

Modelltheorie II

Homework Sheet 2

Deadline: 08.05.2023, 14 Uhr

Exercise 1 (5 Points). Consider the language $\mathcal{L} = \{E\}$ with E a binary relation symbol as well as the \mathcal{L} -theory T whose models are exactly the \mathcal{L} -structures in which E is an equivalence relation with exactly one class of size n for every natural number $n \geq 1$.

- Compute the maximal Cantor-Bendixson rank of a type in the compact Hausdorff space $S_1(\mathcal{M}_0)$, where \mathcal{M}_0 is the prime model of T .
- Compute the Morley rank and degree of the formula $(x \dot{=} x)$.
- Consider now an \aleph_0 -saturated model \mathcal{M} of T and a type p over M of maximal Morley rank. Given a proper elementary extension \mathcal{N} of \mathcal{M} , describe explicitly all the extensions q of p over N with $\text{RM}(q) < \text{RM}(p)$.

Exercise 2 (11 Points). We work inside a sufficiently saturated infinite ambient model \mathbb{U} of a complete first-order theory T with infinite models in a fixed language \mathcal{L} .

- Given a subset A of \mathbb{U} and a type $p(\bar{x})$ over A , show that $\text{CB}(p) \leq \text{RM}(p)$, where the Cantor-Bendixson rank is computed in the compact Hausdorff space $S_{|\bar{x}|}(A)$.
- Assume the instance $\varphi[\bar{x}, \bar{a}]$ has Morley rank α and is α -indecomposable. Show that so is $\varphi[\bar{x}, \bar{b}]$ for every realization \bar{b} of $\text{tp}(\bar{a})$.

Let now \mathcal{M} be an \aleph_0 -saturated elementary substructure of \mathbb{U} .

- Assume now that the instance $\varphi[\bar{x}, \bar{a}]$ of Morley rank α has Morley degree n , and that \bar{a} lies in \mathcal{M} . Show that there is a decomposition of $\varphi[\bar{x}, \bar{a}]$ into n disjoint α -indecomposable instances, each of Morley rank α , with parameters in \mathcal{M} .
- Conclude that every type over M of Morley rank α is *stationary*, that is, of Morley degree 1.
- Show by transfinite induction that $\text{RM}(p) = \text{CB}(p)$, where $\text{CB}(p)$ is the Cantor-Bendixson rank of p computed in $S_{|\bar{x}|}(\mathcal{M})$.

Hint: Every formula belongs to a type with the same Morley rank.

Exercise 3 (4 Points). A natural number m is a prefix of n in \mathbb{N} if m consists of the first digits of n , that is, if $n = m \cdot 10^k + r$ for some $0 \leq r < 10^k$ in \mathbb{N} .

Consider now an infinite subset $A \subset \mathbb{N}$ closed under removing last digits, that is, whenever n belongs to A and has last digit ℓ , then $\frac{n-\ell}{10}$ is also in A . Show that A contains an infinite sequence $(a_n)_{n \in \mathbb{N}}$ of distinct elements such that a_m is a prefix of a_n if $m \leq n$.

Hint: Trees.