Lecture 2 - The Solow Growth Model

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2 The Solow Growth Model

What are the sources of GDP growth, and of GDP differences across countries? As a matter of pure accounting, the supply-side approach to GDP presented in lecture **??**, with GDP given as $Y_t = A_t K_t^\alpha L_t^{1-\alpha}$, implies that growth in GDP may arise from technology, capital, or population. Population is natural, and probably not that interesting from an economist's point of view, since the object of interest for an economist usually is GDP per capita.[1]

So we are left with two factors of potential interest: capital accumulation, and technology growth. In this lecture, we study the Solow-Swan model of economic growth, which deals with the problem of capital accumulation. We study two versions of this model. Section 2.1 considers the case of a Solow growth model with a general, constant returns to scale, production function. Section 2.2 looks at a special case of the Solow growth model for a case of a Cobb-Douglas production function.

2.1 General Production Function

2.1.1 The Solow Model

Production function. Robert Solow, starts from a general production function, giving at any point in time output at time t Y_t as a function of two inputs, capital at time t K_t and labor at time t L_t:

$$Y_t = F\left(K_t, L_t\right).$$

He further assumes **constant returns to scale** with respect to capital and labor, so that for any scaling factor a:

$$F(aK_t, aL_t) = aF(K_t, L_t).$$

For example, for $a = 2$, when the quantity of labor and the quantity of capital are doubled, then we get double the quantity of output.[2]

Because of constant returns to scale, everything "scales with population". For simplicity, we shall assume from now on that the quantity of labor is fixed with $L_t = L$, so that the production function becomes $Y_t = F(K_t, L)$. Because of constant returns to scale with respect to capital and labor (and setting $a = 1/L$

[1] Of course, population is very important for discussions of relative economic or military power.

[2] This is sometimes called the replication argument. By replicating a factory with the same number of workers, and the same machines in them, then the new factory should be able to produce as much.

in the previous expression), we can express everything as a function of output per worker Y_t/L and capital per worker K_t/L:

$$\frac{Y_t}{L} = F\left(\frac{K_t}{L}, 1\right) = f\left(\frac{K_t}{L}\right),$$

where f is defined as a function of F such that:

$$f(x) \equiv F(x, 1).$$

An example of such a production function is the Cobb-Douglas production function, which we started studying in lecture **??**, and which we look at in section 2.2.

Saving and Investment. Robert Solow, in his 1956 contribution, abstracts from public saving, so that **total saving** at time t equals **private saving** at time t, and both are denoted S_t, which also equals investment I_t at time t :

$$S_t = I_t.$$

Saving is assumed to be a constant fraction s of output Y_t, and therefore:

$$S_t = sY_t.$$

This constant saving rate my seem a bit ad-hoc; it is. We will investigate more in detail the determinants of saving and consumption behavior in lecture 3. This implies that:

$$\boxed{I_t = S_t = sY_t}.$$

Law of motion for capital. Depreciation of capital is given by a share δ. Think for example that 8% of the capital stock depreciates each period; the rate of depreciation is much lower for structures, and much higher for computers. The capital stock evolves according to:

$$\boxed{K_{t+1} = (1 - \delta) K_t + I_t}.$$

2.1.2 Law of motion for capital per worker

The law of motion for capital, together with the constant saving rate, implies:

$$K_{t+1} = (1 - \delta) K_t + sY_t.$$

Dividing both sides by L to scale everything by the number of worker (so that we are looking at capital per worker, as well as output per worker):

$$\frac{K_{t+1}}{L} = (1 - \delta)\frac{K_t}{L} + s\frac{Y_t}{L} \quad \Rightarrow \quad \boxed{\frac{K_{t+1}}{L} - \frac{K_t}{L} = s\frac{Y_t}{L} - \delta\frac{K_t}{L}}.$$

The change in the capital stock per worker from t to $t + 1$ has two components:

- investment per worker (equivalently, saving per worker) sY_t/L.
- depreciation per worker $\delta K_t/L$.

