Introducing the GBM model

Definition 2.1. *A stochastic process* $\{S(t)\}_{t_0 \leq t \leq T}$ *is said to follow a Geometric Brownian Motion (GBM) if it is a solution to the stochastic differential equation (SDE), which is a particular case of the Itô process defined in equation (8) [9, 7, 8]:*

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{P}}(t) \tag{14}$$

where $\mathrm{d}W(t)$ *is a differential Wiener process under the* $\mathbb{P}$ *measure, and where* $\mu S(t)$ *and* $\sigma S(t)$ *represent the drift and diffusion terms, respectively, with* $\mu, \sigma \in \mathbb{R}$ *constants. This process is commonly used to model stock prices* $S(t)$ *of underlying assets in derivative contracts, such as no-dividend paying stock options [1, 2, 5].*

Notice that in the GBM definition, both the drift and diffusion terms depend on the stochastic process $S(t)$ itself. As a result, we cannot directly apply the expected value expression (10) or the variance expression (11) derived for general Itô processes with additive coefficients. Consequently, the increment $dS(t)$ does not follow a normal distribution as described in (12). We can then proceed to rewrite equation (14) in the following form:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{P}}(t). \tag{15}$$

Here $\mathrm{d}S(t)/S(t)$ represents the percentage return from the stock [5]: how much the stock price changes from $S(t)$ to $S(t) + \mathrm{d}S(t)$ in a time interval $[t, t + \mathrm{d}t]$. Using equation (14) under the format (15), lets consider the dynamics of the percentage return on the stock $\mathrm{d}S(t)/S(t)$. Evidently the drift and diffusion terms become constant, enabling us to apply (12):

$$\frac{\mathrm{d}S(t)}{S(t)} \sim \mathcal{N}(\mu\mathrm{d}t, \sigma^2\mathrm{d}t). \tag{16}$$

Concluding that although we cannot say that $\mathrm{d}S(t)$ follows a normal distribution, the percentage return on the stock $\mathrm{d}S(t)/S(t)$ does follow one according to (16) [5, 10].

Suppose $\sigma = 0$ in (15), in this scenario we could write (14) as a deterministic equation:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu\mathrm{d}t.$$

Integrating from t_0 to t ($t_0 \leq t \leq T$) yields:

$$S(t) = S(t_0)e^{\mu(t-t_0)}, \tag{17}$$

hence, when there's no uncertainty (described by the diffusion term), the evolution of the stock price is going to be the future value under a continuously compounded rate μ. Here we can see that the coefficient μ is related to the log-return from the stock $S(t)$[4] [5, 10]. From (16) equation's perspective, we have that the coefficient μ can be regarded as the average behavior of the return of the stock $\mathbb{E}[\mathrm{d}S(t)/S(t)] = \mu\mathrm{d}t$ over a small amount of time $\mathrm{d}t$ (from (10)) or, as we expressed when describing the drift term, to the trend movement

[4] Equation (17) shows that $\mu = [\ln S(t) - \ln S(t_0)]/(t - t_0)$ or the average rate of change of the log-price per unit time - also known as the log-return on the stock. Which is essentially different from the percentage return on the stock price $\mathrm{d}S(t)/S(t)$.

of this stochastic process through time. From these properties, we state that the coefficient μ represents under the GBM model for stocks the *expected return* (annualized) earned by an investor on the stock in a short period o time dt [5].

Given μ being the representation of the expected return from the stock of price $S(t)$, we can enumerate its dependencies. If an investment has higher risk (a large chance of failure), the investor will usually require an evenly higher return from the stock, thus μ's value should be regulated by the risk of return from the stock [9, 5]. Another influence in μ is the interest rate (IR) from the economy where this investment is inserted in. A higher level of IR implies a higher expected return from any stock required by the investors [5]. Luckily as we will see, the price of an vanilla option contract won't depend on μ [2, 1].

In practice there will be uncertainty when dealing with stock prices, this statement aligns with the *efficient market hypothesis* which suggests that asset prices changes are essentially random [3]. Under the GBM model this role is portrayed by the diffusion term σ and the Wiener process $dW(t)$ with σ serving as an amplitude for the random fluctuations [9, 5]. When looking under equation (15) framework, the coefficient σ can be associated with the variance of the stock price return: $\mathrm{Var}[dS(t)/S(t)] = \sigma^2 dt$ over a small period of time dt (from (11)) implying that it has an influence over how abruptly the price of a particular stock deviates from the expected value in dt [5]. We conclude that parameter σ measures how volatile a stock price will behave in a short period of time, naming it the *volatility* of the stock.

