Intermediate Macroeconomics

UCLA - Econ 102 - Fall 2018

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Econ 102 Material

This website contains most of the class material for *Intermediate Macro (Econ 102)* I teach at UCLA. The Moodle platform should be used for the discussion board as well as some additional readings.

All lectures and Problem Sets

Below is an up-to-date version to all lectures and problem sets in pdf format, as well as a beta ebook version of the class.

- All lectures and Problem Sets. pdf / epub
- Math Review. pdf1 / pdf2.

Timetable

Below is an html version of all individual lectures, as well as a timetable of the class.

- Oct 1. Lecture 1 - Introduction to Macroeconomics.
- Oct 3. Lecture 2 - The Solow Growth Model.
- Oct 8. Lecture 3 - Two Period Consumption Problem.
- Oct 10. Lecture 4 - The Overlapping Generations Model.
- Oct 15. Lecture 5 - Technological Growth.
- Oct 17. Lecture 6 - The Labor Market and Unemployment.
- Oct 22. Lecture 7 - The Consumption Function and the Multiplier.
- Oct 24. Lecture 8 - The Paradox of Thrift.
- Oct 29. *Review.*
- Oct 31. Lecture 9 - Redistributive Policies.
- Nov 5. *Midterm.*
- Nov 7. Lecture 10 - Public debt, Say's law.
- Nov 12. *No Class (Veteran's day).*
- Nov 14. *Pause (Finishing Lecture 10).*
- Nov 19. Lecture 11 - The Open Economy and the Multiplier.
- Nov 21. *No Class (Thanksgiving).*
- Nov 26. Lecture 12 - Twin Deficits.
- Nov 28. Lecture 13 - Empirics of Fiscal Policy.
- Dec 3. Lecture 14 - Monetary Policy.
- Dec 5. Lecture 15 - Summing Up: A Macroeconomic History of the U.S.

A rough correspondance to chapters in Olivier Blanchard's *Macroeconomics* textbook is provided here.

Problem Sets

Below is an html version of all problem sets and solutions.

- Week 1. Problem Set 1 - Math Review. Solutions.
- Week 2. Problem Set 2 - Solow Growth Model. Solutions.
- Week 3. Problem Set 3 - Two Period. Solutions.
- Week 4. Problem Set 4 - Solow & Labor Market. Solutions.
- Week 5. Problem Set 5 - Keynes and the Multiplier. Solutions.
- Week 6. Problem Set 6 - Keynesian Taxation. Solutions.
- Week 7. Problem Set 7 - Overlapping Generations Model. Solutions.
- Week 8. Problem Set 8 - Neoclassical Taxation. Solutions.
- Week 9. Problem Set 9 - Open Economy. Solutions.
- Week 10. Problem Set 10 - Empirics of Fiscal Policy. Solutions.

Past Exams

Below is a list of past exams.

Warning! The ordering, and the content of the class are different this year compared to previous years, and increasingly so as you go back in time - in 2015 and 2016, I was using Chad Jones' *Macroeconomics* textbook. Anything that I do not go over this quarter is not exam material. In order to succeed in the class, it is best to first and primarily review the material that is found in the lectures and the problem sets - that is probably where "marginal returns" are maximal.

- **Fall 2018.** Midterm / Solution ; Final / Solution.

- **Spring 2018.** Midterm / Solution ; Final / Solution.

- **Winter 2016.** Midterm 1 / Solution ; Midterm 2 / Solution ; Final / Solution.

- **Winter 2015.** Midterm 1 / Solution ; Midterm 2 / Solution ; Final / Solution.

Update! I provide here a list of selected questions from last year's midterm, and final which are potentially relevant for the midterm.

Old Lectures and Problem Sets

Here I include the old lectures which were previously on the website. However, I do not maintain this version of the lectures anymore. For the most up-to-date version as well as the last lectures of the class please go to the lectures directly using the dropdown menu on the left or use the pdf / epub of the class.

- Lecture 1. pdf / slides
- Lecture 2. pdf / slides
- Lecture 3. pdf / slides
- Lecture 4. pdf / slides
- Lecture 5. pdf / slides
- Lecture 6. pdf / slides
- Lecture 7. pdf / slides
- Lecture 8. pdf / slides
- Lecture 9. pdf / slides
- Lecture 10. pdf / slides

Again, I do not maintain this version of the problem sets anymore.

- Problem Set 1. pdf. Solutions. pdf. Spreadsheet. web / ods / xlsx.
- Problem Set 2. pdf. Solutions. pdf. Spreadsheet. web / ods / xlsx.
- Problem Set 3. pdf. Solutions. pdf. Spreadsheet. web / ods / xlsx.
- Problem Set 4. pdf. Solutions. pdf. Spreadsheet. web / ods / xlsx.
- Problem Set 5. pdf. Solutions. pdf.
- Problem Set 6. pdf. Solutions. pdf.
- Problem Set 7. pdf.
- Problem Set 8. pdf.

Syllabus

This online book contains most of the class material for *Intermediate Macro (Econ 102)* I teach at UCLA. The Moodle platform should be used for the discussion board as well as some additional readings.

Lectures: Mondays and Wednesdays, 3:30-4:45pm, Dodd Hall, Room 147.

Office hours: Tuesdays, 4-6pm. (Bunche 8389)

Moodle Website: https://moodle2.sscnet.ucla.edu/course/view/18F-ECON102-1

Graduate Student Instructors (GSIs): Graduate Student Instructors are all graduate students in the UCLA Economics Department. They will teach sections and hold 2 hours of office hours in the Alper Room every week:

- Sections 1E-1I. Paula Beltran. OH: F 11-12; 2-3. pabeltran90@gmail.com
- Sections 1H-1M. Alvaro Boitier. OH: M 2:30-3:30; T 2-3. alvaro.boitier@gmail.com
- Sections 1N-1K. Conor Foley. OH: T 2-4. conor.teaches.econ@gmail.com
- Sections 1D-1J. Kun Hu. OH: R; 9-11. rickhukun@ucla.edu
- Sections 1G-1O. Ivan Lavrov. OH: W 1-3. ilavrov113@gmail.com
- Sections 1B-1C. Anthony Papac. OH: M 10-11; R 12:30-1:30. anthonypapac@g.ucla.edu
- Sections 1A-1F. Mengbo Zhang. OH: W 10-12. zmbruc@gmail.com

Course description. This course is meant to provide an intermediate-level treatment of macroeconomic topics, including the study of economic growth, business cycle fluctuations, unemployment, inflation, as well as open-economy macroeconomic issues such as trade imbalances and exchange rate policy. Although the title of the class is "Macroeconomic Theory", students will learn both the theory as well as some of the empirical evidence behind the theory, and its practical implications. Special emphasis will be placed on the application of economic tools to contemporary economic problems and policies. Competing schools of thought will be presented, with a particular emphasis on Neoclassical and Keynesian theories, and they will be discussed in the light of macroeconomic data. Class meetings will be highly interactive, with many opportunities for you to both ask and answer questions.

Course Objectives. My objective is that, by the end of the course, you will be able to read, and critically assess writings from *The Economist*, *The Wall Street Journal*, or *The New York Times*. Macroeconomics is everywhere in the news, and I want to walk you through the tools you need to understand it better. Economics is ultimately an empirical subject, so as much as possible I will try to convey not just the theory of how the economy works, but also the evidence supporting, or contradicting the theory. We will not always reach definitive conclusions on most of the issues we will examine, but you should have a more informed opinion on each of them and why these questions are hard and debated scientifically.

Prerequisites. A strict prerequisite for the class is that you have taken Econ 101. If you do not meet this prerequisite, you are advised to take this course during another term. You should also be familiar with some elementary mathematics. For example, you need to know what a logarithm is, and how to calculate a geometric sum:

$$1 + c_1 + c_1^2 + ... = \frac{1}{1 - c_1} \quad \text{if} \quad 0 < c_1 < 1,$$

because that is really useful to understand how a Keynesian multiplier works, for example. If you do not know that already, that is fine too, but you should at least be willing to learn. If you want a treatment of Econ 102 which is less heavy on algebra, you are best advised to take this class in another term.

Textbook (optional): Olivier Blanchard's *Macroeconomics*, 7th Edition (previous editions should be fine, too).

Questions? If you have any question about the material covered during the course, you should consider the following options in order:

1. First, you should never refrain from asking questions during class.
2. Second, you may ask questions during recitation sections. The smaller group should allow you to ask questions more freely. Our teaching assistants are all passionate graduate students, writing a phD thesis on macroeconomics or other related subjects, so they will be very happy to help you.
3. Third, TAs will hold their office hours. The times for their office hours is reminded here:
   - Paula Beltran. OH: F 11-12; 2-3. pabeltran90@gmail.com
   - Alvaro Boitier. OH: M 2:30-3:30; T 2-3. alvaro.boitier@gmail.com
   - Conor Foley. OH: T 2-4. conor.teaches.econ@gmail.com
   - Kun Hu. OH: R; 9-11. rickhukun@ucla.edu
   - Ivan Lavrov. OH: W 1-3. ilavrov113@gmail.com
   - Anthony Papac. OH: M 10-11; R 12:30-1:30. anthonypapac@g.ucla.edu
   - Mengbo Zhang. OH: W 10-12. zmbruc@gmail.com
4. Fourth, you should feel free to ask questions on the discussion board (not by email). We will never respond to questions about contents by email (in particular those starting with "is X, Y, Z, test material"), because doing so would be unfair to the rest of the class. In contrast, we commit to respond to all questions on the Moodle Website within 24 hours (either me or the TAs will). Beware ! You should start studying for the midterm exam earlier than November 4 – we will stop answering questions at **6pm the day before each exam** (either the midterm on November 5, or the final on December 14).
5. Finally, I will hold regular office hours on Tuesdays, 4-6pm, in my office 8389. Please send me an email prior if you plan to arrive after 5pm.

Class notes. Class notes will be posted *after* each class, so as to encourage you to take notes. Notes might not always be comprehensive, and everything I will say during class is potentially examinable, even if it does not appear in the notes. Thus, to do well it's best if you attend all lectures.

(Optional) Would-be Data scientists. A lot of what we do in the class involves a fair amount of data. I use the *R statistical software* in order to prepare my lecture notes and input the data from official sources, to provide you with the most up-to-date statistics. I will try to provide the required code to replicate all the analysis available in my lecture notes, as much as possible. For example, lecture 1 has the R code added to the lecture notes available here. An introduction to R statistical software is available here. I think that data science, statistics and economics are very complementary skills (so does the Massachusetts Institute of Technology). However, understanding code is not required at all to succeed in that class. You will not be penalized in any way if you skip this.

Grades. Your final grade has two components: one midterm exam, and a comprehensive final exam. Your final grade will be given by whichever of these two options gives you the best grade: (**Midterm (40%)** + **Final Exam (60%)**) or (**Final Exam (100%)**) at the following dates:

1. November 5, 3:30pm to 4:45pm: Midterm Exam.
2. December 14, 11:30am to 2:30pm: Final Exam.

No make-up exams. In any case, there will be no make-up exams. If a midterm exam is missed due to a documented serious illness or emergency, the final exam will be worth 100 % of your grade. Note that attending the midterm is like an “option value”: you are necessarily better off attending the midterm, no matter what your grade is. Please make sure right away that you can be there on November 5 !

Regrade Policy. Students who wish to have their midterm or their final examinations regraded should submit a request in written form to their assigned Graduate Student Instructor, clearly explaining why they think they deserve a regrade. If a student requests a regrade, the whole exam will be regraded. Therefore, the grade can increase or decrease.

Exam content. Everything that I say during the class, that is covered during recitation sections, is potentially exam material. Exams will be a combination of multiple choice and short essay questions. Therefore, it is absolutely necessary that you attend all lectures! I encourage you to take notes during the class.

Exam practicalities. During exams, sufficient space will be provided on the sheets to answer. No notes, no books, no smartphones, no calculators, will be allowed during the exam. You must bring your UCLA ID in order to take the exam. Without a UCLA ID, you will not be allowed to take the exam. You will not need to bring scantrons, as we will be using Scantrons from the Office of Instructional Development (OID).

Other. For more details about policies regarding grading, exams and other matters please refer to the following link: https://www.econ.ucla.edu/undergraduate/. I will adhere to the guidelines specified in this webpage. If you wish to request an accommodation due to a suspected or documented disability, please contact the Center for Accessible Education as soon as possible at A255 Murphy Hall, (310) 825-1501, (310) 206-6083 (telephone device for the deaf). The website is http://www.cae.ucla.edu/.

Teaching Philosophy. To the extent possible, I will strive to emphasize **facts** over **theories**. This is a major difference with the way that I taught this class in the past. Many of the issues that we will look at are politically charged, and various theories have been developed which usually speak to either ideological views. Theory usually does not allow to conclude definitively. This is unfortunate, because macroeconomic questions are debated on both sides of the political spectrum:

- Do advanced economies have too high levels of public debt?
- Should fiscal stimulus be used to fight recessions?
- What is the cause of unemployment? (how much is voluntary or involuntary?)

… and many other questions. Fortunately, these questions are increasingly studied on the empirical front. Whenever possible, we shall try to “let the data speak”, and put the different theories that we will study to the test. Empirical research is still ongoing, and I will do my best to teach you the most up-to-date findings. In doing so, I will try to be as objective as possible, and try to avoid any conservative or liberal bias. According to this article (link), the latter is more of a risk than the former. I will always try to give you both sides of the debate, and arguments supporting each side. You are welcomed (and even encouraged !) to disagree with what I say during class !

Chapter 1

Introduction to Macroeconomics

GDP is the value of all final goods and services produced in a country within a given period. There are two sides to GDP, the demand side and the supply side:

- On the **demand side**, the **product approach to GDP** recognizes that total aggregate demand is made of four components:
  - Consumption spending by households (C).
  - Investment spending by households and corporations (I).
  - Government purchases (G).
  - Net exports (NX).
- On the **supply side**, the production of output involves the use of factor of production, often limited to capital and labor. These factors of production receive payment for their use, whose sum equals GDP.[1] The **income approach to GDP** consists in dividing up these payments into the different factors of production. Again, this often simply means a division of total value added into capital income, and labor income.

1.1 The Product Approach to GDP

GDP is equal to the total aggregate demand for goods:

$$Y = C + I + G + X - M.$$

We often define net exports as:[2]

$$NX = X - M,$$

so that GDP is simply:

$$Y = C + I + G + NX.$$

We examine each component of aggregate demand in turn:

- Consumption (C)
- Investment (I)
- Government Purchases (G)
- Net Exports (NX)

[1] Note that we are assuming here that there are no "rents" in the economy, that is that nothing can be obtained without either labor or capital. This is not exactly true, as for example oil is clearly more valuable than the costs of extracting it for the soil. Land is another example of something that is preexisting, but nevertheless earns a payment. It is a good first-order approximation however, as most production in fact requires either labor of capital.

[2] In some textbooks (as well as in earlier versions of these lecture notes), imports are denoted by IM instead of M.

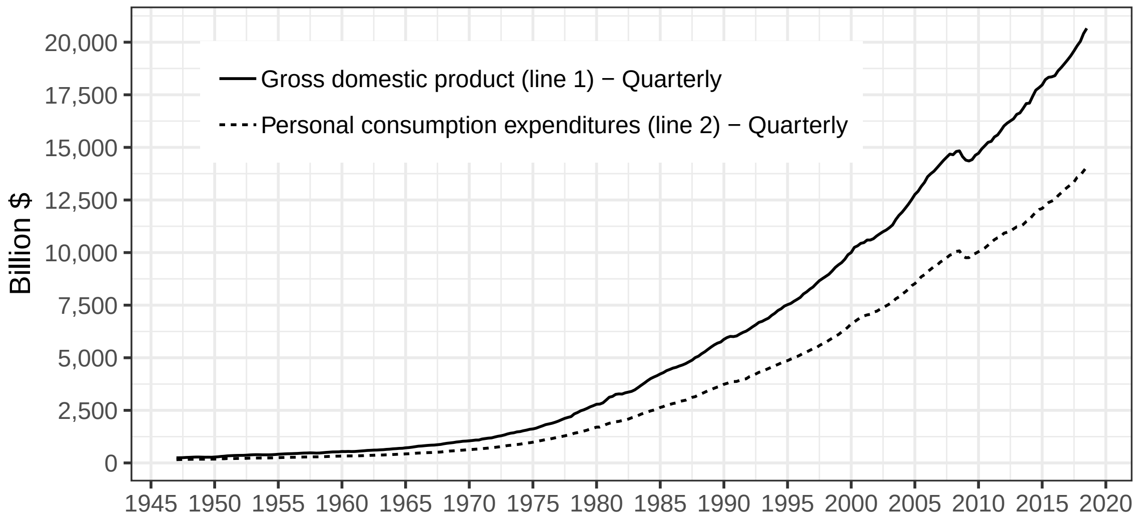

Figure 1.1: US GDP from NIPA (BEA).

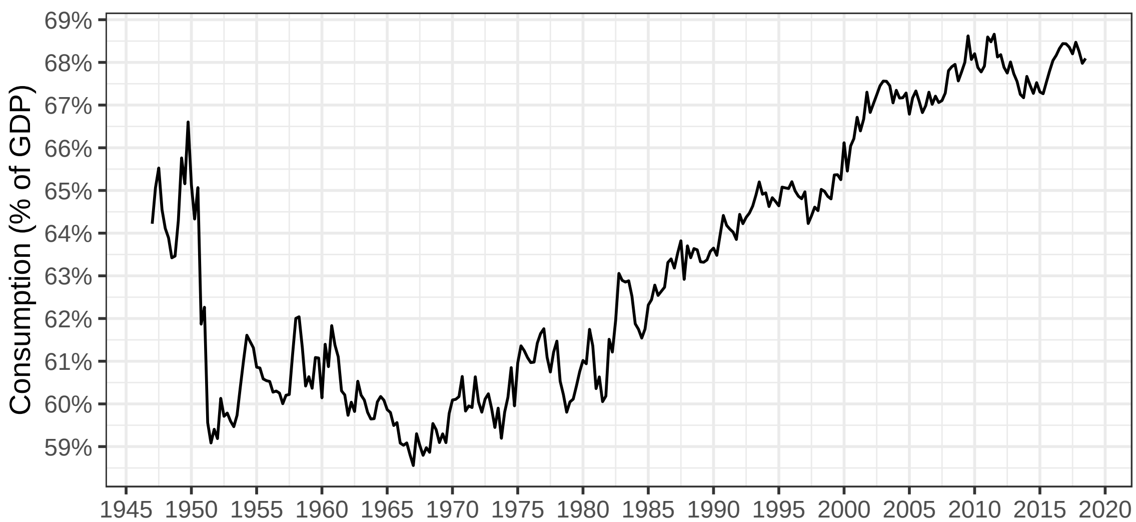

Figure 1.2: Consumption as a share of GDP from NIPA (BEA).

1.1.1 Consumption (C)

Figure 1.1 plots GDP from the National Income and Product Accounts (NIPA) produced by the Bureau of Economic Analysis (BEA), as well as Personal Consumption Expenditures (PCE), in billions of dollars. We can see that Gross Domestic Product is currently in the vicinity of $20 trillion (or $20,000 billion). For this figure, data is retrieved from https://db.nomics.world, a great source of macroeconomic data. For example, data for Gross Domestic Product is available here and data for Personal Consumption Expenditures is available there.

To get a better of sense of how big consumption is as a fraction to GDP (although you may eyeball it on this picture), we might plot consumption as a function of GDP, which is what I do on Figure 1.2. You can see that Personal Consumption Expenditures are approximately **60 to 70 % of GDP**. You can also see that it has been rising since the end of the sixties. We will discuss that.

Personal Consumption Expenditures are divided up into:

- Durable Goods (more than 3 years of durability): e.g. cars.

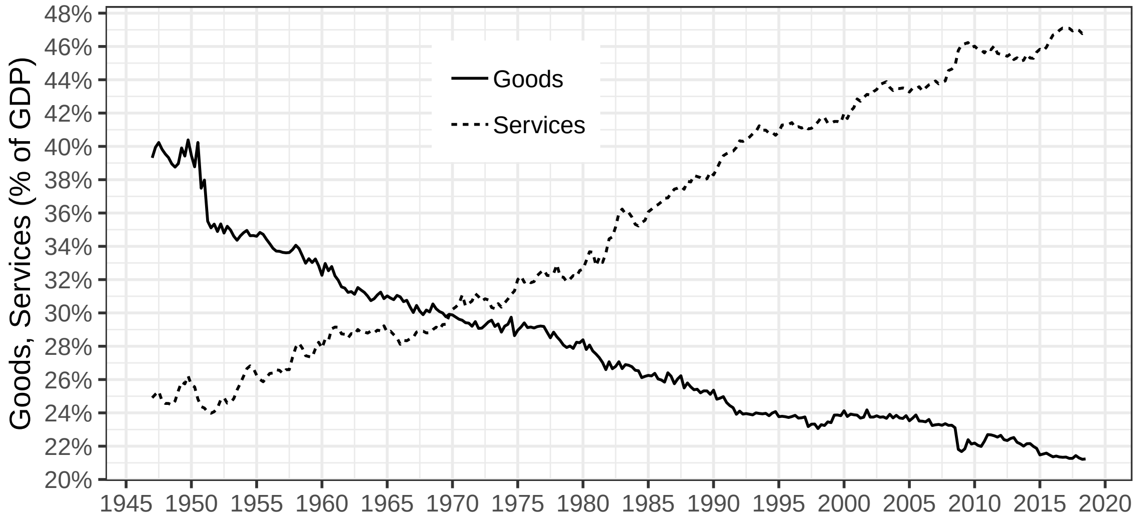

Figure 1.3: Goods and Services Consumption as a share of GDP from NIPA (BEA).

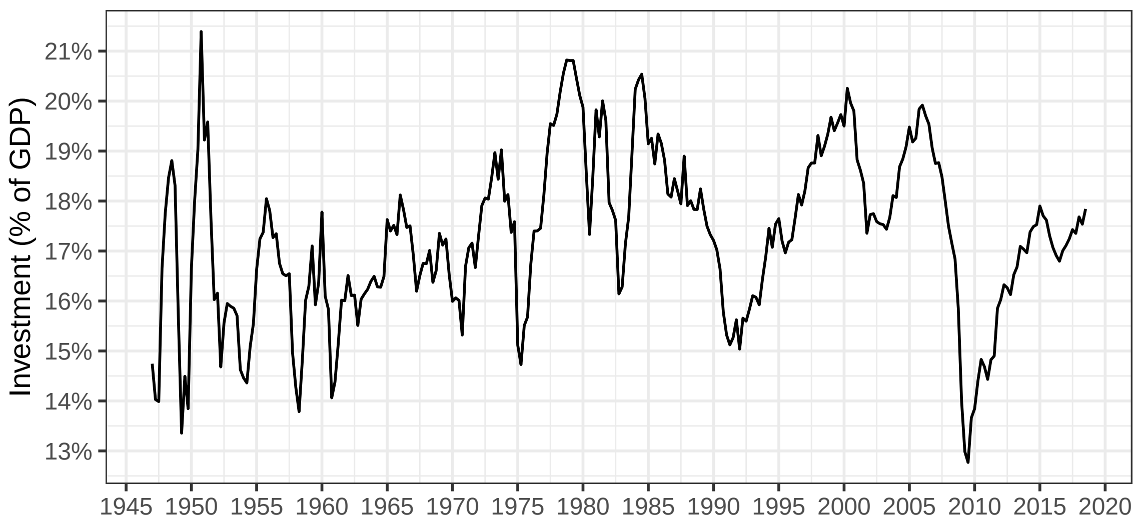

Figure 1.4: Investment as a share of GDP from NIPA (BEA).

- Non-durable Goods (less than 3 years of durability).
- Services.

Services have become more important than Goods in total consumption since the 1970s, as Figure 1.3 shows.

1.1.2 Investment (I)

Investment has two components:

- **non residential investment** is the purchase of new capital goods by firms: structures, new plants.
- **residential investment** is the purchase of new houses.

Gross private domestic investment is approximately **15 to 20 % of GDP**, as you can see on Figure 1.4. It is also very volatile over the cycle.

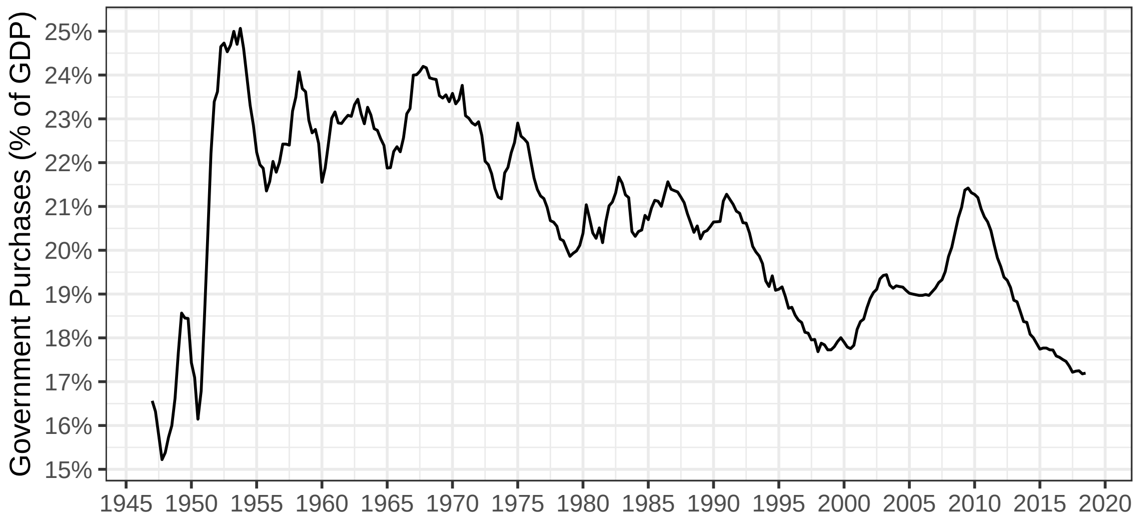

Figure 1.5: Government Purchases as a share of GDP from NIPA (BEA).

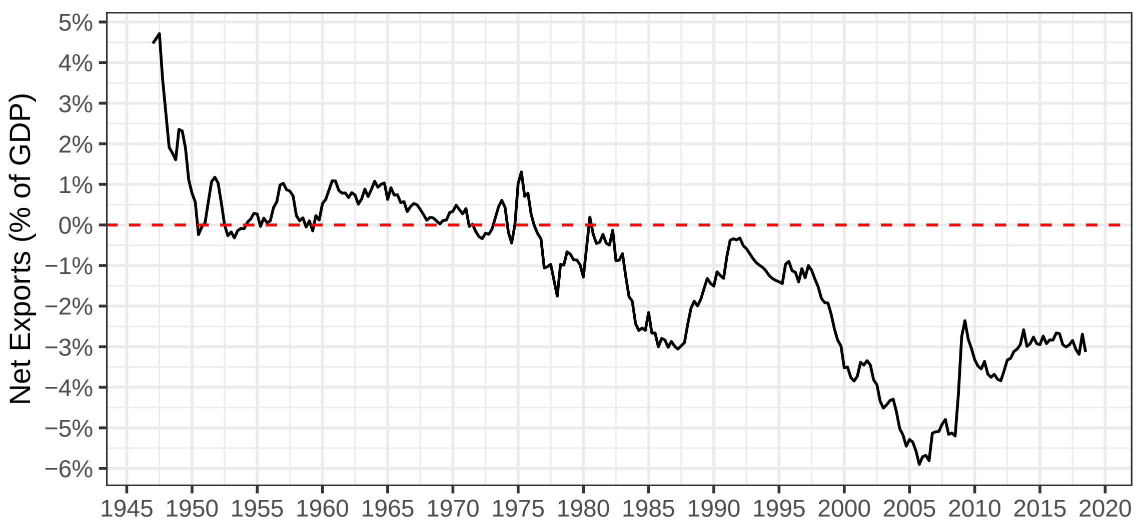

Figure 1.6: Net Exports as a share of GDP from NIPA (BEA).

1.1.3 Government Purchases (G)

Government purchases are composed of purchases of goods by the government plus the compensation of government employees. Overall, they comprise about approximately 20% of GDP, as can be seen on Figure 1.5. Note however that they do not include transfers from the government of interest payments on government debt.

1.1.4 Net Exports (NX)

Net exports of goods and services are approximately **-2 to -6 % of GDP**, at least in the modern period (and in the United States), as you can see on Figure 1.6.

1.2 The Income Approach to GDP

1.2.1 Cobb-Douglas Production function

In order to organize our thinking, let's write out a Cobb-Douglas production function, defined as:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha},$$

where α is a number between 0 and 1. We will see later that this α is related to the share of capital versus labor in value added. In practice, you should think of α as close to 1/3. Again, we shall explain why at the end of this lecture.

To proceed further, it is useful to think of a firm which would choose the amount of labor it uses L_t as well as the amount of capital it uses K_t in order to maximize its profits:[3]

$$\max_{K_t, L_t} \quad A_t K_t^{\alpha} L_t^{1-\alpha} - R_t K_t - w_t L_t,$$

In this expression, R_t is the **rental rate** of capital, also called the **gross return** to capital. This represents how much it costs to rent one unit of capital. This cost actually has two components:

1. It includes a conventional interest rate r_t, which is the cost of borrowing money to invest in capital. You should think of this as the real interest rate which is charged by a bank to borrow money.
2. It also includes a depreciation rate δ, which accounts for the wear and tear of the capital stock, implying that its value drops over time. If capital is bought, then the resale price for each unit of capital is lower by a fraction δ, which is a cost to the investor. If capital is rented, then capital needs to be given back to the owner in its original state.

We have that the rental rate or gross return is equal to the net return plus the depreciation rate:

$$\boxed{R_t = r_t + \delta}.$$

From Econ 11, it should be clear that a way to solve this problem is to set the derivative of the profit function equal to 0 with respect to K_t and L_t:

- Differentiating with respect to K_t implies:

$$\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - R_t = 0 \quad \Rightarrow \quad \boxed{R_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}}.$$

- Differentiating with respect to L_t implies:

$$(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} - w_t = 0 \quad \Rightarrow \quad \boxed{w_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}}.$$

Note that an alternative, and more direct way to get at that same result, would be to use Econ 11 directly, and write that the marginal products have to be equal to prices:

- The **rental rate of capital** R_t is the marginal product of capital. The marginal product of capital is how much more output is obtained when the capital stock is increased by one unit, which is just the derivative of output with respect to capital $\partial Y_t / \partial K_t$:

$$\begin{aligned} R_t &= \frac{\partial Y_t}{\partial K_t} \\ &= \frac{\partial \left(A_t K_t^{\alpha} L_t^{1-\alpha} \right)}{\partial K_t} \\ R_t &= \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \end{aligned}$$

[3] You may be asking yourself what this firm is. We are looking at aggregates here, and there sure isn't just one firm in the economy. However, if all firms were identical, because of constant return to scale, the "representative" firm aggregating the decisions of all individual firms would seem to be solving that problem.

- The **wage** w_t is the marginal product of labor. The marginal product of labor is how much more output is obtained when the quantity of labor is increased by one unit, which is just the derivative of output with respect to labor $\partial Y_t / \partial L_t$:

$$\begin{aligned} w_t &= \frac{\partial Y_t}{\partial L_t} \\ &= \frac{\partial \left(A_t K_t^{\alpha} L_t^{1-\alpha} \right)}{\partial L_t} \\ w_t &= (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} \end{aligned}$$

The total capital income $R_t K_t$ is a fraction α of output Y_t:

$$\begin{aligned} R_t K_t &= \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \cdot K_t \\ &= \alpha A_t K_t^{\alpha} L_t^{1-\alpha} \\ R_t K_t &= \alpha Y_t. \end{aligned}$$

This implies that the share of capital income in output (or equivalently, value added) is:

$$\boxed{\frac{R_t K_t}{Y_t} = \alpha}.$$

The total wage bill $w_t L_t$ is a fraction $1-\alpha$ of output Y_t:

$$\begin{aligned} w_t L_t &= (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} \cdot L_t \\ &= (1-\alpha) A_t K_t^{\alpha} L_t^{1-\alpha} \\ w_t L_t &= (1-\alpha) Y_t \end{aligned}$$

The share of labor income in output $w_t L_t / Y_t$ (or equivalently, value added) is:

$$\boxed{\frac{w_t L_t}{Y_t} = 1-\alpha}.$$

Note that capital income plus labor income equals total output:

$$R_t K_t + w_t L_t = Y_t.$$

Another way to say the same thing is that the share of capital income in output and that of labor income in output add up to one:

$$\boxed{\frac{R_t K_t}{Y_t} + \frac{w_t L_t}{Y_t} = 1}.$$

1.2.2 The Income Side in the Data

In practice, how much goes to the compensation of employees (labor income), and how much goes to the returns to capital (capital income)? The answer is that it goes approximately for 1/3 to capital and for 2/3 to labor. In turn, this implies that we will, in numerical applications of our theories, often assume that:

$$\alpha = \frac{1}{3}$$

The calculations for these are less straightforward than for computing the share of consumption, investment, as we did above. The reason is that in practice, the division between labor and capital is not as clear cut in the national accounts as one might hope: for example, someone who owns her/his own business reports most

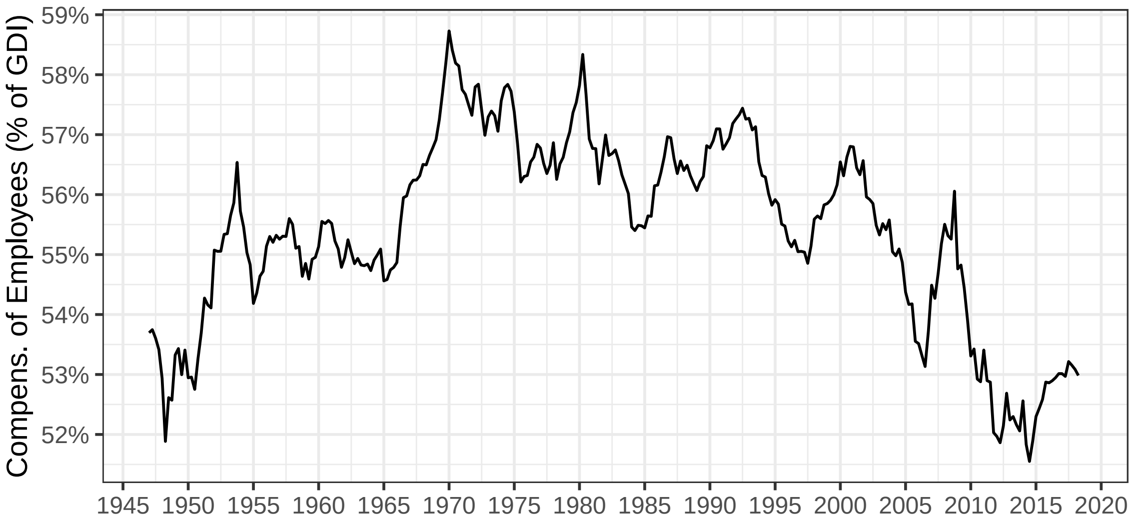

Figure 1.7: Compensation of Employees as a share of GDP from NIPA (BEA).

of her/his income in the form of capital income, even when a large part of it is actually labor income, so that compensation of employees is (vastly) understated. Figure 1.7 shows which results are obtained using this understated measure. It needs to be adjusted upwards by about 10% of GDP, for the reasons mentioned above.

For our purposes, we only need to remember that the share of compensation of employees is approximately 2/3 of value added:

$$1 - \alpha \approx \frac{2}{3} \quad \Rightarrow \quad \boxed{\alpha \approx \frac{1}{3}}.$$

Therefore, we will very often work with a Cobb-Douglas production function where $\alpha = 1/3$, implying that production is given by:

$$Y_t = A_t K_t^{1/3} L_t^{2/3}.$$

Lecture 2 will walk you through the Solow growth model, where we shall make heavy use of that Cobb-Douglas production function.

Readings - To go further

The Economics of Well Being, *Harvard Business Review.*

G.D.P. R.I.P., *The New York Times*, August 9, 2009.

(Gated) Keeping up with the Karumes, *The Economist*, October 29, 2015.

(More Difficult Read) Abraham, Katharine G. “Distinguished Lecture on Economics in Government-What We Don’t Know Could Hurt Us: Some Reflections on the Measurement of Economic Activity.” Journal of Economic Perspectives 19, no. 3 (September 2005): 3–18.

Chapter 2

The Solow Growth Model

The first part of this lecture considers the case of a Solow growth model with a general, constant returns to scale, production function. The second part of the lecture looks at a special case of the Solow growth model for a case of a Cobb-Douglas production function.

2.1 General Production Function

2.1.1 Assumptions

Robert Solow, 1987 Nobel Memorial Prize in Economic Sciences, starts from a general production function, giving at any point in time output Y_t as a function of inputs, capital K_t and labor L_t:

$$Y_t = F\left(K_t, L_t\right).$$

A very important assumption is also **constant returns to scale** with respect to capital and labor, so that for any scaling factor a:

$$F(aK_t, aL_t) = aF(K_t, L_t).$$

For simplicity, we shall assume from now on that the quantity of labor is fixed with $L_t = L$, so that the production function becomes $Y_t = F(K_t, L)$. Because of constant returns to scale with respect to capital and labor (and setting $a = 1/L$ in the previous expression), we have:

$$\frac{Y_t}{L} = F\left(\frac{K_t}{L}, 1\right) = f\left(\frac{K_t}{L}\right)$$

where f is defined as a function of F such that:

$$f(x) \equiv F(x, 1).$$

An example of such a production function is the Cobb-Douglas production function, which we started studying in Lecture 1, and which we look at in the next section.

Robert Solow, in his 1956 contribution, abstracts from public saving, so that **total saving** at time t equals **private saving** at time t, and both are denoted S_t, which also equals investment I_t at time t :

$$S_t = I_t.$$

Saving is assumed to be a constant fraction s of output Y_t, and therefore:

$$S_t = sY_t.$$

This constant saving rate my seem a bit ad-hoc; it is. We will investigate more in detail the determinants of saving and consumption behavior in the next lectures. Depreciation of capital is given by a share δ (think for example that 8% of the capital stock depreciates each period; the rate of depreciation is much lower for structures, and much higher for computers). The capital stock evolves according to:

$$K_{t+1} = (1-\delta)\,K_t + I_t$$

2.1.2 Solution

Replace investment in the previous equation and divide both sides by L:

$$\frac{K_{t+1}}{L} = (1-\delta)\,\frac{K_t}{L} + s\frac{Y_t}{L} \quad \Rightarrow \quad \boxed{\frac{K_{t+1}}{L} - \frac{K_t}{L} = s\frac{Y_t}{L} - \delta\frac{K_t}{L}}$$

The change in the capital stock per person from t to $t+1$ has two components: investment (or saving) and depreciation:

$$\underbrace{\frac{K_{t+1}}{L} - \frac{K_t}{L}}_{\text{Change in capital}} = \underbrace{sf\left(\frac{K_t}{L}\right)}_{\text{Investment}} - \underbrace{\delta\frac{K_t}{L}}_{\text{Depreciation}}.$$

The steady state level of the capital stock K^* is such that $K_{t+1} = K_t = K^*$, and it therefore satisfies:

$$\boxed{sf\left(\frac{K^*}{L}\right) = \delta\frac{K^*}{L}}$$

Note that without further specifying $f(.)$, we can't say much more about the value of K^*/L, we just know it satisfies this implicit equation. The steady-state value of output per worker Y^*/L, as a function of K^*/L is given by:

$$\frac{Y^*}{L} = f\left(\frac{K^*}{L}\right)$$

2.1.3 Three cases

There are 3 cases:

1. If capital per worker is relatively low, that is $K_t/L < K^*/L$, then investment per worker is larger than depreciation per worker, and therefore from the above equation, capital per worker increases:

$$\frac{K_{t+1}}{L} > \frac{K_t}{L}$$

2. If capital per worker is exactly equal to steady state capital per worker, that is $K_t/L = K^*/L$, then investment per worker is equal to depreciation per worker, and therefore from the above equation, capital per worker stays constant:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L}$$

3. If capital per worker is relatively high, that is $K_t/L > K^*/L$, then depreciation per worker is larger than investment per worker, and therefore, capital per worker declines:

$$\frac{K_{t+1}}{L} < \frac{K_t}{L}.$$

2.2 Cobb-Douglas production function

2.2.1 Solving for the model

Assume now that the production function is a Cobb-Douglas production function, so that:

$$F(K,L) = K^\alpha L^{1-\alpha}$$

As we saw during lecture 1, α should be thought of as roughly equal to $\alpha = 1/3$. This implies then that function f defined above is such that:

$$f(x) = x^\alpha$$

The law of motion for capital is given by:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} + s\left(\frac{K_t}{L}\right)^\alpha - \delta\frac{K_t}{L}.$$

Given L, K_0, α, s, δ, we are able to calculate K_1, K_2, ..., as well as K_t for any t, by calculating the quantities of capital successively from the formula above.

If you do so, you will notice that K_t converges to a steady state value K^*. However, you do not need to perform an infinity of operations to get at this K^*. Instead, you can see that capital per worker in steady-state K^*/L solves:

$$s\left(\frac{K^*}{L}\right)^\alpha = \delta\frac{K^*}{L} \quad \Rightarrow \quad \boxed{\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}}$$

What was an implicit equation in the previous section can now be solved for explicitely. The steady-state level of output per worker is then:

$$\frac{Y^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

We are finally able to compute the capital to output ratio K^*/Y^* from the Solow growth model:

$$\begin{aligned}
\frac{K^*}{Y^*} &= \frac{K^*/L}{Y^*/L} \\
&= \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{-\frac{\alpha}{1-\alpha}} \\
\frac{K^*}{Y^*} &= \frac{s}{\delta}.
\end{aligned}$$

Alternatively, you may obtain this expression much more simply by equating saving sY^* to investment δK^* in the steady state:

$$sY^* = \delta K^* \quad \Rightarrow \quad \boxed{\frac{K^*}{Y^*} = \frac{s}{\delta}}.$$

2.2.2 Golden Rule

Most economists believe that policymakers should not care so much about GDP per person, but rather about consumption per person (however, some people hold a different view – we shall talk about that later). The intuition is simple: if an economy was to produce many goods which were only used for investment purposes (which would be the case if $s = 1$), then people in this economy would be starving, even though it was actually producing a lot. Investment, ultimately, should serve to increase future consumption.

