Hence the number of covariants of degree s and order q of the binary quantic of order n is the coefficient of $\rho^{q+1} x^s$ in (1), or the coefficient of ρ^{q+1} in (2).

We may obtain by this method a proof of the formula for the canonical form of an invariant matrix of a matrix with repeated characteristic roots*, without the use of Aitken's method of chains†.

Let A be the matrix $\begin{bmatrix} \mu, & 1 \\ 0, & \mu \end{bmatrix}$. Denote by $[n]_\mu$ the matrix of order n^2 with μ in each position in the leading diagonal, unity in each position in the diagonal next above, and zero elsewhere, so that $A = [2]_\mu$.

Clearly, confining our attention to the canonical form, we have

$$A^{[n]} = [n+1]_{\mu^n}.$$

Then, if
$$[A^{[n]}]^{[s]} = \Sigma\, k_{pq} A^{[p,\, q]},$$

we have
$$[n+1]^{[s]}_{\mu^n} = \Sigma\, k_{pq} [p-q+1]_{\mu^{ns}},$$

and
$$[n+1]^{[s]}_{\mu} = \Sigma\, k_{pq} [p-q+1]_{\mu^s},$$

and the above generating function is applicable. The proof now follows the lines of the preceding paper*.

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THE FINAL PROBLEM: AN ACCOUNT OF THE MOCK THETA FUNCTIONS

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It is not unnatural in one who has held office in this Society for sixteen years that his mode of approach to the preparation of his valedictory Address should have taken the form of an investigation into the procedure of his similarly situated predecessors.

Of the thirty-five previous Presidents, all but three have delivered Addresses on resigning office. Two of the exceptions were the first two Presidents, de Morgan and Sylvester; de Morgan, however, had had his

* D. E. Littlewood, *Proc. London Math. Soc.* (2), 40 (1936), 370–381.

† A. C. Aitken, *Proc. London Math. Soc.* (2), 38 (1935), 354–376.

‡ Presidential Address delivered at the meeting of 14 November, 1935.

say in an inaugural speech at the first meeting of the Society. The third exception was Henrici, who confined himself to thanking the members for "the kind indulgence they had shown him", a sentiment which I would wish to echo to-day.

It had occurred to me that the survival of the Society for seventy years might make an historical topic appropriate for my Address; and, in fact, that a President at a loss for a subject might do far worse than give some account of the Addresses of his remoter predecessors. I was, however, deterred from this course by the examples of those two Presidents, Prof. Love and Prof. Hardy, who, like me, had held the office of Secretary for a lengthy period*; after making two statements on the progress of the Society, I shall follow them by confining myself to a mathematical topic.

In 1928 Prof. Hardy was able to report that the membership had grown to 410 and the annual output to 1,280 pages; the number of members is now 440 and the annual output is 1,440 pages. Each increase in our rate of publication in the last sixteen years has been followed by an increase in the number of papers received, until at last the inelasticity of our financial resources has led the Council reluctantly to announce the adoption of measures tending to limit the amount which we accept for publication.

The topic which I have selected, though unfortunately not too well adapted for oral exposition, will, I hope, be considered to be as characteristic of its author as the choices of most of my predecessors have been. I make no apologies for my subject being what is now regarded as old-fashioned, because, as a friend remarked to me a few months ago, I am an old-fashioned mathematician. Practically everything that I have to say to-night would be immediately comprehensible to Gauss or Jacobi; on the other hand, Euler, though he might enjoy listening, would probably encounter difficulties both of form and substance.

Early in 1920, three months before his death, Ramanujan wrote his last letter to Hardy. In the course of it he said: "I discovered very interesting functions recently which I call 'Mock' ϑ-functions. Unlike the 'False' ϑ-functions (studied partially by Prof. Rogers in his interesting paper†) they enter into mathematics as beautifully as the ordinary ϑ-functions. I am sending you with this letter some examples".

The study of some of the five foolscap pages of notes which accompanied the letter is the subject which I have chosen for my Address; I doubt

* With the exception of Prof. Burnside (Secretary 1902, 1903; President 1906, 1907) no other person has held both offices.

† L. J. Rogers, *Proc. London Math. Soc.* (2), 16 (1917), 315–336. A "false ϑ-function" is a function such as $1-q+q^2-q^5+q^7-\ldots$, which differs from an ordinary theta function in the signs of alternate terms.

whether a more suitable title could be found for it than the title used by John H. Watson, M.D., for what he imagined to be his final memoir on Sherlock Holmes.

The first three pages in which Ramanujan explained what he meant by a "mock ϑ-function" are very obscure. They will be made clearer if I preface them by Hardy's comment that a mock ϑ-function is a function defined by a q-series convergent when $|q|<1$, for which we can calculate asymptotic formulae, when q tends to a "rational point" $e^{2r\pi i/s}$ of the unit circle, of the same degree of precision as those furnished for the ordinary ϑ-functions by the theory of linear transformation.

The three pages of explanation (with a few modifications of the formulae to simplify the type-setting) are as follows:

"If we consider a ϑ-function in the transformed Eulerian form, *e.g.*

$$\text{(A)} \qquad 1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\ldots,$$

$$\text{(B)} \qquad 1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\ldots,$$

and determine the nature of the singularities at the points

$$q=1, \quad q^2=1, \quad q^3=1, \quad q^4=1, \quad q^5=1, \quad \ldots,$$

we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance, when $q=e^{-t}$ and $t\to 0$,

$$\text{(A)} = \sqrt{\left(\frac{t}{2\pi}\right)}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}{}^{*}\right)+o(1)\dagger,$$

$$\text{(B)} = \sqrt{\left(\frac{2}{5-\sqrt{5}}\right)}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}{}^{*}\right)+o(1),$$

and similar results at other singularities.

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* It is not necessary that there should be only one term like this. There may be many terms but *the number of terms must be finite.*

† Also $o(1)$ may turn out to be $O(1)$. That is all. For instance, when $q\to 1$, the function $\{(1-q)(1-q^2)(1-q^3)\ldots\}^{-120}$ is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B), it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance, when $q = e^{-t}$ and $t \to 0$,

$$\text{(C)}\quad 1+\frac{q}{(1-q)^2}+\frac{q^3}{(1-q)^2(1-q^2)^2}+\frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\dots$$

$$= \sqrt{\left(\frac{t}{2\pi\sqrt{5}}\right)} \exp\left[\frac{\pi^2}{5t}{}^{*}+a_1{}^{\dagger}t+a_2t^2+\dots+O(a_kt^k)\right],$$

where $a_1 = 1/8\sqrt{5}$, and so on. The function (C) is a simple example of a function behaving in an unclosed form at the singularities".

At this point I interpose a few explanatory remarks.

The "Eulerian form" of a function apparently refers to the character of the denominators of the terms in the series. The phrase is probably suggested by Euler's formulae

$$\prod_{n=0}^{\infty}(1+q^n z) = \sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n-1)}z^n}{(1-q)(1-q^2)\dots(1-q^n)},$$

$$\prod_{n=0}^{\infty}(1-q^n z)^{-1} = \sum_{n=0}^{\infty}\frac{z^n}{(1-q)(1-q^2)\dots(1-q^n)}.$$

As regards the illustrative functions, (A) is immediately derivable from Heine's formula for basic hypergeometric series,

$${}_2\Phi_1\left[\begin{matrix}a, b; & c\\ c & ab\end{matrix}\right] = \prod_{n=0}^{\infty}\left[\frac{(1-cq^n/a)(1-cq^n/b)}{(1-cq^n)\{1-cq^n/(ab)\}}\right],$$

by making $a \to \infty$, $b \to \infty$, and $c \to q$. It is thus seen that (A) is the partition function

$$\prod_{n=0}^{\infty}(1-q^{n+1})^{-1}.$$

The function (B) is $G(q)$, one of the two functions, $G(q)$ and $H(q)$, which occur in the Rogers-Ramanujan identities. These identities and the

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* The coefficient $1/t$ (*sic*) in the index of e happens to be $\pi^2/5$ in this particular case. It may be some other transcendental numbers in other cases.

† The coefficients of t, t^2, ... happen to be $1/8\sqrt{5}$, ... in this case. In other cases they may turn out to be some other algebraic numbers.

definitions of the functions are contained in the assertions that

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$= [(1-q)(1-q^4)(1-q^6)(1-q^9)\dots]^{-1},$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$= [(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots]^{-1}.$$

The asymptotic formulae for the functions are derivable from known properties of ϑ-functions, the formula for $H(q)$, which corresponds to the formula already quoted for $G(q)$, being

$$H(q) = \sqrt{\left(\frac{2}{5+\sqrt{5}}\right)} \exp\left(\frac{\pi^2}{15t} + \frac{11t}{60}\right) + o(1).$$

The function (C) is more remarkable. It is easy to obtain a first approximation to the value of the function when t is small by the method of estimating the sum of the terms in the neighbourhood of the greatest term of the series*; but the term $t/8\sqrt{5}$, still more the following terms, cannot be determined very satisfactorily in this manner. The only simple procedure which I have devised depends upon making a preliminary transformation of the function.

