Problem Set 4 - Solutions

UCLA - Econ 102 - Fall 2018

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4 Problem Set 4 - Solution

4.1 The Solow Model with Exogenous Growth

1. The saving rate is exogenous and equal to s in the Solow growth model, and the depreciation rate is δ. Therefore, the law of motion for capital is:

$$\Delta K_{t+1} = K_{t+1} - K_t = sY_t - \delta K_t.$$

Using the value for Y_t, we get:

$$\boxed{K_{t+1} = sA_t K_t^\alpha L_t^{1-\alpha} + (1-\delta)K_t}.$$

which is a law of motion for K_t: a value for K_{t+1} as a function of K_t and the exogenous parameters in the model.

2. Defining k_t as:

$$k_t \equiv \frac{K_t}{A_t^{1/(1-\alpha)} L_t},$$

as is suggested, we divide both the left-hand side and the right-hand side of the equation by $A_t^{1/(1-\alpha)} L_t$. This gives:

$$\frac{K_{t+1}}{A_t^{1/(1-\alpha)} L_t} = s\frac{A_t K_t^\alpha L_t^{1-\alpha}}{A_t^{1/(1-\alpha)} L_t} + (1-\delta)\frac{K_t}{A_t^{1/(1-\alpha)} L_t}$$

We may proceed to a simplification of the first term on the right-hand side by putting the A_t and the $L_t^{1-\alpha}$ from the numerator to the denominator (using that $f/g = 1/(g/f)$):

$$\begin{aligned}
\frac{A_t K_t^\alpha L_t^{1-\alpha}}{A_t^{1/(1-\alpha)} L_t} &= \frac{K_t^\alpha}{A_t^{1/(1-\alpha)-1} L_t^{1-(1-\alpha)}} \\
&= \frac{K_t^\alpha}{A_t^{\alpha/(1-\alpha)} L_t^\alpha} \\
\frac{A_t K_t^\alpha L_t^{1-\alpha}}{A_t^{1/(1-\alpha)} L_t} &= \left(\frac{K_t}{A_t^{1/(1-\alpha)} L_t}\right)^\alpha.
\end{aligned}$$

Thus, replacing out the expression for the first term on the right-hand side allows to write:

$$\frac{K_{t+1}}{A_t^{1/(1-\alpha)} L_t} = s\left(\frac{K_t}{A_t^{1/(1-\alpha)} L_t}\right)^\alpha + (1-\delta)\frac{K_t}{A_t^{1/(1-\alpha)} L_t}.$$

And therefore, the right-hand side is now expressed only a function of k_t:

$$\frac{K_{t+1}}{A_t^{1/(1-\alpha)} L_t} = sk_t^\alpha + (1-\delta)k_t.$$

The left-hand side of the equation can also be simplified (we want to express it also only as a function of k_t (or rather, k_{t+1}):

$$\frac{K_{t+1}}{A_t^{1/(1-\alpha)}L_t} = \frac{A_{t+1}^{1/(1-\alpha)}L_{t+1}}{A_t^{1/(1-\alpha)}L_t} \cdot \frac{K_{t+1}}{A_{t+1}^{1/(1-\alpha)}L_{t+1}}$$
$$\frac{K_{t+1}}{A_t^{1/(1-\alpha)}L_t} = (1+g)^{1/(1-\alpha)}(1+n)k_{t+1}.$$

Therefore:

$$(1+g)^{1/(1-\alpha)}(1+n)k_{t+1} = sk_t^\alpha + (1-\delta)k_t.$$

If g and n are small then:

$$(1+g)^{1/(1-\alpha)}(1+n) \approx 1 + \frac{1}{1-\alpha}g + n.$$

Thus:

$$\left(1 + \frac{1}{1-\alpha}g + n\right)k_{t+1} \approx sk_t^\alpha + (1-\delta)k_t.$$

A law of motion for k_{t+1} is thus (we use equal signs now, even though it is really an approximation):

$$\boxed{k_{t+1} = \frac{s}{1+g/(1-\alpha)+n}k_t^\alpha + \frac{1-\delta}{1+g/(1-\alpha)+n}k_t}.$$

