Gamma^n$, we write $\langle u, w\rangle$ for the element $u_1w_1 + \cdots + u_nw_n \in \Gamma$, with the action of $\mathbb{Q}$ on $\Gamma$ defined in the obvious way. This is not to be confused with the operation $v \cdot v'$ on elements $v, v' \in \mathbb{G}_m^n$, which denotes the element $(v_1 \cdot v_1', \ldots, v_n \cdot v_n')$. Finally, if $v \in \mathbb{G}_m^n$ and $u \in \mathbb{Z}^n$, then $v^u = \Pi_{i=1}^n v_i^{u_i}$.

**Definition 2.1.** Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial, $w \in \Gamma^n$. We write $f$ as

$$f = \sum_{u \in S} c_u x^u$$

for some finite set $S \subseteq \mathbb{Z}^n$ (we assume $c_u \neq 0$ for all $u \in S$).

The *initial form* of $f$ with respect to $w$ is $\mathrm{in}_w(f) \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ defined as

$$\mathrm{in}_w(f) = \sum_{u \in S'} \mathrm{res}(c_u \varphi(-\mathrm{val}(c_u)))x^u$$

where

$$S' := \{u \in S \mid \mathrm{val}(c_u) + \langle u, w\rangle \leq \mathrm{val}(c_{u'}) + \langle u', w\rangle \, \forall u' \in S\}.$$

For example, if $\mathrm{val}(c_u) = 0$ for each $u \in S$, then $\mathrm{in}_0(f) = \mathrm{res}(f)$.

Given an ideal $I$ and $w \in \Gamma^n$, we define the *initial ideal*

$$\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) \mid f \in I\rangle.$$

**Lemma 2.2** ([MS15, Lemma 2.6.2(a)])**.** *If $g \in \mathrm{in}_w(I)$, then $g = \mathrm{in}_w(f)$ for some $f \in I$.*

**Definition 2.3.** Let $W$ be an algebraic subvariety of $\mathbb{G}_m^n$ defined over $K$, and let $I$ be the ideal of Laurent polynomials which vanish on $W$.

For $w \in \Gamma^n$, the *initial variety of $W$ with respect to $w$* is the algebraic subvariety of $\mathbb{G}_m^n$ defined over $k$ and consisting of the common zero locus of all polynomials in $\mathrm{in}_w(I)$. It is denoted by $\mathrm{in}_w(W)$.

**Theorem 2.4** ([MS15, Theorem 3.2.3])**.** *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety defined over $K$ and $I$ the ideal of Laurent polynomials which vanish on $W$. Then the following sets coincide:*

(1) $\{\mathrm{val}(v) \in \Gamma^n \mid v \in W(K)\}$;
(2) $\{w \in \Gamma^n \mid \mathrm{in}_w(I) \neq \langle 1\rangle\}$.

**Proposition 2.5** ([MS15, Proposition 3.2.11])**.** *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety defined over $K$, $w \in \Gamma_n$. If $v \in \mathrm{in}_w(W)(k)$, then there is $v' \in W(K)$ such that* $\mathrm{val}(v') = w$ *and* $\mathrm{res}(v' \cdot \varphi(-w)) = v$.

Recall that a $\Gamma$-*rational polyhedron* is a subset $\tau$ of $\Gamma^n$ consisting of the elements $w$ which satisfy finitely many inequalities of the form $\langle u, w\rangle \leq \gamma$, for some $u \in \mathbb{Q}^n$ and $\gamma \in \Gamma$. A *face* of the polyhedron $\tau$ is a polyhedron $\tau'$ for which there exists some $\gamma \in \Gamma^n$ such that $\tau' = \{w \in \tau \mid \langle w, \gamma\rangle \leq \langle w', \gamma\rangle \, \forall w' \in \tau\}$. The *relative interior* of the polyhedron $\tau$, denoted $\mathrm{relint}(\tau)$, is the set of points in $\tau$ which do not lie in any (proper) face of $\tau$.

A $\Gamma$-*rational polyhedral complex* is a finite set $\Sigma$ of $\Gamma$-rational polyhedra that is closed under taking faces, and such that for all $\tau_1, \tau_2 \in \Sigma$ we have that $\tau_1 \cap \tau_2$ is a face of both $\tau_1$ and $\tau_2$ (and hence an element of $\Sigma$.) The *support* $\mathrm{Supp}(\Sigma)$ of the polyhedral complex $\Sigma$ is the union $\bigcup_{\tau \in \Sigma} \tau \subseteq \Gamma^n$.

**Theorem 2.6** ([MS15, Theorem 3.3.8])**.** *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety defined over $K$. The set* $\mathrm{val}(W(K))$ *is the support of a $\Gamma$-rational polyhedral complex, which we denote by* $\mathrm{Trop}(W)$ *and call the tropicalization of $W$.*

Hence we have $\mathrm{val}(W(K)) = \mathrm{Supp}(\mathrm{Trop}(W))$.

By [MS15, Proposition 3.2.8], the polyhedral complex can be chosen so that for all $w_1, w_2 \in \mathrm{Supp}(\mathrm{Trop}(W))$, if $w_1, w_2$ lie in the relative interior of the same polyhedron $\tau \in \mathrm{Trop}(W)$ then $\mathrm{in}_{w_1}(I) = \mathrm{in}_{w_2}(I)$. We fix such a choice, so we are allowed to write $W_\tau$ instead of $W_w$ when $w \in \mathrm{relint}(\tau)$.

**Lemma 2.7.** *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety defined over $K$, $I$ the ideal of Laurent polynomials vanishing on $W$.*

*Let $v \in W(K)$, $w = \mathrm{val}(v) \in \Gamma^n$, and let $W_\tau = \mathrm{in}_w(W)$, $\alpha = \varphi(w)$. Then*

$$\mathrm{in}_w(I) = \{\mathrm{res}(g) \mid g \in \mathcal{O}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],\ g|_{\alpha^{-1}\cdot W} = 0\}$$

*and*

$$W_\tau(k) = \mathrm{res}((\alpha^{-1} \cdot W(K)) \cap (\mathcal{O}^\times)^n).$$

*Proof.* We prove first the equality of the two sets of polynomials.

($\supseteq$) Let $g$ be a Laurent polynomial over $\mathcal{O}$ which vanishes on $\alpha^{-1} \cdot W$; then $g(\alpha^{-1} \cdot x) \in I$. If $\mathrm{res}(g) = 0$ then it is immediate that $\mathrm{res}(g) \in \mathrm{in}_w(I)$, so we assume $\mathrm{res}(g) \neq 0$, that is, $g$ has at least one coefficient of valuation 0.

Write $g = \sum_{u \in S} c_u x^u$. Then $g(\alpha^{-1} \cdot x) = \sum_{u \in S} (c_u \alpha^{-u} x^u)$. Let

$$S' := \{u \in S \mid \mathrm{val}(c_u \alpha^{-u}) + \langle u, w\rangle \leq \mathrm{val}(c_{u'} \alpha^{-u'}) + \langle u', w\rangle\, \forall u' \in S\}.$$

Since $\mathrm{val}(\alpha) = w$, we have that $u \in S'$ if and only if $\mathrm{val}(c_u) \leq \mathrm{val}(c_{u'})$ for all $u' \in S$. By assumption on $g$, this happens if and only if $\mathrm{val}(c_u) = 0$. Hence

$$\begin{aligned}
\mathrm{in}_w(g(\alpha^{-1} \cdot x)) &= \sum_{u \in S'} \mathrm{res}\left(c_u \alpha^u \varphi(-\mathrm{val}(c_u \alpha^u)) x^u\right) \\
&= \sum_{u \in S'} \mathrm{res}\left(c_u \varphi(-\mathrm{val}(c_u)) x^u\right) \\
&= \mathrm{res}\left(\sum_{u \in S} c_u x^u\right) \\
&= \mathrm{res}(g)
\end{aligned}$$

hence $\mathrm{res}(g) \in \mathrm{in}_w(I)$.

($\subseteq$) Let $f \in I$, and write $f = \sum_{u \in S} c_u x^u$. Let

$$S' := \{u \in S \mid \mathrm{val}(c_u) + \langle u, w\rangle \leq \mathrm{val}(c_{u'}) + \langle u', w\rangle\, \forall u' \in S\}.$$

Take some $u_0 \in S'$, and let $\lambda := c_{u_0} \alpha^{u_0}$.