The **Golden Rule level of capital accumulation** is such that the level of steady-state consumption per capita is maximized. The steady-state consumption per capita is given by:

$$\begin{aligned} \frac{C^*}{L} &= (1-s)\frac{Y^*}{L} \\ &= (1-s)\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \\ \frac{C^*}{L} &= \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} \end{aligned}$$

Maximizing this steady state consumption with respect to the saving rate s consists in finding the maximum of that function with respect to s:

$$\frac{d\left(C^*/L\right)}{ds} = 0 \quad \Rightarrow \quad \frac{d\left[(1-s)s^{\frac{\alpha}{1-\alpha}}\right]}{ds} = 0$$

Note that the $1/\delta^{\alpha/(1-\alpha)}$ is just a constant which does not change anything to the maximization. If you are not convinced, then you may also compute the derivative with respect to the whole C^*/L expression. This gives:

$$\begin{aligned} -s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)s^{\frac{\alpha}{1-\alpha}-1} = 0 \quad &\Rightarrow \quad \frac{\alpha}{1-\alpha}\frac{1-s}{s} = 1 \\ \Rightarrow \quad \alpha - \alpha s = s - \alpha s \quad &\Rightarrow \quad \boxed{s = \alpha}. \end{aligned}$$

Therefore, the saving rate corresponding to the Golden Rule level of capital accumulation is equal to α (again, taking α to be equal to roughly 1/3, this would suggest that an economy would optimally need to save about a third of its production every year).

The Golden Rule level of capital accumulation is then such that capital at the steady-state is given as a function of the exogenous parameters by:

$$\frac{K^*}{L} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} \quad \Rightarrow \quad K^* = L\left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$$

The level of GDP corresponding to this Golden rule level is:

$$Y^* = L\left(\frac{\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

2.3 Some Data

Figure 2.1 plots gross saving and investment in the United States from the World Development Indicators put together by the World Bank. Figure 2.2 plots net saving and gross saving for the U.S.: net saving (gross saving net of depreciatin) is slightly higher than 0, but not by much. Figure 2.3 plots investment as a % of GDP on a map of the world, while figure 2.4 plots gross saving as a % of GDP.

Readings - To go further

Humans 1, Robots 0, *Wall Street Journal*, October 6, 2013.

(Gated) Economists understand little about the causes of growth, *The Economist*, April 12, 2018.

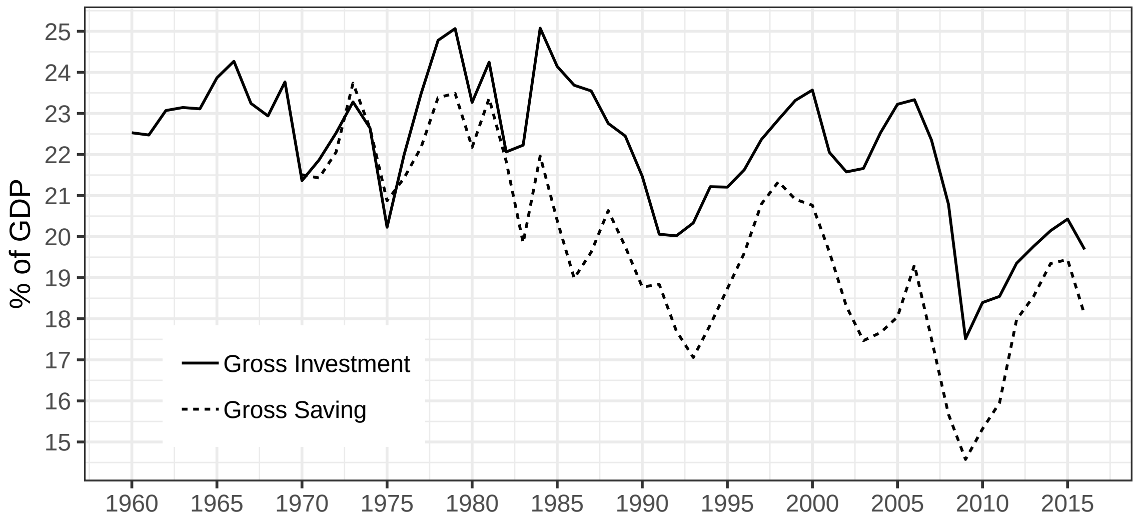

Figure 2.1: Gross Savings and Investment in the U.S. (WDI).

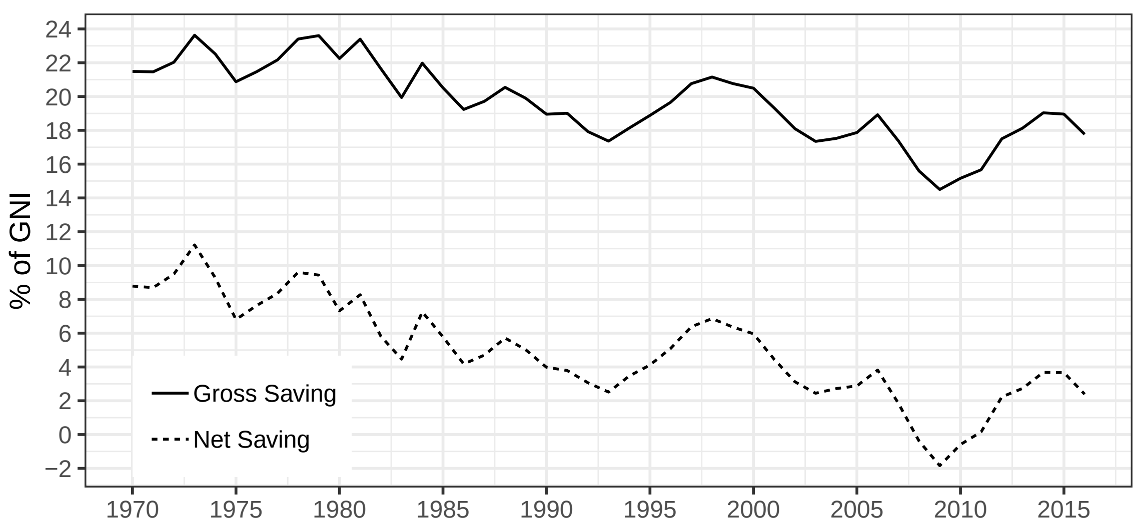

Figure 2.2: Net Savings and Gross savings (WDI).

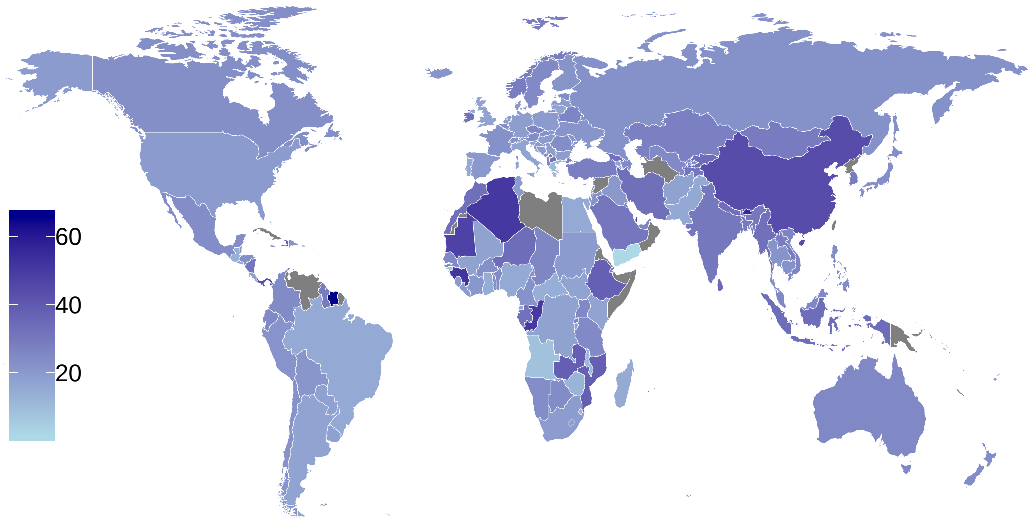

Figure 2.3: Investment (% of GDP), 2016.

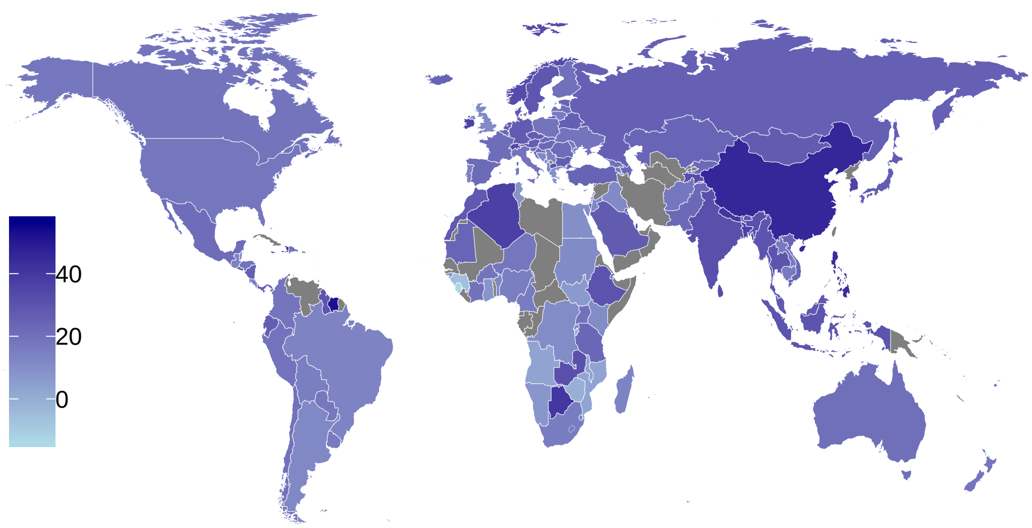

Figure 2.4: Gross Saving (% of GDP), 2016.

Chapter 3

Two Period Consumption Problem

Consumption and saving are perhaps the most important and controversial issues in macroeconomics. In the Solow growth model, saving was a constant fraction s of GDP, by assumption. We now build on *Economics 11* (the one where you learn consumer optimization with Lagrangians and all that), in order to derive saving behavior from microeconomic principles. In other words, we work to make saving "endogenous" (that is, explained by the model), while it was previously taken as exogenous (that is, assumed in the model).

Although this discussion may appear somewhat abstract at first, these calculations are the basis of some of the most important controversies in macroeconomics, which we shall come to in the next lectures.

3.1 The Two-Period Consumption Problem

3.1.1 Assumptions

There are two periods, $t = 0$ (think of this as "today") and $t = 1$ (think of this as "tomorrow"). The consumer values consumption c_0 in period 0 and c_1 in period 1 according to the following utility function:

$$U(c_0, c_1) = u(c_0) + \beta u(c_1).$$

where $u(.)$ is an increasing and concave function, and $\beta \leq 1$. β captures that people typically have a preference for the present. (they are **present-biased**)

Assume that agents earn (labor) income y_0 in period 0, and (labor) income y_1 in period 1. They also are born with some financial wealth f_0 now, and have financial wealth f_1 in period 1, which they consume entirely because this is the last period. (there is no point keeping more money for after period 1, because there is no future at that point) The amount that agents save in this economy is thus $f_1 - f_0$, and the amount of their accumulated savings is the savings they already had plus what they decided to accumulate, so that $f_0 + (f_1 - f_0) = f_1$.

Therefore, consumption in period 0 is given by:

$$c_0 = y_0 - (f_1 - f_0)$$

The second period consumption ($t = 1$) is given by income plus the return to (accumulated !) savings:

$$c_1 = y_1 + (1 + r)f_1.$$

3.1.2 Constrained Optimization Problem

Here, we show that the previous problem can actually be written as a maximization problem, subject to a budget constraint.

Intertemporal budget constraint. Rewriting f_1 from this second equation: $f_1 = (c_1 - y_1)/(1+r)$, and plugging into the first,

$$c_0 = y_0 - \left(\frac{c_1 - y_1}{1+r} - f_0 \right).$$

Rearranging, total wealth is then the sum of financial wealth f_0 and of the present discounted value of human wealth:

$$c_0 + \frac{c_1}{1+r} = \overbrace{f_0 + \underbrace{y_0 + \frac{y_1}{1+r}}_{\text{human wealth}}}^{\text{total wealth}}.$$

The intertemporal budget constraint says that the present discounted value of consumption is equal to total wealth.

Optimization. The problem of the consumer is then simply that of maximizing utility under his budget constraint:

$$\max_{c_0, c_1} \quad u(c_0) + \beta u(c_1)$$
$$\text{s.t.} \quad c_0 + \frac{c_1}{1+r} = f_0 + y_0 + \frac{y_1}{1+r}.$$

3.1.3 4 methods

You may solve this optimization in four different ways:

1. Apply the well known ratio of marginal utilities formula from Econ 11. Let us rewrite this optimization problem as follows:
$$\max_{c_0, c_1} \quad u(c_0) + \beta u(c_1)$$
$$\text{s.t.} \quad p_0 c_0 + p_1 c_1 = B.$$
where we have defined the price of consumption in period 0 by:
$$p_0 \equiv 1,$$
the price of consumption in period 1 by:
$$p_1 \equiv \frac{1}{1+r},$$
and finally the budget B by the present discounted value of lifetime resources:
$$B \equiv f_0 + y_0 + \frac{y_1}{1+r}.$$
Note that the relative price of consumption in period 1 relative to period 0 is given by $1/(1+r)$: when the interest rate becomes higher, consuming in period 1 becomes relatively cheaper, or consuming in period 0 becomes more expensive (it's really expensive to consume now rather than later if the bank is offering me a really high interest rate). Thus, applying the formula from Econ 11 allows to say that

the marginal rate of substitution between consumption in period 1 c_1 and consumption in period 0 c_0 - the ratio of marginal utilities - is equal to the ratio of prices:

$$\frac{\partial U/\partial c_1}{\partial U/\partial c_0} = \frac{p_1}{p_0} = \frac{1}{1+r} \quad \Rightarrow \quad \boxed{\frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+r}}.$$

2. Apply the following intuitive economic argument. The marginal utility from consuming in period 1 is $\beta u'(c_1)$. The marginal utility from consuming in period 0 is $u'(c_0)$. By putting one unit of consumption in the bank, one forgoes 1 unit of consumption in period 0 to get $1+r$ units of consumption in period 1. The two have to be equal, if one is optimizing. If consuming more in period 0 gives a higher marginal utility, or $u'(c_0) > (1+r)\beta u'(c_1)$, then one should consume more and save less. On the contrary, should $u'(c_0) < (1+r)\beta u'(c_1)$, one should consume less and save more. Therefore, in equilibrium, these two options can only be equal:

$$u'(c_0) = (1+r)\beta u'(c_1) \quad \Rightarrow \quad \boxed{\frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+r}}.$$

3. Replace c_0 from the intertemporal budget constraint above and optimize with respect to c_1:

$$\max_{c_1} \quad u\left[\left(f_0 + y_0 + \frac{y_1}{1+r}\right) - \frac{c_1}{1+r}\right] + \beta u(c_1).$$

Taking the derivative of this expression with respect to c_1 leads to:

$$\begin{aligned} &-\frac{1}{1+r}u'\left[\left(f_0 + y_0 + \frac{y_1}{1+r}\right) - \frac{c_1}{1+r}\right] + \beta u'(c_1) = 0 \\ \Rightarrow \quad &-\frac{1}{1+r}u'(c_0) + \beta u'(c_1) = 0 \quad \Rightarrow \quad \boxed{\frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+r}}. \end{aligned}$$

where the first substitution uses the intertemporal budget constraint which implies:

$$\left(f_0 + y_0 + \frac{y_1}{1+r}\right) - \frac{c_1}{1+r} = c_0$$

4. Alternatively, you may substitute c_1 out and optimize with respect to c_0:

$$\max_{c_0} \quad u(c_0) + \beta u\left[(1+r)\left(f_0 + y_0 + \frac{y_1}{1+r}\right) - (1+r)c_0\right].$$

Taking the derivative of this expression with respect to c_0 leads to:

$$\begin{aligned} &u'(c_0) - \beta(1+r)u'\left[(1+r)\left(f_0 + y_0 + \frac{y_1}{1+r}\right) - (1+r)c_0\right] = 0 \\ \Rightarrow \quad &u'(c_0) - \beta(1+r)u'(c_1) = 0 \quad \Rightarrow \quad \boxed{\frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+r}}. \end{aligned}$$

where the first substitution uses the intertemporal budget constraint which implies (pre-multiplying both sides by $1+r$):

$$(1+r)\left(f_0 + y_0 + \frac{y_1}{1+r}\right) - (1+r)c_0 = c_1$$

3.2 Some examples

3.2.1 Log utility, no discounting

Log utility implies that $u(c)$ is given by the natural logarithm. Marginal utility is then just:

$$u'(c) = \frac{1}{c},$$

Since $\beta = 1$, the above optimality condition (derived 4 times) can be written as:

$$\begin{aligned} \frac{u'(c_1)}{u'(c_0)} = \frac{1}{1+r} \quad &\Rightarrow \quad \frac{1/c_1}{1/c_0} = \frac{1}{1+r} \\ \Rightarrow \quad \frac{c_0}{c_1} = \frac{1}{1+r} \quad &\Rightarrow \quad c_0 = \frac{c_1}{1+r} \end{aligned}$$

Substituting out $c_1/(1+r) = c_0$ in the intertemporal budget constraint allows to calculate consumption at time 0 c_0:

$$\begin{aligned} c_0 + \frac{c_1}{1+r} &= f_0 + y_0 + \frac{y_1}{1+r} \\ \Rightarrow \quad c_0 + c_0 &= f_0 + y_0 + \frac{y_1}{1+r} \\ \Rightarrow \quad c_0 &= \frac{1}{2}\left(f_0 + y_0 + \frac{y_1}{1+r}\right) \end{aligned}$$

Finally, we may calculate c_1:

$$c_1 = (1+r)c_0 = \frac{1+r}{2}\left(f_0 + y_0 + \frac{y_1}{1+r}\right).$$

According to this expression, the **Marginal Propensity to Consume (MPC)** out of current wealth f_0 is given by $1/2$. When f_0 rises to $f_0 + \Delta f_0$, the corresponding change in consumption is:

$$\Delta c_0 = \frac{1}{2}\Delta f_0.$$

If we were to study a model with more periods, say T periods, we would find that people Marginal Propensity to Consume is approximately equal to $1/T$, at least according to this model. Whether such is actually the case, and people are that rational, is a subject of fierce debate among macroeconomists, and one that we will take up in the next lectures.

3.2.2 Log utility, with discounting

Marginal utility is then $u'(c) = 1/c$, so that the optimality condition gives:

$$\begin{aligned} \frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+r} \quad &\Rightarrow \quad \frac{\beta/c_1}{1/c_0} = \frac{1}{1+r} \\ \Rightarrow \quad \frac{\beta c_0}{c_1} = \frac{1}{1+r} \quad &\Rightarrow \quad \beta c_0 = \frac{c_1}{1+r} \end{aligned}$$

Substituting out $c_1/(1+r) = \beta c_0$ in the intertemporal budget constraint allows to calculate consumption at time 0 c_0:

$$\begin{aligned} c_0 + \frac{c_1}{1+r} &= f_0 + y_0 + \frac{y_1}{1+r} \\ \Rightarrow \quad c_0 + \beta c_0 &= f_0 + y_0 + \frac{y_1}{1+r} \\ \Rightarrow \quad c_0 &= \frac{1}{1+\beta}\left(f_0 + y_0 + \frac{y_1}{1+r}\right) \end{aligned}$$

Finally, we may calculate c_1:

$$c_1 = \beta(1+r)c_0 = \frac{\beta(1+r)}{1+\beta}\left(f_0 + y_0 + \frac{y_1}{1+r}\right).$$

Because people are more impatient in this case, they consume more, and their Marginal Propensity to Consume (MPC) is **higher** with $\beta < 1$:

$$\Delta c_0 = \frac{1}{1+\beta}\Delta f_0.$$

Note that the solution with no discounting corresponds to that with discounting when $\beta = 1$, which was expected.

3.3 Generalization

Assume that an individual receives wage w in period 0, and that this wage is expected to grow at rate g in the next T years. What is the present value of his human wealth, assuming that the interest rate is given by R? The answer is that his human wealth H is given as follows:

$$H = w + w\frac{1+g}{1+r} + w\left(\frac{1+g}{1+r}\right)^2 + ... + w\left(\frac{1+g}{1+r}\right)^{T-1}$$

$$H = w\frac{1-\left(\frac{1+g}{1+r}\right)^T}{1-\frac{1+g}{1+r}}$$

Chapter 4

The Overlapping Generations Model

In the Solow growth model, we assumed that saving was a constant fraction of GDP. Lecture 2 has shown how to use microeconomics, and optimization, in order to derive saving behavior endogenously (that is, to explain it).

This section presents a very simple version of Peter Diamond's **overlapping-generations model**, published in 1965 - if you would like to read the original paper (you probably don't), it's here.[1] This model is used not just to give microfoundations to Robert Solow's growth model, but also to think about social security, public debt, an endeavor which we will take up in the next lectures as well as in the problem sets. During this lecture, I will present a very simplified version of Peter Diamond's overlapping generations model.

4.1 Assumptions

4.1.1 Time

We assume that people in this economy live only for 2 periods. People are called "young" in the first period of their life, and "old" in the second. Thus, you should really think that the length of a period is a generation (approximately 30 years). However, instead of referring to these two periods as 0 and 1, I shall refer to them as t and $t+1$.

4.1.2 Demographics

People from generation t are young in period t, and old in period $t+1$. We denote their consumption when young by c_t^y and their consumption when old by c_{t+1}^o. In terms of Lecture 3, you should really think of c_t^y as c_0, and of c_{t+1}^o as c_1.

People work when young, and then receive a wage given by w_t. They retire when old, and then do not work. Their lifetime utility is logarithmic with $\beta = 1$:

$$U = \log(c_t^y) + \log(c_{t+1}^o).$$

Their intertemporal budget constraint is given by:

$$c_t^y + \frac{c_{t+1}^o}{1+r} = w_t.$$

[1] Needless to say, you are not responsible for reading the original paper for the exam!

There are always two generations living in period t: the previous period's young, born in period $t-1$, now old, consuming the return from their savings; and this period's young, newly born (in period t).

4.1.3 Production

For simplicity, we shall assume a Cobb-Douglas, constant returns to scale, production function:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}.$$

We assume that the labor force is constant and fixed to unity (this is to avoid carrying L around everywhere - from lecture 2, you should now know that everything can be expressed per capita, because of constant returns to scale), and therefore:

$$L_t = L = 1.$$

Again for simplicity, we shall assume that capital depreciates at rate $\delta = 1 = 100\%$. (that is, capital fully depreciates each period - this is not that unreasonable if you take one unit of time to represent one generation, or about 30 years - remember that the depreciation rate for one year was approximately equal to 2% to 30% depending on the type of capital involved.)

4.2 Solution

4.2.1 Saving

Utility is logarithmic, so that the consumption of the young c_t^y and consumption of the old c_{t+1}^o are given as a function of the wage as follows (this is just an application of Lecture 3):

$$c_t^y = \frac{w_t}{2} \qquad c_{t+1}^o = (1+r)\frac{w_t}{2}.$$

Indeed, if you want to think of this model as the two periods model of Lecture 3, think that everything is as if:

$$f_0 = 0, \qquad y_0 = w_t, \qquad y_1 = 0.$$

4.2.2 Capital accumulation

Saving (and savings) is equal to investment, and therefore we have that:

$$S_t = I_t = w_t - c_t^y = \frac{w_t}{2}.$$

The major difference with the Solow model is that saving is here endogenous, and coming from agents' optimizing choices. In the Solow model in contrast, saving was taken as exogenous and equal to a fraction s.

The wage paid by employers, given that $L = 1$, is:

$$w_t = (1-\alpha)K_t^{\alpha} L^{-\alpha} = (1-\alpha)K_t^{\alpha} = (1-\alpha)Y_t.$$

Finally:

$$\Delta K_{t+1} = \frac{w_t}{2} - \delta K_t = \frac{1-\alpha}{2} Y_t - \delta K_t.$$

This is the capital accumulation equation (or law of motion for capital) of the Solow model, with $s = (1-\alpha)/2$. The new element here of course is to get saving endogenously, from agents' optimal decisions. Note that the value for the saving rate has an economic interpretation: wages are only a fraction $1 - \alpha$ of output, from lecture 1. On the other hand, savers / consumers want to smooth consumption and therefore want to save a half of that. This is why a fraction $(1 - \alpha)/2$ of output is saved.

4.2.3 Numerical Application

Note that if $\alpha = 1/3$, then the saving rate is equal to $s = 1/3$, which happens to be (by coincdence) the Golden Rule level of saving. This does not mean that the Golden Rule level is always satisfied. This only happens by chance in this very stylized model. In particular, saving is not just because of retirement, but also because of precautionary behavior, leaving bequests or simply liking being wealthy. We will come back to these issues in future lectures, but we can look at some data on who owns wealth and how it is divided first, before we move to that.

4.3 Why do people save?

In Peter Diamond's overlapping generations model, saving behavior only has one source: planning for retirement. Reality is a bit more nuanced. This section provides data which is suggestive that much of the wealth does not in fact come from young workers saving to provide for their old age. Thus, the overlapping generations model, in which most saving is lifecycle saving, does not capture an important part of the motive to save. We propose other factors at the end of this note.

4.3.1 Some data

Figure 4.1 from Emmanuel Saez and Gabriel Zucman, two economists at the University of California, Berkeley, working on the world distribution of wealth, shows the composition of aggregate US household wealth from 1913 to 2013.[2] The US tax code includes provisions which strongly encourage retirement saving in the form of retirement accounts. However, houses are also clearly a potential source of revenue for older people – because of the flow of rents that owner-occupied housing provides, but also because there is always an option to liquidite one's house when old.

Figure 4.2 shows the saving rate by wealth class, which echoes the evidence on saving rate by income shown previously in Lecture 3.

Figure 4.3 shows the top 10% wealth share. As you can see, nearly 75% of household wealth is held by the top 10% wealth owners. This is more concentrated than labor income (the top 10% in the United States gets about 50% of pre-tax income, and much less after-tax), and therefore does not appear to be solely accounted for by saving for retirement.

Figure 4.4 shows the top 1% wealth share, and the top 1-10% wealth share. As you can see, the top 1% now owns nearly 40% of the wealth in the United States, while it only accounts for about 20-25% of pre-tax income. Again, it does not seem like saving for retirement is the whole story.

4.3.2 Saving of the rich

Understanding the sources of the capital stock amounts to a first order to undertand the saving of people with very high net worth. What leads high income and high net worth people to save so much? A number

[2] This picture is taken from work by Emmanuel Saez and Gabriel Zucman, "Wealth Inequality in the United States since 1913: Evidence from Capitalized Income Tax Data," published in 2016 in *The Quarterly Journal of Economics*. The paper is available here: https://doi.org/10.1093/qje/qjw004.

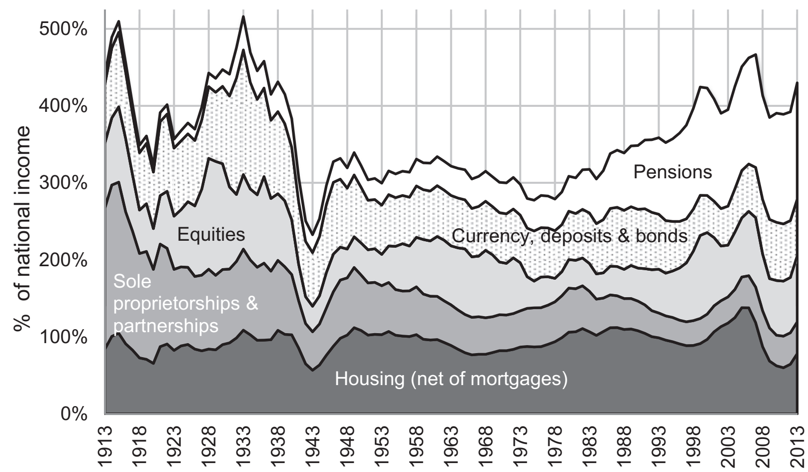

Figure 4.1: Aggregate US Household Wealth, 1913–2013.

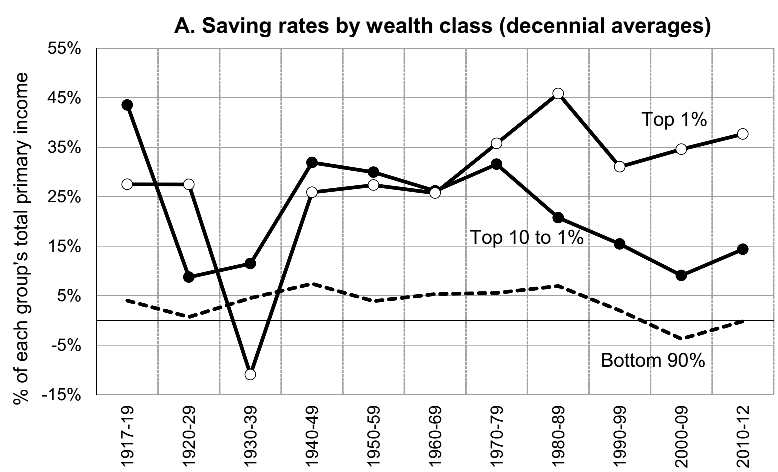

Figure 4.2: Saving Rate by Wealth Class.

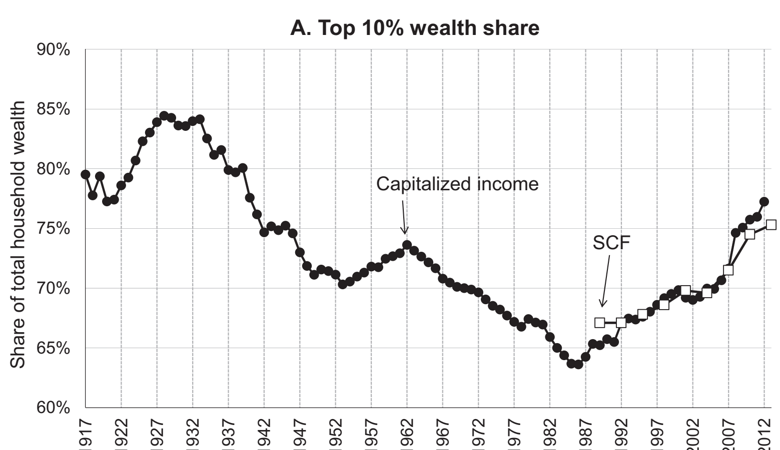

Figure 4.3: Top 10 per cent wealth share.

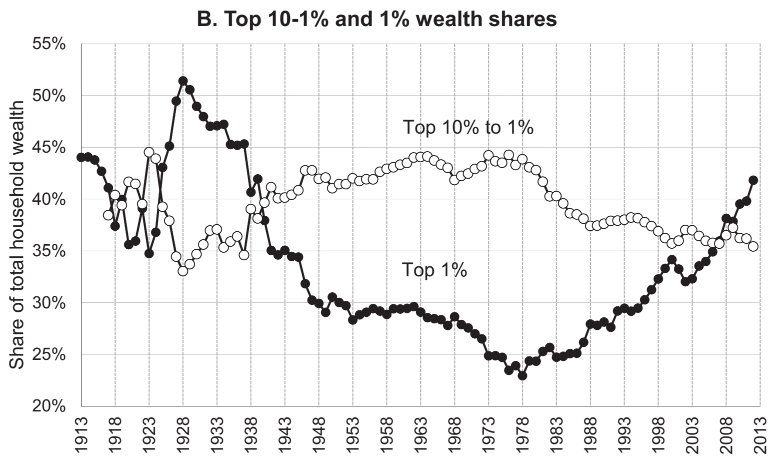

Figure 4.4: Top 1-10 per cent and Top 1 per cent wealth share.

of explanations have been proposed:

1. **Leaving bequests.** One reason why people might want to save over and above what they need to provide for retirement, is to leave bequests. However, it has been shown that even high income workers without children save a lot, more than warranted by their retirement needs.

2. **Prestige.** Wealth brings prestige. Adam Smith has a great passage in The *Theory of Moral Sentiments*, published in 1759:

   To what purpose is all the toil and bustle of the world?... lt is our vanity which urges us on... It is not wealth that men desire, but the consideration and good opinion that wait upon riches.

3. **Concern for relative wealth.** Related to this explanation is a concern not for the absolute level of wealth per se, but for a relative standing compared to others in society. This is for example echoed in an academic article, published in 1992:[3]

   But think for a moment about an already very rich agent such as Donald Trump. Why does he continue to work long days, endure substantial amounts of stress, and take enormous risks? Surely it cannot be that he is savoring the prospect of going to the grocery store with a looser budget constraint next year. He seems to have more money than he could spend in several lifetimes. Even if we are wrong about Trump's net worth, there clearly seem to be wealthy individuals that continue to work very hard and take large risks to increase their net worth. It is hard to reconcile such behavior with the underlying decision making in traditional growth models. We propose that people like Trump continue to care about increasing their net worth because their utility depends not only on the absolute level of their wealth but also on their wealth relative to that of other very rich people.

4. **Religious beliefs and work ethic.** Max Weber has famously proposed the protestant work ethic in The Protestant Ethic and the Spirit of Capitalism as one explanation for the emergence of capitalism, and the importance of hard work and saving. John Maynard Keynes, in the Economic Consequences of the Peace published in 1919, was thinking very much in these terms:

   Europe was so organised socially and economically as to secure the maximum accumulation of capital. While there was some continuous improvement in the daily conditions of life of the mass of the population, society was so framed as to throw a great part of the increased income into the control of the class least likely to consume it. The new rich of the nineteenth century were not brought up to large expenditures, and preferred the power which investment gave them to the

[3] Harold L. Cole, George J. Mailath, and Andrew Postlewaite, "Social Norms, Savings Behavior, and Growth," Journal of Political Economy 100, no. 6 (December 1, 1992): 1092–1125, https://doi.org/10.1086/261855.

> pleasures of immediate consumption. In fact, it was precisely the inequality of the distribution of wealth which made possible those vast accumulations of fixed wealth and of capital improvements which distinguished that age from all others. Herein lay, in fact, the main justification of the capitalist system. If the rich had spent their new wealth on their own enjoyments, the world would long ago have found such a régime intolerable. But like bees they saved and accumulated, not less to the advantage of the whole community because they themselves held narrower ends in prospect.
>
> The immense accumulations of fixed capital which, to the great benefit of mankind, were built up during the half century before the war, could never have come about in a society where wealth was divided equitably. The railways of the world, which that age built as a monument to posterity, were, not less than the pyramids of Egypt, the work of labour which was not free to consume in immediate enjoyment the full equivalent of its efforts.
>
> Thus this remarkable system depended for its growth on a double bluff or deception. On the one hand the labouring classes accepted from ignorance or powerlessness, or were compelled, persuaded, or cajoled by custom, convention, authority, and the well-established order of society into accepting, a situation in which they could call their own very little of the cake that they and nature and the capitalists were co-operating to produce. And on the other hand the capitalist classes were allowed to call the best part of the cake theirs and were theoretically free to consume it, on the tacit underlying condition that they consumed very little of it in practice. The duty of 'saving' became nine-tenths of virtue and the growth of the cake the object of true religion. There grew round the non-consumption of the cake all those instincts of puritanism which in other ages has withdrawn itself from the world and has neglected the arts of production as well as those of enjoyment. And so the cake increased; but to what end was not clearly contemplated. Individuals would be exhorted not so much to abstain as to defer, and to cultivate the pleasures of security and anticipation. Saving was for old age or for your children; but this was only in theory – the virtue of the cake was that it was never to be consumed, neither by you nor by your children after you.

5. **A final hypothesis.** An even more mundane explanation (which does not make it wrong!) has been proposed by Lee Iacocca, former CEO from Chrysler. According to him, the rich simply do not know what to do with their money:

   > Once you reach a certain level in a material way, what more can you do? You can't eat more than three meals a day; you'll kill yourself. You can't wear two suits one over the other. You might now have three cars in your garage-but six! Oh, you can indulge yourself, but only to a point.

Most economists are however general skeptical of this type of explanations. What they find puzzling is that high net worth individuals keep working even when they have achieved a sufficient amount of wealth.[4]

All this discussion may seem like armchair theorizing. At the same time, these are probably the most important questions facing macroeconomics. They actually determine the stance that should be taken on optimal capital accumulation, the optimal level of public debt, etc. We shall come back to these issues repeatedly in the following lectures.

[4] I personally am less sure, as culture and norms certainly play a bigger role than many economists imagine - for example, as explained above, the protestant work ethic, might be one answer.

Chapter 5

Technological Change

In this lecture, we start by reviewing some facts on long-run economic growth. We then review an endogenous growth model which tries to account for long-run economic growth.

5.1 Long-run Economic Growth

In this section, we remind ourselves some basic facts on technological growth since the 1200s. We in particular use Angus Maddison's long run data series on GDP per capita in major advanced economies, to show that growth as we know it is a rather recent phenomenon, one that starts at the beginning of the nineteenth century.

5.1.1 The Facts of Growth

The Figures below show GDP per capita in France, Germany, Italy, the United Kingdom, and the United States, starting respectively in 1200, 1700, 1800, and 1900. These are presented on a semi-logarithmic scale (that is, the y-axis is in log), also sometimes called a ratio scale. Do not worry too much about the unit in which GDP per capita is expressed in this Maddison data: the unit is called the Geary–Khamis dollar, but we shall come back to it when we discuss the issue of real exchange rates and purchasing power parity comparisons. The reason for using such a semi-logarithmic scale is that growth is roughly exponential, so it is approximately linear on a semi-logarithmic scale.

5.1.2 The Mathematics of Growth

The mathematical orders of magnitude with exponential growth are sometimes hard to grasp intuitively. This section provides you with some maths to help get an intuitive feel for exponential growth. Let's consider an economic quantity y_t, growing at a constant rate g. For our purposes in this lecture, y_t is GDP per capita, but it could also veru well be a price level, consumption, or some other economic quantity. Iterating on the difference equation allows to get y_t as a function of the initial value:

$$y_t = (1+g)y_{t-1} \quad \Rightarrow \quad y_t = (1+g)^t y_0.$$

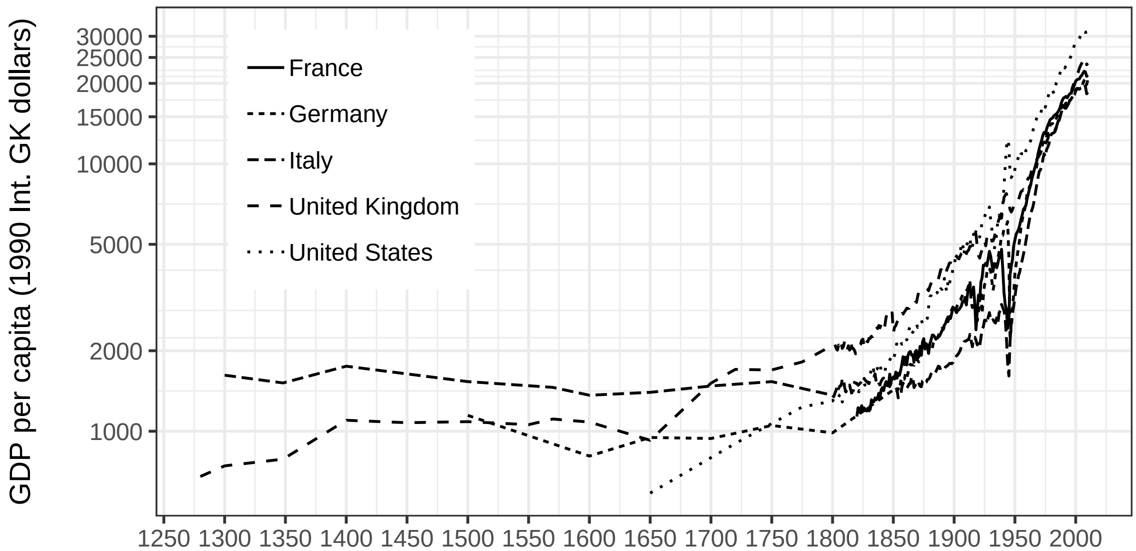

Figure 5.1: 1200-2010 GDP PER CAPITA (MADDISON DATA).

