[DERIVATION OF BLACK-SCHOLES FORMULA]

ACHIEVING EUROPEAN CALL OPTION PRICE ANALYTICALLY THROUGH THE HEAT EQUATION

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Abstract: In this text, we present the derivation of the well-known Black-Scholes formula for a European call option by mapping it to the heat equation from Physics through a suitable change of variables. We begin by introducing the classical heat equation and derive its general solution using Fourier transform techniques [7]. Then, we introduce the Black-Scholes partial differential equation (PDE) and apply a transformation of variables that reduces it to the heat equation, enabling us to obtain its general solution. By applying appropriate terminal conditions, we recover the European call option pricing formula. Finally, we draw a conceptual and graphical comparison between the two solutions, highlighting the analogy between heat diffusion and the temporal decay of option value under uncertainty [5, 4].

Solving the heat equation

For the purpose of this text, we consider the unidimensional heat equation for the function $u = u(x,t)$ with $-\infty < x < \infty$ and $t > 0$, where $\lim_{x\to\pm\infty} u(x,t)e^{-ikx} = 0$ and $\lim_{x\to\pm\infty} \frac{\partial u(x,t)}{\partial x} e^{-ikx} = 0$ for $-\infty < k < \infty$:

$$\alpha \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{1}$$

with the initial condition $u(x,0) = u_0(x)$. This equation is a fundamental PDE from physics introduced by Joseph Fourier in 1807 and received this name due to its capability of describing how heat diffuses through a given region over time [1]. In the physics context, the function $u = u(x,t)$ represents the temperature of the system at position x and instant t. The parameter α is usually called thermal diffusivity and it measures how quickly the heat transfer from a warm environment to a cold environment (related to the nature of the system where the diffusion happens) [6].

The heat equation can be solved using Fourier transform (FT), that is defined operating over a generic function $g = g(x,t)$, as

$$\mathcal{F}[g(x,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x,t)e^{-ikx}\,\mathrm{d}x. \tag{2}$$

We proceed using the FT (2) on both sides of equation (1):

$$\alpha\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = \mathcal{F}\left[\frac{\partial u}{\partial t}\right].$$

Developing the right hand side of the equation, using integration by parts and the assump-

[1] In this particular case a linear system as an iron rod for example.

tions $\lim_{x\to\pm\infty} u(x,t)e^{-ikx} = 0$ and $\lim_{x\to\pm\infty} \frac{\partial u(x,t)}{\partial x}e^{-ikx} = 0$ for $k \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\frac{\partial u}{\partial x}e^{-ikx}\Big|_{-\infty}^{\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}e^{-ikx}\,\mathrm{d}x \\
&= ik\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}e^{-ikx}\,\mathrm{d}x \\
&= ik\left[\frac{1}{2\pi}ue^{-ikx}\Big|_{-\infty}^{\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty}u(-ik)e^{-ikx}\,\mathrm{d}x\right] \\
&= (ik)^2\frac{1}{2\pi}\int_{-\infty}^{\infty}ue^{-ikx}\,\mathrm{d}x \\
&= -k^2\mathcal{F}[u(x,t)],
\end{aligned}$$

we can use the result to write the PDE as follows

$$-\alpha k^2\mathcal{F}[u(x,t)] = \frac{\partial}{\partial t}\mathcal{F}[u(x,t)].$$

Calling $\tilde{u}(k,t) = \mathcal{F}[u(x,t)]$, we now have a simpler PDE:

$$\frac{\partial}{\partial t}\tilde{u}(k,t) + \alpha k^2\tilde{u}(k,t) = 0.$$

Multiplying both sides by $e^{\alpha k^2 t}$ allow us to write the left side of the equation as a derivative through the product rule:

$$\begin{aligned}
e^{\alpha k^2 t}\frac{\partial \tilde{u}}{\partial t} + (\alpha k^2)e^{\alpha k^2 t}\tilde{u} &= 0 \\
\frac{\partial}{\partial t}\left(e^{\alpha k^2 t}\tilde{u}\right) &= 0,
\end{aligned}$$

which means that $e^{\alpha k^2 t}\tilde{u}$ must equal a constant A:

$$\tilde{u}(k,t) = Ae^{-\alpha k^2 t}.$$

Taking the initial condition under the FT $\tilde{u}(k,0) = \tilde{u}_0(k)$, we finding out that $A = \tilde{u}_0(k)$ arriving at the equation,

$$\tilde{u}(k,t) = \tilde{u}_0(k)e^{-\alpha k^2 t}.$$

The next step is to search for a function $u = u(x,t)$ whose FT is given by $\tilde{u}_0(k)e^{-\alpha k^2 t}$. For this purpose we search for function $U = U(x,t)$ whose FT equals $\tilde{U}(k,t) = e^{\alpha k^2 t}$:

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{U}(k,t)e^{ikx}\,\mathrm{d}k = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-[(\alpha t)k^2 + (-ix)k]}\,\mathrm{d}k.$$

The solution of the integral is given by the generalized Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{b^2}{4a} - c\right), \tag{3}$$

when $a = \alpha t$, $b = -ix$ and $c = 0$:

$$U(x,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha t}}e^{-x^2/4\alpha t} = \frac{1}{\sqrt{4\pi\alpha t}}e^{-x^2/4\alpha t}.$$

However, what we want to know is the inverse FT of

$$\tilde{u}(k,t) = \tilde{u}_0(k)\tilde{U}(k,t).$$

From the convolution theorem, a convolution of two functions $f(x,t)$ and $g(x,t)$ is equal to the inverse FT of the product of these functions:

$$f(x,t) \circ g(x,t) = \mathcal{F}^{-1}[\tilde{f}(k,t) \cdot \tilde{g}(k,t)] \tag{4}$$

where the convolution of $f(x,t)$ and $g(x,t)$ is defined as

$$(f(x,t) \circ g(x,t))(y) = \int_{-\infty}^{\infty} f(x-y,t)g(x,t)\ \mathrm{d}x = \int_{-\infty}^{\infty} f(x,t)g(x-y,t)\ \mathrm{d}x \tag{5}$$

and the inverse FT of $\tilde{f}(k,t)$ as

$$f(x,t) = \mathcal{F}^{-1}[\tilde{f}(k,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k,t)e^{ikx}\mathrm{d}k. \tag{6}$$

Since $\mathcal{F}^{-1}[\tilde{u}(k,t)] = u(x,t)$, we then have that

$$\mathcal{F}^{-1}[\tilde{u}(k,t)] = \mathcal{F}^{-1}[\tilde{u}_0(k)\tilde{U}(k,t)] = u(x,t) = u_0(x) \circ U(x,t)$$

according to (4). This is, using (5):

$$u(x,t) = u_0(x) \circ U(x,t) = \int_{-\infty}^{\infty} U(x-y,t)u_0(y)\ \mathrm{d}y$$

and replacing the function U we arrive at the solution of the heat equation:

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4\alpha t}u_0(y)\ \mathrm{d}y. \tag{7}$$

Change of variables in Black-Scholes PDE

The Black-Scholes model or Black-Scholes-Merton model is a mathematical framework widely used to determine the theoretical price of option contracts. The equation was first introduced by Fisher Black and Myron Scholes in 1973 [1, 2] and later improved by Robert Merton [3]. The Black-Scholes PDE is given by

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = rV \tag{8}$$

for $V = V(S,t)$ being the derivative value, $S = S(t)$ the underlying asset price at time t (such as a non dividend paying stocks), r the risk-free interest rate and σ the volatility of the stock price (both the risk-free IR and volatility are constants). It is used to calculate the theoretical values of option prices under certain assumptions such as the Geometric Brownian Motion (GBM) behavior of the underlying asset price $S = S(t)$:

$$\mathrm{d}\ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}W(t),$$

which leads to a lognormal distribution of the underlying asset prices $S = S(t)$:

$$\ln S(t) \sim \mathcal{N}\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right],$$

for S_0 being the value of the stock price at the beginning of the contract (at $t = 0$), $t = T$ the maturity of the contract, $\mathcal{N}(\cdot)$ the normal CDF and $W(t)$ a wiener process (random walk process of the form $\sqrt{t}Z$ with $Z \sim \mathcal{N}(0, 1)$). Other assumptions are made by the Black-Scholes model such as: the market allows participants to continuously trade with no transaction costs, there is no limit on short selling, no borrowing constrains, the risk free interest rate and stock price volatility are constant during the option lifetime and that the market is free from arbitrage opportunities [5, 4, 8].