Then, for all $u \in S$,

$$\begin{aligned}
\mathrm{val}\left(\frac{c_u \alpha^u}{\lambda}\right) &= \mathrm{val}(c_u) + \mathrm{val}\left(\alpha^u\right) - \mathrm{val}(\lambda) \\
&= \mathrm{val}(c_u) + \langle u, w\rangle - \mathrm{val}(\lambda) \\
&= \mathrm{val}(c_u) + \langle u, w\rangle - \mathrm{val}(c_{u_0}) - \langle u_0, w\rangle \\
&\geq 0
\end{aligned}$$

with equality if and only if $u \in S'$, and hence the coefficients of the polynomial

$$\begin{aligned}\frac{f(\alpha \cdot x)}{\varphi(\mathrm{val}(\lambda))} &= \frac{\sum_{u\in S} c_u(\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))} \\ &= \sum_{u\in S} \frac{c_u\alpha^u}{\varphi(\mathrm{val}(\lambda))} x^u\end{aligned}$$

have non-negative valuation, strictly positive for all $u \in S \setminus S'$. Then

$$\begin{aligned}\mathrm{res}\left(\frac{f(\alpha \cdot x)}{\varphi(\mathrm{val}(\lambda))}\right) &= \mathrm{res}\left(\frac{\sum_{u\in S} c_u(\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))}\right) \\ &= \mathrm{res}\left(\frac{\sum_{u\in S'} c_u(\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))}\right) + \mathrm{res}\left(\frac{\sum_{u\in S\setminus S'} c_u(\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))}\right) \\ &= \mathrm{res}\left(\sum_{u\in S'} \frac{c_u(\alpha\cdot x)^u}{\varphi(\mathrm{val}(\lambda))}\right) + 0 \\ &= \mathrm{res}\left(\sum_{u\in S'} c_u\varphi(-\mathrm{val}(c_u))x^u\right) \\ &= \mathrm{in}_w(f).\end{aligned}$$

Hence if $f \in I$, then $\mathrm{in}_w(f) = \mathrm{res}\left(\frac{f(\alpha\cdot x)}{\varphi(\mathrm{val}(\lambda))}\right)$, and $\frac{f(\alpha\cdot x)}{\varphi(\mathrm{val}(\lambda))}$ is a polynomial over $\mathcal{O}$ which vanishes on $\alpha^{-1}\cdot W$.

We now prove the second equality.

($\subseteq$) Let $v \in W_\tau(k)$. Then by Proposition 2.5, there is $v' \in W(K)$ such that $\mathrm{val}(v') = w$ and $\mathrm{res}\left(\frac{v'}{\varphi(w)}\right) = \mathrm{res}(\alpha^{-1}\cdot v') = v$, so $v \in \mathrm{res}\left(\alpha^{-1}\cdot W(K) \cap (\mathcal{O}^\times)^n\right)$.

($\supseteq$) Let $v \in \alpha^{-1}\cdot W(K)\cap(\mathcal{O}^\times)^n$, $f \in \mathrm{in}_w(I)$. By the first part of the proof there is $g \in \mathcal{O}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ such that $g$ vanishes on $\alpha^{-1}\cdot W$ (so in particular $g(v) = 0$) and $\mathrm{res}(g) = f$. Then, $f(\mathrm{res}(v)) = \mathrm{res}(g)(\mathrm{res}(v)) = \mathrm{res}(g(v)) = 0$. Thus $\mathrm{res}(v) \in W_\tau(k)$. □

**Definition 2.8.** Let $\tau \leq \Gamma^n$ be a polyhedron. The *affine span* of $\tau$, denoted $\mathrm{aff}(\tau)$, is the smallest affine subspace of $\Gamma^n$ containing $\tau$. We denote by $\mathrm{lin}(\tau)$ the linear subspace of $\Gamma^n$ that is parallel to $\mathrm{aff}(\tau)$.

**Definition 2.9.** Let $\tau \leq \Gamma^n$ be a polyhedron, $A \in M_{n\times d}(\mathbb{Z})$ a matrix with integer entries such that $\mathrm{aff}(\tau) = \{x \in \Gamma^n \mid Ax = b\}$ for some $b \in \Gamma^d$.

We denote by $J_\tau$ the connected component of the algebraic subgroup of $\mathbb{G}_m^n$ defined by $\{y \mid y^A = 1\}$.

**Lemma 2.10.** *Let $J \leq \mathbb{G}_m^n$ be an algebraic subgroup defined by $J := \{x \in \mathbb{G}_m^n \mid x^A = 1\}$ for some matrix $A \in M_{n,d}(\mathbb{Z})$.*

*Then* $\mathrm{Supp}(\mathrm{Trop}(J))$ *is the polyhedron $\tau$ defined by $\{x \in \Gamma^n \mid Ax = 0\}$.*

*Proof.* We use description (1) of the support of the tropicalization in Theorem 2.4, that is, $\mathrm{Supp}(\mathrm{Trop}(J)) = \mathrm{val}(J(K))$.

Let $v \in J(K)$; then, $v^A = 1$ implies that $A(\mathrm{val}(v)) = 0$, so $\mathrm{Supp}(\mathrm{Trop}(J)) \subseteq \tau$.

Conversely, if $w \in \tau$ then it satisfies $Aw = 0$, and thus $A\varphi(w) = 1$, so $\varphi(w) \in W(K)$. Since $\mathrm{val}(\varphi(w)) = w$, we get that $w \in \mathrm{Supp}(\mathrm{Trop}(J))$. □

In the next proof we will take initial forms of initial forms. To be able to do this, we see the residue field $k$ as a valued field with the trivial valuation. Hence, for $f = \sum_{u \in S} c_u x^u \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $w \in \Gamma^n$ we have that $\mathrm{in}_w(f) := \sum_{u \in S'} c_u x^u$ where $S' := \{u \in S \mid \langle u, w\rangle \leq \langle u', w\rangle \, \forall u' \in S\}$.

**Proposition 2.11.** *Let $W \subseteq \mathbb{G}_m^n$ be an algebraic subvariety, $\tau \in \mathrm{Trop}(W)$ a polyhedron. Then $J_\tau \leq \mathrm{Stab}(W_\tau)$.*

*Proof.* Let $I$ be the ideal of Laurent polynomials vanishing on $W$; let $f_0 \in \mathrm{in}_\tau(I)$. Let $w \in \mathrm{relint}(\tau)$, and let $f = \sum_{u \in S} c_u x^u$ be a polynomial such that $f_0 = \mathrm{in}_w(f)$ (which exists by Lemma 2.2). We denote by $S'$ the set

$$S' := \{u \in S \mid \mathrm{val}(c_u) + \langle u, w\rangle \leq \mathrm{val}(c_{u'}) + \langle u', w\rangle \, \forall u' \in S\}.$$

The set of points in $J_\tau$ which lie in a 1-dimensional algebraic subgroup of $J_\tau$ is Zariski-dense in $J_\tau$; hence, it is sufficient to check that $W_\tau$ is stabilized by these points, as the stabilizer is a (Zariski closed) algebraic subgroup of $\mathbb{G}_m^n$.

Assume then that $z \in J_\tau(k)$ lies in some 1-dimensional algebraic subgroup $H$ of $J_\tau$. Let $z_1 \in H(K)$ be a point with valuation $v \neq 0$. By Lemma 2.10 $\mathrm{val}(J_\tau(K)) = \mathrm{lin}(\tau)$, so $v \in \mathrm{lin}(\tau)$; since $w \in \mathrm{relint}(\tau)$, we may then choose $z_1$ so that $w + v \in \tau$. Arguing as in [MS15, Lemma 3.3.6] we have that, possibly after replacing $v$ by some other element of $\mathrm{Trop}(H)$ closer to 0, $\mathrm{in}_{w+v}(I) = \mathrm{in}_v(\mathrm{in}_w(I))$. Note that in that argument $\Gamma$ is assumed to be an $\mathbb{R}$-vector space rather than a divisible ordered abelian group, and so the result is stated in a slightly different form; however, since our $v$ varies in a 1-dimensional subgroup of $\Gamma^n$ it makes sense to say that we can take it "sufficiently close to 0", say after fixing an isomorphism between $\Gamma$ and $\mathrm{Supp}(\mathrm{Trop}(H))$, and the same proofs go through.

Let

$$S'' := \{u \in S' \mid \langle u, v\rangle \leq \langle u', v\rangle \, \forall u' \in S'\}.$$

For $v \in \Gamma^n$, we then have $\mathrm{in}_v(\mathrm{in}_w(f)) := \sum_{u \in S''} \mathrm{res}(c_u \varphi(-\mathrm{val}(c_u)) x^u$ (where the residue and valuation are taken with respect to the valued field structure on $K$).

Fix some $u_0 \in S''$: then for every $u \in S''$ we have $\langle u - u_0, v\rangle = 0$. By the choice of $v$, this implies that $z^{u-u_0} = 1$. Then we have

$$\begin{aligned} f_0(z \cdot x) &= \sum_{u \in S'} \mathrm{res}(c_u \varphi(-\mathrm{val}(c_u)) z^u x^u \\ &= z^{u_0} \left( \mathrm{in}_v(f_0) + \sum_{u \in S' \setminus S''} \mathrm{res}(c_u \varphi(-\mathrm{val}(c_u))) z^{u-u_0} x^u \right) \end{aligned}$$

If $S' \setminus S'' = \varnothing$, then we are done: $f_0(z \cdot x) = z^{u_0} \mathrm{in}_v(f_0) \in \mathrm{in}_v(\mathrm{in}_w(I)) = \mathrm{in}_{w+v}(I)$ and since $v \in \mathrm{lin}(\tau)$ and $w \in \mathrm{relint}(\tau)$, this implies $f_0(z \cdot x) \in \mathrm{in}_\tau(I)$.