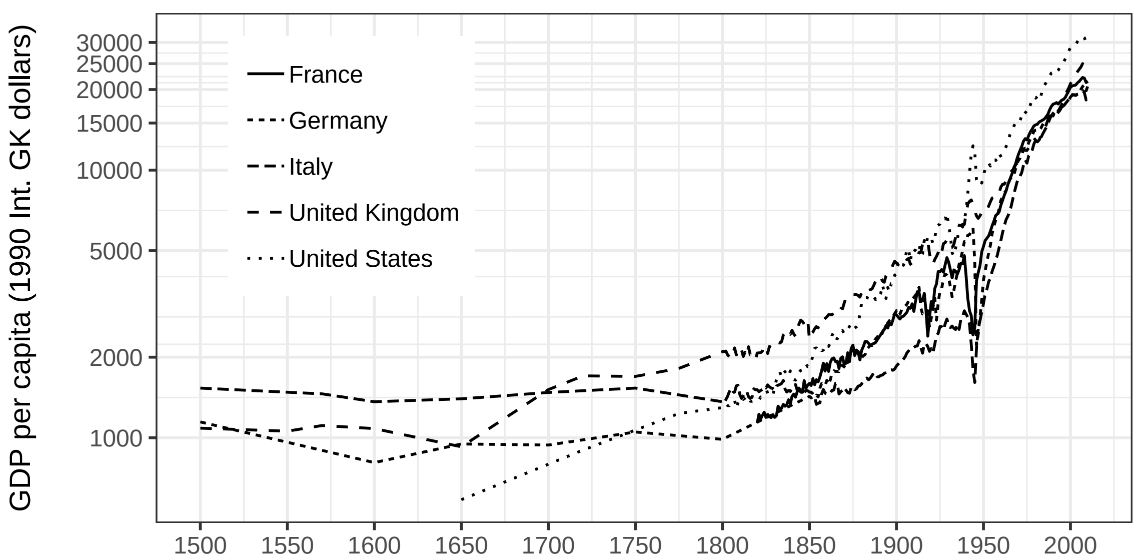

Figure 5.2: 1700-2010 GDP PER CAPITA (MADDISON DATA).

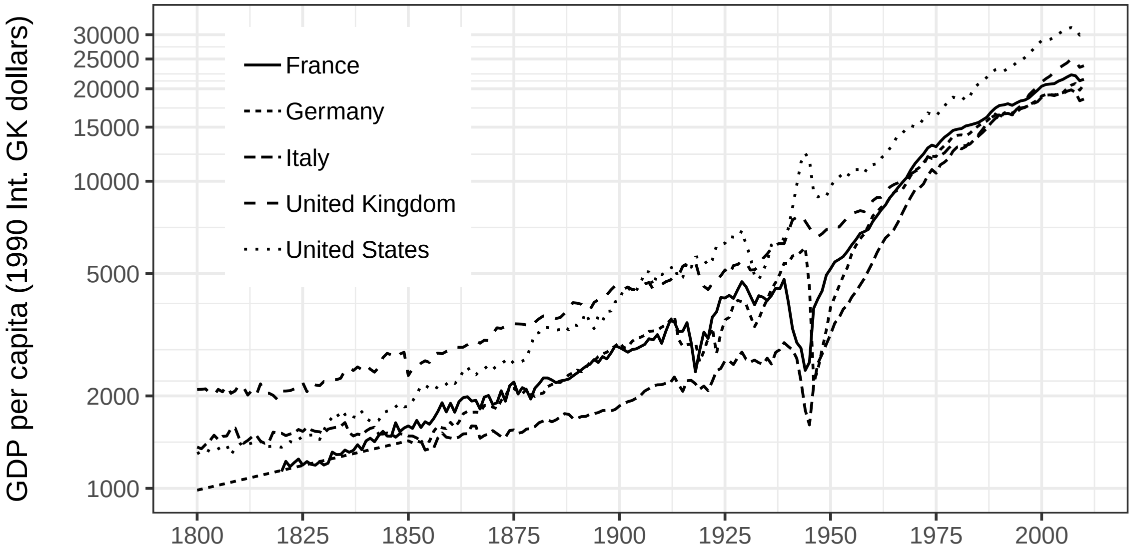

Figure 5.3: 1800-2010 GDP per capita (Maddison Data).

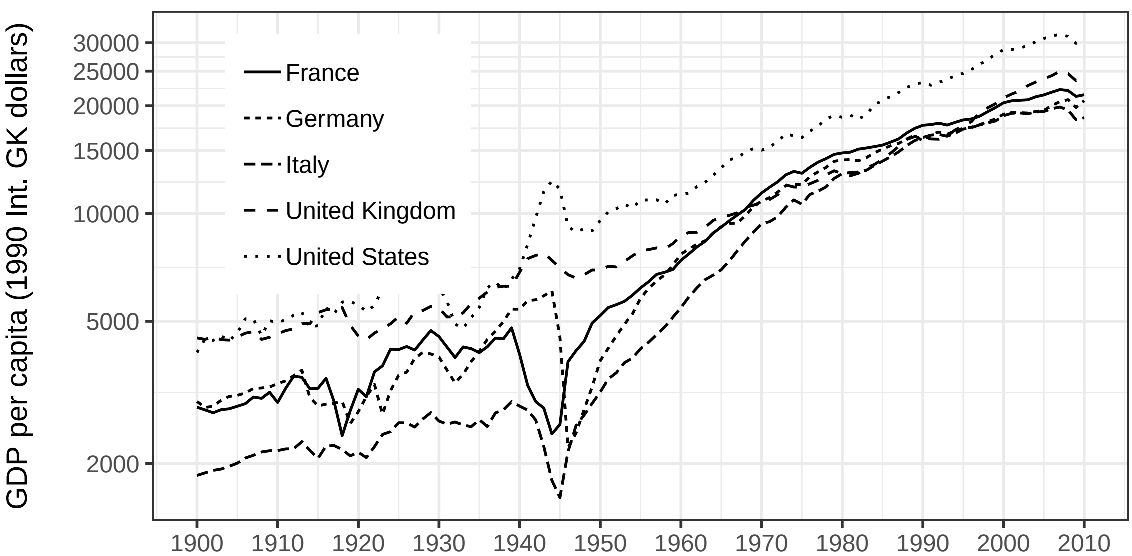

Figure 5.4: 1900-2010 GDP per capita (Maddison Data).

How fast of a growth is that? The number of years T it takes for GDP per capita to be multiplied by a factor F (where $F = 2$, if we are looking for GDP to double) is given by :

$$
\begin{aligned}
y_t = F y_0 \quad &\Rightarrow \quad (1+g)^t y_0 = F y_0 \quad \Rightarrow \quad (1+g)^T = F \\
&\Rightarrow \quad \ln(1+g)^T = \ln F \quad \Rightarrow \quad T \ln(1+g) = \ln F \\
&\Rightarrow \quad T = \frac{\ln F}{\ln(1+g)}.
\end{aligned}
$$

For small rates of growth, we know that a Taylor expansion of the ln gives $\ln(1+g) = g$, so that this formula can be approximated by:

$$T = \frac{\ln F}{g}.$$

Rule of 70. This formula gives the "rule of 70" for small growth rates. The "rule of 70" states that a quantity which grows at a rate g in percentage terms, takes approximately $70/g$ years to double. This comes from the fact that $100 * ln(2) \approx 69.3147181$. For example, if GDP per capita is growing at 2% per year, then it takes approximately 35 years for it to double: approximately one generation only. If it is growing at 1% per year, it takes approximately 70 years for it to double, so two generations. Whether you grow to be twice as rich as your parents, or twice as rich as your grandparents on average makes a big difference.

Rule of 230. The same formula also gives the "rule of 230" for small growth rates. The "rule of 230" states that a quantity which grows at a rate g in percentage terms, takes approximately $230/g$ years to be multiplied by 10. This is because: $100 * ln(10) \approx 230.2585093$. Thus, with a 2% growth rate, GDP per capita is multiplied by 10 over (as an exercise, you may create your own rules…)

5.1.3 Questions

The previous section has shown that economic growth has been 2% on average in advanced economics, in the last two centuries. This raises a number of questions.

Why is there GDP per capita growth at all? One major shortcoming of Bob Solow's Growth model of Lecture 2 and of Peter Diamond's growth model of Lecture 4 is that they fail to account for long run economic growth - rather ironically. According to these models, the process of capital accumulation reaches a steady-state at some point, such that capital per capita, GDP per capita, remain forever constant. This happens for intuitive reasons: if the labor force is given, then growth cannot be sustained through capital accumulation alone. In fact, in the Solow model, even a saving rate equal to 100% would lead to very high GDP, but not to constant GDP growth. At one point, the capital stock would be so large that merely maintaining it would consume all output. Output would then be equal to gross saving, and depreciation. This is coming from the fact that at the steady state of the Solow model, gross saving is given by sY^* which would then be equal to Y^* (if $s = 1 = 100\%$), which is also equal to depreciation as:

$$sY^* = \delta K^*.$$

The Cross-section of Countries: Rich versus Poor Countries. The power of compounding is such that even a few decades of lower growth makes a big difference. Given persistent differences in growth rates, per capita GDP is far from being equalized across countries. Contrary to the predictions in the Solow growth model, there has hardly been a strong catch up of poor countries towards the level of rich countries. On one hand of the spectrum, many countries have very small GDP per capita levels.

On the other, some countries have very high GDP per capita. Table 5.2 below shows the countries that have a GDP per capita higher than $40K.

Why are poor countries so poor, and largely failing to catch up, and why advanced economies are growing at an average of 2% per year since the beginning of the nineteenth century, are perhaps the potentially most impactful questions for economics, but also those for which economists perhaps know the least. This lecture presents an attempt at endogenizing GDP per capita growth, based on the a very simplified version of Paul Romer's endogenous growth model. This may allow us to shed light on some of these issues.

Table 5.1: Countries with GDP per capita lower than $1.5K.

Country	GDP per capita
Ethiopia	$ 1,489
Togo	$ 1,446
Sierra Leone	$ 1,353
Guinea-Bissau	$ 1,257
Madagascar	$ 1,237
Mozambique	$ 1,211
D.R. of the Congo	$ 1,199
Malawi	$ 971
Liberia	$ 877
Niger	$ 868
Burundi	$ 840
Central African Republic	$ 599

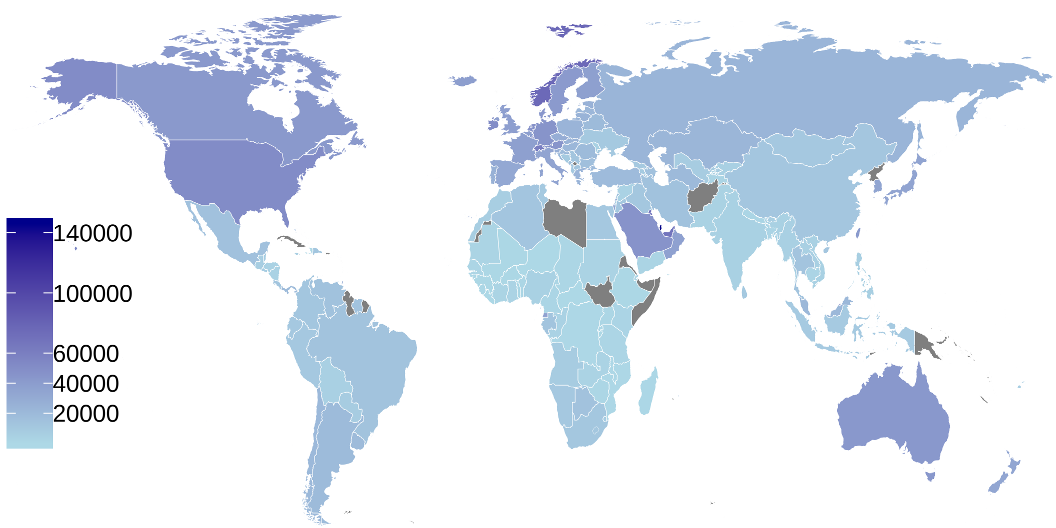

Figure 5.5: 2014 GDP per capita, by Country (Penn World Tables).

Table 5.2: Countries with GDP per capita higher than $40K.

Country	GDP per capita
Qatar	$ 146,037
China, Macao SAR	$ 130,758
Norway	$ 75,920
United Arab Emirates	$ 68,021
Kuwait	$ 67,432
Brunei Darussalam	$ 66,968
Singapore	$ 66,050
Luxembourg	$ 65,842
Switzerland	$ 61,570
United States	$ 51,623
Ireland	$ 51,224
Netherlands	$ 47,392
Saudi Arabia	$ 46,772
Germany	$ 46,190
China, Hong Kong SAR	$ 45,399
Austria	$ 45,158
Denmark	$ 43,733
Australia	$ 43,590
Canada	$ 42,794
Sweden	$ 42,117
Taiwan	$ 41,514

5.2 Endogenous Growth Model

Paul Romer was awarded the Nobel Prize in Economic Sciences this year. The Economist is a good coverage of his work. In this section, we present a very simplified version of his much more subtle ideas.

5.2.1 Objects VS Ideas

A crucial distinction for understanding endogenous growth theory, is that of objects versus ideas.

Objects. Objects like houses, food, or cell phones are **rivalrous**. That is, one person's use of one of these particular objects prevents its use by someone else. Most goods are rivalrous, and this is what leads to scarcity, the central topic of economics.

Ideas and recipes. In contrast, ideas are **nonrival**. My use of an idea does not prevent someone else's use of that idea. For example, after it has been shot, a movie can be shown in any theatre in the world, and for a cost equal to zero on the internet (apart for the cost of electricity, and of storing the movie on servers). Similarly, a recipe for a pharmaceutical drug takes years to develop. However, once the drug has been invented, the cost of producing this drug is typically very small, much smaller than the associated initial fixed cost. Finally, Antonín Dvořák's 9th Symphony "From the New World" can be performed by any orchestra in the world, Gustavo Dudamel's or Herbert Von Karajan's, once Antonín Dvořák has composed it.

According to Paul Romer, ideas and recipes have the potential to be a source of indefinite economic growth:

> Every generation has perceived the limits to growth that finite resources and undesirable side effects would pose if no new recipes or ideas were discovered. And every generation has underes-

> timated the potential for finding new recipes and ideas. We consistently fail to grasp how many ideas remain to be discovered.

As Chad Jones puts it describing Paul Romer's contributions in a Vox Eu article:

> Once you've got increasing returns, growth follows naturally. Output per person then depends on the total stock of knowledge; the stock doesn't need to be divided up among all the people in the economy. Contrast this with capital in a Solow model. If you add one computer, you make one worker more productive. If you add a new idea – think of the computer code for the first spreadsheet or word processor or even the internet itself – you can make any number of workers more productive. With non-rivalry, growth in income per person is tied to growth in the total stock of ideas – an aggregate – not to growth in ideas per person.

However, ideas implies increasing returns and therefore, potentially inefficient market outcomes.

5.2.2 Increasing Returns

Because of increasing returns, a competitive market economy may not lead to an efficient level of development of ideas. The issue with pure competition is that if a movie director was forced to charge a price equal to the marginal cost of showing a movie, then he would have to charge a zero price for this movie, as it would then be available for everyone to watch on Youtube. Similarly, once a drug against AIDS (for example) has been invented, one would ideally like to give this drug to the largest possible number of people. Indeed, the marginal cost of producing a new drug is much lower than the cost at which it is commercialized. However, this would decrease the incentives to innovate in the first place.

Patents. One way that this market failure has historically been addressed, is by granting patents (property rights) on newly invented ideas. (in biology, scientists are sometimes granted 12 years of exclusivity; in chemistry, they may be granted 5 years) Similarly, intellectual property rights, as well as laws against piracy, have the same purpose: they seek to encourage innovation by preventing 0 marginal cost pricing. Note however that patents should not be too strong either, or they discourage the use of new ideas, potentially at a suboptimal level. In fact, historically, especially in periods of high innovation and economic growth, intellectual property rights were relatively weak. When the US was a rising economic power, it was also a notorious pirate of intellectual property, very much like what China is being accused of today.

Government funded research. One alternative way through which R&D can be incentivized, while allowing for large dissemination, is to have the allow for government funded research. For example, the Department of Defense's ARPANET was a precursor of today's World Wide Web.

Prizes. Another way for the government to incentivize fundamental research is to give out prizes, which are a rather inexpensive way to motivate researchers hoping for recognition. The Economics Nobel Prize is one such example. In general, the motivations of researchers are complex, and probably not purely driven by profit seeking. For example, a very large community of developers contributes to open-source software, which is hard to rationalize from an orthodox (individualist) economic standpoint.

5.2.3 Model

In Paul Romer's endogenous growth model, there are two types of workers: research workers, whose number is L_{at}, and production workers, whose number is L_{yt}. We denote the share of research workers in total labor by l, so that:

$$L_{at} = lL$$

The number of production workers is simply the complement of that, so that:

$$L_{yt} = (1 - l)L.$$

As an example, if $l = 0.10$, then 10% of the labor force works in the R&D sector. For simplicity, we neglect the role of capital and capital accumulation, an issue that you will take up during sections. Therefore, production is simply given by:

$$Y_t = A_t L_{yt}$$

Finally, productivity is assumed to grow at a rate that is function of the productivity of the research sector z, overall productivity A_t, and the number of researchers in the research sector L_{at}:

$$\Delta A_{t+1} = z A_t L_{at}.$$

5.2.4 Solution

In order to solve this model, we may simply iterate on the production function for new ideas, as a function of the initial value for productivity A_0 (just as there was an initial value for capital K_0 in the Solow growth model), and using that the number of research workers is $L_{at} = lL$:

$$\begin{aligned}
\Delta A_{t+1} = A_{t+1} - A_t &= z A_t L_{at} \\
\Rightarrow \quad A_{t+1} &= (1 + z L_{at}) A_t \\
\Rightarrow \quad A_{t+1} &= (1 + zlL)\, A_t \\
\Rightarrow \quad A_t &= (1 + zlL)^t\, A_0.
\end{aligned}$$

Replacing then L_{yt} in the production function with $L_{yt} = (1 - l)L$, as well as this endogenous level of productivity, we get:

$$\begin{aligned}
Y_t &= A_t L_{yt} \\
&= (1 + zlL)^t\, A_0 L_{yt} \\
&= (1 + zlL)^t\, A_0 (1 - l) L \\
Y_t &= A_0 (1 - l) L\, (1 + zlL)^t
\end{aligned}$$

Change in l. A change on the research share of workers has too opposing effect. On the one hand, there are less production workers available for production Thus, there is a fall in L_{yt}, which tends to reduce Y_t. On the other, overall growth, which is here given by $g = zlL$, is then permanently higher.

5.2.5 Shortcomings

In Paul Romer's model, all economic growth comes from R&D. Despite its successes, this model however leaves a number of questions unanswered. For example, it is unclear why poor countries are not able to benefit from "ideas" which are available in rich countries. Ideas being a public good, they are potentially available for anyone to use. Then, productivity A_t should be quite similar in rich and poor countries. Moreover, the endogenous growth model does not really explain why economic growth took off where and when it took off (in the UK, at the beginning of the nineteenth century), a question we started out with.

If you want to know more, this Economist article gives a number of limitations on the theory of economic growth. In particular, economic historians are still debating the origins of the Industrial Revolution. Important factors that the endogenous growth model neglects are the relative importance of **secure property rights**, the extent to which cultures tolerate **personal ambition**, and other cultural or political factors, all of which certainly matter for economic growth. You may refer to the next section, if you wish to read more about economic growth.

Readings - To go further

Michele Boldrin and David K. Levine, “The Case against Patents,” Journal of Economic Perspectives 27, no. 1 (February 2013): 3–22.

Chad Jones, New ideas about new ideas: Paul Romer, Nobel laureate, *Vox Eu*, October 12, 2018.

Paul Krugman, Notes on Global Convergence (Wonkish and Off-Point), *New York Times*, October 20, 2018.

(Gated) Time to fix patents, *The Economist*, August 8 2015.

(Gated) A question of utility, *The Economist*, August 8, 2015.

(Gated) Economists understand little about the causes of growth, *The Economist*, April 12, 2018.

(Gated) Paul Romer and William Nordhaus win the economics Nobel, *The Economist*, Oct 13, 2018.

Chapter 6

The Labor Market and Unemployment

This lecture goes over three different models of the labor market, each of which has a different explanation for the phenomenon of unemployment: the Neoclassical model, the Keynesian model, the Bathtub model. In this lecture, we examine each of them in turn.

6.1 The Neoclassical Model

6.1.1 Assumptions

Neoclassical economic theory is fundamentally based on the laws of supply and demand. In this theory, labor is treated as another good, which enters negatively in the utility function of workers because people would rather not work (therefore, labor is a “bad” rather than a “good”). Labor is also a useful input into production, which allows firms to sell consumption goods to workers. In the neoclassical theory, the real wage is determined at the intersection of supply and demand.

Labor Demand. Firms hire labor at price w, and sell the consumption good a price p. Assume that the production function is decreasing returns to scale with respect to labor (think for example, that the quantity of capital is fixed):

$$y = f(l).$$

Then, firms maximize their profits, and therefore solve:

$$\max_l \quad pf(l) - wl.$$

Labor Supply. Neoclassical theory of labor supply starts from a static problem of a consumer-worker choosing how much to work, and how much to consume. For example, assume that utility is strictly increasing in consumption, and strictly decreasing in the amount of labor supplied:

$$U(c,l) = u(c) - v(l),$$

so that the consumer-worker likes to consume, but does not like working. If the price of consumption is p, and assuming a static problem, the budget constraint of a worker/consumer is given by:

$$pc = wl.$$

You may think of w as the hourly wage, for example \$15/hour. Then l would be expressed in terms of the number of hours.

Warning. Labor Supply and Labor Demand are sometimes being mixed up. In the language of microeconomics, workers supply labor, and firms demand labor. What workers do demand is jobs, or job vacancies, which are supplied by firms.

6.1.2 Solution

Labor Demand. The problem of labor demand shown above leads to the following first-order condition:

$$pf'(l) - w = 0 \quad \Rightarrow \quad f'(l) = \frac{w}{p}.$$

This equation has a straightforward interpretation: at the optimum, the marginal product of labor - that is, how much is gained from using one more unit of labor in production $f'(l)$ - needs to be equal to how much that additional unit of labor costs to the firm, the real wage w/p. J.M. Keynes calls it the first fundamental postulate of classical economics in Chapter 2 of the General Theory.

Labor Supply. On the other hand, the problem of the consumer consists in maximizing utility under his budget constraint:

$$\max_{c,l} \quad u(c) - v(l),$$
$$\text{s.t.} \quad p \cdot c = w \cdot l.$$

Again, similarly to the two-period optimization problem of Lecture 3, you may solve this optimization in four different ways:

1. You may compute the ratio of marginal utilities (the marginal rate of substitution between consumption and labor) and state that it is equal to minus the real wage (because labor slackens the intertemporal budget constraint, and it appears on the right hand-side of the equal sign):

$$\frac{\partial U/\partial l}{\partial U/\partial c} = -\frac{w}{p} \quad \Rightarrow \quad \boxed{\frac{v'(l)}{u'(c)} = \frac{w}{p}}$$

2. You may apply the following intuitive economic argument. The marginal disutility from supplying one more unit of labor is $v'(l)$, for a worker already supplying l units of them. The marginal utility which is gained from doing so is given by the number of additional units of consumption one gets out of it, given by the real wage w/p, and by how much I value each one of these additional utilities of consumption is valued, given by marginal utility $u'(c)$. The total gain in utility from consumption is the unit value $u'(c)$ times the number of units w/p, which gives the result:

$$v'(l) = \frac{w}{p}u'(c) \quad \Rightarrow \quad \boxed{\frac{v'(l)}{u'(c)} = \frac{w}{p}.}$$

   J.M. Keynes, who did not write one equation in his General Theory, calls this the second postulate of the classical economics, in Chapter 2: "The utility of the wage when a given volume of labour is employed is equal to the marginal disutility of that amount of employment."

3. You may substitute out l from the budget constraint, and optimize over the choice of consumption:

$$\max_{c} \quad u(c) - v\left(\frac{p}{w}c\right).$$

   This implies:

$$u'(c) - \frac{p}{w}v'(l) = 0 \quad \Rightarrow \quad \frac{v'(l)}{u'(c)} = \frac{w}{p}.$$

4. You may substitute out c from the budget constraint, and optimize over the choice of labor:

$$\max_{l} \quad u\left(\frac{w}{p}l\right) - v(l).$$

This implies:

$$\frac{w}{l}u'(c) - v'(l) = 0 \quad \Rightarrow \quad \boxed{\frac{v'(l)}{u'(c)} = \frac{w}{p}}.$$

6.1.3 A Simple Example

Assumptions. Assume a Cobb-Douglas production function for $f(l)$, such that:

$$f(l) = Al^{1-\alpha}.$$

In the background, you can really think that the capital stock is exogenous and taken to be equal to $K = 1$, which would lead to this production function exactly.

Let us also assume linear utility for consumption (that is, people enjoy increasing utility equally, regardless of whether it is coming from the first dollar or the last one - this assumption is not realistic and is really made for simplicity), as well as a power function of disutility for work:

$$u(c) = c, \quad v(l) = B\frac{l^{1+\epsilon}}{1+\epsilon}$$

so that:

$$U(c,l) = c - B\frac{l^{1+\epsilon}}{1+\epsilon}.$$

Results. Using the above functional forms for $u(.)$ and $v(.)$ allows to write:

$$v'(l) = Bl^{\epsilon}, \quad u'(c) = 1, \quad \Rightarrow \quad Bl^{\epsilon} = \frac{w}{p}.$$

Labor supply $L^s(.)$ as a function of the real wage w/p is thus given by:

$$l = \frac{1}{B^{1/\epsilon}}\left(\frac{w}{p}\right)^{1/\epsilon} \equiv L^s\left(\frac{w}{p}\right).$$

Moreover, using the above functional form for $f(.)$ allows to write:

$$f'(l) = A(1-\alpha)l^{-\alpha} \quad \Rightarrow \quad A(1-\alpha)l^{-\alpha} = \frac{w}{p}.$$

Therefore, labor demand $L^d(.)$ is given as a function of the real wage w/p by:

$$l = A^{1/\alpha}(1-\alpha)^{1/\alpha}\left(\frac{w}{p}\right)^{-1/\alpha} \equiv L^d\left(\frac{w}{p}\right).$$

To sum up, the neoclassical labor makret is composed of the following labor supply and labor demand equations:

$$L^d\left(\frac{w}{p}\right) = A^{1/\alpha}(1-\alpha)^{1/\alpha}\left(\frac{w}{p}\right)^{-1/\alpha},$$
$$L^s\left(\frac{w}{p}\right) = \frac{1}{B^{1/\epsilon}}\left(\frac{w}{p}\right)^{1/\epsilon}.$$

Market clearing implies that labor supply equals labor demand:

$$
\begin{aligned}
& L^d\left(\frac{w}{p}\right) = L^s\left(\frac{w}{p}\right) \\
\Rightarrow \quad & A^{1/\alpha}(1-\alpha)^{1/\alpha}\left(\frac{w}{p}\right)^{-1/\alpha} = \frac{1}{B^{1/\epsilon}}\left(\frac{w}{p}\right)^{1/\epsilon} \\
\Rightarrow \quad & \left(\frac{w}{p}\right)^{\frac{1}{\alpha}+\frac{1}{\epsilon}} = (1-\alpha)^{1/\alpha}A^{1/\alpha}B^{1/\epsilon} \\
\Rightarrow \quad & \frac{w}{p} = (1-\alpha)^{\frac{\epsilon}{\alpha+\epsilon}}A^{\frac{\epsilon}{\alpha+\epsilon}}B^{\frac{\alpha}{\alpha+\epsilon}}.
\end{aligned}
$$

We may use either labor supply or labor demand in order to express the equilibrium quantity of labor l. For example, let us use labor demand (as an exercise, you may check that using labor supply instead leads to the same expression):

$$
\begin{aligned}
l &= \frac{1}{B^{1/\epsilon}}\left(\frac{w}{p}\right)^{1/\epsilon} \\
&= \frac{1}{B^{1/\epsilon}}(1-\alpha)^{\frac{1}{\alpha+\epsilon}}A^{\frac{1}{\alpha+\epsilon}}B^{\frac{\alpha}{\epsilon(\alpha+\epsilon)}} \\
&= (1-\alpha)^{\frac{1}{\alpha+\epsilon}}A^{\frac{1}{\alpha+\epsilon}}B^{\frac{\alpha}{\epsilon(\alpha+\epsilon)}-\frac{1}{\epsilon}} \\
l &= (1-\alpha)^{\frac{1}{\alpha+\epsilon}}A^{\frac{1}{\alpha+\epsilon}}B^{-\frac{1}{\alpha+\epsilon}}
\end{aligned}
$$

6.1.4 An Example: A Shock to Labor Demand

Assume that firms all of a sudden become less productive: that is, A declines. This corresponds to a shift in the labor demand curve to the left (because A appears in the labor demand curve). As a consequence, the graph shows that clearly the equilibrium number of hours declines, and the real wage falls. The algebra above can also be used in order to show that a reduction in A both leads to a fall in the number of hours as well as to a fall in the real wage. Note that in the neoclassical model, there is nothing that explains why the fall in hours worked goes through the extensive margin (the number of people employed) rather than at the intensive margin (how intensively everyone works). In practice, some countries such as Germany have put in place work sharing programs during the depression, in order to share the reduction in employment equally among workers, rather than proceed to lay-off workers.

6.2 The “Keynesian” Model

There are two ways in which the neoclassical model above is not satisfying. Empirically, the real wage moves very little during recessions, compared to the increase in unemployment, an observation which is usually viewed as inconsistent with the neoclassical model. Moreover, and probably more importantly, the neoclassical model assumes that all unemployment is voluntary. In contrast, intuitively, the level of employment appears “too low” during recessions, in the sense that many people seem to be looking for work, but do not find one.

One potential explanation might be that wages are sticky (at least downwards). In that case, the graph shows that following a shock to labor demand, labor demand might be lower than labor supply: there is involuntary unemployment, in the sense that some workers want to work more than they do at the prevailing market wage. Workers are said to be “off their labor supply curve”, because it is the level of the demand for labor that determines the employment level.

The graph shows clearly that “Keynesian” unemployment is larger than Classical Unemployment: employment falls by more than if wages were flexible.

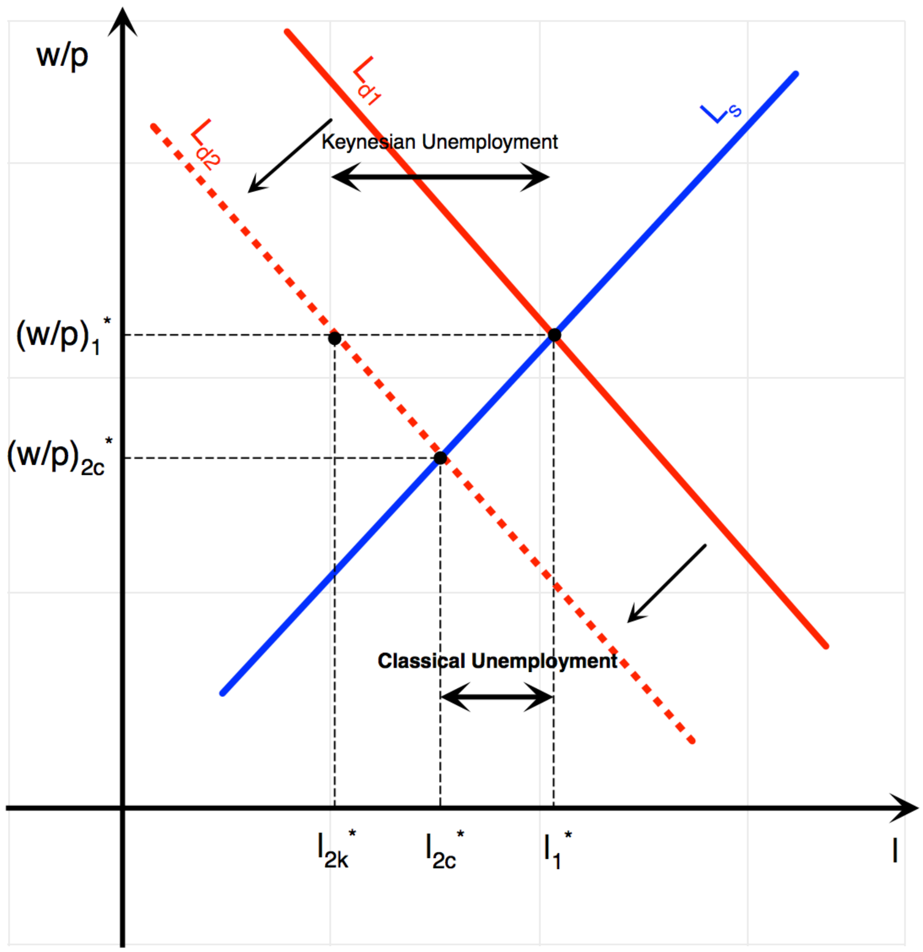

Figure 6.1: Shock to Labor Demand in the Neoclassical and “Keynesian” Models.

Finally, I use quotes for "Keynesian" because although John Maynard Keynes mentioned sticky wages in *The General Theory of Employment, Interest and Money* as a potential cause for unemployment, his thought was much more complex, and he did not see a reduction in real wages as a cure for unemployment. Although rigid wages have become synonymous with Keynes' thought in many textbooks, you should keep in mind that John Maynard Keynes' thought was much more complex than this, and that J.M. Keynes actually was not in favor of a reduction in wages to cure unemployment problems (you can see for yourself directly in The General Theory). If you want to know more, *The Economist* has a great briefing on the natural rate of unemployment - however, we will not investigate these notions any further, so you are not responsible for the content of this article.

6.3 The Bathtub Model

A final view of unemployment consists in a mechanical, almost statistical model of the labor market, called the "bathtub" model of unemployment. This view recognizes that the labor market involves constant churning, which implies that there always are some workers who are in between jobs. It is called the bathtub model of unemployment because in some ways, the unemployment rate is quite similar to the height of the water in a bathtub.

6.3.1 Model

In the bathtub model of unemployment, there is a number L of people in the labor force. Jobs separate at a rate s, often called the **job separation rate**, and the rate at which unemployed people find new jobs is f, the **job finding rate**. For example, assuming that in a typical month, 20% of the unemployed find new jobs, then $f = 20\%$. If, on the other hand, 1% of the employed lose their jobs, then $s = 1\%$. Finally, the number of people unemployed is denoted by U_t and the number of people employed as E_t. The law of motion for the unemployment population is given by:

$$\Delta U_{t+1} = sE_t - fU_t.$$

6.3.2 Solution

Using the fact that people are either employed or unemployed:

$$E_t + U_t = L,$$

and plugging in the law of motion for U_t:

$$\begin{aligned} \Delta U_{t+1} = U_{t+1} - U_t &= s\left(L - U_t\right) - fU_t \\ \Rightarrow \quad U_{t+1} &= (1-s-f)U_t + sL. \end{aligned}$$

In the long run, we have that, the number of unemployed people U^* satisfies the following equation:

$$U_{t+1} = U_t = U^* \quad \Rightarrow \quad U^* = \frac{sL}{s+f}.$$

The corresponding long run unemployment rate u^*, that is the number of unemployed as a function of the total population, is then given by:

$$u^* = \frac{U^*}{L} = \frac{s}{f+s}.$$

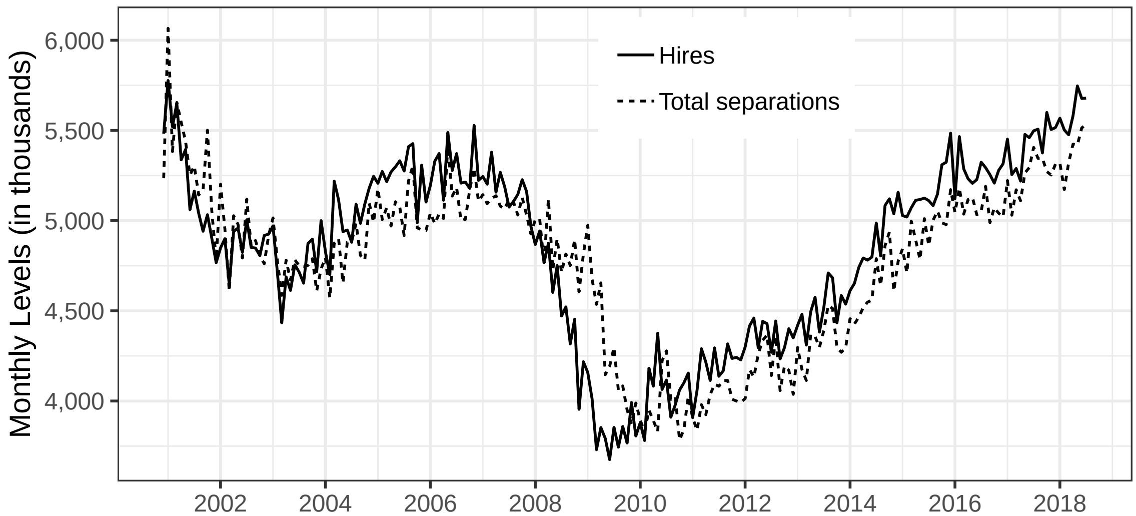

Figure 6.2: Monthly Hires and Separations, in Thousands. Source: BLS-JOLTS.

Numerical Application. Using the above numerical values, a job separation rate of about $s = 1\%$, as well as a job finding rate of about $f = 20\%$, we get a value for long-run unemployment, sometimes called the natural rate of unemployment, equal to:

$$u^* = \frac{0.01}{0.2 + 0.01} \approx 4.8\%$$

.

Transition dynamics. Unlike with the Solow growth model of lecture 2, we can do more and ask how the unemployment rate converges to this steady-state level, using the law of motion for unemployment (this was impossible for the Solow growth model because the law of motion for capital was too complex):

$$\begin{aligned} U_{t+1} &= (1 - s - f)\, U_t + sL \\ &\Rightarrow \quad U_{t+1} - U^* = (1 - s - f)\,(U_t - U^*) \\ &\Rightarrow \quad U_t = U^* + (1 - s - f)^t\,(U_0 - U^*)\,. \end{aligned}$$

6.3.3 Data on Job Churning

Data on job churning is collected by the Bureau of Labor Statistics (the Job Opening and Labor Turnover Survey, also known as JOLTS). You may in particular view the latest data on Job openings, Hires levels, Total separations, Quits, Layoffs and discharges.

The evolution of job openings and hire levels on the one hand, and total separations, quits, layoffs and discharges are plotted on the two figures below.

This data shows that there are many ways in which the bathtub model of unemployment is an oversimplification. First, the separation and job finding rates both fall substantially during recessions. Second, the notion of a "separation" is a bit more subtle in the real world: some separations are chosen from workers (quits), some of them are chosen by firms (layoffs). Third, the job finding rate is also endogenous, because not all job openings lead to hires.

Increased unemployment during recessions is explained by an increase in layoffs, but more importantly by a decline in job openings.

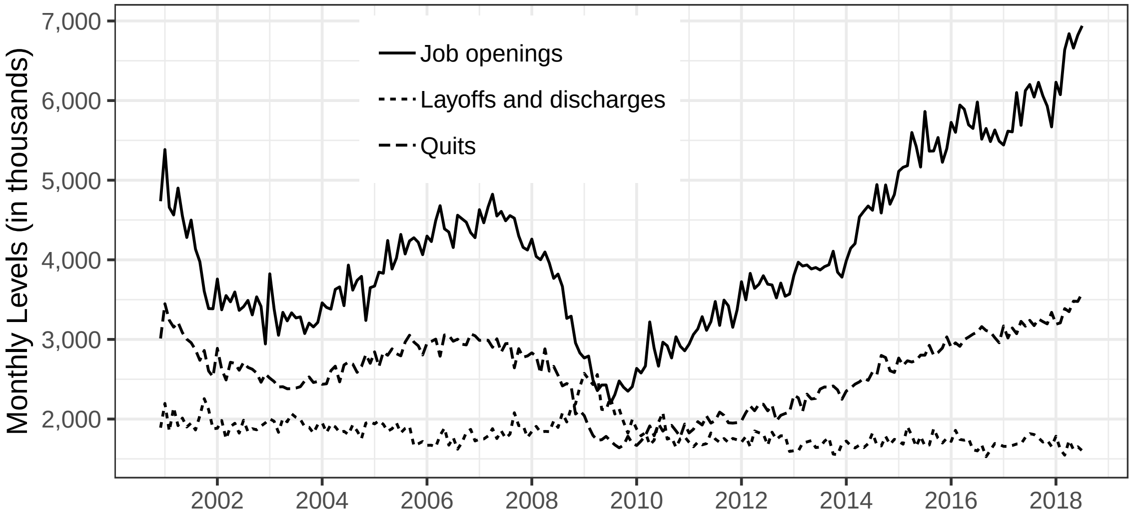

Figure 6.3: Monthly Job Openings, Layoffs and Quits, in Thousands. Source: BLS-JOLTS.

Readings - To go further

Several quotes in search of a theory, *The Economist*, January 17, 2011.

Blanchard Olivier, and Wolfers Justin. "The Role of Shocks and Institutions in the Rise of European Unemployment: The Aggregate Evidence." *The Economic Journal* 110, no. 462 (December 25, 2001): 1–33.

What We Know About the 92 Million Americans Who Aren't in the Labor Force, *Wall Street Journal*, Oct 21, 2015.

(Gated) "Central bankers' holy grail: The Natural rate of unemployment", *The Economist*, August 26, 2017.

Chapter 7

The Consumption Function and the Multiplier

This lecture opens a set of lectures on Keynesian economics. The neoclassical models of consumption, saving, investment, and the labor market that we have studied so far are quite close to what the mainstream paradigm was teaching when John Maynard Keynes started to think about these issues. J.M. Keynes refers to this paradigm as “classical”. Yet, according to him, this paradigm was not a *general* one, in that it did not apply to a case where there was an insufficient use of resources. According to him, the U.S. after the Great Depression was in such a situation, where the teaching of traditional economics was therefore “misleading and disastrous”. The first chapter of the General Theory starts with these words:

> I have called this book the General Theory of Employment, Interest and Money, placing the emphasis on the prefix general. The object of such a title is to contrast the character of my arguments and conclusions with those of the classical theory of the subject, upon which I was brought up and which dominates the economic thought, both practical and theoretical, of the governing and academic classes of this generation, as it has for a hundred years past. I shall argue that the postulates of the classical theory are applicable to a special case only and not to the general case, the situation which it assumes being a limiting point of the possible positions of equilibrium. Moreover, the characteristics of the special case assumed by the classical theory happen not to be those of the economic society in which we actually live, with the result that its teaching is misleading and disastrous if we attempt to apply it to the facts of experience.

