Lecture 3 - Consumption - Intertemporal Optimization

UCLA - Econ 102 - Fall 2018

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Contents

1 The Two-Period Consumption Problem {#2-period}

1.1 Most general calculations

There are two periods, $t = 0$ (think of this as "today") and $t = 1$ (think of this as "tomorrow"). There is one consumption good. The consumer values utility in both periods according to the following utility function:

$$U(c_0, c_1) = u(c_0) + \beta u(c_1).$$

Utility is strictly increasing in both arguments, consumption in period 0 (c_0) and consumption in period 1 (c_1).

People earn income y_0 in period 0, and income y_1 in period 1. They also are born with some financial wealth f_0 now, and have financial wealth f_1 in period 1, which they consume entirely because this is the last period. (there is no point keeping more money for after period 1, because there is no future at that point)

The amount that agents save in this economy is thus $f_1 - f_0$, and the amount of their accumulated savings is the savings they already had plus what they decided to accumulate, so that $f_0 + (f_1 - f_0) = f_1$.

Consumption in period 0 is given by:

$$c_0 = y_0 - (f_1 - f_0)$$

The second period consumption ($t = 1$) is given by income plus the return to (accumulated !) savings:

$$c_1 = y_1 + (1 + R)f_1.$$

Rewriting f_1 from this second equation: $f_1 = \frac{c_1 - y_1}{1+R}$, and plugging into the first, one gets that total wealth is the sum of financial wealth and of human wealth:

$$c_0 = y_0 - \left(\frac{c_1 - y_1}{1+R} - f_0\right) \quad \Rightarrow \quad c_0 + \frac{c_1}{1+R} = \overbrace{f_0 + \underbrace{y_0 + \frac{y_1}{1+R}}_{\text{human wealth}}}^{\text{total wealth}}.$$

Thus the problem of the consumer is the following problem of constrained optimization:

$$\max_{c_0, c_1} \quad u(c_0) + \beta u(c_1)$$
$$\text{s.t.} \quad c_0 + \frac{c_1}{1+R} = f_0 + y_0 + \frac{y_1}{1+R}.$$

You can solve this optimization in three ways:

1. Apply the well known ratio of marginal utilities formula from a basic Microeconomic class and say that:

$$\boxed{\frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+R}}$$

2. Apply the following intuitive argument. The marginal utility from consuming in period 1 is $\beta u'(c_1)$. The marginal utility from consuming in period 0 is $u'(c_0)$. By putting one unit of consumption in the bank, one forgoes 1 unit of consumption in period 0 to get $1 + R$ units of consumption in period 1. The two have to be equal:

$$u'(c_0) = (1+R)\beta u'(c_1) \quad \Rightarrow \quad \frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+R}.$$

3. Replace c_0 and optimize with respect to c_1:

$$c_0 = \left(f_0 + y_0 + \frac{y_1}{1+R}\right) - \frac{c_1}{1+R} \qquad \Rightarrow \qquad \max_{c_1} \quad u\left[\left(f_0 + y_0 + \frac{y_1}{1+R}\right) - \frac{c_1}{1+R}\right] + \beta u(c_1)$$

This leads to the following First-Order Condition:

$$-\frac{1}{1+R}u'(c_0) + \beta u'(c_1) = 0 \quad \Rightarrow \quad \frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+R}.$$

4. Finally, you can replace c_1 and optimize with respect to c_0:

$$c_1 = (1+R)\left(f_0 + y_0 + \frac{y_1}{1+R}\right) - (1+R)c_0$$
$$\Rightarrow \quad \max_{c_0} \quad u(c_0) + \beta u\left[(1+R)\left(f_0 + y_0 + \frac{y_1}{1+R}\right) - (1+R)c_0\right]$$

This leads to the following First-Order Condition:

$$u'(c_0) - \beta(1+R)u'(c_1) = 0 \quad \Rightarrow \quad \frac{\beta u'(c_1)}{u'(c_0)} = \frac{1}{1+R}.$$

1.2 Log utility, no discounting

With log utility, and no discounting, we have:

$$u(c) = \log(c) \qquad \beta = 1.$$

Marginal utility is then:

$$u'(c) = \frac{1}{c},$$

so that the optimality condition gives:

$$c_0 = \frac{c_1}{1+R} \quad \Rightarrow \quad c_0 = \frac{1}{2}\left(f_0 + y_0 + \frac{y_1}{1+R}\right) \quad \text{and} \quad c_1 = \frac{1+R}{2}\left(f_0 + y_0 + \frac{y_1}{1+R}\right).$$

Consumption decreases when the interest rate rises, all the more that income will arise late into the future. This is a way to "microfound" the assumptions about aggregate demand that we made before.