The purpose of this text is to derive the European Call option price solution from Black-Scholes PDE (8) [7]:

$$C(S, t) = S(t)\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \tag{9}$$

for $0 \leq t \leq T$,

$$d_1 = \frac{\ln(\frac{S}{K}) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = d_1 - \sigma\sqrt{T - t}, \tag{10}$$

and $\mathcal{N}(\cdot)$ begin the standard normal CDF:

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\mathrm{d}z, \tag{11}$$

imposing the European call option terminal condition:

$$V(S, T) = \max(S - K, 0) \tag{12}$$

where K is the strike price of the option.

For us to arrive at (9), first we must introduce sequence of two change of variables into (8) in order to realize its heat equation format (1) and obtain a solution of the form (7). The first change of variables is the following:

$$x = \ln\left(\frac{S}{K}\right), \quad \tau = (T - t)\frac{\sigma^2}{2} \quad \text{and} \quad v(x, \tau) = \frac{V(S, t)}{K}. \tag{13}$$

Applying them to the Black-Scholes Equation derivatives we get

- $\boldsymbol{\frac{\partial V}{\partial t}}$:

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = K\frac{\partial v}{\partial \tau}\frac{\sigma^2}{2} = -\frac{\sigma^2}{2}K\frac{\partial v}{\partial \tau}.$$

- $\boldsymbol{\frac{\partial V}{\partial S}}$:

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}S} = K\frac{\partial v}{\partial x}\frac{\mathrm{d}}{\mathrm{d}S}\ln\left(\frac{S}{K}\right) = K\frac{\partial v}{\partial x}\frac{K}{S}\frac{1}{K} = \frac{K}{S}\frac{\partial v}{\partial x}.$$

- $\frac{\partial^2 V}{\partial S^2}$:

$$\begin{aligned}\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial S}\right)\frac{\mathrm{d}x}{\mathrm{d}S} = \frac{\partial}{\partial x}\left(\frac{K}{S}\frac{\partial v}{\partial x}\right)\frac{1}{S} = \frac{K}{KS}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial v}{\partial x}\right)\\ &= \frac{K}{KS}\left(e^{-x}\frac{\partial^2 v}{\partial x^2} - e^{-x}\frac{\partial v}{\partial x}\right) = \frac{K}{S(Ke^x)}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) = \frac{K}{S^2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right).\end{aligned}$$

No we must use this derivative substitutions on the Black-Scholes PDE (8) arranging it into a differential equation with constant coefficients:

$$\begin{aligned}\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} &= rV\\ \left(-\frac{\sigma^2}{2}K\frac{\partial v}{\partial \tau}\right) + rS\left(\frac{K}{S}\frac{\partial v}{\partial x}\right) + \frac{\sigma^2}{2}S^2\left[\frac{K}{S^2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right)\right] &= rKv\\ -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + r\frac{\partial v}{\partial x} + \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) &= rv\\ -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + r\frac{\partial v}{\partial x} + \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) &= rv\\ -\frac{\partial v}{\partial \tau} + \frac{2r}{\sigma^2}\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} &= \frac{2r}{\sigma^2}v\end{aligned}$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} - kv, \tag{14}$$

where $k = \frac{2r}{\sigma^2}$. We also need to apply the change of variables (13) into the terminal condition (12):

$$\begin{aligned}V(S,T) &= \max(S-K,0)\\ Kv(x,0) &= \max(Ke^x - K,0)\end{aligned}$$

$$v(x,0) = \max(e^x - 1, 0). \tag{15}$$

Finally, for this first change of variables, is important for us to determine how they impact in the limits of the variables from $S \to x$ (where $0 < S < \infty$) and $t \to \tau$ (where $0 \leq t \leq T$). When $S \to 0 \ \Rightarrow \ x = \ln(S/K) \to -\infty$ and when $S \to \infty \ \Rightarrow \ x = \ln(S/K) \to -\infty$[2] thus $-\infty < x < \infty$. To find the τ limits we use the inequality $0 \leq t \leq T$:

$$\begin{aligned}0 &\geq -t \geq -T\\ T &\geq T - t \geq 0\\ 0 &\leq (T-t)\frac{\sigma^2}{2} \leq T\frac{\sigma^2}{2}\\ 0 &\leq \tau \leq T\frac{\sigma^2}{2}.\end{aligned}$$

