Lecture 2 - The Solow Growth Model

UCLA - Econ 102 - Fall 2018

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Contents

The first part of this note considers the case of a Solow [1956] growth model with a general production function. The second part of the note looks at a special case of the Solow [1956] growth model for a case of a Cobb-Douglas production function.

1 General Production Function

1.1 Assumptions

Solow [1956] starts from a general production function, giving at any point in time output Y_t as a function of inputs, capital K_t and labor L_t:

$$Y_t = F\left(K_t, L_t\right).$$

For simplicity, we shall assume from now on that the quantity of labor is fixed with $L_t = L$, so that the production function becomes $Y_t = F(K_t, L)$. A very important assumption is also constant returns to scale with respect to capital and labor, so that for any factor a:

$$F(aK_t, aL_t) = aF(K_t, L_t).$$

Because of constant returns to scale with respect to capital and labor, we have:

$$\frac{Y_t}{L} = F\left(\frac{K_t}{L}, 1\right) = f\left(\frac{K_t}{L}\right)$$

with

$$f(x) \equiv F(x, 1).$$

An example of such a production function is the Cobb and Douglas [1928] production function, which we started studying in Lecture 1. This function has constant returns to scale.

Solow [1956] abstracts from public saving, so that **total saving** equals **private saving** S_t at time t, which also equals investment I_t at time t :

$$S_t = I_t.$$

Saving is assumed to be a constant fraction s of output Y_t, and therefore:

$$S_t = sY_t.$$

This constant saving rate my seem a bit ad-hoc; it is. We will investigate more in detail the determinants of saving and consumption behavior in the next lectures.

Depreciation of capital is given by δ. Therefore, the capital stock evolves according to:

$$K_{t+1} = (1-\delta)\, K_t + I_t$$

1.2 Solution

Replace investment in the previous equation and divide both sides by N:

$$\frac{K_{t+1}}{L} = (1-\delta)\,\frac{K_t}{L} + s\frac{Y_t}{L} \quad \Rightarrow \quad \boxed{\frac{K_{t+1}}{L} - \frac{K_t}{L} = s\frac{Y_t}{L} - \delta\frac{K_t}{L}}$$

$$\underbrace{\frac{K_{t+1}}{L} - \frac{K_t}{L}}_{\text{Change in capital}} = \underbrace{sf\left(\frac{K_t}{L}\right)}_{\text{Investment}} - \underbrace{\delta\frac{K_t}{L}}_{\text{Depreciation}}$$

The steady state level of the capital stock K^* is such that $K_{t+1} = K_t = K^*$, and it verifies:

$$sf\left(\frac{K^*}{L}\right) = \delta\frac{K^*}{L}$$

The steady-state value of output per worker Y^*/N, as a function of K^*/N is given by:

$$\frac{Y^*}{L} = f\left(\frac{K^*}{L}\right)$$

1.3 Three cases

There are 3 cases:

1. If capital per worker is relatively low initially, that is $K_t/N < K^*/N$, then investment per worker is larger than depreciation per worker, and therefore from the above equation, capital per worker increases:

$$\frac{K_{t+1}}{L} > \frac{K_t}{L}$$

2. If capital per worker is exactly equal to steady state capital per worker, that is $K_t/N = K^*/N$, then investment per worker is equal to depreciation per worker, and therefore from the above equation, capital per worker stays constant:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L}$$

3. If capital per worker is relatively high initially, that is $K_t/N > K^*/N$, then depreciation per worker is larger than investment per worker, and therefore, capital per worker declines:

$$\frac{K_{t+1}}{L} < \frac{K_t}{L}.$$

2 Cobb-Douglas production function

2.1 Solving for the model

Assume now that the production function is a Cobb-Douglas production function, so that:

$$F(K,N) = K^\alpha L^{1-\alpha}$$

As we saw during lecture 1, α should be taken as roughly equal to $\alpha = 1/3$.

This implies then that:

$$f(x) = x^\alpha$$

The law of motion for capital is given by:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} + s\left(\frac{K_t}{L}\right)^\alpha - \delta\frac{K_t}{L}.$$

Given L, K_0, α, s, δ, you are able to calculate K_1, K_2, $\ldots$, as well as K_t for any t, by calculating the quantities of capital successively from the formula above.

If you do so, you will notice that K_t indeed converges to a steady state value K^*. However, you do not need to perform an infinity of operations to get at this K^*. Instead, you should see that capital per worker in steady-state K^*/L solves:

$$s\left(\frac{K^*}{L}\right)^\alpha = \delta\frac{K^*}{L} \quad \Rightarrow \quad \frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$$

The steady-state level of output per worker is then:

$$\frac{Y^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

2.2 Golden Rule

The Golden Rule level of capital accumulation is such that the level of steady-state consumption per capita is maximized. The steady-state consumption per capita is given by:

$$\frac{C^*}{L} = (1-s)\frac{Y^*}{L} = (1-s)\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} = \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}$$

Maximizing this steady state consumption with respect to the saving rate s consists in finding the maximum of that function with respect to s:

$$\frac{d\left(C^*/L\right)}{ds} = 0 \quad \Rightarrow \quad \frac{d\left[(1-s)s^{\frac{\alpha}{1-\alpha}}\right]}{ds} = 0$$

This gives:

$$-s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)s^{\frac{\alpha}{1-\alpha}-1} = 0 \quad \Rightarrow \quad \frac{\alpha}{1-\alpha}\frac{1-s}{s} = 1$$

$$\Rightarrow \quad \alpha - \alpha s = s - \alpha s \quad \Rightarrow \quad \boxed{s = \alpha}.$$

Therefore, the saving rate corresponding to the Golden Rule level of capital accumulation is equal to α. The Golden Rule level of capital accumulation is then such that:

$$\frac{K^*}{L} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} \quad \Rightarrow \quad K^* = L\left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$$

The level of GDP corresponding to this Golden rule level is:

$$Y^* = L\left(\frac{\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

References

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