Lecture 2 - The Solow Growth Model

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Contents

The first part of this note considers the case of a Solow [1956] growth model with a general, constant returns to scale, production function. The second part of the note looks at a special case of the Solow [1956] growth model for a case of a Cobb and Douglas [1928] production function.

1 General Production Function

1.1 Assumptions

Solow [1956] starts from a general production function, giving at any point in time output Y_t as a function of inputs, capital K_t and labor L_t:

$$Y_t = F(K_t, L_t).$$

For simplicity, we shall assume from now on that the quantity of labor is fixed with $L_t = L$, so that the production function becomes $Y_t = F(K_t, L)$. A very important assumption is also constant returns to scale with respect to capital and labor, so that for any scaling factor a:

$$F(aK_t, aL_t) = aF(K_t, L_t).$$

Because of constant returns to scale with respect to capital and labor (and setting $a = 1/L$ in the previous expression), we have:

$$\frac{Y_t}{L} = F\left(\frac{K_t}{L}, 1\right) = f\left(\frac{K_t}{L}\right)$$

where f is defined as a function of F such that

$$f(x) \equiv F(x, 1).$$

An example of such a production function is the Cobb and Douglas [1928] production function, which we started studying in Lecture 1, and which we look at in the next section.

Solow [1956] abstracts from public saving, so that **total saving** at time t equals **private saving** at time t, and both are denoted S_t, which also equals investment I_t at time t :

$$S_t = I_t.$$

Saving is assumed to be a constant fraction s of output Y_t, and therefore:

$$S_t = sY_t.$$

This constant saving rate my seem a bit ad-hoc; it is. We will investigate more in detail the determinants of saving and consumption behavior in the next lectures. Depreciation of capital is given by a share δ (think for example that 8% of the capital stock depreciates each period; the rate of depreciation is much lower for structures, and much higher for computers). The capital stock evolves according to:

$$K_{t+1} = (1-\delta)\, K_t + I_t$$

1.2 Solution

Replace investment in the previous equation and divide both sides by L:

$$\frac{K_{t+1}}{L} = (1-\delta)\,\frac{K_t}{L} + s\frac{Y_t}{L} \quad \Rightarrow \quad \boxed{\frac{K_{t+1}}{L} - \frac{K_t}{L} = s\frac{Y_t}{L} - \delta\frac{K_t}{L}}$$

The change in the capital stock per person from t to $t+1$ has two components: investment (or saving) and depreciation:

$$\underbrace{\frac{K_{t+1}}{L} - \frac{K_t}{L}}_{\text{Change in capital}} = \underbrace{sf\left(\frac{K_t}{L}\right)}_{\text{Investment}} - \underbrace{\delta\frac{K_t}{L}}_{\text{Depreciation}}.$$

The steady state level of the capital stock K^* is such that $K_{t+1} = K_t = K^*$, and it therefore satisfies:

$$\boxed{sf\left(\frac{K^*}{L}\right) = \delta\frac{K^*}{L}}$$

Note that without further specifying $f(.)$, we can't say much more about the value of K^*/L, we just know it satisfies this implicit equation. The steady-state value of output per worker Y^*/L, as a function of K^*/L is given by:

$$\frac{Y^*}{L} = f\left(\frac{K^*}{L}\right)$$

1.3 Three cases

There are 3 cases:

1. If capital per worker is relatively low, that is $K_t/L < K^*/L$, then investment per worker is larger than depreciation per worker, and therefore from the above equation, capital per worker increases:

$$\frac{K_{t+1}}{L} > \frac{K_t}{L}$$

2. If capital per worker is exactly equal to steady state capital per worker, that is $K_t/L = K^*/L$, then investment per worker is equal to depreciation per worker, and therefore from the above equation, capital per worker stays constant:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L}$$

3. If capital per worker is relatively high, that is $K_t/L > K^*/L$, then depreciation per worker is larger than investment per worker, and therefore, capital per worker declines:

$$\frac{K_{t+1}}{L} < \frac{K_t}{L}.$$

2 Cobb and Douglas [1928] production function

2.1 Solving for the model

Assume now that the production function is a Cobb and Douglas [1928] production function, so that:

$$F(K, L) = K^\alpha L^{1-\alpha}$$

As we saw during lecture 1, α should be thought of as roughly equal to $\alpha = 1/3$. This implies then that function f defined above is such that:

$$f(x) = x^\alpha$$

The law of motion for capital is given by:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} + s\left(\frac{K_t}{L}\right)^\alpha - \delta\frac{K_t}{L}.$$

Given L, K_0, α, s, δ, we are able to calculate K_1, K_2, $\dots$, as well as K_t for any t, by calculating the quantities of capital successively from the formula above.

If you do so, you will notice that K_t converges to a steady state value K^*. However, you do not need to perform an infinity of operations to get at this K^*. Instead, you can see that capital per worker in steady-state K^*/L solves:

$$s\left(\frac{K^*}{L}\right)^\alpha = \delta\frac{K^*}{L} \quad \Rightarrow \quad \boxed{\frac{K^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}}$$

What was an implicit equation in the previous section can now be solved for explicitely. The steady-state level of output per worker is then:

$$\frac{Y^*}{L} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

2.2 Golden Rule

Most economists believe that policymakers should not care so much about GDP per person, but rather about consumption per person (however, some people hold a different view – we shall talk about that later). The intuition is simple: if an economy was to produce many goods which were only used for investment purposes (which would be the case if $s = 1$), then people in this economy would be starving, even though it was actually producing a lot. Investment, ultimately, should serve to increase future consumption.

The **Golden Rule level of capital accumulation** is such that the level of steady-state consumption per capita is maximized. The steady-state consumption per capita is given by:

$$\frac{C^*}{L} = (1-s)\frac{Y^*}{L} = (1-s)\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} = \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}$$

Maximizing this steady state consumption with respect to the saving rate s consists in finding the maximum of that function with respect to s:

$$\frac{d\,(C^*/L)}{ds} = 0 \quad \Rightarrow \quad \frac{d\left[(1-s)s^{\frac{\alpha}{1-\alpha}}\right]}{ds} = 0$$

Note that the $1/\delta^{\alpha/(1-\alpha)}$ is just a constant which does not change anything to the maximization. If you are not convinced, then you may also compute the derivative with respect to the whole C^*/L expression. This gives:

$$-s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)s^{\frac{\alpha}{1-\alpha}-1} = 0 \quad \Rightarrow \quad \frac{\alpha}{1-\alpha}\frac{1-s}{s} = 1$$

$$\Rightarrow \quad \alpha - \alpha s = s - \alpha s \quad \Rightarrow \quad \boxed{s = \alpha}.$$

Therefore, the saving rate corresponding to the Golden Rule level of capital accumulation is equal to α (again, taking α to be equal to roughly 1/3, this would suggest that an economy would optimally need to save about a third of its production every year).

The Golden Rule level of capital accumulation is then such that capital at the steady-state is given as a function of the exogenous parameters by:

$$\frac{K^*}{L} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} \quad \Rightarrow \quad K^* = L\left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$$

The level of GDP corresponding to this Golden rule level is:

$$Y^* = L\left(\frac{\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

References

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