Martingales and the $\mathbb{Q}$-Measure

We may ask ourselves if the stochastic process $S(t)$ is a martingale under definition (1.2). The short answer is no, stock prices usually are not martingales [1, 9], meaning there are at some level a degree of predictability of the future states of $S(t)$ [14, 12]. Up to this point we've being using exclusively the Wiener process $\mathrm{d}W^{\mathbb{P}}(t)$ under probability measure $\mathbb{P}$ when describing $S(t)$, which is usually called *physical probability measure* or *real-world probability measure*. It reflects actual probabilities of events based on observed frequencies, considering investor's risk preferences in the expected return parameter μ [12, 9].

In order to see that under the physical probability measure $\mathbb{P}$, the GBM process of $S(t)$ (22) isn't a martingale, we utilize the definition (1.2) [12]. Property **(1)** from the martingale definition (1.2) must hold since it implies that the expected value of the process must exist, which is something crucial for the usage of expected values in derivatives payoffs such as in option contracts [8, 2]. Recalling equation (23), we figure out that the conditional expectation of $S(t)$ under sigma-algebra $\mathcal{F}(t_0)$ is given by [9]

$$\mathbb{E}[S(t+\Delta t)|\mathcal{F}(t)] = S(t_0)e^{\mu\Delta t} \tag{26}$$

in the same way as (23). This is because $S(t_0)$ is already $\mathcal{F}(t_0)$-measurable [14, 9] and $W^{\mathbb{P}}(t) - W^{\mathbb{P}}(t_0)$ is independent (by the same arguments explained when proving that $W(t)$ is a martingale), making the conditional expectation into a ordinary expected value. Equation (26) shows that

$$\begin{cases} \mathbb{E}[S(t+\Delta t)|\mathcal{F}(t)] > S(t), & \text{if } \mu > 0 \\ \mathbb{E}[S(t+\Delta t)|\mathcal{F}(t)] < S(t), & \text{if } \mu < 0, \end{cases} \tag{27}$$

implying that property **(3)** doesn't hold for $S(t)$ under physical probability measure $\mathbb{P}$ and thus it is not a martingale [12]. In (27) we have a super-martingale for positive μ and a sub-martingale for negative μ reflecting the investor preferences innate to the expected return and the risk premium associated to it [9].

Consider r being the risk-free interest rate. This would be the return an investor would get in a completely riskless asset such as government bond (like U.S. Treasuries) maturing in short time with no default risk [1, 4]. In contrast with the expected return μ, which takes investors risk preferences into account, in summary:

$$\mu = r + \rho, \tag{28}$$

where $\rho > 0$ stands for the risk premium or the compensation for a riskier asset (higher return), such as stocks. Notice that for the investment to be a rational decision for the investor we must have $\mu > r$ (which can be stated from (28) since $\rho > 0$).

With the goal of establishing a probability measure framework capable of describing the process $S(t)$ as a martingale, we define the *risk-neutral* measure $\mathbb{Q}$, under which we have the same Geometric Brownian Motion SDE (14) with the expected return μ replaced by the risk-free rate r [12, 7]:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t), \tag{29}$$

from which all our previous results can be derived changing μ by r. To show that (14) becomes (29) by changing measure $\mathbb{P}$ to $\mathbb{Q}$ we introduce Girsanov theorem [7].

Theorem 3.1. *Roughly speaking*[7]*, the Girsanov theorem formulates how stochastic processes $X(t)$ change under changes in its measure [7]. In particular, for a Wiener process $W(t)$, the change in measure reflects in the addition of a drift term of coefficient θ:*

$$\tilde{W}(t) = W(t) + \theta \mathrm{d}t, \tag{30}$$

where $\tilde{W}(t)$ represents the process under the new probability measure.