It is easy to see that

$$\sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m+1)}}{(1-q)^2(1-q^2)^2\dots(1-q^m)^2}$$

$$= \prod_{r=0}^{\infty} (1-q^{r+1})^{-1} . \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m+1)}}{(1-q)(1-q^2)\dots(1-q^m)} (1-q^{m+1})(1-q^{m+2})\dots$$

$$= \prod_{r=0}^{\infty} (1-q^{r+1})^{-1} . \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}m(m+1)} . (-)^n q^{\frac{1}{2}n(n+1)+mn}}{(1-q)(1-q^2)\dots(1-q^m) . (1-q)(1-q^2)\dots(1-q^n)}.$$

We sum this double series by diagonals. The sum of the terms for which $m+n$ has a given odd value, say $2p+1$, is zero; the sum of the terms for which $m+n$ has a given even value, say $2p$, is

$$\frac{q^{p(2p+1)}}{(1-q^2)(1-q^4)\dots(1-q^{2p})}.$$

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* Perhaps this method played a larger part in Ramanujan's work on mock ϑ-functions than my own studies of the subject would suggest.

We thus find that the function (C) is equal to

$$\prod_{r=0}^{\infty} (1-q^{r+1})^{-1} G_{\frac{1}{2}}(q^2),$$

where $G_\mu(q)$ denotes the function

$$\sum_{n=0}^{\infty} \frac{q^{n(n+\mu)}}{(1-q)(1-q^2)\dots(1-q^n)},$$

which occurs in Ramanujan's proof of his identities. Now, for integral values of μ, it can easily be shown that

$$G_\mu(q) = \frac{1}{\sqrt[4]{5}} \left(\frac{\sqrt{5}-1}{2}\right)^{\mu-\frac{1}{2}} \exp\left[\frac{\pi^2}{15t} + \frac{(5\mu^2-\mu)-(\mu^2-\mu)\sqrt{5}-\frac{1}{3}}{20}\, t + O(t^2)\right],$$

from a consideration of the difference equation

$$G_\mu(q) = G_{\mu+1}(q) + q^{\mu+1} G_{\mu+2}(q),$$

combined with the asymptotic values of $G_0(q) \equiv G(q)$ and $G_1(q) \equiv H(q)$. If we make the plausible assumption that the formula is valid for any fixed μ, whether an integer or not, we can obtain Ramanujan's asymptotic formula for his function (C) by taking $\mu = \frac{1}{2}$.

I now revert to Ramanujan's notes; they continue thus:

"Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say: Suppose there is a function in the Eulerian form and suppose that all or an infinity of points are exponential singularities, and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: Is the function taken the sum of two functions one of which is an ordinary ϑ-function and the other a (trivial) function which is $O(1)$ at *all* the points $e^{2m\pi i/n}$? The answer is *it is not necessarily so*. When it is not so, I call the function a Mock ϑ-function. I have not proved rigorously that *it is not necessarily so*. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function. Also I have shown that if *it is necessarily so* then it leads to the following assertion:— viz. it is possible to construct two power series in x, namely $\Sigma a_n x^n$ and $\Sigma b_n x^n$, both of which have *essential singularities* on the unit circle, are convergent when $|x| < 1$, and tend to *finite limits at every point* $x = e^{2r\pi i/s}$, and that at the same time the limit of $\Sigma a_n x^n$ at the point $x = e^{2r\pi i/s}$ is equal to the limit of $\Sigma b_n x^n$ at the point $x = e^{-2r\pi i/s}$.

This assertion seems to be untrue. Anyhow, we shall go to the examples and see how far our assertions are true.

I have proved that, if

$$f(q) = 1+\frac{q}{(1+q)^2}+\frac{q^4}{(1+q)^2(1+q^2)^2}+\ldots,$$

then

$$f(q)+(1-q)(1-q^3)(1-q^5)\ldots(1-2q+2q^4-2q^9+\ldots) = O(1)$$

at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1$, ...; and at the same time

$$f(q)-(1-q)(1-q^3)(1-q^5)\ldots(1-2q+2q^4-2q^9+\ldots) = O(1)$$

at all the points $q^2=-1$, $q^4=-1$, $q^6=-1$, ... Also, obviously, $f(q)=O(1)$ at all the points $q=1$, $q^3=1$, $q^5=1$, ... And so $f(q)$ is a Mock ϑ-function.

When $q=-e^{-t}$ and $t\to 0$,

$$f(q)+\sqrt{\left(\frac{\pi}{t}\right)}\exp\left(\frac{\pi^2}{24t}-\frac{t}{24}\right)\to 4.$$

The coefficient of q^n in $f(q)$ is

$$(-1)^{n-1}\frac{\exp\{\pi\sqrt{(\frac{1}{6}n-\frac{1}{144})}\}}{2\sqrt{(n-\frac{1}{24})}}+O\left(\frac{\exp\{\frac{1}{2}\pi\sqrt{(\frac{1}{6}n-\frac{1}{144})}\}}{\sqrt{(n-\frac{1}{24})}}\right).$$

It is inconceivable that a single ϑ-function could be found to cut out the singularities of $f(q)$".

This completes Ramanujan's general description of mock ϑ-functions. His remarks about lack of rigorous proof indicate that he was not completely convinced that the functions which he had constructed actually cannot be expressed in terms of ϑ-functions and "trivial" functions. It would therefore seem that his work on the transformation theory of mock ϑ-functions did not lead him to the precise formulae (such as I shall describe presently) for transformations of mock ϑ-functions of the third order. The precise forms of the transformation formulae make it clear that the behaviour of mock ϑ-functions near the unit circle is of a more complex character than that of ordinary ϑ-functions.

The subsequent results about $f(q)$ which I have quoted are all immediate consequences of my transformation formulae, except for the approximation for the general coefficient of the expansion of $f(q)$ in power series. I have not troubled to verify this approximation; it is presumably derivable from the transformation formulae in the manner in which Hardy and Ramanujan* obtained the corresponding formula for $p(n)$, the number of partitions of n.

The last two pages of Ramanujan's notes consist of lists of definitions of four sets of mock ϑ-functions with statements of relations connecting members of each of the first three sets; for fairly obvious reasons the functions in the various sets are described as being of orders 3, 5, 5, and 7 respectively. These lists and statements have already been published†.

On this occasion I propose to restrict myself to the consideration of functions of order 3. In addition to the function $f(q)$ defined above, Ramanujan has discovered three such functions. Rather strangely‡ he seems to have overlooked the existence of the set of functions which I call $\omega(q)$, $\nu(q)$, $\rho(q)$.

The definitions of the complete set of functions are as follows:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \dots (1+q^n)^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4) \dots (1+q^{2n})},$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^3) \dots (1-q^{2n-1})},$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4) \dots (1-q^n+q^{2n})},$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2(1-q^5)^2 \dots (1-q^{2n+1})^2},$$

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1+q)(1+q^3) \dots (1+q^{2n+1})},$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6) \dots (1+q^{2n+1}+q^{4n+2})}.$$

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* *Proc. London Math. Soc.* (2), 17 (1918), 75–115.

† *Collected papers of S. Ramanujan* (1927), 354–355.

‡ Particularly in view of his having discovered both sets of functions of order 5.

These several functions are connected by the following relations:

$$2\phi(-q)-f(q)=f(q)+4\psi(-q)=\vartheta_4(0,\,q)\prod_{r=1}^{\infty}(1+q^r)^{-1},$$

$$4\chi(q)-f(q)=3\vartheta_4{}^2(0,\,q^3)\prod_{r=1}^{\infty}(1-q^r)^{-1},$$

$$2\rho(q)+\omega(q)=3[\tfrac{1}{2}q^{-\frac{3}{8}}\vartheta_2(0,\,q^{\frac{3}{2}})]^2\prod_{r=1}^{\infty}(1-q^{2r})^{-1},$$

$$v(\pm q)\pm q\omega(q^2)=\tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,\,q)\prod_{r=1}^{\infty}(1+q^{2r}),$$

$$f(q^8)\pm 2q\omega(\pm q)\pm 2q^3\omega(-q^4)=\vartheta_3(0,\,\pm q)\,\vartheta_3{}^2(0,\,q^2)\prod_{r=1}^{\infty}(1-q^{4r})^{-2}.$$

Whether Ramanujan's proofs of the relations involving $f(q)$, $\phi(q)$, and $\psi(q)$ are the same as mine must remain unknown.

