Lecture 3 - Consumption - Intertemporal Optimization

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Consumption and saving are perhaps the most important and controversial issues in macroeconomics. In the Solow [1956] growth model, saving was a constant fraction s of GDP, by assumption. We now build on Economics 101, in order to derive saving behavior from microeconomic principles. In other words, we work to make saving "endogenous" (that is, explained by the model), while it was previously taken as exogenous (that is, assumed in the model).

Although this discussion may appear somewhat abstract at first, these calculations are the basis of some of the most important controversies in macroeconomics, which we shall come to in the next lectures.

1 The Two-Period Consumption Problem

1.1 Assumptions

There are two periods, $t = 0$ (think of this as "today") and $t = 1$ (think of this as "tomorrow"). The consumer values consumption c_0 in period 0 and c_1 in period 1 according to the following utility function:

$$U(c_0, c_1) = u(c_0) + \beta u(c_1).$$

where $u(.)$ is an increasing and concave function, and $\beta \leq 1$. β captures that people typically have a preference for the present. (they are **present-biased**)

Assume that agents earn (labor) income y_0 in period 0, and (labor) income y_1 in period 1. They also are born with some financial wealth f_0 now, and have financial wealth f_1 in period 1, which they consume entirely because this is the last period. (there is no point keeping more money for after period 1, because there is no future at that point) The amount that agents save in this economy is thus $f_1 - f_0$, and the amount of their accumulated savings is the savings they already had plus what they decided to accumulate, so that $f_0 + (f_1 - f_0) = f_1$.

Therefore, consumption in period 0 is given by:

$$c_0 = y_0 - (f_1 - f_0)$$

The second period consumption ($t = 1$) is given by income plus the return to (accumulated !) savings:

$$c_1 = y_1 + (1 + R) f_1.$$

1.2 Solution

Intertemporal budget constraint. Rewriting f_1 from this second equation: $f_1 = (c_1 - y_1)/(1+R)$, and plugging into the first,

$$c_0 = y_0 - \left(\frac{c_1 - y_1}{1+R} - f_0\right).$$

Rearranging, total wealth is then the sum of financial wealth f_0 and of the present discounted value of human wealth:

$$c_0 + \frac{c_1}{1+R} = \overbrace{f_0 + \underbrace{y_0 + \frac{y_1}{1+R}}_{\text{human wealth}}}^{\text{total wealth}}.$$

The intertemporal budget constraint says that the present discounted value of consumption is equal to total wealth.

Optimization. The problem of the consumer is then simply a problem of maximizing utility:

$$\max_{c_0,c_1} \quad u(c_0) + \beta u(c_1)$$

under a budget constraint:

$$\boxed{c_0 + \frac{c_1}{1+R} = f_0 + y_0 + \frac{y_1}{1+R}}.$$

You may solve this optimization in four different ways:

1. Apply the well known ratio of marginal utilities formula from Econ 101 and say that:

$$\frac{\partial U/\partial c_1}{\partial U/\partial c_0} = \frac{1}{1+R} \quad \Rightarrow \quad \frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+R}$$

2. Apply the following intuitive economic argument. The marginal utility from consuming in period 1 is $\beta u'(c_1)$. The marginal utility from consuming in period 0 is $u'(c_0)$. By putting one unit of consumption in the bank, one forgoes 1 unit of consumption in period 0 to get $1+R$ units of consumption in period 1. The two have to be equal:

$$u'(c_0) = (1+R)\beta u'(c_1) \quad \Rightarrow \quad \frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+R}.$$

3. Replace c_0 from the intertemporal budget constraint above and optimize with respect to c_1:

$$\max_{c_1} \quad u\left[\left(f_0 + y_0 + \frac{y_1}{1+R}\right) - \frac{c_1}{1+R}\right] + \beta u(c_1)$$

This leads to the following First-Order Condition:

$$-\frac{1}{1+R}u'(c_0) + \beta u'(c_1) = 0 \quad \Rightarrow \quad \frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+R}.$$

4. Alternatively, you may substitute c_1 out and optimize with respect to c_0

$$\max_{c_0} \quad u(c_0) + \beta u \left[(1+R)\left(f_0 + y_0 + \frac{y_1}{1+R} \right) - (1+R)c_0 \right]$$

This again leads to the same First-Order Condition:

$$u'(c_0) - \beta(1+R)u'(c_1) = 0 \quad \Rightarrow \quad \frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+R}.$$

1.3 Some examples

Log utility, no discounting ($\beta = 1$). Log utility implies that $u(c)$ is given by the natural logarithm. Marginal utility is then just:

$$u'(c) = \frac{1}{c},$$

Therefore, the above optimality condition (derived 4 times) can be written as:

$$c_0 = \frac{c_1}{1+R}$$

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Substituting out in the intertemporal budget constraint allows to calculate consumption at time 0 c_0 as well as consumption at time 1 c_1:

$$c_0 = \frac{1}{2}\left(f_0 + y_0 + \frac{y_1}{1+R} \right),$$
$$c_1 = \frac{1+R}{2}\left(f_0 + y_0 + \frac{y_1}{1+R} \right).$$

According to this expression, the **Marginal Propensity to Consume (MPC)** out of current wealth f_0 is given by $1/2$. When f_0 rises to $f_0 + \Delta f_0$, the corresponding change in consumption is:

$$\Delta c_0 = \frac{1}{2}\Delta f_0.$$

If we were to study a model with more periods, say T periods, we would find that people Marginal Propensity to Consume is approximately equal to $1/T$, at least according to this model. Whether such is actually the case, and people are that rational, is a subject of fierce debate among macroeconomists, and one that we will take up in the next lectures.