Subsequently, we apply Itô lemma to a function $f(S(t), t) = \ln S(t)$, taking the result into account[4]:

$$\begin{aligned}
d\,[\ln S(t)] &= \left[\frac{\partial}{\partial t}(\ln S) + \mu S \frac{\partial}{\partial S}(\ln S) + \frac{1}{2}\sigma^2 S \frac{\partial^2}{\partial S^2}(\ln S)\right] dt + \sigma S \frac{\partial}{\partial S}(\ln S) dW^{\mathbb{P}}(t) \\
&= \left[0 + \mu S\left(\frac{1}{S}\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\right)\right] dt + \left(\frac{1}{S}\right)\sigma S dW^{\mathbb{P}}(t)
\end{aligned}$$

$$d\,[\ln S(t)] = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW^{\mathbb{P}}(t). \tag{18}$$

Since we have deterministic drift $(\mu - \sigma^2/2)$ and diffusion σ terms, we can apply (12) yielding

$$d\,[\ln S(t)] \sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right) dt, \sigma^2 dt\right]. \tag{19}$$

When the logarithm of a random variable has a normal distribution $\ln X \sim \mathcal{N}(\cdot,\cdot)$, it's said that it follows a lognormal distribution. Hence we can say that the GBM dynamics $dS(t)$ is lognormally distributed according to (19) [9, 7], with mean $(\mu - \sigma^2/2)dt$ and variance $\sigma^2 dt$.

To find $\ln S(t)$ according to the GBM model (14), we solve the SDE (19). Integrating (19) from t_0 to t ($t_0 \leq t \leq T$) produces the following result[5]:

$$\begin{aligned}
\int_{t_0}^{t} d\,[\ln S(t')] &= \int_{t_0}^{t}\left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \int_{t_0}^{t} dW^{\mathbb{P}}(t') \\
\ln S(t) - \ln S(t_0) &= \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma(W^{\mathbb{P}}(t) - W^{\mathbb{P}}(t_0))
\end{aligned}$$

[5]Notice that the following equation is valid for an Itô's stochastic process such as $\{X(t)\}_{t_0 \leq t \leq T}$ according to Itô calculus [9, 7]:

$$\int_{t_0}^{t} dX(t') = X(t) - X(t_0)$$

$$\ln S(t) = \ln S(t_0) + \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma(W^{\mathbb{P}}(t) - W^{\mathbb{P}}(t_0)), \tag{20}$$

yielding a lognormal distribution for the stock $S(t)$:

$$\ln S(t) \sim \mathcal{N}\left[\ln S(t_0) + \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0),\ \sigma^2(t - t_0)\right]. \tag{21}$$

Equivalently, equation (20) can be written as:

$$S(t) = S(t_0) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma(W^{\mathbb{P}}(t) - W^{\mathbb{P}}(t_0))\right], \tag{22}$$

where, for $\sigma = 0$, we retrieve the deterministic result (17).

The expected value of $S(t)$ (22) can then be calculated as follows (the derivation is displayed in the appendices):

$$\mathbb{E}[S(t)] = S(t_0)e^{\mu(t-t_0)}. \tag{23}$$

Here we notice that the average behavior of the stock price over time is an analogous result for the deterministic version of the $S(t)$ process proposed in equation (17), with $\sigma = 0$. Concluding that $\mathbb{E}[S(t)]$ can be regarded as the future value of $S(t_0)$ with rate μ from t_0 to t, not being dependent on the volatility of the process [9, 4]. The variance of $S(t)$ is given by the following equation (the derivation is displayed in the appendices):

$$\mathrm{Var}[S(t)] = S(t_0)^2 e^{2\mu(t-t_0)}(e^{\sigma^2(t-t_0)} - 1), \tag{24}$$

where for $\sigma = 0$ or no uncertainty, we evidently get a null result as it would be expected. In summary, $S(t)$ follows a lognormal distribution of the form[6] [9, 7]:

$$S(t) \sim \mathrm{LN}\left[S(t_0)e^{\mu(t-t_0)},\ S(t_0)^2 e^{2\mu(t-t_0)}(e^{\sigma^2(t-t_0)} - 1)\right]. \tag{25}$$

Appendices

Derivation expected value and variance of $S(t)$

First we derive the expected value for a lognormal process $Y(t) = e^{X(t)}$ with $X(t)$ being an integrated from t_0 to t Itô process (8) with parameters a and b non-stochastic:

$$X(t) = a(t - t_0) + b[W(t) - W(t_0)]. \tag{26}$$

Noticing that $X(t)$ follows a normal distribution $X(t) \sim \mathcal{N}[a(t-t_0), b^2\sqrt{(t-t_0)}]$ for a and b non-stochastic (from property **(3)** in Wiener's process definition (1.3)). We can use the moment generating function for a Gaussian distribution $Z(t) \sim \mathcal{N}(\bar{\mu}, \bar{\sigma}^2)$:

$$M_Z(\tau) = \mathbb{E}\left(e^{\tau Z}\right) = e^{\bar{\mu}\tau + \frac{\bar{\sigma}^2}{2}\tau^2}, \tag{27}$$

[6] Here the notation used is such that for a lognormally distributed random variable X of mean $\bar{\mu}$ and variance $\bar{\sigma}^2$, we can say that $X \sim \mathrm{LN}(\bar{\mu}, \bar{\sigma}^2)$.

