Problem Set 7 - Solutions

UCLA - Econ 102 - Fall 2018

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7 Problem Set 7 - Solution

Another overlapping-generations model with government debt

1. We could use the expressions derived in lecture 3 for a 2-period consumption problem, since utility is logarithmic with $\beta = 2$. However, we will instead derive the formulas from scratch, using the same techniques we used in that lecture (and so should you during an exam). The problem we are looking to solve is the following:

$$\max_{c_t^y, c_{t+1}^o} \quad \log(c_t^y) + 2\log(c_{t+1}^o)$$
$$\text{s.t} \quad c_t^y + \frac{c_{t+1}^o}{1+r_t} = w_t.$$

   Again, there are many methods through which we could potentially solve this problem. We can just use the ratio os marginal utilities to get the optimality condition:

$$\frac{1/c_t^y}{2/c_{t+1}^o} = 1 + r_t \quad \Rightarrow \quad c_{t+1}^o = 2(1+r_t)c_t^y.$$

   Plugging back in the intertemporal budget constraint:

$$c_t^y + \frac{c_{t+1}^o}{1+r_t} = w_t \quad \Rightarrow \quad c_t^y + 2c_t^y = w_t$$
$$\Rightarrow \quad \boxed{c_t^y = \frac{w_t}{3}}.$$

   And finally, plugging this in the optimality condition:

$$c_{t+1}^o = 2(1+r_t)c_t^y \quad \Rightarrow \quad \boxed{c_{t+1}^o = (1+r_t)\frac{2w_t}{3}}.$$

2. Saving is given by $w_t - c_t^y = 2w_t/3$ and depreciation is $\delta = 1$, and therefore the law of motion of the capital stock is:

$$K_{t+1} - K_t = \frac{2w_t}{3} - K_t.$$

   From firms' optimality condition, the wage is just the marginal product of labor:

$$w_t = \frac{\partial Y_t}{\partial L_t} = \frac{2}{3}K_t^{1/3}L_t^{-1/3} = \frac{2}{3}K_t^{1/3}.$$

   Therefore, we get finally the following law of motion for capital:

$$K_{t+1} = \frac{4}{9}K_t^{1/3}.$$

3. The steady-state capital stock K^* is such that:

$$K^* = \frac{4}{9}(K^*)^{1/3} \quad \Rightarrow \quad (K^*)^{2/3} = \frac{4}{9}$$
$$\Rightarrow \quad K^* = \left(\frac{4}{9}\right)^{3/2} \quad \Rightarrow \quad K^* = \left(\frac{2}{3}\right)^3 \quad \Rightarrow \quad \boxed{K^* = \frac{8}{27}}.$$

The (net) steady-state real interest rate r^* is given by the fact that the marginal product of capital is equal to $r^* + \delta$, which is $1 + r^*$ here:

$$1 + r^* = \frac{1}{3}(K^*)^{-2/3} \quad \Rightarrow \quad r^* = \frac{1}{3}\frac{1}{(K^*)^{2/3}} - 1$$

Thus, using the above expression that $(K^*)^{2/3} = \frac{4}{9}$, we get:

$$r^* = \frac{1}{3} \cdot \frac{9}{4} - 1 \quad \Rightarrow \quad \boxed{r^* = -25\%}$$

Comment: you may think that this interest rate is counterfactually way too low. However, remember that one period is a generation here. (that is, the age at which people have a child on average, about 30 years) The corresponding annual interest rate is thus:

$$(1 + r^*)^{1/30} - 1 \approx -0.95\%.$$

Steady-state output Y^* is given by the production function:

$$Y^* = (K^*)^{1/3} = \left(\frac{8}{27}\right)^{1/3} \quad \Rightarrow \quad \boxed{Y^* = \frac{2}{3}}$$

the steady-state wage w^* is given by the marginal product of labor evaluated at the steady-state capital stock:

$$w^* = \frac{2}{3}(K^*)^{1/3} = \frac{2}{3}Y^* \quad \Rightarrow \quad \boxed{w^* = \frac{4}{9}}.$$

The steady-state consumption of the young is:

$$(c^y)^* = \frac{1}{3}w^* \quad \Rightarrow \quad \boxed{(c^y)^* = \frac{4}{27}}.$$

The steady-state consumption of the old is:

$$(c^o)^* = (1 + r^*)\frac{2w^*}{3} = \frac{3}{4} \cdot \frac{8}{27} = \frac{3}{9} \quad \Rightarrow \quad \boxed{(c^o)^* = \frac{2}{9}}$$