Log utility, with discounting ($\beta < 1$). Marginal utility is then $u'(c) = 1/c$, so that the optimality condition gives:

$$\frac{c_1}{1+R} = \beta c_0.$$

Substituting out in the intertemporal budget constraint, you can check that:

$$c_0 = \frac{1}{1+\beta}\left(f_0 + y_0 + \frac{y_1}{1+R} \right)$$
$$c_1 = \frac{\beta(1+R)}{1+\beta}\left(f_0 + y_0 + \frac{y_1}{1+R} \right).$$

Because people are more impatient in this case, their Marginal Propensity to Consume (MPC) is lower with $\beta < 1$:

$$\Delta c_0 = \frac{1}{1+\beta}\Delta f_0.$$

Note that the solution with no discounting corresponds to that with discounting when $\beta = 1$, which was expected.

2 (YET TO BE DONE IN CLASS) The Overlapping Generations Model

In the Solow [1956] growth model, we assumed that saving was a constant fraction of GDP. The previous section has shown how to use microeconomics, and optimization, in order to derive saving behavior endogenously (that is, to explain it). This section presents a very simple version of the Diamond [1965] so-called **overlapping-generations model** (for reasons that will soon become clear). This model is used not just to give microfoundations to the Solow [1956] model, but also to think about social security, public debt, which we shall take up in the next lectures.

2.1 Assumptions

We assume that people in this economy live only for two periods. However, instead of referring to these two periods as 0 and 1, we now call refer to them as t and $t+1$. People are called "young" in the first period of their life, and "old" in the second. (thus the length of one period is really long !)

People from generation t are young in period t, and old in period $t+1$. We denote their consumption when young by c_t^y and their consumption when old by c_{t+1}^o. In terms of the previous section, you should really think of c_t^y as c_0, and of c_{t+1}^o as c_1.

People work when young, and then receive a wage given by w_t. They retire when old, and then do not work. Their lifetime utility is logarithmic with $\beta = 1$:

$$U = \log(c_t^y) + \log(c_{t+1}^o).$$

Their intertemporal budget constraint is given by:

$$c_t^y + \frac{c_{t+1}^o}{1+R} = w_t.$$

There is always two generations living in period t: the previous period's young, born in period $t-1$, now old, consuming the return from their savings; and this period's young, newly born (in period t).

Production. There is a Cobb-Douglas, constant returns to scale, production function:

$$Y_t = K_t^\alpha L_t^{1-\alpha}.$$

We assume that the labor force is constant and fixed to unity (this is to avoid carrying L around everywhere - from lecture 2, you should now know that everything can be expressed per capita, because of constant returns to scale):

$$L_t = L = 1.$$

Again for simplicity, we shall assume that capital depreciates at rate $\delta = 1 = 100\%$. (that is, capital fully depreciates each period - this is not that unreasonable if you take one unit of time to represent one generation, or about 30 years - remember that the depreciation rate for one year was approximately equal to 10%.)

2.2 Solution

Utility is logarithmic, so that the consumption of the young c_t^y and consumption of the old c_{t+1}^o are given as a function of the wage as follows (this is just an application of the previous section):

$$c_t^y = \frac{w_t}{2} \qquad c_{t+1}^o = (1+R)\frac{w_t}{2}.$$

Indeed, if you want to think of this model as the two periods model of the previous section, think that everything is as if:

$$f_0 = 0, \qquad y_0 = w_t, \qquad y_1 = 0.$$

Saving (and savings) is equal to investment, and given by:

$$S_t = I_t = w_t - c_t^y = \frac{w_t}{2}.$$

Therefore, saving is here endogenous, and coming from agents' optimizing choices. In the Solow model in contrast, saving was taken as exogenous and equal to a fraction s.

The wage paid by employers, given that $L = 1$, is:

$$w_t = (1-\alpha)K_t^\alpha L^{-\alpha} = (1-\alpha)K_t^\alpha = (1-\alpha)Y_t.$$

Finally:

$$\Delta K_{t+1} = \frac{w_t}{2} - \delta K_t = \frac{1-\alpha}{2} Y_t - \delta K_t.$$

This is the capital accumulation equation (or law of motion for capital) of the Solow model, with $s = (1-\alpha)/2$. The new element here of course is to get savings endogenously, from agents' optimal decisions. Note that if $\alpha = 1/3$, then the saving rate is equal to $s = 1/3$, which happens to be (by coincdence) the Golden Rule level of saving.

References

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Robert M. Solow. A Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics*, 70(1):65–94, 1956. ISSN 0033-5533. doi: 10.2307/1884513. URL http://www.jstor.org/stable/1884513.