Problem Set 3 - Solutions

UCLA - Econ 102 - Fall 2018

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3 Problem Set 3 - Solution

3.1 Two-period Intertemporal Optimization

1. Given the expression for the utility function:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

we know that marginal utility is:

$$u'(c) = c^{-\sigma},$$

while the derivative of marginal utility is:

$$u''(c) = -\sigma c^{-\sigma-1}.$$

Thus, because $u''(.)$ must be negative for the function to be concave, we have $\sigma > 0$.

2. This is straight from lecture **??**.

3. Using the equation from question 2, we can write:

$$\frac{\beta c_1^{-\sigma}}{c_0^{-\sigma}} = \frac{1}{1+r} \qquad \Rightarrow \qquad \frac{c_1}{c_0} = \beta^{1/\sigma}(1+r)^{1/\sigma}$$

4. The intertemporal budget constraint is:

$$c_0 + \frac{c_1}{1+r} = f_0 + y_0 + \frac{y_1}{1+r},$$

and therefore:

$$\left(1 + \beta^{1/\sigma}(1+r)^{1/\sigma-1}\right) c_0 = f_0 + y_0 + \frac{y_1}{1+r}$$
$$\Rightarrow \quad c_0 = \frac{1}{1+\beta^{1/\sigma}(1+r)^{1/\sigma-1}} \left(f_0 + y_0 + \frac{y_1}{1+r}\right).$$

which implies:

$$c_1 = \frac{\beta^{1/\sigma}(1+r)^{1/\sigma}}{1+\beta^{1/\sigma}(1+r)^{1/\sigma-1}} \left(f_0 + y_0 + \frac{y_1}{1+r}\right)$$

5. Assume $\sigma = 1/2$. If $r = 1\%$, then according to this Google Spreadsheet, c_0 is equal to \$44,776, and c_1 is equal to \$45,676. If $r = 2\%$, then c_0 is equal to \$44,554 and c_1 is equal to \$46,354. Consumption c_0 thus falls by \$222, approximately -0.5% in percentage terms.

6. Assume $\sigma = 1$. If $r = 1\%$, then according to this Google Spreadsheet, c_0 is equal to \$45,000 and c_1 is \$45,450. If $r = 2\%$, then c_0 is equal to \$45,000 and c_1 is equal to \$45,900. Consumption c_0 does not change, this is the case we have seen in class. **Remark.** Note that this case is the one we saw in the class, because when σ approaches 1, we have:

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \log(c).$$

You can see this in many different ways. The simplest way is to write that:

$$c^{1-\sigma} = e^{(1-\sigma)\log(c)} = \exp\left((1-\sigma)\log(c)\right).$$

Then, we use that:

$$\lim_{x\to 0} \frac{e^{ax}-1}{x} = a$$

Indeed, the limit of $(e^{ax}-1)/x$ when x goes to 0 is by definition the derivative of e^{ax} at $x=0$. Thus, since the derivative of e^{ax} is ae^{ax}, we get that the derivative at $x=0$ of e^{ax} is a. Using that formula for $x = 1-\sigma$ and $a = \log(c)$ allows to show:

$$\lim_{(1-\sigma)\to 0} \frac{e^{\log(c)(1-\sigma)}-1}{1-\sigma} = \log(c)$$

Therefore, we get:

$$\lim_{\sigma\to 1} \frac{c^{1-\sigma}-1}{1-\sigma} = \log(c).$$

7. Assume $\sigma = 2$. If $r = 1\%$, then according to this Google Spreadsheet, c_0 is equal to \$45,112 and c_1 is equal to \$45,337. If $r = 2\%$, then c_0 is equal to \$45,223 and c_1 is equal to \$45,673. Consumption c_0 increases by \$111, or approximately 0.25%.

8. Whether an increase in real interest rates leads to a fall or an increase in consumption depends on σ, which can be seen on this formula (it is crucial for this that $y_1 = 0$, or that second-period income is zero):

$$c_0 = \frac{1}{1+\beta^{1/\sigma}(1+r)^{1/\sigma-1}}\left(f_0 + y_0\right).$$

   When $1/\sigma - 1 > 0$, or $\sigma < 1$, an increase in the real interest rate leads to lower consumption today, and more saving. Conversely, when $1/\sigma - 1 < 0$, or $\sigma > 1$, an increase in the real interest rate leads to higher consumption today, and less saving. Finally, when $\sigma = 1$, the interest rate has no effect on current consumption c_0 or saving.

3.2 Another Overlapping Generations model

1. Agents care only about old age consumption, so they save everything, regarless of what the utility function is.

2. Since they save everything, saving is equal to the wage, and thus:

$$S_t = w_t.$$

   The wage paid by employers, given that $L = 1$, is:

$$w_t = (1-\alpha)K_t^\alpha L^{-\alpha} = (1-\alpha)K_t^\alpha = (1-\alpha)Y_t.$$

   This implies, in turn, the following law of motion for the capital stock:

$$\Delta K_{t+1} = S_t - \delta K_t = (1-\alpha)Y_t - \delta K_t.$$

3. The corresponding value of the saving rate in the Solow model is:

$$s = 1-\alpha.$$

4. The Golden rule level of capital accumulation is characterized by a level of the saving rate equal to α. Thus, to be below the Golden Rule level of capital accumulation, the saving rate must be lower than that:
$$1 - \alpha < \alpha.$$
This, in turn, implies:
$$\alpha > \frac{1}{2}.$$
5. This condition is likely not satisfied, as we saw in lecture **??**. Thus, there is too much saving in this situation.