Assume then that $S' \setminus S'' \neq \varnothing$. Then, we have $\mathrm{in}_v(f_0) \in \mathrm{in}_\tau(I)$ and $0 \neq f_0 - \mathrm{in}_v(f_0) \in \mathrm{in}_\tau(I)$. Hence, $\mathrm{in}_{w+v}(f_0 - \mathrm{in}_v(f_0)) \in \mathrm{in}_v(\mathrm{in}_w(I)) = \mathrm{in}_\tau(I)$, so in particular it is not a monomial; so, the set $S''' := \{u \in S' \setminus S'' \mid \langle u, v\rangle \leq \langle u', v\rangle \, \forall u' \in S\}$ has at least two elements.

Then, we get

$$\begin{aligned}
f_0(z\cdot x) - z^{u_0}\mathrm{in}_v(f_0) &= \sum_{u\in S'\setminus S''} \mathrm{res}(c_u\varphi(-\mathrm{val}(c_u)))z^u x^u \\
&= z^{u_1}\left(\sum_{u\in S'''} \mathrm{res}(c_u\varphi(-\mathrm{val}(c_u)))x^u\right) \\
&+ \sum_{u\in S'\setminus(S''\cup S''')} \mathrm{res}(c_u\varphi(-\mathrm{val}(c_u)))z^{u-u_1}x^u \\
&= z^{u_1}(\mathrm{in}_v(f_0-\mathrm{in}_v(f_0))) + \sum_{u\in S'\setminus(S''\cup S''')} \mathrm{res}(c_u\varphi(-\mathrm{val}(c_u)))z^{u-u_1}x^u
\end{aligned}$$

Again, if $S'\setminus(S''\cup S''')=\varnothing$ we are done: we get that $f_0(z\cdot x)-z^{u_0}\mathrm{in}_v(f)\in \mathrm{in}_\tau(I)$, and thus $f_0(z\cdot x)\in \mathrm{in}_\tau(I)$ as we wanted. Otherwise, we iterate the process as above; $S'$ is finite so it will terminate at some point. □

**Definition 2.12.** Let $\Sigma$ be a polyhedral complex, $\tau\in\Sigma$ a polyhedron. For every $\sigma\in\Sigma$ such that $\tau$ is a face of $\sigma$, consider the polyhedron $\bigcup_{x\in\sigma} x+\mathrm{lin}(\tau)$. These form a polyhedral complex $\Sigma'$, invariant under translation by $\mathrm{lin}(\tau)$.

The *star* of $\tau$ in $\Sigma$, denoted $\mathrm{star}_\Sigma(\tau)$, is the polyhedral complex $\Sigma'-w$ for any $w\in\mathrm{relint}(\tau)$.

**Proposition 2.13** ([MS15, Lemma 3.3.6])**.** *Let $\tau\in\mathrm{Trop}(W)$ be a polyhedron, $w\in\mathrm{relint}(\tau)$, $W_\tau$ the corresponding initial variety.*

*Then* $\mathrm{Trop}(W_\tau)=\mathrm{star}_\Sigma(\tau)$.

**Remark 2.14.** Every polyhedron $\sigma\in\mathrm{star}(\Sigma)(\tau)$ contains $\mathrm{lin}(\tau)$, and thus $\mathrm{Trop}(W_\tau)$ is invariant under translation by $\tau$. This is the tropical counterpart to Proposition 2.11.

Finally, we recall the definition of *amoeba*, and the fact that tropicalizations are limits of amoebas.

**Definition 2.15.** Let $\mathrm{Log}:(\mathbb{C}^\times)^n\to\mathbb{R}^n$ denote the map

$$(z_1,\dots,z_n)\mapsto(\log|z_1|,\dots,\log|z_n|).$$

Given an algebraic subvariety $W\subseteq\mathbb{G}_m^n$ defined over $\mathbb{C}$, the *amoeba* $\mathcal{A}_W$ of $W$ is the image of $W(\mathbb{C})$ under Log.

If $W\subseteq\mathbb{G}_m^n$ is defined over $\mathbb{C}$, we may see it as a subvariety of some algebraically closed valued field extension of $\mathbb{C}$, for example the Puiseux series field, with the trivial valuation induced on $\mathbb{C}$. In particular then the polyhedral complex $\mathrm{Trop}(W)$ will consist of polyhedra defined by equations of the form $Ax\le 0$, for $A\in\mathbb{Z}^n$, and as such we may look at the set $\mathrm{Supp}(\mathrm{Trop}(W))(\mathbb{R})$ of points in $\mathbb{R}^n$ satisfying the given conditions.

**Theorem 2.16** ([Jon16, Theorem A])**.** *Let $W\subseteq\mathbb{G}_m^n$ be an algebraic subvariety defined over $\mathbb{C}$. In the Hausdorff topology on subsets of $\mathbb{R}^n$, we have that*

$$\lim_{r\to+\infty}\frac{\mathcal{A}_W}{r}=\mathrm{Supp}(\mathrm{Trop}(W))(\mathbb{R}).$$

2.3. **Embedding $\mathbb{C}$ in a large valued field.** We use an approach based on nonstandard analysis: we embed $\mathbb{C}$ into a larger field, naturally endowed with a valuation. This field will contain infinite and infinitesimal elements, which intuitively correspond to the various order of growths of sequences in $\mathbb{C}$ going to infinity or 0; most of the statements and proofs in the next section could be formulated in the standard language, but that would make them far less readable.

Consider the model-theoretic structure $\mathbb{R}_{\exp,\sin}$, that is the field of real numbers in the language of ordered rings expanded by constants for all the elements of $\mathbb{R}$ and function symbols for exp and sin (here by sin we mean the **total** sine function, not the restricted version, so this structure is not o-minimal, and in fact not tame in any sense.) We may identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, and using exp and sin we may define the complex exponential function in this structure.

We denote by $\mathfrak{R}$ a proper elementary extension of $\mathbb{R}_{\exp,\sin}$, and by $\mathfrak{C} = \mathfrak{R} + i\mathfrak{R}$ its algebraic closure.

We may define an absolute value $|\cdot| : \mathfrak{C} \to \mathfrak{R}$ by setting $|x+iy| = \sqrt{x^2+y^2}$. We use this to define the *Archimedean valuation* on $\mathfrak{C}$ as the quotient by the relation $z_1 \sim z_2$ if and only if there is $n \in \mathbb{N}$ such that $n|z_1| \geq |z_2|$ and $|z_2| \geq n|z_1|$. We denote by $\Gamma$ the set $\mathfrak{C}/\sim$, and by $\mathrm{val} : \mathfrak{C} \twoheadrightarrow \Gamma$ the valuation. Endowing $\Gamma$ with the ordering induced by the reverse ordering on the absolute values of elements of $\mathfrak{C}$, and with the sum induced by the product on $\mathfrak{C}$, makes $\Gamma$ into an ordered abelian group and val into a valuation.

The field $\mathfrak{C}$ is then an algebraically closed valued field. By [MS15, Lemma 2.1.15] the valuation has a splitting, which as in the previous section we denote by $\varphi$. By composing with the absolute value, we may and will assume that $\varphi(\Gamma) \subseteq \mathfrak{R}_{>0}$.

Since $\mathfrak{R}$ is an elementary extension of $\mathbb{R}$ in the language with exp and sin, it has an exponential function $\mathfrak{R} \to \mathfrak{R}_{>0}$, which may be extended to an exponential $\mathfrak{C} \to \mathfrak{C}$ by the usual formula $\exp(x+iy) = \exp(x)(\cos(y) + i\sin(y))$. For every $r \in \mathfrak{R}$ we have a well-defined power function on the positive elements defined by $x \mapsto x^r := \exp(r\log(x))$, where log here means the (well-defined) function $\mathfrak{R}_{>0} \to \mathfrak{R}$.

We can make $\Gamma$ into an ordered $\mathfrak{R}$-vector space by setting $r \cdot \gamma = \mathrm{val}(\varphi(\gamma)^r)$ for $\gamma \in \Gamma$ and $r \in \mathfrak{R}$. This will allow us, given a vector subspace $L$ of $\mathfrak{R}^n$ defined by some matrix $A$ with entries in $\mathfrak{R}$, to look at the set of $\Gamma$-points of $L$, denoted by $L(\Gamma)$ and defined by the same matrix $A$. Note that $\dim_{\mathfrak{R}} \Gamma = 1$, because given $\gamma_1, \gamma_2 \in \Gamma$ we have that $\frac{\log(\varphi(\gamma_1)}{\log(\varphi(\gamma_2))} \cdot \gamma_2 = \mathrm{val}\left(\varphi(\gamma_2)^{\frac{\log(\varphi(\gamma_1))}{\log(\varphi(\gamma_2))}}\right) = \mathrm{val}(\varphi(\gamma_1)) = \gamma_1$.