Choosing the Girsanov theorem (3.1) parameter θ to be such that

$$\mathrm{d}W^{\mathbb{Q}}(t) = \mathrm{d}W^{\mathbb{P}}(t) + \theta \mathrm{d}t = \mathrm{d}W^{\mathbb{P}}(t) + \left(\frac{\mu - r}{\sigma}\right)\mathrm{d}t \tag{31}$$

allow us to see how the change from real-world measure $\mathbb{P}$ to risk-neutral measure $\mathbb{Q}$ implies the Geometric Brownian Motion SDE (14) to become (29) [12, 7]:

$$\begin{aligned}
\mathrm{d}S(t) &= \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{P}}(t) \\
&= \mu S(t)\mathrm{d}t + \sigma S(t)[\mathrm{d}W^{\mathbb{Q}}(t) - \theta \mathrm{d}t] \\
&\overset{(31)}{=} \mu S(t)\mathrm{d}t + \sigma S(t)\left[\mathrm{d}W^{\mathbb{Q}}(t) - \frac{\mu - r}{\sigma}\mathrm{d}t\right] \\
&= \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t) - S(t)(\mu - r)\mathrm{d}t \\
&= rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t).
\end{aligned}$$

Using the new probability measure $\mathbb{Q}$ we may attempt to check if $S(t)$ can now be stated as a martingale process. Since practically the change from measure $\mathbb{P}$ to $\mathbb{Q}$ only changes the drift term making $\mu \to r$, the third property from martingale definition (1.2) will still not be satisfied:

$$\mathbb{E}[S(t + \Delta t)|\mathcal{F}(t)] = S(t_0)e^{r\Delta t}, \tag{32}$$

and thus

$$\begin{cases} \mathbb{E}[S(t + \Delta t)|\mathcal{F}(t)] > S(t), & \text{if } r > 0 \\ \mathbb{E}[S(t + \Delta t)|\mathcal{F}(t)] < S(t), & \text{if } r < 0, \end{cases} \tag{33}$$

analogously to (27).

Observe though that if we instead use a discounted by the risk-free rate version of the stock price $e^{-r\Delta t}S(t)$, we satisfy property **(3)** from (1.2), achieving a martingale [14, 12] (properties **(1)** and **(2)** where already satisfied under probability measure $\mathbb{P}$ as we've seen):

$$\mathbb{E}[e^{-r\Delta t}S(t + \Delta t)|\mathcal{F}(t)] = e^{-r\Delta t}[S(t_0)e^{r\Delta t}] = S(t_0). \tag{34}$$

Notice that the same wouldn't hold for the process under physical probability measure $\mathbb{P}$, since the exponential factors wouldn't cancel out [9].

When pricing derivatives, one of the main goals is to avoid arbitrage opportunities [1], i.e. guarantee risk-free profits without taking on any risk and without committing any capital [1, 4], these opportunities undermine the market. In an efficient market, prices fully reflect all available information [6], meanwhile arbitrage opportunities suggest that assets are mispriced relative to each other.

For a martingale such as the discounted stock price $\tilde{S}(t) = e^{-r\Delta t}S(t)$, arbitrage is prevented [14, 12]. If $S(t)$ were expected to rise, one could buy the stock and expect a

[7] The formal Girsanov theorem can be seen in the original reference [7].

profit without risk, if the opposite was expected to happen, one could short sell the asset and expect a profit, either case there would be an arbitrage opportunity, since the profit is riskless and with no initial investment. By enforcing (34) we are saying "Given what I known now, I can't expect the discounted price to go up or down in the future", which removes the possibility of arbitrage via buying, selling or holding the discounted stock price $\tilde{S}(t)$ without taking risk.

The connection between martingales and no-arbitrage opportunities such as the GBM modeled discounted stock price under risk-neutral measure $\mathbb{Q}$ can be extended through the Fundamental Theorem of Asset Pricing (FTAP) [5].

Theorem 3.2. *The Fundamental Theorem of Asset Pricing (FTAP) states that a financial market is arbitrage-free if and only if there exists an equivalent martingale measure $\mathbb{Q}$ under which the discounted price processes $e^{-rt}S(t)$ (adjusted by the risk-free rate r) of all tradable assets are martingales [5]. Where equivalency stands for measures $\mathbb{P}$ and $\mathbb{Q}$ agreeing on what events are possible (both assign 0 probability to the the same sets).*

The FTAP theorem (3.2) then concludes that if discounted prices are martingales under some measure $\mathbb{Q}$, then there's no way to guarantee a profitable operation without involving risk and therefore the market is arbitrage-free [5].