4. The Golden Rule (net) interest rate r_g^* is given by $r_g^* = 0$, again because there is no growth. Using this and plugging back to get an expression for $(K^*)_g$:

$$1 + r_g^* = \frac{1}{3}(K_g^*)^{-2/3} \quad \Rightarrow \quad K_g^* = \frac{1}{3^{3/2}} \quad \Rightarrow \quad \boxed{K_g^* = \frac{1}{3\sqrt{3}}}$$

The Golden Rule output Y_g^* is then given by the production function:

$$Y_g^* = (K_g^*)^{1/3} = \left(\frac{1}{3^{3/2}}\right)^{1/3} = \frac{1}{3^{1/2}} \quad \Rightarrow \quad \boxed{Y_g^* = \frac{1}{\sqrt{3}}}$$

The Golden Rule wage w_g^* is:

$$w_g^* = \frac{2}{3}(K_g^*)^{1/3} = \frac{2}{3}Y_g^* \quad \Rightarrow \quad \boxed{w_g^* = \frac{2}{3\sqrt{3}}}.$$

The Golden Rule consumption of the young is:

$$(c^y)_g^* = \frac{1}{3}w_g^* \quad \Rightarrow \quad \boxed{(c^y)_g^* = \frac{2}{9\sqrt{3}}}.$$

The Golden Rule consumption of the old is:

$$(c^o)_g^* = \frac{2}{3}w_g^* \quad \Rightarrow \quad \boxed{(c^o)_g^* = \frac{4}{9\sqrt{3}}}.$$

5. We have the following inequalities:

$$
\begin{aligned}
r_g^* &> r^* \\
K_g^* &< K^* \\
Y_g^* &< Y^* \\
w_g^* &< w^* \\
(c^y)_g^* &< (c^y)^* \quad \text{since} \quad \frac{2}{9\sqrt{3}} < \frac{4}{27} \quad \Leftrightarrow \quad \sqrt{3} < 2 \\
(c^o)_g^* &> (c^o)^* \quad \text{since} \quad \frac{4}{9\sqrt{3}} > \frac{2}{9} \quad \Leftrightarrow \quad 2 > \sqrt{3}
\end{aligned}
$$

The economic intuition is that the capital stock is at a lower steady-state under the Golden-Rule, so that the marginal product of capital is higher, output is lower, and the wage is lower. For consumption, it is higher when old under the Golden Rule (because the return is higher, which more than compensates for the lowest wage) and lower when young because the wage is lower. Overall, the Golden-Rule steady-state utility is given by:

$$U_g^* = \log(c^y)_g^* + 2\log(c^o)_g^* = \log\frac{2}{9\sqrt{3}} + 2\log\frac{4}{9\sqrt{3}}$$

While the steady-state utility is:

$$U^* = \log(c^y)^* + 2\log(c^o)^* = \log\frac{4}{27} + 2\log\frac{2}{9}$$

We compute $U_g^* - U^*$ to see which steady-state utility is greater:

$$
\begin{aligned}
U_g^* - U^* &= \log\frac{2}{9\sqrt{3}} + 2\log\frac{4}{9\sqrt{3}} - \log\frac{4}{27} - 2\log\frac{2}{9} \\
&= \log\left[\frac{2}{9\sqrt{3}} \cdot \left(\frac{4}{9\sqrt{3}}\right)^2 \cdot \frac{27}{4} \cdot \left(\frac{9}{2}\right)^2\right] \\
&= \log\left[\frac{2 \cdot 4^2 \cdot 27 \cdot 9^2}{9\sqrt{3} \cdot 9^2 \cdot 3 \cdot 4 \cdot 4}\right] \\
U_g^* - U^* &= \log\frac{2}{\sqrt{3}} > \log\frac{2}{\sqrt{4}} = \log 1 = 0
\end{aligned}
$$

Thus, we conclude that the Golden Rule level of steady-state utility is higher than the equilibrium level of steady-state utility since:

$$U_g^* - U^* > 0 \quad \Rightarrow \quad \boxed{U_g^* > U^*}.$$

The economic intuition for these results is as follows. First, the steady-state level of the interest rate was $r^* = -25$ % - so, again, around -1% per year - which as we know now is below the Golden Rule interest rate, which is 0, in an economy which has zero growth in the long run. Moreover, we also know that the gross interest rate $R^* = 1 + r^*$ is decreasing in the quantity of capital, because of decreasing returns to capital. Therefore, if the net interest rate is higher at the Golden Rule, then necessarily the capital stock is smaller. But if the capital stock is smaller then by virtue of $Y = K^{1/3}$ we know that output is also smaller. For the same reason, we know that the wage, which is a fraction 2/3 of output, is also smaller.