The valuation ring $\mathcal{O}$ is the convex hull of the complex numbers, and the residue field is isomorphic to $\mathbb{C}$. The maximal ideal $\mathfrak{m}$ consists of the elements whose absolute value is smaller than any positive real. Any element $z \in \mathcal{O}$ can be written uniquely as $\mathrm{res}(z) + \varepsilon$, where $\mathrm{res}(z) \in \mathbb{C}$ is the complex number closest to $z$ and $\varepsilon \in \mathfrak{m}$. From this it is easy to see that for $z \in \mathcal{O}$ we have

$$\exp(z) = \exp(\mathrm{res}(z) + \varepsilon) = \exp(\mathrm{res}(z))\exp(\varepsilon) \in \exp(\mathrm{res}(z)) \cdot (1 + \mathfrak{m}) \subseteq \mathcal{O},$$

from which it follows that exp induces a function $\mathcal{O} \to \mathcal{O}^\times$ which satisfies $\operatorname{res}(\exp(z)) = \exp(\operatorname{res}(z))$.

2.4. **Rotundity.** Let $J$ be an algebraic subgroup of $\mathbb{G}_m^n$. We denote by $TJ$ its tangent bundle, which we identify with a subgroup of $\mathbb{G}_a^n \times \mathbb{G}_m^n$, and by $\pi_{TJ} : \mathbb{G}_a^n \times \mathbb{G}_m^n \twoheadrightarrow (\mathbb{G}_a^n \times \mathbb{G}_m^n)/TJ \cong \mathbb{G}_a^{n-\dim J} \times \mathbb{G}_m^{n-\dim J}$ the quotient mapping.

**Definition 2.17.** Let $V \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ an algebraic subvariety. $V$ is *rotund* if for any algebraic subgroup $J \leq \mathbb{G}_m^n$,

$$\dim \pi_{TJ}(V) \geq n - \dim J.$$

The following proposition gives some characterizations of rotundity for complex subvarieties of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ which split as the product $L \times W$ for some linear subspace $L \leq \mathbb{G}_a^n$ and algebraic subvariety $W \subseteq \mathbb{G}_m^n$. If $W$ is defined over $\mathbb{C}$, we may also see it as a subvariety of $\mathbb{G}_m^n$ defined over the algebraically closed valued field $\mathfrak{C}$, so that we may take its tropicalization.

**Proposition 2.18.** *Let $L \leq \mathbb{G}_{a,\mathbb{C}}^n$ be a linear subspace defined over $\mathbb{R}$, and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an algebraic subvariety such that $\dim L + \dim W = n$.*

*The following are equivalent:*

1. *$L \times W$ is rotund.*
2. *There is a point $w \in W$ such that the map $\delta : (L \times W)(\mathbb{C}) \to (\mathbb{C}^\times)^n$ defined by $(l, w) \mapsto \frac{w}{\exp(l)}$ is open in a (Euclidean) neighbourhood of $(l, w)$ for all $l \in L$.*
3. *There is a Zariski-open subset $W^\circ \neq \varnothing$ of $W$ such that the map $\delta : (L \times W^\circ)(\mathbb{C}) \to (\mathbb{C}^\times)^n$ defined by $(l, w) \mapsto \frac{w}{\exp(l)}$ is open.*
4. *$\mathcal{A}_W + \operatorname{Re}(L) = \mathbb{R}^n$.*
5. *$\bigcup_{\tau \in \operatorname{Trop}(W)} \tau(\Gamma) + \operatorname{Re}(L)(\Gamma) = \Gamma^n$.*
6. *There is $\tau \in \operatorname{Trop}(W)$ such that $\dim(\tau + \operatorname{Re}(L)(\Gamma)) = n$.*
7. *For all $z \in (\mathbb{C}^\times)^n$ there exist $z' \in z \cdot \mathbb{S}_1^n, \varepsilon > 0$ such that $B(z', \varepsilon) \subseteq \frac{W(\mathbb{C})}{\exp(L(\mathbb{C}))}$.*

*Proof.* For $(1 \Rightarrow 3)$ see [Kir19, Proposition 6.2 and Remark 6.3] and [Gal23, Proposition 3.7]. $(3 \Rightarrow 2)$ is obvious. $(2 \Rightarrow 1)$ is [Gal23, Proposition 3.8].

$(1 \Rightarrow 4)$ is [Gal23, Lemma 6.15].

We show that $(1 \Rightarrow 7)$.

Assume then that $L \times W$ is rotund, and let $W^\circ$ be the Zariski-open subset given by (3). Let $W' := W \setminus W^\circ$. Let $F : W \to \mathbb{C}$ be some non-zero algebraic function which vanishes on all points of $W'$.

Consider the algebraic subvariety $W_1$ of $(\mathbb{C}^\times)^{n+1}$ given by

$$\{(w_1, \ldots, w_{n+1}) \in (\mathbb{C}^\times)^{n+1} \mid (w_1, \ldots, w_n) \in (\mathbb{C}^\times)^n \wedge w_{n+1} = F(w_1, \ldots, w_n)\}.$$

Then $(L \times \mathbb{C}) \times W_1$ satisfies condition (3), with $W_1^\circ = W_1$; hence, $(L \times \mathbb{C}) \times W_1$ is rotund, so by (4), $\mathcal{A}_{W_1} - (\operatorname{Re}(L) \times \mathbb{R}) = \mathbb{R}^{n+1}$, which together with $W_1^\circ = W_1$ easily implies that $(L \times \mathbb{C}) \times W_1$ satisfies (7). But then $L \times W$ also does.

For $(7 \Rightarrow 2)$, note that, because of [Chi89, Proposition on p. 41 in Section 3.8], (7) implies that the map $L \times W \to (\mathbb{C}^\times)^n$ defined by $(l, w) \mapsto \frac{w}{\exp(l)}$ has a zero-dimensional fiber and this in turn implies (2) by the Open Mapping Theorem [GR12, p. 107]. ($5 \Rightarrow 6$) is obvious.

For ($4 \Rightarrow 5$) we use Theorem 2.16: if $\mathcal{A}_W + \mathrm{Re}(L) = \mathbb{R}^n$, then

$$\begin{aligned}\bigcup_{\tau \in \mathrm{Trop}(W)} \tau(\mathbb{R}) + \mathrm{Re}(L)(\mathbb{R}) &= \left(\lim_{r \to +\infty} \frac{\mathcal{A}_W}{r}\right) + \mathrm{Re}(L) \\ &= \lim_{r \to +\infty} \left(\frac{\mathcal{A}_W}{r} + \mathrm{Re}(L)\right) \\ &= \lim_{r \to +\infty} \left(\frac{\mathcal{A}_W + \mathrm{Re}(L)}{r}\right) \\ &= \mathbb{R}^n.\end{aligned}$$

Hence we have that the first-order sentence

$$\forall (x_1, \ldots, x_n) \exists \gamma \in \mathrm{Supp}(\mathrm{Trop}(W)), l \in \mathrm{Re}(L)\, (\gamma + l = (x_1, \ldots, x_n))$$

holds in the $\mathbb{R}$-vector space $\mathbb{R}$. Hence, since the theory of infinite ordered vector spaces over a fixed field is complete, it also holds in $\Gamma$.

Finally, let us prove that ($6 \Rightarrow 1$) (more precisely, we prove the contrapositive ($\neg 1 \Rightarrow \neg 6$)).

**Claim**: If $W$ has an initial variety $W_\tau$ such that $L \times W_\tau$ is rotund, then $L \times W$ is rotund.

**Proof of Claim**:

Assume that $L \times W_\tau$ is rotund for some $\tau \in \mathrm{Trop}(W)$, and let $w \in \mathrm{relint}(\tau)$. By Lemma 2.7, $W_\tau(\mathbb{C}) = \mathrm{res}((\varphi(w^{-1}) \cdot W(\mathfrak{C})) \cap (\mathcal{O}^\times)^n)$. Let $\Omega$ be a semialgebraic subset of some Cartesian power of $\mathbb{C}$, such that there exists a family $\{W_a \mid a \in \Omega\}$, such that $W_\tau = W_a$ for some $a \in \Omega(\mathbb{C})$, and $\varphi(w^{-1}) \cdot W(\mathfrak{C}) = W_b$ for some $b \in \Omega(\mathfrak{C})$. Shrinking $\Omega$ if necessary, we may find some $z_0 \in (\mathbb{C}^\times)^n$ and some $\varepsilon \in \mathbb{R}_{>0}$ such that the model-theoretic structure $\mathbb{C}$ (in the language $\mathcal{L}_{\exp,\sin}$) satisfies the first-order sentence

$$\forall z \in \Omega \left( B(z_0, \varepsilon) \subseteq \frac{W}{\exp(L)} \right).$$

Since $\mathfrak{C}$ is an elementary extension of $\mathbb{C}$, this also holds in $\mathfrak{C}$, and therefore $(\varphi(w^{-1}) \cdot W)/\exp(L)$ has non-empty interior; hence, $L \times W$ is rotund by Condition (2), again using [Chi89, Proposition on p. 41 in Section 3.8].