Therefore, the law of motion for capital per worker is:

$$\underbrace{\frac{K_{t+1}}{L} - \frac{K_t}{L}}_{\text{Change in capital}} = \underbrace{sf\left(\frac{K_t}{L}\right)}_{\text{Investment}} - \underbrace{\delta\frac{K_t}{L}}_{\text{Depreciation}}.$$

2.1.3 Steady-state

The steady state level of the capital stock K^* is such that $K_{t+1} = K_t = K^*$, and it therefore satisfies:

$$\boxed{sf\left(\frac{K^*}{L}\right) = \delta\frac{K^*}{L}}.$$

Note that without further specifying $f(.)$, we can't say much more about the value of K^*/L, we just know it satisfies this implicit equation. The steady-state value of output per worker Y^*/L, as a function of K^*/L is given by:

$$\frac{Y^*}{L} = f\left(\frac{K^*}{L}\right).$$

2.1.4 Convergence to the steady-state

Without further specifying the shape of the production function, we may identify three cases. In all cases, there is convergence to the steady-state level of capital per worker K^*/L which was identified in the previous section.

1. If capital per worker is initially relatively low, that is $K_t/L < K^*/L$, then investment per worker is larger than depreciation per worker, and therefore from the above equation, capital per worker increases:

$$\boxed{\frac{K_t}{L} < \frac{K^*}{L} \quad \Rightarrow \quad \frac{K_{t+1}}{L} > \frac{K_t}{L}}.$$

2. If capital per worker is exactly equal to steady state capital per worker, that is $K_t/L = K^*/L$, then investment per worker is equal to depreciation per worker, and therefore from the above equation, capital per worker stays constant:

$$\boxed{\frac{K_t}{L} = \frac{K^*}{L} \quad \Rightarrow \quad \frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L}}.$$

3. If capital per worker is relatively high, that is $K_t/L > K^*/L$, then depreciation per worker is larger than investment per worker, and therefore, capital per worker declines:

$$\boxed{\frac{K_t}{L} > \frac{K^*}{L} \quad \Rightarrow \quad \frac{K_{t+1}}{L} < \frac{K_t}{L}}.$$

2.2 Cobb-Douglas production function

2.2.1 Production function

Assume now that the production function is a Cobb-Douglas production function, so that:

$$F(K, L) = K^\alpha L^{1-\alpha}.$$

As we saw during lecture **??**, α should be thought of as roughly equal to 1/3. This implies then that function f defined above is such that:

$$f(x) = x^\alpha.$$

2.2.2 Law of motion for capital

The law of motion for capital is given by:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} + s\left(\frac{K_t}{L}\right)^{\alpha} - \delta\frac{K_t}{L}.$$

Given L, K_0, α, s, δ, we are able to calculate K_1, K_2, $\dots$, as well as K_t for any t, by calculating the quantities of capital successively from the formula above. If you do so, you will notice that K_t converges to a steady state value K^*. However, you do not need to perform an infinity of operations to get at this K^*.

2.2.3 Steady-state

Capital per worker. In the Cobb-Douglas case, there is a simpler way to get steady-state capital per worker: what was an implicit equation in section 2.1 can now be solved for explicitely. Capital per worker in steady-state K^*/L solves:

$$s\left(\frac{K^*}{L}\right)^{\alpha} = \delta\frac{K^*}{L} \quad \Rightarrow \quad \boxed{\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}}.$$

Output per worker. The steady-state level of output per worker is then:

$$\boxed{\frac{Y^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}}.$$

Capital-Output Ratio. We are also able to compute the capital to output ratio K^*/Y^* from the Solow growth model:

$$\begin{aligned}
\frac{K^*}{Y^*} &= \frac{K^*/L}{Y^*/L} \\
&= \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{-\frac{\alpha}{1-\alpha}} \\
\frac{K^*}{Y^*} &= \frac{s}{\delta}.
\end{aligned}$$

Alternatively, you may obtain this expression much more simply by equating saving sY^* to investment δK^* in the steady state:

$$sY^* = \delta K^* \quad \Rightarrow \quad \boxed{\frac{K^*}{Y^*} = \frac{s}{\delta}}.$$

MPK. Another object of potential interest is the steady-state marginal product of capital (also called Rental Rate of capital R^*, see lecture **??**:

$$\begin{aligned}
R^* &= \alpha\left(K^*\right)^{\alpha-1}L^{1-\alpha} \\
&= \alpha\left(\frac{K^*}{L}\right)^{\alpha-1} \\
&= \alpha\left(\frac{s}{\delta}\right)^{\frac{\alpha-1}{1-\alpha}} \\
R^* &= \frac{\alpha\delta}{s}
\end{aligned}$$

