Problem Set 2 - Solutions

UCLA - Econ 102 - Fall 2018

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2 Problem Set 2 - Solution

2.1 Solow Growth Model if $\alpha = 1/2$ and $A = 1/2$

1. We write the law of motion for capital:

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t \\ &= (1-\delta)K_t + sY_t \\ K_{t+1} &= (1-\delta)K_t + \frac{s}{2}\sqrt{K_t}\sqrt{L} \\ \frac{K_{t+1}}{L} &= (1-\delta)\frac{K_t}{L} + \frac{s}{2}\sqrt{\frac{K_t}{L}} \end{aligned}$$

   In steady state, we can then get steady-state capital per worker:

$$\delta\frac{K^*}{L} = \frac{s}{2}\sqrt{\frac{K^*}{L}} \quad \Rightarrow \quad \boxed{\frac{K^*}{L} = \left(\frac{s}{2\delta}\right)^2}.$$

   This implies an expression for steady-state output per worker:

$$\frac{Y^*}{L} = \frac{1}{2}\sqrt{\frac{K^*}{L}} \quad \Rightarrow \quad \boxed{\frac{Y^*}{L} = \frac{s}{4\delta}}.$$

2. From the previous expression for steady-state output per worker, we can get steady-state consumption per worker:

$$\frac{C^*}{L} = (1-s)\frac{Y^*}{L} \quad \Rightarrow \quad \boxed{\frac{C^*}{L} = \frac{s(1-s)}{4\delta}}.$$

3. I used Google Sheets, and the result is available here. The intuition for why steady-state output per worker is a monotone function of the saving rate is that more investment always leads to a higher capital stock, which leads to higher output per worker. However, the effect of saving on steady-state consumption is ambiguous. It should be intuitive that if saving is equal to 0%, or 100%, consumption per worker is zero: in the first case, because there is no capital and therefore no production; in the second case, because everything is saved and there is nothing left to consumption. Thus, there is a limit to how much capital should be accumulated, at least for consumption purposes.

s	Y/L	C/L
0	0	0
0.1	0.5	0.45
0.2	1	0.8
1	5	0

4. Again, I used Google Sheets, and the result is available here.

5. It should be clear from the Google Sheet that $s = 1$ maximizes output per worker, and that $s = 0.5$

maximizes consumption per worker.

2.2 Solow Growth Model if $\alpha = 1/3$

1. Yes, there are constant returns to scale. When one doubles all inputs, one gets double the output. This is true more generally for any x:

$$F(xK, xL) = (xK)^{\alpha}(xL)^{1-\alpha} = xK^{\alpha}L^{1-\alpha} = xF(K, L).$$

2. Yes, returns are decreasing with respect to capital. The reason is that the derivative of the production function with respect to the capital stock, which is:

$$\frac{\partial F}{\partial K} = \alpha K^{\alpha-1}L^{1-\alpha},$$

is decreasing in the amount of capital (indeed, since $\alpha = 1/3$ we have that the exponent on K is $-2/3$, so that this is a decreasing function). This implies that the "gross returns to capital", defined as the additional production allowed by one additional unit of capital, given by the derivative $\partial F/\partial K$, are decreasing in K. Another way to see this is to note that the production function is concave in the capital stock, as the second derivative is negative:

$$\frac{\partial^2 F}{\partial K^2} = \alpha(\alpha - 1)K^{\alpha-2}L^{1-\alpha} < 0$$

which is just another characterization of decreasing returns. This is because $\alpha - 1 = -2/3$ which is negative. See lecture **??** for more detail.

3. Yes, returns are decreasing with respect to labor, for the same reason as returns are decreasing with respect to capital. The reason is that the derivative of the production function with respect to the amount of labor (number of employees, or number of hours), which is:

$$\frac{\partial F}{\partial L} = (1 - \alpha)K^{\alpha}L^{-\alpha},$$

is decreasing in the amount of labor (indeed, since $\alpha = 1/3$ we have that the exponent on L is $-1/3$, so that this is a decreasing function). This implies that the "returns to labor", defined as the additional production allowed by one additional unit of labor, given by the derivative $\partial F/\partial L$, are decreasing in L. Another way to see this is to note that the production function is concave in the stock of labor, as the second derivative is negative:

$$\frac{\partial^2 F}{\partial L^2} = -(1 - \alpha)\alpha K^{\alpha}L^{-\alpha-1} < 0$$

which is just another characterization of decreasing returns. See Lecture 2 for more detail.

4. Dividing the LHS and the RHS by L:

$$\frac{Y}{L} = \frac{K^{\alpha}L^{1-\alpha}}{L} = \left(\frac{K}{L}\right)^{\alpha} = F\left(\frac{K}{L}, 1\right).$$

Definining the intensive form of the production function by $f(.)$:

$$f(k) \equiv F(k, 1),$$

we can then write:

$$\frac{Y}{L} = f\left(\frac{K}{L}\right).$$

5. Again, we write the lax of motion of the capital stock as:

$$K_{t+1} = (1-\delta)K_t + I_t$$
$$K_{t+1} = (1-\delta)K_t + sK_t^{1/3}L^{2/3}$$

Dividing both sides by L:

$$\frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\left(\frac{K_t}{L}\right)^{1/3}.$$

In steady state,

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L},$$

This implies:

$$\delta\frac{K^*}{L} = s\left(\frac{K^*}{L}\right)^{1/3} \quad \Rightarrow \quad \boxed{\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{3/2}}$$

