Problem Set 7 - Solutions

UCLA - Econ 102 - Fall 2018

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Another overlapping-generations model with government debt

1. We could use the expressions derived in lecture 3 for a 2-period consumption problem, since utility is logarithmic with $\beta = 2$. However, we will instead derive the formulas from scratch, using the same techniques we used in that lecture (and so should you during an exam). The problem we are looking to solve is the following:

$$\max_{c_t^y, c_{t+1}^o} \quad \log(c_t^y) + 2\log(c_{t+1}^o)$$

$$\text{s.t} \quad c_t^y + \frac{c_{t+1}^o}{1+r_t} = w_t.$$

Again, there are many methods through which we could potentially solve this problem. We can just use the ratio os marginal utilities to get the optimality condition:

$$\frac{1/c_t^y}{2/c_{t+1}^o} = 1 + r_t \quad \Rightarrow \quad c_{t+1}^o = 2(1+r_t)c_t^y.$$

Plugging back in the intertemporal budget constraint:

$$c_t^y + \frac{c_{t+1}^o}{1+r_t} = w_t \quad \Rightarrow \quad c_t^y + 2c_t^y = w_t$$

$$\Rightarrow \quad \boxed{c_t^y = \frac{w_t}{3}}.$$

And finally, plugging this in the optimality condition:

$$c_{t+1}^o = 2(1+r_t)c_t^y \quad \Rightarrow \quad \boxed{c_{t+1}^o = (1+r_t)\frac{2w_t}{3}}.$$

2. Saving is given by $w_t - c_t^y = 2w_t/3$ and depreciation is $\delta = 1$, and therefore the law of motion of the capital stock is:

$$K_{t+1} - K_t = \frac{2w_t}{3} - K_t.$$

From firms' optimality condition, the wage is just the marginal product of labor:

$$w_t = \frac{\partial Y_t}{\partial L_t} = \frac{2}{3}K_t^{1/3}L_t^{-1/3} = \frac{2}{3}K_t^{1/3}.$$

Therefore, we get finally the following law of motion for capital:

$$K_{t+1} = \frac{4}{9}K_t^{1/3}.$$

3. The steady-state capital stock K^* is such that:

$$K^* = \frac{4}{9}(K^*)^{1/3} \quad \Rightarrow \quad (K^*)^{2/3} = \frac{4}{9}$$

$$\Rightarrow \quad K^* = \left(\frac{4}{9}\right)^{3/2} \quad \Rightarrow \quad K^* = \left(\frac{2}{3}\right)^3 \quad \Rightarrow \quad \boxed{K^* = \frac{8}{27}}.$$

The (net) steady-state real interest rate r^* is given by the fact that the marginal product of capital is equal to $r^* + \delta$, which is $1 + r^*$ here:

$$1 + r^* = \frac{1}{3}(K^*)^{-2/3} \quad \Rightarrow \quad r^* = \frac{1}{3}\frac{1}{(K^*)^{2/3}} - 1$$

Thus, using the above expression that $(K^*)^{2/3} = \frac{4}{9}$, we get:

$$r^* = \frac{1}{3} \cdot \frac{9}{4} - 1 \quad \Rightarrow \quad \boxed{r^* = -25\%}$$

Steady-state output Y^* is given by the production function:

$$Y^* = (K^*)^{1/3} = \left(\frac{8}{27}\right)^{1/3} \quad \Rightarrow \quad \boxed{Y^* = \frac{2}{3}}$$

the steady-state wage w^* is given by the marginal product of labor evaluated at the steady-state capital stock:

$$w^* = \frac{2}{3}(K^*)^{1/3} = \frac{2}{3}Y^* \quad \Rightarrow \quad \boxed{w^* = \frac{4}{9}}.$$

The steady-state consumption of the young is:

$$(c^y)^* = \frac{1}{3}w^* \quad \Rightarrow \quad \boxed{(c^y)^* = \frac{4}{27}}.$$

The steady-state consumption of the old is:

$$(c^o)^* = (1 + r^*)\frac{2w^*}{3} = \frac{3}{4} \cdot \frac{8}{27} = \frac{3}{9} \quad \Rightarrow \quad \boxed{(c^o)^* = \frac{2}{9}}$$