Therefore, the steady-state marginal product of capital is given by:

$$\boxed{R^* = \frac{\alpha\delta}{s}}.$$

Net MPK. The net marginal product of capital r^*, which is the gross marginal product of capital minus depreciation (relevant for calculating the real return to an investor) is thus given by:

$$r^* = R^* - \delta$$
$$r^* = \frac{\alpha\delta}{s} - \delta$$

Therefore, the steady-state **net** marginal product of capital is given by:

$$\boxed{r^* = \frac{\alpha\delta}{s} - \delta}.$$

Consumption per worker. Finally, steady-state consumption per worker is given by:

$$\begin{aligned}\frac{C^*}{L} &= (1-s)\frac{Y^*}{L}\\ &= (1-s)\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}\\ \frac{C^*}{L} &= \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}\end{aligned}$$

Thus, consumption per capita at the steady-state is:

$$\boxed{\frac{C^*}{L} = \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}}.$$

2.2.4 Golden Rule

Most economists believe that policymakers should not care so much about GDP per person, but rather about consumption per person (however, some people hold a different view - we shall talk about that later). The intuition is simple: if an economy was to produce many goods which were only used for investment purposes (which would be the case if $s = 1$), then people in this economy would be starving, even though it was actually producing a lot. Investment, ultimately, should serve to increase future consumption.

The **Golden Rule level of capital accumulation** is such that the level of steady-state consumption per capita is maximized. Again, from the previous section, steady-state consumption per capita is given by:

$$\frac{C^*}{L} = \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}.$$

Maximizing this steady state consumption with respect to the saving rate s consists in finding the maximum of that function with respect to s:

$$\frac{d\left(C^*/L\right)}{ds} = 0 \quad \Rightarrow \quad \frac{d\left[(1-s)s^{\frac{\alpha}{1-\alpha}}\right]}{ds} = 0.$$

Note that $1/\delta^{\alpha/(1-\alpha)}$ is a constant which does not change anything to the maximization. If you are not convinced that it is equivalent to maximize the function of s without the constant, then you may as well compute the derivative with respect to the whole C^*/L expression. The first order condition is therefore:

$$-s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)s^{\frac{\alpha}{1-\alpha}-1} = 0.$$

Noting that we can divide by $s^{\alpha/(1-\alpha)}$ and then doing some algebra allows to get at the Golden Rule saving rate:

$$
\begin{aligned}
& -1 + \frac{\alpha}{1-\alpha}(1-s)s^{-1} = 0 \\
\Rightarrow \quad & \frac{\alpha}{1-\alpha}\frac{1-s}{s} = 1 \\
\Rightarrow \quad & \alpha(1-s) = (1-\alpha)s \\
\Rightarrow \quad & \alpha - \alpha s = s - \alpha s \\
\Rightarrow \quad & \boxed{s = \alpha}.
\end{aligned}
$$

Therefore, the saving rate corresponding to the Golden Rule level of capital accumulation is equal to α (again, taking α to be equal to roughly $1/3$, this would suggest that an economy would optimally need to save about a third of its production every year).

We can then use the expressions that were found in section 2.2.3, substituting out the saving rate s with α.

Capital per worker. The Golden Rule level of capital accumulation is then such that capital per worker at the steady-state is:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}} \quad \Rightarrow_{s=\alpha} \quad \boxed{\frac{K_g^*}{L} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}}.$$

Output per worker. At the Golden Rule, steady-state GDP per worker is:

$$\frac{Y^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \quad \Rightarrow_{s=\alpha} \quad \boxed{\frac{Y_g^*}{L} = \left(\frac{\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}}.$$

Capital-Output Ratio. At the Golden Rule, the steady-state capital to output ratio is:

$$\frac{K^*}{Y^*} = \frac{s}{\delta} \quad \Rightarrow_{s=\alpha} \quad \boxed{\frac{K_g^*}{Y_g^*} = \frac{\alpha}{\delta}}.$$

MPK. At the Golden Rule, the steady-state Marginal Product of Capital is:

$$R^* = \frac{\alpha\delta}{s} \quad \Rightarrow_{s=\alpha} \quad \boxed{R_g^* = \delta}.$$

Note: Another way to see that for the Golden Rule level of capital, the marginal product of capital R_g^* necessarily equals the rate of depreciation δ is to note that maximizing consumption C^* is the same as maximizing output minus saving $(1-s)Y^* = Y^* - sY^*$, and that saving in the steady-state equals depreciation in the steady-state (see above) so that $sY^* = \delta K^*$. Therefore, maximizing consumption C^* is maximizing $Y^* - \delta K^*$. But the derivative of that expression implies that:

$$
\begin{aligned}
& \frac{\partial C^*}{\partial K^*} = 0 \quad \Rightarrow \quad \frac{\partial(Y^* - \delta K^*)}{\partial K^*} = 0 \\
\Rightarrow \quad & \frac{\partial Y^*}{\partial K^*} - \delta = 0 \quad \Rightarrow \quad R_g^* = \delta.
\end{aligned}
$$