Let then $L \times W$ be a non-rotund variety. We show that

$$\dim \left( \bigcup_{\tau \in \mathrm{Trop}(W)} \tau(\Gamma) + \mathrm{Re}(L)(\Gamma) \right) \neq n$$

by induction on $\dim W$.

If $\dim W = 0$, then $\mathrm{Trop}(W) = \{\{0\}\}$ and $L = \mathbb{G}_{a,\mathbb{C}}^n$, so $\cup_{\tau \in \mathrm{Trop}(W)} \tau(\Gamma) + \mathrm{Re}(L)(\Gamma) = \mathrm{Re}(L)(\Gamma) = \Gamma^n$.

If $\dim W = d > 0$, let $\tau \in \mathrm{Trop}(W)$. Since $\dim W > 0$, we have $\dim L < n$ and so we can assume that $\dim \tau > 0$. By assumption and the above claim, $L \times W_\tau$ is not rotund. Let $J_\tau$ be the algebraic subgroup that $W_\tau$ is invariant by, and $LJ_\tau$ its Lie algebra. Then $\pi_{TJ_\tau}(L \times W_\tau)$ is also not rotund, and $\dim(\pi_{J_\tau}(W_\tau)) < \dim W$, so we can apply the induction hypothesis and get that

$$\dim\left(\bigcup_{\tau'\in\mathrm{Trop}(\pi_{J_\tau}(W_\tau))}\tau'(\Gamma)+\mathrm{Re}(\pi_{LJ_\tau}(L))(\Gamma)\right)\neq n-\dim J_\tau$$

and hence

$$\dim\left(\bigcup_{\tau'\in\mathrm{Trop}(W),\tau\subseteq\tau'}\tau'(\Gamma)+\mathrm{Re}(L)(\Gamma)\right)\neq n.$$

Making this vary among all $\tau$'s we obtain the result. □

## 3. Geometrical non-degeneracy

**Definition 3.1.** Let $W\subseteq\mathbb{G}_m^n$ be an irreducible algebraic subvariety. $W$ is *geometrically non-degenerate* if for any algebraic subgroup $J\leq\mathbb{G}_m^n$,

$$\dim\pi_J(W)=\min\{\dim W,n-\dim J\}.$$

**Notation 3.2.** For a field $k$, we denote by $G_k(d,n)$ the Grassmannian of linear subspaces $L\leq\mathbb{G}_a^n$ defined over $k$ of dimension $d$.

**Remark 3.3.** We will see the Grassmannian $G_{\mathbb{R}}(d,n)$ as a definable set in $\mathbb{R}_{\exp,\sin}$, by identifying each linear subspace $L$ with the matrix that defines it, seen as an element in $\mathbb{R}^{dn}$. Hence, $G_{\mathfrak{R}}(d,n)$ is the corresponding set in $\mathfrak{R}$.

It is straighforward to verify that if $W$ is defined over $\mathbb{C}$, geometrically non-degenerate and of codimension $d$ then $L\times W$ is rotund for all $L\in G_{\mathbb{R}}(d,n)$. Our goal in this section is to show that under these assumptions Condition 7 in Proposition 2.18 holds uniformly in $z$ and $L$: there is an $\varepsilon>0$ such that for all $z\in(\mathbb{C}^\times)^n$ and $L\in G_{\mathbb{R}}(d,n)$, there is $z'\in z\cdot\mathbb{S}_1^n(\mathbb{C})$ such that $B(z',\varepsilon)\subseteq\frac{W(\mathbb{C})}{\exp(L(\mathbb{C}))}$.

**Lemma 3.4.** *Let $L\leq\mathbb{G}_{a,\mathfrak{C}}^n$ be a linear subspace and let*

$$I=\left\{(a_1,\dots,a_n)\in\mathcal{O}^n\mid\sum_{i=1}^n a_ix_i=0\forall(x_1,\dots,x_n)\in L(\mathfrak{C})\right\}$$

*and*

$$\mathrm{res}(I)=\{b\in\mathbb{C}^n\mid b=\mathrm{res}(a),a\in I\}.$$

*Set $L(\mathcal{O})=L(\mathfrak{C})\cap\mathcal{O}^n$. Then*

$$\mathrm{res}(L(\mathcal{O}))=\left\{(v_1,\dots,v_n)\in\mathbb{C}^n\mid\sum_{i=1}^n b_iv_i=0\forall b=(b_1,\dots,b_n)\in\mathrm{res}(I)\right\}.$$

*Furthermore,* $\dim\mathrm{res}(L(\mathcal{O}))=\dim L$.

*Proof.* We induct on $\dim L$, the case $L=\{0\}$ being trivial.

If $L\neq\{0\}$, then $L$ contains some non-zero vector $v=(v_1,\dots,v_n)$. After rescaling and permuting the coordinates, we can assume without loss of generality that $v\in\{1\}\times\mathcal{O}^{n-1}$. There exists an invertible matrix $A\in\mathrm{GL}_n(\mathcal{O})$ such that $Av=(1,0,\dots,0)^t$.

Set $L' = A \cdot L$, then $L' = \mathfrak{C} \times L''$ for some linear subspace $L''$ of $\mathfrak{C}^{n-1}$ and $L'(\mathcal{O}) = A \cdot L(\mathcal{O})$. Furthermore, if

$$I' = \left\{ (a_1, \ldots, a_n) \in \mathcal{O}^n \mid \sum_{i=1}^{n} a_i x_i = 0 \forall (x_1, \ldots, x_n) \in L'(\mathfrak{C}) \right\},$$

then

$$I' = \{a \cdot A^{-1} \mid a \in I\}.$$

Furthermore, $I' \subseteq \{0\} \times \mathcal{O}^{n-1}$ and in fact $I' = \{0\} \times I''$, where

$$I'' = \left\{ (a_1, \ldots, a_{n-1}) \in \mathcal{O}^{n-1} \mid \sum_{i=1}^{n-1} a_i x_i = 0 \forall (x_1, \ldots, x_{n-1}) \in L''(\mathfrak{C}) \right\}.$$

By induction,

$$\mathrm{res}(L(\mathcal{O})) = \mathrm{res}(A)^{-1} \cdot \mathrm{res}(L'(\mathcal{O})) = \mathrm{res}(A)^{-1} \cdot (\mathbb{C} \times \mathrm{res}(L''(\mathcal{O})))$$

is the linear subspace defined by the linear equations with coefficient vectors in the set

$$\{(0, b) \cdot \mathrm{res}(A) \mid b \in \mathrm{res}(I'')\} = \{\mathrm{res}(a \cdot A) \mid a \in I'\} = \mathrm{res}(I).$$

Furthermore, $\dim \mathrm{res}(L(\mathcal{O})) = 1 + \dim \mathrm{res}(L''(\mathcal{O})) = 1 + \dim L'' = \dim L' = \dim L$. □

**Lemma 3.5.** *Let $L \leq \mathbb{G}_{a,\mathfrak{C}}^n$ be a linear subspace of dimension $d$ defined over $\mathfrak{R}$. Then* $\mathrm{val}(\exp(L(\mathfrak{C}))) = \mathrm{Re}(L)(\Gamma)$.

*Proof.* We fix a $(n-d) \times n$-matrix $A$ of maximal rank with coefficients in $\mathfrak{R}$ such that $L$ is the kernel of $A$. After dividing each row by its entry of minimal valuation, we can and will assume that $A$ has coefficients in $\mathcal{O} \cap \mathfrak{R}$.

We start by proving "$\subseteq$". Let $\ell \in L(\mathfrak{C})$. We want to show that

$$A \cdot \mathrm{val}(\exp(\ell)) = 0.$$

By definition

$$\mathrm{val}(\exp(\ell)) = \mathrm{val}(|\exp(\ell)|) = \mathrm{val}(\exp(\mathrm{Re}(\ell))).$$

We know that $A \cdot \mathrm{Re}(\ell) = 0$. By definition

$$\exp(\mathrm{Re}(\ell))^A = \exp(A \cdot \mathrm{Re}(\ell)) = \exp(0) = 1.$$

It remains to be shown that

$$\mathrm{val}(\exp(\mathrm{Re}(\ell))^A) = A \cdot \mathrm{val}(\exp(\mathrm{Re}(\ell))).$$

But by definition

$$A \cdot \mathrm{val}(\exp(\mathrm{Re}(\ell))) = \mathrm{val}(\varphi(\mathrm{val}(\exp(\mathrm{Re}(\ell))))^A)$$

and

$$\mathrm{val}(\varphi(\mathrm{val}(\exp(\mathrm{Re}(\ell)))) \cdot \exp(\mathrm{Re}(\ell))^{-1}) = 0.$$

Since the coefficients of $A$ are in $\mathcal{O} \cap \mathfrak{R}$, it follows that

$$\mathrm{val}(\varphi(\mathrm{val}(\exp(\mathrm{Re}(\ell))))^A \cdot \exp(\mathrm{Re}(\ell))^{-A}) = 0$$

or equivalently

$$A \cdot \mathrm{val}(\exp(\mathrm{Re}(\ell))) - \mathrm{val}(\exp(\mathrm{Re}(\ell))^A) = 0.$$

We next prove "$\supseteq$". We take $\gamma \in \mathrm{Re}(L)(\Gamma)$, so $A\cdot\gamma = 0$. We set $x = \varphi(\gamma)$. Then

$$\mathrm{val}(x^A) = A\cdot\gamma = 0,$$

so $x^A \in (\mathcal{O}^\times)^{n-d}$. It follows that there exists $y \in (\mathcal{O}^\times)^n$ such that $x\cdot y \in \exp(L(\mathfrak{C}))$ and so $\gamma = \mathrm{val}(x) = \mathrm{val}(x\cdot y) \in \mathrm{val}(\exp(L(\mathfrak{C})))$. $\square$

**Lemma 3.6.** *Let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an algebraic subvariety of codimension $d$ that is geometrically non-degenerate and let $L \leq \mathbb{G}_{a,\mathfrak{C}}^n$ be a linear subspace of dimension $d$ defined over $\mathfrak{R}$. Then* $\mathrm{Supp}(\mathrm{Trop}(W))(\Gamma) + \mathrm{Re}(L)(\Gamma) = \Gamma^n$.