3. The steady-state is such that:

$$\left(1 + \frac{1}{1-\alpha}g + n\right)k^* = s(k^*)^\alpha + (1-\delta)k^*.$$

Therefore:

$$\left(\delta + \frac{1}{1-\alpha}g + n\right)k^* = s(k^*)^\alpha.$$

Finally, this gives k^*:

$$\boxed{k^* = \left(\frac{s}{\delta + g/(1-\alpha)+n}\right)^{\frac{1}{1-\alpha}}}.$$

4. In this exercise, we make intensive use of the following rules on growth rates:

$$g_{XY} = g_X + g_Y$$
$$g_{X/Y} = g_X - g_Y$$
$$g_{X^a} = ag_X.$$

On the balanced growth path:

$$\frac{K_t}{A_t^{1/(1-\alpha)}L_t} = k^* \quad \Rightarrow \quad K_t = k^* A_t^{1/(1-\alpha)}L_t.$$

We may apply the rule above on products ($g_{XY} = g_X + g_Y$) to see that the growth rate of K_t is the growth rate of $A_t^{1/(1-\alpha)}$ plus the growth rate of L_t, since k^* is simply a constant which does not grow. In turn, using the rule on "powers" (that is $g_{X^a} = ag_X$, with $a = 1/(1-\alpha)$) we get that the growth rate of $A_t^{1/(1-\alpha)}$ is the growth rate of A_t times $1/(1-\alpha)$. Finally, the growth rate of A_t is g and the growth rate of L_t is n by assumption. Thus, finally:

$$\begin{aligned} g_K &= g_{A^{1/(1-\alpha)}L} \\ &= g_{A^{1/(1-\alpha)}} + g_L \\ &= \frac{1}{1-\alpha}g_A + g_L \\ g_K &= \frac{1}{1-\alpha}g + n \end{aligned}$$

Output is given by:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

Therefore, the rate of growth of output is:

$$\begin{aligned}
g_Y &= g + \alpha g_K + (1-\alpha) g_L \\
&= g + \alpha \left(n + \frac{1}{1-\alpha} g \right) + (1-\alpha) n \\
&= g + \alpha n + \frac{\alpha}{1-\alpha} g + (1-\alpha) n \\
&= [\alpha n + (1-\alpha) n] + \left[g + \frac{\alpha}{1-\alpha} g \right] \\
g_Y &= n + \frac{1}{1-\alpha} g.
\end{aligned}$$

C_t grows at the same rate as Y_t since $C_t = (1-s) Y_t$, thus:

$$\begin{aligned}
g_C &= g_Y \\
g_C &= n + \frac{1}{1-\alpha} g.
\end{aligned}$$

The rate of growth of K_t/Y_t is zero since Y_t and k_t grow at the same rate:

$$\begin{aligned}
g_{K/Y} &= g_K - g_Y \\
&= \left(\frac{1}{1-\alpha} g + n \right) - \left(\frac{1}{1-\alpha} g + n \right) \\
g_{K/Y} &= 0
\end{aligned}$$

The rate of growth of K_t/L_t is:

$$\begin{aligned}
g_{K/L} &= g_K - g_L \\
&= \left(\frac{1}{1-\alpha} g + n \right) - n \\
g_{K/L} &= \frac{1}{1-\alpha} g.
\end{aligned}$$

The wage is equal to the marginal product of labor from firms' optimality condition, as in lecture **??**:

$$\begin{aligned}
w_t &= \frac{\partial Y_t}{\partial L_t} \\
&= (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} \\
w_t &= (1-\alpha) A_t \left(\frac{K_t}{L_t} \right)^{\alpha}.
\end{aligned}$$

Thus, the rate of growth of w_t is:

$$\begin{aligned}
g_w &= g_A + \alpha g_{K/L} \\
&= g + \frac{\alpha}{1-\alpha} g \\
g_w &= \frac{1}{1-\alpha} g.
\end{aligned}$$

The rate of growth of $w_t L_t$ is the sum of that of w and that of L_t thus:

$$\begin{aligned}
g_{wL} &= g_w + g_L \\
g_{wL} &= \frac{1}{1-\alpha} g + n.
\end{aligned}$$

The marginal product of capital R_t is:

$$\begin{aligned} R_t &= \frac{\partial Y_t}{\partial K_t} \\ &= \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \\ R_t &= \alpha A_t \left(\frac{K_t}{L_t}\right)^{\alpha-1}. \end{aligned}$$