Intuitively, at the steady-state optimum, the annual gain of an additional unit of capital, $\partial Y^*/\partial K^*$ in additional output, must equal the cost of paying the resulting annual maintainance cost δ for this additional unit. If the marginal gain is higher than the marginal cost, then there is not enough capital. Should the marginal gain be lower than the marginal cost, then there would be too much capital. At the optimum, the marginal gain equals the marginal cost, so $R_g^* = \delta$.

Net MPK. At the Golden Rule, the steady-state net Marginal Product of Capital is:

$$r^* = \frac{\alpha\delta}{s} - \delta \quad \Rightarrow_{s=\alpha} \quad \boxed{r_g^* = 0}.$$

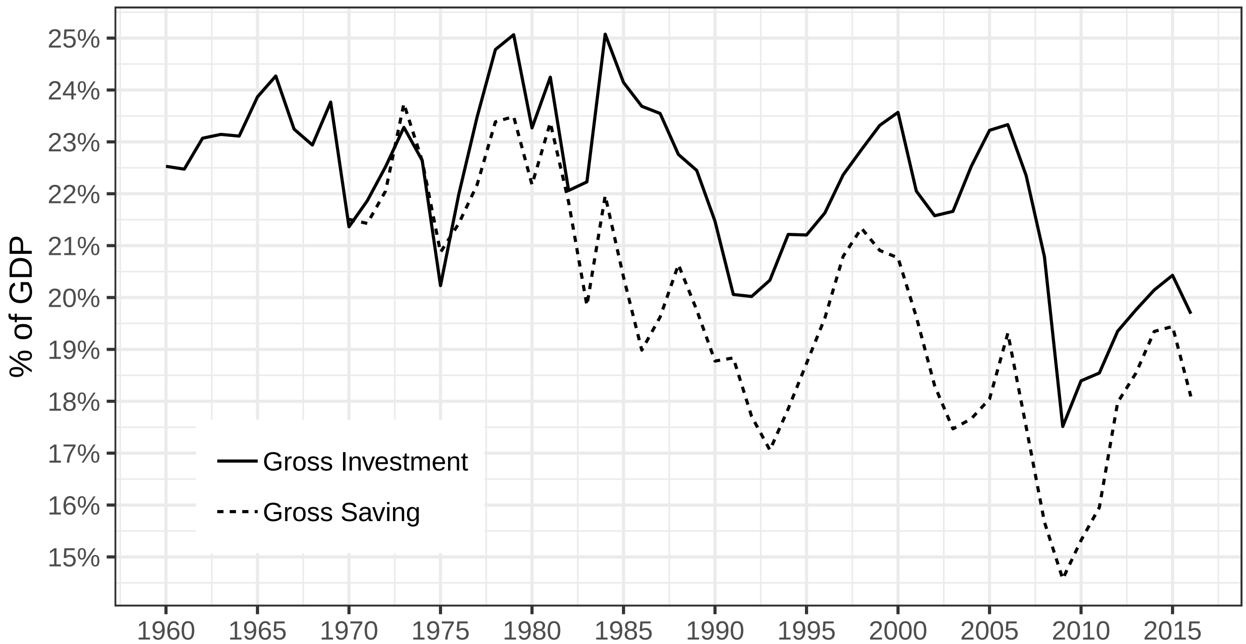

Figure 1: Gross Savings and Investment in the U.S. (WDI).

Therefore, at the Golden Rule level of capital accumulation, the net marginal product of capital is equal to 0. *Note*: Again, this should be obvious from the previous discussion on R_g^*. Maximizing C^* is maximizing $Y^* - \delta K^*$, and the derivative of this expression is equal to r^*. Therefore, at the Golden Rule, the net return is equal to 0:

$$\frac{\partial C^*}{\partial K^*} = 0 \quad \Rightarrow \quad \frac{\partial Y^*}{\partial K^*} - \delta = 0 \quad \Rightarrow \quad r_g^* = 0.$$

Consumption per worker. Finally, at the Golden Rule, consumption per worker is:

$$\frac{C^*}{L} = \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} \quad \Rightarrow_{s=\alpha} \quad \boxed{\frac{C_g^*}{L} = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}}.$$

2.3 Some Data

Figure 1 plots gross saving and investment in the United States from the World Development Indicators put together by the World Bank. Figure 2 plots net saving and gross saving for the U.S.: net saving (gross saving net of depreciation) is slightly higher than 0, but not by much. Figure 3 plots investment as a % of GDP on a map of the world, while figure 4 plots gross saving as a % of GDP.