*Proof.* We consider the two-sorted structure $(\mathfrak{R}, \Gamma)$, where $\Gamma$ is seen as a structure in the language of ordered vector spaces over a real closed field; the field itself is not fixed in this language, and therefore we may consider the substructure $(\mathbb{R}, \Gamma)$. Applying the Tarski-Vaught test [Hod97, Theorem 2.5.1] this is easily seen to be an elementary substructure, so every first-order sentence satisfied by $(\mathbb{R}, \Gamma)$ is also true in $(\mathfrak{R}, \Gamma)$.

Every polyhedron $\tau \in \mathrm{Trop}(W)$ is a definable subset of $\Gamma^n$ with parameters from $\Gamma$. Since $W$ is geometrically non-degenerate, for every $L' \leq \mathbb{G}_{a,\mathbb{C}}^n$ we have by Proposition 2.18(5) that $\mathrm{Supp}(\mathrm{Trop}(W))(\Gamma) + \mathrm{Re}(L')(\Gamma) = \Gamma^n$, and hence $(\mathbb{R}, \Gamma)$ satisfies the first-order sentence

$$\forall L' \in G_{\mathbb{R}}(d,n) \forall \gamma \in \Gamma \exists l \in L',\, x \in \bigcup_{\tau\in\mathrm{Trop}(W)} \tau \text{ s.t. } x + l = \gamma.$$

Since $(\mathfrak{R}, \Gamma)$ is an elementary extension of $(\mathbb{R}, \Gamma)$, this sentence also holds in $(\mathfrak{R}, \Gamma)$, so we are done. $\square$

**Proposition 3.7.** *Let $L \leq \mathbb{G}_{a,\mathfrak{C}}^n$ be a linear subspace of dimension $d$ defined over $\mathfrak{R}$ and $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ an algebraic subvariety such that $W$ is geometrically non-degenerate and* $\dim L + \dim W = n$.

*Then there is a finite set of polyhedra $S = \{\tau_1, \dots, \tau_k\} \subseteq \mathrm{Trop}(W)$ such that the following hold:*

1. *For all $z \in (\mathfrak{C}^\times)^n$ there is $z' \in z \cdot \exp(L(\mathfrak{C})) \cdot (\mathcal{O}^\times)^n \cap W(\mathfrak{C})$ such that $\mathrm{val}(z') \in \tau$ for some $\tau \in S$.*
2. *$\mathrm{res}(L) \times W_\tau$ is rotund for all $\tau \in S$.*

*Proof.* Since $W$ is geometrically non-degenerate, it follows from Lemma 3.6 that there exists $\tau \in \mathrm{Trop}(W)$ such that $\dim(\tau + \mathrm{Re}(L)(\Gamma)) = n$. Let

$$S := \{\tau \in \mathrm{Trop}(W) \mid \dim(\mathrm{Re}(L)(\Gamma) + \tau) = n\}.$$

Then for each $\tau \in S$ we have that $\dim(\mathrm{Re}(L)(\Gamma) + \tau) = n$, so using Proposition 2.18(6), we have that $\mathrm{res}(L) \times W_\tau$ is rotund.

Moreover, using Lemma 3.6, it is easy to verify that we have

$$\bigcup_{\tau\in S} \tau + \mathrm{Re}(L)(\Gamma) = \Gamma^n$$

so for every $z \in \mathfrak{C}^n$ there exists $\tau \in S$ such that $(\mathrm{val}(z) + \mathrm{Re}(L)(\Gamma)) \cap \tau \neq \varnothing$. Let $\alpha$ be a point in this intersection. By Lemma 3.5, there is $z_0 \in z\cdot\exp(L(\mathfrak{C}))$ such that $\mathrm{val}(z_0) = \alpha \in \tau$.

Since $\alpha \in \tau$, by Theorem 2.4, there exists $z' \in W(\mathfrak{C})$ with $\mathrm{val}(z') = \alpha$. It follows that $\mathrm{val}\left(\frac{z_0}{z'}\right) = \mathrm{val}\left(\frac{z'}{z_0}\right) = 0$ and so $\frac{z'}{z_0} \in (\mathcal{O}^\times)^n$. Therefore

$$z' = \frac{z'}{z_0} \cdot z_0 \in z \cdot \exp(L(\mathfrak{C})) \cdot (\mathcal{O}^\times)^n$$

and we are done. □

**Lemma 3.8.** *Let $L \leq \mathbb{G}_{a,\mathbb{C}}^n$ be a linear subspace of dimension $d$ defined over $\mathbb{R}$ and let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an algebraic subvariety of codimension $d$. Let $\tau \in \mathrm{Trop}(W)$ such that $L \times W_\tau$ is rotund. Then, for every $\tau' \in \mathrm{Trop}(W)$ such that $\tau' \subseteq \tau$, the variety $L \times W_{\tau'}$ is rotund.*

*Proof.* Every polyhedron in $\mathrm{Trop}(W_\tau)$ contains $\tau$. By Proposition 2.11, $W_\tau$ is invariant under translation by $J_\tau$, which implies that each of its irreducible components is.

By rotundity of $L \times W_\tau$,

$$\begin{aligned} n - \dim J_\tau &\leq \dim \pi_{TJ_\tau}(L \times W_\tau) \\ &\leq \dim L + \dim W - \dim J_\tau \\ &= n - \dim J_\tau \end{aligned}$$

As rotundity is preserved by taking quotients, $\pi_{TJ_\tau}(L \times W_\tau)$ is rotund; by the above it has dimension $n - \dim J_\tau$.

Then there is a polyhedron $\sigma_0 \in \mathrm{Trop}(\pi_{J_\tau}(W_\tau))$ such that

$$\dim \sigma_0 + \mathrm{Re}(\pi_{LJ_\tau}(L))(\Gamma) = n - \dim J_\tau$$

and then there is $\sigma \supseteq \tau$ in $\mathrm{Trop}(W_\tau)$ such that $\dim \sigma + \mathrm{Re}(L)(\Gamma) = n$.

Now let $\tau' \subseteq \tau$ be in $\mathrm{Trop}(W)$. By Proposition 2.13, $\mathrm{Trop}(W_{\tau'}) = \mathrm{star}_{\mathrm{Trop}(W)}(\tau')$. Since $\tau' \subseteq \tau \subseteq \sigma$, $\mathrm{Trop}(W_{\tau'})$ contains then a face $\sigma'$ that has affine span parallel to $\sigma$. Then $\sigma' + \mathrm{Re}(L)$ has dimension $n$, so $L \times W_{\tau'}$ satisfies Proposition 2.18(6) and is rotund. □

**Theorem 3.9.** *Let $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ be an algebraic subvariety of codimension $d$ that is geometrically non-degenerate.*

*There is $\varepsilon > 0$ such that for every linear subspace $L \in G_{\mathbb{R}}(d, n)$ and every $z \in (\mathbb{C}^\times)^n$ there is $s \in \mathbb{S}_1^n(\mathbb{C})$ such that $B(s, \varepsilon) \subseteq \frac{z^{-1} \cdot W(\mathbb{C})}{\exp(L(\mathbb{C}))}$.*

*Proof.* Let $L \in G_{\mathfrak{R}}(d, n)$. We will show that for all $z \in (\mathfrak{C}^\times)^n$, there are $s \in \mathbb{S}_1^n(\mathfrak{C})$ and $\varepsilon \in \mathbb{R}_{>0}$ such that $B(s, \varepsilon) \subseteq \frac{z^{-1} \cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))}$.