According to J.M. Keynes, the type of models we have seen so far are *special* in that they only apply in a case where there is full employment of resources. In contrast, underutilization of resources requires a different approach. In the model that we will study today, consumption does not result from intertemporal optimization, as in Lecture 3 and Lecture 4 but from a mechanical “consumption function” relating consumption to current income. Investment is not be determined by the supply of saving as in the Solow growth model of Lecture 2, but by the demand for investment by firms. Finally, there may exist involuntary unemployment, unlike in the neoclassical model of Lecture 6. Workers are potentially off their labor supply curve: they would be willing to take a job at the current wage. In Keynes’ view of the world, an increase in demand can therefore potentially be met by an increase in supply, because some labor is idle and ready to be used in production.

During this lecture, I first go through a presentation of the simple Keynesian cross, and then present a couple of variations on the simple Keynesian cross: automatic stabilizers, and “accelerator” effects of output on investment. The assumptions and logic of the Keynesian theory may be a little disturbing at first:

> The difficulty lies, not in the new ideas, but in escaping from the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds.

7.1 The Demand for Goods

Consider the first accounting identity of lecture 1, showing the different components of the demand for goods:

$$Y = C + I + G + X - M.$$

The total demand for goods is determined by consumption demand, investment demand, government spending, and net exports (for example, China's total demand includes US' demand for Chinese products). For simplicity, we shall assume for now that we are considering a closed economy - think, for example, of the world economy - so that there are no exports or imports ($X = M = 0$). This assumption, which is not satisfying as the policies we shall consider are national (fiscal policy, etc.), will be relaxed starting in lecture11.

In lecture 2 and lecture 4, output was also determined by the amount of available technology, as well as by the supply of labor L, and of capital K in the economy (the "supply side"):

$$Y = F(K, L).$$

For example, in the Solow growth model of lecture 2, where government spending was zero ($G = 0$), GDP was simultaneously equal to $F(K, L)$ as well as to $C + I$. This is what allowed us to derive the dynamics in the economy.

In the Keynesian model, output is determined only by demand. The underlying assumption is that there exists some idle resources which allow to accommodate any increase in demand by an increase in labor utilization. In order to show that output is determined by demand, we shall denote the total demand for goods by Z:

$$Z = C + I + G.$$

The Keynesian assumption then is that output is demand determined, so that:

$$Y = Z.$$

We now investigate these components of demand in turn.

7.1.1 Consumption Demand

The first (and largest) component of the demand for goods comes from consumption. Rather than starting from microeconomic principles of optimization under constraints, J.M. Keynes postulates a consumption function relating disposable income to consumption. In its most general form, consumption is simply a function of disposable income Y_D, disposable income being the income that remains once consumers have received transfers from the government and paid their taxes:

$$C = C(Y_D), \quad \frac{dC}{dY_D} > 0.$$

The positive derivative implies that when disposable income goes up, people buy more goods; when it goes down, they buy fewer goods. It is often useful to specify a functional form for this consumption function, such as:

$$C(Y_D) = c_0 + c_1 Y_D.$$

In this case, the consumption function has a linear relation. This linear relation has two parameters:

1. c_1 is called the **marginal propensity to consume**. It gives the effect of an additional dollar of income on consumption. For instance, if $c_1 = 0.7$ ($c_1 = 70\%$), then an additional dollar of disposable income increases consumption by 1 dollar times 0.7, or 70 cents. A natural restriction on c_1 is that it is positive: an increase in disposable income should (at least averaging across individuals) lead to an increase in consumption. Another natural restriction on c_1 is that it be less than 1: people are likely to save some of their increase in disposable income. Keynes presents this as follows in Chapter 10 of the General Theory:

Our normal psychological law that, when the real income of the community increases or decreases, its consumption will increase or decrease but not so fast, can, therefore, be translated - not, indeed, with absolute accuracy but subject to qualifications which are obvious and can easily be stated in a formally complete fashion into the propositions that ΔC_w and ΔY_w have the same sign, but $\Delta Y_w > \Delta C_w$, where C_w is the consumption in terms of wage-units. This is merely a repetition of the proposition already established in Chapter 3 above. Let us define, then, dC_w/dY_w as the marginal propensity to consume.

2. c_0 corresponds to what people would consume if their disposable income was equal to 0 (for instance, they would still need to eat). In most instances, they would still be able to eat by running down their saving, or borrowing from banks. In terms of the model, changes in c_0 should be thought of as everything that moves consumption without going through disposable income directly. This might therefore proxy for consumer confidence, the willingness of banks to lend to consumers, etc.

Finally, disposable income is defined as:

$$Y_D \equiv Y - T,$$

where Y is income and T is taxes paid minus government transfers received by consumers. Very often, we will refer to T as "taxes", which will always imply "net taxes", unless otherwise specified.

Finally, note that "income" includes here both labor income as well as capital income, and is therefore equal to GDP. The underlying assumption is that c_1 represents a weighted average of marginal propensity to consume across people with different incomes, and different types of incomes.

7.1.2 Investment Demand

The second component of the demand for goods is coming from investment demand. In the Solow growth model, investment was determined by saving. For the Keynesian model, we shall make two alternative assumptions about investment:

1. In the baseline version of the model, we shall assume that investment is fixed:

$$I = \bar{I}.$$

   In order to remind ourselves that investment is fixed (or "exogenous", that is, determined outside of the model), we shall denote investment by "I bar" $\bar{I}$.

2. In the investment accelerator version of the model, we shall assume in contrast that investment depends on output. This assumption is more realistic in practice: an Italian restaurant which has many customers is more likely to renovate it and to buy a new pizza oven. In this case, we will assume that investment depends on output in a linear way:

$$I = b_0 + b_1 Y.$$

7.1.3 Government Spending

The third and final component of demand in our closed economy model is government spending G. Together with net taxes T representing the tax-and-transfer system, G is a component of fiscal policy. We shall also take G as exogenous, but we will study the impact of alternative values for G for the determination of GDP, for example (when we talk about the multiplier).

7.2 The Simple Goods Market Model

The Simple Goods Market Model, also called the **(ZZ)-(YY) model**, assumes a closed economy, that investment is exogenous and equal to $\bar{I}$, an exogenous government spending G, and a consumption function given as above by:

$$C = c_0 + c_1 (Y - T).$$

7.2.1 Algebraic derivation of the multiplier

(ZZ) curve. With these assumptions, and assuming a closed economy, the value of the demand for goods Z is:

$$\begin{aligned} Z &= C + \bar{I} + G \\ &= c_0 + c_1 (Y - T) + \bar{I} + G \\ Z &= \left(c_0 + \bar{I} + G - c_1 T\right) + c_1 Y \end{aligned}$$

This determines a value for the demand for goods Z, as a function of aggregate income, or GDP Y, which defines the (ZZ) curve. The part of this demand which does not depend on income is called *autonomous spending* z_0 defined as:

$$z_0 = c_0 + \bar{I} + G - c_1 T.$$

This autonomous spending z_0 is also the value of demand when income is equal to $Y = 0$. Therefore, the demand for goods Z is given as:

$$Z = \underbrace{\left(c_0 + \bar{I} + G - c_1 T\right)}_{\text{Autonomous Spending } z_0} + \underbrace{c_1}_{\text{MPC}} Y$$

(YY) curve. Equilibrium in the goods market requires that:

$$Z = Y.$$

Indeed, income is determined by the total demand for goods.

Equilibrium. Equilibrium is determined by the intersection of the (YY) and the (ZZ) curves, represented by point A on the Figure below. Replacing out $Z = Y$ in the expression for Z above indeed implies:

$$Y = \left(c_0 + \bar{I} + G - c_1 T\right) + c_1 Y.$$

Therefore, putting all Y terms of the left-hand side:

$$Y = \underbrace{\frac{1}{1 - c_1}}_{\text{Multiplier}} \times \underbrace{\left(c_0 + \bar{I} + G - c_1 T\right)}_{\text{Autonomous Spending } z_0}.$$

Multiplier. Consider a change in autonomous spending $\Delta z_0 = z_0' - z_0$ coming from a change in government spending $\Delta z_0 = \Delta G$, or from a change in net taxes $\Delta z_0 = -c_1 \Delta T$. We have:

$$\Delta Y = \frac{\Delta z_0}{1 - c_1}.$$

By definition, the multiplier gives the increase in income which is brought about by the increase in autonomous spending. Therefore, the multiplier is given by:

$$\text{Multiplier} = \frac{1}{1 - c_1}.$$

As a consequence steepness of the (ZZ) curve determines the value for the multiplier. The closer to one the slope c_1 is, the higher the value of the Keynesian multiplier.

Government expenditure multiplier; tax multiplier. An increase in government spending $\Delta G > 0$ leads to an increase in output given by:

$$\Delta Y = \frac{\Delta G}{1 - c_1}.$$

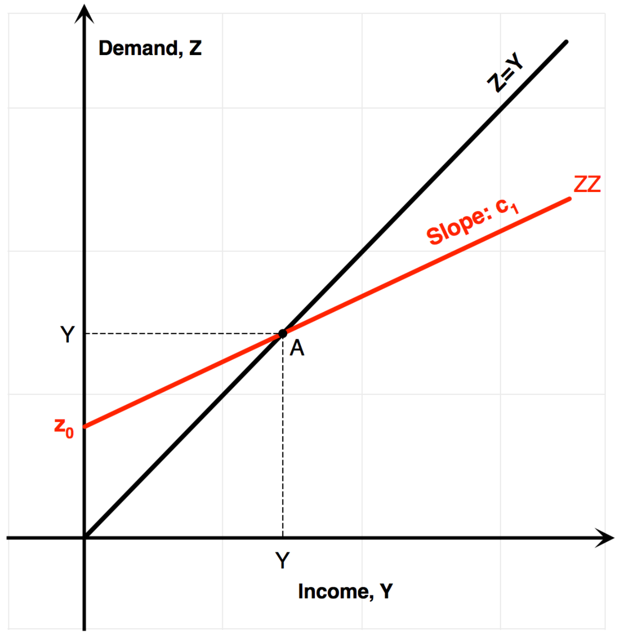

Figure 7.1: THE SIMPLE GOODS MARKET (ZZ)-(YY) MODEL.

Thus, $1/(1-c_1)$ is usually called the *government expenditure multiplier*.

On the other hand, a decrease in net taxes $\Delta T < 0$ leads to an increase in output given by:

$$\Delta Y = -\frac{c_1 \Delta T}{1-c_1}.$$

For this reason, $c_1/(1-c_1)$, or $-c_1/(1-c_1)$ is sometimes called the tax multiplier.

7.2.2 Four interpretations

Why is the change in output higher than the change in autonomous spending? And why is it equal to $1/(1-c_1)$, where c_1 is the marginal propensity to consume. There are actually four different ways to see this, 2 are algebraic, and 2 are geometric:

1. **Algebra.** The first way to see this is just, as above to equate output to demand $Y = Z$, which allows to get at the result.
2. **Infinite sum of a geometric series.** 1$ of additional autonomous spending brings in a second round c_1\$ increase in consumption, a third round c_1^2\$ increase in consumption, which add up to:

$$\sum_{i=0}^{n} c_1^i = 1 + c_1 + c_1^2 + ... + c_1^n = \frac{1-c_1^{n+1}}{1-c_1}.$$

Therefore, because $0 < c_1 < 1$, we have:

$$\text{Multiplier} = \sum_{i=0}^{+\infty} c_1^i = \frac{1}{1-c_1}.$$

3. **Graphical interpretation 1.** The left-hand panel of Figure 7.2 gives a graphical interpretation to

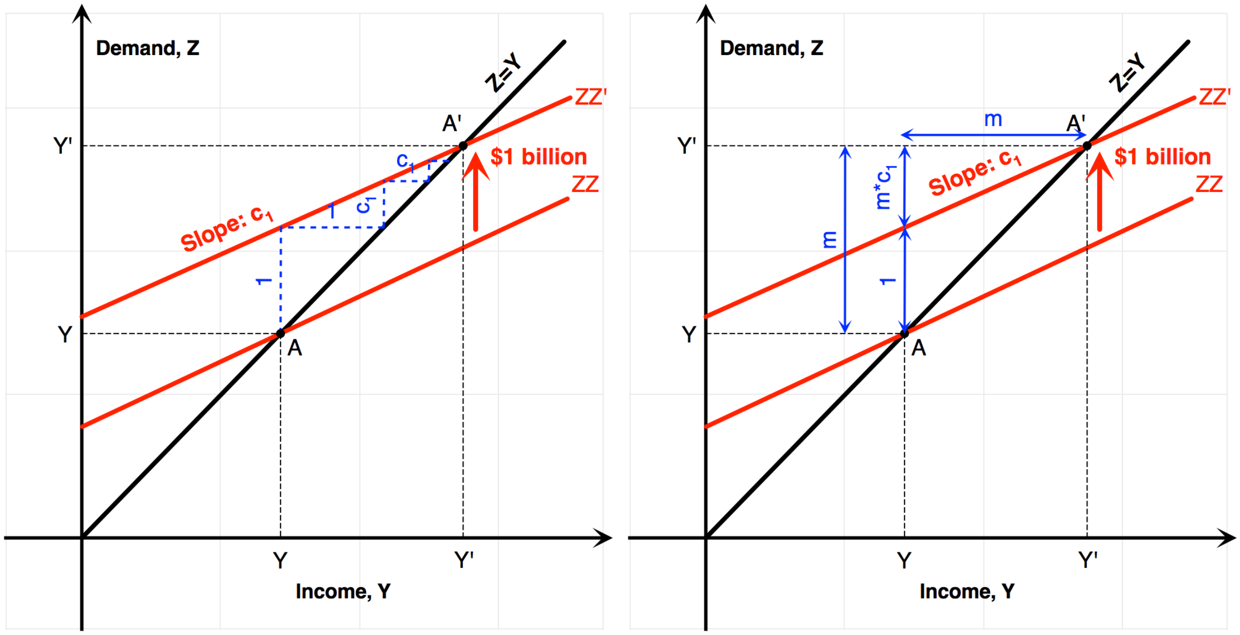

Figure 7.2: Simple Keynesian Cross: Graphical Interpretations.

this infinite geometric sum. This graphical interpretation makes it clear that:

$$Y' - Y = \sum_{i=0}^{+\infty} c_1^i = \frac{1}{1-c_1}.$$

This graph should make clear why the model is called a "Keynesian cross". This is because the (ZZ) and (YY) curves cross to determine output. This pedagogical device to visualize the Keynesian multiplier was introduced in 1948 by Paul Samuelson in his famous *Economics* textbook, to provide a geometric intuitive interpretation of Keynes' ideas.

4. **Graphical interpretation 2.** The right-hand panel of Figure 7.2 gives a graphical interpretation which does not use a geometric sum. If m is the unknown value of the multiplier, then the geometry makes clear that m has to satisfy $m = 1 + mc_1$ which also gives the value for the multiplier:

$$m = 1 + mc_1 \quad \Rightarrow \quad m = \frac{1}{1-c_1}.$$

7.3 Extensions of the Goods Market Model

There are multiple extensions of the Goods Market Model: one with automatic stabilizers, one with an accelerator effect of demand on investment.

7.3.1 Automatic Stabilizers

In practice, the tax and transfer system is designed in such a way that taxes strongly depend on the level of income: net taxes are said to be **procyclical**:

1. Many taxes, such taxes on wage income, profits, capital gains, mechanically collect more in revenues when income rises: taxes are procyclical (they vary positively with the state of the business cycle).

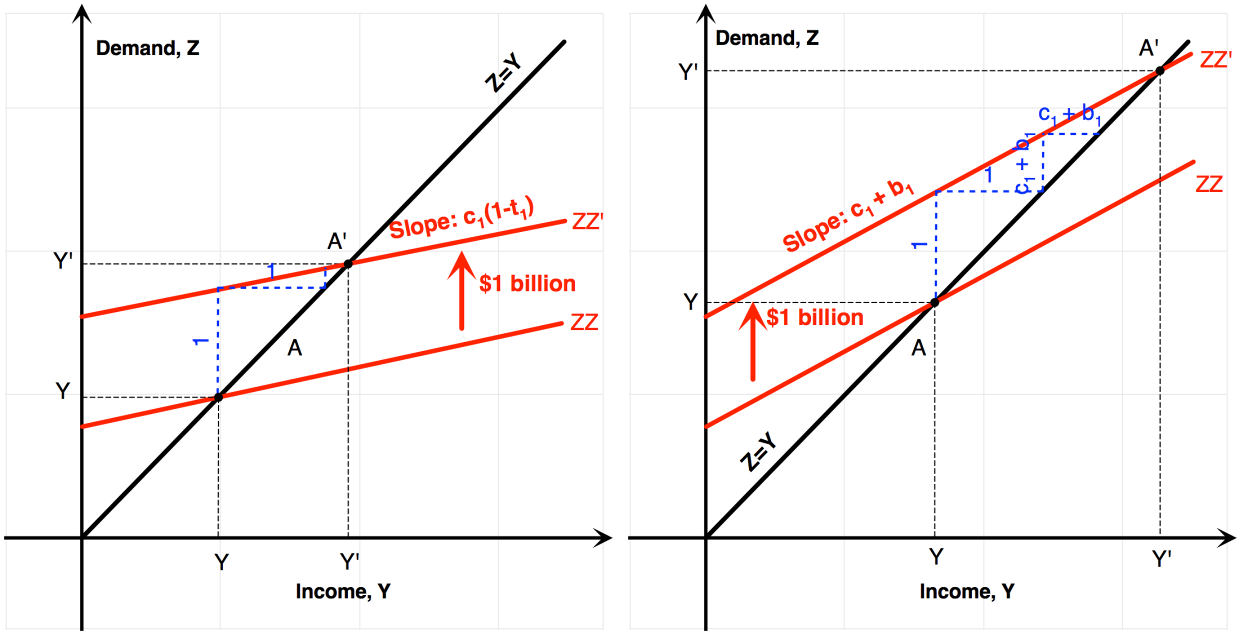

Figure 7.3: Variations: Automatic Stabilizers, Accelerator Effect.

2. On the contrary, many transfers are countercyclical (they vary negatively with the state of the business cycle): when unemployment is high and income is low, more unemployment benefits or food stamps are being paid.

Overall, these two effects reinforce themselves so that net taxes are procyclical. In terms of the model, we model this by postulating a linear function for net taxes which depends on the value for income:

$$T = t_0 + t_1 Y,$$

where $t_1 \in [0, 1]$ indexes the cyclicality of net taxes with GDP.

For example, assume that $t_1 = 0.3$ or 30%. Then if income rises by 1 dollar, net taxes rise by 30 cents. Using this expression for taxes allows us to write:

$$\begin{aligned} Z &= C + \bar{I} + G \\ &= c_0 + c_1 \left(Y - t_0 - t_1 Y\right) + \bar{I} + G \\ Z &= \left(c_0 - c_1 t_0 + \bar{I} + G\right) + \left(\left(1 - t_1\right) c_1\right) Y \end{aligned}$$

Using that $Y = Z$ we see that the multiplier is now $1/(1 - c_1(1 - t_1))$:

$$Y = \frac{1}{1 - c_1 \left(1 - t_1\right)} \left(c_0 - c_1 t_0 + \bar{I} + G\right)$$

The (ZZ) curve is less steep, and therefore the multiplier $Y' - Y$ is smaller.

7.3.2 "Accelerator" Effect

Another variation on the Goods Market model consists in recognizing that investment demand might also depend on income. Firms will invest more if they expect high income, and therefore high demand for their goods:

$$I = b_0 + b_1 Y.$$

Thus, the (ZZ)-curve now has a slope equal to $c_1 + b_1$:

$$\begin{aligned} Z &= C + I + G \\ &= c_0 + c_1 (Y - T) + b_0 + b_1 Y + G \\ Z &= (c_0 - c_1 T + b_0 + G) + (c_1 + b_1) Y. \end{aligned}$$

Using $Z = Y$, we can conclude that the multiplier is $1/(1 - c_1 - b_1)$ if $c_1 + b_1 < 1$:

$$Y = \frac{1}{1 - (c_1 + b_1)} (c_0 - c_1 T + b_0 + G)$$

In this case, the (ZZ) curve is steeper, and thus the multiplier $Y' - Y$ is larger. Note that if $c_1 + b_1 \geq 1$, then the multiplier is infinite. This, of course, shows the limits of the Keynesian model: it remains valid as long as output is constrained by aggregate demand. In this case, output will increase to a level such as it becomes constrained by supply, as in lecture 2.

Required Readings

Mankiw, N. Gregory. What would Keynes have done? *New York Times*, November 30, 2008

Readings - To go further

Keynes, John Maynard. The Economic Consequences of the Peace, 1919.

Keynes, John Maynard. Chapter 10 “The Marginal Propensity to Consume and the Multiplier”, The General Theory of Employment, Interest, and Money, 1936.

Robert Barro. Keynes Is Still Dead. *Wall Street Journal*. October 29, 1992.

Bruce Bartlett. Keynes Was Really A Conservative. *Forbes*. August 14, 2009.

What Is Keynesian Economics?, *Finance & Development*, September 2014, Sarwat Jahan, Ahmed Saber Mahmud, and Chris Papageorgiou.

Chapter 8

The Paradox of Thrift

The idea that thrift is always virtuous is very deeply ingrained in our culture. It is a matter of philosophy, morals, and sometimes even religion. For example, in the Walt Disney movie Mary Poppins, Michael is being lectured by a banker that he should not be "feeding the birds" but instead invest his tuppence "wisely in the bank" to "be part of railways through Africa; Dams across the Nile, fleets of ocean Greyhounds; Majestic, self-amortizing canals; Plantations of ripening tea" (interestingly, these capital investments are all abroad; we shall come back to this later).

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# [1] "Sorry I don't know how to embed videos in PDF: https://www.youtube.com/watch?v=XxyB29bDbBA"
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However, the model of lecture 7 implies that saving might be detrimental to the economy, at least when the economy has some slack. This phenomenon was explained by J.M. Keynes in the *General Theory*:

> For although the amount of his own saving is unlikely to have any significant influence on his own income, the reactions of the amount of his consumption on the incomes of others makes it impossible for all individuals simultaneously to save any given sums. Every such attempt to save more by reducing consumption will so affect incomes that the attempt necessarily defeats itself. It is, of course, just as impossible for the community as a whole to save less than the amount of current investment, since the attempt to do so will necessarily raise incomes to a level at which the sums which individuals choose to save add up to a figure exactly equal to the amount of investment.

In this lecture, we use the models of lecture 7 to understand that argument better. From the outset, we should note that the paradox of thrift was known before J.M. Keynes, perhaps in the Book of Proverbs:

> There is that scattereth, and yet increaseth; and there is that withholdeth more than is meet, but it tendeth to poverty. (Proverbs 11:24)

More certainly, it was present as early as in Bernard Mandeville's *The Fable of the Bees: or, Private Vices, Public Benefits* (1714):

> As this prudent economy, which some people call Saving, is in private families the most certain method to increase an estate, so some imagine that, whether a country be barren or fruitful, the same method if generally pursued (which they think practicable) will have the same effect upon a whole nation, and that, for example, the English might be much richer than they are, if they would be as frugal as some of their neighbours. This, I think, is an error.

This idea was also stated by Thomas Malthus:

> Adam Smith has stated that capitals are increased by parsimony, that every frugal man is a public benefactor, and that the increase of wealth depends upon the balance of produce above consumption. That these propositions are true to a great extent is perfectly unquestionable… But it is quite obvious that they are not true to an indefinite extent, and that the principles of

> saving, pushed to excess, would destroy the motive to production. If every person were satisfied with the simplest food, the poorest clothing, and the meanest houses, it is certain that no other sort of food, clothing, and lodging would be in existence.
>
> While it is quite certain that an adequate passion for consumption may fully keep up the proper proportion between supply and demand, whatever may be the powers of production, it appears to be quite as certain that a passion for accumulation must inevitably lead to a supply of commodities beyond what the structure and habits of such a society will permit to be consumed.

Thomas Malthus really was a forerunner of J.M. Keynes. He set himself out to explain why unemployment could occur, as well as to suggest steps which might be taken to eliminate it. He was inspired by events surrounding the post-Napoleonic wars period, during which time industrial depression in Britain was causing serious unemployment of labor and capital.

That this part of his thinking was not that original was in fact recognized by J.M. Keynes in Chapter 23 of the General Theory - Notes on Mercantilism, the usury laws, stamped money and theories of under-consumption, which I strongly encourage you to read (although you are not responsible for it). We will come back to it when we talk about open economy macroeconomics, starting in lecture 11

Back to our Keynesian goods market model of lecture 7, we shall now study the paradox of thrift in more detail. We will consider three ways in which an economy might save more, which are all relevant in practice:

1. an increase in the desire to save through a fall in autonomous consumption $\Delta c_0 < 0$.
2. an increase in public saving, also called **deficit reduction**, through a decrease in government spending $\Delta G < 0$.
3. an increase in public saving, also called **deficit reduction**, through an increase in taxes or a decrease in transfers $\Delta T > 0$.

We will show that these acts of saving have very similar detrimental effect on output, and therefore saving. We shall consider two models in which such a paradox of thrift arises:

1. in the simple goods market model of lecture 7, we will show that attempts to save more, either by private individuals or by the government, are self-defeating, in the sense that saving does not move.
2. in the variation on the goods market model with an accelerator effect of investment where $I = b_0 + b_1 Y$, we will even show that attemps to save more are more self-defeating: they lead to lower saving in the aggregate. The "paradox of thrift" will appear very clearly: attempts to save more lead to less saving.

8.1 Simple goods market model

Let us start from the simple goods market model:

$$\begin{aligned} C &= c_0 + c_1 Y_D \\ Y_D &= Y - T, \end{aligned}$$

where c_1 is the marginal propensity to consume out of disposable income Y_D, disposable income is income minus taxes, and investment $I = \bar{I}$ and government spending G are taken as given, as well as taxes T. There are several ways to show the "paradox of thrift" in this model. We investigate an increase in the desire to save, modeled as a reduction in c_0 $\Delta c_0 < 0$, as well as two attempts at deficit reduction in turn.

8.1.1 $\Delta c_0 < 0$

There are two ways to see that a fall in the desire to consume $\Delta c_0 < 0$, or equivalently an increase in the desire to save, may lead to an equal level of aggregate saving. One way is the most straightforward, but does not give much intuition for the result. The second way is more complex, but gives a very nice economic intuition.

Direct proof. The first way to see the paradox of thrift is simply to notice that investment is fixed and equal to $\bar{I}$ by assumption. Since total saving equals private saving S plus public saving $T-G$, we can express private saving as a function of only fixed variables:

$$\bar{I} = S + (T - G) \quad \Rightarrow \quad S = \bar{I} - (T - G)\,.$$

This proves the result ! In particular, even if there is a change in consumption of $\Delta c_0 < 0$, then private saving does has to be equal to $\bar{I} - (T - G)$ always and thus cannot move. However, this proof is a little bit disappointing and does not provide much intuition.

Intuitive Proof. For the more intuitive proof, we need to write the equations from the goods market model as well as equate output to demand $Y = Z$:

$$\begin{aligned} Y = Z &= C + I + G \\ Y &= c_0 + c_1(Y - T) + \bar{I} + G \end{aligned}$$

This leads to equilibrium output:

$$Y = \frac{1}{1 - c_1}\left(c_0 - c_1 T + \bar{I} + G\right).$$

This is the usual multiplier: for a given change in Δc_0, the change in output is given by the direct effect on output, but also by all the successive new rounds, which add up to $\frac{\Delta c_0}{1-c_1}$ in total. Thus, such a change in Δc_0 leads to a change in output of:

$$\Delta Y = \frac{\Delta c_0}{1 - c_1}.$$

Private saving is given by disposable income $Y - T$ minus consumption (what is earned, not paid in taxes, nor consumed, is saved), and therefore:

$$\begin{aligned} S &= Y - T - C \\ &= Y - T - c_0 - c_1\,(Y - T) \\ S &= -c_0 + (1 - c_1)\,(Y - T)\,. \end{aligned}$$

What happens when people attempt to save more, say by lowering c_0? A change in consumption of $\Delta c_0 < 0$ clearly leads to:

- on the one hand, a *direct effect* on private saving that is given by $-\Delta c_0 > 0$ (private saving rises).
- on the other hand, an *indirect effect* going through the change in output whose magnitude was calculated above given by $\Delta\left[(1 - c_1)\,(Y - T)\right]$.

In other words, we may write:

$$\Delta S = \underbrace{\Delta(-c_0)}_{\text{direct effect}} + \underbrace{\Delta\left[(1 - c_1)\,(Y - T)\right]}_{\text{indirect effect}}.$$

Now, how large is the indirect effect? Some algebra allows to conclude that it is exactly the opposite of the direct effect:

$$\begin{aligned} \Delta\left[(1 - c_1)\,(Y - T)\right] &= (1 - c_1)\Delta Y \\ &= (1 - c_1)\frac{\Delta c_0}{1 - c_1} \\ \Delta\left[(1 - c_1)\,(Y - T)\right] &= \Delta c_0. \end{aligned}$$

Therefore, the total effect on saving is:

$$\begin{aligned}\Delta S &= \Delta(-c_0) + \Delta\left[(1-c_1)(Y-T)\right] \\ &= -\Delta c_0 + \Delta c_0 \\ \Delta S &= 0\end{aligned}$$

8.1.2 Fall in spending $\Delta G < 0$

Again, there exists both a straightforward proof which does not explain much, and a proof providing more economic intuition. We start with the direct proof.

Direct Proof. The direct proof simply uses the investment equals saving identity:

$$\bar{I} = S + (T - G)$$

This implies that a fall in expenditure $\Delta G < 0$, and resulting increase in public saving must be matched by a fall in private saving. However, this proof is again, somewhat disappointing.

Intuitive Proof. Denote the fall in government spending by $\Delta G < 0$. This leads to a rise in public saving:

$$\Delta(T - G) = -\Delta G > 0.$$

However, this fall also leads to a fall in output, whose magnitude is given by the government spending multiplier. Indeed, we know that:

$$Y = \frac{1}{1-c_1}\left(c_0 - c_1 T + \bar{I} + G\right),$$

which implies that the fall in output is:

$$\Delta Y = \frac{\Delta G}{1-c_1}.$$

Private saving is given by disposable income $Y - T$ minus consumption (what is earned, not paid in taxes, nor consumed, is saved), and therefore:

$$\begin{aligned}S &= Y - T - C \\ &= Y - T - c_0 - c_1(Y - T) \\ S &= -c_0 + (1-c_1)(Y-T).\end{aligned}$$

Therefore:

$$\Delta S = (1-c_1)\Delta Y = \Delta G.$$

Thus, we have a fall in private saving whose magnitude is exactly matching the rise in public saving. Overall, the effect on total saving, and therefore investment is zero in this model:

$$\Delta I = \Delta S + \Delta(T - G) = 0.$$

8.1.3 Increase in net taxes $\Delta T > 0$

An increase in net taxes $\Delta T > 0$ can come both from an increase in taxes or a reduction in transfers. Again, there are two proofs. The direct proof is exactly the same as the one with $\Delta G < 0$, so we do not go over it. On the other hand, the intuitive proof is a bit different because taxes impact disposable income too.

Intuitive proof. If the government chooses to engage in deficit reduction through tax increases (or by reducing transfers), then denoting by $\Delta T > 0$ the increase in aggregate taxes, we have a rise in public saving given by: $\Delta(T - G) = \Delta T > 0$.

Again, this leads to a fall in private saving through two channels: a direct channel which goes through the mechanic reduction in disposable income, and a second channel which goes through the reduction in output, which lowers income. Again, the magnitude of the second channel can be computed using the above equation for output:

$$\begin{aligned} Y &= \frac{1}{1-c_1}\left(c_0 - c_1 T + \bar{I} + G\right) \\ \Rightarrow \quad \Delta Y &= -\frac{c_1}{1-c_1}\Delta T \end{aligned}$$

Again, given the above expression for private saving:

$$S = -c_0 + (1-c_1)(Y-T).$$

we have:

$$\begin{aligned} \Delta S &= (1-c_1)(\Delta Y - \Delta T) \\ &= (1-c_1)\left(-\frac{c_1}{1-c_1}\Delta T\right) - (1-c_1)\Delta T \\ \Delta S &= \underbrace{-c_1 \Delta T}_{\text{Effect through output}} - \underbrace{(1-c_1)\Delta T}_{\text{Reduction in disposable income}} \end{aligned}$$

Therefore:

$$\Delta S = -\Delta T.$$

Thus, we have a fall in private saving whose magnitude is exactly equal to the rise in public saving. Overall, the effect on total saving, and therefore investment is:

$$\Delta I = \Delta S + \Delta(T-G) = 0.$$

8.2 Extended goods market model

In the consumption and investment multiplier model, we get an even stronger paradox of thrift in that efforts by consumers to save more lead to declining saving. We thus start from the extended goods market model of lecture 7:

$$\begin{aligned} C &= c_0 + c_1(Y-T) \\ I &= b_0 + b_1 Y. \end{aligned}$$

Again, we investigate a fall in private saving first, and then two attempts at deficit reduction.

8.2.1 $\Delta c_0 < 0$

Direct proof. Again, let us write the investment = total saving identity:

$$I = S + (T-G) \quad \Rightarrow \quad S = I - (T-G).$$

We know that a fall in consumption of $\Delta c_0 < 0$ leads to a decline in output $\Delta Y < 0$, and therefore through the equation giving investment as a function of output, to a decline in investment since $\Delta I = b_1 \Delta Y$:

$$I = b_0 + b_1 Y \quad \Rightarrow \quad \Delta I = b_1 \Delta Y.$$

Because T and G are assumed to be fixed (so that public saving is fixed), the change in private saving is equal to the change in investment, and is therefore negative. Therefore, a fall in consumption, leads to a fall in private saving ! Again, that proof is probably not very intuitive. We now turn to the longer proof.

Intuitive proof. Again, we write that output equals demand, which allows to get an expression for output:

$$Y = \frac{1}{1 - c_1 - b_1}\left(c_0 + b_0 - c_1 T + G\right)$$

We have the usual multiplier, compounding the consumption and investment effects (it is assumed here that $c_1 + b_1 < 1$). Therefore, a given change in $\Delta c_0 < 0$ leads to decline in output of:

$$\Delta Y = \frac{\Delta c_0}{1 - c_1 - b_1}.$$

Again, one can show that $S = -c_0 + (1 - c_1)(Y - T)$ – see the previous section:

$$\Delta S = \underbrace{\Delta(-c_0)}_{\text{direct effect}} + \underbrace{\Delta\left[(1 - c_1)(Y - T)\right]}_{\text{indirect effect}}$$

However, this time, the two effects do not exactly cancel out as the indirect effect is:

$$\begin{aligned}
\Delta\left[(1 - c_1)(Y - T)\right] &= (1 - c_1)\Delta Y \\
&= (1 - c_1)\frac{\Delta c_0}{1 - c_1 - b_1} \\
\Delta\left[(1 - c_1)(Y - T)\right] &= \frac{1 - c_1}{1 - c_1 - b_1}\Delta c_0.
\end{aligned}$$

Therefore, the total effect on saving is negative:

$$\begin{aligned}
\Delta S &= \Delta(-c_0) + \Delta\left[(1 - c_1)(Y - T)\right] \\
&= -\Delta c_0 + \frac{1 - c_1}{1 - c_1 - b_1}\Delta c_0 \\
\Delta S &= \frac{b_1}{1 - c_1 - b_1}\Delta c_0 < 0
\end{aligned}$$

8.2.2 Fall in spending $\Delta G < 0$

Direct proof. We know that a rise in public saving, arising from either a decrease in government spending, or a rise in taxes, or a decrease in transfers, leads to a decline in output $\Delta Y < 0$, and therefore through the above equation giving investment as a function of output, to a decline in investment since $\Delta I = b_1 \Delta Y$:

$$I = b_0 + b_1 Y \quad \Rightarrow \quad \Delta I = b_1 \Delta Y < 0.$$

Therefore, a deficit reduction is clearly bad for investment. However, once again, this calculation does not really help understand what the above reasoning was wrong.

Intuitive proof. The fall in government spending $\Delta G < 0$ leads to a rise in public saving:

$$\Delta(T - G) = -\Delta G > 0.$$

However, this fall also leads to a fall in output, whose magnitude is given by the government spending multiplier. We write that output equals demand, to get an expression for output to get, once again, that:

$$Y = \frac{1}{1 - c_1 - b_1}\left(c_0 + b_0 - c_1 T + G\right)$$

So the fall in output is:

$$\Delta Y = \frac{\Delta G}{1 - c_1 - b_1}.$$

Once again, private saving is given by disposable income $Y - T$ minus consumption (what is earned, not paid in taxes, nor consumed, is saved), and therefore (see the previous sections):

$$S = -c_0 + (1 - c_1)(Y - T).$$

Therefore:

$$\Delta S = (1 - c_1)\Delta Y = \frac{1 - c_1}{1 - c_1 - b_1}\Delta G.$$

Thus, we have a fall in private saving whose magnitude is larger than the rise in public saving. Overall, the effect on total saving, and therefore investment is negative:

$$\begin{aligned}
\Delta I &= \Delta S + \Delta(T - G) \\
&= \frac{1 - c_1}{1 - c_1 - b_1}\Delta G - \Delta G \\
&= \frac{1 - c_1}{1 - c_1 - b_1}\Delta G - \frac{1 - c_1 - b_1}{1 - c_1 - b_1}\Delta G \\
&= \frac{1 - c_1 - (1 - c_1 - b_1)}{1 - c_1 - b_1}\Delta G \\
\Delta I &= \frac{b_1}{1 - c_1 - b_1}\Delta G < 0.
\end{aligned}$$

8.2.3 Increase in net taxes $\Delta T > 0$

The direct proof is exactly similar at the one in the previous section. However, the intuitive proof is a bit difference.

Intuitive proof. If the government chooses to engage in deficit reduction through tax increases (or by reducing transfers), then denoting by $\Delta T > 0$ the increase in aggregate taxes, we have a rise in public saving given by:

$$\Delta(T - G) = \Delta T > 0.$$

Again, this leads to a fall in private saving through two channels: a direct channel which goes through the mechanic reduction in disposable income, and a second channel which goes through the reduction in output, which lowers income. Again, the magnitude of the second channel can be computed using the above equation for output:

$$\Delta Y = -\frac{c_1}{1 - c_1 - b_1}\Delta T.$$

Again, given the above expression for private saving:

$$S = -c_0 + (1 - c_1)(Y - T).$$

we have:

$$\begin{aligned}
\Delta S &= (1 - c_1)(\Delta Y - \Delta T) \\
&= (1 - c_1)\left(-\frac{c_1}{1 - c_1 - b_1}\Delta T\right) - (1 - c_1)\Delta T \\
\Delta S &= \underbrace{-\frac{c_1(1 - c_1)}{1 - c_1 - b_1}\Delta T}_{\text{Effect through output}} - \underbrace{(1 - c_1)\Delta T}_{\text{Reduction in disposable income}}
\end{aligned}$$

Therefore:

$$\Delta S = -\frac{1 - b_1 - c_1 + b_1 c_1}{1 - c_1 - b_1}\Delta T.$$

Overall, the effect on total saving, and therefore investment is decreasing:

$$\Delta I = \Delta S + \Delta(T - G) = -\frac{b_1 c_1}{1 - c_1 - b_1}\Delta T < 0.$$

Readings - To go further

O'Leary, James J. "Malthus and Keynes." Journal of Political Economy 50, no. 6 (1942): 901–19.

Wikipedia, Paradox of thrift.

(Gated) Paul Krugman, The Paradox of Thrift – for real, *New York Times*, July 7, 2008.

Paul Krugman, When Consumers Capitulate, *New York Times*, October 31, 2008.

Robert J. Barro, Government Spending Is No Free Lunch, *Wall Street Journal*, January 22, 2009.

Paul Krugman, A Dark Age of Macroeconomics (wonkish), *New York Times*, Jan 27, 2009

Paul Krugman, Paradox of Thrift, *New York Times Blog*, February 3, 2009

Robert J. Barro, Stimulus Spending Keeps Failing; If austerity is so terrible, how come Germany and Sweden have done so well?, *Wall Street Journal*, May 9, 2012.

Paul Krugman, Greece As Victim, *New York Times*, June 18, 2012.

Chapter 9

Redistributive Policies

Keynesian economics provides a mechanism through which more redistribution might actually increase output overall, at the same time as it reduces inequality. The idea that the economy suffers from a shortage of aggregate demand coming from increases in inequality has been put forward recently by mainstream academics such as Raghuram Rajan, former chief economist of the IMF, and now governor at the Bank of England, as well as by Robert Reich, US Secretary of Labor from 1993 to 1997.

The idea that the MPC is influenced by the distribution of income and wealth comes back to J.M. Keynes in the *General Theory*:

> Since the end of the nineteenth century significant progress towards the removal of very great disparities of wealth and income has been achieved through the instrument of direct taxation—income tax and surtax and death duties—especially in Great Britain. Many people would wish to see this process carried much further, but they are deterred by two considerations; partly by the fear of making skilful evasions too much worth while and also of diminishing unduly the motive towards risk-taking, but mainly, I think, by the belief that the growth of capital depends upon the strength of the motive towards individual saving and that for a large proportion of this growth we are dependent on the savings of the rich out of their superfluity. Our argument does not affect the first of these considerations. But it may considerably modify our attitude towards the second. For we have seen that, up to the point where full employment prevails, the growth of capital depends not at all on a low propensity to consume but is, on the contrary, held back by it; and only in conditions of full employment is a low propensity to consume conducive to the growth of capital. Moreover, experience suggests that in existing conditions saving by institutions and through sinking funds is more than adequate, and that measures for the redistribution of incomes in a way likely to raise the propensity to consume may prove positively favourable to the growth of capital.