Note also that the "marginal propensity to consume" out of current wealth f_0 is given by $1/2$. If one looks at a model with more periods, say T periods, one can actually find that people's marginal propensity to consume are approximately equal to $1/T$.

1.3 Log utility, with discounting

Marginal utility is then $u'(c) = 1/c$, so that the optimality condition gives:

$$\frac{c_1}{1+R} = \beta c_0 \quad \Rightarrow \quad c_0 = \frac{1}{1+\beta}\left(f_0 + y_0 + \frac{y_1}{1+R}\right) \quad \text{and} \quad c_1 = \frac{\beta(1+R)}{1+\beta}\left(f_0 + y_0 + \frac{y_1}{1+R}\right).$$

You can check that the solution with no discounting corresponds to that with discounting when $\beta = 1$.

2 The Overlapping Generations Model

In the Solow [1956] growth model, we assumed that saving was a constant fraction of GDP. The previous section has shown how to "endogenize" the saving behavior. This section presents a very simple version of the Diamond [1965] so-called overlapping-generations model (for reasons that will soon become clear). This model is used by policymakers to think about social security, public debt, and intergenerational transfers more generally.

2.1 Assumptions

We assume that people in this economy live only for two periods. However, instead of referring to these two periods as 0 and 1, we now call refer to them as t and $t+1$. People are called "young" in the first period of their life, and "old" in the second. (thus the length of one period is really long !)

People from generation t are young in period t, and old in period $t+1$. We denote their consumption when young by c_t^y and their consumption when old by c_{t+1}^o. In terms of the previous section, you should really think of c_t^y as c_0, and of c_{t+1}^o as c_1.

People work when young, and then receive a wage given by w_t. They retire when old, and then do not work. Their lifetime utility is logarithmic with $\beta = 1$:

$$U = \log(c_t^y) + \log(c_{t+1}^o).$$

Their intertemporal budget constraint is given by:

$$c_t^y + \frac{c_{t+1}^o}{1+R} = w_t.$$

There is always two generations living in period t: the previous period's young, born in period $t-1$, now old, consuming the return from their savings; and this period's young, newly born (in period t).

Production. There is a Cobb-Douglas, constant returns to scale, production function:

$$Y_t = K_t^{1/3} L_t^{2/3}.$$

Assume that the labor force is constant and fixed to unity (this is to avoid carrying L around everywhere - from lecture 2, you should now know that everything can be expressed per capita, because of constant returns to scale):

$$L_t = L = 1.$$

Again for simplicity, we shall assume that capital depreciates at rate $\delta = 1 = 100\%$. (that is, capital fully depreciates each period - this is not that unreasonable if you take one unit of time to represent one generation, or about 30 years - remember that the depreciation rate for one year was approximately equal to 10%.)

2.2 Solution

Utility is logarithmic, so that the consumption of the young c_t^y and consumption of the old c_{t+1}^o are given as a function of the wage as follows:

$$c_t^y = \frac{w_t}{2} \qquad c_{t+1}^o = (1+R)\frac{w_t}{2}.$$

Indeed, if you want to think of this model as the two periods model of the previous section, think that everything is as if: $f_0 = 0, \qquad y_0 = w_t, \qquad y_1 = 0$. Saving (and savings) is equal to investment, and given by:

$$S_t = I_t = w_t - c_t^y = \frac{w_t}{2}.$$

Therefore, savings are here endogenous, and coming from agents' optimizing choices. In the Solow model in contrast, they were assumed to be exogenous and equal to a fraction s.

The wage paid by employers, given that $\bar{L} = 1$, is given by:

$$w_t = \frac{2}{3} K_t^{1/3} L^{-1/3} = \frac{2}{3} K_t^{1/3} = \frac{2}{3} Y_t.$$

Finally:

$$\Delta K_{t+1} = \frac{w_t}{2} - \delta K_t = \frac{Y_t}{3} - \delta K_t.$$

This is the capital accumulation equation (or law of motion for capital) of the Solow model, with $s = 1/3$. The new element here of course is to get savings endogenously, from agents' optimal decisions.

References

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Robert M. Solow. A Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics*, 70(1):65–94, 1956. ISSN 0033-5533. doi: 10.2307/1884513. URL http://www.jstor.org/stable/1884513.