With our PDE (14) in hands after the first change of variables we proceed to the second one:

$$u(x,\tau) = e^{-(\alpha x + \beta\tau)}v(x,\tau), \tag{16}$$

applying it to the derivatives of the equation (14):

[2]This two conclusions becomes evident when looking at the graphic of the logarithmic function $x = x(S)$.

- $\frac{\partial v}{\partial \tau}$:

$$\frac{\partial v}{\partial \tau} = \beta e^{\alpha u+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} = e^{\alpha u+\beta\tau}\left(\beta u + \frac{\partial u}{\partial \tau}\right).$$

- $\frac{\partial v}{\partial x}$:

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(u e^{\alpha x+\beta\tau}\right) = \alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} = e^{\alpha x+\beta\tau}\left(\alpha u + \frac{\partial u}{\partial x}\right).$$

- $\frac{\partial^2 v}{\partial x^2}$:

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right) \\ &= \left(\alpha^2 e^{\alpha x+\beta\tau} u + \alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right) + \left(\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}\right) \\ &= e^{\alpha x+\beta\tau}\left(\alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right).\end{aligned}$$

We proceed then with the replacement of the derivatives into (14):

$$\begin{aligned}&\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} - kv \\ &e^{\alpha u+\beta\tau}\left(\beta u + \frac{\partial u}{\partial \tau}\right) = e^{\alpha x+\beta\tau}\left(\alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) + e^{\alpha x+\beta\tau}\left(\alpha u + \frac{\partial u}{\partial x}\right)(k-1) - k e^{\alpha x+\beta\tau} u \\ &\left(\beta u + \frac{\partial u}{\partial \tau}\right) = \left(\alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) + \left(\alpha u + \frac{\partial u}{\partial x}\right)(k-1) - ku\end{aligned}$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + [2\alpha + (k-1)]\frac{\partial u}{\partial x} + [\alpha^2 + \alpha(k-1) - \beta - k]u. \tag{17}$$

Since α and β are purely arbitrary (in contrast to $k = \frac{2r}{\sigma}$ that depends on r and σ), we can choose them in such a way to vanish the u and $\frac{\partial u}{\partial x}$ coefficients on (17), this is

$$\begin{cases} 2\alpha - 1 + k = 0 \\ \alpha^2 + \alpha(k-1) - \beta - k = 0. \end{cases}$$

For the first equation we'll have

$$\alpha = \frac{1-k}{2},$$

and the second equation will give us β in terms of k:

$$\begin{aligned}\beta &= \left(\frac{1-k}{2}\right)^2 + \left(\frac{1-k}{2}\right)(k-1) - k \\ &= \frac{(1-k)^2}{4} - \frac{(1-k)^2}{2} - k \\ &= -\frac{(1-k)^2}{4} - k \\ &= -\frac{1-2k+k^2+4k}{4} \\ &= -\frac{1+2k+k^2}{4} \\ &= -\frac{(1+k)^2}{4}.\end{aligned}$$

