Lecture 8: A Theory of Pareto Distributions

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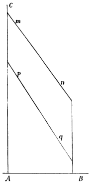
Pareto distributions

Schedule D — Année 1893-94.

x £	N — GREAT BRITAIN	N — IRELAND
150	400 648	17 717
200	234 185	9 365
300	121 996	4 592
400	74 041	2 684
500	54 419	1 898
600	42 072	1 428
700	34 269	1 104
800	29 311	940
900	25 033	771
1000	22 896	684
2000	9 880	271
3000	6 069	142
4000	4 161	88
5000	3 081	68
10000	1 104	22



- 1890s, tax tabulations: Pareto plots N of people with incomes $\geq x$:

$$\boxed{\log N_{\text{income} \geq x} = C - \alpha \log x}.$$

- Same log linear relationship, differing $\alpha \in [1, 3]$:
  - Semifeudal Prussia
  - Victorian England
  - Capitalist but highly diversified Italian cities
  - Communist-like regime of the Jesuits in Peru under Spanish rule, etc.

- With Pareto:
  - No scale. US: $y_{50} = \$51,939 < y_{av} = \$72,641$.
  - Long tails. Top 1% gets $\approx$ 20% of pre-tax income.
  - Constant elasticity: $\boxed{d \log N_{\geq x} / d \log x = -\alpha}$

Pareto $\neq$ bell-shaped curve. Few empirical regularities in economics.

Pareto tail for US labor incomes, 2008

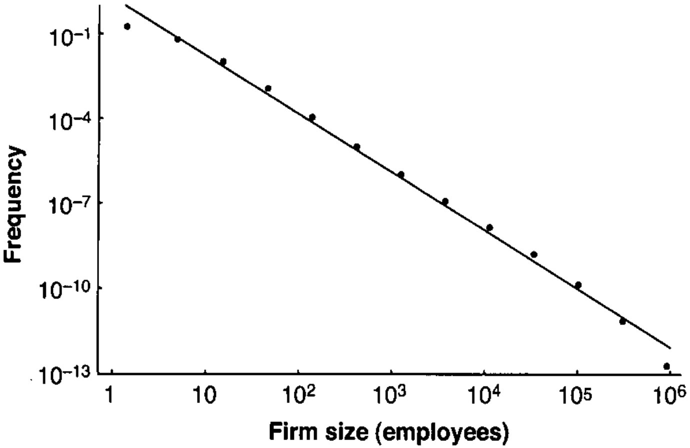
Source: Statistics of Income, Public Use Sample

Pareto tail for US labor incomes, 1968

Zipf's law for firm sizes

Distribution of US firm sizes. Source: Axtell (2001).

Slope: 2.059 (density) $\Rightarrow$ Tail coeff: $\boxed{\mathbf{1.059}}$. "**Zipf's law**".



Theories of Pareto distributions in Economics

Why Pareto? May reflect some fundamental economic principle:

1. **Pareto distributed primitives.** Explain one Pareto with another Pareto.
   - Lucas (1978), Kortum (1997), Melitz / Chaney (2008), Gabaix, Landier (2008), etc.
2. **Paretos from random growth models.**
   - Champernowne (1953), Simon, Bonini (1958), Kesten (1973), Gabaix (1999), Gabaix, Lasry, Lions, Moll (2016), Jones, Kim (2016), etc.
3. **New from this paper: Paretos from production functions.** Assignment models with positive sorting, with a special form of production function.
   - Presentation: Garicano (2000) model.
   - Property of the production function, not of specific microfoundations.
   - Another example: Geerolf (2015).

This paper

- Production function derives from **a particular version** of **Garicano (2000)**. Under limited assumptions on the skill distribution:
  - **L layers of hierarchy** = **Pareto tail** for span of control with coefficient:

    $$\boxed{\alpha_L = 1 + \frac{1}{L-1}}, \qquad \boxed{\alpha_2 = 2}, \qquad \boxed{\alpha_{+\infty} = 1}.$$

    $\Rightarrow$ a new theory of **Zipf's law for firm sizes**.
  - **Pareto tail for labor incomes**, with $\beta_L \in [1, +\infty]$, when top skills are scarce enough.
- Data supports these predictions: French matched employer-employee / known US data.
- Taking competitive assignment models to the extreme, where wages are a convex function of skills. (Sattinger (1975)) Here: **wages are Pareto with a bounded support for skills.**

Literature

- **Pareto distributions.** Pareto (1896), Zipf (1949).
- **Competitive assignment models.** Roy (1950), Becker (1973, 1974), Rosen (1981), Sattinger (1975), Kremer (1993), Terviö (2008), Gabaix, Landier (2008).
- **Span of control.** Lucas (1978), Rosen (1981), Rosen (1982), Rossi-Hansberg, Wright (2007).
- **Organizational structure.** Calvo, Wellisz (1978,1979), Garicano (2000), Garicano, Rossi-Hansberg (2004, 2006), Antras, Garicano, Rossi-Hansberg (2006), Caliendo, Monte, Rossi-Hansberg (2015).
- **Literature in Physics.** Sornette (2002), Newman (2005), Sornette (2006).
- **Random growth.** Champernowne (1953), Simon, Bonini (1958), Kesten (1973), Sutton (1997), Gabaix (1999), Axtell (2001), Luttmer (2007), Gabaix, Lasry, Lions, Moll (2016).