Let $z \in (\mathfrak{C}^\times)^n$. Then by Proposition 3.7, there is $z' \in z \cdot \exp(L(\mathfrak{C})) \cdot (\mathcal{O}^\times)^n \cap W(\mathfrak{C})$ such that $\alpha := \mathrm{val}(z') \in \tau$ for some $\tau \in \mathrm{Trop}(W)$ and $\mathrm{res}(L) \times W_\tau$ is rotund. There exists $\tau' \in \mathrm{Trop}(W)$ such that $\tau' \subseteq \tau$ and $\alpha \in \mathrm{relint}(\tau')$. By Lemma 3.8, also $\mathrm{res}(L) \times W_{\tau'}$ is rotund and so, after replacing $\tau$ by $\tau'$, we can assume without loss of generality that $\mathrm{val}(z') \in \mathrm{relint}(\tau')$. Since $z' \in z \cdot \exp(L) \cdot (\mathcal{O}^\times)^n$, there exists $\ell \in L(\mathfrak{C})$ such that $y = z' \cdot z^{-1} \cdot \exp(-\ell) \in (\mathcal{O}^\times)^n$ and so

$$\mathrm{res}(y) \in (\mathbb{C}^\times)^n.$$

We deduce that

$$\frac{z^{-1} \cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))} = \frac{z'^{-1} \cdot y \cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))}.$$

By Proposition 2.18(7), there are $s \in \mathbb{S}_1^n(\mathbb{C})$ and $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B(s,\varepsilon) \subseteq \frac{\operatorname{res}(\varphi(\alpha)z'^{-1}y)\cdot W_\tau(\mathbb{C})}{\exp(\operatorname{res}(L)(\mathbb{C}))} = \operatorname{res}\left(\frac{\left(W(\mathfrak{C})\cdot z'^{-1}\cdot y\right)\cap(\mathcal{O}^\times)^n}{\exp(L(\mathcal{O}))}\right)$$

(the equality holds by Lemmas 2.7 and 3.4).

Let $\Omega$ be an open semialgebraic subset of some Cartesian power of $\mathbb{C}$ such that there is a family of algebraic subvarieties $\{W_a \mid a \in \Omega\}$, definable over $\mathbb{C}$ in the structure $\mathbb{R}_{\text{exp,sin}}$, such that $W_a = \operatorname{res}(\varphi(\alpha)z'^{-1}y)\cdot W_\tau$ for some $a \in \Omega$. Let $\Omega'$ be an open semialgebraic subset of some Cartesian power of $\mathbb{R}$ such that there is a family $\{L_b \mid b \in \Omega'\}$ of linear subspaces, definable over $\mathbb{R}$ in $\mathbb{R}_{\text{exp,sin}}$, such that $L_b = \operatorname{res}(L)$ for some $b \in \Omega'$. We see $\Omega$ and $\Omega'$ as definable sets, so we can consider $\Omega(\mathbb{C})$, $\Omega(\mathfrak{C})$, $\Omega'(\mathbb{R})$ and $\Omega'(\mathfrak{R})$. Restricting $\Omega$ and $\Omega'$ if necessary, we may assume:

(1) that for all $(a,b) \in \Omega(\mathbb{C})\times\Omega'(\mathbb{R})$,
$$B\left(s,\frac{\varepsilon}{2}\right) \subseteq \frac{W_a(\mathbb{C})}{\exp(L_b(\mathbb{C}))}.$$
(2) that there exist $a \in \Omega(\mathfrak{C})$ such that $W_a = W\cdot z'^{-1}\cdot y$ and $b \in \Omega'(\mathfrak{R})$ such that $L_b = L$.

Let $\theta = \theta(x)$ be the first-order formula, with parameters from $\mathbb{C}$,

$$\forall (a,b) \in \Omega\times\Omega'\left(B\left(x,\frac{\varepsilon}{2}\right) \subseteq \frac{W_a}{\exp(L_b)}\right).$$

By item (1) above, $\theta(s)$ holds true in $\mathbb{R}_{\text{exp,sin}}$. Since $\mathfrak{R}$ is an elementary extension, $\theta(s)$ also holds in the bigger structure; hence, by item (2),

$$B\left(s,\frac{\varepsilon}{2}\right) \subseteq \frac{z'^{-1}\cdot y\cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))} = \frac{z^{-1}\cdot W(\mathfrak{C})}{\exp(L(\mathfrak{C}))}.$$

Then the model-theoretic structure $\mathfrak{R}$ satisfies the first-order sentence

$$\exists\varepsilon > 0\left(\forall z \in \mathbb{G}_m^n \forall L \in G(d,n)\left(\exists s \in \mathbb{S}_1^n\left(B(s,\varepsilon) \subseteq \frac{z^{-1}\cdot W}{\exp(L)}\right)\right)\right)$$

because any infinitesimal element is a witness for it. Hence, this sentence must also hold true in $\mathbb{R}_{\text{exp,sin}}$, completing the proof. □

**Remark 3.10.** If $W \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ is not geometrically non-degenerate then there is $L \leq \mathbb{G}_a^n$ defined over $\mathbb{Q}$ such that $\frac{W}{\exp(L)}$ has empty interior, so there is no such $\varepsilon$. Therefore the converse of Theorem 3.9 holds as well, giving a characterization of geometrically non-degenerate subvarieties of $\mathbb{G}_{m,\mathbb{C}}^n$ in a similar spirit as the characterization of rotundity in Proposition 2.18.

## 4. An equidistribution argument

In this section, we will apply the following well-known equidistribution theorem for Galois orbits of torsion points in algebraic tori. We denote the set of roots of unity in $\mathbb{C}^\times$ by $\mu_\infty$, and the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ by $\overline{\mathbb{Q}}$. For an element $z = (z_1,\dots,z_n) \in (\mathbb{C}^\times)^n$ and a set $S$ of field automorphisms of $\mathbb{C}$, we define

$$S\cdot z = \{\sigma(z) \mid \sigma \in S\},$$

where $\sigma(z)$ is defined to be

$$(\sigma(z_1), \ldots, \sigma(z_n))$$

for a field automorphism $\sigma$ of $\mathbb{C}$.

**Theorem 4.1.** *Let $K \subset \mathbb{C}$ be a subfield that is finitely generated over $\mathbb{Q}$, let $n \in \mathbb{N}$, and let $B \subset (\mathbb{C}^\times)^n$ be some open Euclidean ball such that $B \cap \mathbb{S}_1^n(\mathbb{C}) \neq \emptyset$.*

*Let $(\zeta_j)_{j\in\mathbb{N}}$ be a sequence in $\mu_\infty^n$ such that for every algebraic subgroup $G \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$, the set of $j \in \mathbb{N}$ such that $\zeta_j \in G(\mathbb{C})$ is finite.*

*Then there exists $N \in \mathbb{N}$ such that for all $j \in \mathbb{N}$:*

$$j \geq N \implies \mathrm{Aut}(\mathbb{C}/K) \cdot \zeta_j \cap B \cap \mathbb{S}_1^n(\mathbb{C}) \neq \emptyset.$$

*Proof.* Set $L = \overline{\mathbb{Q}} \cap K$. Then $L$ is a finite extension of $\mathbb{Q}$ by [Isa09, Theorem 24.9] and the homomorphism $\mathrm{Aut}(\mathbb{C}/K) \to \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ is surjective since $\mathrm{Gal}(\overline{\mathbb{Q}}K/K) \to \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ is surjective by [Lan02, Chapter VI, Theorem 1.12] and $\mathrm{Aut}(\mathbb{C}/K) \to \mathrm{Gal}(\overline{\mathbb{Q}}K/K)$ is surjective as well. Hence, we can assume without loss of generality that $K$ is a finite extension of $\mathbb{Q}$.

The theorem then follows from Bilu's equidistribution theorem; see [Bil97], where the theorem is formulated over $\mathbb{Q}$, and see [Küh22] for a proof of a much more general statement over an arbitrary number field. □

**Lemma 4.2.** *Let $K \subset \mathbb{C}$ be a subfield that is finitely generated over $\mathbb{Q}$, let $n \in \mathbb{N}$, and let $\varepsilon > 0$.*

*Let $\mathscr{N} : \mu_\infty^n \to \mathbb{Z}_{>0}$ be defined by*

$$\mathscr{N}(\underline{x}) = \min\{\|\underline{N}\|_1; \underline{N} \in \mathbb{Z}^n \backslash \{0\} \text{ such that } \underline{x}^{\underline{N}} = 1\}.$$

*There exists $N = N(n, K, \varepsilon) \in \mathbb{Z}_{>0}$ such that for every $\xi \in \mu_\infty^n$ with $\mathscr{N}(\xi) > N$, the set $\mathrm{Aut}(\mathbb{C}/K) \cdot \xi$ intersects every open Euclidean ball of radius $\varepsilon$ centered at a point of $\mathbb{S}_1^n(\mathbb{C})$.*

*Proof.* We argue by contradiction and assume that the lemma is false. Hence, there is a sequence $(\xi_j)_{j\in\mathbb{N}}$ in $\mu_\infty^n$ and a sequence of open Euclidean balls $B_j$ ($j \in \mathbb{N}$) of radius $\varepsilon$, each centered at some point of $\mathbb{S}_1^n(\mathbb{C})$, such that $\lim_{j\to\infty} \mathscr{N}(\xi_j) = \infty$ and

$$\mathrm{Aut}(\mathbb{C}/K) \cdot \xi_j \cap B_j = \emptyset$$

for all $j \in \mathbb{N}$.