Hence, setting $\alpha = \frac{1-k}{2}$ and $\beta = -\frac{(1+k)^2}{4}$ on (17) will yield us the following PDE:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{18}$$

with $-\infty < x < \infty$ and $0 \leq \tau \leq \frac{2T}{\sigma^2}$. As in the first change of variables, here we too need to find the correspondent terminal condition, using (15):

$$\begin{aligned} v(x,0) &= \max(e^x - 1, 0) \\ e^{\alpha x} u(x,0) &= \max(e^x - 1, 0) \\ u(x,0) &= \max[e^{(1-\alpha)x} - e^{-\alpha x}, 0] \end{aligned}$$

but $\alpha = \frac{1-k}{2}$ and $1 - \alpha = \frac{k+1}{2}$, thus

$$u(x,0) = \max[e^{(k+1)x/2} - e^{(k-1)x/2}, 0]. \tag{19}$$

Deriving European Call option solution

Comparing equations (18) and (1), we can see that for $\alpha = 1$ on the former we have an equivalence (noticing that for (18) τ has an upper boundary $\frac{2T}{\sigma^2}$). Since we showed that the Black-Scholes PDE can be turned into a heat diffusion equation of the format (1), we can use the solution (7) for $\alpha = 1$ [7]:

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4\tau} u_0(s) \mathrm{d}s,$$

where $u_0(s)$ is the terminal condition. Further developing this solution, we use an integral change of variables simplifying the exponential argument:

$$w = \frac{x-s}{\sqrt{2\tau}}, \quad \mathrm{d}w = \frac{\mathrm{d}s}{\sqrt{2\tau}} \tag{20}$$

obtaining

$$u(x,\tau) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(\sqrt{2\tau} + x) e^{-w^2/2} \mathrm{d}w. \tag{21}$$

Now lets make considerations about the terminal condition under change of variables (19):

$$u_0(s) = u(s,0) = \max[e^{(k+1)s/2} - e^{(k-1)s/2}, 0].$$

First suppose $u_0(s) \neq 0$, for this case we'll have

$$e^{(k+1)s/2} - e^{(k-1)s/2} > 0,$$

otherwise $u_0(s) = 0 \;\Rightarrow\; u(x,\tau) = 0$. Considering $s < 0$ or in other words $s = -y$:

$$\begin{gathered} e^{-(k+1)y/2} > e^{-(k-1)y/2} \\ -(k+1)y/2 > -(k-1)y/2 \\ -k-1 > -k+1 \\ -1 > +1 \end{gathered}$$

that is obviously false, thus we must have $s > 0$, or in the integral change of variables:

$$s = \sqrt{2\tau}w + x > 0 \;\Rightarrow\; w > -\frac{x}{\sqrt{2\tau}}.$$

With this in mind we replace the terminal condition in (19):

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \left[e^{(k+1)/2(\sqrt{2\tau}w+x)-w^2/2} - e^{(k-1)/2(\sqrt{2\tau}w+x)-w^2/2} \right] \mathrm{d}w$$

$$u(x,\tau) = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{(k+1)/2(\sqrt{2\tau}w+x)-w^2/2} \mathrm{d}w}_{\mathcal{I}_1} - \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{(k-1)/2(\sqrt{2\tau}w+x)-w^2/2} \mathrm{d}w}_{\mathcal{I}_2}. \tag{22}$$

Next we solve integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ at once

$$\mathcal{I}_{1,2} = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{(k\pm1)/2(\sqrt{2\tau}w+x)-w^2/2} \mathrm{d}w,$$

by simplifying the argument of the exponential inside of the integral by completing the square:

$$\begin{aligned} \frac{(k\pm1)}{2}(\sqrt{2\tau}w + x) - \frac{w^2}{2} &= -\frac{1}{2}[w^2 - (k\pm1)\sqrt{2\tau}w] + \frac{(1\pm k)}{2}x \\ &= -\frac{1}{2}\left[w^2 - 2\frac{(k\pm1)}{2}\sqrt{2\tau}w + \frac{(k\pm1)^2 2\tau}{4}\right] + \frac{(1\pm k)}{2}x + \frac{(k\pm1)^2\tau}{4} \\ &= -\frac{1}{2}\left[w - \frac{(k\pm1)}{2}\sqrt{2\tau}\right]^2 + \frac{(1\pm k)}{2}x + \frac{(k\pm1)^2\tau}{4}. \end{aligned}$$

Naming m the content of the brackets, we use it for a second integral change of variables:

$$m = w - \frac{(k\pm1)\sqrt{2\tau}}{2}, \quad \mathrm{d}m = \mathrm{d}w. \tag{23}$$