Overview

Environment

Span of control with 2 layers

Span of control with L layers - Zipf's law

Empirics

Labor income distribution

Conclusion

Environment

Span of control with 2 layers

Span of control with L layers - Zipf's law

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Conclusion

A Garicano (2000) Economy 1/2

- Agents: continuum, measure 1. 1 unit of time.
- 1 good. 1 unit of time $\rightarrow$ 1 good.
- Agents: different exogenous **skills**. Agent with skill x can solve "problems" in $[0, x]$.
- Distribution of skills x: c.d.f. $F(.)$, density $f(.)$ on $[1-\Delta, 1]$.
  Δ: Heterogeneity in Skills.
  $F(.)$: Skill Distribution.
- Workers encounter **problems** in production. Draw a unit continuum of different problems on $[0, 1]$ in c.d.f. $G(.)$, uniform w.l.o.g. :
  - When they know the solution: produce 1 unit of the good.
  - When they don't: can ask someone else for a solution. $h < 1$: manager's time cost to listen to one problem.
    h: Helping Time.

A Garicano (2000) Economy 2/2

- Assumption 1: x unknown.
- Assumption 2: h low enough: always hierarchies.
- Assumption 3: one manager with time 1 at the top.

Environment

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Imposing 2 layers

- **Planner's problem.** Planner maximizes total output.
- Occupational cutoff: z_2 splits managers (high x) and workers (low x).
- Workers x fail to solve $1 - x$ problems. Time supervising worker x: $h(1 - x)$. Span of control of a manager hiring workers with skill x:

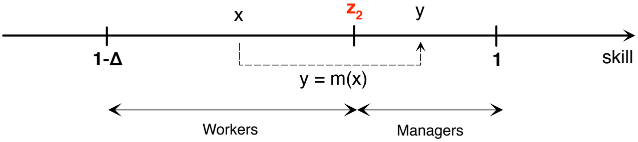
$$\boxed{n = \frac{1}{h(1-x)}}$$

- Output $Q(x, y)$ jointly produced by manager with skill y hiring workers with skill x:

$$Q(x, y) = \frac{y}{h(1-x)} \quad \Rightarrow \quad \frac{\partial^2 Q(x, y)}{\partial x \partial y} = \frac{1}{h(1-x)^2} > 0$$

- Complementarities $\Rightarrow$ **Positive sorting**. $y = m(x)$, $m'(x) > 0$.

Uniform distribution



- $m(.)$ ensures market clearing for time:

  $$f(y)dy = h(1-x)f(x)dx \quad \Rightarrow \quad f(m(x))m'(x) = h(1-x)f(x).$$

- z_2, $m(.)$ unknowns. Boundary value problem:

  $$m(1-\Delta) = z_2, \qquad m(z_2) = 1.$$

- Assume for a moment that $f(x) = 1/\Delta$ on $[1-\Delta, 1]$. Then 1-x is a uniform distribution on $[1-z_2, \Delta]$. What is the distribution of span of control:

  $$n(y) = \frac{1}{h(1-x)}.$$

Mathematical Result: Inverse of a Uniform on $[\Delta^2, \Delta]$

Lemma

If $\mathbf{U} \sim$ *Uniform* $([\Delta^2, \Delta])$, *then*
$\mathbf{1/U} \sim$ *Truncated Pareto* $(1, 1/\Delta, 1/\Delta^2)$.

- Assume $f_U(u) = 1/(\Delta - \Delta^2)$ on $[\Delta^2, \Delta]$. The "tail function" (complementary c.d.f) of $1/U$ is:

$$\begin{aligned}
\bar{F}_{1/U}(x) \equiv 1 - F_{1/\mathrm{U}}(x) &= \mathbb{P}\left[\frac{1}{U} \geq x\right] \\
&= \mathbb{P}\left[U \leq \frac{1}{x}\right] \\
&= \int_{\Delta^2}^{1/x} f_U(u)du \\
\bar{F}_{1/U}(x) \equiv 1 - F_{1/\mathrm{U}}(x) &= \frac{\frac{1}{x} - \Delta^2}{\Delta - \Delta^2}.
\end{aligned}$$

- Inverse of a Uniform on $[0, \Delta]$ = **full Pareto** with tail coefficient 1.

