PDF and CDF for the GBM model

Through our progress of the understanding of the GBM model for the stock price $S(t)$, we figured out that $\ln S(t)$ follows a normal distribution according to (21) [9, 7]. Lets then use this distribution to determine the CDF (cumulative density function) and PDF (probability density function) for $\ln S(t)$. For $X(t) \sim \mathcal{N}(\bar{\mu}, \bar{\sigma})$ a random process under probability measure $\mathbb{P}$, the CDF is given by

$$F_{X(t)}(x) = \mathbb{P}[X(t) \leq x] = \frac{1}{\sqrt{2\pi\bar{\sigma}}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{x' - \bar{\mu}}{\bar{\sigma}}\right)^2\right] \mathrm{d}x'. \tag{26}$$

with PDF

$$f_{X(t)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X(t)}(x) = \frac{1}{\sqrt{2\pi\bar{\sigma}}} \exp\left[-\frac{1}{2}\left(\frac{x - \bar{\mu}}{\bar{\sigma}}\right)^2\right]. \tag{27}$$

for $\ln S(t)$ normal distribution (21), we'll have [4]

$$X(t) = \ln S(t),\ \ \bar{\mu} = \ln S(t_0) + \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0) \text{ and } \bar{\sigma}^2 = \sigma^2(t - t_0)$$

yielding the CDF:

$$F_{\ln S(t)}(x) = \mathbb{P}[\ln S(t) \leq x] = \frac{1}{\sigma\sqrt{2\pi(t - t_0)}} \int_{-\infty}^{x} \exp\left\{-\frac{\left[x' - \ln S(t_0) - \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0)\right]^2}{2\sigma^2(t - t_0)}\right\} \mathrm{d}x' \tag{28}$$

with PDF

$$f_{\ln S(t)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{\ln S(t)}(x) = \frac{1}{\sigma\sqrt{2\pi(t - t_0)}} \exp\left\{-\frac{\left[x - \ln S(t_0) - \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0)\right]^2}{2\sigma^2(t - t_0)}\right\}. \tag{29}$$

If instead of $\ln S(t)$ we want to express the CDF and PDF of $S(t)$ itself, first we consider $S(t) = e^{\ln S(t)} = e^{X(t)}$ keeping in mind that

$$\begin{aligned} S(t) &\leq x \\ e^{X(t)} &\leq x \\ X(t) &\leq \ln x \end{aligned}$$

or in other words

$$F_{S(t)}(x) = \mathbb{P}[S(t) \leq x] = \mathbb{P}[X(t) \leq \ln x] = F_{X(t)}(\ln x).$$

Which is exactly (28) for $x \to \ln x$ [4]:

$$F_{S(t)}(x) = \frac{1}{\sigma\sqrt{2\pi(t - t_0)}} \int_{-\infty}^{\ln x} \frac{1}{x'} \exp\left\{-\frac{\left[\ln\left(\frac{x'}{S(t_0)}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0)\right]^2}{2\sigma^2(t - t_0)}\right\} \mathrm{d}x' \tag{30}$$

noticing that $d(\ln x') = dx'/x'$. The PDF could then be derived analogously to (29) [4]:

$$f_{S(t)}(x) = \frac{d}{dx} F_{S(t)}(x) = \frac{1}{\sigma x \sqrt{2\pi(t-t_0)}} \exp\left\{ -\frac{\left[\ln\left(\frac{x}{S(t_0)}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t-t_0)\right]^2}{2\sigma^2(t-t_0)} \right\}. \tag{31}$$

In figure (2) we have in the xy-plane several GBM paths following accordingly to (20). For different fixed dates (from $t = 0$ to $t = 1$ year with timestep 0.25 or a quarter), the PDF from $\ln S(t)$ (29) is shown [4].