Thus, the rate of growth of the marginal product of capital R_t is:

$$\begin{aligned} g_R &= g_A + (\alpha-1) g_{K/L} \\ &= g + \frac{\alpha-1}{1-\alpha} g \\ &= g - g \\ g_R &= 0 \end{aligned}$$

The rate of growth of capital income $R_t K_t$ is given by:

$$\begin{aligned} g_{RK} &= g_R + g_K \\ g_{RK} &= n + \frac{g}{1-\alpha}. \end{aligned}$$

Finally, the growth in the labor share $w_t L_t$ and that in the capital share $R_t K_t$ are equal to zero which can be inferred from the fact that they are constant with a Cobb-Douglas production function, or that the growth of $w_t L_t$ and $r_t K_t$ are equal to that of output.

5. Using the expression for y_t allows to write what is called the *intensive form* of the production function:

$$\begin{aligned} y_t &= \frac{Y_t}{A_t^{1/(1-\alpha)} L_t} \\ &= \left(\frac{K_t}{A_t^{1/(1-\alpha)} L_t}\right)^{\alpha} \\ y_t &= k_t^{\alpha}. \end{aligned}$$

This implies that the relationship applies also to the steady state, and allows us to calculate steady-state y^* corresponding to steady-state k^*:

$$y^* = (k^*)^{\alpha}$$

From question 3, we replace out k^* in the equation above which gives directly:

$$\boxed{y^* = \left(\frac{s}{\delta + g/(1-\alpha) + n}\right)^{\frac{\alpha}{1-\alpha}}}.$$

Finally, using that $C_t = (1-s)Y_t$ and dividing on both sides by $A_t^{1/(1-\alpha)} L_t$ gives:

$$\frac{C_t}{A_t^{1/(1-\alpha)} L_t} = (1-s)\frac{Y_t}{A_t^{1/(1-\alpha)} L_t}.$$

Using the given definitions for c_t and y_t, this implies that:

$$c_t = (1-s) y_t.$$

From this we can see that this relationship applies also to steady states so that:

$$c^* = (1-s) y^*.$$

Thus, replacing y^* by its expression from previously:

$$\boxed{c^* = (1-s)\left(\frac{s}{\delta + g/(1-\alpha) + n}\right)^{\frac{\alpha}{1-\alpha}}}.$$

6. Just as in the Solow growth model of lecture **??**, we see that we have a constant times a function of s, which simplifies the maximization problem a lot:

$$c^* = \frac{1}{(\delta + g/(1-\alpha) + n)^{\frac{\alpha}{1-\alpha}}}(1-s)s^{\frac{\alpha}{1-\alpha}}$$

We are thus led to maximize only the part which depends on s (if you are not convinced, you can leave the constant there, your calculations will just be more complicated!):

$$\max_s \quad (1-s)s^{\frac{\alpha}{1-\alpha}}.$$

Taking the first-order condition as in lecture **??**:

$$-s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)s^{\frac{\alpha}{1-\alpha}-1} = 0 \quad \Rightarrow \quad \frac{\alpha}{1-\alpha}\frac{1-s}{s} = 1$$
$$\Rightarrow \quad \alpha - \alpha s = s - \alpha s \quad \Rightarrow \quad \boxed{s = \alpha}.$$

7. The marginal product of capital R_t is then equal to:

$$\begin{aligned} R_t &= \alpha A_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} \\ &= \alpha \left(\frac{K_t}{A_t^{1/(1-\alpha)} L_t}\right)^{\alpha-1} \\ R_t &= \alpha k_t^{\alpha-1}. \end{aligned}$$

Therefore, in the steady state, using the above expression for k^* (question 3) we get:

$$\begin{aligned} R^* &= \alpha \left[\left(\frac{s}{\delta + g/(1-\alpha) + n}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha-1} \\ &= \alpha \frac{\delta + g/(1-\alpha) + n}{s} \\ &= \delta + g/(1-\alpha) + n \\ R^* &= \frac{\alpha}{s}\left(\delta + g_Y\right). \end{aligned}$$

where we have used that the rate of growth of output g_Y is given by $g_Y = g/(1-\alpha) + n$ which was proved in an earlier question. Using that $s = \alpha$ at the Golden Rule, we get an expression for the steady-state marginal product of capital:

$$s = \alpha \quad \Rightarrow \quad \boxed{R^* = \delta + g_Y}.$$

Finally, note that the net interest rate at the Golden Rule $R^* - \delta$, which is often denoted by r^* needs to be equal to the rate of growth of output g_Y, to be at the Golden Rule level of capital accumulation:

$$\boxed{r^* = R^* - \delta = g_Y}.$$

We shall encounter this condition again in lecture **??** when we study the sustainability of public debt.