This passage from J.M. Keynes in the *General Theory* is intuitive: as long as saving propensities are no longer an impediment to capital accumulation, redistributing income or wealth from low to high **Marginal Propensity to Consume (MPC)** should lead to higher output. According to J.M. Keynes, this is in fact one reason for restricting the increase in inequality:

> The State will have to exercise a guiding influence on the propensity to consume partly through its scheme of taxation. (…) Whilst, therefore, the enlargement of the functions of government, involved in the task of adjusting to one another the propensity to consume and the inducement to invest, would seem to a nineteenth-century publicist or to a contemporary American financier to be a terrific encroachment on individualism, I defend it, on the contrary, both as the only practicable means of avoiding the destruction of existing economic forms in their entirety and as the condition of the successful functioning of individual initiative.

During this lecture, we derive this result using the Keynesian model that was developed in lecture 7 and lecture 8. One appeal of writing the equations is that we are not able to prove these assertions qualitatively, but we are also able to understand how important they are quantitatively. As we go along, we therefore attempt to put some actual numbers on all these arguments, to get a sense of the orders of magnitude. We shall investigate two types of policies:

- Income redistribution, from high to low income earners.
- Deficit-financed decreases in taxes, on high income earners or low income earners, financed by public debt.

9.1 Assumptions

Some minor modifications to the goods market model underlying lecture 7 and lecture 8 are in order, in order to think about stimulus policies in the presence of inequality. Instead of assuming one type of consumer, with the average income Y and a given MPC c_1, we shall assume two types of workers. In total, there are N workers:

- There is a fraction λ of low income earners, who earn income $\underline{y}$, pay net taxes $\underline{t}$, and the MPC of the low income earners is $\underline{c}_1$:
$$\underline{c} = \underline{c}_0 + \underline{c}_1(\underline{y} - \underline{t}).$$
- There is a fraction $1 - \lambda$ of high income earners, they get a higher income $\bar{y}$ which is a multiple γ of low income earners' income, given by:
$$\bar{y} = \gamma \underline{y},$$
where γ indexes inequality. They each pay net taxes $\bar{t}$, and the MPC of the high income earners is $\bar{c}_1$:
$$\bar{c} = \bar{c}_0 + \bar{c}_1(\bar{y} - \bar{t})$$
Moreover, we assume that high income earners have a lower MPC than low income earners, so that:
$$\bar{c}_1 < \underline{c}_1$$

We assume that investment depends on output:

$$I = b_0 + b_1 Y$$

We also assume that taxes depend on output, both for low income earners:

$$\underline{t} = \underline{t}_0 + t_1 \underline{y}$$

as well as for high income earners:

$$\bar{t} = \bar{t}_0 + t_1 \bar{y}.$$

9.2 Solving the model

9.2.1 Income

Since there is a fraction λ of low income earners, and the total population is N, the total income $\underline{Y}$ captured by low income earners $\underline{Y}$ is:

$$\underline{Y} = \lambda N \underline{y}$$

Symmetrically, the total income $\bar{Y}$ captured by high income earners is:

$$\bar{Y} = (1 - \lambda) N \bar{y}.$$

Total income is given by the sum of $\underline{Y}$, and $\bar{Y}$, which allows to express low income earners as a function of total income:

$$\begin{aligned} Y &= \underline{Y} + \bar{Y} \\ &= \lambda N \underline{y} + (1-\lambda) N \bar{y} \\ &= \lambda N \underline{y} + (1-\lambda) N \gamma \underline{y} \\ Y &= (\lambda + (1-\lambda)\gamma) N \underline{y} \end{aligned}$$

This implies that low income earners' $\underline{y}$ is given as a function of output per person Y/N and the parameters of the model by:

$$\underline{y} = \frac{1}{\lambda + (1-\lambda)\gamma} \frac{Y}{N}$$

As a consequence, high income' individual $\bar{y}$ is given as a function of output per person Y/N and the parameters of the model by:

$$\bar{y} = \gamma \underline{y} = \frac{\gamma}{\lambda + (1-\lambda)\gamma} \frac{Y}{N}.$$

The share of income captured by low income earners is:

$$\begin{aligned} \frac{\underline{Y}}{Y} &= \frac{\lambda N \underline{y}}{(\lambda + (1-\lambda)\gamma) N \underline{y}} \\ \frac{\underline{Y}}{Y} &= \frac{\lambda}{\lambda + (1-\lambda)\gamma}. \end{aligned}$$

The share of income captured by high income earners is:

$$\begin{aligned} \frac{\bar{Y}}{Y} &= \frac{(1-\lambda) N \bar{y}}{(\lambda + (1-\lambda)\gamma) N \underline{y}} \\ &= \frac{(1-\lambda)\gamma N \underline{y}}{(\lambda + (1-\lambda)\gamma) N \underline{y}} \\ \frac{\bar{Y}}{Y} &= \frac{(1-\lambda)\gamma}{\lambda + (1-\lambda)\gamma}. \end{aligned}$$

Numerical Application: Approximate the number of workers in the US to about 150 million:

$$N = 150,000,000$$

and that GDP is 20 trillion. Therefore, GDP per worker on average is:

$$\frac{Y}{N} = \$133,333.33$$

Let us divide the population in two groups, the top 10% income share, and the bottom 90% income share, so that: $\lambda = 0.9$. Since the top 10% get approximately 50% of the income in the U.S. (the exact data is available here), this implies, using the formula above $\underline{Y}/Y = \lambda / (\lambda + (1-\lambda)\gamma)$:

$$\frac{0.9}{0.9 + 0.1 \cdot \gamma} = 0.5 \quad \Rightarrow \quad \gamma = 9.$$

This is actually very intuitive: if 90% of the population have the same income as 10% of the population (half of total income), then on average they are 9 times poorer. The average income for someone in the top 10% is then:

$$\bar{y} = \gamma \underline{y} = \frac{\gamma}{\lambda + (1-\lambda)\gamma} \frac{Y}{N} = \$666,666.66$$

They are:

$$(1-\lambda) N = 15,000,000.$$

While the average income for someone in the bottom 90% is:

$$\underline{y} = \frac{1}{\lambda + (1-\lambda)\gamma} \frac{Y}{N} = \$74,074.07.$$

They are:

$$\lambda N = 135,000,000.$$

9.2.2 Taxes

Aggregate taxes T are the sum of taxes paid by the low income earners $\underline{T}$ and the high income earners $\bar{T}$:

$$\begin{aligned}
T &= \underline{T} + \bar{T} \\
&= \lambda N \underline{t} + (1-\lambda) N \bar{t} \\
&= \lambda N \left(\underline{t}_0 + t_1 \underline{y}\right) + (1-\lambda) N \left(\bar{t}_0 + t_1 \bar{y}\right) \\
&= \left(\lambda N \underline{t}_0 + (1-\lambda) N \bar{t}_0\right) + t_1 \left(\lambda N \underline{y} + (1-\lambda) N \bar{y}\right) \\
T &= \left(\underline{T}_0 + \bar{T}_0\right) + t_1 Y
\end{aligned}$$

The aggregate baseline level of taxes T_0 is:

$$T_0 \equiv \underline{T}_0 + \bar{T}_0 = \lambda N \underline{t}_0 + (1-\lambda) N \bar{t}_0,$$

where baseline level of taxes for low and high income earners is given by:

$$\underline{T}_0 \equiv \lambda N \underline{t}_0, \qquad \bar{T}_0 \equiv (1-\lambda) N \bar{t}_0.$$

To conclude, total aggregate taxes are:

$$\boxed{T = \left(\underline{T}_0 + \bar{T}_0\right) + t_1 Y}.$$

9.2.3 Consumption

The challenging part, which differs from the models seen in lecture 7 and lecture 8, is to calculate aggregate consumption, which is composed both of the consumption by low income earners, and that by high income earners. Total consumption by the low income earners $\underline{C}$ is such that:

$$\begin{aligned}
\underline{C} &= \lambda N \underline{c} \\
&= \lambda N \left(\underline{c}_0 + \underline{c}_1 (\underline{y} - \underline{t})\right) \\
&= \lambda N \underline{c}_0 + \lambda N (1-t_1) \underline{c}_1 \underline{y} - \lambda N \underline{c}_1 \underline{t}_0 \\
\underline{C} &= [\lambda N \underline{c}_0 - \lambda N \underline{c}_1 \underline{t}_0] + \frac{\lambda \underline{c}_1}{\lambda + (1-\lambda)\gamma} (1-t_1) Y
\end{aligned}$$

Symmetrically, consumption by the high income earners $\bar{C}$ is such that:

$$\begin{aligned}
\bar{C} &= (1-\lambda) N \bar{c} \\
&= (1-\lambda) N \left(\bar{c}_0 + \bar{c}_1 (\bar{y} - \bar{t})\right) \\
&= (1-\lambda) N \bar{c}_0 + (1-\lambda) N (1-t_1) \bar{c}_1 \bar{y} - (1-\lambda) N \bar{c}_1 \bar{t}_0 \\
\bar{C} &= \left[(1-\lambda) N \bar{c}_0 - (1-\lambda) N \bar{c}_1 \bar{t}_0\right] + \frac{(1-\lambda)\gamma \bar{c}_1}{\lambda + (1-\lambda)\gamma} (1-t_1) Y
\end{aligned}$$

Therefore, aggregate consumption $C = \underline{C} + \bar{C}$ is given by:

$$\begin{aligned}
C &= \underline{C} + \bar{C} \\
&= [\lambda N \underline{c}_0 - \lambda N \underline{c}_1 \underline{t}_0] + \frac{\lambda \underline{c}_1}{\lambda + (1-\lambda)\gamma}(1-t_1)Y + \left[(1-\lambda)N\bar{c}_0 - (1-\lambda)N\bar{c}_1\bar{t}_0\right] + \frac{(1-\lambda)\gamma\bar{c}_1}{\lambda+(1-\lambda)\gamma}(1-t_1)Y \\
&= (\lambda N \underline{c}_0 + (1-\lambda)N\bar{c}_0) - \left(\lambda N \underline{c}_1 \underline{t}_0 + (1-\lambda)N\bar{c}_1\bar{t}_0\right) + \frac{\lambda \underline{c}_1 + (1-\lambda)\,\gamma\bar{c}_1}{\lambda+(1-\lambda)\gamma}(1-t_1)Y \\
C &= C_0 - \left(\underline{c}_1 \underline{T}_0 + \bar{c}_1 \bar{T}_0\right) + c_1(1-t_1)Y.
\end{aligned}$$

where we have defined the average MPC c_1 by:

$$c_1 \equiv \frac{\lambda \underline{c}_1 + (1-\lambda)\,\gamma\bar{c}_1}{\lambda+(1-\lambda)\gamma}.$$

the aggregate baseline level of consumption C_0 as:

$$C_0 \equiv \underline{C}_0 + \bar{C}_0 = \lambda N \underline{c}_0 + (1-\lambda)N\bar{c}_0,$$

where the baseline level of consumption for low and high income earners is given by:

$$\underline{C}_0 \equiv \lambda N \underline{c}_0, \quad \bar{C}_0 \equiv (1-\lambda)N\bar{c}_0.$$

To conclude, aggregate consumption is:

$$\boxed{C = C_0 - \left(\underline{c}_1 \underline{T}_0 + \bar{c}_1 \bar{T}_0\right) + c_1(1-t_1)Y}.$$

9.3 Aggregate Demand

We want to compute aggregate demand:

$$Z = C + I + G.$$

We know that I has a very straightforward expression:

$$I = b_0 + b_1 Y.$$

Using the expression for aggregate consumption C as well and plugging in into total demand yields:

$$\begin{aligned}
Z &= C + I + G \\
&= C_0 - \left(\underline{c}_1 \underline{T}_0 + \bar{c}_1 \bar{T}_0\right) + c_1(1-t_1)Y + b_0 + b_1 Y + G \\
Z &= \left[C_0 - \left(\underline{c}_1 \underline{T}_0 + \bar{c}_1 \bar{T}_0\right) + b_0 + G\right] + \left(c_1(1-t_1) + b_1\right) Y
\end{aligned}$$

Equating output to demand $Z = Y$ gives the value for output:

$$\boxed{Y = \frac{1}{1-(1-t_1)\,c_1 - b_1}\left[C_0 - \underline{c}_1 \underline{T}_0 - \bar{c}_1 \bar{T}_0 + b_0 + G\right]}$$

9.4 Different Fiscal Policies

9.4.1 Redistribution from high income to low income earners

Assume that transfers to the low income earners are increased (or taxes decreased), so that $\Delta \underline{T}_0 < 0$, with an offsetting increase in taxes on high income earners such that $\Delta T_0 = \Delta \underline{T}_0 + \Delta \bar{T}_0 = 0$. We then have that $\Delta \bar{T}_0 = -\Delta \underline{T}_0 > 0$. This leads to a change in autonomous spending:

$$\Delta z_0 = -\underline{c}_1 \Delta \underline{T}_0 - \bar{c}_1 \Delta \bar{T}_0 \quad \Rightarrow \quad \Delta z_0 = \left(\underline{c}_1 - \bar{c}_1\right) \Delta \bar{T}_0 > 0.$$

That impulse leads to an increase in output given by:

$$\Delta Y = \frac{\underline{c}_1 - \bar{c}_1}{1 - (1 - t_1)\, c_1 - b_1} \Delta \bar{T}_0 > 0.$$

Using the value for aggregate taxes:

$$\begin{aligned} T &= \left(\underline{T}_0 + \bar{T}_0\right) + t_1 Y \\ \Rightarrow \quad \Delta T &= \underbrace{\Delta \underline{T}_0 + \Delta \bar{T}_0}_{\Delta T_0 = 0} + t_1 \Delta Y. \end{aligned}$$

Finally:

$$\Delta T = \frac{t_1 \left(\underline{c}_1 - \bar{c}_1\right)}{1 - (1 - t_1)\, c_1 - b_1} \Delta \bar{T}_0.$$

Thus, public saving increase, there is a reduction in the deficit, in public debt, and therefore:

$$\Delta \left(T - G\right) = \frac{t_1 \left(\underline{c}_1 - \bar{c}_1\right)}{1 - (1 - t_1)\, c_1 - b_1} \Delta \bar{T}_0$$

Numerical Application: On top of the above numerical values, we assume that the marginal tax rate is 25% so that $t_1 = 1/4$. Therefore:

$$\underline{c}_1 = 1, \quad \bar{c}_1 = 1/3, \quad \gamma = 9, \quad \lambda = 0.9, \quad b_1 = 1/6, \quad t_1 = 1/4.$$

As shown above, this implies an average MPC given by $c_1 = 2/3$. Thus, a tax cut to low income earners financed by tax increases to high income earners leads to an increase in output given by the following multiplier:

$$\begin{aligned} \frac{\underline{c}_1 - \bar{c}_1}{1 - (1 - t_1) c_1 - b_1} &= \frac{1 - 1/3}{1 - (1 - 1/4) * 2/3 - 1/6} \\ &= \frac{2/3}{1 - 1/2 - 1/6} \\ \frac{\underline{c}_1 - \bar{c}_1}{1 - (1 - t_1) c_1 - b_1} &= 2 \end{aligned}$$

This implies that if \$1 billion is transfered from high to low income earners, GDP rises by \$2 billion. Importantly, this does not necessarily imply it should be done: first, high income earners are clearly worse off. Second, this calculation based on high income earners' lower MPC does not take into account that they may be discouraged to create jobs and become entrepreneurs if they are taxed too much.

Because output increases, we get an improvement in the budget surplus as well, given by:

$$\begin{aligned} \Delta \left(T - G\right) &= t_1 * \frac{\left(\underline{c}_1 - \bar{c}_1\right)}{1 - (1 - t_1)\, c_1 - b_1} * \Delta \bar{T}_0 \\ &= \frac{1}{4} * 2 * 1 \\ \Delta \left(T - G\right) &= \frac{1}{2} \end{aligned}$$

or, 500 million dollar improvement.

Therefore, **\$1 billion dollar transfer from high to low income earners** leads to an improvement in the public **surplus** of **\$0.5 billion dollars**.

9.4.2 Deficit-financed tax cuts for high income earners

Assume tax cuts for high income earners $\Delta\bar{T}_0 < 0$, then output increases:

$$\Delta Y = -\frac{\bar{c}_1}{1-(1-t_1)\,c_1-b_1}\Delta\bar{T}_0 > 0$$

The impact on aggregate taxes is however ambiguous:

$$\begin{aligned}\Delta T &= \Delta\bar{T}_0 + \underbrace{\Delta\underline{T}_0}_{=0} + t_1\Delta Y\\ \Delta T &= \left(1-\frac{t_1\bar{c}_1}{1-(1-t_1)\,c_1-b_1}\right)\Delta\bar{T}_0\end{aligned}$$

Therefore, the impact on public saving is similarly ambiguous:

$$\Delta\left(T-G\right) = \left(1-\frac{t_1\bar{c}_1}{1-(1-t_1)\,c_1-b_1}\right)\Delta\bar{T}_0$$

There is an increase in output and, depending on parameters, there can be a government surplus or a government deficit.

Numerical Application: We assume:

$$\underline{c}_1 = 1, \quad \bar{c}_1 = 1/3, \quad \gamma = 9, \quad \lambda = 0.9, \quad b_1 = 1/6, \quad t_1 = 1/4.$$

This implies $c_1 = 2/3$. Thus, we may calculate the tax multiplier for tax cuts to high income earners, which is given by:

$$\begin{aligned}\frac{\bar{c}_1}{1-(1-t_1)c_1-b_1} &= \frac{1/3}{1-(1-1/4)*2/3-1/6}\\ &= \frac{1/3}{1-1/2-1/6}\\ \frac{\bar{c}_1}{1-(1-t_1)c_1-b_1} &= 1\end{aligned}$$

Therefore, the impact on public saving is given by:

$$\begin{aligned}\Delta\left(T-G\right) &= \left(1-\frac{t_1\bar{c}_1}{1-(1-t_1)\,c_1-b_1}\right)\Delta\bar{T}_0\\ &= \left(1-t_1*\frac{\bar{c}_1}{1-(1-t_1)\,c_1-b_1}\right)\Delta\bar{T}_0\\ &= \left(1-\frac{1}{4}*1\right)*(-1)\\ \Delta\left(T-G\right) &= -\frac{3}{4}\end{aligned}$$

Therefore, **\$1 billion dollar tax cut on high income earners** leads to an increase in the public **deficit** of **\$0.75 billion dollars**.

9.4.3 Deficit-financed tax cuts for low income earners

Assume tax cuts for low income earners $\Delta\underline{T}_0 < 0$, then output increases:

$$\Delta Y = -\frac{\underline{c}_1}{1-(1-t_1)\,c_1-b_1}\Delta\underline{T}_0 > 0.$$

The impact on aggregate taxes is however ambiguous:

$$\Delta T = \Delta \underline{T}_0 + \underbrace{\Delta \bar{T}_0}_{=0} + t_1 \Delta Y$$

$$\Delta T = \left(1 - \frac{t_1 \underline{c}_1}{1 - (1 - t_1)\, c_1 - b_1}\right) \Delta \underline{T}_0$$

The impact on public saving is similarly ambiguous:

$$\Delta\left(T - G\right) = \left(1 - \frac{t_1 \underline{c}_1}{1 - (1 - t_1)\, c_1 - b_1}\right) \Delta \underline{T}_0$$

There is an increase in output and, depending on parameters, there can be a government surplus or a government deficit.

Numerical Application: We assume:

$$\underline{c}_1 = 1, \quad \bar{c}_1 = 1/3, \quad \gamma = 9, \quad \lambda = 0.9, \quad b_1 = 1/6, \quad t_1 = 1/4.$$

This implies $c_1 = 2/3$. Thus, we may calculate the tax multiplier for tax cuts to low income earners, which is given by:

$$\begin{aligned}
\frac{\underline{c}_1}{1 - (1 - t_1) c_1 - b_1} &= \frac{1}{1 - (1 - 1/4) * 2/3 - 1/6} \\
&= \frac{1}{1 - 1/2 - 1/6} \\
\frac{\underline{c}_1}{1 - (1 - t_1) c_1 - b_1} &= 3
\end{aligned}$$

Therefore, the impact on public saving is given by:

$$\begin{aligned}
\Delta\left(T - G\right) &= \left(1 - \frac{t_1 \underline{c}_1}{1 - (1 - t_1)\, c_1 - b_1}\right) \Delta \underline{T}_0 \\
&= \left(1 - t_1 * \frac{\underline{c}_1}{1 - (1 - t_1)\, c_1 - b_1}\right) \Delta \underline{T}_0 \\
&= \left(1 - \frac{1}{4} * 3\right) * (-1) \\
\Delta\left(T - G\right) &= -\frac{1}{4}
\end{aligned}$$

Therefore, **\$1 billion dollar tax cut on low income earners** leads to an increase in the public **deficit** of **\$0.25 billion dollars**.

Readings - To go further

"Secular Stagnation, Coalmines, Bubbles, and Larry Summers", Paul Krugman, *New York Times* Blog Post, November 16, 2013.

"The Economic Hokum of 'Secular Stagnation' ", John B. Taylor, *Wall Street Journal*, January 2, 2014.

"The Age of Secular Stagnation", Larry Summers, *Foreign Affairs*, February 15, 2016.

"Inequality Is Slowing US Economic Growth: Faster Wage Growth for Low- and Middle-Wage Workers Is the Solution", Josh Bivens, Economic Policy Institute, December 12, 2017.

Chapter 10

Public Debt, Say's Law

Until now, we have been talking about government spending and taxes as if the government could take on as much debt as it wants. But then, why doesn't the government just engage in more tax cutting and more government spending, or both? We alluded to a first reason when we talked about the consequences of having $1 - c_1 < b_1$, or the propensity to save be less than the propensity to invest. We argued then that we would never be in a Keynesian situation of deficient aggregate demand, so that multiplier effects would stop when facing constraints on supply. Similarly, if the government started to make public saving very negative (running a budget deficit), then it would similarly start facing constraints on what the economy can supply. For instance, if fiscal policy was too accommodative, and to the limit if G was set at too high a value, then supply constraints would start to bite: one example was given historically in the 1940s when the U.S. engaged in World War II. However, these levels of spending are clearly out of the question, and this is perhaps not what constrains the government from doing a little bit more spending, or a little bit more tax reductions.

Another potentially more pressing issue is that of the government deficit, and the impact of government debt on future generations. The Trump tax cuts which have just been enacted have reduced unemployment to historically low levels, and pushed GDP growth up to a level which has not been seen in a long time, as predicted by the Keynesian model; however, it also has raised U.S. public debt, and is being criticized mostly on these grounds. This makes sense: when government spending increases $\Delta G > 0$, this leads to a government deficit of equal magnitude: $\Delta(T - G) = -\Delta G < 0$. Similarly, a tax cut $\Delta T < 0$ leads to increased deficits given by $\Delta(T - G) = \Delta T < 0$ [1]. One might worry that this debt will someday have to be repaid, and that the current generation is simply putting a burden on future generations. In this case, higher GDP today might only be thought of as leading to lower GDP in the future, when aggregate demand will be diminished.

During this lecture, we make three related points concerning government deficits and government debt:

1. We show first, without using any economic model, that simple accounting suggests that public debt is on a sustainable path whenever the real interest rate on public debt is lower than the rate of growth of GDP ($r < g$), a situation called "dynamic inefficiency" for reasons that will become clear later. (from problem set 4 you may already remember that the Golden Rule level of capital accumulation corresponded to $r = g$). I shall argue that real interest rates appear to be below the rate of growth of GDP, at least for now, so there does not seem to be cause for alarm - at least, until interest rates don't rise more.

2. Second, I illustrate using an economic model that it is not true that public debt necessarily will need to be repaid eventually, so that government debt is not necessarily a burden on future generations - an

[1] Note however that automatic stabilizers go against that: in problem set 6, we even saw that tax cuts sometimes can pay for themselves. However, this happens only if the multiplier is really very high. For example, if tax cuts benefit agents with high marginal propensities to consume.

argument which is often made in the public debate. In the overlapping generations model of lecture 4, and provided that capital accumulation is above the Golden Rule level ($r < g$), so that there is **dynamic inefficiency**, public debt is never repaid, as there are always new generations coming along, who buy government debt when they are young and sell it to the next generation when old. This is sustainable if $r < g$, for reasons laid out in part one.

3. Third and last, we shall discuss the effects of larger government deficits on the economy, and constrast the Keynesian and Neoclassical views on this issue. In particular, Keynesian and Neoclassical economists have very different predictions for the impact of higher public deficits on investment spending. You may already have understood that by contrasting lectures 2 and 4 with lectures 7, 8 and 9. We discuss this and related issues surrounding the so-called Treasury View and Say's law in the last section of this lecture.

10.1 Sustainability of Public Debt

10.1.1 Law of motion for Public Debt

In this lecture, we denote everything in terms of goods, to avoid thinking about the complicated issues surrounding inflation. Let us denote by G_t the government spending at period t, and by T_t the taxes in period t. Let us also denote by $(G_t - T_t)$ the government (primary) deficit in period t, which is the excess of government expenditures over taxes levied by the government (thus, when $G_t - T_t > 0$, there is a deficit in the budget, so that the government must borrow). If the interest rate that the government pays is given by r_t, then the law of motion of government debt is given by:

$$B_t = (1 + r_t)B_{t-1} + G_t - T_t$$

Thefore, the law of motion for government debt is given by the sum of the **primary deficit** and **interest payments** on the debt.

The **total** government deficit, which is equal to the change in government debt ΔB_t, is equal to the sum of interest payments and the primary deficit $G_t - T_t$.

$$\text{Deficit}_t = \Delta B_t = B_t - B_{t-1} = \underbrace{r_t B_{t-1}}_{\text{Interest Payments}} + \underbrace{G_t - T_t}_{\text{Primary Deficit}}$$

From the above equation, the evolution of the debt to GDP ratio B_t/Y_t:

$$\frac{B_t}{Y_t} = (1 + r_t)\frac{Y_{t-1}}{Y_t}\frac{B_{t-1}}{Y_{t-1}} + \frac{G_t - T_t}{Y_t}$$

Let us denote the debt to GDP ratio by b_t:

$$b_t \equiv \frac{B_t}{Y_t}.$$

Therefore:

$$b_t = (1 + r_t)\frac{Y_{t-1}}{Y_t}b_{t-1} + \frac{G_t - T_t}{Y_t}.$$

Assuming that GDP grows at rate g_Y, we have that:

$$\frac{Y_t}{Y_{t-1}} = 1 + g_Y$$

Therefore:

$$\boxed{b_t = \frac{1 + r_t}{1 + g_Y}b_{t-1} + \frac{G_t - T_t}{Y_t}}.$$

10.1.2 Condition for Sustainability

A thought experiment is useful to think about the sustainability of public debt in this environment. Imagine that all future primary surpluses were equal to zero after $t = t_0$, that is:

$$\text{for all } t \geq t_0, \quad G_t = T_t$$

and that real interest rates are constant after $t \geq t_0$:

$$r_t = r.$$

We then have that:

$$\text{for all } t \geq t_0, \quad b_t = \frac{1+r}{1+g_Y} b_{t-1}$$

Then the debt to GDP ratio would be given by:

$$\boxed{\text{for all } t \geq t_0, \quad b_t = \left(\frac{1+r}{1+g_Y}\right)^{t-t_0} b_{t_0}}$$

There are three possible cases:

1. If $r < g_Y$ - a situation called dynamic inefficiency [2] - the debt to GDP ratio goes to 0. (Indeed, when $a < 1$, $a^t \to 0$ when $t \to +\infty$.) Therefore, the **debt to GDP ratio goes to zero**.
2. If $r = g_Y$, the debt to GDP ratio stays constant. Then, the **debt to GDP ratio stays constant**.
3. If $r > g_Y$, the debt to GDP ratio goes to infinity. Indeed, when $a > 1$, $a^t \to +\infty$ when $t \to +\infty$. Then, the **debt to GDP ratio goes to infinity**.

10.1.3 Is public debt sustainable in the U.S.?

Which of these three cases is relevant for the U.S. economy? Is public debt sustainable in the U.S.? How do the real interest rate r and the growth rate of GDP g_Y compare? Up until now, I would argue that it's fair to say that $r < g_Y$.

The real interest rate r can be measured in two ways:

1. Either using the nominal interest rate, and substracting an average expected (or realized) inflation rate in order to get to a real interest rate. Figure 10.1 shows that the nominal interest rate has averaged around 2 to 3% recently, while inflation has been from 1 to 2% on average. This implies a real interest rate which is around 1%, perhaps 2%.
2. Or, one can measure the real interest rate by using the rate on **Treasury Inflation Protected Securities (TIPS)**. Figure 10.2 from FRED (the Federal Reserve Economic Data) shows that the real interest rate has recently been around 1% per year.

On the other hand, real GDP growth seems to be hovering around $g_Y \approx 2.5\%$. Real GDP per capita growth is variable, but it is usually estimated to be **around 1.5%**: $g_{Y/L} \approx 1.5\%$. It varies over time, however: it was around 3.0% per year on average in the 1960s, 2.1% in the 1970s, 2.4% in the 1980s, 2.2% in the 1990s,

[2] It might seem a bit contradictory that a situation where the debt to GDP ratio goes to 0 automatically is called inefficient - this seems like a rather positive state of affairs. However, we shall see in the next chapter through the overlapping-generations model, that "dynamic inefficiency" means here that the government should in fact be taking on even *more* government debt, to restore an equality between r and g_Y. You may also remember from Exercise 1 of problem set 4 - the solution is posted here - that an interest rate lower than the rate of growth implies that we are below the Golden Rule interest rate, so that the capital stock is too high, and consumption is too low.

Figure 10.1: 10-Year Treasury Rate (Source: FRED).

Figure 10.2: 10-Year Treasury Inflation Protected Securities Rate (Source: FRED).

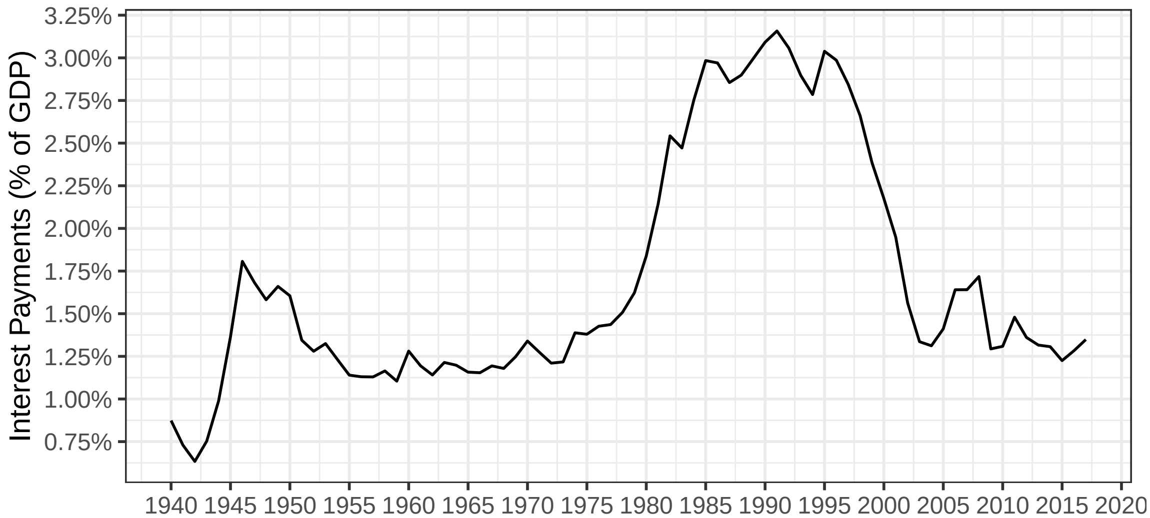

Figure 10.3: Interest Payments on Government Debt as % of GDP (Source: FRED).

0.7% in the 2000s, and 0.9% from 2010 to 2017. On the other hand, the growth rate of population is **around 1%**: $g_L \approx 1.5\%$. Therefore:

$$\begin{aligned} g_Y &= g_{Y/L} + g_L \\ &\approx 1.5\% + 1\% \\ g_Y &\approx 2.5\%. \end{aligned}$$

Therefore, the ratio of government debt to GDP does not appear to be on an unsustainable path so far.

A final way to see that U.S. debt is not yet on an unsustainable path is to note that the ratio of interest payments to GDP is not particularly high historically, which is shown on Figure 10.3. This implies that if the primary deficit was reduced to zero, the debt to GDP ratio would not be on an explosive trajectory.

10.2 Public Debt in the Overlapping Generations Model

In this section, I illustrate using the overlapping-generations model of lecture 4 that public debt does not necessarily need to be repaid eventually, so that government debt is not necessarily a burden on future generations - an argument which we nonetheless often hear in the public debate. However, one precondition for this is naturally that the debt to GDP has to be stable. In other words, we need that $r^* \leq g_Y$. In the overlapping-generations model of lecture 4, we had $g_Y = 0$, since there was no long-run growth. So we want $r^* \leq 0$. In order to have a role for public debt, we will look at the model that we studied in Problem Set 3 called "Another Overlapping Generations Model" - the solution to this problem set was available here.

10.2.1 Overlapping Generations Model

Let us look at a simplified version of the overlapping generations model we looked at in Lecture 4. For this model, we shall assume that people only care about old age consumption, and that they work only when young, receiving wage w_t. It does not really matter what the form of their utility function is with respect to old age consumption, because they will save everything anyway:

$$U = u(c_{t+1}^o).$$

Denoting by r_t the (net) real interest rate, their intertemporal budget constraint is then given by:

$$c_t^y + \frac{c_{t+1}^o}{1 + r_t} = w_t.$$

In this very simple environment, and because consumption in young age will always optimally be set to zero ($c_t^y = 0$), this implies:

$$c_{t+1}^o = (1 + r_t)w_t.$$

Similarly to the previous time, we assume that the labor force is fixed to unity ($L_t = \bar{L} = 1$). The production function is Cobb-Douglas:

$$Y_t = K_t^\alpha L_t^{1-\alpha}.$$

Together with the previous assumption of constant labor $L_t = 1$, this implies that:

$$Y_t = K_t^\alpha.$$

From firms' optimality conditions, the wage is just the marginal product of labor:

$$w_t = \frac{\partial Y_t}{\partial L_t} = (1 - \alpha)K_t^\alpha L_t^{-\alpha} = (1 - \alpha)K_t^\alpha.$$

Similarly as previously, we also get through firms' optimization on the amount of capital that:

$$r_t + \delta = \frac{\partial Y_t}{\partial K_t} = \alpha K_t^{\alpha-1}.$$

Finally, for simplicity, we shall sometimes assume that the depreciation rate is equal to $\delta = 1 = 100\%$. In other words, capital fully depreciates each period - this is reasonable if you take one unit of time to represent one generation, or about 30 years - remember that the depreciation rate for one year was ranging from 5% to 10%.

10.2.2 Without Government Debt

Let us first remind ourselves what happens in the absence of government debt in this model, as in lecture 4. In the absence of a government, we get even simpler expressions than the previous time. The law of motion for capital is given as follows:

$$\Delta K_{t+1} = w_t - \delta K_t.$$

Since w_t is a fraction $1 - \alpha$ of output, this law of motion corresponds to the Solow growth model with $s = 1 - \alpha$. The law of motion for capital is:

$$K_{t+1} = (1 - \alpha)K_t^\alpha + (1 - \delta)K_t.$$

This is a difference equation for sequence K_t which converges to a steady state value for the capital stock K^* such that:

$$\delta K^* = (1 - \alpha)(K^*)^\alpha$$
$$\Rightarrow \quad K^* = \frac{(1 - \alpha)^{\frac{1}{1-\alpha}}}{\delta^{\frac{1}{1-\alpha}}}.$$

The steady state value for the interest rate r^* is then such that:

$$\begin{aligned} r^* + \delta &= \alpha(K^*)^{\alpha-1} \\ &= \alpha \left[\left(\frac{1-\alpha}{\delta} \right)^{\frac{1}{1-\alpha}} \right]^{\alpha-1} \\ r^* + \delta &= \frac{\delta\alpha}{1-\alpha} \end{aligned}$$

Therefore, the steady-state value of the interest rate r^*:

$$r^* = \frac{2\alpha - 1}{1 - \alpha}\delta.$$

which, note, is negative for $\alpha < 1/2$. The steady state value for output Y^* is then:

$$Y^* = (K^*)^{\alpha}$$
$$Y^* = \frac{(1-\alpha)^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}.$$

The value for the wage w^* is:

$$\begin{aligned} w^* &= (1-\alpha)(K^*)^{\alpha} \\ &= (1-\alpha)\left(\frac{1-\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \\ w^* &= \frac{(1-\alpha)^{\frac{1}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} \end{aligned}$$

Steady-state consumption of the old $(c^o)^*$ is thus given by:

$$\begin{aligned} (c^o)^* &= (1+r^*)w^* \\ (c^o)^* &= \left(1+\frac{2\alpha-1}{1-\alpha}\delta\right)(1-\alpha)^{\frac{1}{1-\alpha}} \end{aligned}$$

Full depreciation ($\delta = 1$). Since one period here is one generation, a useful assumption is that capital fully depreciates in one period, so that $\delta = 1$. Then, the previous expressions are considerably more simple to work with:

$$\begin{aligned} K^* &= \frac{(1-\alpha)^{\frac{1}{1-\alpha}}}{\delta^{\frac{1}{1-\alpha}}} = (1-\alpha)^{\frac{1}{1-\alpha}} \\ r^* &= \frac{2\alpha-1}{1-\alpha}\delta = \frac{2\alpha-1}{1-\alpha} \\ Y^* &= \frac{(1-\alpha)^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} = (1-\alpha)^{\frac{\alpha}{1-\alpha}} \\ w^* &= \frac{(1-\alpha)^{\frac{1}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} = (1-\alpha)^{\frac{1}{1-\alpha}} \\ (c^o)^* &= \left(1+\frac{2\alpha-1}{1-\alpha}\delta\right)(1-\alpha)^{\frac{1}{1-\alpha}} \\ &= \frac{\alpha}{1-\alpha}(1-\alpha)^{\frac{1}{1-\alpha}} \\ (c^o)^* &= \alpha(1-\alpha)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Numerical Application. With $\delta = 1$ and $\alpha = 1/3$:

$$\begin{aligned} K^* &= (1-\alpha)^{\frac{1}{1-\alpha}} = \left(\frac{2}{3}\right)^{3/2} = \frac{2\sqrt{2}}{3\sqrt{3}} \\ r^* &= \frac{2\alpha-1}{1-\alpha} = \frac{-1/3}{2/3} = -\frac{1}{2} = -50\% \\ Y^* &= (1-\alpha)^{\frac{\alpha}{1-\alpha}} = \left(\frac{2}{3}\right)^{1/2} = \frac{\sqrt{2}}{\sqrt{3}} \\ w^* &= (1-\alpha)^{\frac{1}{1-\alpha}} = \frac{2\sqrt{2}}{3\sqrt{3}} \\ (c^o)^* &= \alpha(1-\alpha)^{\frac{\alpha}{1-\alpha}} = \frac{1}{3}\left(\frac{2}{3}\right)^{1/2} = \frac{\sqrt{2}}{3\sqrt{3}} \end{aligned}$$

10.2.3 With Government Debt

As we saw in lecture 2, and then again in lecture 4, because $r^* < 0$ we have that the quantity of capital is higher than the Golden Rule level of the capital stock, which is such that $r_g^* = 0$. Why, again, is the Golden rule interest rate equal to 0 when there is no growth ? One quick way to understand it is to write that in the steady-state, the consumption of the old $(c^o)^*$ is supposed to be maximized (since the old are the only ones consuming). However, we also know that in the steady-state, total production $F(K^*, 1) = (K^*)^\alpha$ is used for two things: consuming (for the old) and repairing the capital stock (saving equals investment equals depreciation, as in the Solow growth model), so that:

$$(c^o)^* + \delta K^* = (K^*)^\alpha \quad \Rightarrow \quad (c^o)^* = (K^*)^\alpha - \delta K^*$$

Therefore, the Golden rule capital stock K_g^*, which maximizes $(c^o)^*$ solves:

$$\max_{K^*} \quad (K^*)^\alpha - \delta K^*,$$

which implies that:

$$\alpha (K_g^*)^{\alpha-1} = \delta.$$

Moreover, we know that the marginal product of capital $\partial F(K^*, 1)/\partial K^* = \alpha (K_g^*)^{\alpha-1}$ is also equal to $r_g^* + \delta$, the gross return, from the firms' problem, which gives the following optimality condition:

$$\alpha (K_g^*)^{\alpha-1} = r_g^* + \delta.$$

Using these two last equalities allows to conclude that in this situation with no growth, the Golden rule interest rate is equal to zero since $r_g^* + \delta = \alpha (K_g^*)^{\alpha-1} = \delta$:

$$r_g^* = 0.$$

However, we saw in the previous section that the *equilibrium* interest rate r^* was negative, equal to -50%. Therefore, the capital stock is too high here. As we saw in lectures 2 and 4, one way to solve this problem would be to force everyone to save less, in order to decrease private saving; however this might be thought of as a little bit too intrusive. Another way to solve this problem is to use public debt (decrease public saving, to reduce total saving) in order to solve this problem of excess saving and excess investment.