Mathematical Result 2: Inverse of a Uniform on $[\Delta^2, \Delta]$

- Span of control of manager y hiring workers with skill x:

$$n(y) = \frac{1}{h(1-x)}$$

- If $f(.)$ is uniform, $1-x$ is a uniform distribution over $[1-z_2, \Delta]$.
- I show that:

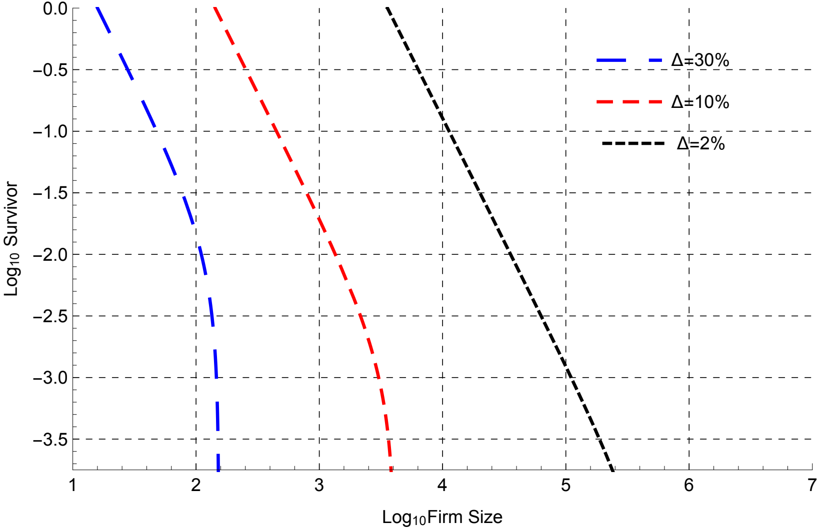
$$1 - z_2 = \frac{\sqrt{1+h^2\Delta^2} - 1}{h} \sim_{\Delta \to 0} \frac{h}{2}\Delta^2.$$

- Thus the **size-biased distribution** is a Truncated Pareto (1).
- Size-biased distribution: a firm with 100 employees is counted 100 times. $\Rightarrow$ Overstating fattailedness.
- **Size-biased distribution** is Truncated Pareto (1) $\Rightarrow$ **distribution** is Truncated Pareto (2):

$$f_S(x) \sim \frac{1}{x^2} \quad \text{and} \quad f_S(x) \sim x f(x) \quad \Rightarrow \quad f(x) \sim \frac{1}{x^3}.$$

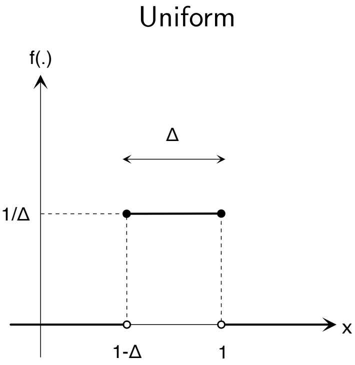
Pareto plot

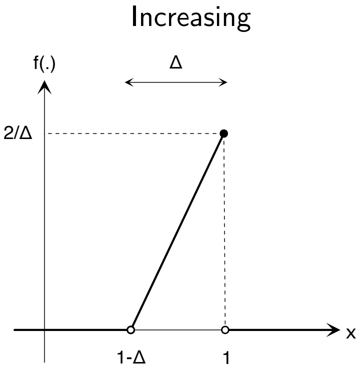
Example: $h = 70\%$, $f(.)$ uniform with $\Delta = 30\%, 10\%, 2\%$.



Non-uniform distribution

- What happens if f is not uniform?
- Example with an increasing distribution.



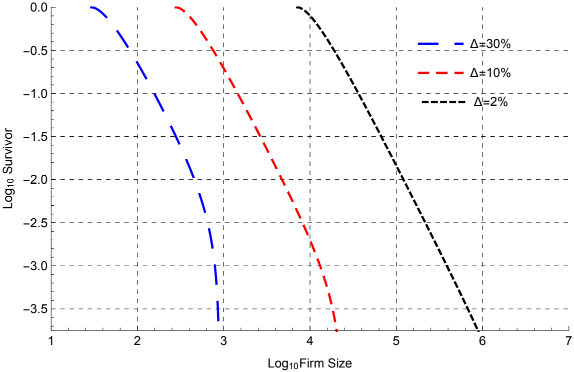


Non-uniform distribution

- "blowing up" of the denominator $\Rightarrow$ under some regularity conditions on $f(.)$, works also if not uniform.
- If $f_X(0) \neq 0$ (some mass at 0), then **Pareto tail**:

$$1 - F_{1/X}(x) = \int_0^{1/x} f_X(u)du \sim_{+\infty} \frac{f_X(0)}{x}.$$

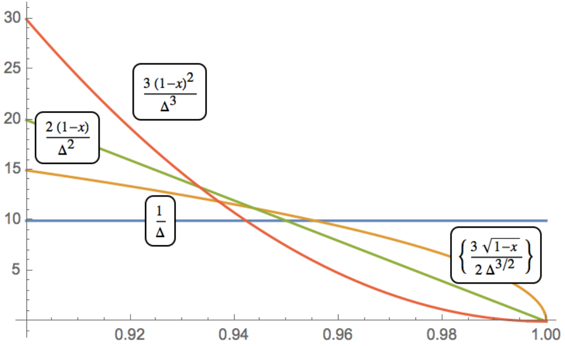
- Example with a linear increasing density.