4.2 The Neoclassical Labor Market Model

1. This is straight from lecture **??**. Labor demand is:

$$l = A^{1/\alpha}(1-\alpha)^{1/\alpha}\left(\frac{w}{p}\right)^{-1/\alpha}.$$

Note: If asked about this during an exam, you are required to provide the different steps. And you are not supposed to memorize this formula.

2. See the spreadsheet.
3. Taking logs on both sides leads to:

$$\log(l) = \frac{1}{\alpha}\log A + \frac{1}{\alpha}\log(1-\alpha) - \frac{1}{\alpha}\log\left(\frac{w}{p}\right).$$

   Expressing the log of the real wage as a function of the log of labor demand, since the real wage is on the y-axis:

$$\boxed{\log\left(\frac{w}{p}\right) = [\log A + \log(1-\alpha)] - \alpha\log(l)}.$$

   Therefore, it is clear that the slope of the labor demand curve is given by α. If α is higher, then the labor demand curve is steeper. This result is intuitive: as α is higher, returns to scale become more and more decreasing with respect to labor (if $\alpha = 0$, technology is constant returns in labor in constrast). Therefore, a higher quantity of labor is hired by the firm only if the real wage becomes substantially lower. (in order to "make up for" the decreasing returns) An increase in A clearly shifts the labor demand curve to the **right**: for a given amount of labor hired, a higher productivity implies a higher real wage, both intuitively as well as in the algebra. When the labor demand curve moves to the right, then we move *along the labor supply curve*, towards higher values of employment and higher real wages. A decrease in A, in contrast, shifts the labor demand curve to the **left**, then we move down lower values of employment and lower real wages, along the labor supply curve.
4. Again, this is straight from lecture **??**. Labor supply is:

$$l = \frac{1}{B^{1/\epsilon}}\left(\frac{w}{p}\right)^{1/\epsilon}.$$

   Note: If asked about this during an exam, you are required to provide the different steps. And you are not supposed to memorize this formula.
5. See the spreadsheet. However, in order to plot the two curves on the same graphs, it is best to invert these relationship and to express the real wage as a function of labor demand

$$\frac{w}{p} = A(1-\alpha)l^{-\alpha},$$

   and the real wage as a function of labor supply:

$$\frac{w}{p} = Bl^{\epsilon}.$$

   Again, see the second sheet of the spreadsheet for a plot where both labor supply and labor demand appear.
6. The labor supply curve is also a line in a $(\log(l), \log(w/p))$ plane, because we have a linear relationship between the log labor supply and the log real wage:

$$\boxed{\log\left(\frac{w}{p}\right) = \log B + \epsilon\log(l)}.$$

   The slope of this supply curve on a log-log graph is given by ϵ. If ϵ is larger, the slope is larger. This is intuitive: if the disutility of labor is more convex, then people dislike more working extra hours, and need to be compensated by a much higher real wage to do it. Clearly, if B increases, the labor supply curve moves to the left: people get more disutility from working, and they need to be compensated by a higher real wage to work the same number of hours. On the contrary, when B decreases, people enjoy working much more, and so employers may pay them a low wage to do so.

7. There are many ways to answer this question. I will provide just two. One is to derive the expressions in lecture **??**, using the original versions of labor supply and demand (without logs). The real wage is:

$$\frac{w}{p} = (1-\alpha)^{\frac{\epsilon}{\alpha+\epsilon}} A^{\frac{\epsilon}{\alpha+\epsilon}} B^{\frac{\alpha}{\alpha+\epsilon}}.$$

The level of employment:

$$l = (1-\alpha)^{\frac{1}{\alpha+\epsilon}} A^{\frac{1}{\alpha+\epsilon}} B^{-\frac{1}{\alpha+\epsilon}}$$

Using the spreadsheet, and plugging in the values for $A_1 = 2$, $A_2 = 1.9$, we get:

$$l_1 = 0.9267933073, \qquad \left(\frac{w}{p}\right)_1 = 1.367553862$$
$$l_2 = 0.9179226047, \qquad \left(\frac{w}{p}\right)_2 = 1.303347791$$