Readings - To go further

Humans 1, Robots 0, *Wall Street Journal*, October 6, 2013.

(Gated) Economists understand little about the causes of growth, *The Economist*, April 12, 2018.

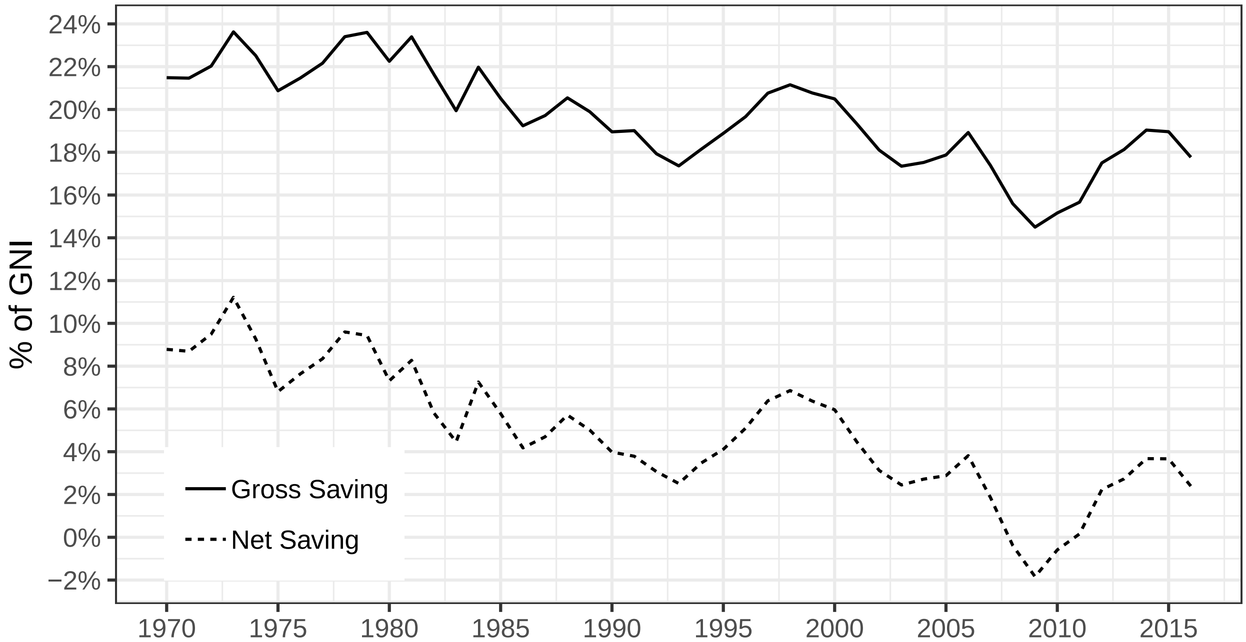

Figure 2: Net Savings and Gross savings (WDI).