The first stage in my discussion of the functions consists in obtaining new definitions of the functions by transforming the series by which they are defined into series more amenable to manipulation. For this purpose I use a limiting case of a general formula connecting basic hypergeometric series which I discovered some years ago* in the course of the construction of the seventh proof of the Rogers-Ramanujan identities.

If we write

$${}_r\Phi_s\left[\begin{matrix}\alpha,\,\beta,\,\gamma,\,\ldots;\;x\\ \delta,\,\epsilon,\,\ldots;\;\end{matrix}\right]=1+\sum_{n=1}^{\infty}\prod_{m=0}^{n-1}\left\{\frac{(1-\alpha q^m)(1-\beta q^m)(1-\gamma q^m)\ldots}{(1-q^{m+1})(1-\delta q^m)(1-\epsilon q^m)\ldots}\right\}x^n,$$

where r is the number of the symbols $\alpha,\,\beta,\,\gamma,\,\ldots$, and s is the number of the symbols $\delta,\,\epsilon,\,\ldots$, the general formula is

$$\begin{aligned}{}_8\Phi_7&\left[\begin{matrix}a,\;q\sqrt{a},\;-q\sqrt{a},\;c,\;d,\;e,\;f,\;g;\;\dfrac{a^2q^2}{cdefg}\\ \sqrt{a},\;-\sqrt{a},\;aq/c,\;aq/d,\;aq/e,\;aq/f,\;aq/g;\end{matrix}\right]\\ &=\prod_{n=1}^{\infty}\left[\frac{\{1-aq^n\}\{1-aq^n/(fg)\}\{1-aq^n/(ge)\}\{1-aq^n/(ef)\}}{\{1-aq^n/e\}\{1-aq^n/f\}\{1-aq^n/g\}\{1-aq^n/(efg)\}}\right]\\ &\qquad\times{}_4\Phi_3\left[\begin{matrix}aq/(cd),\;e,\;f,\;g;\;q\\ efg/a,\;aq/c,\;aq/d;\end{matrix}\right],\end{aligned}$$

provided that $e,f,$ or g is of the form q^{-N}, where N is a positive integer.

* G. N. Watson, *Journal London Math. Soc.*, 4 (1929), 4–9.

Make $a \to 1$, $e \to \infty$, $f \to \infty$, $g \to \infty$; and let

$$c = \exp i\theta, \quad d = \exp(-i\theta);$$

we find that*

$$1 + \sum_{n=1}^{\infty} \frac{(-)^n (1+q^n)(2-2\cos\theta)\, q^{\frac{1}{2}n(3n+1)}}{1-2q^n\cos\theta+q^{2n}}$$

$$= \prod_{r=1}^{\infty} (1-q^r) \left[1 + \sum_{n=1}^{n} \frac{q^{n^2}}{\prod_{m=1}^{n} (1-2q^m\cos\theta+q^{2m})} \right].$$

This is the general relation which is fundamental in the construction of the new definitions of the mock ϑ-functions. In this relation take successively

$$\theta = \pi, \quad \theta = \tfrac{1}{2}\pi, \quad \theta = \tfrac{1}{3}\pi,$$

and we get immediately

$$f(q) \prod_{r=1}^{\infty} (1-q^r) = 1 + 4 \sum_{n=1}^{\infty} \frac{(-)^n q^{\frac{1}{2}n(3n+1)}}{1+q^n},$$

$$\phi(q) \prod_{r=1}^{\infty} (1-q^r) = 1 + 2 \sum_{n=1}^{\infty} \frac{(-)^n (1+q^n)\, q^{\frac{1}{2}n(3n+1)}}{1+q^{2n}},$$

$$\chi(q) \prod_{r=1}^{\infty} (1-q^r) = 1 + \sum_{n=1}^{\infty} \frac{(-)^n (1+q^n)\, q^{\frac{1}{2}n(3n+1)}}{1-q^n+q^{2n}}.$$

These are the formulae which will henceforth be adopted as the definitions of $f(q)$, $\phi(q)$, and $\chi(q)$. They render obvious the connection between $\chi(q)$ and $f(q)$; for, by combining like terms of the series on the right, we have

$$[4\chi(q) - f(q)] \prod_{r=1}^{\infty} (1-q^r) = 3 \left[1 + 4 \sum_{n=1}^{\infty} \frac{(-)^n q^{\frac{3}{2}n(n+1)}}{1+q^{3n}} \right],$$

and the required relation follows from the formula†

$$1 + 4 \sum_{n=1}^{\infty} \frac{(-)^n q^{\frac{1}{2}n(n+1)}}{1+q^n} = \vartheta_4^2(0, q).$$

* This formula may also be obtained by expressing the series on the right (*qua* function of $\cos\theta$) as a sum of partial fractions.

† This simple formula (well known to Ramanujan) apparently is not given explicitly by Tannery and Molk; to obtain it, take $z = \frac{1}{2}\pi$ in the expression for $1/\vartheta_1(z, \sqrt{q})$ as a sum of partial fractions. *Cf.* J. Tannery et J. Molk, *Théorie des fonctions elliptiques*, 3 (1898), 136.

The new definition of $\psi(q)$ is not such a direct consequence of my formula. To transform $\psi(q)$, take $\exp i\theta = q^{\frac{1}{2}}$ and then replace q by q^4; we thus have

$$1+\sum_{n=1}^{\infty} \frac{(-)^n(1+q^{4n})(2-q-q^{-1})q^{2n(3n+1)}}{(1-q^{4n-1})(1-q^{4n+1})}$$

$$= \prod_{r=1}^{\infty}(1-q^{4r})\left[1+\sum_{n=1}^{\infty}\frac{q^{4n^2}}{(1-q^3)(1-q^5)(1-q^7)\dots(1-q^{4n+1})}\right],$$

that is to say

$$1+\sum_{n=1}^{\infty}(-)^n q^{2n(3n+1)}\left\{\frac{1-q^{-1}}{1-q^{4n-1}}+\frac{1-q}{1-q^{4n+1}}\right\}$$

$$= \prod_{r=1}^{\infty}(1-q^{4r})\left[1+\sum_{n=1}^{\infty}\frac{q^{4n^2}(1-q^{4n+1})+q^{(2n+1)^2}}{(1-q^3)(1-q^5)\dots(1-q^{4n+1})}\right]$$

$$= \prod_{r=1}^{\infty}(1-q^{4r})\left[1+\sum_{m=2}^{\infty}\frac{q^{m^2}}{(1-q^3)(1-q^5)\dots(1-q^{2m-1})}\right]$$

$$(m = 2n \text{ or } 2n+1)$$

$$= \prod_{r=1}^{\infty}(1-q^{4r})[1-q+(1-q)\psi(q)].$$

Hence we have

$$\psi(q)\prod_{r=1}^{\infty}(1-q^{4r})$$

$$= \frac{1}{1-q}+\sum_{n=1}^{\infty}(-)^n q^{2n(3n+1)}\left\{\frac{1}{1-q^{4n+1}}-\frac{q^{-1}}{1-q^{4n-1}}\right\}-\prod_{r=1}^{\infty}(1-q^{4r})$$

$$= \frac{1}{1-q}-1+\sum_{n=1}^{\infty}(-)^n q^{2n(3n+1)}\left\{\frac{1}{1-q^{4n+1}}-\frac{q^{-1}}{1-q^{4n-1}}-(1+q^{-4n})\right\}$$

$$= \frac{q}{1-q}+\sum_{n=1}^{\infty}(-)^n q^{2n(3n+1)}\left\{\frac{q^{4n+1}}{1-q^{4n+1}}+\frac{q^{1-8n}}{1-q^{-4n+1}}\right\}$$

$$= \sum_{n=-\infty}^{\infty}\frac{(-)^n q^{6n(n+1)+1}}{1-q^{4n+1}}.$$

We accordingly adopt the formula

$$\psi(q)\prod_{r=1}^{\infty}(1-q^{4r}) = \sum_{n=-\infty}^{\infty}\frac{(-)^n q^{6n(n+1)+1}}{1-q^{4n+1}}$$

as the modified definition of $\psi(q)$.

We next take another limiting case of the formula connecting basic hypergeometric series. Make $e\to\infty$, $f\to\infty$, $g\to\infty$, and let $a=q$ and $cd=q$, writing

$$c=q^{\frac{1}{2}}\exp i\theta, \quad d=q^{\frac{1}{2}}\exp(-i\theta).$$

On reduction we find that

$$\sum_{n=0}^{\infty}\frac{(-)^n(1-q^{2n+1})\,q^{\frac{3}{2}n(n+1)}}{1-2q^{n+\frac{1}{2}}\cos\theta+q^{2n+1}}=\prod_{r=1}^{\infty}(1-q^r)\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\prod_{m=0}^{n}(1-2q^{m+\frac{1}{2}}\cos\theta+q^{2m+1})}.$$

In this relation take successively

$$\theta=0, \quad \theta=\tfrac{1}{2}\pi, \quad \theta=\tfrac{2}{3}\pi;$$

and at the same time replace q by q^2 in the first and third of the results which are obtained.