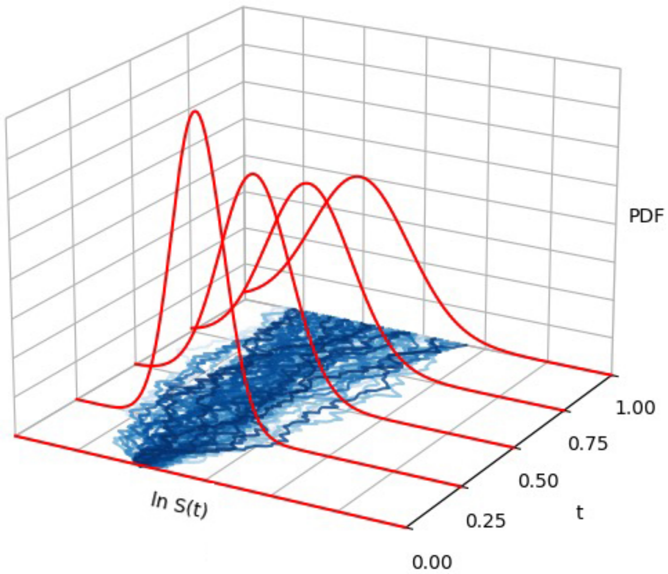

Figure 2: GBM paths from $\ln S(t)$ (xy-plane) and their probability density on the z-axis for fixed dates. The figure utilizes $S(t_0) = e^4$, $\mu = 0.05$, $\sigma = 0.4$, $t_0 = 0$, $T = 1$, number of paths = 100 and $dt = 1/100$.

As we can see in figure (2), uncertainty raises as time passes by, which can be noticed from the tendency of divergence of the paths in later times, meanwhile the variance (width) of the PDF increases [9]. This widening happens due to the linear time dependence of the variance of the distribution (21): $\sigma^2(t - t_0)$. This conclusion implies is that in later horizons, there's a broader set of possible values of $\ln S(t)$ and consequently $S(t)$, becoming less possible to address the value of $S(t)$ as time progresses.

The PDF (29) gets flatter as time passes by in figure (2), or in other words, the probability of x being a specific value $\ln S(t)$ in (29) becomes similar for a wider range of values of x in the real line. Given that $\ln S(t)$'s variance also depends on the volatility σ, this parameters will also have a heavy influence in how fast the spread of the paths will happen.

Although hard to notice from figure (2) the normal distributions (29) mean tends to shift towards smaller values in the $\ln S(t)$-axis as time progresses (not reaching negative values since $\ln S(t) > 0$ always). Given that $\ln S(t)$ mean, showed in (21), is linear dependent on time with an argument $(\mu - \sigma^2/2)$, in the scenario from figure (2), the mean of $\ln S(t)$ follows a decrescent line $\ln S(t_0) + (\mu - \sigma^2/2)(t - t_0)$ since $(\mu - \sigma^2/2) < 0$. This happens since we chose $\mu = 0.05$ and $\sigma = 0.4$ yielding a negative drift of -0.03 and a mean of $\ln(e^4) - 0.03t = 4 - 0.03t$.

Keeping $\mu = 0.05$ and setting a smaller volatility such as $\sigma = 0.2$, would give us a positive drift value of $+0.03$ and a crescent line for $\ln S(t)$ mean: $4 + 0.03t$. Since the drift $4 - 0.03t$ is a line with a small angular parameter, the change in the average direction of the paths is almost unnoticed in figure (2).

Lets now drawn conclusions regarding $S(t)$ behavior in a similar manner as we did for figure (2), recalling that it follows a lognormal distribution (25) with PDF (31), thus having a skewed shape when compared to the normal distribution of $\ln S(t)$ as we can see from figure (3).

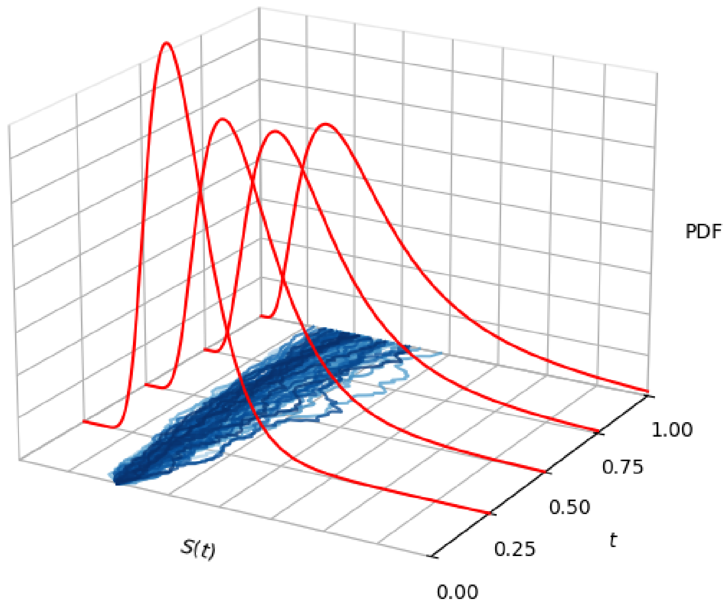

Figure 3: GBM paths from $S(t)$ (xy-plane) and their probability density on the z-axis for fixed dates. The figure utilizes $S(t_0) = 100$, $\mu = 0.08$, $\sigma = 0.25$, $t_0 = 0$, $T = 1$, number of paths = 100 and $\mathrm{d}t = 1/100$.