In order to better understand which level of public debt is warranted, we look at the level of capital such that $r_g^* = 0$ - which again, is the golden rule interest rate, since the rate of growth of output is $g_Y = 0$. Thus, the corresponding Golden Rule level of the capital stock K_g^* is such that:

$$r_g^* + \delta = \alpha (K_g^*)^{\alpha-1} \quad \Rightarrow_{r_g^*=0} \quad \delta = \alpha (K_g^*)^{\alpha-1}$$

Therefore:

$$K_g^* = \frac{\alpha^{\frac{1}{1-\alpha}}}{\delta^{\frac{1}{1-\alpha}}}.$$

The Golden rule steady-state value for output Y_g^* would be then:

$$Y_g^* = \left(K_g^*\right)^\alpha$$
$$Y_g^* = \frac{\alpha^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}.$$

The value for the steady-state wage w_g^* is then:

$$w_g^* = (1-\alpha)\,(K^*)^\alpha$$
$$w_g^* = (1-\alpha)\frac{\alpha^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}$$

Steady-state consumption of the old $(c^o)_g^*$ is thus given by:

$$(c^o)_g^* = (1+r^*)w_g^*$$
$$(c^o)_g^* = (1-\alpha)\frac{\alpha^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}$$

If $\delta = 1$, then:

$$(c^o)_g^* = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$$

The question is how to we achieve this quantity of capital K_g^*? The answer is that some public debt needs to be taken on by the government. Again, saving is equal to the wage w_g^*, and to the purchase of total assets, which includes both public debt whose quantity is given by B_g^*, and the capital stock whose quantity is K_g^*. Therefore, we may compute the level of the public debt which allows to reach this Golden-Rule level of capital accumulation:

$$B_g^* + K_g^* = w_g^* \quad \Rightarrow \quad B_g^* = w_g^* - K_g^*.$$

Substituting:

$$\begin{aligned} B_g^* &= w_g^* - K_g^* \\ &= (1-\alpha)\frac{\alpha^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} - \frac{\alpha^{\frac{1}{1-\alpha}}}{\delta^{\frac{1}{1-\alpha}}} \\ B_g^* &= \frac{\alpha^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}\left(1-\alpha-\frac{\alpha}{\delta}\right). \end{aligned}$$

Note that with $\delta = 1$, then this level of public debt is strictly positive when $\alpha < 1/2$.

Numerical Application. With $\alpha = 1/3$ and $\delta = 1$:

$$\begin{aligned} K_g^* &= \frac{\alpha^{\frac{1}{1-\alpha}}}{\delta^{\frac{1}{1-\alpha}}} = \left(\frac{1}{3}\right)^{3/2} = \frac{1}{3\sqrt{3}} \\ r_g^* &= 0 \\ Y_g^* &= \frac{\alpha^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} = \left(\frac{1}{3}\right)^{1/2} = \frac{1}{\sqrt{3}} \\ w_g^* &= (1-\alpha)\frac{\alpha^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{3\sqrt{3}} \\ (c^o)_g^* &= (1+r_g^*)w_g^* = w_g^* = \frac{2}{3\sqrt{3}} \end{aligned}$$

Note that the steady-state consumption of the old $(c^o)_g^*$ is greater than the level of consumption achieved by the old without government debt since $2 > \sqrt{2}$. But what is amazing is that the level of capital in this case is actually lower than the level of capital in the previous section. The government can force the economy into this level of capital accumulation by taking on debt. The level of debt B_g^* that corresponds to that level of capital accumulation is given by:

$$\begin{aligned} B_g^* &= w_g^* - K_g^* \\ &= \frac{2}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} \\ B_g^* &= \frac{1}{3\sqrt{3}}. \end{aligned}$$

The government can reach that level of debt by giving a transfer to the first generation of old, like the war veterans, who will then consume in the first period $t = 0$ an amount equal to:

$$c_0^o = \frac{\sqrt{2}}{3\sqrt{3}} + \frac{1}{3\sqrt{3}} = \frac{1+\sqrt{2}}{3\sqrt{3}}.$$

All future generations thus consume more. With a lot of capital, there is such a thing as a free lunch! Public debt is a Ponzi scheme, but a beneficial one. Public debt allows to increase consumption for everyone, and it can be rolled over every period (note that the debt to GDP ratio stays constant, as GDP growth is zero in the long run, just as the long run interest rate is zero). This is true more generally even in the neoclassical model, provided that there is **dynamic inefficiency** ($r^* < g_Y$) to begin with.

10.3 The Treasury View, and Say's Law

The most controversial and also most important questions in macroeconomics revolve around the issue of the so-called **Treasury View**, and **Say's law**. These are probably the most difficult, controversial, but also the most important questions for macroeconomics.

10.3.1 Treasury View: The Effects of Deficit Spending on Investment

The *Treasury View* asserts that more government spending, either in the form of government purchases or of tax reductions, and therefore lower saving, necessarily leads to *crowd out* (reduce) an equal amount of investment spending. Conversely more saving, either by the government or by households, leads to more investment. The logic of this argument is rather straightforward: if there exists a finite amount of resources in the economy - in other words, output is supply-determined - then whatever is being saved goes to increase investment. Output is simply the sum of consumption, investment, and government spending, so that "necessarily" increasing government spending leads to crowd out either of every component:

$$
\begin{aligned}
& Y = C + I + G \\
\Rightarrow \quad & (Y - C - T) = I + (G - T) \\
\Rightarrow \quad & (Y - C - T) + (T - G) = I
\end{aligned}
$$

Therefore investment I equals total saving, private saving $S = Y - C - T$ plus public saving $T - G$:

$$
\boxed{I = \underbrace{(Y - C - T)}_{\text{Private Saving}} + \underbrace{(T - G)}_{\text{Public Saving}}}
$$

The reason why this view is called the **Treasury View** is that it was advanced in the 1930s, during the Great Depression, by the staff of the British Chancellor of the Exchequer, Winston Churchill. When defending his 1929 budget, Winston Churchill explained:

> The orthodox Treasury view… is that when the Government borrows in the money market it becomes a new competitor with industry and engrosses to itself resources which would otherwise have been employed by private enterprise, and in the process raises the rent of money to all who have need of it.

What we have seen so far leads us to take a very constrasted perspective on the Treasury View:

- In the **Keynesian model** of lectures 7, 8, and 9, investment is not crowded out by public debt - in the simplest model of the goods market, investment is in fact fixed. In the accelerator model, $I = b_0 + b_1 Y$ so that investment depends only on sales, not on saving. According to this model, what the Treasury View misses is that output is not determined by supply, but it is instead determined by demand. Therefore, one cannot reason as if GDP was fixed: GDP is precisely what adjusts when saving is reduced, to maintain the equality between investment and total saving.
- In the **neoclassical model** of lectures 2 and 4 in contrast, investment is determined by total saving, and it moves flexibly in response to saving. According to this view, investment is indeed crowded out by public deficits. Note however that this does not mean that in the neoclassical model, government deficits are always bad. As we just saw, public deficits may be a good thing if the economy has too much capital to begin with.

This issue of the Treasury view was discussed a lot during the U.S. 2008 financial crisis, when policymakers were turning to economists for advice on the appropriate policy response. You can see some discussion of this issue in "Readings - To go further". While Chicago economists were articulating the Treasury view in various different flavors, Keynesian economists were rejecting this notion very strongly - most notably Paul Krugman. Of course, whether the Treasury View is correct or not is ultimately an empirical question. We will present some empirical evidence on this issue during lecture 13.

10.3.2 Say's law: supply creates its own demand

Say's law, named after Jean-Baptiste Say (1767 - 1832), has been summarized by J.M. Keynes as a statement that "supply creates its own demand". Jean-Baptiste Say's reasoning was straightforward:

> It is worthwhile to remark that a product is no sooner created than it, from that instant, affords a market for other products to the full extent of its own value. When the producer has put the finishing hand to his product, he is most anxious to sell it immediately, lest its value should diminish in his hands. Nor is he less anxious to dispose of the money he may get for it; for the value of money is also perishable. But the only way of getting rid of money is in the purchase of some product or other. Thus the mere circumstance of creation of one product immediately opens a vent for other products.

Thus, in Say's opinion, supply created its own demand, and there could never be any aggregate demand shortages. In a briefing on Say's law, *The Economist* magazine writes:

> In Say's time, as nowadays, the world economy combined strong technological progress with fitful demand, spurts of innovation with bouts of austerity. (…) On the other hand, global demand was damaged by failed ventures in South America and debilitated by the eventual downfall of Napoleon. In Britain government spending was cut by 40% after the Battle of Waterloo in 1815. Some 300,000 discharged soldiers and sailors were forced to seek alternative employment. The result was a tide of overcapacity, what Say's contemporaries called a "general glut". Britain was accused of inundating foreign markets, from Italy to Brazil, much as China is blamed for dumping products today. In 1818 a visitor to America found "not a city, nor a town, in which the quantity of goods offered for sale is not infinitely greater than the means of the buyers". It was this "general overstock of all the markets of the universe" that came to preoccupy Say and his critics. In trying to explain it, Say at first denied that a "general" glut could exist. Some goods can be oversupplied, he conceded. But goods in general cannot. His reasoning became known as Say's law: "it is production which opens a demand for products", or, in a later, snappier formulation: supply creates its own demand.

Once again, we may contrast two very different perspectives on the Say's law:

- In the **Keynesian model** of lectures 7, 8, and 9, supply does not create its own demand, as some resources (labor or capital) are idle. This allows government spending or tax cuts to have an effect on GDP, by utilizing some of these resources. The reason is that in this model, the desire to save does not necessarily translate into more investment, since investment is fixed or given by the overall level of GDP, which is depressed by more saving. This was the paradox of thrift of lecture 8: instead of increasing investment, a higher desire to save actually reduces output, which ends up depressing saving.
- In the **neoclassical model** of lectures 2 and 4 in contrast, investment is determined by total saving, and it moves flexibly in response to saving. According to this view, there can indeed never be a general glut: consumption increases aggregate demand, but saving increases it too, by stimulating more investment. True, there might be "too much capital", when the capital stock is higher than the Golden rule level, but this is not a "general glut". Thus, supply can be thought of as indeed "creating its own demand".

How you stand on those two issues (the Treasury View and Say's law) determines whether you are more a neoclassical or a Keynesian economist. As we have seen however, Keynesians and neoclassicals agree on one thing: consumption is the sole purpose of all production, and there exists such a thing as "too much capital", when $r^* < g_Y$. Whether that is the situation we are in, even after Donald Trump's massive tax cuts, is still a controversial question.

Readings - To go further

Uriel H. Crocker and S. M. Macvane, "General Overproduction," The Quarterly Journal of Economics 1, no. 3 (1887): 362–66.

Robert J. Barro, Government Spending Is No Free Lunch, *Wall Street Journal*, January 22, 2009.

Paul Krugman, A Dark Age of Macroeconomics (wonkish), *New York Times*, Jan 27, 2009

Paul Krugman, Multipliers and Reality, New York Times Blog Post, June 3, 2015.

In Japan, the Government Gets Paid to Borrow Money, *Wall Street Journal*, March 1, 2016.

(Gated) Say's Law: Supply Creates its Own Demand. *The Economist*, August 10, 2017.

(Gated) Overlapping Generations: Kicking the road down an endless road. *The Economist*, August 31, 2017.

(Gated) Why is macroeconomics so hard to teach? *The Economist*, August 9, 2018.

"U.S. on a course to spend more on debt than defense", *Wall Street Journal*, November 11, 2018.

Appendix A

Problem Set 1

A.1 Geometric Sums

1. Calculate the following geometric sum, when $x \neq 1$, as a function of x^{n+1} in particular:

$$\sum_{i=0}^{n} x^i = 1 + x + x^2 + ... + x^n$$

2. State a condition on x such that the infinite geometric sum (when $n \to \infty$) has a finite value.
3. Assuming that this geometric sum is finite, calculate:

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + ... + x^n + ...$$

4. More generally, for m a positive integer, calculate:

$$\sum_{i=m}^{n} x^i = x^m + x^{m+1} + ... + x^n$$

5. What is the present discounted value of an infinite stream of incomes, which grows at rate $g = 2$ %, starts at $y_0 = 90000$, if the interest rate is $i = 3$ %?

A.2 Taylor Approximations

1. If x is small, show that:

$$(1 + x)^n \approx 1 + nx.$$

2. If x and y are small, then show that:

$$(1 + x)(1 + y) \approx 1 + x + y.$$

3. If x and y are small, then show that:

$$\frac{1 + x}{1 + y} \approx 1 + x - y.$$

4. Your savings account offers a nominal interest rate of 1%. Meanwhile, annual inflation is 1.5%. What is an exact value for the real interest rate, defined as the rate of increase in your purchasing power if you leave your money in the bank? What is an approximate value for this real interest rate, using one of the previous formulas?

A.3 Growth Rates

1. If y_t grows at a constant rate g during a given period, then show that the growth G of y_t after T periods is:
$$G = \frac{y_T}{y_0} - 1 = (1+g)^T - 1.$$

2. Conversely, if the growth rate of y_t after T periods is G, then show that the average growth rate of y_t per period is:
$$g = \frac{y_{t+1}}{y_t} - 1 = (1+G)^{1/T} - 1.$$

3. If your annual return on your savings rate is 1%, what is your daily return (assuming a year is 365 days)? How much do you then give up each day by leaving $100K on a zero interest checking account?

Appendix B

Problem Set 2

B.1 Solow Growth Model if $\alpha = 1/2$ and $A = 1/2$

Suppose that the production function is such that $A = 1/2$ (often, we assume that $A = 1$) and $\alpha = 1/2$:

$$Y = \frac{1}{2}\sqrt{K}\sqrt{L}$$

1. Derive the steady-state levels of output per worker and capital per worker in terms of the saving rate, s, and the depreciation rate, δ.
2. Derive the equation for steady-state output per worker and steady-state consumption per worker in terms of s and δ.
3. Suppose that $\delta = 0.05$. With your favorite spreadsheet software, compute steady-state output per worker and steady-state consumption per worker for $s = 0$; $s = 0.1$; $s = 0.2$; $s = 1$. Explain the intuition behind your results.
4. Use your favorite spreadsheet software to graph the steady-state level of output per worker and the steady-state level of consumption per worker as a function of the saving rate (i.e., measure the saving rate on the horizontal axis of your graph and the corresponding values of output per worker and consumption per worker on the vertical axis).
5. Does the graph show that there is a value of s that maximizes output per worker? Does the graph show that there is a value of s that maximizes consumption per worker? If so, what is this value?

B.2 Solow Growth Model if $\alpha = 1/3$

Suppose that the economy's production function is given by

$$Y = K^{\alpha}L^{1-\alpha}$$

and assume that $\alpha = 1/3$.

1. Is this production function characterized by constant returns to scale? Explain.
2. Are there decreasing returns to capital?
3. Are there decreasing returns to labor?
4. Transform the production function into a relation between output per worker and capital per worker.

5. For a given saving rate, s, and depreciation rate, δ, give an expression for capital per worker in the steady state.
6. Give an expression for output per worker in the steady state.
7. Solve for the steady-state level of output per worker when $s = 0.32$ and $\delta = 0.08$.
8. Suppose that the depreciation rate remains constant at $\delta = 0.08$, while the saving rate is reduced by half, to $s = 0.16$. What is the new steady-state output per worker?

B.3 An increase in the depreciation rate

Continuing with the logic from the previous problem, suppose that the economy's production function is given by

$$Y = K^{\alpha} L^{1-\alpha}$$

with $\alpha = 1/3$ and that both the saving rate, s, and the depreciation rate, δ are equal to 0.10.

1. What is the steady-state level of capital per worker?
2. What is the steady-state level of output per worker? Suppose that the economy is in steady state and that, in period t, the depreciation rate increases permanently from 0.10 to 0.20.
3. What will be the new steady-state levels of capital per worker and output per worker?
4. Compute the path of capital per worker and output per worker over the first three periods after the change in the depreciation rate.

B.4 Deficits and the capital stock

Suppose that the production function is given by:

$$Y = \sqrt{K}\sqrt{L}$$

1. Show that the steady-state capital stock per worker and output per worker are given by:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 \quad \text{and} \quad \frac{Y^*}{L} = \frac{s}{\delta}.$$

2. Suppose that the saving rate, s, is initially 15 % per year, and the depreciation rate, δ, is 7.5 %. What is the steady-state capital stock per worker? What is steady-state output per worker?
3. Suppose that there is a government deficit of 5% of GDP and that the government eliminates this deficit. Assume that private saving is unchanged so that total saving increases to 20%. What is the new steady-state capital stock per worker? What is the new steady-state output per worker? How does this compare to your answer to part 2?

B.5 U.S. saving and government deficits

This question continues the logic of the previous question, to explore the implications of the U.S. government budget deficit for the long-run capital stock.

1. The World Bank reports gross domestic saving rate by country and year. The Web site is http://data.worldbank.org/indicator/NY.GDS.TOTL.ZS. Find the most recent number for the United States. What is the total saving rate in the United States as a percentage of GDP? Using the depreciation rate and the logic from the previous problem, what would be the steady-state capital stock per worker? What would be steady-state output per worker?

2. Go to the most recent Economic Report of the President (ERP) and find the most recent federal deficit as a percentage of GDP. In the 2015 ERP, this is found in Table B-20. Using the reasoning from the previous problem, suppose that the federal budget deficit was eliminated and there was no change in private saving. What would be the effect on the long-run capital stock per worker? What would be the effect on long-run output per worker?

3. Return to the World Bank table of gross domestic saving rates. How does the saving rate in China compare to the saving rate in the United States?

Appendix C

Problem Set 3

C.1 Two-period Intertemporal Optimization

Consider the model of lecture 3 again. Instead of logarithmic preferences, assume that preferences are given by:

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma},$$

1. Under what condition on σ is this an increasing and concave utility function?
2. Show using 4 different methods that:
$$\frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+r}$$
3. Using the equation in question 1, what is the ratio c_1/c_0?
4. Replace in the intertemporal budget constraint to find an implicit equation for c_1. Do the same for c_0.
5. Assume that $\sigma = 1/2$, and $f_0 = 0$, $y_0 = \$90,000$, $y_1 = 0$, $\beta = 1$. What are c_0 and c_1 if $r = 1\%$? What about if $r = 2\%$? How much does c_0 change then? How much in percentage terms?
6. Same questions if $\sigma = 1$.
7. Same questions if $\sigma = 2$.
8. Compare the changes in c_0 following an increase in the real interest rates in questions 5, 6, 7. Comment.

C.2 Another Overlapping Generations model

Consider the model of lecture 4 again, with one small twist: agents care only about old age consumption. In other words, their utility functions are given by:

$$U = u(c^o_{t+1}).$$

Our goal is to derive the law of motion for the capital stock K_t, that is, a function relating K_{t+1} to K_t and the parameters of the model.

1. Why can the utility function be left unspecified for computing the level of saving?
2. Derive the law of motion for capital.

3. What is the corresponding value of the saving rate in the Solow model?
4. Provide a condition on α such that the capital stock is below the Golden Rule level.
5. Is that condition likely to be satisfied?

Appendix D

Problem Set 4

D.1 The Solow Growth Model with Exogenous Growth

Consider the Solow growth model of Lecture 2, with however two small changes. Assume that the production function is given by:

$$F(K_t, L_t) = A_t K_t^\alpha L_t^{1-\alpha},$$

where productivity A_t grows exogenously at rate g and $A_0 = 1$:

$$A_t = (1+g)^t.$$

Moreover, assume that the labor force also grows at a rate n and $L_0 = 1$, so that at any time t:

$$L_t = (1+n)^t.$$

1. Write the law of motion for capital K_t.
2. Define k_t as:
$$k_t \equiv \frac{K_t}{A_t^{1/(1-\alpha)} L_t},$$
and write a law of motion for k_t. Assume that n, and g are small in order to simplify this law of motion. *Hint*: if n and g are small then: $(1+g)^{1/(1-\alpha)}(1+n) \approx 1 + \frac{1}{1-\alpha} g + n$.
3. Show that k_t converges to a steady-state k^*. Compute k^*.
4. When k_t has reached a steady-state, the economy is said to be on a **balanced growth path**. On this balanced growth path, what is the rate of growth of Y_t, C_t, K_t, K_t/Y_t, K_t/L_t, w_t, $w_t L_t$ and $w_t L_t / Y_t$? Denoting by R_t the marginal product of capital, what is the rate of growth of R_t, $R_t K_t$, and $R_t K_t / Y_t$?
5. Compute y^* and c^* corresponding to steady-state k^* with:
$$y_t \equiv \frac{Y_t}{A_t^{1/(1-\alpha)} L_t} \quad \text{and} \quad c_t \equiv \frac{C_t}{A_t^{1/(1-\alpha)} L_t}.$$
6. What is the saving rate which maximizes c^* ? (Golden Rule level of capital accumulation)
7. What is then the value of the marginal product of capital R^* ?

D.2 The Neoclassical Labor Market Model

Consider the neoclassical labor market model of lecture 6. Assume that preferences and the production function are as in lecture 6:

$$U(c,l) = c - B\frac{l^{1+\epsilon}}{1+\epsilon}, \qquad f(l) = Al^{1-\alpha}.$$

Denote the wage by w, and the price of consumption by p.

1. Derive the Labor Demand curve.
2. Assume that $\alpha = 1/3$ and $A = 2$. Using your favorite spreadsheet software, plot this demand curve in a $(l, w/p)$ plane - that is, putting l on the x-axis and w/p on the y-axis.
3. Take logs of both sides. What does the demand curve look like in a $(\log(l), \log(w/p))$ plane? What is the slope of the demand curve equal to? If α is higher, is the demand curve steeper or flatter? What shifts the demand curve to the left or to the right?
4. Derive the Labor Supply curve.
5. Assume that $\epsilon = 5$ and $B = 2$. Using your favorite spreadsheet software, plot this supply curve in a $(l, w/p)$ plane - that is, putting l on the x-axis and w/p on the y-axis. Add the supply curve to the demand curve of question 2.
6. Take logs of both sides. What does the supply curve look like in a $(\log(l), \log(w/p))$ plane? What is the slope of the supply curve equal to? If ϵ is higher, is the supply curve steeper or flatter? What shifts the supply curve to the left, or to the right?
7. Assume that productivity A decreases by 5%, to $A = 1.9$. What is the effect on the quantity of employment, and on the real wage? If α is higher, is that effect larger or smaller? What is the economic intuition?
8. Assume that leisure becomes relatively more attractive relative to working (think of Facebook, Netflix, etc.), so that B increases by 10% (the disutility of work increases). What is the effect on the quantity of employment, and on the real wage? If ϵ is higher, is that effect larger or smaller? What is the economic intuition for this?

D.3 The "Keynesian" Labor Market Model

Consider the neoclassical labor market model of the previous problem.

1. Assume that productivity A decreases by 5%, but that real wages w/p are rigid. Compute the change in the quantity of employment following a fall in productivity.
2. Compare the effect with question 7 in the previous problem. Explain.
3. Assume that leisure becomes relatively more attractive relative to working, so that B increases by 10%. Compute the change in the quantity of employment following a increase in leisure attractiveness.
4. Compare the effect with question 8 in the previous problem. Explain.

D.4 The Bathtub model

Consider the bathtub model of lecture 6. Assume a monthly job separation rate equal to $s = 1\%$, and a monthly job finding rate equal to $f = 20\%$. Assume that the labor force is given by $L = 159$ million.

1. Derive the steady-state unemployment rate. How many people are unemployed in the steady-state? How many people lose their jobs every month? How many people find a job every month?

2. Assume that the economy starts with an unemployment rate equal to $u_0 = 8\%$. Using your favorite spreadsheet software, show the evolution of the unemployment rate over time. How long before the unemployment rate reaches 5%?
3. If $s = 2\%$ instead, which job finding rate f gives the same steady-state unemployment rate?
4. Assuming the separation rate and the job finding rate are given from question 3, answer question 2 again.
5. Explain why an economy with more churning (that is, faster reallocation) - think of the US versus Europe - has a faster recovery in terms of unemployment after a recession. *Note:* A recession could be coming from a temporary increase in the job separation rate, or a temporary decrease in the job finding rate, which then goes back to its original value.

Appendix E

Problem Set 5

E.1 Gregory N. Mankiw - NYT - Nov 30, 2008

This exercise is based on the following *New York Times* article, which Gregory N. Mankiw wrote a bit more than 2 months after the bankruptcy of Lehman Brothers (to which we shall come back at the end of the class). The *New York Times* articles are gated after you have read your monthly quota, but archives are available through UCLA on ProQuest:

Mankiw, N. Gregory. What would Keynes have done? *New York Times*, November 30, 2008

1. According to Gregory N. Mankiw, which factors contributed to hold back consumption? Can you interpret these factors in terms of changes in c_0?
2. Gregory N. Mankiw mentions the "paradox of thrift". Which model that we saw in the class makes most sense of his explanations?
3. Concerning saving, one of the arguments that Gregory N. Mankiw makes is more neoclassical. Which one is it?
4. What is Gregory N. Mankiw most concerned about though?

E.2 Procyclical Government spending

Consider the basic goods market model of Lecture 7: consumption is linear in disposable income, disposable income is income minus taxes, investment is exogenous and equal to $\bar{I}$, and taxes are exogenous as well. However, government spending depends on the level of output. For example, the government systematically spends more when GDP is higher (it builds new roads, hires new teachers, etc.), and conversely when GDP is lower (it then stops construction projects, fires teachers, etc.). Thus, government spending is given by the following equation, with $g_1 > 0$:

$$G = g_0 + g_1 Y$$

1. Solve for equilibrium output.
2. If $g_1 + c_1 < 1$, what is the value of the tax multiplier? (the tax multiplier is equal to the increase in output following from a \$1 decrease in taxes) If $g_1 > 0$, is the multiplier higher or lower than when government spending does not depend on GDP ($g_1 = 0$)? What is the intuition for this?
3. Does this kind of policy appear like a good policy?
4. Give both a graphical as well as an algebraic justification for the value of the multiplier.

5. What happens if $g_1 + c_1 > 1$? Explain using the multiplier intuition.

E.3 Accelerator and Automatic Stabilizer

Consider the basic goods market model of Lecture 7: consumption is linear in disposable income with a Marginal Propensity to Consume equal to c_1, disposable income is income minus taxes. However, we assume an accelerator effect of demand on investment (investment depends on sales):

$$I = b_0 + b_1 Y,$$

as well as the presence of automatic stabilizers:

$$T = t_0 + t_1 Y.$$

1. Solve for equilibrium output.
2. Find a condition on b_1, c_1, and t_1 such that the multiplier stays finite.
3. What happens if the multiplier is infinite? Does GDP become infinite?
4. Give both a graphical as well as an algebraic justification for the value of the multiplier.

Appendix F

Problem Set 6

Another Numerical Example

During lecture 9, we studied the aggregate demand effects of tax cuts on the bottom 90% financed by tax increases on the top 10%. In this problem, we study another example of these aggregate demand effects, looking at redistribution from the top 1% income share to the bottom 99%. *An important warning*: Again, note that this model only has Keynesian, aggregate demand effects. However, raising taxes on high income earners certainly has effects on the supply side as well. Raising taxes on the top 1% probably also has effects on entrepreneurship, incentives to take risk and create jobs, which are not taken into account here. (Symmetrically, changing taxes on the bottom 99% also may have effects on their incentives to work.) Whether these supply effects are sufficiently large to offset and perhaps even overturn the aggregate demand effects we focus on here is controversial, subject to heated debates and outside of the scope of the class.

To illustrate the effects on aggregate demand of redistributive policies between the top 1% and the bottom 99%, we now use the same notations as in lecture 9. There is a share $\lambda = 99\%$ of population N who are in the bottom 99%, who earn individual income $\underline{y}$, pay net taxes $\underline{t} = \underline{t}_0 + t_1\underline{y}$, have an MPC $\underline{c}_1$, baseline consumption $\underline{c}_0$. Notations for high income are similar, but with bars: $\bar{y}$, $\bar{t} = \bar{t}_0 + t_1\bar{y}$, $\bar{c}_1$, $\bar{c}_0$. Total GDP is Y, investment is $I = b_0 + b_1 Y$, government spending is exogenous and equal to G.

1. Use Google to find out how much income would put you in the top 1%.
2. The World Income Database suggests that the top 1% captures approximately 20% of total U.S. income in 2017, while it was approximately 10% in 1980. Using the notations of the class, what is $\gamma = \bar{y}/\underline{y}$?
3. Compute aggregate consumption $C = \underline{C} + \bar{C}$, as in lecture 9, using the following notations:

$$
\begin{aligned}
c_1 &\equiv \frac{\lambda\underline{c}_1 + (1-\lambda)\,\gamma\bar{c}_1}{\lambda + (1-\lambda)\gamma} \\
C_0 &\equiv \lambda N\underline{c}_0 + (1-\lambda)N\bar{c}_0 \\
\underline{T}_0 &\equiv \lambda N\underline{t}_0 \\
\bar{T}_0 &\equiv (1-\lambda)N\bar{t}_0.
\end{aligned}
$$

4. What is an economic interpretation for c_1 ? Calculate c_1 if $\underline{c}_1 = 1$ and $\bar{c}_1 = 1/4$.
5. Using the expression for I, and for aggregate consumption C, compute Y.
6. Assume that $t_1 = 1/4$ and $b_1 = 1/6$. Compute the impact on GDP of a 100 billion dollars tax cut on the top 1%. What is the impact on the government deficit of such a cut?
7. Assume that $t_1 = 1/4$ and $b_1 = 1/6$. Compute the impact on GDP of a 100 billion dollars tax cut on the bottom 99%. What is the impact on the government deficit of such a cut?

8. Assume that $t_1 = 1/4$ and $b_1 = 1/6$. Compute the impact on GDP of a transfer of 100 billion dollars from the top 1% to the bottom 99%. What is the impact on the government deficit of such a transfer?

9. What happens if there are no automatic stabilizers in this economy ($t_1 = 0$)? Explain.

Appendix G

Problem Set 7

Another overlapping-generations model with government debt

In this exercise, we consider the same problem as in lecture 10, except that lifetime utility is logarithmic with $\beta = 2$ (that is, people are patient instead of impatient, so they tend to save a lot):

$$U = \log(c_t^y) + 2\log(c_{t+1}^o)$$

We denote the (net) real interest rate by r_t so that the intertemporal budget constraint is:

$$c_t^y + \frac{c_{t+1}^o}{1+r_t} = w_t.$$

Other than that, we still assume a Cobb-Douglas production function with $\alpha = 1/3$, so that:

$$Y_t = K_t^{1/3} L_t^{2/3}.$$

We assume that the labor force is constant so that $L_t = 1$. The depreciation rate is still $\delta = 1 = 100\%$.

1. Compute c_{t+1}^o and c_t^y as a function of the wage w_t.
2. What is the law of motion for the capital stock?
3. Compute the steady-state capital stock K^*, the (net) steady-state real interest rate r^*, the steady-state output Y^*, the steady-state wage w^*, and the steady-state consumption of the young $(c^y)^*$ and of the old $(c^o)^*$.
4. Compute the Golden Rule (net) interest rate r_g^*, the Golden Rule capital stock K_g^*, the Golden Rule output Y_g^*, the Golden Rule wage w_g^*, and the Golden Rule consumption of the young $(c^y)_g^*$ and of the old $(c^o)_g^*$.
5. Compare the Golden Rule and steady-state levels, and give an economic intuition.
6. What level of government debt B_g^* brings the capital stock to the Golden Rule level ?
7. Starting from the steady-state situation of question 3, assume that the government gives this money to retirees, taking on government debt. How much is this (lucky) generation of retirees able to consume ?
8. Why is national debt a Ponzi scheme here? Is it bad ?
9. Assume that the government puts in place a pay-as-you-go system, such as Social Security (think of OASDI), giving retirees an amount B_g^* each period (where B_g^* is the same level of government debt as the one found in question 6), and taxing the young an equal amount B_g^*. Compare this situation to question 7. What are the differences and similarities?

10. What is the difference between pay-as-you-go financing and deficit financing ? Explain why government debt is not a very meaningful statistic.

Appendix H

Problem Set 8

H.1 Optimal Taxation: The "Supply Side" (Neoclassical) View

Let us go back to the labor market model of lecture 6. Denoting the (hourly) wage by w, the price of consumption by p, and the number of hours (per year) by l, assume that moreover $\alpha = 0$ so that:

$$U(c,l) = c - B\frac{l^{1+\epsilon}}{1+\epsilon}, \qquad f(l) = Al^{1-\alpha} = Al.$$

1. Derive the Labor Demand curve.
2. For labor supply, assume that the tax and transfer system is such that τ is the marginal tax rate, and c_0 is a real subsistence level of income given by the government: $pc = (1-\tau)wl + pc_0$. Derive the Labor Supply curve.
3. Express the number of hours worked per year $\underline{l}$, as well real annual pre-tax income $y = (w/p) \cdot l$, as a function of the parameters of the model.
4. Compute the number of hours worked per year $\underline{l}$, as well as real pre-tax income $\underline{y}$ if $\epsilon = 2$, $\underline{\tau} = 1/4$, $\underline{A} = 1000000/28188 \approx 35.5$, $\underline{B} = 3000000/491569855488 \approx 6.1 \cdot 10^{-6}$.
5. Compute the number of hours worked per year $\bar{l}$, as well as real pre-tax income $\bar{y}$ if $\epsilon = 2$, $\bar{\tau} = 1/2$, $\bar{A} = 1000000/6264 \approx 159.6$, $\bar{B} = 3000000/655426473984 \approx 4.6 \cdot 10^{-6}$.
6. Assume that the labor force is $N = 150$ million, with a fraction $\lambda = 0.9$ of the type described in question 4, and a fraction $1 - \lambda = 0.1$ of the type described in question 5. What is total output in the economy?
7. Assume that the tax system has two marginal tax rates: one 25% marginal tax rate above 25K, one 50% marginal tax rate above 200K and that $c_0 = 5K$. In other words, if income is y then taxes are:

   $$T(y) = -5K + 0.25 \cdot \max\{y - 25K, 0\} + 0.25 \cdot \max\{y - 200K, 0\}.$$

   Assume a tax reform that lowers the marginal tax rate on the richest by 5 points. How much is the total tax cut? What is the impact on GDP?
8. Assume a tax reform that lowers the marginal tax rate on the poorest by 5 points (and on the poorest only, even though in reality it would also change how much the rich pay in taxes). How much is the total tax cut? What is the impact on GDP?

H.2 Paul Krugman VS Robert Barro on the Bush tax cuts

Watch this debate between Paul Krugman and Robert Barro, moderated by Charlie Rose.

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# [1] "Sorry I don't know how to embed videos in PDF"
# [1] "Use the html version, or the link: https://www.youtube.com/embed/O77G0SaJv5M"
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1. Explain Robert Barro's view on taxes, using the previous exercise.
2. What is Paul Krugman's take on taxes?
3. What is the discussion on public debt, and the deficit ? Explain.
4. From 11:26 Robert Barro suggests to move from pay-as-you-go to "personal accounts". Explain their discussion in terms of problem set 7.

Appendix I

Problem Set 9

Appendix J

Problem Set 10

Appendix K

Problem Set 1 - Solution

K.1 Geometric Sums

1. Let us denote the geometric sum of interest by S:

$$S = \sum_{i=0}^{n} x^i = 1 + x + x^2 + ... + x^n.$$

The trick to calculate this sum is to multiply it by x, which allows to get:

$$xS = x + x^2 + x^3 + ... + x^{n+1}.$$

We can see that this is almost the same sum as the previous one except for the first term, which is missing, and the last term, which was absent from S, therefore:

$$\begin{aligned} xS &= -1 + 1 + x + x^2 + ... + x^n + x^{n+1} \\ xS &= -1 + S + x^{n+1} \end{aligned}$$

This implies (for $x \neq 1$):

$$1 - x^{n+1} = (1-x)S \quad \Rightarrow \quad S = \frac{1 - x^{n+1}}{1 - x}.$$

2. For S to have a finite value when $n \to \infty$, we need that x^{n+1} stays finite. This happens when:

$$|x| < 1.$$

In the knife edge case when $x = 1$, the sum goes to infinity since it is then equal to $n+1$. If $x = -1$, then the sum oscillates between 1 and -1 and does not have a limit when n goes to infinity.

3. From question 1 and 2, we know that when $|x| < 1$, $x^{n+1} \to 0$ when $n \to \infty$, and therefore:

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + ... + x^n + ... = \frac{1}{1-x}.$$

4. We just factor in x^m and then use the formula in question 1:

$$\begin{aligned}\sum_{i=m}^{n} x^i &= x^m + x^{m+1} + ... + x^n \\ &= x^m \left(1 + x + ... + x^{n-m}\right) \\ &= x^m \frac{1 - x^{n-m+1}}{1-x} \\ \sum_{i=m}^{n} x^i &= \frac{x^m - x^{n+1}}{1-x}.\end{aligned}$$

5. The present discounted value of an infinite stream of incomes, which grows at rate $g = 2\%$, starts at $y_0 = 90000$, if the interest rate is $i = 3\%$ is:

$$y_0 + y_0 \frac{1+g}{1+i} + y_0 \frac{(1+g)^2}{(1+i)^2} + ...$$

Using the formula found in question 3 with $x = (1+g)/(1+i)$, we get:

$$y_0 + y_0 \frac{1+g}{1+i} + y_0 \frac{(1+g)^2}{(1+i)^2} + ... = y_0 \frac{1}{1 - \frac{1+g}{1+i}} = \frac{y_0(1+i)}{i-g}$$

A numerical application gives (see Google Sheet):

$$\frac{y_0(1+i)}{i-g} = \frac{90000 * (1+0.03)}{0.03 - 0.02} = 9270000.$$

K.2 Taylor Approximations

1. We have:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

When x is small, all the x^k terms for $k \geq 2$ are negligible, and therefore:

$$(1+x)^n \approx 1 + \binom{n}{1} x = 1 + nx.$$

If you do not know what $\binom{n}{k}$ means, that is fine too. You can prove the same result using recursion. For $n = 1$, we know that $(1+x)^1 = 1 + x$ (obviously). Assume that the approximation is true for n, or that $(1+x)^n \approx 1 + nx$, let's prove that it is true for $n+1$:

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x) \\ &\approx (1+nx)(1+x) \\ &\approx 1 + (n+1)x + nx^2 \\ (1+x)^{n+1} &\approx 1 + (n+1)x\end{aligned}$$

which proves the proposition for $n+1$. Thus, the Taylor approximation is true for any $n \in \mathbb{N}$.

2. We have:

$$(1+x)(1+y) = 1 + x + y + xy.$$

When x and y are both small, then xy is negligible, which gives the result:

$$(1+x)(1+y) \approx 1 + x + y.$$

3. Using the formula proven in Problem 1, we get that:

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + ...$$

When x and y are both small, all terms of the product are negligible except for first-order terms:

$$\frac{1+x}{1+y} \approx 1 + x - y.$$

4. Denote the price level by p_t (that is, in dollars, the price of a representative basket of goods). Inflation π_t at time t is defined as the rate of growth of this price level between t and $t+1$:

$$\frac{p_{t+1}}{p_t} = 1 + \pi_t.$$

If you leave one dollar at the bank, and if the nominal interest rate is given by i_t, then you end up at the end of the period with $1 + i_t$ dollars at the bank. With this, you can buy a quantity of goods given by $(1 + i_t)/p_{t+1}$. If you buy a quantity of goods at time t, then you get a number of goods equal to $1/p_t$. Thus, the rate of increase in your purchasing power if you leave your money in the bank is given by:

$$\frac{(1+i_t)/p_{t+1}}{1/p_t} = \frac{1+i_t}{1+\pi_t}.$$

An exact value for the real interest rate is thus:

$$\begin{aligned}\frac{1+i_t}{1+\pi_t} - 1 &= \frac{1+0.01}{1+0.015} - 1\\ &= -0.00492610837\\ \frac{1+i_t}{1+\pi_t} - 1 &= -0.492610837\%.\end{aligned}$$

An approximate value from the above formula is:

$$\begin{aligned}\frac{1+i_t}{1+\pi_t} - 1 &\approx 1 + i_t - \pi_t - 1\\ &\approx 0.01 - 0.015\\ \frac{1+i_t}{1+\pi_t} - 1 &\approx -0.5\%.\end{aligned}$$

This is not such a bad approximation.