We get immediately

$$\omega(q)\prod_{r=1}^{\infty}(1-q^{2r})=\sum_{n=0}^{\infty}(-)^n q^{3n(n+1)}\frac{1+q^{2n+1}}{1-q^{2n+1}},$$

$$\upsilon(q)\prod_{r=1}^{\infty}(1-q^{r})=\sum_{n=0}^{\infty}(-)^n q^{\frac{3}{2}n(n+1)}\frac{1-q^{2n+1}}{1+q^{2n+1}},$$

$$\rho(q)\prod_{r=1}^{\infty}(1-q^{2r})=\sum_{n=0}^{\infty}(-)^n q^{3n(n+1)}\frac{1-q^{4n+2}}{1+q^{2n+1}+q^{4n+2}}.$$

These are the formulae which will be adopted as the definitions of $\omega(q)$, $\upsilon(q)$, and $\rho(q)$. They render obvious the connection between $\rho(q)$ and $\omega(q)$; for, by combining like terms of the series on the right, we have

$$[2\rho(q)-\omega(q)]\prod_{r=1}^{\infty}(1-q^{2r})=3\sum_{n=0}^{\infty}(-)^n q^{3n(n+1)}\frac{1+q^{6n+3}}{1-q^{6n+3}};$$

and the required relation follows from the formula*

$$\sum_{n=0}^{\infty}(-)^n q^{n(n+1)}\frac{1+q^{2n+1}}{1-q^{2n+1}}=\frac{\vartheta_2(0,q)\,\vartheta_3(0,q)}{2q^{\frac{1}{4}}}$$

$$=\left[\frac{\vartheta_2(0,\sqrt{q})}{2q^{\frac{1}{8}}}\right]^2.$$

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* This formula is immediately derivable from the expression for $1/\vartheta_4(z)$ as a sum of partial fractions. *Cf.* J. Tannery et J. Molk, *Théorie des fonctions elliptiques*, 3 (1898), 136.

Now that the new definitions of the mock ϑ-functions have been constructed, it is a fairly easy matter to establish the relations which connect the functions.

Apart from the relations connecting $\chi(q)$ with $f(q)$ and $\rho(q)$ with $\omega(q)$ which have been obtained already, these relations are special cases of an expansion of the reciprocal of the product of three ϑ-functions. Easy though this expansion is to establish, I do not remember having encountered it previously. It may be stated as follows:

Let r be any integer (positive, zero, or negative) and let α, β, γ be any constants such that

$$\vartheta_1(\beta-\gamma)\,\vartheta_1(\gamma-\alpha)\,\vartheta_1(\alpha-\beta) \neq 0.$$

Then the function

$$\frac{e^{(2r-1)iz}}{\vartheta_2(z-\alpha)\,\vartheta_2(z-\beta)\,\vartheta_2(z-\gamma)}$$

is expressible as the sum of partial fractions

$$\sum_{m=-\infty}^{\infty} \frac{A_m}{e^{2iz}+q^{2m}\,e^{2i\alpha}} + \sum_{m=-\infty}^{\infty} \frac{B_m}{e^{2iz}+q^{2m}\,e^{2i\beta}} + \sum_{m=-\infty}^{\infty} \frac{C_m}{e^{2iz}+q^{2m}\,e^{2i\gamma}},$$

where
$$A_m = \frac{2(-)^{m+r}\,q^{m(3m+1)+2mr}\,e^{2mi(2\alpha-\beta-\gamma)+(2r+1)i\alpha}}{\vartheta_1'(0)\,\vartheta_1(\alpha-\beta)\,\vartheta_1(\alpha-\gamma)},$$

with corresponding values for B_m and C_m. The expansion is valid for all values of z except the poles of the function under consideration.

First observe that, if $q = e^{\pi i\tau}$, it is easy enough to verify that

$$\lim_{z\to\alpha+\frac{1}{2}\pi+m\pi\tau} \frac{(e^{2iz}+q^{2m}\,e^{2i\alpha})\,e^{(2r-1)iz}}{\vartheta_2(z-\alpha)\,\vartheta_2(z-\beta)\,\vartheta_2(z-\gamma)}$$

is equal to the value of A_m given above.

We now proceed to establish the expansion by obtaining it as a limiting case of a similar expansion in which the functions concerned are algebraic. Write

$$\vartheta_{2;\,N}(z) = 2q^{\frac{1}{4}}\cos z \prod_{n=1}^{\infty}(1-q^{2n}) \prod_{n=1}^{N}(1+2q^{2n}\cos 2z+q^{4n})$$

so that
$$\frac{e^{(2r-1)iz}}{\vartheta_{2;\,N}(z-\alpha)\,\vartheta_{2;\,N}(z-\beta)\,\vartheta_{2;\,N}(z-\gamma)}$$

is the quotient of two polynomials in e^{2iz}. The elementary theory of partial fractions then shows that, when $3N+r+1 \geqslant 0$, we have

$$\frac{e^{(2r-1)iz}}{\vartheta_{2;N}(z-\alpha)\,\vartheta_{2;N}(z-\beta)\,\vartheta_{2;N}(z-\gamma)}$$

$$= \sum_{m=-N}^{N} \frac{A_{m;N}}{e^{2iz}+q^{2m}e^{2i\alpha}} + \sum_{m=-N}^{N} \frac{B_{m;N}}{e^{2iz}+q^{2m}e^{2i\beta}} + \sum_{m=-N}^{N} \frac{C_{m;N}}{e^{2iz}+q^{2m}e^{2i\gamma}},$$

where

$$A_{m;N} = A_m \prod_{n=N-m+1}^{\infty} [(1-q^{2n})(1-q^{2n}e^{-2i(\alpha-\beta)})(1-q^{2n}e^{-2i(\alpha-\gamma)})]$$

$$\times \prod_{n=N+m+1}^{\infty} [(1-q^{2n})(1-q^{2n}e^{2i(\alpha-\beta)})(1-q^{2n}e^{2i(\alpha-\gamma)})]$$

with corresponding values for $B_{m;N}$ and $C_{m;N}$.

We now make $N \to \infty$. The observation that

$$A_{m;N}/A_m$$

is a bounded function of m and N which, for any fixed m, tends to unity as $N \to \infty$, combined with the obvious remark that the series

$$\sum_{m=-\infty}^{\infty} \frac{A_m}{e^{2iz}+q^{2m}e^{2i\alpha}}$$

is absolutely convergent, justifies an appeal to Tannery's theorem; we thus have

$$\lim_{N\to\infty} \sum_{m=-N}^{N} \frac{A_{m;N}}{e^{2iz}+q^{2m}e^{2i\alpha}} = \sum_{m=-\infty}^{\infty} \frac{A_m}{e^{2iz}+q^{2m}e^{2i\alpha}},$$

and the other details of the passage to the limit present no difficulties. We have therefore established the expansion

$$\frac{e^{(2r-1)iz}}{\vartheta_2(z-\alpha)\,\vartheta_2(z-\beta)\,\vartheta_2(z-\gamma)}$$

$$= \sum_{\alpha,\beta,\gamma} \sum_{m=-\infty}^{\infty} \frac{2(-)^{m+r}\,q^{m(3m+1)+2mr}\,e^{2mi(2\alpha-\beta-\gamma)+(2r+1)i\alpha}}{\vartheta_1'(0)\,\vartheta_1(\alpha-\beta)\,\vartheta_1(\alpha-\gamma)\,(e^{2iz}+q^{2m}e^{2i\alpha})}.$$

In this expansion take

$$r=0, \quad z=0, \quad \alpha=\tfrac{1}{4}\pi, \quad \beta=-\tfrac{1}{4}\pi, \quad \gamma=0;$$

we get

$$\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)} = \frac{2\vartheta_2(\frac14\pi,q)}{\vartheta_2(0,q)} \sum_{m=-\infty}^{\infty} (-)^m q^{m(3m+1)} \left\{\frac{e^{(\frac32 m+\frac14)\pi i}}{1+iq^{2m}} + \frac{e^{-(\frac32 m+\frac14)\pi i}}{1-iq^{2m}}\right\}$$

$$-2 \sum_{m=-\infty}^{\infty} \frac{(-)^m q^{m(3m+1)}}{1+q^{2m}}.$$

Now it is easy to verify that

$$e^{(\frac32 m+\frac14)\pi i}(1-iq^{2m})+e^{-(\frac32 m+\frac14)\pi i}(1+iq^{2m})$$

$$=2\cos(\tfrac12 m-\tfrac14)\pi-2q^{2m}\sin(\tfrac12 m-\tfrac14)\pi$$

$$=(-)^{\frac12 m(3m+1)}\{1+(-q^2)^m\}\sqrt{2},$$

and hence we have

$$\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)} = \frac{2\vartheta_2(\frac14\pi,q)\sqrt{2}}{\vartheta_2(0,q)} \phi(-q^2)\prod_{r=1}^{\infty}\{1-(-q^2)^r\}-f(q^2)\prod_{r=1}^{\infty}(1-q^{2r}).$$

