

## Modelltheorie II

### Homework Sheet 3

Deadline: 15.05.2023, 14 Uhr

**Exercise 1** (4 Points). Consider a sufficiently saturated ambient model  $\mathbb{U}$  of a complete first-order theory with infinite models in a fixed language  $\mathcal{L}$ .

Given an instance  $\varphi[\bar{x}, \bar{a}]$  with parameters in  $\mathbb{U}$ , assume that there are a natural number  $k \geq 2$  and a sequence  $(\psi_n)_{n \in \mathbb{N}}$  of definable subsets of  $\varphi[\bar{x}, \bar{a}]$  with parameters in  $\mathbb{U}$ , each of Morley rank at least  $\alpha$ , such that the intersection of  $k$  many instances among the  $\psi_n$ 's has Morley rank strictly less than  $\alpha$ , that is,

$$\text{RM}\left(\bigcap_{j=1}^k \psi_{n_j}\right) < \alpha$$

for every distinct  $n_1, \dots, n_k$ . Show that  $\varphi[\bar{x}, \bar{a}]$  has Morley rank at least  $\alpha + 1$ .

**Hint:** A ranked definable set is the finite disjoint union of its irreducible components.

**Exercise 2** (3 Points). Consider a sufficiently saturated ambient model  $\mathbb{U}$  of a complete first-order theory with infinite models in a fixed language  $\mathcal{L}$ . Two tuples  $\bar{a}$  and  $\bar{b}$  of the same length  $n$  have the same *strong type* over  $C$  if they belong to the same class with respect to every  $C$ -definable equivalence relation on  $\mathbb{U}^n \times \mathbb{U}^n$  with only finitely many equivalence classes.

- If  $\bar{a}$  and  $\bar{b}$  have the same strong type over  $C$ , show that  $\bar{a} \equiv_C \bar{b}$ .
- Show that all elements of an indiscernible sequence  $(a_n)_{n \in \mathbb{N}}$  have the same strong type over  $C$ .

**Exercise 3** (6 Points). Consider the language  $\mathcal{L} = \{P_n\}_{n \in \mathbb{N}}$ , with each  $P_n$  a unary predicate, and the  $\mathcal{L}$ -Theory  $T$  of *independent predicates* whose models are exactly the  $\mathcal{L}$ -structures  $\mathcal{A}$  such that for every collection of indexes  $n_1, \dots, n_r$  and  $m_1, \dots, m_s$  with  $\{n_i\}_{i=1}^r \cap \{m_j\}_{j=1}^s = \emptyset$ , the definable set

$$\bigcap_{i=1}^r P_{n_i}^{\mathcal{A}} \cap \bigcap_{j=1}^s A \setminus P_{m_j}^{\mathcal{A}}$$

is infinite.

- Does  $T$  have Vaught pairs?
- Does  $T$  eliminate  $\exists^\infty$ ?
- For which cardinals  $\kappa \geq \aleph_0$  is the theory  $\kappa$ -stable?
- Describe explicitly the types with ordinal-valued Morley rank over an  $\aleph_0$ -saturated model of  $T$ .

**Hint:** Prove inductively on  $\alpha$  a certain property of formulae first.

**Please turn over the page!!**

DIE ÜBUNGSBLÄTTER KÖNNEN ZU ZWEIT EINGEREICHT WERDEN. ABGABE DER ÜBUNGSBLÄTTER IM FACH 3.07 IM KELLER DES MATHEMATISCHEN INSTITUTS.

**Exercise 4** (7 Points). The language  $\mathcal{L} = \{f, P, Q, P_n \mid 2 \leq n \in \mathbb{N}\}$  consists of a unary function symbol  $f$ , as well as unary predicates  $P$ ,  $Q$  and  $P_n$  for every  $n \geq 2$  in  $\mathbb{N}$ .

We consider the  $\mathcal{L}$ -theory  $T$  whose models are exactly the  $\mathcal{L}$ -structures  $\mathcal{A}$  such that the universe is a disjoint union of  $P^{\mathcal{A}}$  and  $Q^{\mathcal{A}}$ ; the function  $f^{\mathcal{A}}$  restricted to the infinite set  $P^{\mathcal{A}}$  is the identity and maps  $Q^{\mathcal{A}}$  surjectively onto  $P^{\mathcal{A}}$ ; moreover, the set  $P_n^{\mathcal{A}}$  consists of the elements  $a$  in  $P$  whose fibre  $(f^{\mathcal{A}})^{-1}(a)$  has size exactly  $n$ . Notice that  $T$  is complete and has quantifier elimination.

- a) Show that  $T$  is totally transcendental.
- b) Compute the Morley ranks and degrees of  $P(y)$  and  $Q(x)$ .
- c) Compute the Morley rank of  $f^{-1}(a)$  where  $a$  realizes the type of maximal Morley rank in  $P$ .
- d) Is Morley rank additive in  $T$ ?

**Hint:**  $f(b) = a$ .