The effects of employment of a change in A given by $(A_2 - A_1)/A_1 = -5\%$, are thus a fall in employment and in real wages given by:

$$\frac{l_2 - l_1}{l_1} = -0.96\%, \qquad \frac{\left(\frac{w}{p}\right)_2 - \left(\frac{w}{p}\right)_1}{\left(\frac{w}{p}\right)_1} = -4.69\%.$$

In log changes:

$$\log(l_2) - \log(l_1) = -0.96\%, \qquad \log\left(\frac{w}{p}\right)_2 - \log\left(\frac{w}{p}\right)_1 = -4.81\%.$$

Or we combine the logged versions of these same equations:

$$\log\left(\frac{w}{p}\right) = [\log A + \log(1-\alpha)] - \alpha \log(l).$$

and labor supply:

$$\log\left(\frac{w}{p}\right) = \log B + \epsilon \log(l)$$

we get that:

$$\log B + \epsilon \log(l) = [\log A + \log(1-\alpha)] - \alpha \log(l)$$
$$\Rightarrow \quad \log(l) = \frac{1}{\epsilon+\alpha}[\log A + \log(1-\alpha) - \log B]$$

We may use either the labor demand curve or the labor supply curve to compute the real wage (if everything goes well, they should both give the same answer). We can plug it back in the supply curve:

$$\log\left(\frac{w}{p}\right) = \log B + \frac{\epsilon}{\epsilon+\alpha}[\log A + \log(1-\alpha) - \log B]$$
$$\log\left(\frac{w}{p}\right) = \frac{\alpha}{\epsilon+\alpha}\log B + \frac{\epsilon}{\epsilon+\alpha}\log A + \frac{\epsilon}{\epsilon+\alpha}\log(1-\alpha)$$

Therefore, we get the equilibrium employment:

$$\boxed{\log(l) = \frac{1}{\epsilon+\alpha}[\log A + \log(1-\alpha) - \log B]},$$

as well as the equilibrium real wage:

$$\boxed{\log\left(\frac{w}{p}\right) = \frac{\alpha}{\epsilon+\alpha}\log B + \frac{\epsilon}{\epsilon+\alpha}\log A + \frac{\epsilon}{\epsilon+\alpha}\log(1-\alpha)}.$$

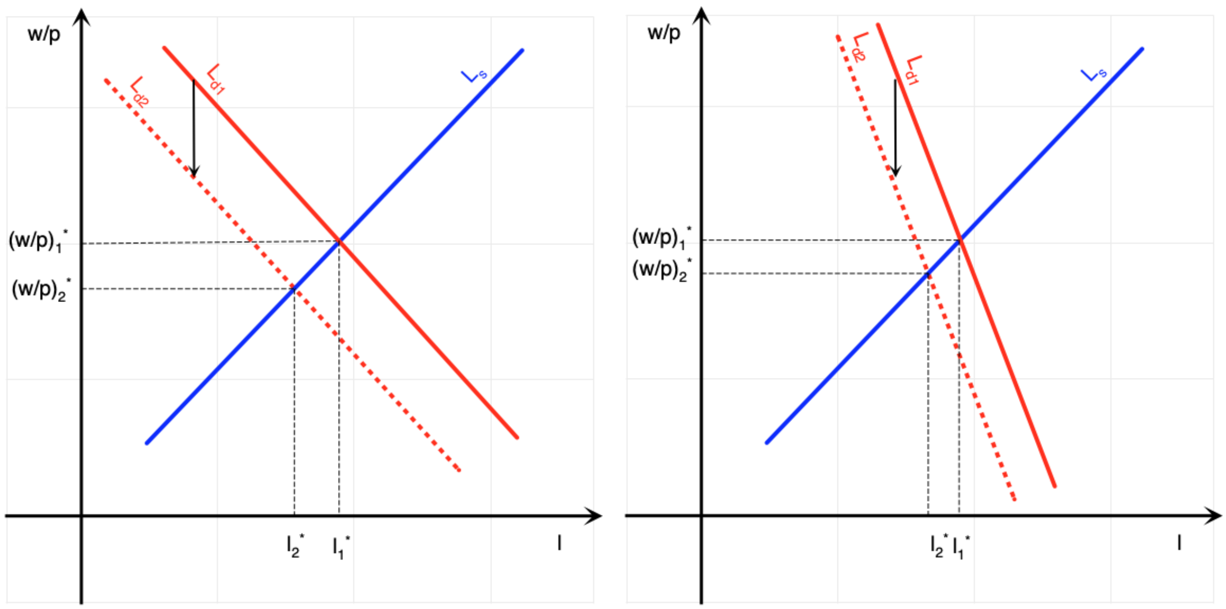

Figure 1: LABOR MARKET: PRODUCTIVITY SHOCK.