Further,

$$\frac{\vartheta_2(\frac14\pi,q)\sqrt{2}}{\vartheta_2(0,q)}\prod_{r=1}^{\infty}\{1-(-q^2)^r\} = \prod_{n=1}^{\infty}\left[\frac{1+q^{4n}}{(1+q^{2n})^2}\right]\prod_{n=1}^{\infty}[(1+q^{4n-2})(1-q^{4n})]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1+q^{2n})(1-q^{4n})}{(1+q^{2n})^2}\right]$$

$$=\prod_{n=1}^{\infty}(1-q^{2n}),$$

so that we have

$$2\phi(-q^2)-f(q^2)=\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)}\prod_{n=1}^{\infty}(1-q^{2n})^{-1}$$

$$=\prod_{n=1}^{\infty}\left[\frac{1-q^{2n}}{(1+q^{2n})^2}\right]=\vartheta_4(0,q^2)\prod_{n=1}^{\infty}(1+q^{2n})^{-1},$$

whence Ramanujan's relation connecting $\phi(-q)$ with $f(q)$ follows immediately.

Again, in the partial fraction formula take

$$r=1,\quad z=0,\quad \alpha=0,\quad \beta=\tfrac12\pi\tau,\quad \gamma=\tfrac14\pi\tau;$$

we get

$$\frac{\vartheta_1'(0,q)\,\vartheta_1^2(\tfrac14\pi\tau,q)}{\vartheta_2(0,q)\,\vartheta_2(\tfrac14\pi\tau,q)\,\vartheta_2(\tfrac12\pi\tau,q)}$$

$$= -\frac{2\vartheta_1(\tfrac14\pi\tau,q)}{\vartheta_1(\tfrac12\pi\tau,q)}\left[\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+\frac12)}}{1+q^{2m}}+\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3(m+\frac12)(m+1)}}{1+q^{2m+1}}\right]$$

$$+2\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+1)+\frac34}}{1+q^{2m+\frac12}}$$

$$= -\frac{2\vartheta_1(\tfrac14\pi\tau,q)}{\vartheta_1(\tfrac12\pi\tau,q)}\sum_{n=-\infty}^{\infty}\frac{(-)^{\frac12 n(3n+1)}q^{\frac34 n(n+1)}}{1+q^n}+2q^{\frac14}\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+1)+\frac12}}{1+q^{2m+\frac12}}$$

$$(n = 2m \text{ or } 2m+1)$$

$$= -\frac{\vartheta_1(\tfrac14\pi\tau,q)}{\vartheta_1(\tfrac12\pi\tau,q)}\,\phi(-\sqrt{q})\prod_{r=1}^{\infty}\{1-(-\sqrt{q})^r\}-2q^{\frac14}\psi(-\sqrt{q})\prod_{r=1}^{\infty}(1-q^{2r}).$$

Now evidently

$$\frac{q^{-\frac14}\vartheta_1(\tfrac14\pi\tau,q)}{\vartheta_1(\tfrac12\pi\tau,q)}\prod_{r=1}^{\infty}\{1-(-\sqrt{q})^r\}=\prod_{n=1}^{\infty}\left[\frac{1-q^{n-\frac12}}{(1-q^{2n-1})^2}\right]\prod_{n=1}^{\infty}[(1+q^{n-\frac12})(1-q^n)]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})(1-q^n)}{(1-q^{2n-1})^2}\right]$$

$$=\prod_{n=1}^{\infty}(1-q^{2n}).$$

Hence we have

$$\phi(-\sqrt{q})+2\psi(-\sqrt{q}) = -\frac{q^{-\frac14}\vartheta_1'(0,q)\,\vartheta_1^2(\tfrac14\pi\tau,q)}{\vartheta_2(0,q)\,\vartheta_2(\tfrac14\pi\tau,q)\,\vartheta_2(\tfrac12\pi\tau,q)}\prod_{n=1}^{\infty}(1-q^{2n})^{-1}$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})(1-q^{n-\frac12})^2}{(1+q^{2n})^2(1+q^{n-\frac12})(1+q^{2n-1})^2}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})(1-q^{n-\frac12})^2}{(1+q^n)^2(1+q^{n-\frac12})}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^n)(1-q^{n-\frac12})^2}{(1+q^n)(1+q^{n-\frac12})}\right]$$

$$=\prod_{n=1}^{\infty}\left(\frac{1-q^{\frac12 n}}{1+q^{\frac12 n}}\right)\prod_{n=1}^{\infty}(1-q^{n-\frac12})$$

$$=\vartheta_4(0,\sqrt{q})\prod_{n=1}^{\infty}(1+q^{\frac12 n})^{-1}.$$

We have therefore obtained the two results

$$\vartheta_4(0,q)\prod_{n=1}^{\infty}(1+q^n)^{-1}=2\phi(-q)-f(q)=\phi(-q)+2\psi(-q),$$

and from them the formula

$$f(q)+4\psi(-q)=\vartheta_4(0,\,q)\prod_{n=1}^{\infty}(1+q^n)^{-1}$$

follows at once.

Next, in the partial fraction formula take

$$r=1,\quad z=0,\quad \alpha=\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\tau,\quad \beta=\tfrac{1}{4}\pi\tau,\quad \gamma=\pi+\tfrac{3}{4}\pi\tau;$$

we get

$$\frac{\vartheta_1'(0,\,q)}{2q^{\frac{3}{8}}\vartheta_2(\frac{3}{4}\pi\tau,\,q)}=-\frac{iq^{\frac{3}{8}}\vartheta_1(\frac{1}{2}\pi\tau,\,q)}{\vartheta_2(\frac{1}{4}\pi\tau,\,q)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}$$

$$+\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+\frac{1}{2})}}{1+q^{2m+\frac{1}{2}}}-\sum_{m=-\infty}^{\infty}\frac{q^{3(m+\frac{1}{2})(m+1)}}{1+q^{2m+\frac{3}{2}}}$$

$$=\frac{q^{\frac{1}{2}}\vartheta_4(0,\,q)}{\vartheta_2(\frac{1}{4}\pi\tau,\,q)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{\frac{3}{4}n(n+1)}}{1+q^{n+\frac{1}{2}}}$$

$$(n=2m \text{ or } 2m+1)$$

$$=\frac{q^{\frac{1}{2}}\vartheta_4(0,\,q)}{\vartheta_2(\frac{1}{4}\pi\tau,\,q)}\,\omega(q)\prod_{r=1}^{\infty}(1-q^{2r})+v(\sqrt{q})\prod_{r=1}^{\infty}(1-q^{\frac{1}{2}r}).$$

Now evidently

$$\frac{\vartheta_4(0,\,q)}{\vartheta_2(\frac{1}{4}\pi\tau,\,q)}\prod_{r=1}^{\infty}\left(\frac{1-q^{2r}}{1-q^{\frac{1}{2}r}}\right)=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})^2}{1+q^{n-\frac{1}{2}}}\right]\prod_{n=1}^{\infty}\left[\frac{1-q^{2n}}{(1-q^{n-\frac{1}{2}})(1-q^n)}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})(1-q^{2n})}{1-q^n}\right]=1.$$

Hence we have

$$v(\sqrt{q})+q^{\frac{1}{2}}\omega(q)=\frac{\vartheta_1'(0,\,q)}{2q^{\frac{3}{8}}\vartheta_2(\frac{3}{4}\pi\tau,\,q)}\prod_{r=1}^{\infty}(1-q^{\frac{1}{2}r})^{-1}$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})^2}{(1+q^{n-\frac{1}{2}})(1-q^{n-\frac{1}{2}})(1-q^n)}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})^2}{(1-q^{2n-1})(1-q^n)}\right]$$

$$=\prod_{n=1}^{\infty}\left[(1+q^n)^3(1-q^n)\right]$$

$$=\tfrac{1}{2}q^{-\frac{1}{8}}\vartheta_2(0,\,\sqrt{q})\prod_{n=1}^{\infty}(1+q^n).$$

Since the expression on the right is a one-valued function of q, we immediately deduce the pair of formulae

$$v(q)+q\omega(q^2)=v(-q)-q\omega(q)^2=\tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,q)\prod_{n=1}^{\infty}(1+q^{2n}).$$