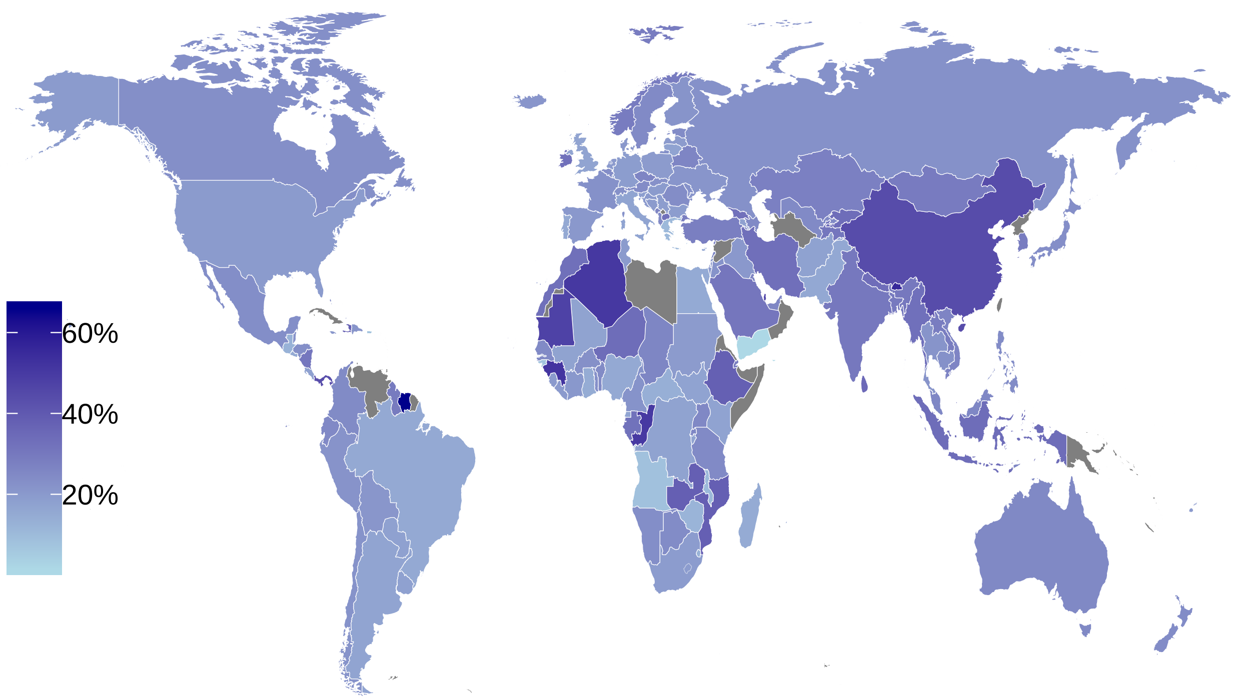

Figure 3: Investment (% of GDP), 2016.

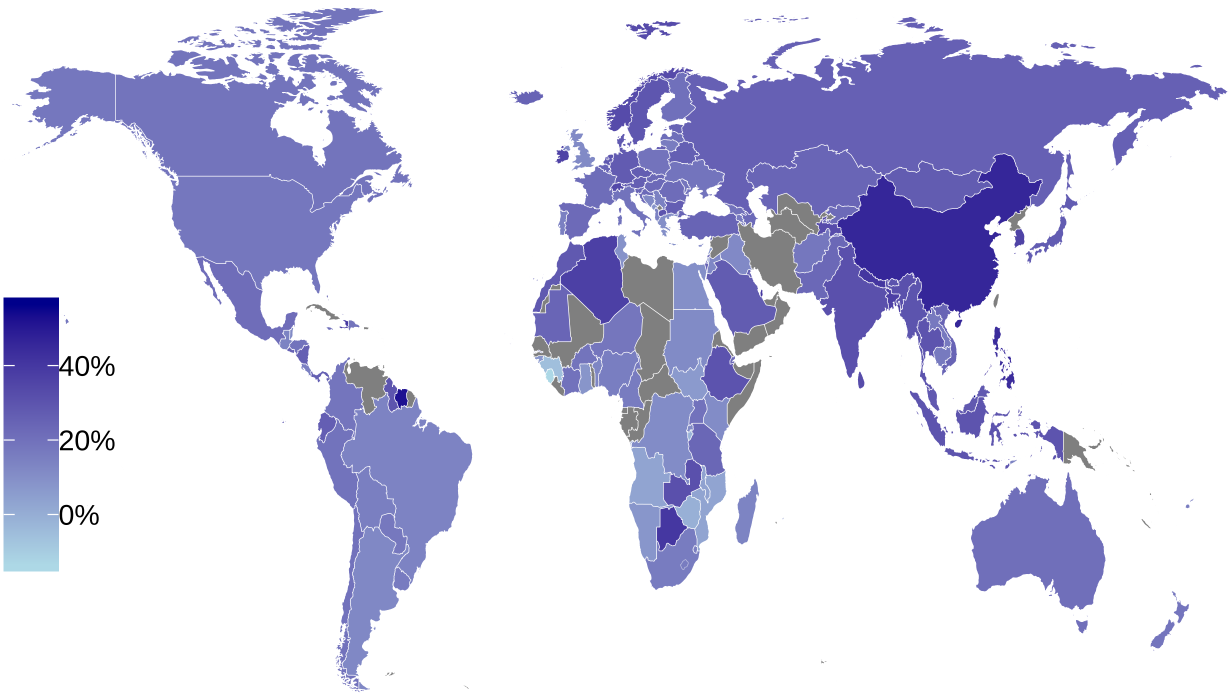

Figure 4: Gross Saving (% of GDP), 2016.