setting $\tau = 1$ for our Itô's process $X(t)$ with $\bar{\mu} = a(t - t_0)$ and $\bar{\sigma} = b\sqrt{(t - t_0)}$ yielding:

$$\mathbb{E}(Y) = \mathbb{E}\left(e^X\right) = e^{a(t-t_0)+\frac{b^2}{2}(t-t_0)}. \tag{28}$$

We can then apply (28) for the particular case of the stock price $S(t)$ in (22), starting by taking the expected value on both sides:

$$\begin{aligned}
\mathbb{E}[S(t)] &= S(t_0)e^{\left(\mu-\frac{\sigma^2}{2}\right)(t-t_0)}\mathbb{E}\left[e^{\sigma(W^{\mathbb{P}}(t)-W^{\mathbb{P}}(t_0))}\right] \\
&\overset{(28)}{=} S(t_0)e^{\left(\mu-\frac{\sigma^2}{2}\right)(t-t_0)}e^{0+\frac{\sigma^2}{2}(t-t_0)} \\
&= S(t_0)e^{\mu(t-t_0)},
\end{aligned}$$

where $\sigma[W^{\mathbb{P}}(t)-W^{\mathbb{P}}(t_0)] \sim \mathcal{N}[0, \sigma^2(t-t_0)]$ according to Wiener's process definition property **(3)**, matching the presented result (23).

Next we derive the variance of the lognormal process $Y(t) = e^{X(t)}$ using (28) and noticing that $\mathbb{E}(2X) = 2\mathbb{E}(X)$ and $\text{Var}(2X) = 4\text{Var}(X)$:

$$\begin{aligned}
\text{Var}(e^X) &= \mathbb{E}\left(e^{2X}\right) - \mathbb{E}\left(e^X\right)^2 \\
&\overset{(28)}{=} e^{2a(t-t_0)+\frac{4b^2}{2}(t-t_0)} - e^{2a(t-t_0)+2\frac{b^2}{2}(t-t_0)} \\
&= e^{2a(t-t_0)}\left[e^{2b^2(t-t_0)} - e^{b^2(t-t_0)}\right]
\end{aligned}$$

or

$$\text{Var}(Y) = \text{Var}\left(e^X\right) = e^{(2a+b^2)(t-t_0)}\left[e^{b^2(t-t_0)} - 1\right]. \tag{29}$$

This result can then be used to achieve the variance in the particular case $S(t)$, starting by taking the variance on both sides and keeping only the stochastic terms inside the $\text{Var}(\cdot)$:

$$\begin{aligned}
\text{Var}[S(t)] &= S(t_0)^2 e^{2\left(\mu-\frac{\sigma^2}{2}\right)(t-t_0)}\text{Var}\left[e^{\sigma(W^{\mathbb{P}}(t)-W^{\mathbb{P}}(t_0))}\right] \\
&= S(t_0)^2 e^{(2\mu-\sigma^2)(t-t_0)}\left\{e^{(2\cdot 0+\sigma^2)(t-t_0)}\left[e^{\sigma^2(t-t_0)} - 1\right]\right\} \\
&= S(t_0)^2 e^{(2\mu-\sigma^2)(t-t_0)}\left\{e^{\sigma^2(t-t_0)}\left[e^{\sigma^2(t-t_0)} - 1\right]\right\} \\
&= S(t_0)^2 e^{2\mu(t-t_0)}\left[e^{\sigma^2(t-t_0)} - 1\right],
\end{aligned}$$

again we used $\sigma[W^{\mathbb{P}}(t) - W^{\mathbb{P}}(t_0)] \sim \mathcal{N}[0, \sigma^2(t - t_0)]$ matching (24).

References

[1] Fischer Black and Myron Scholes. “The pricing of options and corporate liabilities”. In: *Journal of political economy* 81.3 (1973), pp. 637–654.

[2] Fischer Black and Myron Scholes. “The valuation of option contracts and a test of market efficiency”. In: *The Journal of finance* 27.2 (1972), pp. 399–417.

[3] Eugene F. Fama. “Efficient Capital Markets: A Review of Theory and Empirical Work”. In: *The Journal of Finance* 25.2 (1970), pp. 383–417. DOI: 10.2307/2325486.

[4] Lech A. Grzelak and Cornelis W. Oosterlee. *Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*. World Scientific Publishing Europe Ltd, 2019. ISBN: 978-1786348050.

[5] John C. Hull. *Options, Futures, and Other Derivatives*. 10th ed. Pearson, 2022. ISBN: 978-1-292-40856-2.

[6] Kiyosi Ito et al. *On stochastic differential equations*. Vol. 4. American Mathematical Society New York, 1951.

[7] Ioannis Karatzas and Steven E Shreve. *Brownian motion*. Springer, 2021, pp. 47–127.

[8] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[9] Steven E Shreve et al. *Stochastic calculus for finance II: Continuous-time models*. Vol. 11. Springer, 2004.

[10] Paul Wilmott, Jeff Dewynne, and Sam Howison. *Option Pricing: Mathematical Models and Computation*. Oxford Financial Press, 1994. ISBN: 978-0952208204.