This implies that, following a change in productivity A::

$$\Delta \log(l) = \frac{1}{\epsilon + \alpha} \Delta \log A$$

$$\Delta \log \left(\frac{w}{p}\right) = \frac{\epsilon}{\epsilon + \alpha} \Delta \log A$$

The change in A in log points is:

$$\Delta \log A = \log(A_2) - \log(A_1) = -5.13\%.$$

Therefore, in log changes:

$$\log(l_2) - \log(l_1) = -0.96\%, \qquad \log\left(\frac{w}{p}\right)_2 - \log\left(\frac{w}{p}\right)_1 = -4.81\%.$$

If α is higher, then from the above formula the change in employment and in the real wage is lower:

$$\Delta \log(l) = \frac{1}{\epsilon + \alpha} \Delta \log A$$

$$\Delta \log \left(\frac{w}{p}\right) = \frac{\epsilon}{\epsilon + \alpha} \Delta \log A$$

The economic intuition is that a change in A shifts the labor demand curve, and leads to a movement along the labor supply curve. However, the size of this shock is dampened, the larger the amount of decreasing returns to scale. Graphically, the shift in the labor demand curve from a given change shift along the y-axis is lower when the slope is larger. This is shown on the two figures below, showing the shift in the labor demand curve from a given change in A (along the y-axis), when α is low (low decreasing returns) on the left hand side and when α is high (high decreasing returns) on the right hand side. *Note*: You may play around with the spreadsheet to see what happens when parameters are changed.

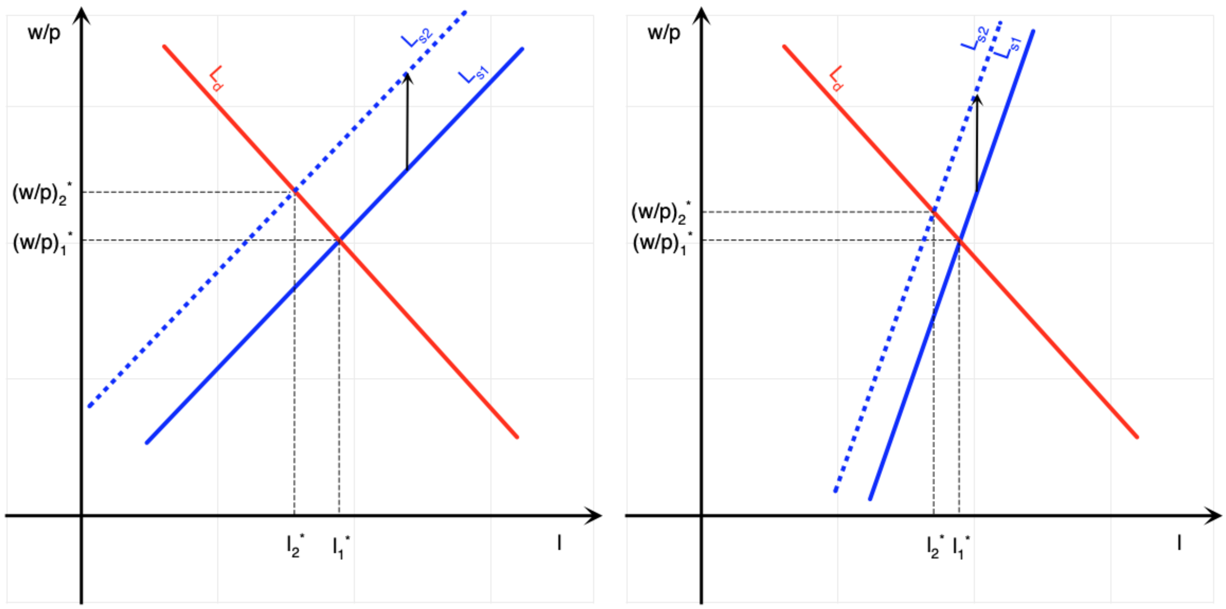

Figure 2: Labor Market: Laziness Shock.