K.3 Growth Rates

1. Iterating on the formula:

$$y_{t+1} = (1+g)y_t \quad \Rightarrow \quad y_T = (1+g)^T y_0,$$

allows to find the result:

$$G = \frac{y_T}{y_0} - 1 = (1+g)^T - 1.$$

2. Again, inverting the previous relation:

$$G = (1+g)^T - 1,$$

allows to find:

$$g = \frac{y_{t+1}}{y_t} - 1 = (1+G)^{1/T} - 1.$$

3. Applying the previous formula allows to get (see Google Sheet):

$$\begin{aligned} g &= (1+0.01)^{1/365} - 1 \\ &= 0.00002726155 \\ g &= 0.0027\% \end{aligned}$$

You give up approximately \$2.7 every day (your bank most likely is investing this money on your behalf, so you are rather giving this money to your bank):

$$0.00002726155 * 100000 = 2.7.$$

Appendix L

Problem Set 2 - Solution

L.1 Solow Growth Model if $\alpha = 1/2$ and $A = 1/2$

1. We write the capital accumulation equation:

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
&= (1-\delta)K_t + sY_t \\
K_{t+1} &= (1-\delta)K_t + \frac{s}{2}\sqrt{K_t}\sqrt{L} \\
\frac{K_{t+1}}{L} &= (1-\delta)\frac{K_t}{L} + \frac{s}{2}\sqrt{\frac{K_t}{L}}
\end{aligned}$$

In steady state, we can then get steady-state capital per worker:

$$\delta\frac{K^*}{L} = \frac{s}{2}\sqrt{\frac{K^*}{L}} \quad \Rightarrow \quad \boxed{\frac{K^*}{L} = \left(\frac{s}{2\delta}\right)^2}.$$

This implies an expression for steady-state output per worker:

$$\frac{Y^*}{L} = \frac{1}{2}\sqrt{\frac{K^*}{L}} \quad \Rightarrow \quad \boxed{\frac{Y^*}{L} = \frac{s}{4\delta}}.$$

2. From the previous expression for steady-state output per worker, we can get steady-state consumption per worker:

$$\frac{C^*}{L} = (1-s)\frac{Y^*}{L} \quad \Rightarrow \quad \boxed{\frac{C^*}{L} = \frac{s(1-s)}{4\delta}}.$$

3. I used Google Sheets, and the result is available here. The intuition for why steady-state output per worker is a monotone function of the saving rate is that more investment always leads to a higher capital stock, which leads to higher output per worker. However, the effect of saving on steady-state consumption is ambiguous. It should be intuitive that if saving is equal to 0%, or 100%, consumption per worker is zero: in the first case, because there is no capital and therefore no production; in the second case, because everything is saved and there is nothing left to consumption. Thus, there is a limit to how much capital should be accumulated, at least for consumption purposes.

s	Y/L	C/L
0	0	0

s	Y/L	C/L
0.1	0.5	0.45
0.2	1	0.8
1	5	0

4. Again, I used Google Sheets, and the result is available here.
5. It should be clear from the Google Sheet that $s = 1$ maximizes output per worker, and that $s = 0.5$ maximizes consumption per worker.

L.2 Solow Growth Model if $\alpha = 1/3$

1. Yes, there are constant returns to scale. When one doubles all inputs, one gets double the output. This is true more generally for any x:
$$F(xK, xL) = (xK)^\alpha (xL)^{1-\alpha} = xK^\alpha L^{1-\alpha} = xF(K, L).$$
2. Yes, returns are decreasing with respect to capital. The reason is that the derivative of the production function with respect to the capital stock, which is:
$$\frac{\partial F}{\partial K} = \alpha K^{\alpha-1} L^{1-\alpha},$$
is decreasing in the amount of capital (indeed, since $\alpha = 1/3$ we have that the exponent on K is $-2/3$, so that this is a decreasing function). This implies that the "returns to capital", defined as the additional production allowed by one additional unit of capital, given by the derivative $\partial F/\partial K$, are decreasing in K. Another way to see this is to note that the production function is concave in the capital stock, as the second derivative is negative:
$$\frac{\partial^2 F}{\partial K^2} = \alpha(\alpha - 1)K^{\alpha-2} L^{1-\alpha} < 0$$
which is just another characterization of decreasing returns. This is because $\alpha - 1 = -2/3$ which is negative. See Lecture 2 for more detail.
3. Yes, returns are decreasing with respect to labor, for the same reason as returns are decreasing with respect to capital. The reason is that the derivative of the production function with respect to the amount of labor (number of employees, or number of hours), which is:
$$\frac{\partial F}{\partial L} = (1 - \alpha)K^\alpha L^{-\alpha},$$
is decreasing in the amount of labor (indeed, since $\alpha = 1/3$ we have that the exponent on L is $-1/3$, so that this is a decreasing function). This implies that the "returns to labor", defined as the additional production allowed by one additional unit of labor, given by the derivative $\partial F/\partial L$, are decreasing in L. Another way to see this is to note that the production function is concave in the stock of labor, as the second derivative is negative:
$$\frac{\partial^2 F}{\partial L^2} = -(1 - \alpha)\alpha K^\alpha L^{-\alpha-1} < 0$$
which is just another characterization of decreasing returns. See Lecture 2 for more detail.
4. Dividing the LHS and the RHS by L:
$$\frac{Y}{L} = \frac{K^\alpha L^{1-\alpha}}{L} = \left(\frac{K}{L}\right)^\alpha = F\left(\frac{K}{L}, 1\right).$$

Defininining the intensive form of the production function by $f(.)$:

$$f(k) \equiv F(k,1),$$

we can then write:

$$\frac{Y}{L} = f\left(\frac{K}{L}\right).$$

5. Again, we write the evolution of the capital stock as:

$$K_{t+1} = (1-\delta)K_t + I_t$$
$$K_{t+1} = (1-\delta)K_t + sK_t^{1/3}L^{2/3}$$

Dividing both sides by L:

$$\frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\left(\frac{K_t}{L}\right)^{1/3}.$$

In steady state,

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L},$$

so we have

$$\delta\frac{K^*}{L} = s\left(\frac{K^*}{L}\right)^{1/3} \quad \Rightarrow \quad \boxed{\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{3/2}}$$

6. Using that: $Y^* = K^{*1/3}L^{2/3}$, we have:

$$\frac{Y^*}{L} = \left(\frac{K^*}{L}\right)^{1/3} \quad \Rightarrow \quad \boxed{\frac{Y^*}{L} = \sqrt{\frac{s}{\delta}}}.$$

7. A straightforward numerical application gives:

$$\frac{Y^*}{L} = \sqrt{\frac{0.32}{0.08}} = \sqrt{4} = 2$$

8. If the saving rate declines to 16, then:

$$\frac{Y^*}{L} = \sqrt{\frac{s}{\delta}} = \sqrt{\frac{0.16}{0.08}} = \sqrt{2}$$

L.3 An increase in the depreciation rate in the Solow growth model

1. The steady-state level of capital per worker is:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{3/2} = \left(\frac{0.10}{0.10}\right)^{3/2} = 1.$$

2. The steady-state level of output per worker is:

$$\frac{Y^*}{L} = \sqrt{\frac{s}{\delta}} = \sqrt{\frac{0.10}{0.10}} = 1$$

3. The new steady-state levels of capital per worker and output per worker will be:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{3/2} = \left(\frac{0.10}{0.20}\right)^{3/2} \approx 0.35,$$

$$\frac{Y^*}{L} = \sqrt{\frac{s}{\delta}} = \sqrt{\frac{0.10}{0.20}} \approx 0.71.$$

4. We know the evolution of capital per worker is:

$$\frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\left(\frac{K_t}{L}\right)^{1/3}$$

So starting from $\frac{K_0}{L} = 1$, with $s = 0.10$, $\delta = 0.20$, we have:

$$\frac{K_1}{L} = (1-\delta)\frac{K_0}{L} + s\left(\frac{K_0}{L}\right)^{1/3} = 0.9$$
$$\frac{K_2}{L} = (1-\delta)\frac{K_1}{L} + s\left(\frac{K_1}{L}\right)^{1/3} \approx 0.82$$
$$\frac{K_3}{L} = (1-\delta)\frac{K_2}{L} + s\left(\frac{K_2}{L}\right)^{1/3} \approx 0.75$$

For more iterations, you may use Google Sheets: the result is available here. You should see that it indeed converges to the above values. From there, we may calculate the path of output per worker:

$$\frac{Y_1}{L} = \left(\frac{K_1}{L}\right)^{1/3} \approx 0.97$$
$$\frac{Y_2}{L} = \left(\frac{K_2}{L}\right)^{1/3} \approx 0.93$$
$$\frac{Y_3}{L} = \left(\frac{K_3}{L}\right)^{1/3} \approx 0.91.$$

For more iterations, you may use Google Sheets: the result is available here.

L.4 Deficits and the capital stock

1. Using the law of motion for the capital stock:

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t \\ &= (1-\delta)K_t + sY_t \\ K_{t+1} &= (1-\delta)K_t + s\sqrt{K_t}\sqrt{L}, \end{aligned}$$

Dividing both sides by L:

$$\frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\sqrt{\frac{K_t}{L}}$$

In steady state,

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L},$$

so we have:

$$\delta\frac{K^*}{L} = s\sqrt{\frac{K^*}{L}}$$

Therefore:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2$$

Using the production function

$$Y^* = \sqrt{K^*}\sqrt{L} : \frac{Y^*}{L} = \sqrt{\frac{K^*}{L}} = \frac{s}{\delta}.$$

2. The steady-state capital stock per worker is given by:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 = \left(\frac{15\%}{7.5\%}\right)^2 = 4$$

The steady-state output per worker is given by:

$$\frac{Y^*}{L} = \frac{s}{\delta} = \frac{15\%}{7.5\%} = 2.$$

3. The new steady-state capital stock per worker is:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 = \left(\frac{20\%}{7.5\%}\right)^2 \approx 7.11$$

The new steady-state output per worker is:

$$\frac{Y^*}{L} = \frac{s}{\delta} = \frac{20\%}{7.5\%} \approx 2.67.$$

Therefore, both the capital per worker and the output per worker increase.

L.5 U.S. saving and government deficits

1. According to http://data.worldbank.org/indicator/NY.GDS.TOTL.ZS, the national saving rate was approximately 16.9% in 2016. The steady-state capital stock per worker is given by:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 = \left(\frac{16.9\%}{7.5\%}\right)^2 \approx 5.08$$

The steady-state output per worker is:

$$\frac{Y^*}{L} = \frac{s}{\delta} = \frac{16.9\%}{7.5\%} \approx 2.25.$$

2. For fiscal year 2017, the federal fiscal deficit was 3.5% percent of GDP. Assuming that the federal budget deficit was eliminated and there was no change in private saving, the saving rate would change from 16.9% to 16.9%+3.5%=20.4%. The new steady-state capital stock per worker would be:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 = \left(\frac{20.4\%}{7.5\%}\right)^2 \approx 7.40$$

which increases by 45.67%. The new steady-state output per worker would be:

$$\frac{Y}{L} = \frac{s}{\delta} = \frac{20.4\%}{7.5\%} = 2.72$$

which increases by 20.89%.

3. The saving rate in China was 46.54% in year 2016, which is much higher than the saving rate in the United States. This is perhaps not that surprising according to the Solow model, as China is still in the process of catching up. At the same time, it is not clear that what China was lacking before was capital, rather than market-oriented economic reforms.

Appendix M

Problem Set 3 - Solution

M.1 Two-period Intertemporal Optimization

1. Given the expression for the utility function:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

   we know that marginal utility is:

$$u'(c) = c^{-\sigma},$$

   while the derivative of marginal utility is:

$$u''(c) = -\sigma c^{-\sigma-1}.$$

   Thus, because $u''(.)$ must be negative for the function to be concave, we have $\sigma > 0$.

2. This is straight from lecture 3.

3. Using the equation from question 2, we can write:

$$\frac{\beta c_1^{-\sigma}}{c_0^{-\sigma}} = \frac{1}{1+r} \qquad \Rightarrow \qquad \frac{c_1}{c_0} = \beta^{1/\sigma}(1+r)^{1/\sigma}$$

4. The intertemporal budget constraint is:

$$c_0 + \frac{c_1}{1+r} = f_0 + y_0 + \frac{y_1}{1+r},$$

   and therefore:

$$\left(1 + \beta^{1/\sigma}(1+r)^{1/\sigma-1}\right) c_0 = f_0 + y_0 + \frac{y_1}{1+r}$$

$$\Rightarrow \quad c_0 = \frac{1}{1 + \beta^{1/\sigma}(1+r)^{1/\sigma-1}} \left(f_0 + y_0 + \frac{y_1}{1+r}\right).$$

   which implies:

$$c_1 = \frac{\beta^{1/\sigma}(1+r)^{1/\sigma}}{1 + \beta^{1/\sigma}(1+r)^{1/\sigma-1}} \left(f_0 + y_0 + \frac{y_1}{1+r}\right)$$

5. Assume $\sigma = 1/2$. If $r = 1\%$, then according to this Google Spreadsheet, c_0 is equal to \$44,776, and c_1 is equal to \$45,676. If $r = 2\%$, then c_0 is equal to \$44,554 and c_1 is equal to \$46,354. Consumption c_0 thus falls by \$222, approximately -0.5% in percentage terms.

6. Assume $\sigma = 1$. If $r = 1\%$, then according to this Google Spreadsheet, c_0 is equal to \$45,000 and c_1 is \$45,450. If $r = 2\%$, then c_0 is equal to \$45,000 and c_1 is equal to \$45,900. Consumption c_0 does not change, this is the case we have seen in class. **Remark.** Note that this case is the one we saw in the class, because when σ approaches 1, we have:

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \log(c).$$

You can see this in many different ways. The simplest way is to write that:

$$c^{1-\sigma} = e^{(1-\sigma)\log(c)} = \exp\left((1-\sigma)\log(c)\right).$$

Then, we use that:

$$\lim_{x \to 0} \frac{e^{ax} - 1}{x} = a$$

Indeed, the limit of $(e^{ax} - 1)/x$ when x goes to 0 is by definition the derivative of e^{ax} at $x = 0$. Thus, since the derivative of e^{ax} is ae^{ax}, we get that the derivative at $x = 0$ of e^{ax} is a. Using that formula for $x = 1 - \sigma$ and $a = \log(c)$ allows to show:

$$\lim_{(1-\sigma) \to 0} \frac{e^{\log(c)(1-\sigma)} - 1}{1-\sigma} = \log(c)$$

Therefore, we get:

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \log(c).$$

7. Assume $\sigma = 2$. If $r = 1\%$, then according to this Google Spreadsheet, c_0 is equal to \$45,112 and c_1 is equal to \$45,337. If $r = 2\%$, then c_0 is equal to \$45,223 and c_1 is equal to \$45,673. Consumption c_0 increases by \$111, or approximately 0.25%.

8. Whether an increase in real interest rates leads to a fall or an increase in consumption depends on σ, which can be seen on this formula (it is crucial for this that $y_1 = 0$, or that second-period income is zero):

$$c_0 = \frac{1}{1 + \beta^{1/\sigma}(1+r)^{1/\sigma - 1}}\left(f_0 + y_0\right).$$

When $1/\sigma - 1 > 0$, or $\sigma < 1$, an increase in the real interest rate leads to lower consumption today, and more saving. Conversely, when $1/\sigma - 1 < 0$, or $\sigma > 1$, an increase in the real interest rate leads to higher consumption today, and less saving. Finally, when $\sigma = 1$, the interest rate has no effect on current consumption c_0 or saving.

M.2 Another Overlapping Generations model

1. Agents care only about old age consumption, so they save everything, regarless of what the utility function is.

2. Since they save everything, saving is equal to the wage, and thus:

$$S_t = w_t.$$

The wage paid by employers, given that $L = 1$, is:

$$w_t = (1-\alpha)K_t^{\alpha} L^{-\alpha} = (1-\alpha)K_t^{\alpha} = (1-\alpha)Y_t.$$

This implies, in turn, the following law of motion for the capital stock:

$$\Delta K_{t+1} = S_t - \delta K_t = (1-\alpha)Y_t - \delta K_t.$$

3. The corresponding value of the saving rate in the Solow model is:

$$s = 1 - \alpha.$$

4. The Golden rule level of capital accumulation is characterized by a level of the saving rate equal to α. Thus, to be below the Golden Rule level of capital accumulation, the saving rate must be lower than that:

$$1 - \alpha < \alpha.$$

   This, in turn, implies:

$$\alpha > \frac{1}{2}.$$

5. This condition is likely not satisfied, as we saw in Lecture 1. Thus, there is too much saving in this situation.

Appendix N

Problem Set 4 - Solution

N.1 The Solow Model with Exogenous Growth

1. The saving rate is exogenous and equal to s in the Solow growth model, and the depreciation rate is δ. Therefore, the law of motion for capital is:

$$\Delta K_{t+1} = K_{t+1} - K_t = sY_t - \delta K_t.$$

Using the value for Y_t, we get:

$$\boxed{K_{t+1} = sA_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t}.$$

which is a law of motion for K_t: a value for K_{t+1} as a function of K_t and the exogenous parameters in the model.

2. Defining k_t as:

$$k_t \equiv \frac{K_t}{A_t^{1/(1-\alpha)} L_t},$$

as is suggested, we divide both the left-hand side and the right-hand side of the equation by $A_t^{1/(1-\alpha)} L_t$. This gives:

$$\frac{K_{t+1}}{A_t^{1/(1-\alpha)} L_t} = s\frac{A_t K_t^{\alpha} L_t^{1-\alpha}}{A_t^{1/(1-\alpha)} L_t} + (1-\delta)\frac{K_t}{A_t^{1/(1-\alpha)} L_t}$$

We may proceed to a simplification of the first term on the right-hand side by putting the A_t and the $L_t^{1-\alpha}$ from the numerator to the denominator (using that $f/g = 1/(g/f)$):

$$\begin{aligned}
\frac{A_t K_t^{\alpha} L_t^{1-\alpha}}{A_t^{1/(1-\alpha)} L_t} &= \frac{K_t^{\alpha}}{A_t^{1/(1-\alpha)-1} L_t^{1-(1-\alpha)}} \\
&= \frac{K_t^{\alpha}}{A_t^{\alpha/(1-\alpha)} L_t^{\alpha}} \\
\frac{A_t K_t^{\alpha} L_t^{1-\alpha}}{A_t^{1/(1-\alpha)} L_t} &= \left(\frac{K_t}{A_t^{1/(1-\alpha)} L_t}\right)^{\alpha}.
\end{aligned}$$

Thus, replacing out the expression for the first term on the right-hand side allows to write:

$$\frac{K_{t+1}}{A_t^{1/(1-\alpha)} L_t} = s\left(\frac{K_t}{A_t^{1/(1-\alpha)} L_t}\right)^{\alpha} + (1-\delta)\frac{K_t}{A_t^{1/(1-\alpha)} L_t}.$$

And therefore, the right-hand side is now expressed only a function of k_t:

$$\frac{K_{t+1}}{A_t^{1/(1-\alpha)}L_t} = sk_t^\alpha + (1-\delta)k_t.$$

The left-hand side of the equation can also be simplified (we want to express it also only as a function of k_t (or rather, k_{t+1}):

$$\begin{aligned}\frac{K_{t+1}}{A_t^{1/(1-\alpha)}L_t} &= \frac{A_{t+1}^{1/(1-\alpha)}L_{t+1}}{A_t^{1/(1-\alpha)}L_t}\cdot\frac{K_{t+1}}{A_{t+1}^{1/(1-\alpha)}L_{t+1}}\\ \frac{K_{t+1}}{A_t^{1/(1-\alpha)}L_t} &= (1+g)^{1/(1-\alpha)}(1+n)k_{t+1}.\end{aligned}$$

Therefore:

$$(1+g)^{1/(1-\alpha)}(1+n)k_{t+1} = sk_t^\alpha + (1-\delta)k_t.$$

If g and n are small then:

$$(1+g)^{1/(1-\alpha)}(1+n) \approx 1 + \frac{1}{1-\alpha}g + n.$$

Thus:

$$\left(1+\frac{1}{1-\alpha}g+n\right)k_{t+1} \approx sk_t^\alpha + (1-\delta)k_t.$$

A law of motion for k_{t+1} is thus (we use equal signs now, even though it is really an approximation):

$$\boxed{k_{t+1} = \frac{s}{1+g/(1-\alpha)+n}k_t^\alpha + \frac{1-\delta}{1+g/(1-\alpha)+n}k_t}.$$

3. The steady-state is such that:

$$\left(1+\frac{1}{1-\alpha}g+n\right)k^* = s(k^*)^\alpha + (1-\delta)k^*.$$

Therefore:

$$\left(\delta+\frac{1}{1-\alpha}g+n\right)k^* = s(k^*)^\alpha.$$

Finally, this gives k^*:

$$\boxed{k^* = \left(\frac{s}{\delta+g/(1-\alpha)+n}\right)^{\frac{1}{1-\alpha}}}.$$

4. In this exercise, we make intensive use of the following rules on growth rates:

$$\begin{aligned}g_{XY} &= g_X + g_Y\\ g_{X/Y} &= g_X - g_Y\\ g_{X^a} &= ag_X.\end{aligned}$$

On the balanced growth path:

$$\frac{K_t}{A_t^{1/(1-\alpha)}L_t} = k^* \quad\Rightarrow\quad K_t = k^*A_t^{1/(1-\alpha)}L_t.$$

We may apply the rule above on products ($g_{XY} = g_X + g_Y$) to see that the growth rate of K_t is the growth rate of $A_t^{1/(1-\alpha)}$ plus the growth rate of L_t, since k^* is simply a constant which does not grow. In turn, using the rule on "powers" (that is $g_{X^a} = ag_X$, with $a = 1/(1-\alpha)$) we get that the growth rate of $A_t^{1/(1-\alpha)}$ is the growth rate of A_t times $1/(1-\alpha)$. Finally, the growth rate of A_t is g and the growth rate of L_t is n by assumption. Thus, finally:

$$\begin{aligned} g_K &= g_{A^{1/(1-\alpha)}L} \\ &= g_{A^{1/(1-\alpha)}} + g_L \\ &= \frac{1}{1-\alpha} g_A + g_L \\ g_K &= \frac{1}{1-\alpha} g + n \end{aligned}$$

Output is given by:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

Therefore, the rate of growth of output is:

$$\begin{aligned} g_Y &= g + \alpha g_K + (1-\alpha) g_L \\ &= g + \alpha \left(n + \frac{1}{1-\alpha} g \right) + (1-\alpha) n \\ &= g + \alpha n + \frac{\alpha}{1-\alpha} g + (1-\alpha) n \\ &= [\alpha n + (1-\alpha) n] + \left[g + \frac{\alpha}{1-\alpha} g \right] \\ g_Y &= n + \frac{1}{1-\alpha} g. \end{aligned}$$

C_t grows at the same rate as Y_t since $C_t = (1-s)Y_t$, thus:

$$\begin{aligned} g_C &= g_Y \\ g_C &= n + \frac{1}{1-\alpha} g. \end{aligned}$$

The rate of growth of K_t/Y_t is zero since Y_t and k_t grow at the same rate:

$$\begin{aligned} g_{K/Y} &= g_K - g_Y \\ &= \left(\frac{1}{1-\alpha} g + n \right) - \left(\frac{1}{1-\alpha} g + n \right) \\ g_{K/Y} &= 0 \end{aligned}$$

The rate of growth of K_t/L_t is:

$$\begin{aligned} g_{K/L} &= g_K - g_L \\ &= \left(\frac{1}{1-\alpha} g + n \right) - n \\ g_{K/L} &= \frac{1}{1-\alpha} g. \end{aligned}$$

The wage is equal to the marginal product of labor from firms' optimality condition, as in lecture 1:

$$\begin{aligned} w_t &= \frac{\partial Y_t}{\partial L_t} \\ &= (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} \\ w_t &= (1-\alpha) A_t \left(\frac{K_t}{L_t} \right)^{\alpha}. \end{aligned}$$

Thus, the rate of growth of w_t is:

$$\begin{aligned} g_w &= g_A + \alpha g_{K/L} \\ &= g + \frac{\alpha}{1-\alpha} g \\ g_w &= \frac{1}{1-\alpha} g. \end{aligned}$$

The rate of growth of $w_t L_t$ is the sum of that of w and that of L_t thus:

$$\begin{aligned} g_{wL} &= g_w + g_L \\ g_{wL} &= \frac{1}{1-\alpha} g + n. \end{aligned}$$

The marginal product of capital R_t is:

$$\begin{aligned} R_t &= \frac{\partial Y_t}{\partial K_t} \\ &= \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \\ R_t &= \alpha A_t \left(\frac{K_t}{L_t}\right)^{\alpha-1}. \end{aligned}$$

Thus, the rate of growth of the marginal product of capital R_t is:

$$\begin{aligned} g_R &= g_A + (\alpha - 1) g_{K/L} \\ &= g + \frac{\alpha-1}{1-\alpha} g \\ &= g - g \\ g_R &= 0 \end{aligned}$$

The rate of growth of capital income $R_t K_t$ is given by:

$$\begin{aligned} g_{RK} &= g_R + g_K \\ g_{RK} &= n + \frac{g}{1-\alpha}. \end{aligned}$$

Finally, the growth in the labor share $w_t L_t$ and that in the capital share $R_t K_t$ are equal to zero which can be inferred from the fact that they are constant with a Cobb-Douglas production function, or that the growth of $w_t L_t$ and $r_t K_t$ are equal to that of output.

5. Using the expression for y_t allows to write what is called the *intensive form* of the production function:

$$\begin{aligned} y_t &= \frac{Y_t}{A_t^{1/(1-\alpha)} L_t} \\ &= \left(\frac{K_t}{A_t^{1/(1-\alpha)} L_t}\right)^{\alpha} \\ y_t &= k_t^{\alpha}. \end{aligned}$$

This implies that the relationship applies also to the steady state, and allows us to calculate steady-state y^* corresponding to steady-state k^*:

$$y^* = (k^*)^{\alpha}$$

From question 3, we replace out k^* in the equation above which gives directly:

$$\boxed{y^* = \left(\frac{s}{\delta + g/(1-\alpha) + n}\right)^{\frac{\alpha}{1-\alpha}}}.$$

Finally, using that $C_t = (1-s) Y_t$ and dividing on both sides by $A_t^{1/(1-\alpha)} L_t$ gives:

$$\frac{C_t}{A_t^{1/(1-\alpha)} L_t} = (1-s) \frac{Y_t}{A_t^{1/(1-\alpha)} L_t}.$$

Using the given definitions for c_t and y_t, this implies that:

$$c_t = (1-s) y_t.$$

From this we can see that this relationship applies also to steady states so that:

$$c^* = (1-s)y^*.$$

Thus, replacing y^* by its expression from previously:

$$\boxed{c^* = (1-s)\left(\frac{s}{\delta + g/(1-\alpha) + n}\right)^{\frac{\alpha}{1-\alpha}}}.$$

6. Just as in the Solow growth model of lecture 2, we see that we have a constant times a function of s, which simplifies the maximization problem a lot:

$$c^* = \frac{1}{(\delta + g/(1-\alpha) + n)^{\frac{\alpha}{1-\alpha}}}(1-s)s^{\frac{\alpha}{1-\alpha}}$$

We are thus led to maximize only the part which depends on s (if you are not convinced, you can leave the constant there, your calculations will just be more complicated!):

$$\max_s \quad (1-s)s^{\frac{\alpha}{1-\alpha}}.$$

Taking the first-order condition as in lecture 2:

$$\begin{aligned} -s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)s^{\frac{\alpha}{1-\alpha}-1} = 0 \quad &\Rightarrow \quad \frac{\alpha}{1-\alpha}\frac{1-s}{s} = 1 \\ \Rightarrow \quad \alpha - \alpha s = s - \alpha s \quad &\Rightarrow \quad \boxed{s = \alpha}. \end{aligned}$$

7. The marginal product of capital R_t is then equal to:

$$\begin{aligned} R_t &= \alpha A_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} \\ &= \alpha \left(\frac{K_t}{A_t^{1/(1-\alpha)} L_t}\right)^{\alpha-1} \\ R_t &= \alpha k_t^{\alpha-1}. \end{aligned}$$

Therefore, in the steady state, using the above expression for k^* (question 3) we get:

$$\begin{aligned} R^* &= \alpha \left[\left(\frac{s}{\delta + g/(1-\alpha) + n}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha-1} \\ &= \alpha \frac{\delta + g/(1-\alpha) + n}{s} \\ &= \delta + g/(1-\alpha) + n \\ R^* &= \frac{\alpha}{s}\left(\delta + g_Y\right). \end{aligned}$$

where we have used that the rate of growth of output g_Y is given by $g_Y = g/(1-\alpha) + n$ which was proved in an earlier question. Using that $s = \alpha$ at the Golden Rule, we get an expression for the steady-state marginal product of capital:

$$s = \alpha \quad \Rightarrow \quad \boxed{R^* = \delta + g_Y}.$$

Finally, note that the net interest rate at the Golden Rule $R^* - \delta$, which is often denoted by r^* needs to be equal to the rate of growth of output g_Y, to be at the Golden Rule level of capital accumulation:

$$\boxed{r^* = R^* - \delta = g_Y}.$$

We shall encounter this condition again in lecture 10 when we study the sustainability of public debt.

N.2 The Neoclassical Labor Market Model

1. This is straight from lecture 6. Labor demand is:

$$l = A^{1/\alpha}(1-\alpha)^{1/\alpha}\left(\frac{w}{p}\right)^{-1/\alpha}.$$

   Note: If asked about this during an exam, you are required to provide the different steps. And you are not supposed to memorize this formula.

2. See the spreadsheet.

3. Taking logs on both sides leads to:

$$\log(l) = \frac{1}{\alpha}\log A + \frac{1}{\alpha}\log(1-\alpha) - \frac{1}{\alpha}\log\left(\frac{w}{p}\right).$$

   Expressing the log of the real wage as a function of the log of labor demand, since the real wage is on the y-axis:

$$\boxed{\log\left(\frac{w}{p}\right) = [\log A + \log(1-\alpha)] - \alpha\log(l)}.$$

   Therefore, it is clear that the slope of the labor demand curve is given by α. If α is higher, then the labor demand curve is steeper. This result is intuitive: as α is higher, returns to scale become more and more decreasing with respect to labor (if $\alpha = 0$, technology is constant returns in labor in constrast). Therefore, a higher quantity of labor is hired by the firm only if the real wage becomes substantially lower. (in order to "make up for" the decreasing returns) An increase in A clearly shifts the labor demand curve to the **right**: for a given amount of labor hired, a higher productivity implies a higher real wage, both intuitively as well as in the algebra. When the labor demand curve moves to the right, then we move *along the labor supply curve*, towards higher values of employment and higher real wages. A decrease in A, in contrast, shifts the labor demand curve to the **left**, then we move down lower values of employment and lower real wages, along the labor supply curve.

4. Again, this is straight from lecture 6. Labor supply is:

$$l = \frac{1}{B^{1/\epsilon}}\left(\frac{w}{p}\right)^{1/\epsilon}.$$

   Note: If asked about this during an exam, you are required to provide the different steps. And you are not supposed to memorize this formula.

5. See the spreadsheet. However, in order to plot the two curves on the same graphs, it is best to invert these relationship and to express the real wage as a function of labor demand

$$\frac{w}{p} = A(1-\alpha)l^{-\alpha},$$

   and the real wage as a function of labor supply:

$$\frac{w}{p} = Bl^{\epsilon}.$$

   Again, see the second sheet of the spreadsheet for a plot where both labor supply and labor demand appear.

6. The labor supply curve is also a line in a $(\log(l), \log(w/p))$ plane, because we have a linear relationship between the log labor supply and the log real wage:

$$\boxed{\log\left(\frac{w}{p}\right) = \log B + \epsilon\log(l)}.$$

The slope of this supply curve on a log-log graph is given by ϵ. If ϵ is larger, the slope is larger. This is intuitive: if the disutility of labor is more convex, then people dislike more working extra hours, and need to be compensated by a much higher real wage to do it. Clearly, if B increases, the labor supply curve moves to the left: people get more disutility from working, and they need to be compensated by a higher real wage to work the same number of hours. On the contrary, when B decreases, people enjoy working much more, and so employers may pay them a low wage to do so.

7. There are many ways to answer this question. I will provide just two. One is to derive the expressions in lecture 6, using the original versions of labor supply and demand (without logs). The real wage is:

$$\frac{w}{p} = (1-\alpha)^{\frac{\epsilon}{\alpha+\epsilon}} A^{\frac{\epsilon}{\alpha+\epsilon}} B^{\frac{\alpha}{\alpha+\epsilon}}.$$

The level of employment:

$$l = (1-\alpha)^{\frac{1}{\alpha+\epsilon}} A^{\frac{1}{\alpha+\epsilon}} B^{-\frac{1}{\alpha+\epsilon}}$$

Using the spreadsheet, and plugging in the values for $A_1 = 2$, $A_2 = 1.9$, we get:

$$l_1 = 0.9267933073, \qquad \left(\frac{w}{p}\right)_1 = 1.367553862$$
$$l_2 = 0.9179226047, \qquad \left(\frac{w}{p}\right)_2 = 1.303347791$$

The effects of employment of a change in A given by $(A_2 - A_1)/A_1 = -5\%$, are thus a fall in employment and in real wages given by:

$$\frac{l_2 - l_1}{l_1} = -0.96\%, \qquad \frac{\left(\frac{w}{p}\right)_2 - \left(\frac{w}{p}\right)_1}{\left(\frac{w}{p}\right)_1} = -4.69\%.$$

In log changes:

$$\log(l_2) - \log(l_1) = -0.96\%, \qquad \log\left(\frac{w}{p}\right)_2 - \log\left(\frac{w}{p}\right)_1 = -4.81\%.$$

Or we combine the logged versions of these same equations:

$$\log\left(\frac{w}{p}\right) = [\log A + \log(1-\alpha)] - \alpha \log(l).$$

and labor supply:

$$\log\left(\frac{w}{p}\right) = \log B + \epsilon \log(l)$$

we get that:

$$\log B + \epsilon \log(l) = [\log A + \log(1-\alpha)] - \alpha \log(l)$$
$$\Rightarrow \quad \log(l) = \frac{1}{\epsilon+\alpha}[\log A + \log(1-\alpha) - \log B]$$

We may use either the labor demand curve or the labor supply curve to compute the real wage (if everything goes well, they should both give the same answer). We can plug it back in the supply curve:

$$\log\left(\frac{w}{p}\right) = \log B + \frac{\epsilon}{\epsilon+\alpha}[\log A + \log(1-\alpha) - \log B]$$
$$\log\left(\frac{w}{p}\right) = \frac{\alpha}{\epsilon+\alpha}\log B + \frac{\epsilon}{\epsilon+\alpha}\log A + \frac{\epsilon}{\epsilon+\alpha}\log(1-\alpha)$$

Therefore, we get the equilibrium employment:

$$\boxed{\log(l) = \frac{1}{\epsilon+\alpha}[\log A + \log(1-\alpha) - \log B]},$$

as well as the equilibrium real wage:

$$\boxed{\log\left(\frac{w}{p}\right) = \frac{\alpha}{\epsilon+\alpha}\log B + \frac{\epsilon}{\epsilon+\alpha}\log A + \frac{\epsilon}{\epsilon+\alpha}\log(1-\alpha)}.$$

This implies that, following a change in productivity A::

$$\Delta\log(l) = \frac{1}{\epsilon+\alpha}\Delta\log A$$

$$\Delta\log\left(\frac{w}{p}\right) = \frac{\epsilon}{\epsilon+\alpha}\Delta\log A$$

The change in A in log points is:

$$\Delta\log A = \log(A_2) - \log(A_1) = -5.13\%.$$

Therefore, in log changes:

$$\log(l_2) - \log(l_1) = -0.96\%, \qquad \log\left(\frac{w}{p}\right)_2 - \log\left(\frac{w}{p}\right)_1 = -4.81\%.$$

If α is higher, then from the above formula the change in employment and in the real wage is lower:

$$\Delta\log(l) = \frac{1}{\epsilon+\alpha}\Delta\log A$$

$$\Delta\log\left(\frac{w}{p}\right) = \frac{\epsilon}{\epsilon+\alpha}\Delta\log A$$

The economic intuition is that a change in A shifts the labor demand curve, and leads to a movement along the labor supply curve. However, the size of this shock is dampened, the larger the amount of decreasing returns to scale. Graphically, the shift in the labor demand curve from a given change shift along the y-axis is lower when the slope is larger. This is shown on the two figures below, showing the shift in the labor demand curve from a given change in A (along the y-axis), when α is low (low decreasing returns) on the left hand side and when α is high (high decreasing returns) on the right hand side. *Note*: You may play around with the spreadsheet to see what happens when parameters are changed.

8. (Warning! if this idea of an increase in leisure attractiveness seems a bit peculiar to you, it also seems odd to me. But it has been proposed by some economists as an explanation for unemployment, to explain why the real wage did not fall that much during the recession.) Similar calculations on the spreadsheet and using the same formulas:

$$\frac{w}{p} = (1-\alpha)^{\frac{\epsilon}{\alpha+\epsilon}} A^{\frac{\epsilon}{\alpha+\epsilon}} B^{\frac{\alpha}{\alpha+\epsilon}}.$$

The level of employment is:

$$l = (1-\alpha)^{\frac{1}{\alpha+\epsilon}} A^{\frac{1}{\alpha+\epsilon}} B^{-\frac{1}{\alpha+\epsilon}}$$

imply that a 10% increase in B leads to a reduction in employment and an increase in real wages given by:

$$\log(l_2) - \log(l_1) = -1.79\%, \qquad \log\left(\frac{w}{p}\right)_2 - \log\left(\frac{w}{p}\right)_1 = 0.60\%.$$

If ϵ is higher, then the effect on both employment and real wages is smaller in absolute value. Again, this is intuitive: if labor supply is steeper to begin with, then a given increase in B does not lead to as much of a shift in the labor supply curve. Thus, the move along the labor demand curve towards lower employment and higher wages is not as important then. This is shown on the two figures below, where the increase in leisure attractiveness is the same across the two experiments: on the left-hand side, epsilon is low, while on the right hand side, epsilon is high. *Note*: Again, you may play around with the spreadsheet to see what happens when parameters are changed.

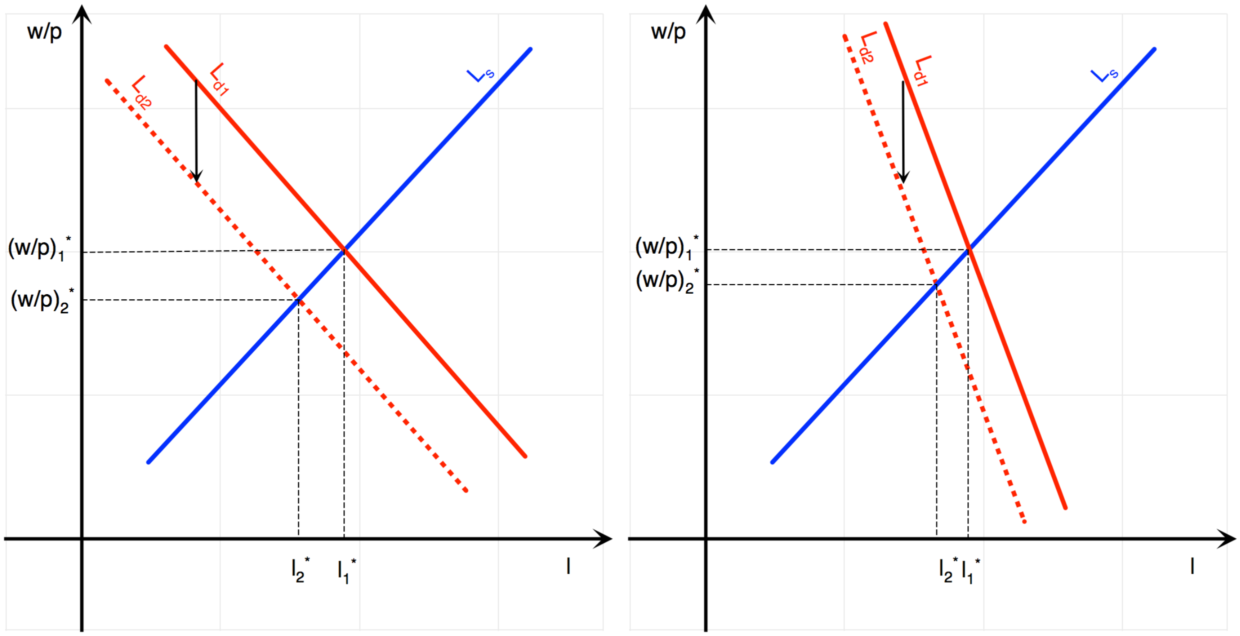

Figure N.1: Labor Market: Productivity Shock.

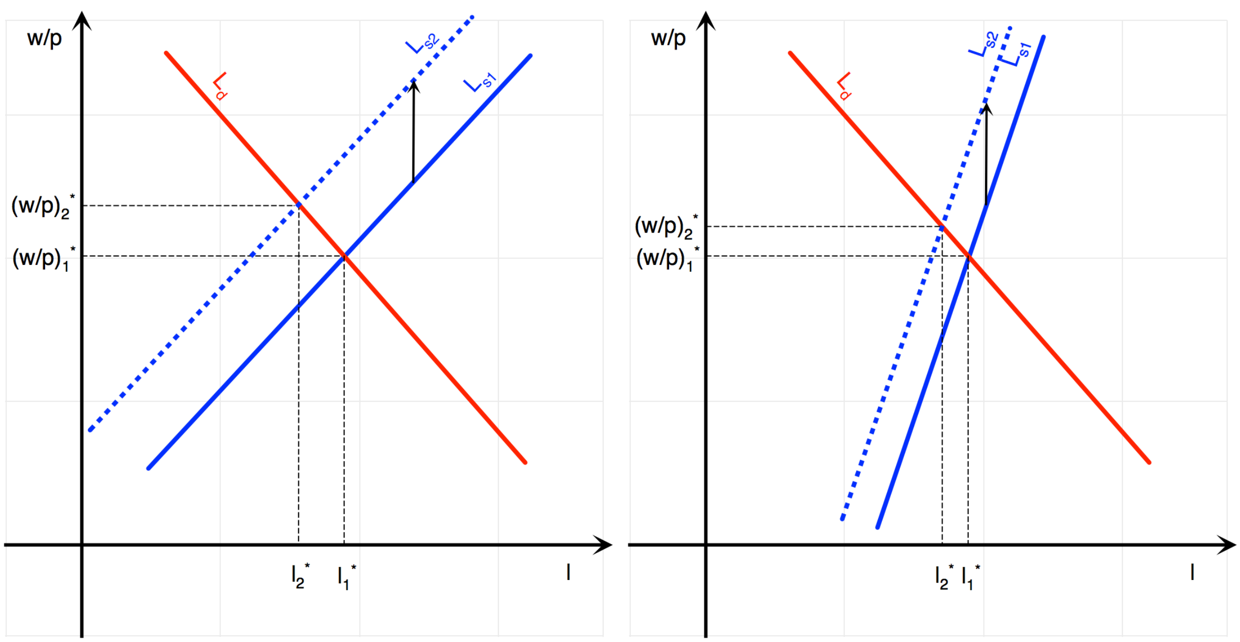

Figure N.2: Labor Market: Laziness Shock.