Lastly, in the partial fraction formula take

$$r=1,\quad z=0,\quad \alpha=\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\tau,\quad \beta=\tfrac{1}{4}\pi,\quad \gamma=\tfrac{3}{4}\pi+\pi\tau;$$

we get

$$\frac{e^{-\frac{1}{4}\pi i}\vartheta_1'(0)\vartheta_1(\frac{1}{4}\pi+\frac{1}{2}\pi\tau)\vartheta_2(\pi\tau)}{2\vartheta_1(\frac{1}{2}\pi\tau)\vartheta_2(\frac{1}{4}\pi)\vartheta_2(\frac{3}{4}\pi+\pi\tau)}=\frac{e^{\frac{1}{4}\pi i}q^{\frac{3}{4}}\vartheta_1(\frac{1}{2}\pi+\pi\tau)}{\vartheta_1(\frac{1}{4}\pi+\frac{1}{2}\pi\tau)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}$$
$$+i\sum_{m=-\infty}^{\infty}\frac{q^{3m^2}e^{-\frac{1}{2}m\pi i}}{1+iq^{2m}}+\sum_{m=-\infty}^{\infty}\frac{q^{3(m+1)^2}e^{\frac{1}{2}m\pi i}}{1-iq^{2m+2}},$$

so that, replacing m by n or $n-1$, we have

$$\frac{\vartheta_1'(0)\vartheta_4(\frac{1}{4}\pi)\vartheta_2(0)}{2\vartheta_4(0)\vartheta_2^2(\frac{1}{4}\pi)}$$
$$=\frac{q^{\frac{3}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+i\sum_{n=-\infty}^{\infty}q^{3n^2}\left\{\frac{e^{-\frac{1}{2}n\pi i}}{1+iq^{2n}}-\frac{e^{\frac{1}{2}n\pi i}}{1-iq^{2n}}\right\}$$
$$=\frac{q^{\frac{3}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+2\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{4m(3m+1)}}{1+q^{8m}}$$
$$+2q^3\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{12m(m+1)}}{1+q^{8m+4}}$$
$$(n=2m\ \text{ or }\ 2m+1)$$

$$=\frac{q^{\frac{3}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\omega(q)\prod_{r=1}^{\infty}(1-q^{2r})+[f(q^8)+2q^3\omega(-q^4)]\prod_{r=1}^{\infty}(1-q^{8r}).$$

When we reduce this in the usual manner, we find that

$$f(q^8)+2q\,\omega(q)+2q^3\omega(-q^4)=\vartheta_3(0,q)\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n})^{-2},$$

and hence, by changing the sign of q,

$$f(q^8)-2q\,\omega(-q)-2q^3\omega(-q^4)=\vartheta_4(0,q)\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n})^{-2}.$$

Any other relations of this kind connecting mock ϑ-functions of order 3 would appear to be derivable from the relations now obtained.

It is now feasible to construct the linear transformations of the mock ϑ-functions. Since any substitution of the modular group can be resolved into a number of substitutions of the forms

$$\tau' = \tau + 1, \quad \tau' = -1/\tau,$$

it is sufficient to construct the transformations which express the fourteen functions* $f(\pm q)$, ... in terms of similar functions of q_1 (or powers of q_1), where q and q_1 are connected by the relations†

$$q = e^{-\alpha}, \quad \alpha\beta = \pi^2, \quad q_1 = e^{-\beta}.$$

The general similarity between the series involved in the new definitions of the mock ϑ-functions and the series which are generating functions of class-numbers of binary quadratic forms suggests that it may be possible to construct the required transformations by means of functional equations such as have been used by Mordell‡ in connection with class-numbers. Since, however, I lacked the ingenuity necessary for the construction of the functional equations (if indeed they exist), I decided to use the more prosaic methods of contour integration by which a writer subsequent to Mordell has treated the generating functions of class-numbers§.

It is unnecessary to work out all the fourteen transformation formulae by contour integration; when the transformation formulae for $f(q)$ and $\phi(q)$ have been constructed, the remainder can be deduced immediately from the relations connecting the various mock ϑ-functions.

First consider $f(q)$. We have, by Cauchy's theorem,

$$f(q) \prod_{r=1}^{\infty} (1-q^r) = \frac{1}{2\pi i} \left\{ \int_{-\infty-ic}^{\infty-ic} + \int_{\infty+ic}^{-\infty+ic} \right\} \frac{\pi}{\sin \pi z} \frac{\exp(-\frac{3}{2}\alpha z^2)}{\cosh \frac{1}{2}\alpha z} dz,$$

where c is a positive number so small that the zeros of $\sin \pi z$ are the only poles of the integrand between the lines forming the contour. On the higher of these two lines we write

$$\frac{1}{\sin \pi z} = -2i \sum_{n=0}^{\infty} e^{(2n+1)\pi i z},$$

* Actually I do not trouble to deal with the functions $\chi(\pm q)$ and $\rho(\pm q)$ which are less interesting than the rest.

† The numbers α and β, which are positive when q is positive, are slightly easier to work with than the complex τ.

‡ L. J. Mordell, *Quart. J. of Math.*, 48 (1920), 329–342.

§ G. N. Watson, *Compositio Math.*, 1 (1934), 39–68.

so that

$$\frac{1}{2\pi i}\int_{\infty+ic}^{-\infty+ic} \frac{\pi}{\sin \pi z}\,\frac{\exp(-\frac{3}{2}az^2)}{\cosh \frac{1}{2}az}\,dz$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic} 4\pi i \exp\{(2n+1)\pi i z - \tfrac{3}{2}az^2\}\,\frac{e^{az}+e^{-az}-1}{e^{\frac{3}{2}az}+e^{-\frac{3}{2}az}}\,dz$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic} F_n(z)\,dz,$$

say. We calculate these integrals in the following manner. The poles of $F_n(z)$ are (at most) simple poles at the points

$$z_m = \frac{(2m+1)\pi i}{3a} \quad (m = -\infty, \ldots, -1, 0, 1, \ldots, +\infty),$$

and the residue at z_m is

$$\frac{4\pi}{3a}(-)^m \exp\{(2n+1)\pi i z_m - \tfrac{3}{2}a{z_m}^2\}\,.\,(2\cosh a z_m - 1) = \lambda_{n,m},$$

say. Now, by Cauchy's theorem,

$$\frac{1}{2\pi i}\left\{\int_{-\infty+ic}^{\infty+ic} - P\int_{-\infty+z_n}^{\infty+z_n}\right\} F_n(z)\,dz = \lambda_{n,0}+\lambda_{n,1}+\ldots+\lambda_{n,n-1}+\tfrac{1}{2}\lambda_{n,n},$$

where P denotes the "principal value" of the integral. Next, by rearrangement of repeated series,

$$\tfrac{1}{2}\lambda_{0,0}+\sum_{n=1}^{\infty}(\lambda_{n,0}+\lambda_{n,1}+\ldots+\lambda_{n,n-1}+\tfrac{1}{2}\lambda_{n,n})$$

$$= \sum_{m=0}^{\infty}(\tfrac{1}{2}\lambda_{m,m}+\lambda_{m+1,m}+\lambda_{m+2,m}+\ldots)$$

$$= \tfrac{1}{2}\sum_{m=0}^{\infty}\lambda_{m,m}\,\frac{1+\exp 2\pi i z_m}{1-\exp 2\pi i z_m}$$

$$= \frac{2\pi}{3a}\sum_{m=0}^{\infty}(-)^m\{2\cos\tfrac{1}{3}(2m+1)\pi - 1\}\,q_1^{\frac{1}{6}(2m+1)^2}\,\frac{1+q_1^{\frac{2}{3}(2m+1)}}{1-q_1^{\frac{2}{3}(2m+1)}}$$

$$= \frac{2\pi}{a}\sum_{p=0}^{\infty}(-)^p\,q_1^{\frac{3}{2}(2p+1)^2}\,\frac{1+q_1^{4p+2}}{1-q_1^{4p+2}},$$

where $m = 3p+1$, the terms for which $m \neq 3p+1$ vanishing.

Further, we have

$$P\int_{-\infty+z_n}^{\infty+z_n} F_n(z)\,dz$$

$$= P\int_{-\infty}^{\infty} F_n(z_n+x)\,dx$$

$$= P\int_{-\infty}^{\infty} 4\pi i \exp\left\{-\frac{(2n+1)^2\pi^2}{6\alpha}-\frac{3\alpha x^2}{2}\right\}\frac{\cosh\{\alpha x+\frac{1}{3}(2n+1)\pi i\}-\frac{1}{2}}{(-)^n i\sinh\frac{3}{2}\alpha x}\,dx$$

$$= 4\pi i(-)^n \sin\tfrac{1}{3}(2n+1)\pi \,.\, q_1^{\frac{1}{6}(2n+1)^2}\int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2}\frac{\sinh \alpha x}{\sinh\frac{3}{2}\alpha x}\,dx.$$

This simplification in the integral under consideration is due to the modified contour having been chosen to pass through the stationary point of the function

$$\exp\{(2n+1)\pi i z-\tfrac{3}{2}\alpha z^2\},$$

which occurs in the integrand, in the manner of the "method of steepest descents".