8. (Warning! if this idea of an increase in leisure attractiveness seems a bit peculiar to you, it also seems odd to me. But it has been proposed by some economists as an explanation for unemployment, to explain why the real wage did not fall that much during the recession.) Similar calculations on the spreadsheet and using the same formulas:

$$\frac{w}{p} = (1-\alpha)^{\frac{\epsilon}{\alpha+\epsilon}} A^{\frac{\epsilon}{\alpha+\epsilon}} B^{\frac{\alpha}{\alpha+\epsilon}}.$$

The level of employment is:

$$l = (1-\alpha)^{\frac{1}{\alpha+\epsilon}} A^{\frac{1}{\alpha+\epsilon}} B^{-\frac{1}{\alpha+\epsilon}}$$

imply that a 10% increase in B leads to a reduction in employment and an increase in real wages given by:

$$\log(l_2) - \log(l_1) = -1.79\%, \qquad \log\left(\frac{w}{p}\right)_2 - \log\left(\frac{w}{p}\right)_1 = 0.60\%.$$

If ϵ is higher, then the effect on both employment and real wages is smaller in absolute value. Again, this is intuitive: if labor supply is steeper to begin with, then a given increase in B does not lead to as much of a shift in the labor supply curve. Thus, the move along the labor demand curve towards lower employment and higher wages is not as important then. This is shown on the two figures below, where the increase in leisure attractiveness is the same across the two experiments: on the left-hand side, epsilon is low, while on the right hand side, epsilon is high. *Note*: Again, you may play around with the spreadsheet to see what happens when parameters are changed.

4.3 The "Keynesian" Labor Market Model

1. One way to answer this question is to note that with a fall in productivity, the labor demand curve will shift to the left (as in lecture **??**). If real wages are rigid, then workers are off their labor supply curve (they would like to work more at the current wage) but still on firms' labor demand curve:

$$\log\left(\frac{w}{p}\right) = [\log A + \log(1-\alpha)] - \alpha \log(l).$$

For a given change in $\Delta \log A$, the change in employment is therefore simply given by the previous expression through:

$$\Delta \log(l) = \frac{\Delta \log A}{\alpha}.$$

In contrast, in the previous case, with flexible wages, the change in employment following a productivity shock was only:

$$\Delta \log(l) = \frac{\Delta \log A}{\epsilon + \alpha}.$$

Given a change in productivity of 5%, which goes from 2 to 1.9, or $\log(1.9) - \log(2) = 5.13\%$ in log points, we get a drop of 15.39% in log points in employment:

$$\log(l_2) - \log(l_1) = -15.39\%.$$

2. In question 7 of the previous exercise, we got in contrast a change:

$$\log(l_2) - \log(l_1) = -0.96\%,$$

a much smaller number. The intuition was that the wage falling incentivizes employers to hire more workers. Here, in contrast, because the wage is "too high" and cannot fall by definition, employers do not want to hire them.

3. We know from lecture **??** that if leisure becomes more attractive, then employment must fall and the real wage must rise. We imagine here that wages are sticky upwards. Then, people will be off firms' labor demand curves, but on their labor supply curves. Thus, it is still true that:

$$\log\left(\frac{w}{p}\right) = \log B + \epsilon \log(l).$$

For a given change in $\Delta \log B$, given that wages are sticky, the change in employment is therefore simply given by the previous expression through:

$$\Delta \log(l) = -\frac{\Delta \log B}{\epsilon}.$$

In contrast, in the previous case, with flexible wages, the change in employment following a B shock was only:

$$\Delta \log(l) = -\frac{\Delta \log B}{\epsilon + \alpha}.$$

Given an increase in B of 10%, which goes from 2 to 2.2, or $\log(2.2) - \log(2) = 9.53\%$ in log points, we get a drop of 1.91% in log points in employment:

$$\log(l_2) - \log(l_1) = -1.91\%.$$

4. In question 8 of the previous exercise, we got in contrast a change:

$$\log(l_2) - \log(l_1) = -1.79\%,$$

a somewhat smaller number. The intuition was that the wage increasing would have incentivized workers to work more. But because wages are sticky, this did not happen.