6. As in lecture 10, we know that the level of government debt B_g^* which brings the capital stock to the Golden Rule level is that which implies that when savers save 2/3 of their young age wage, they are able to exactly buy the quantity of capital corresponding to the Golden Rule level, as well as that public debt. Thus, we get an equation that B_g^* must satisfy (note the difference with the lecture notes: this time not all the wage is saved, but only a fraction 2/3):

$$B_g^* + K_g^* = \frac{2}{3}w_g^* \quad \Rightarrow \quad B_g^* = \frac{2}{3}w_g^* - K_g^*.$$

Substituting:

$$\begin{aligned} B_g^* &= \frac{2}{3} w_g^* - K_g^* \\ &= \frac{2}{3} \cdot \frac{2}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} \\ &= \left(\frac{4}{3} - 1 \right) \cdot \frac{1}{3\sqrt{3}} \\ B_g^* &= \frac{1}{9\sqrt{3}}. \end{aligned}$$

Note that this is considerably less government debt than in the lecture notes: this makes sense, as the dynamic inefficiency problem is considerably less severe here.

7. The lucky generation of retirees is able to consume the sum of what that generation would have consumed anyway, since these savings come to fruition when they are old, plus what the government gives them:

$$c_0^o = \frac{2}{9} + \frac{1}{9\sqrt{3}} = \frac{2\sqrt{3}+1}{9\sqrt{3}}.$$

8. Public debt is often criticized as a "Ponzi scheme", a negative term which is named after Charles Ponzi, who became famous for creating this fraudulent system. (although apparently, he was not really the first) A Ponzi scheme is a form of fraud, through which a money manager pays very big profits to earlier investors (higher than the market interest rate), making them think he is a really good investor, while in fact he is just able to pay these returns by raising funds from new investors. A more recent example is Bernie Madoff (*The Wizard of Lies* is an ok HBO movie where De Niro plays Bernie Madoff). Public debt is also a Ponzi scheme in the overlapping generations model, in the sense that the government in fact never intends to fully repay all investors: it is always raising new money from the young generation, and using this money to reimburse the old one. However, this Ponzi scheme is actually good because it solves the issue of overaccumulation of capital, and allows everyone to consume more.

9. If the government puts in place a pay-as-you-go (PAYG) system, giving retirees an amount B_g^* each period (where B_g^* is the same level of government debt as the one found in question 6), and taxing the young an equal amount B_g^*, then the money which is left for the young to invest in capital is given by $2/3 \cdot w_g^* - B_g^*$, the income after tax, so that the level of capital accumulation is still K_g^*. When old, the government gives them what they would have gotten from investing in government debt had they not been taxes. Therefore, in terms of consumption, capital accumulation, everything "real", it is exactly the same situation. In terms of financial accounting however, the young used to have savings on their books (in the form of government debt), which they do not have anymore: nothing appears on their bank accounts. They have "implicit assets" in the sense that the government owes them whatever they gave contributed. Symmetrically, it does not look like the government has any debt, while it actually has "implicit liabilities": it owes the young whatever it has been taxing them.

10. Here, there really is no difference between pay-as-you-go financing and deficit financing, except that one appears on the government's books (deficit financing), while the other does not (pay-as-you-go financing). Pay-as-you-go is thus an implicit government liability: instead of having promised to repay creditors, the government has promised to repay future retirees who have put money in the system when they were working. This helps understand why government debt is not a very meaningful statistic: by reclassifying some of its liabilities as pension liabilities, the government is able to reduce its government debt. A forceful proponent of including pay-as-you-go liabilities into government debt numbers is Lawrence Kotlikoff, an economist at Boston University. You can read here his very dismal assessment of the current U.S. fiscal outlook. As you now know, I am not as concerned as he is. As I said during the class, what matters really is the value for real interest rates compared to growth rates, in order to understand whether this debt can actually be rolled over indefinitely or not. Don't hesitate to let me know if you feel strongly in the opposite direction, and find my arguments unpersuasive !