N.3 The "Keynesian" Labor Market Model

1. One way to answer this question is to note that with a fall in productivity, the labor demand curve will shift to the left (as in lecture 6). If real wages are rigid, then workers are off their labor supply curve (they would like to work more at the current wage) but still on firms' labor demand curve:

   $$\log\left(\frac{w}{p}\right) = [\log A + \log(1-\alpha)] - \alpha \log(l).$$

   For a given change in $\Delta \log A$, the change in employment is therefore simply given by the previous expression through:

   $$\Delta \log(l) = \frac{\Delta \log A}{\alpha}.$$

   In contrast, in the previous case, with flexible wages, the change in employment following a productivity shock was only:

   $$\Delta \log(l) = \frac{\Delta \log A}{\epsilon + \alpha}.$$

   Given a change in productivity of 5%, which goes from 2 to 1.9, or $\log(1.9) - \log(2) = 5.13\%$ in log points, we get a drop of 15.39% in log points in employment:

   $$\log(l_2) - \log(l_1) = -15.39\%.$$

2. In question 7 of the previous exercise, we got in contrast a change:

   $$\log(l_2) - \log(l_1) = -0.96\%,$$

   a much smaller number. The intuition was that the wage falling incentivizes employers to hire more workers. Here, in contrast, because the wage is "too high" and cannot fall by definition, employers do not want to hire them.

3. We know from lecture 6 that if leisure becomes more attractive, then employment must fall and the real wage must rise. We imagine here that wages are sticky upwards. Then, people will be off firms' labor demand curves, but on their labor supply curves. Thus, it is still true that:

   $$\log\left(\frac{w}{p}\right) = \log B + \epsilon \log(l).$$

   For a given change in $\Delta \log B$, given that wages are sticky, the change in employment is therefore simply given by the previous expression through:

   $$\Delta \log(l) = -\frac{\Delta \log B}{\epsilon}.$$

   In contrast, in the previous case, with flexible wages, the change in employment following a B shock was only:

   $$\Delta \log(l) = -\frac{\Delta \log B}{\epsilon + \alpha}.$$

   Given an increase in B of 10%, which goes from 2 to 2.2, or $\log(2.2) - \log(2) = 9.53\%$ in log points, we get a drop of 1.91% in log points in employment:

   $$\log(l_2) - \log(l_1) = -1.91\%.$$

4. In question 8 of the previous exercise, we got in contrast a change:

   $$\log(l_2) - \log(l_1) = -1.79\%,$$

   a somewhat smaller number. The intuition was that the wage increasing would have incentivized workers to work more. But because wages are sticky, this did not happen.

N.4 The Bathtub model

Assume a monthly job separation rate equal to $s = 1\%$, and a monthly job finding rate equal to $f = 20\%$. Assume that the labor force is given by $L = 159$ million.

1. The steady state unemployment rate is obtained by equating separations and job findings in the steady state, so that:
$$sE^* = fU^* \quad \Rightarrow \quad sL - sU^* = fU^*$$
$$\Rightarrow \quad U^* = \frac{s}{s+f}L \quad \Rightarrow \quad u^* = \frac{s}{s+f}.$$
The steady state unemployment rate u^*, number of people unemployed U^*, and number of people losing or finding a job each month, are given by (see the spreadsheet):
$$u^* = 4.76\%, \quad U^* = 7,571,429, \quad fU^* = 1,514,286.$$

2. Again, there are many ways we can proceed here. One is to use the spreadsheet to iterate on the law of motion, that is calculate u_{t+1} as a function of u_t and then see graphically when the given unemployment rate is reached. The law of motion is:
$$U_{t+1} - U_t = s(L - U_t) - fU_t.$$
Thus (see lecture 6):
$$U_{t+1} = sL + (1 - s - f)\, U_t.$$
Dividing both sides by L, and denoting by $u_t = U_t/L$ the *rate* of unemployment:
$$u_{t+1} = s + (1 - s - f)\, u_t.$$
We find that the unemployment rate reaches 5% after approximately **11 months** (the unit of time is one month). A second method is to do a bit of algebra before using the computer. We write the law of motion for unemployment (again, see lecture 6 for details):
$$U_{t+1} - U_t = s(L - U_t) - fU_t$$
$$\Rightarrow \quad U_t - U^* = (1 - s - f)^t\, (U_0 - U^*)\,.$$
Dividing everything by L gives everything in terms of unemployment *rates*:
$$u_t = (1 - s - f)^t\, (u_0 - u^*) + u^*.$$
Thus, starting from an unemployment rate u_0 it is possible to get a value for all subsequent u_t, using the above formula. We may then use the spreadsheet to compute this and find that the unemployment rate reaches 5% after approximately. Again find that the unemployment rate reaches 5% after approximately **11 months** (the unit of time is one month). A third method is to in fact do all the algebra and calculate the time T we are looking for explicitely. We are looking for T such that $u_t \leq \bar{u} = 5\%$ for $t \geq T$. This implies:
$$(1 - s - f)^t\, (u_0 - u^*) + u^* \leq \bar{u}$$
$$\Rightarrow \quad (1 - s - f)^t \leq \frac{\bar{u} - u^*}{u_0 - u^*}$$
$$\Rightarrow \quad t \log(1 - s - f) \leq \log \frac{\bar{u} - u^*}{u_0 - u^*}$$
$$\Rightarrow \quad t \geq \frac{\log \frac{\bar{u} - u^*}{u_0 - u^*}}{\log(1 - s - f)}.$$
(be careful, $\log(1 - s - f)$ is negative because $1 - s - f$ is lower than 1 so you have to change the inequality from $\leq$ to $\geq$). A numerical application using the spreadsheet shows that this condition means:
$$t \geq 11.07.$$

The advantage of this method is we know exactly when the unemployment rate reaches 5%. After 11.07 months ! (given the simplicity of the model, displaying the second digit does not make much sense, though)

3. We are looking for f such that:

$$u^* = \frac{s}{s+f} \quad \Rightarrow \quad f = \frac{s}{u^*} - s,$$

which implies using these numbers that:

$$f = 40\%.$$

This is intuitive: you have double the separation rate, you want double the job finding rate for the unemployment rate to be the same in the steady-state. Indeed, in the steady state:

$$sE^* = fU^*.$$

Therefore, if the unemployment rate is the same, then if s doubles f must double as well.

4. See question 2. The spreadsheet should give all the answers. We find:

$$t \geq 4.79.$$

Thus, the unemployment rate reaches 5% after approximately 5 months.

5. The answer is very much contained in questions 2 and 4. If the rates of job separations and job finding are higher like they typically are in America, the unemployment rate reaches its steady-state value faster. This may explain why the United States are able to recover faster from shocks than, say, Spain or Italy (at least in terms of unemployment rates). Here is some supporting evidence that the unemployment rate is more persistent in Europe than in America:
https://db.nomics.world/OECD/EO/USA.UNR.Q
https://db.nomics.world/OECD/EO/ESP.UNR.Q
https://db.nomics.world/OECD/EO/ITA.UNR.Q

Appendix O

Problem Set 5 - Solution

O.1 Gregory N. Mankiw - NYT - Nov 30, 2008

1. According to Gregory N. Mankiw, the factors contributing to hold back consumption are low consumer confidence and "wait and see" behavior caused by falling house price values, shrinking 401(k) balances (due to the fall of the stock market, my addition) and increased unemployment. Yes, these factors can be interpreted in terms of a fall in c_0, since they reduce consumption for a given level of income Y.

2. Gregory N. Mankiw writes: "Keynesian theory suggests a"paradox of thrift." If all households try to save more, a short-run result could be lower aggregate demand and thus lower national income. Reduced incomes, in turn, could prevent households from reaching their new saving goals." This is exactly the type of phenomenon which we explained in lecture 8: if there is a fall in c_0, then output falls and saving falls as a result. Because investment is fixed and equal to $\bar{I}$, output decreases until saving equals investment again.

3. The neoclassical comment of the article is: "In normal times, a fall in consumption could be met by an increase in investment, which includes spending by businesses on plant and equipment and by households on new homes." This is the usual logic of the Solow (1956) model, for instance. What happens in this model is that saving in fact determines investment entirely. The way this happens through market mechanisms is that the interest rate falls to equate demand and supply of capital. The cost of capital falls down to the point where firms and households want to invest enough to make productive use of all this saving. This logic is somewhat contradictory with the "paradox of thrift" logic, according to which investment is fixed or even increasing with demand (and the interest rate does not clear markets).

4. Gregory N. Mankiw is very concerned about "the long-term fiscal picture. Increased government spending may be a good short-run fix, but it would add to the budget deficit. The baby boomers are now starting to retire and claim Social Security and Medicare benefits. Any increase in the national debt will make fulfilling those unfunded promises harder in coming years." We will talk about this potential issue during lecture 10.

O.2 Procyclical Government Spending

1. We write that Output = Demand:

$$\begin{aligned} Y = Z &= C + \bar{I} + G \\ Y &= c_0 + c_1 (Y - T) + \bar{I} + g_0 + g_1 Y \\ Y &= \left(c_0 - c_1 T + g_0 + \bar{I}\right) + (c_1 + g_1) Y \\ \Rightarrow &\quad \boxed{Y = \frac{1}{1 - c_1 - g_1}\left(c_0 - c_1 T + g_0 + \bar{I}\right)}. \end{aligned}$$

2. The tax multiplier is found by computing the change ΔY in output corresponding to a given change ΔT in taxes:

$$\Delta Y = -\frac{c_1}{1 - c_1 - g_1}\Delta T$$

Therefore, if $\Delta T = -\$1$, the change in output is $\frac{c_1}{1-c_1-g_1}$. Therefore:

$$\boxed{\text{Tax Multiplier} = \frac{c_1}{1 - c_1 - g_1}}.$$

The multiplier is higher in this economy than when government spending does not depend on GDP since:

$$\frac{c_1}{1 - c_1 - g_1} > \frac{c_1}{1 - c_1}.$$

The intuition is that government spending automatically increases when GDP increases, which increases demand further. Thus, the multiplier is higher.

3. This policy appears to be the opposite of what automatic stabilizers are doing, which is to stabilize the economy. The government is having a very procyclical policy, which means that when things go well, it is spending more and therefore helping things go even better; while when things go wrong, it is making it worse by cutting spending. This is clearly not a good policy ! Note however that this is the policy you end up having if you follow a fixed deficit rule, for instance. With a fixed deficit rule, $T - G$ needs to be constant. If T depend on GDP through automatic stabilizers, through $T = t_0 + t_1 Y$ then by construction G needs to respond as well, and it needs to be that $g_1 = t_1$.

4. The (ZZ) curve in this problem has a slope equal to $c_1 + g_1$. The impulse to autonomous spending is given by $\$c_1$, since one dollar of decreased taxes leads to an increase in consumption equal c_1. This increase leads to a second round of increased consumption and investment $c_1(c_1 + g_1)$, and so on:

$$\begin{aligned} \text{Tax Multiplier} &= c_1 + c_1(c_1 + g_1) + c_1(c_1 + g_1)^2 + ... + c_1(c_1 + g_1)^n + ... \\ &= c_1 \left(1 + (c_1 + g_1) + (c_1 + g_1)^2 + ... + (c_1 + g_1)^n + ...\right) \\ \text{Tax Multiplier} &= c_1 \sum_{i=0}^{+\infty} (c_1 + g_1)^i = \frac{c_1}{1 - c_1 - g_1} \end{aligned}$$

A graphical justification for the multiplier is given below. The initial impulse is given by c_1. A fraction $c_1 + g_1$ of the additional income injected in the economy is being consumed, so that adds $c_1(c_1 + g_1)$. This repeats itself in a loop, and all of these effects are summed up, so that the total effect is:

$$c_1 + c_1(c_1 + g_1)$$

5. If $g_1 + c_1 > 1$, then each new round of spending leads to an even greater new round of new income and new spending. Therefore, the above infinite sum is then infinite, and the tax multiplier is infinite:

$$\text{Tax Multiplier} = c_1 \sum_{i=0}^{+\infty} (c_1 + g_1)^i = +\infty.$$

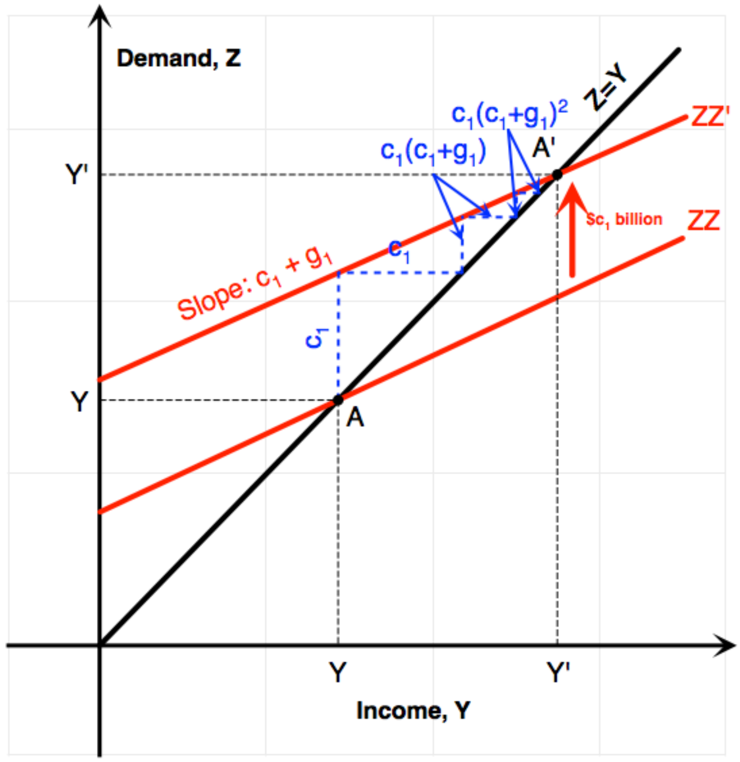

Figure O.1: GRAPHICAL INTERPRETATION FOR THE MULTIPLIER.

Of course, this is not possible. Therefore, this just means that output cannot be demand-determined and that it is always constrained by supply (the amount of inputs there is in the economy).

O.3 Accelerator and Automatic Stabilizer

1. We have both an automatic stabilizer as well as an accelerator effect of investment. We write, again, the total demand for goods Z and then equate demand to output so that $Y = Z$:

$$
\begin{aligned}
Y = Z = C + \bar{I} + G \\
Y &= c_0 + c_1 Y - c_1 t_0 - c_1 t_1 Y + b_0 + b_1 Y + G \\
Y &= (c_0 - c_1 t_0 + b_0 + G) + (c_1(1-t_1) + b_1) Y \\
\Rightarrow &\boxed{Y = \frac{1}{1 - b_1 - c_1(1-t_1)} (c_0 - c_1 t_0 + b_0 + G)}.
\end{aligned}
$$

2. The condition is now given by:

$$b_1 + c_1(1 - t_1) < 1.$$

3. If this condition isn't satisfied, then the multiplier becomes infinite: one dollar of stimulus leads to more than 1 dollar in additional spending, etc. This leads to an infinite sum. This does not, of course, mean that GDP becomes infinite. It just means that output cannot be constrained by demand, and that the Keynesian model no longer applies. Such an economy should always be neoclassical, in other words GDP should always be limited by available factors of production (capital and labor).

4. The graph is very similar to the usual one, only that the slope is $b_1 + c_1(1 - t_1)$. (if asked for it in an exam, you need to provide it) The geometric sum is:

$$1 + (b_1 + c_1(1 - t_1)) + (b_1 + c_1(1 - t_1))^2 + ... = \frac{1}{1 - b_1 - c_1(1 - t_1)}.$$

(again, you will need to provide the multiplier intuition when asked for it in an exam)

Appendix P

Problem Set 6 - Solution

P.1 Another Numerical Example

1. For a household, the threshold to be in the top 1% is around **$421,926** according to the Economic Policy Institute. Let me Google that for you: http://bfy.tw/KhF4.

2. Using the given notations (and those of lecture 9), total income is the sum of the top 1% income and that of the bottom 99%:
$$Y = \lambda N \underline{y} + (1-\lambda) N \bar{y}.$$
Since $\bar{y} = \gamma \underline{y}$ we have:
$$Y = \lambda N \underline{y} + (1-\lambda) N \gamma \underline{y}.$$
The total income for the bottom 99% $\underline{Y}$ is given by:
$$\underline{Y} = \lambda N \underline{y}.$$
Therefore, the share of total income captured by the bottom 99% is:
$$\frac{\underline{Y}}{Y} = \frac{\lambda N \underline{y}}{\lambda N \underline{y} + (1-\lambda) N \gamma \underline{y}}$$
$$\frac{\underline{Y}}{Y} = \frac{\lambda}{\lambda + (1-\lambda)\gamma}$$
Denoting by ν the share of income going to the low income:
$$\nu \equiv \frac{\underline{Y}}{Y}$$
$$\nu = \frac{\lambda}{\lambda + (1-\lambda)\gamma}$$
Solving for γ:
$$\frac{\lambda}{\lambda + (1-\lambda)\gamma} = \nu \quad \Rightarrow \quad \lambda = \lambda \cdot \nu + (1-\lambda)\gamma \cdot \nu$$
$$\Rightarrow \quad \lambda \cdot (1-\nu) = \gamma \cdot (1-\lambda) \cdot \nu \quad \Rightarrow \quad \boxed{\gamma = \frac{\lambda}{1-\lambda}\frac{1-\nu}{\nu}}$$
A numerical application is $\nu = 0.8$ and $\lambda = 0.99$ so that:
$$\gamma = \frac{0.99}{1-0.99}\frac{1-0.8}{0.8}$$
$$= \frac{99}{4}$$
$$\gamma = 24.75$$

This implies that on average, high income earners in the top 1% are approximately **25 times richer** (exactly 24.75 times richer) than low income earners in the bottom 99% (note that you can use the above formula to recover the $\gamma = 9$ from the class, using $\lambda = 0.9$ and $\nu = 0.5$ since $\gamma = 0.9/0.1 \cdot 0.5/0.5 = 9$).

3. Total consumption by the low income earners $\underline{C}$ is such that:

$$\begin{aligned}
\underline{C} &= \lambda N \underline{c} \\
&= \lambda N \left(\underline{c}_0 + \underline{c}_1(\underline{y} - \underline{t})\right) \\
&= \lambda N \underline{c}_0 + \lambda N(1-t_1)\underline{c}_1 \underline{y} - \lambda N \underline{c}_1 \underline{t}_0 \\
\underline{C} &= [\lambda N \underline{c}_0 - \lambda N \underline{c}_1 \underline{t}_0] + \frac{\lambda \underline{c}_1}{\lambda + (1-\lambda)\gamma}(1-t_1)Y
\end{aligned}$$

Symmetrically, consumption by the high income earners $\bar{C}$ is such that:

$$\begin{aligned}
\bar{C} &= (1-\lambda)N\bar{c} \\
&= (1-\lambda)N\left(\bar{c}_0 + \bar{c}_1(\bar{y} - \bar{t})\right) \\
&= (1-\lambda)N\bar{c}_0 + (1-\lambda)N(1-t_1)\bar{c}_1\bar{y} - (1-\lambda)N\bar{c}_1\bar{t}_0 \\
\bar{C} &= \left[(1-\lambda)N\bar{c}_0 - (1-\lambda)N\bar{c}_1\bar{t}_0\right] + \frac{(1-\lambda)\gamma\bar{c}_1}{\lambda + (1-\lambda)\gamma}(1-t_1)Y
\end{aligned}$$

Therefore, aggregate consumption $C = \underline{C} + \bar{C}$ is given by:

$$\begin{aligned}
C &= \underline{C} + \bar{C} \\
&= [\lambda N \underline{c}_0 - \lambda N \underline{c}_1 \underline{t}_0] + \frac{\lambda \underline{c}_1}{\lambda + (1-\lambda)\gamma}(1-t_1)Y + \left[(1-\lambda)N\bar{c}_0 - (1-\lambda)N\bar{c}_1\bar{t}_0\right] + \frac{(1-\lambda)\gamma\bar{c}_1}{\lambda + (1-\lambda)\gamma}(1-t_1)Y \\
&= (\lambda N \underline{c}_0 + (1-\lambda)N\bar{c}_0) - \left(\lambda N \underline{c}_1 \underline{t}_0 + (1-\lambda)N\bar{c}_1\bar{t}_0\right) + \frac{\lambda \underline{c}_1 + (1-\lambda)\gamma\bar{c}_1}{\lambda + (1-\lambda)\gamma}(1-t_1)Y \\
&= [\lambda N \underline{c}_0 + (1-\lambda)N\bar{c}_0] - \left[\underline{c}_1(\lambda N \underline{t}_0) + \bar{c}_1((1-\lambda)N\bar{t}_0)\right] + \frac{\lambda \underline{c}_1 + (1-\lambda)\gamma\bar{c}_1}{\lambda + (1-\lambda)\gamma}(1-t_1)Y \\
C &= C_0 - \left(\underline{c}_1 \underline{T}_0 + \bar{c}_1 \bar{T}_0\right) + c_1(1-t_1)Y.
\end{aligned}$$

where we have used the suggested notations:

$$\begin{aligned}
C_0 &\equiv \lambda N \underline{c}_0 + (1-\lambda)N\bar{c}_0 \\
\underline{T}_0 &\equiv \lambda N \underline{t}_0 \\
\bar{T}_0 &\equiv (1-\lambda)N\bar{t}_0 \\
c_1 &\equiv \frac{\lambda \underline{c}_1 + (1-\lambda)\gamma\bar{c}_1}{\lambda + (1-\lambda)\gamma}.
\end{aligned}$$

Therefore, aggregate consumption is given by:

$$\boxed{C = C_0 - \left(\underline{c}_1 \underline{T}_0 + \bar{c}_1 \bar{T}_0\right) + c_1(1-t_1)Y}.$$

4. c_1 is the average marginal propensity to consume, where the marginal propensity to consume of each group $\underline{c}_1$ and $\bar{c}_1$ is weighted by their share of income in the population $\underline{Y}/Y$ and $\bar{Y}/Y$:

$$\begin{aligned}
c_1 &= \frac{\lambda \underline{c}_1 + (1-\lambda)\gamma\bar{c}_1}{\lambda + (1-\lambda)\gamma} \\
&= \frac{\lambda}{\lambda + (1-\lambda)\gamma}\underline{c}_1 + \frac{(1-\lambda)\gamma}{\lambda + (1-\lambda)\gamma}\bar{c}_1 \\
c_1 &= \frac{\underline{Y}}{Y}\underline{c}_1 + \frac{\bar{Y}}{Y}\bar{c}_1
\end{aligned}$$

This has a straightforward economic interpretation: for each additional dollar of output, a fraction $\underline{Y}/Y$ goes to low income earners who consume a fraction $\underline{c}_1$, and a fraction $\bar{Y}/Y$ goes to high income earners who consume a fraction $\bar{c}_1$. The average propensity to consume is the sum of these two fractions. We can then simply compute the average marginal propensity to consume when $\underline{c}_1 = 1$ and $\bar{c}_1 = 1/4$:

$$\begin{aligned} c_1 &= \frac{\underline{Y}}{Y}\underline{c}_1 + \frac{\bar{Y}}{Y}\bar{c}_1 \\ &= \frac{4}{5} \cdot 1 + \frac{1}{5} \cdot \frac{1}{4} \\ c_1 &= \frac{17}{20} \end{aligned}$$

Therefore the average propensity to consume is:

$$\boxed{c_1 = 0.85}.$$

5. Using the expression for aggregate consumption C in question 3., and that $I = b_0 + b_1 Y$, and plugging it into total aggregate demand Z yields:

$$\begin{aligned} Z &= C + I + G \\ &= C_0 - \left(\underline{c}_1 \underline{T}_0 + \bar{c}_1 \bar{T}_0\right) + c_1(1-t_1)Y + b_0 + b_1 Y + G \\ Z &= \left[C_0 - \left(\underline{c}_1 \underline{T}_0 + \bar{c}_1 \bar{T}_0\right) + b_0 + G\right] + \left(c_1(1-t_1) + b_1\right) Y \end{aligned}$$

Equating aggregate demand to aggregate income $Z = Y$ gives the value for output (see the lecture notes for details):

$$\boxed{Y = \frac{1}{1 - (1-t_1)\, c_1 - b_1} \left[C_0 - \underline{c}_1 \underline{T}_0 - \bar{c}_1 \bar{T}_0 + b_0 + G\right]}$$

6. As in lecture 9, a 100 billion dollars tax cut on the top 1% $\Delta \bar{T}_0 = -100$ leads to an increase in GDP given by:

$$\begin{aligned} \Delta Y &= \frac{-\bar{c}_1 \Delta \bar{T}_0}{1 - c_1(1-t_1) - b_1} \\ &= \frac{-1/4 * (-100 \text{ billion})}{1 - 0.85 \cdot 0.75 - 1/6} \\ \Delta Y &\approx 127.6 \text{ billion.} \end{aligned}$$

Thus, according to these numbers, we get a **127.6 billion dollars** increase in GDP. The impact on the government surplus is given by:

$$\begin{aligned} \Delta (T - G) &= \Delta T \\ &= \Delta \bar{T}_0 + \Delta \underline{T}_0 + t_1 \Delta Y \\ &= \Delta \bar{T}_0 + t_1 \Delta Y \\ &\approx -100 + \frac{1}{4} \cdot 127.6 \\ \Delta (T - G) &\approx -68.1 \text{ billion} \end{aligned}$$

Thus, we get a **68.1 billion dollars** increase in the government deficit.

7. A 100 billion dollars tax cut on the bottom 99% $\Delta \underline{T}_0 = -100$ leads to an increase in GDP given by:

$$\begin{aligned} \Delta Y &= \frac{-\underline{c}_1 \Delta \underline{T}_0}{1 - c_1(1-t_1) - b_1} \\ &= \frac{-1 * (-100 \text{ billion})}{1 - 0.85 \cdot 0.75 - 1/6} \\ \Delta Y &\approx 510.6 \text{ billion.} \end{aligned}$$

Thus, according to these numbers, we get a **510.6 billion dollars** increase in GDP. The impact on the government surplus is given by:

$$\begin{aligned}
\Delta(T-G) &= \Delta T \\
&= \Delta\bar{T}_0 + \Delta\underline{T}_0 + t_1\Delta Y \\
&= \Delta\underline{T}_0 + t_1\Delta Y \\
&\approx -100 + \frac{1}{4}\cdot 510.6 \\
\Delta(T-G) &\approx 27.6 \text{ billion}
\end{aligned}$$

Thus, despite the 100 billion dollars tax cut, we get a **27.6 billion dollars** increase in the government surplus, or a reduction in the government deficit. In this situation, tax cuts **more than pay for themselves**. This seems like a much better policy than the tax reduction on the rich. However, we will see in the next lectures that things are not so straightforward.

8. Finally, a 100 billion dollars tax cut on the bottom 99% $\Delta\underline{T}_0 = -100$ financed by a 100 billion tax increase on the top 1% $\Delta\bar{T}_0 = 100$ leads to an increase in GDP given by:

$$\begin{aligned}
\Delta Y &= \frac{(\underline{c}_1 - \bar{c}_1)\Delta\bar{T}_0}{1 - c_1(1-t_1) - b_1} \\
&= \frac{(1-1/4)*(100 \text{ billion})}{1 - 0.85\cdot 0.75 - 1/6} \\
\Delta Y &\approx 383.0 \text{ billion}
\end{aligned}$$

Thus, according to these numbers, we get a **383.0 billion dollars** increase in GDP. The impact on the government surplus is given by:

$$\begin{aligned}
\Delta(T-G) &= \Delta T \\
&= \Delta\bar{T}_0 + \Delta\underline{T}_0 + t_1\Delta Y \\
&= t_1\Delta Y \\
&\approx \frac{1}{4}\cdot 383.0 \\
\Delta(T-G) &\approx 95.7 \text{ billion}
\end{aligned}$$

We get a **95.7 billion dollars** increase in the government surplus, or a reduction in the government deficit.

9. If there are no automatic stabilizers ($t_1 = 0$), then the multiplier apparently becomes infinite since $1 - c_1 - b_1 = 1 - 0.85 - 1/6 < 0$. However, this is impossible, as there are only finite resources in the economy. Therefore, this implies that output is determined by supply constraints (amount of labor, capital, and technology), as in the Solow growth model, and that the Keynesian type of analysis no longer applies.
In fact, as I said during the lecture, this result is intuitive. If $1 - c_1 < b_1$, then this implies that the marginal propensity to invest b_1 is higher than the marginal propensity to save $1 - c_1$. As a result, we never get to a Keynesian situation of "too much saving".

Appendix Q

Problem Set 7 - Solution

Q.1 Another overlapping-generations model with government debt

1. We could use the expressions derived in lecture 3 for a 2-period consumption problem, since utility is logarithmic with $\beta = 2$. However, we will instead derive the formulas from scratch, using the same techniques we used in that lecture (and so should you during an exam). The problem we are looking to solve is the following:

$$\max_{c_t^y, c_{t+1}^o} \quad \log(c_t^y) + 2\log(c_{t+1}^o)$$
$$\text{s.t} \quad c_t^y + \frac{c_{t+1}^o}{1+r_t} = w_t.$$

   Again, there are many methods through which we could potentially solve this problem. We can just use the ratio os marginal utilities to get the optimality condition:

$$\frac{1/c_t^y}{2/c_{t+1}^o} = 1 + r_t \quad \Rightarrow \quad c_{t+1}^o = 2(1+r_t)c_t^y.$$

   Plugging back in the intertemporal budget constraint:

$$c_t^y + \frac{c_{t+1}^o}{1+r_t} = w_t \quad \Rightarrow \quad c_t^y + 2c_t^y = w_t$$
$$\Rightarrow \quad \boxed{c_t^y = \frac{w_t}{3}}.$$

   And finally, plugging this in the optimality condition:

$$c_{t+1}^o = 2(1+r_t)c_t^y \quad \Rightarrow \quad \boxed{c_{t+1}^o = (1+r_t)\frac{2w_t}{3}}.$$

2. Saving is given by $w_t - c_t^y = 2w_t/3$ and depreciation is $\delta = 1$, and therefore the law of motion of the capital stock is:

$$K_{t+1} - K_t = \frac{2w_t}{3} - K_t.$$

   From firms' optimality condition, the wage is just the marginal product of labor:

$$w_t = \frac{\partial Y_t}{\partial L_t} = \frac{2}{3}K_t^{1/3}L_t^{-1/3} = \frac{2}{3}K_t^{1/3}.$$

Therefore, we get finally the following law of motion for capital:

$$K_{t+1} = \frac{4}{9} K_t^{1/3}.$$

3. The steady-state capital stock K^* is such that:

$$K^* = \frac{4}{9}(K^*)^{1/3} \quad \Rightarrow \quad (K^*)^{2/3} = \frac{4}{9}$$
$$\Rightarrow \quad K^* = \left(\frac{4}{9}\right)^{3/2} \quad \Rightarrow \quad K^* = \left(\frac{2}{3}\right)^3 \quad \Rightarrow \quad \boxed{K^* = \frac{8}{27}}.$$

The (net) steady-state real interest rate r^* is given by the fact that the marginal product of capital is equal to $r^* + \delta$, which is $1 + r^*$ here:

$$1 + r^* = \frac{1}{3}(K^*)^{-2/3} \quad \Rightarrow \quad r^* = \frac{1}{3}\frac{1}{(K^*)^{2/3}} - 1$$

Thus, using the above expression that $(K^*)^{2/3} = \frac{4}{9}$, we get:

$$r^* = \frac{1}{3} \cdot \frac{9}{4} - 1 \quad \Rightarrow \quad \boxed{r^* = -25\%}$$

Comment: you may think that this interest rate is counterfactually way too low. However, remember that one period is a generation here. (that is, the age at which people have a child on average, about 30 years) The corresponding annual interest rate is thus:

$$(1 + r^*)^{1/30} - 1 \approx -0.95\%.$$

Steady-state output Y^* is given by the production function:

$$Y^* = (K^*)^{1/3} = \left(\frac{8}{27}\right)^{1/3} \quad \Rightarrow \quad \boxed{Y^* = \frac{2}{3}}$$

the steady-state wage w^* is given by the marginal product of labor evaluated at the steady-state capital stock:

$$w^* = \frac{2}{3}(K^*)^{1/3} = \frac{2}{3}Y^* \quad \Rightarrow \quad \boxed{w^* = \frac{4}{9}}.$$

The steady-state consumption of the young is:

$$(c^y)^* = \frac{1}{3}w^* \quad \Rightarrow \quad \boxed{(c^y)^* = \frac{4}{27}}.$$

The steady-state consumption of the old is:

$$(c^o)^* = (1 + r^*)\frac{2w^*}{3} = \frac{3}{4} \cdot \frac{8}{27} = \frac{3}{9} \quad \Rightarrow \quad \boxed{(c^o)^* = \frac{2}{9}}$$

4. The Golden Rule (net) interest rate r_g^* is given by $r_g^* = 0$, again because there is no growth. Using this and plugging back to get an expression for $(K^*)_g$:

$$1 + r_g^* = \frac{1}{3}(K_g^*)^{-2/3} \quad \Rightarrow \quad K_g^* = \frac{1}{3^{3/2}} \quad \Rightarrow \quad \boxed{K_g^* = \frac{1}{3\sqrt{3}}}$$

The Golden Rule output Y_g^* is then given by the production function:

$$Y_g^* = (K_g^*)^{1/3} = \left(\frac{1}{3^{3/2}}\right)^{1/3} = \frac{1}{3^{1/2}} \quad \Rightarrow \quad \boxed{Y_g^* = \frac{1}{\sqrt{3}}}$$

The Golden Rule wage w_g^* is:

$$w_g^* = \frac{2}{3}(K_g^*)^{1/3} = \frac{2}{3}Y_g^* \quad \Rightarrow \quad \boxed{w_g^* = \frac{2}{3\sqrt{3}}}.$$

The Golden Rule consumption of the young is:

$$(c^y)_g^* = \frac{1}{3}w_g^* \quad \Rightarrow \quad \boxed{(c^y)_g^* = \frac{2}{9\sqrt{3}}}.$$

The Golden Rule consumption of the old is:

$$(c^o)_g^* = \frac{2}{3}w_g^* \quad \Rightarrow \quad \boxed{(c^o)_g^* = \frac{4}{9\sqrt{3}}}.$$

5. We have the following inequalities:

$$\begin{aligned}
&r_g^* > r^* \\
&K_g^* < K^* \\
&Y_g^* < Y^* \\
&w_g^* < w^* \\
&(c^y)_g^* < (c^y)^* \quad \text{since} \quad \frac{2}{9\sqrt{3}} < \frac{4}{27} \quad \Leftrightarrow \quad \sqrt{3} < 2 \\
&(c^o)_g^* > (c^o)^* \quad \text{since} \quad \frac{4}{9\sqrt{3}} > \frac{2}{9} \quad \Leftrightarrow \quad 2 > \sqrt{3}
\end{aligned}$$

The economic intuition is that the capital stock is at a lower steady-state under the Golden-Rule, so that the marginal product of capital is lower, output is lower, and the wage is lower. For consumption, it is higher when old under the Golden Rule (because the return is higher, which more than compensates for the lowest wage) and lower when young because the wage is lower. Overall, the Golden-Rule steady-state utility is given by:

$$U_g^* = \log(c^y)_g^* + 2\log(c^o)_g^* = \log\frac{2}{9\sqrt{3}} + 2\log\frac{4}{9\sqrt{3}}$$

While the steady-state utility is:

$$U^* = \log(c^y)^* + 2\log(c^o)^* = \log\frac{4}{27} + 2\log\frac{2}{9}$$

We compute $U_g^* - U^*$ to see which steady-state utility is greater:

$$\begin{aligned}
U_g^* - U^* &= \log\frac{2}{9\sqrt{3}} + 2\log\frac{4}{9\sqrt{3}} - \log\frac{4}{27} - 2\log\frac{2}{9} \\
&= \log\left[\frac{2}{9\sqrt{3}} \cdot \left(\frac{4}{9\sqrt{3}}\right)^2 \cdot \frac{27}{4} \cdot \left(\frac{9}{2}\right)^2\right] \\
&= \log\left[\frac{2 \cdot 4^2 \cdot 27 \cdot 9^2}{9\sqrt{3} \cdot 9^2 \cdot 3 \cdot 4 \cdot 4}\right] \\
U_g^* - U^* &= \log\frac{2}{\sqrt{3}} > \log\frac{2}{\sqrt{4}} = \log 1 = 0
\end{aligned}$$

Thus, we conclude that the Golden Rule level of steady-state utility is higher than the equilibrium level of steady-state utility since:

$$U_g^* - U^* > 0 \quad \Rightarrow \quad \boxed{U_g^* > U^*}.$$

The economic intuition for these results is as follows. First, the steady-state level of the interest rate was $r^* = -25$ % - so, again, around -1% per year - which as we know now is below the Golden Rule interest rate, which is 0, in an economy which has zero growth in the long run. Moreover, we also know that the gross interest rate $R^* = 1 + r^*$ is decreasing in the quantity of capital, because of decreasing returns to capital. Therefore, if the net interest rate is higher at the Golden Rule, then necessarily the capital stock is smaller. But if the capital stock is smaller then by virtue of $Y = K^{1/3}$ we know that output is also smaller. For the same reason, we know that the wage, which is a fraction 2/3 of output, is also smaller.

6. As in lecture 10, we know that the level of government debt B_g^* which brings the capital stock to the Golden Rule level is that which implies that when savers save 2/3 of their young age wage, they are able to exactly buy the quantity of capital corresponding to the Golden Rule level, as well as that public debt. Thus, we get an equation that B_g^* must satisfy (note the difference with the lecture notes: this time not all the wage is saved, but only a fraction 2/3):

$$B_g^* + K_g^* = \frac{2}{3} w_g^* \quad \Rightarrow \quad B_g^* = \frac{2}{3} w_g^* - K_g^*.$$

Substituting:

$$\begin{aligned} B_g^* &= \frac{2}{3} w_g^* - K_g^* \\ &= \frac{2}{3} \cdot \frac{2}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} \\ &= \left(\frac{4}{3} - 1\right) \cdot \frac{1}{3\sqrt{3}} \\ B_g^* &= \frac{1}{9\sqrt{3}}. \end{aligned}$$

Note that this is considerably less government debt than in the lecture notes: this makes sense, as the dynamic inefficiency problem is considerably less severe here.

7. The lucky generation of retirees is able to consume the sum of what that generation would have consumed anyway, since these savings come to fruition when they are old, plus what the government gives them:

$$c_0^o = \frac{2}{9} + \frac{1}{9\sqrt{3}} = \frac{2\sqrt{3} + 1}{9\sqrt{3}}.$$

8. Public debt is often criticized as a "Ponzi scheme", a negative term which is named after Charles Ponzi, who became famous for creating this fraudulent system. (although apparently, he was not really the first) A Ponzi scheme is a form of fraud, through which a money manager pays very big profits to earlier investors (higher than the market interest rate), making them think he is a really good investor, while in fact he is just able to pay these returns by raising funds from new investors. A more recent example is Bernie Madoff (*The Wizard of Lies* is an ok HBO movie where De Niro plays Bernie Madoff). Public debt is also a Ponzi scheme in the overlapping generations model, in the sense that the government in fact never intends to fully repay all investors: it is always raising new money from the young generation, and using this money to reimburse the old one. However, this Ponzi scheme is actually good because it solves the issue of overaccumulation of capital, and allows everyone to consume more.

9. If the government puts in place a pay-as-you-go (PAYG) system, giving retirees an amount B_g^* each period (where B_g^* is the same level of government debt as the one found in question 6), and taxing the young an equal amount B_g^*, then the money which is left for the young to invest in capital is given by $2/3 \cdot w_g^* - B_g^*$, the income after tax, so that the level of capital accumulation is still K_g^*. When old, the government gives them what they would have gotten from investing in government debt had they not been taxes. Therefore, in terms of consumption, capital accumulation, everything "real", it is exactly the same situation. In terms of financial accounting however, the young used to have savings on their books (in the form of government debt), which they do not have anymore: nothing appears on their bank accounts. They have "implicit assets" in the sense that the government owes them whatever they

gave contributed. Symmetrically, it does not look like the government has any debt, while it actually has "implicit liabilities": it owes the young whatever it has been taxing them.

10. Here, there really is no difference between pay-as-you-go financing and deficit financing, except that one appears on the government's books (deficit financing), while the other does not (pay-as-you-go financing). Pay-as-you-go is thus an implicit government liability: instead of having promised to repay creditors, the government has promised to repay future retirees who have put money in the system when they were working. This helps understand why government debt is not a very meaningful statistic: by reclassifying some of its liabilities as pension liabilities, the government is able to reduce its government debt. A forceful proponent of including pay-as-you-go liabilities into government debt numbers is Lawrence Kotlikoff, an economist at Boston University. You can read here his very dismal assessment of the current U.S. fiscal outlook. As you now know, I am not as concerned as he is. As I said during the class, what matters really is the value for real interest rates compared to growth rates, in order to understand whether this debt can actually be rolled over indefinitely or not. Don't hesitate to let me know if you feel strongly in the opposite direction, and find my arguments unpersuasive !

Appendix R

Problem Set 8 - Solution

Appendix S

Problem Set 9 - Solution

Appendix T

Problem Set 10 - Solution