With m we can rewrite the exponential argument as

$$\frac{(1\pm k)}{2}x + \frac{(k\pm1)^2\tau}{4} - \frac{m^2}{2}.$$

Notice that with this change of variables, we were able to isolate the integral variable $m = m(w)$ into a standard normal exponent in m, with $\mathrm{d}w = \mathrm{d}m$. Regarding the limits of m, the only difference appears in the lower limit:

$$m_{\text{lower}} = -d_{1,2} = -\left[\frac{x}{\sqrt{2\tau}} + \frac{(k \pm 1)\sqrt{2\tau}}{2}\right] \tag{24}$$

Applying this second integral change of variables provides us with

$$\mathcal{I}_{1,2} = \frac{1}{\sqrt{2\pi}} e^{(k\pm1)x/2+(k\pm1)^2\tau/4} \int_{-d_{1,2}}^{\infty} e^{-m^2/2}\mathrm{d}m.$$

Using a third and last integral change of variables

$$m = -z, \quad \mathrm{d}m = -\mathrm{d}z \tag{25}$$

allow us to replace and invert the limits of the integral:

$$\mathcal{I}_{1,2} = e^{(k\pm1)x/2+(k\pm1)^2\tau/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_{1,2}} e^{-z^2/2}\mathrm{d}z = e^{(k\pm1)x/2+(k\pm1)^2\tau/4}\mathcal{N}(d_{1,2}),$$

where the standard normal CDF $\mathcal{N}(d_{1,2})$ was introduced in (11). Using the integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ on (22) we arrive at:

$$u(x,\tau) = \mathcal{I}_1 - \mathcal{I}_2 = e^{(k+1)x/2+(k+1)^2\tau/4}\mathcal{N}(d_1) - e^{(k-1)x/2+(k-1)^2\tau/4}\mathcal{N}(d_2). \tag{26}$$

We have our solution, we now only need to retrieve our original variables by doing the two change of variables (13) and (16) in reverse, first using (16):

$$\begin{aligned} v(x,\tau) &= e^{\alpha x+\beta\tau}u(x,\tau) = e^{(1-k)x/2-(1+k)^2\tau/4} \\ &= e^{-(k-1)x/2-(1+k)^2\tau/4}\left[e^{(k+1)x/2+(k+1)^2\tau/4}\mathcal{N}(d_1) - e^{(k-1)x/2+(k-1)^2\tau/4}\mathcal{N}(d_2)\right] \end{aligned}$$

$$v(x,\tau) = e^{x}\mathcal{N}(d_1) - e^{-k\tau}\mathcal{N}(d_2). \tag{27}$$

Recalling the first change of variables (13) and that $k = 2r/\sigma^2$, we can see that

$$e^x = e^{\ln(S/K)} = \frac{S}{K} \quad \text{and} \quad k\tau = \frac{r}{(\sigma^2/2)} \cdot (T-t)(\sigma^2/2) = r(T-t).$$

Using this on (27) we obtain [7],

$$KV(S,t) = \frac{S}{K}\mathcal{N}(d_1) - e^{-r(T-t)}\mathcal{N}(d_2)$$

$$C(S,t) = V(S,t) = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)$$

with $d_{1,2}$ given by:

$$\begin{aligned} d_{1,2} &= \frac{x}{\sqrt{2\tau}} + \frac{(k\pm1)\sqrt{2\tau}}{2} \cdot \frac{\sqrt{2\tau}}{\sqrt{2\tau}} \\ &= \frac{\ln\left(\frac{S}{K}\right)}{\sqrt{\sigma^2(T-t)}} + \frac{\frac{1}{2}\left(\frac{2r}{\sigma^2} \pm 1\right)}{\sqrt{\sigma^2(T-t)}}\left(\sqrt{\sigma^2(T-t)}\right)^2 \\ &= \frac{\ln\left(\frac{S}{K}\right) + \frac{1}{2}\left(\frac{2r}{\sigma^2} \pm 1\right)\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}} \end{aligned}$$

$$d_{1,2} = \frac{\ln\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

concluding the demonstration of the European call option price solution (9) from Black-Scholes equation (8). The European put option price solution