6. Using that: $Y^* = K^{*1/3}L^{2/3}$, we have:

$$\frac{Y^*}{L} = \left(\frac{K^*}{L}\right)^{1/3} \quad \Rightarrow \quad \boxed{\frac{Y^*}{L} = \sqrt{\frac{s}{\delta}}}.$$

7. A straightforward numerical application gives:

$$\frac{Y^*}{L} = \sqrt{\frac{0.32}{0.08}} = \sqrt{4} = 2$$

8. If the saving rate declines to 16, then:

$$\frac{Y^*}{L} = \sqrt{\frac{s}{\delta}} = \sqrt{\frac{0.16}{0.08}} = \sqrt{2}$$

2.3 An increase in the depreciation rate in the Solow growth model

1. The steady-state level of capital per worker is:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{3/2} = \left(\frac{0.10}{0.10}\right)^{3/2} = 1.$$

2. The steady-state level of output per worker is:

$$\frac{Y^*}{L} = \sqrt{\frac{s}{\delta}} = \sqrt{\frac{0.10}{0.10}} = 1.$$

3. The new steady-state levels of capital per worker and output per worker will be:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{3/2} = \left(\frac{0.10}{0.20}\right)^{3/2} \approx 0.35,$$

$$\frac{Y^*}{L} = \sqrt{\frac{s}{\delta}} = \sqrt{\frac{0.10}{0.20}} \approx 0.71.$$

4. We know the evolution of capital per worker is:

$$\frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\left(\frac{K_t}{L}\right)^{1/3}.$$

Starting from $\frac{K_0}{L} = 1$, with $s = 0.10$, $\delta = 0.20$, we have:

$$\begin{aligned}
\frac{K_1}{L} &= (1-\delta)\frac{K_0}{L} + s\left(\frac{K_0}{L}\right)^{1/3} = 0.9 \\
\frac{K_2}{L} &= (1-\delta)\frac{K_1}{L} + s\left(\frac{K_1}{L}\right)^{1/3} \approx 0.82 \\
\frac{K_3}{L} &= (1-\delta)\frac{K_2}{L} + s\left(\frac{K_2}{L}\right)^{1/3} \approx 0.75
\end{aligned}$$

For more iterations, you may use Google Sheets: the result is available here. You should see that it indeed converges to the above values. From there, we may calculate the path of output per worker:

$$\begin{aligned}
\frac{Y_1}{L} &= \left(\frac{K_1}{L}\right)^{1/3} \approx 0.97 \\
\frac{Y_2}{L} &= \left(\frac{K_2}{L}\right)^{1/3} \approx 0.93 \\
\frac{Y_3}{L} &= \left(\frac{K_3}{L}\right)^{1/3} \approx 0.91.
\end{aligned}$$

For more iterations, you may use Google Sheets: the result is available here.

2.4 Deficits and the capital stock

1. Using the law of motion for the capital stock:

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
&= (1-\delta)K_t + sY_t \\
K_{t+1} &= (1-\delta)K_t + s\sqrt{K_t}\sqrt{L},
\end{aligned}$$

Dividing both sides by L:

$$\frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\sqrt{\frac{K_t}{L}}$$

In the steady state:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L},$$

so we have:

$$\delta\frac{K^*}{L} = s\sqrt{\frac{K^*}{L}}$$

Therefore:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2$$

Using the production function:

$$\frac{Y^*}{L} = \sqrt{\frac{K^*}{L}} = \frac{s}{\delta}.$$

2. The steady-state capital stock per worker is given by:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 = \left(\frac{15\%}{7.5\%}\right)^2 = 4$$

The steady-state output per worker is given by:

$$\frac{Y^*}{L} = \frac{s}{\delta} = \frac{15\%}{7.5\%} = 2.$$

3. The new steady-state capital stock per worker is:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 = \left(\frac{20\%}{7.5\%}\right)^2 \approx 7.11$$

The new steady-state output per worker is:

$$\frac{Y^*}{L} = \frac{s}{\delta} = \frac{20\%}{7.5\%} \approx 2.67.$$

Therefore, both the capital per worker and the output per worker increase.

2.5 U.S. saving and government deficits

1. According to http://data.worldbank.org/indicator/NY.GDS.TOTL.ZS, the national saving rate was approximately 16.9% in 2016. The steady-state capital stock per worker is given by:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 = \left(\frac{16.9\%}{7.5\%}\right)^2 \approx 5.08.$$

The steady-state output per worker is:

$$\frac{Y^*}{L} = \frac{s}{\delta} = \frac{16.9\%}{7.5\%} \approx 2.25.$$

2. For fiscal year 2017, the federal fiscal deficit was 3.5% percent of GDP. Assuming that the federal budget deficit was eliminated and there was no change in private saving, the saving rate would change from 16.9% to 16.9%+3.5%=20.4%. The new steady-state capital stock per worker would be:

$$\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^2 = \left(\frac{20.4\%}{7.5\%}\right)^2 \approx 7.40$$

which increases by 45.67%. The new steady-state output per worker would be:

$$\frac{Y}{L} = \frac{s}{\delta} = \frac{20.4\%}{7.5\%} = 2.72,$$

which increases by 20.89%.

3. The saving rate in China was 46.54% in year 2016, which is much higher than the saving rate in the United States. This is perhaps not that surprising according to the Solow model, as China is still in the process of catching up. At the same time, it is not clear that what China was lacking before was capital, rather than market-oriented economic reforms.