Just as the physical measure $\mathbb{P}$ represents the real world—where asset dynamics reflect investors' risk preferences through the expected return μ (with higher risk demanding higher return), we can define a fictional probabilistic framework known as the *risk-neutral world* [4, 1], governed by the risk-neutral measure $\mathbb{Q}$. In this world, all investors are indifferent to risk, and consequently, every asset is expected to grow at the risk-free rate r. Under this perspective, risky assets such as stocks carry no risk premium, since investors require no additional compensation for bearing uncertainty [4].

Under the real-world measure $\mathbb{P}$, the expected return μ is generally difficult to estimate, as it depends on investor preferences and market expectations [4]. Moreover, as previously discussed, neither the stock price process $S(t)$ or its discounted version $e^{-rT}S(t)$ are martingales under $\mathbb{P}$, which can lead to inconsistencies and arbitrage opportunities in pricing. According to the Fundamental Theorem of Asset Pricing (3.2), the risk-neutral world associated with the measure $\mathbb{Q}$ ensures that all discounted asset prices follow martingale dynamics [5], thereby eliminating arbitrage and providing a consistent and robust framework for pricing derivatives.

Of course, the real world does not operate under risk-neutral assumptions, investors are generally risk-averse, and asset returns reflect varying degrees of risk premia. Market imperfections, transaction costs, and behavioral factors all contribute to deviations from the idealized dynamics prescribed by the risk-neutral framework [4]. Nonetheless, the risk-neutral world remains an invaluable mathematical construct: it allows us to compute fair prices for contingent claims based solely on the absence of arbitrage [1], without requiring knowledge of individual risk preferences or the true expected return μ. In practice, this framework forms the basis for most modern derivative pricing models [3, 2], offering internally consistent results even if the underlying assumptions do not fully capture market reality.

References

[1] Tomas Björk. *Arbitrage theory in continuous time*. Oxford university press, 2009.

[2] Fischer Black and Myron Scholes. “The pricing of options and corporate liabilities”. In: *Journal of political economy* 81.3 (1973), pp. 637–654.

[3] Fischer Black and Myron Scholes. “The valuation of option contracts and a test of market efficiency”. In: *The Journal of finance* 27.2 (1972), pp. 399–417.

[4] Merton Robert C. “Theory of Rational Option Pricing”. In: *The Bell Journal of Economics and Management Science* 4.3 (1973), pp. 167–183.

[5] Freddy Delbaen and Walter Schachermayer. “A General Version of the Fundamental Theorem of Asset Pricing”. In: *Mathematische Annalen* 300.1 (1994), pp. 463–520. DOI: 10.1007/BF01450498.

[6] Eugene F. Fama. “Efficient Capital Markets: A Review of Theory and Empirical Work”. In: *The Journal of Finance* 25.2 (1970), pp. 383–417. DOI: 10.2307/2325486.

[7] Igor Vladimirovich Girsanov. “On transforming a certain class of stochastic processes by absolutely continuous substitution of measures”. In: *Theory of Probability & Its Applications* 5.3 (1960), pp. 285–301.

[8] Paul Glasserman. *Monte Carlo methods in financial engineering*. Vol. 53. Springer, 2004.

[9] Lech A. Grzelak and Cornelis W. Oosterlee. *Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*. World Scientific Publishing Europe Ltd, 2019. ISBN: 978-1786348050.

[10] John C. Hull. *Options, Futures, and Other Derivatives*. 10th ed. Pearson, 2022. ISBN: 978-1-292-40856-2.

[11] Kiyosi Ito et al. *On stochastic differential equations*. Vol. 4. American Mathematical Society New York, 1951.

[12] Ioannis Karatzas and Steven E Shreve. *Brownian motion*. Springer, 2021, pp. 47–127.

[13] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[14] Steven E Shreve et al. *Stochastic calculus for finance II: Continuous-time models*. Vol. 11. Springer, 2004.

[15] Paul Wilmott, Jeff Dewynne, and Sam Howison. *Option Pricing: Mathematical Models and Computation*. Oxford Financial Press, 1994. ISBN: 978-0952208204.