$$P(S,T) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1) \tag{28}$$

could also be derived similarly, applying the terminal condition

$$V(S,T) = \max(K-S,0) \tag{29}$$

or simply derived from the European call combined with the put-call parity:

$$P(S,t) = C(S,t) - S + Ke^{-r(T-t)}. \tag{30}$$

Resemblances between the Black-Scholes PDE and Heat equation

Both equations, (1) and (8), describe diffusion processes. The heat equation governs the diffusion of temperature through a medium, while the Black-Scholes PDE describes the diffusion of option prices under uncertainty (volatility of the underlying asset). An insightful analogy can be drawn between the flow of heat from a hotter to a cooler region and the erosion of an option's value over time — a phenomenon commonly referred to as time decay [5, 4].

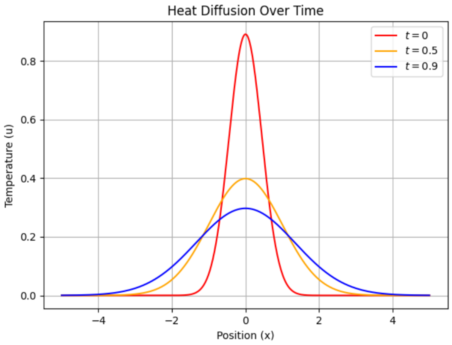
To understand this analogy, it is first necessary to define the time value of an option. For an European call option, the premium can be decomposed as follows:

$$C(S,T) = \mathrm{IV} + \epsilon = \max(S-K,0) + \epsilon(S,t), \tag{31}$$

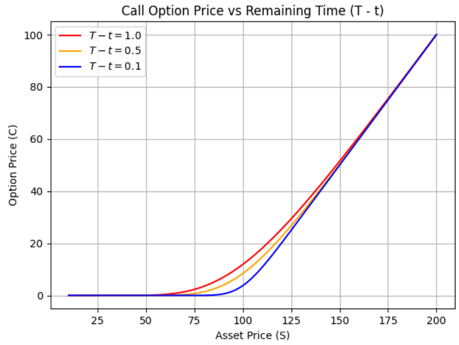
where the first term is the intrinsic value of the option, and $\epsilon(S,t)$ denotes its time value. The time value reflects the premium attributed to the remaining time until expiration and the potential for favorable movements in the underlying asset's price. As the maturity date approaches, this value decreases — capturing the diminishing chance of the option ending up in-the-money if it is not already profitable.

This reduction in time value as expiration nears is known as option price erosion, and the associated process is called time decay. It is quantified by the Greek letter theta (θ), defined as the negative partial derivative of the option price with respect to time (the speed of the erosion):

$$\theta = -\frac{\partial V}{\partial t}. \tag{32}$$



(a) Heat diffusion over time.



(b) Call option price evolution as $T-t$ decreases.

Figure 1: Comparison between heat diffusion and option price decay.

As an illustration, consider the solution of the heat equation for a metal rod initially heated at its center ($x = 0$) and kept cool at the ends. The initial temperature profile resembles a sharp Gaussian peak that gradually spreads and flattens over time due to thermal diffusion, as shown in Figure (1a). Similarly, for an out-of-the-money (OTM) call option, the price initially has a greater time value localized around the strike. Over time, as $T-t$ decreases, the value erodes toward zero (becoming the exclusively the intrinsic value in (31)) due to the vanishing time value, as illustrated in Figure (1b).

The connection between time decay and the curvature of both solutions arises from the structure of the PDEs themselves. Both (1) and (8) link the first time derivative ($\partial V/\partial t$) to the second spatial derivative (either $\partial^2 u/\partial x^2$ in the heat equation or $\partial^2 V/\partial S^2$ in the Black-Scholes equation). In the option price context, this second derivative corresponds to the option's gamma:

$$\Gamma = \frac{\partial^2 V}{\partial S^2}, \tag{33}$$

which quantifies the curvature of the option price with respect to the underlying asset price. Steeper gamma (more convexity) leads to faster changes in theta — just as sharper temperature gradients accelerate heat diffusion [5, 4].

Thus, both equations capture the essence of diffusion: the progressive smoothing of gradients over time, whether in physical temperature or in an option value.

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