

# Problem Set 1 - Solutions

UCLA - Econ 102 - Fall 2018

*François Geerolf*

## 1 Geometric Sums

1. Let us denote the geometric sum of interest by  $S$ :

$$S = \sum_{i=0}^n x^i = 1 + x + x^2 + \dots + x^n.$$

The trick to calculate this sum is to multiply it by  $x$ , which allows to get:

$$xS = x + x^2 + x^3 + \dots + x^{n+1}.$$

We can see that this is almost the same sum as the previous one except for the first term, which is missing, and the last term, which was absent from  $S$ , therefore:

$$\begin{aligned} xS &= -1 + 1 + x + x^2 + \dots + x^n + x^{n+1} \\ xS &= -1 + S + x^{n+1} \end{aligned}$$

This implies (for  $x \neq 1$ ):

$$1 - x^{n+1} = (1 - x)S \quad \Rightarrow \quad S = \frac{1 - x^{n+1}}{1 - x}.$$

2. For  $S$  to have a finite value when  $n \rightarrow \infty$ , we need that  $x^{n+1}$  stays finite. This happens when:

$$|x| < 1.$$

In the knife edge case when  $x = 1$ , the sum goes to infinity since it is then equal to  $n + 1$ . If  $x = -1$ , then the sum oscillates between 1 and  $-1$  and does not have a limit when  $n$  goes to infinity.

3. From question 1 and 2, we know that when  $|x| < 1$ ,  $x^{n+1} \rightarrow 0$  when  $n \rightarrow \infty$ , and therefore:

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1 - x}.$$

4. We just factor in  $x^m$  and then use the formula in question 1:

$$\begin{aligned} \sum_{i=m}^n x^i &= x^m + x^{m+1} + \dots + x^n \\ &= x^m (1 + x + \dots + x^{n-m}) \\ &= x^m \frac{1 - x^{n-m+1}}{1 - x} \\ \sum_{i=m}^n x^i &= \frac{x^m - x^{n+1}}{1 - x}. \end{aligned}$$

5. The present discounted value of an infinite stream of incomes, which grows at rate  $g = 2\%$ , starts at  $y_0 = 90000$ , if the interest rate is  $i = 3\%$  is:

$$y_0 + y_0 \frac{1+g}{1+i} + y_0 \frac{(1+g)^2}{(1+i)^2} + \dots$$

Using the formula found in question 3 with  $x = (1 + g)/(1 + i)$ , we get:

$$y_0 + y_0 \frac{1 + g}{1 + i} + y_0 \frac{(1 + g)^2}{(1 + i)^2} + \dots = y_0 \frac{1}{1 - \frac{1 + g}{1 + i}} = \frac{y_0(1 + i)}{i - g}$$

A numerical application gives (see Google Sheet):

$$\frac{y_0(1 + i)}{i - g} = \frac{90000 * (1 + 0.03)}{0.03 - 0.02} = 9270000.$$

## 2 Taylor Approximations

1. We have:

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} \cdot x^k$$

When  $x$  is small, all the  $x^k$  terms for  $k \geq 2$  are negligible, and therefore:

$$(1 + x)^n \approx 1 + \binom{n}{1}x = 1 + nx.$$

If you do not know what  $\binom{n}{k}$  means, that is fine too. You can prove the same result using recursion. For  $n = 1$ , we know that  $(1 + x)^1 = 1 + x$  (obviously). Assume that the approximation is true for  $n$ , or that  $(1 + x)^n \approx 1 + nx$ , let's prove that it is true for  $n + 1$ :

$$\begin{aligned} (1 + x)^{n+1} &= (1 + x)^n(1 + x) \\ &\approx (1 + nx)(1 + x) \\ &\approx 1 + (n + 1)x + nx^2 \\ (1 + x)^{n+1} &\approx 1 + (n + 1)x \end{aligned}$$

which proves the proposition for  $n + 1$ . Thus, the Taylor approximation is true for any  $n \in \mathbb{N}$ .

2. We have:

$$(1 + x)(1 + y) = 1 + x + y + xy.$$

When  $x$  and  $y$  are both small, then  $xy$  is negligible, which gives the result:

$$(1 + x)(1 + y) \approx 1 + x + y.$$

3. Using the formula proven in Problem 1, we get that:

$$\frac{1}{1 + y} = 1 - y + y^2 - y^3 + \dots$$

When  $x$  and  $y$  are both small, all terms of the product are negligible except for first-order terms:

$$\frac{1 + x}{1 + y} \approx 1 + x - y.$$

4. Denote the price level by  $p_t$  (that is, in dollars, the price of a representative basket of goods). Inflation  $\pi_t$  at time  $t$  is defined as the rate of growth of this price level between  $t$  and  $t + 1$ :

$$\frac{p_{t+1}}{p_t} = 1 + \pi_t.$$

If you leave one dollar at the bank, and if the nominal interest rate is given by  $i_t$ , then you end up at the end of the period with  $1 + i_t$  dollars at the bank. With this, you can buy a quantity of goods given

by  $(1 + i_t)/p_{t+1}$ . If you buy a quantity of goods at time  $t$ , then you get a number of goods equal to  $1/p_t$ . Thus, the rate of increase in your purchasing power if you leave your money in the bank is given by:

$$\frac{(1 + i_t)/p_{t+1}}{1/p_t} = \frac{1 + i_t}{1 + \pi_t}.$$

An exact value for the real interest rate is thus:

$$\begin{aligned} \frac{1 + i_t}{1 + \pi_t} - 1 &= \frac{1 + 0.01}{1 + 0.015} - 1 \\ &= -0.00492610837 \\ \frac{1 + i_t}{1 + \pi_t} - 1 &= -0.492610837\%. \end{aligned}$$

An approximate value from the above formula is:

$$\begin{aligned} \frac{1 + i_t}{1 + \pi_t} - 1 &\approx 1 + i_t - \pi_t - 1 \\ &\approx 0.01 - 0.015 \\ \frac{1 + i_t}{1 + \pi_t} - 1 &\approx -0.5\%. \end{aligned}$$

This is not such a bad approximation.

### 3 Growth Rates

1. Iterating on the formula:

$$y_{t+1} = (1 + g)y_t \quad \Rightarrow \quad y_T = (1 + g)^T y_0,$$

allows to find the result:

$$G = \frac{y_T}{y_0} - 1 = (1 + g)^T - 1.$$

2. Again, inverting the previous relation:

$$G = (1 + g)^T - 1,$$

allows to find:

$$g = \frac{y_{t+1}}{y_t} - 1 = (1 + G)^{1/T} - 1.$$

3. Applying the previous formula allows to get (see Google Sheet):

$$\begin{aligned} g &= (1 + 0.01)^{1/365} - 1 \\ &= 0.00002726155 \\ g &= 0.0027\% \end{aligned}$$

You give up approximately \$2.7 every day (your bank most likely is investing this money on your behalf, so you are rather giving this money to your bank):

$$0.00002726155 * 100000 = 2.7.$$