

Problem Set 1 - Solutions

UCLA - Econ 102 - Fall 2018

François Geerolf

1 Geometric Sums

1. Let us denote the geometric sum of interest by S :

$$S = \sum_{i=0}^n x^i = 1 + x + x^2 + \dots + x^n.$$

The trick to calculate this sum is to multiply it by x , which allows to get:

$$xS = x + x^2 + x^3 + \dots + x^{n+1}.$$

We can see that this is almost the same sum as the previous one except for the first term, which is missing, and the last term, which was absent from S , therefore:

$$\begin{aligned} xS &= -1 + 1 + x + x^2 + \dots + x^n + x^{n+1} \\ xS &= -1 + S + x^{n+1} \end{aligned}$$

This implies (for $x \neq 1$):

$$1 - x^{n+1} = (1 - x)S \quad \Rightarrow \quad S = \frac{1 - x^{n+1}}{1 - x}.$$

2. For S to have a finite value when $n \rightarrow \infty$, we need that x^{n+1} stays finite. This happens when:

$$|x| < 1.$$

In the knife edge case when $x = 1$, the sum goes to infinity since it is then equal to $n + 1$. If $x = -1$, then the sum oscillates between 1 and -1 and does not have a limit when n goes to infinity.

3. From question 1 and 2, we know that when $|x| < 1$, $x^{n+1} \rightarrow 0$ when $n \rightarrow \infty$, and therefore:

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1 - x}.$$

4. We just factor in x^m and then use the formula in question 1:

$$\begin{aligned} \sum_{i=m}^n x^i &= x^m + x^{m+1} + \dots + x^n \\ &= x^m (1 + x + \dots + x^{n-m}) \\ &= x^m \frac{1 - x^{n-m+1}}{1 - x} \\ \sum_{i=m}^n x^i &= \frac{x^m - x^{n+1}}{1 - x}. \end{aligned}$$

5. The present discounted value of an infinite stream of incomes, which grows at rate $g = 2\%$, starts at $y_0 = 90000$, if the interest rate is $i = 3\%$ is:

$$y_0 + y_0 \frac{1+g}{1+i} + y_0 \frac{(1+g)^2}{(1+i)^2} + \dots$$

Using the formula found in question 3 with $x = (1 + g)/(1 + i)$, we get:

$$y_0 + y_0 \frac{1 + g}{1 + i} + y_0 \frac{(1 + g)^2}{(1 + i)^2} + \dots = y_0 \frac{1}{1 - \frac{1 + g}{1 + i}} = \frac{y_0(1 + i)}{i - g}$$

A numerical application gives (see Google Sheet):

$$\frac{y_0(1 + i)}{i - g} = \frac{90000 * (1 + 0.03)}{0.03 - 0.02} = 9270000.$$

2 Taylor Approximations

1. We have:

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} \cdot x^k$$

When x is small, all the x^k terms for $k \geq 2$ are negligible, and therefore:

$$(1 + x)^n \approx 1 + \binom{n}{1}x = 1 + nx.$$

If you do not know what $\binom{n}{k}$ means, that is fine too. You can prove the same result using recursion. For $n = 1$, we know that $(1 + x)^1 = 1 + x$ (obviously). Assume that the approximation is true for n , or that $(1 + x)^n \approx 1 + nx$, let's prove that it is true for $n + 1$:

$$(1 + x)^{n+1} = (1 + x)^n \approx (1 + nx)(1 + x) \approx 1 + (n + 1)x + nx^2 \approx 1 + (n + 1)x,$$

which proves the proposition for $n + 1$. Thus, the Taylor approximation is true for any $n \in \mathbb{N}$.

2. We have:

$$(1 + x)(1 + y) = 1 + x + y + xy.$$

When x and y are both small, then xy is negligible, which gives the result:

$$(1 + x)(1 + y) \approx 1 + x + y.$$

3. Using the formula proven in Problem 1, we get that:

$$\frac{1}{1 + y} = 1 - y + y^2 - y^3 + \dots$$

When x and y are both small, all terms of the product are negligible except for first-order terms:

$$\frac{1 + x}{1 + y} \approx 1 + x - y.$$

4. Denote the price level by p_t (that is, in dollars, the price of a representative basket of goods). Inflation π_t at time t is defined as the rate of growth of this price level between t and $t + 1$:

$$\frac{p_{t+1}}{p_t} = 1 + \pi_t.$$

If you leave one dollar at the bank, and if the nominal interest rate is given by i_t , then you end up at the end of the period with $1 + i_t$ dollars at the bank. With this, you can buy a quantity of goods given by $(1 + i_t)/p_{t+1}$. If you buy a quantity of goods at time t , then you get a number of goods equal to $1/p_t$. Thus, the rate of increase in your purchasing power if you leave your money in the bank is given by:

$$\frac{(1 + i_t)/p_{t+1}}{1/p_t} = \frac{1 + i_t}{1 + \pi_t}.$$

An exact value for the real interest rate is thus:

$$\begin{aligned}\frac{1+i_t}{1+\pi_t} - 1 &= \frac{1+0.01}{1+0.015} - 1 \\ &= -0.00492610837 \\ \frac{1+i_t}{1+\pi_t} - 1 &= -0.492610837\%.\end{aligned}$$

An approximate value from the above formula is:

$$\begin{aligned}\frac{1+i_t}{1+\pi_t} - 1 &\approx 1+i_t - \pi_t - 1 \\ &\approx 0.01 - 0.015 \\ \frac{1+i_t}{1+\pi_t} - 1 &\approx -0.5\%.\end{aligned}$$

This is not such a bad approximation.

3 Growth Rates

1. Iterating on the formula:

$$y_{t+1} = (1+g)y_t \quad \Rightarrow \quad y_T = (1+g)^T y_0,$$

allows to find the result:

$$G = \frac{y_T}{y_0} - 1 = (1+g)^T - 1.$$

2. Again, inverting the previous relation:

$$G = (1+g)^T - 1,$$

allows to find:

$$g = \frac{y_{t+1}}{y_t} - 1 = (1+G)^{1/T} - 1.$$

3. Applying the previous formula allows to get (see Google Sheet):

$$\begin{aligned}g &= (1+0.01)^{1/365} - 1 \\ &= 0.00002726155 \\ g &= 0.0027\%\end{aligned}$$

You give up approximately \$2.7 every day (your bank most likely is investing this money on your behalf, so you are rather giving this money to your bank):

$$0.00002726155 * 100000 = 2.7.$$