Observing figure (3), we can see that the red lognormal distributions (31) allow a wider range of values of $S(t)$ than $\ln S(t)$ normal distributions in figure (2), shifting some of the paths to be under its long tail. This is caused by a positive skewness of the PDFs, implying a higher probability of movements towards larger values of $S(t)$ than for the smaller ones [9, 7]. This can be illustrated by some of the paths right below the tail of the curve in the figure.

The range of values expressed by the distributions width increases with time t, drawing the same conclusion we had for $\ln S(t)$ distributions: the uncertainty or the set of possible values of $S(t)$ raises as time progresses due to a time dependence in the variance of $S(t)$ [5]. Although this time, variance of $S(t)$ doesn't change in a linear way, instead varying exponentially according to (24) as we can see from figure (4). Since (24) depends on the square of the volatility's value in the exponential argument, it will also have a high influence in the spreading of the paths.

As we can see from figure (4), for usually small values of μ, the expected value of $S(t)$ (23) tends to become closer to a constant line, even though it is exponential. Meaning μ

won't have that much influence in the behavior of the paths and their PDFs represented in figure (4). The consequence of a larger value of μ would be to create a small tendency in the paths by on average shifting them towards larger values of $S(t)$ and attenuating the peaks from the lognormal PDFs, making the probabilities of different possible values of $S(t)$ more and more homogeneous with time [4].

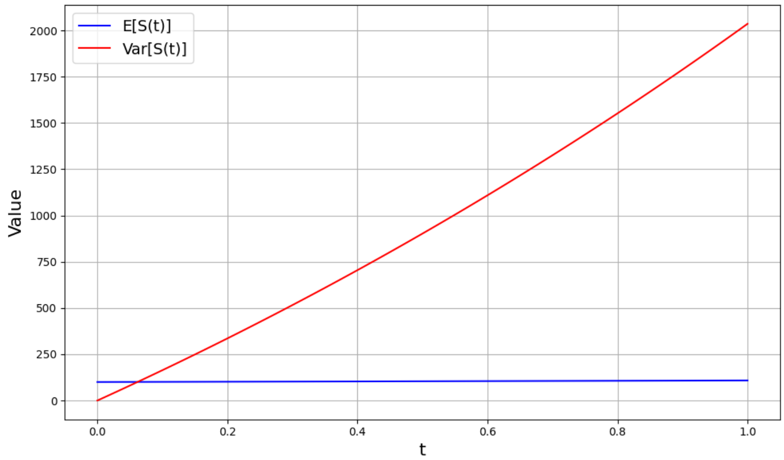

Figure 4: Graphic of the mean and variance of $S(t)$ (lognormally distributed) according to (23) and (24). The figure utilizes $S(t_0) = 100$, $\mu = 0.08$, $\sigma = 0.40$, $t_0 = 0$, $T = 1$, number of paths = 100.

References

[1] Fischer Black and Myron Scholes. “The pricing of options and corporate liabilities”. In: *Journal of political economy* 81.3 (1973), pp. 637–654.

[2] Fischer Black and Myron Scholes. “The valuation of option contracts and a test of market efficiency”. In: *The Journal of finance* 27.2 (1972), pp. 399–417.

[3] Eugene F. Fama. “Efficient Capital Markets: A Review of Theory and Empirical Work”. In: *The Journal of Finance* 25.2 (1970), pp. 383–417. DOI: 10.2307/2325486.

[4] Lech A. Grzelak and Cornelis W. Oosterlee. *Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*. World Scientific Publishing Europe Ltd, 2019. ISBN: 978-1786348050.

[5] John C. Hull. *Options, Futures, and Other Derivatives*. 10th ed. Pearson, 2022. ISBN: 978-1-292-40856-2.

[6] Kiyosi Ito et al. *On stochastic differential equations*. Vol. 4. American Mathematical Society New York, 1951.

[7] Ioannis Karatzas and Steven E Shreve. *Brownian motion*. Springer, 2021, pp. 47–127.

[8] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[9] Steven E Shreve et al. *Stochastic calculus for finance II: Continuous-time models*. Vol. 11. Springer, 2004.

[10] Paul Wilmott, Jeff Dewynne, and Sam Howison. *Option Pricing: Mathematical Models and Computation*. Oxford Financial Press, 1994. ISBN: 978-0952208204.