4.4 The Bathtub model

Assume a monthly job separation rate equal to $s = 1\%$, and a monthly job finding rate equal to $f = 20\%$. Assume that the labor force is given by $L = 159$ million.

1. The steady state unemployment rate is obtained by equating separations and job findings in the steady state, so that:

$$sE^* = fU^* \quad \Rightarrow \quad sL - sU^* = fU^*$$
$$\Rightarrow \quad U^* = \frac{s}{s+f}L \quad \Rightarrow \quad u^* = \frac{s}{s+f}.$$

The steady state unemployment rate u^*, number of people unemployed U^*, and number of people losing or finding a job each month, are given by (see the spreadsheet):

$$u^* = 4.76\%, \quad U^* = 7,571,429, \quad fU^* = 1,514,286.$$

2. Again, there are many ways we can proceed here. One is to use the spreadsheet to iterate on the law of motion, that is calculate u_{t+1} as a function of u_t and then see graphically when the given unemployment rate is reached. The law of motion is:

$$U_{t+1} - U_t = s(L - U_t) - fU_t.$$

Thus (see lecture **??**):

$$U_{t+1} = sL + (1 - s - f)\, U_t.$$

Dividing both sides by L, and denoting by $u_t = U_t/L$ the *rate* of unemployment:

$$u_{t+1} = s + (1 - s - f)\, u_t.$$

We find that the unemployment rate reaches 5% after approximately **11 months** (the unit of time is one month). A second method is to do a bit of algebra before using the computer. We write the law of motion for unemployment (again, see lecture **??** for details):

$$U_{t+1} - U_t = s(L - U_t) - fU_t$$
$$\Rightarrow \quad U_t - U^* = (1 - s - f)^t \left(U_0 - U^*\right).$$

Dividing everything by L gives everything in terms of unemployment *rates*:

$$u_t = (1 - s - f)^t \left(u_0 - u^*\right) + u^*.$$

Thus, starting from an unemployment rate u_0 it is possible to get a value for all subsequent u_t, using the above formula. We may then use the spreadsheet to compute this and find that the unemployment rate reaches 5% after approximately. Again find that the unemployment rate reaches 5% after approximately **11 months** (the unit of time is one month). A third method is to in fact do all the algebra and calculate the time T we are looking for explicitely. We are looking for T such that $u_t \leq \bar{u} = 5\%$ for $t \geq T$. This implies:

$$(1 - s - f)^t \left(u_0 - u^*\right) + u^* \leq \bar{u}$$
$$\Rightarrow \quad (1 - s - f)^t \leq \frac{\bar{u} - u^*}{u_0 - u^*}$$
$$\Rightarrow \quad t \log(1 - s - f) \leq \log \frac{\bar{u} - u^*}{u_0 - u^*}$$
$$\Rightarrow \quad t \geq \frac{\log \frac{\bar{u} - u^*}{u_0 - u^*}}{\log(1 - s - f)}.$$

(be careful, $\log(1 - s - f)$ is negative because $1 - s - f$ is lower than 1 so you have to change the inequality from $\leq$ to $\geq$). A numerical application using the spreadsheet shows that this condition means:

$$t \geq 11.07.$$

The advantage of this method is we know exactly when the unemployment rate reaches 5%. After 11.07 months ! (given the simplicity of the model, displaying the second digit does not make much sense, though)

3. We are looking for f such that:

$$u^* = \frac{s}{s+f} \quad \Rightarrow \quad f = \frac{s}{u^*} - s,$$

which implies using these numbers that:

$$f = 40\%.$$

This is intuitive: you have double the separation rate, you want double the job finding rate for the unemployment rate to be the same in the steady-state. Indeed, in the steady state:

$$sE^* = fU^*.$$

Therefore, if the unemployment rate is the same, then if s doubles f must double as well.

4. See question 2. The spreadsheet should give all the answers. We find:

$$t \geq 4.79.$$

Thus, the unemployment rate reaches 5% after approximately 5 months.

5. The answer is very much contained in questions 2 and 4. If the rates of job separations and job finding are higher like they typically are in America, the unemployment rate reaches its steady-state value faster. This may explain why the United States are able to recover faster from shocks than, say, Spain or Italy (at least in terms of unemployment rates). Here is some supporting evidence that the unemployment rate is more persistent in Europe than in America:
https://db.nomics.world/OECD/EO/USA.UNR.Q
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