4. The Golden Rule (net) interest rate r_g^* is given by $r_g^* = 0$, again because there is no growth. Using this and plugging back to get an expression for $(K^*)_g$:

$$1 + r_g^* = \frac{1}{3}(K_g^*)^{-2/3} \quad \Rightarrow \quad K_g^* = \frac{1}{3^{3/2}} \quad \Rightarrow \quad \boxed{K_g^* = \frac{1}{3\sqrt{3}}}$$

The Golden Rule output Y_g^* is then given by the production function:

$$Y_g^* = (K_g^*)^{1/3} = \left(\frac{1}{3^{3/2}}\right)^{1/3} = \frac{1}{3^{1/2}} \quad \Rightarrow \quad \boxed{Y_g^* = \frac{1}{\sqrt{3}}}$$

The Golden Rule wage w_g^* is:

$$w_g^* = \frac{2}{3}(K_g^*)^{1/3} = \frac{2}{3}Y_g^* \quad \Rightarrow \quad \boxed{w_g^* = \frac{2}{3\sqrt{3}}}.$$

The Golden Rule consumption of the young is:

$$(c^y)_g^* = \frac{1}{3}w_g^* \quad \Rightarrow \quad \boxed{(c^y)_g^* = \frac{2}{9\sqrt{3}}}.$$

The Golden Rule consumption of the old is:

$$(c^o)_g^* = \frac{2}{3}w_g^* \quad \Rightarrow \quad \boxed{(c^o)_g^* = \frac{4}{9\sqrt{3}}}.$$

5. We have the following inequalities:

$$
\begin{aligned}
r_g^* &> r^* \\
K_g^* &< K^* \\
Y_g^* &< Y^* \\
w_g^* &< w^* \\
(c^y)_g^* &< (c^y)^* \quad \text{since} \quad \frac{2}{9\sqrt{3}} < \frac{4}{27} \quad \Leftrightarrow \quad \sqrt{3} < 2 \\
(c^o)_g^* &> (c^o)^* \quad \text{since} \quad \frac{4}{9\sqrt{3}} > \frac{2}{9} \quad \Leftrightarrow \quad 2 > \sqrt{3}
\end{aligned}
$$

The economic intuition is that the capital stock is at a lower steady-state under the Golden-Rule, so that the marginal product of capital is lower, output is lower, and the wage is lower. For consumption, it is higher when old under the Golden Rule (because the return is higher, which more than compensates for the lowest wage) and lower when young because the wage is lower. Overall, the Golden-Rule steady-state utility is given by:

$$U_g^* = \log(c^y)_g^* + 2\log(c^o)_g^* = \log\frac{2}{9\sqrt{3}} + 2\log\frac{4}{9\sqrt{3}}$$

While the steady-state utility is:

$$U^* = \log(c^y)^* + 2\log(c^o)^* = \log\frac{4}{27} + 2\log\frac{2}{9}$$

We compute $U_g^* - U^*$ to see which steady-state utility is greater:

$$
\begin{aligned}
U_g^* - U^* &= \log\frac{2}{9\sqrt{3}} + 2\log\frac{4}{9\sqrt{3}} - \log\frac{4}{27} - 2\log\frac{2}{9} \\
&= \log\left[\frac{2}{9\sqrt{3}} \cdot \left(\frac{4}{9\sqrt{3}}\right)^2 \cdot \frac{27}{4} \cdot \left(\frac{9}{2}\right)^2\right] \\
&= \log\left[\frac{2 \cdot 4^2 \cdot 27 \cdot 9^2}{9\sqrt{3} \cdot 9^2 \cdot 3 \cdot 4 \cdot 4}\right] \\
U_g^* - U^* &= \log\frac{2}{\sqrt{3}} > \log\frac{2}{\sqrt{4}} = \log 1 = 0
\end{aligned}
$$

Thus, we conclude that the Golden Rule level of steady-state utility is higher than the equilibrium level of steady-state utility.

6. What level of government debt B_g^* brings the capital stock to the Golden Rule level ?

7. Starting from the steady-state situation of question 3, assume that the government gives this money to retirees, taking on government debt. How much is this (lucky) generation of retirees able to consume ?

8. Why is national debt a Ponzi scheme here? Is it bad ?

9. Assume that the government puts in place a pay-as-you-go system, such as Social Security (think of OASDI), giving retirees an amount B_g^* each period (where B_g^* is the same level of government debt as the one found in question 6), and taxing the young an equal amount B_g^*. Compare this situation to question 7. What are the differences and similarities?

10. What is the difference between pay-as-you-go financing and deficit financing ? Explain why government debt is not a very meaningful statistic.