The integral along the lower line can be evaluated at once by changing the sign of i throughout the previous work. On combining the results we get

$$f(q)\prod_{r=1}^{\infty}(1-q^r)=\frac{4\pi}{\alpha}q_1^{\frac{4}{3}}\omega(q_1^2)\prod_{r=1}^{\infty}(1-q_1^{4r})+4\vartheta_1(\tfrac{1}{3}\pi,\, q_1^{\frac{2}{3}})\int_0^{\infty} e^{-\frac{3}{2}\alpha x^2}\frac{\sinh \alpha x}{\sinh\frac{3}{2}\alpha x}\,dx.$$

By Jacobi's imaginary transformation this reduces to

$$q^{-\frac{1}{24}}f(q)=2\sqrt{\left(\frac{2\pi}{\alpha}\right)}\,q_1^{\frac{4}{3}}\omega(q_1^2)+4\sqrt{\left(\frac{3\alpha}{2\pi}\right)}\int_0^{\infty}e^{-\frac{3}{2}\alpha x^2}\frac{\sinh \alpha x}{\sinh\frac{3}{2}\alpha x}\,dx,$$

which is the transformation for $f(q)$. We shall consider the integral on the right presently.

We now turn to $\phi(q)$. We have, by Cauchy's theorem,

$$\phi(q)\prod_{r=1}^{\infty}(1-q^r)=\frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic}+\int_{\infty+ic}^{-\infty+ic}\right\}\frac{\pi}{\sin\pi z}\,\frac{\cosh\frac{1}{2}\alpha z}{\cosh \alpha z}\exp(-\tfrac{3}{2}\alpha z^2)\,dz,$$

and, as before, we get

$$\frac{1}{2\pi i}\int_{\infty+ic}^{-\infty+ic}\frac{\pi}{\sin\pi z}\,\frac{\cosh\frac{1}{2}\alpha z}{\cosh \alpha z}\exp(-\tfrac{3}{2}\alpha z^2)\,dz=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic}\Phi_n(z)\,dz,$$

where

$$\Phi_n(z)=2\pi i\exp\{(2n+1)\pi i z-\tfrac{3}{2}\alpha z^2\}\,\frac{\cosh\frac{1}{2}\alpha z\{2\cosh 2\alpha z-1\}}{\cosh 3\alpha z}.$$

The poles of $\Phi_n(z)$ are (at most) simple poles at the points

$$\zeta_m = \frac{(4m+1)\pi i}{6a}, \quad \eta_m = \frac{(4m-1)\pi i}{6a} \quad (m = -\infty, \ldots, -1, 0, 1, \ldots, \infty),$$

and the residues at ζ_m and η_m are

$$\frac{2\pi}{3a} \exp\{(2n+1)\pi i \zeta_m - \tfrac{3}{2}a\zeta_m^2\} . \cosh \tfrac{1}{2}a\zeta_m \{2 \cosh 2a\zeta_m - 1\} = \mu_{n,m},$$

$$-\frac{2\pi}{3a} \exp\{(2n+1)\pi i \eta_m - \tfrac{3}{2}a\eta_m^2\} . \cosh \tfrac{1}{2}a\eta_m \{2 \cosh 2a\eta_m - 1\} = \nu_{n,m},$$

say. Now, by Cauchy's theorem,

$$\frac{1}{2\pi i} \left\{ \int_{-\infty+ic}^{\infty+ic} - \int_{-\infty+z_n}^{\infty+z_n} \right\} \Phi_n(z)\, dz$$

$$= \mu_{n,0} + \mu_{n,1} + \mu_{n,2} + \ldots + \mu_{n,n} + \nu_{n,1} + \nu_{n,2} + \ldots + \nu_{n,n},$$

and, as before, by rearrangement of repeated series,

$$\sum_{n=0}^{\infty} (\mu_{n,0} + \mu_{n,1} + \ldots + \mu_{n,n}) + \sum_{n=1}^{\infty} (\nu_{n,1} + \nu_{n,2} + \ldots + \nu_{n,n})$$

$$= \sum_{m=0}^{\infty} (\mu_{m,m} + \mu_{m+1,m} + \mu_{m+2,m} + \ldots) + \sum_{m=1}^{\infty} (\nu_{m,m} + \nu_{m+1,m} + \nu_{m+2,m} + \ldots)$$

$$= \sum_{m=0}^{\infty} \frac{\mu_{m,m}}{1 - \exp 2\pi i \zeta_m} + \sum_{m=1}^{\infty} \frac{\nu_{m,m}}{1 - \exp 2\pi i \eta_m}$$

$$= \frac{2\pi}{3a} \sum_{m=0}^{\infty} \frac{q_1^{(4m+1)(4m+3)/24}}{1 + q_1^{(4m+1)/3}} \cos \frac{(4m+1)\pi}{12} \left\{ 2 \cos \frac{(4m+1)\pi}{3} - 1 \right\}$$

$$- \frac{2\pi}{3a} \sum_{m=1}^{\infty} \frac{q_1^{(4m-1)(4m+5)/24}}{1 - q_1^{(4m-1)/3}} \cos \frac{(4m-1)\pi}{12} \left\{ 2 \cos \frac{4m-1)\pi}{3} - 1 \right\}$$

$$= \frac{\pi\sqrt{2}}{a} q_1^{\frac{3}{8}} \left[\sum_{n=0}^{\infty} \frac{(-)^n q_1^{2n(3n+5)+1}}{1 - q_1^{4n+3}} + \sum_{p=0}^{\infty} \frac{(-)^p q_1^{6p(p+1)}}{1 - q_1^{4p+1}} \right]$$

$$(m = 3n+2;\ m = 3p+1).$$

Further, we have

$$\int_{-\infty+z_n}^{\infty+z_n} \Phi_n(z)\, dz = \int_{-\infty}^{\infty} \Phi_n(z_n + x)\, dx$$

$$= 2\pi i \int_{-\infty}^{\infty} \exp\left\{ -\frac{(2n+1)^2\pi^2}{6a} - \frac{3ax^2}{2} \right\}$$

$$\times \frac{\cosh \tfrac{5}{2}az + \cosh \tfrac{3}{2}az - \cosh \tfrac{1}{2}az}{-\cosh 3ax}\, dx$$

$$= 2\pi i q_1^{(2n+1)^2/6} \cos \frac{(2n+1)\pi}{6} \int_{-\infty}^{\infty} e^{-\frac{3}{2}ax^2} \frac{\cosh \tfrac{5}{2}ax + \cosh \tfrac{1}{2}ax}{\cosh 3ax}\, dx.$$

The integral along the lower line can be evaluated by changing the sign of i throughout the previous work. On combining the results we get.

$$\phi(q) \prod_{r=1}^{\infty} (1-q^r)$$

$$= \frac{2\pi\sqrt{2}}{\alpha} q_1^{\frac{1}{3}} \psi(q_1) \prod_{r=1}^{\infty} (1-q_1^{4r}) + \vartheta_2(\tfrac{1}{6}\pi, q_1^{\frac{8}{3}}) \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x} \, dx.$$

This reduces to

$$q^{-\frac{1}{24}} \phi(q) = 2\sqrt{\left(\frac{\pi}{\alpha}\right)} q_1^{-\frac{1}{24}} \psi(q_1) + \sqrt{\left(\frac{6\alpha}{\pi}\right)} \int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x} \, dx,$$

which is the transformation for $\phi(q)$.

We next consider the integrals on the right of the transformation formulae. Let

$$\int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x} \, dx = J(\alpha);$$

then it is easy to see that

$$J(\alpha) = \sqrt{\left(\frac{6\beta}{\pi}\right)} \int_0^{\infty} \int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \cos 3\pi xy \, \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x} \, dy \, dx$$

$$= \frac{\pi}{3\alpha} \sqrt{\left(\frac{6\beta}{\pi}\right)} \int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \left\{ \frac{\cosh\frac{1}{2}\beta y \cos\frac{5}{12}\pi}{\cosh\beta y + \cos\frac{5}{6}\pi} + \frac{\cosh\frac{1}{2}\beta y \cos\frac{1}{12}\pi}{\cosh\beta y + \cos\frac{1}{6}\pi} \right\} dy$$

$$= \sqrt{\left(\frac{\beta\pi}{\alpha^2}\right)} \int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \frac{2\cosh\frac{3}{2}\beta y \cosh\beta y}{\cosh 3\beta y} \, dy,$$

so that

$$J(\alpha) = \sqrt{\left(\frac{\pi^3}{\alpha^3}\right)} J(\beta).$$

Next let

$$\int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\sinh\alpha x}{\sinh\frac{3}{2}\alpha x} \, dx = J_1(\alpha),$$

and it is found in a similar manner that

$$J_1(\alpha) = \sqrt{\left(\frac{2\pi^3}{\beta^3}\right)} J_2(\beta),$$

where

$$J_2(\beta) = \int_0^{\infty} e^{-\frac{3}{2}\beta x^2} \frac{\cosh\beta x}{\cosh 3\beta x} \, dx.$$

It is easy to obtain asymptotic expansions for $J(\alpha)$, $J_1(\alpha)$, and $J_2(\alpha)$ proceeding in ascending powers of α and valid when $|\alpha|$ is small and the real part of a is positive; the first few terms of the expansions are

$$J(a) = \sqrt{\left(\frac{2\pi}{3a}\right)}\,[1-\tfrac{23}{24}a+\tfrac{3985}{1152}a^2-\ldots],$$

$$J_1(a) = \sqrt{\left(\frac{2\pi}{27a}\right)}\,[1-\tfrac{5}{72}a+\tfrac{17}{1152}a^2-\ldots],$$

$$J_2(a) = \sqrt{\left(\frac{\pi}{6a}\right)}\,[1-\tfrac{4}{3}a+\tfrac{44}{9}a^2-\ldots].$$

It can be proved that these expansions possess the property that (for a complex) the error due to stopping at any term never exceeds in absolute value the first term neglected; in addition, for a positive, the error is of the same sign as that term*.