Since $\mathbb{S}_1^n(\mathbb{C})$ is compact, we can find finitely many open Euclidean balls $\widetilde{B}_1, \ldots, \widetilde{B}_M$ in $(\mathbb{C}^\times)^n$ centered at points of $\mathbb{S}_1^n(\mathbb{C})$ and of radius $\varepsilon/2$ such that

$$\mathbb{S}_1^n(\mathbb{C}) \subseteq \widetilde{B}_1 \cup \cdots \cup \widetilde{B}_M.$$

For each $j \in \mathbb{N}$, there exists an $i(j) \in \{1, \ldots, M\}$ such that $\widetilde{B}_{i(j)}$ contains the center of $B_j$. It follows that $\widetilde{B}_{i(j)} \subseteq B_j$.

We obtain that *a fortiori*

$$\mathrm{Aut}(\mathbb{C}/K) \cdot \xi_j \cap \widetilde{B}_{i(j)} = \emptyset$$

for all $j \in \mathbb{N}$. After passing to a subsequence of $(\xi_j)_{j\in\mathbb{N}}$, we can assume that $i(j) = i_0$ for some $i_0 \in \{1, \ldots, M\}$ and all $j \in \mathbb{N}$.

It follows that

$$\mathrm{Aut}(\mathbb{C}/K) \cdot \xi_j \cap \widetilde{B}_{i_0} = \emptyset$$

for all $j \in \mathbb{N}$. At the same time, we deduce from $\lim_{j\to\infty} \mathscr{N}(\xi_j) = \infty$ that for every algebraic subgroup $G \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$, the set of $j \in \mathbb{N}$ such that $\xi_j \in G(\mathbb{C})$ is finite. We have found a contradiction with Theorem 4.1, which finishes the proof. $\square$

**Lemma 4.3.** *Let $n \in \mathbb{N}$, let $K \subset \mathbb{C}$ be a subfield that is finitely generated over $\mathbb{Q}$, and let $\varepsilon > 0$. There exists a finite set $\mathcal{G} = \{G_1, \ldots, G_N\}$, depending only on $n$, $K$, and $\varepsilon$, such that $G_i \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ is an algebraic subgroup for all $i = 1, \ldots, N$ and such that for every subtorus $J \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ and every torsion point $\zeta \in \mathbb{G}_{m,\mathbb{C}}^n(\mathbb{C})$, one of the following holds:*

- *(i) for every Euclidean ball of radius $\varepsilon$ centered at a point of $\mathbb{S}_1^n(\mathbb{C})$, the intersection of $\mathbb{S}_1^n(\mathbb{C}) \cap \sigma(\zeta) \cdot J(\mathbb{C})$ and the ball is not empty for some $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ or*
- *(ii) $\zeta \cdot J \subseteq G_i$ for some $i \in \{1, \ldots, N\}$.*

*Proof.* Let $N = N(n, K, \varepsilon)$ be the positive integer provided by Lemma 4.2.

Let $J$ be a subtorus of $\mathbb{G}_{m,\mathbb{C}}^n$ and let $\zeta \in \mathbb{G}_{m,\mathbb{C}}^n(\mathbb{C})$ be a torsion point. Since $\zeta \cdot J(\mathbb{C}) \cap \mu_\infty^n$ is Zariski dense in $\zeta J$, it follows that either $\zeta J$ itself is contained in an algebraic subgroup defined by an equation $\underline{x}^{\underline{N}} = 1$ with $\underline{N} \in \mathbb{Z}^n$ such that $0 < \|\underline{N}\|_1 \le N$ or there is a point $\xi \in \zeta \cdot J(\mathbb{C}) \cap \mu_\infty^n$ with $\mathscr{N}(\xi) > N$. In the second case, it follows from Lemma 4.2 that the first alternative in the conclusion of Lemma 4.3 holds (note that the set $J(\mathbb{C})$ is invariant under any field automorphism of $\mathbb{C}$). We are now done by setting

$$\mathcal{G} = \{G;\ G \text{ algebraic subgroup defined by an equation } \underline{x}^{\underline{N}} = 1 \text{ with } \underline{N} \in \mathbb{Z}^n \text{ such that } 0 < \|\underline{N}\|_1 \le N\}.$$

$\square$

*Proof of Theorem 1.1.* By Theorem 3.9, there exists $\varepsilon > 0$ such that for every $z \in (\mathbb{C}^\times)^n$ and every subtorus $J \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim J + \dim W \ge n$, the image of the map

$$\psi_{W,J,z} : W(\mathbb{C}) \times J(\mathbb{C}) \to (\mathbb{C}^\times)^n, \quad (w, y) \mapsto wyz$$

contains a Euclidean ball of radius $\varepsilon$ centered at a point of $\mathbb{S}_1^n(\mathbb{C})$.

Let $x \in (\mathbb{C}^\times)^n$ be an arbitrary point. Note that

$$x \in \psi_{W,J,1}(W(\mathbb{C}) \times J(\mathbb{C})) \Leftrightarrow 1 \in \psi_{W,J,x^{-1}}(W(\mathbb{C}) \times J(\mathbb{C})).$$

We know that the image of $\psi_{W,J,x^{-1}}$ contains a Euclidean ball of radius $\varepsilon$ centered at a point of $\mathbb{S}_1^n(\mathbb{C})$. Since

$$J(\mathbb{C}) \cdot \psi_{W,J,x^{-1}}(W(\mathbb{C}) \times J(\mathbb{C})) = \psi_{W,J,x^{-1}}(W(\mathbb{C}) \times J(\mathbb{C})),$$

the theorem follows from Lemma 4.3 applied with $\zeta = (1, \ldots, 1)$. $\square$

*Proof of Theorem 1.2.* By Theorem 3.9, there exists $\varepsilon > 0$ such that for every $z \in (\mathbb{C}^\times)^n$ and every subtorus $J \subseteq \mathbb{G}_{m,\mathbb{C}}^n$ with $\dim J + \dim W \ge n$, the image of the map

$$\psi_{W,J,z} : W(\mathbb{C}) \times J(\mathbb{C}) \to (\mathbb{C}^\times)^n, \quad (w, y) \mapsto wyz$$

contains a Euclidean ball of radius $\varepsilon$ centered at a point of $\mathbb{S}_1^n(\mathbb{C})$.

Let now $J$ be a fixed subtorus of $\mathbb{G}_{m,\mathbb{C}}^n$ such that $\dim J + \dim W \geq n$ and let $\zeta \in \mathbb{G}_{m,\mathbb{C}}^n(\mathbb{C})$ be a torsion point. We deduce that the image of $\psi_{W,J,1}$ contains a Euclidean ball of radius $\varepsilon$ centered at a point of $\mathbb{S}_1^n(\mathbb{C})$. We have

$$W(\mathbb{C}) \cap \zeta \cdot J(\mathbb{C}) \neq \emptyset \Leftrightarrow 1 \in \psi_{W,J,\zeta^{-1}}(W(\mathbb{C}) \times J(\mathbb{C})).$$

The subvariety $W$ of $\mathbb{G}_{m,\mathbb{C}}^n$ is defined by equations with coefficients in some field $K \subseteq \mathbb{C}$ that is finitely generated over $\mathbb{Q}$. Since

$$z^{-1} \cdot J(\mathbb{C}) \cdot \psi_{W,J,1}(W(\mathbb{C}) \times J(\mathbb{C})) = \psi_{W,J,z^{-1}}(W(\mathbb{C}) \times J(\mathbb{C}))$$

for all $z \in (\mathbb{C}^\times)^n$, it follows from Lemma 4.3 that either $\zeta \cdot J$ is contained in one of finitely many algebraic subgroups $G \subsetneq \mathbb{G}_{m,\mathbb{C}}^n$ or $W(\mathbb{C}) \cap \sigma(\zeta) \cdot J(\mathbb{C}) \neq \emptyset$ for some $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. We can assume without loss of generality that the second alternative holds.

But since $W$ is defined over $K$, the set $W(\mathbb{C})$ is invariant under $\sigma^{-1}$ (and the same holds for $J(\mathbb{C})$). It follows that $W(\mathbb{C}) \cap \zeta \cdot J(\mathbb{C}) \neq \emptyset$ and we are done. $\square$


## Acknowledgements

TODO.

The first author thanks Thomas Scanlon for helpful discussions.

Gabriel Dill has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement n° 945714).

This material is based upon work supported by the National Science Foundation under Grant No. DMS–1928930 while the first author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2023 semester. The first author also thanks the DFG for its support (grant no. EXC-2047/1 - 390685813). The second author was partially supported by the program GeoMod ANR-19-CE40-0022-01 (ANR-DFG) while in Freiburg and by the PRIN 2022-Modelli, Insiemi e Classificazioni while in Pisa.


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