The Solow (1956) growth model

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These are notes corresponding to derivations made on the board on May 7, 2018. The Solow (1956) growth model considers an economy whose production function has constant returns to scale with respect to capital and labor, and the saving rate is constant. Time is discrete with $t = 0, 1, 2, ...$, and the economy starts with an amount of capital K_0 at $t = 0$. Capital depreciates at a rate δ. Section 1 considers the case of a general production function. Section 2 looks at the case of a Cobb-Douglas production function.

1 General production function

We started from the following production function, with constant returns to scale with respect to capital and labor:

$$\frac{Y}{N} = F\left(\frac{K}{N}, 1\right) = f\left(\frac{K}{N}\right)$$

with $f(x) \equiv F(x, 1)$.

There is no public saving, therefore total saving, which is equal to private saving et time t S_t equals investment at time t I_t:

$$S_t = I_t$$

Saving is assumed to be a constant fraction s of output Y_t, therefore:

$$S_t = sY_t$$

Depreciation of capital is given by δ. Therefore, the capital stock evolves according to:

$$K_{t+1} = (1 - \delta) K_t + I_t \tag{1}$$

Replace investment in equation (1) and divide both sides by N:

$$\frac{K_{t+1}}{N} = (1 - \delta) \frac{K_t}{N} + s\frac{Y_t}{N} \quad \Rightarrow \quad \boxed{\frac{K_{t+1}}{N} - \frac{K_t}{N} = s\frac{Y_t}{N} - \delta\frac{K_t}{N}}$$

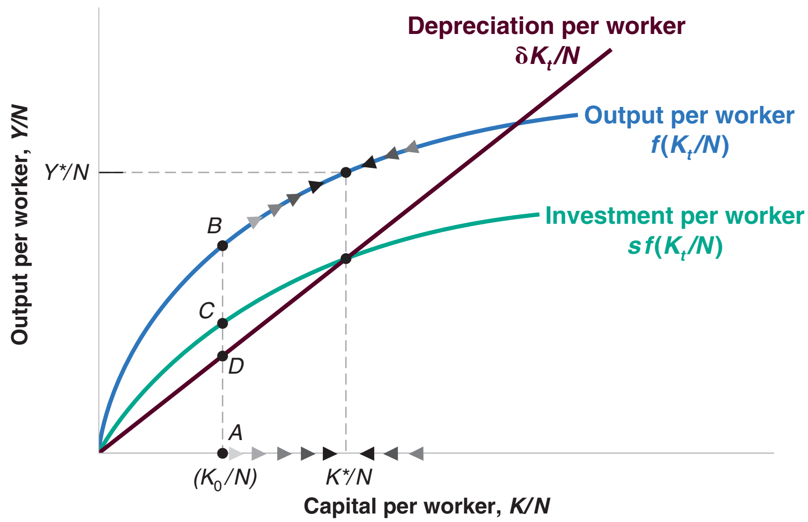
$$\underbrace{\frac{K_{t+1}}{N} - \frac{K_t}{N}}_{\text{Change in capital}} = \underbrace{sf\left(\frac{K_t}{N}\right)}_{\text{Investment}} - \underbrace{\delta\frac{K_t}{N}}_{\text{Depreciation}}$$

The steady state level of the capital stock K^* is such that $K_{t+1} = K_t = K^*$, and it verifies:

$$sf\left(\frac{K^*}{N}\right) = \delta\frac{K^*}{N}$$

The steady-state value of output per worker Y^*/N, as a function of K^*/N is given by:

$$\frac{Y^*}{N} = f\left(\frac{K^*}{N}\right)$$



There are 3 cases:

1. If capital per worker is relatively low initially, that is $K_t/N < K^*/N$, then the green curve is above the maroon curve, which means that investment per worker is larger than depreciation per worker, and therefore from the above equation, capital per worker increases:
$$\frac{K_{t+1}}{N} > \frac{K_t}{N}$$

2. If capital per worker is exactly equal to steady state capital per worker, that is $K_t/N = K^*/N$, then the green curve crosses the maroon curve, which means that investment per worker is equal to depreciation per worker, and therefore from the above equation, capital per worker stays constant:
$$\frac{K_{t+1}}{N} = \frac{K_t}{N} = \frac{K^*}{N}$$

3. If capital per worker is relatively high initially, that is $K_t/N > K^*/N$, then the maroon curve is above the green curve, which means that depreciation per worker is larger than investment per worker, and therefore, capital per worker declines:
$$\frac{K_{t+1}}{N} < \frac{K_t}{N}.$$

2 Cobb and Douglas (1928) production function

Assume now that the production function is a Cobb-Douglas production function, so that:

$$F(K,N) = K^{\alpha}N^{1-\alpha}$$

This implies then that:

$$f(x) = x^{\alpha}$$

The law of motion for capital is given by:

$$\frac{K_{t+1}}{N} = \frac{K_t}{N} + s\left(\frac{K_t}{N}\right)^{\alpha} - \delta\frac{K_t}{N}.$$

Note that given N, K_0, α, s, δ, you are able to calculate K_1, K_2, ..., as well as K_t for any t, by calculating the quantities of capital successively from the formula above. If you do so, you will notice that K_t indeed converges to a steady state value K^*. However, you do not need to perform an infinity of operations to get at this K^*. Instead, you should see that capital per worker in steady-state K^*/N solves:

$$s\left(\frac{K^*}{N}\right)^{\alpha} = \delta\frac{K^*}{N} \quad \Rightarrow \quad \frac{K^*}{N} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$$

The steady-state level of output per worker is then:

$$\frac{Y^*}{N} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Golden Rule. The Golden Rule level of capital accumulation is such that the level of steady-state consumption per capita is maximized. The steady-state consumption per capita is given by:

$$\frac{C^*}{N} = (1-s)\frac{Y^*}{N} = (1-s)\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} = \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}$$

Maximizing this steady state consumption with respect to the saving rate s consists in finding the maximum of that function with respect to s:

$$\frac{d\left(C^*/N\right)}{ds} = 0 \quad \Rightarrow \quad \frac{d\left[(1-s)s^{\frac{\alpha}{1-\alpha}}\right]}{ds} = 0$$

This gives:

$$-s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)s^{\frac{\alpha}{1-\alpha}-1} = 0 \quad \Rightarrow \quad \frac{\alpha}{1-\alpha}\frac{1-s}{s} = 1$$
$$\Rightarrow \quad \alpha - \alpha s = s - \alpha s \quad \Rightarrow \quad \boxed{s = \alpha}.$$

Therefore, the saving rate corresponding to the Golden Rule level of capital accumulation is equal to α. The Golden Rule level of capital accumulation is then such that:

$$\frac{K^*}{N} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} \quad \Rightarrow \quad K^* = N\left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$$

The level of GDP corresponding to this Golden rule level is:

$$Y^* = N\left(\frac{\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

References

Cobb, Charles W. and Paul H. Douglas, "A Theory of Production," *The American Economic Review*, 1928, *18* (1), 139–165.

Solow, Robert M., "A Contribution to the Theory of Economic Growth," *The Quarterly Journal of Economics*, 1956, *70* (1), 65–94.