I now revert to the construction of the set of transformation formulae; there is no difficulty in verifying that

$$q^{-\frac{1}{24}}f(q)-2\sqrt{\left(\frac{2\pi}{a}\right)}\,q_1^{\frac{4}{3}}\,\omega(q_1^2) = 2\sqrt{\left(\frac{6a}{\pi}\right)}\,J_1(a) = \frac{4\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{-\frac{1}{24}}f(-q)+\sqrt{\left(\frac{\pi}{a}\right)}\,q_1^{-\frac{1}{24}}f(-q_1) = 2\sqrt{\left(\frac{6a}{\pi}\right)}\,J(a) = \frac{2\beta\sqrt{6}}{\pi}\,J(\beta).$$

$$q^{-\frac{1}{24}}\phi(q)-2\sqrt{\left(\frac{\pi}{a}\right)}\,q_1^{-\frac{1}{24}}\psi(q_1) = \sqrt{\left(\frac{6a}{\pi}\right)}\,J(a) = \frac{\beta\sqrt{6}}{\pi}\,J(\beta),$$

$$q^{-\frac{1}{24}}\phi(-q)-\sqrt{\left(\frac{2\pi}{a}\right)}\,q_1^{\frac{1}{3}}\,v(-q_1) = \sqrt{\left(\frac{6a}{\pi}\right)}\,J_1(a) = \frac{2\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{-\frac{1}{24}}\psi(-q)-\sqrt{\left(\frac{\pi}{2a}\right)}\,q_1^{\frac{1}{3}}\,v(q_1) = -\sqrt{\left(\frac{3a}{2\pi}\right)}\,J_1(a) = -\frac{\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{\frac{2}{3}}\,\omega(q)-\sqrt{\left(\frac{\pi}{4a}\right)}\,q_1^{-\frac{1}{12}}f(q_1^2) = -\sqrt{\left(\frac{3a}{\pi}\right)}\,J_2(\tfrac{1}{2}a) = -\frac{2\beta\sqrt{3}}{\pi}\,J_1(2\beta).$$

The transformation formula for $\omega(-q)$ is a little more troublesome; we need the two relations

$$f(q^8)+2q\,\omega(q)+2q^3\,\omega(-q^4) = \vartheta_3(0,q)\,\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n-2})^{-2},$$

$$f(q^8)+q\,\omega(q)-q\omega(-q) = \vartheta_3(0,q^4)\,\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n-2})^{-2}.$$

* This property is established by the method given by G. N. Watson, *Compositio Math.*, 1 (1934), 39–68 (64–65). It is the fact that these expansions are asymptotic (and not terminating series) which shows that mock ϑ-functions are of a more complex character than ordinary ϑ-functions.

From these relations we have

$$q_1^{\frac{8}{3}}\,\omega(-q_1^4) = \tfrac{1}{2}q_1^{-\frac{1}{3}}\vartheta_3(0,\,q_1)\,\vartheta_3^2(0,\,q_1^2)\prod_{n=1}^{\infty}(1-q_1^{4n})^{-2} - \tfrac{1}{2}q_1^{-\frac{1}{3}}f(q_1^8) - q_1^{\frac{2}{3}}\,\omega(q_1)$$

$$= \sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{-\frac{1}{12}}\vartheta_3(0,\,q)\,\vartheta_3^2(0,\,q^{\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)^{-2}$$

$$-\left[\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{\frac{1}{6}}\,\omega(q^{\frac{1}{4}}) + 4\sqrt{\left(\frac{3\beta}{\pi}\right)}\,J_1(8\beta)\right]$$

$$-\left[\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{-\frac{1}{12}}f(q^2) - \sqrt{\left(\frac{3\beta}{\pi}\right)}\,J_2(\tfrac{1}{2}\beta)\right]$$

$$= -\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{\frac{1}{6}}\,\omega(-q^{\frac{1}{4}}) + \sqrt{\left(\frac{3\beta}{\pi}\right)}\,[J_2(\tfrac{1}{2}\beta) - 4J_1(8\beta)].$$

Hence, replacing q_1 by $q^{\frac{1}{4}}$, we get, as the last of the required transformations,

$$q^{\frac{2}{3}}\,\omega(-q) + \sqrt{\left(\frac{\pi}{\alpha}\right)}\,q_1^{\frac{2}{3}}\,\omega(-q_1) = 2\sqrt{\left(\frac{3\alpha}{\pi}\right)}\,J_3(\alpha),$$

where

$$J_3(\alpha) = \tfrac{1}{4}J_2(\tfrac{1}{8}\alpha) - J_1(2\alpha)$$

$$= \tfrac{1}{4}\int_0^{\infty} e^{-\frac{3}{16}\alpha y^2}\,\frac{\cosh\frac{1}{8}\alpha y}{\cosh\frac{3}{8}\alpha y}\,dy - \int_0^{\infty} e^{-3\alpha x^2}\,\frac{\sinh 2\alpha x}{\sinh 3\alpha x}\,dx$$

$$= \int_0^{\infty} e^{-3\alpha x^2}\left\{\frac{\cosh\frac{1}{2}\alpha x}{\cosh\frac{3}{2}\alpha x} - \frac{\sinh 2\alpha x}{\sinh 3\alpha x}\right\}dx,$$

i.e.

$$J_3(\alpha) = \int_0^{\infty} e^{-3\alpha x^2}\,\frac{\sinh \alpha x}{\sinh 3\alpha x}\,dx.$$

It is easy to prove that

$$J_3(\beta) = \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} J_3(\alpha),$$

and that $J_3(\alpha)$ possesses the asymptotic expansion

$$J_3(\alpha) = \tfrac{1}{6}\sqrt{\left(\frac{\pi}{3\alpha}\right)}\,[1 - \tfrac{2}{9}\alpha + \tfrac{17}{108}\alpha^2 - \ldots],$$

for small values of α, this expansion having the same general properties as the asymptotic expansions previously obtained.

Now that I have no more to say about the functions of order 3, I conclude with a brief mention of the functions of orders 5 and 7. The basic hypergeometric series which has been used hitherto is of no avail for these functions, and other means must be sought to establish Ramanujan's relations which connect functions of order 5. After spending a fortnight on fruitless attempts, I proceeded to attack the problem by the most elementary methods available, namely applications of Euler's formulae mingled with rearrangements of repeated series; and within the day I had proved not only the five relations set out by Ramanujan but also five other relations whose existence he had merely stated. My proofs of these relations are all so long that I took the trouble to analyse one of the longest in the hope of being able to say that it involved "thirty-nine steps"; it was, however, disappointing to a student of John Buchan to find that a moderately liberal count revealed only twenty-four.

The functions of order 7 seem to possess fewer features of interest, though a study of their behaviour near the unit circle by the process of estimating the sum of those terms of the series by which they are defined which are in the neighbourhood of the greatest terms has raised one question for which it was fascinating to seek the answer.

The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, "on a la chair de poule". I would express my own attitude with more prolixity by saying that such a formula as

$$\int_0^\infty e^{-3\pi x^2} \frac{\sinh \pi x}{\sinh 3\pi x}\, dx = \frac{1}{e^{\frac{2}{3}\pi}\sqrt{3}} \sum_{n=0}^{\infty} \frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2 (1+e^{-3\pi})^2 \dots (1+e^{-(2n+1)\pi})^2}$$

gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing "Day", "Night", "Evening", and "Dawn" which Michelangelo has set over the tombs of Giuliano de' Medici and Lorenzo de' Medici.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine where

"Pale, beyond porch and portal,
Crowned with calm leaves, she stands
Who gathers all things mortal
With cold immortal hands".