Relaxing $f(1) > 0$

- If $f(1) = 0$. Illustration: polynomial functions:

$$f(x) = \frac{\rho + 1}{\Delta^{\rho+1}}(1 - x)^{\rho} \quad \text{if } x \in [1 - \Delta, 1]$$



Relaxing $f(1) > 0$

- Closed-form for span of control. Truncated Pareto$(2+\rho)$:

$$n(y) = \frac{1}{h}\left[\frac{1}{h}\frac{\rho+2}{\rho+1}(1-y)^{\rho+1} + (1-z_2)^{\rho+2}\right]^{-\frac{1}{\rho+2}}.$$

- Do not appear in the upper tail, as smaller however. Maximum size $\bar{n}$ is such that:

$$\frac{\bar{n}(\rho>0)}{\bar{n}(\rho=0)} = \Delta^{\frac{\rho}{\rho+1}} \rightarrow_{\Delta\to 0} 0.$$

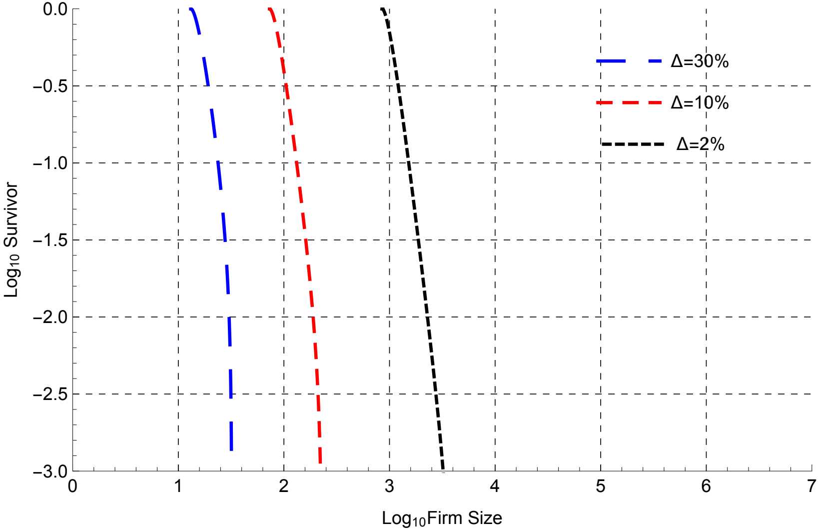
- If $f(1) = 0$, sufficiently regular, i.e. Taylor with $\rho < +\infty$:

$$f(x) = A(1-x)^{\rho} + O\left((1-x)^{\rho+1}\right), \quad \text{with} \quad A > 0.$$

  then similarly, weak form of truncated Pareto$(2+\rho)$. Also smaller.

Relaxing $f(1) > 0$ - example: Beta(1,1)

- Higher tail coefficients, but smaller firms which do not appear in the upper tail.



Environment

Span of control with 2 layers

Span of control with L layers - Zipf's law

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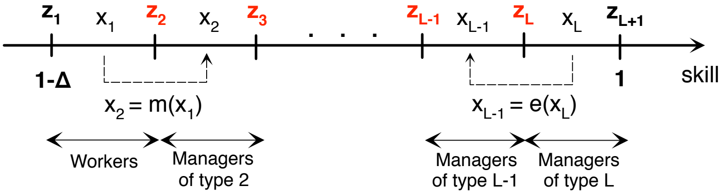
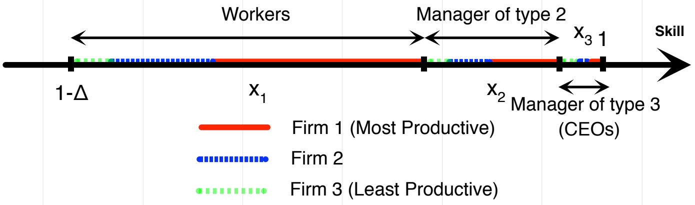
Conclusion

Occupational choice, given L

- In equilibrium, agents split into L types according to their skills in $[1 - \Delta, 1] = [z_1, z_{L+1}]$, and form a hierarchical organization:



Firm with $L = 3$ layers



- Positive Sorting.
- Span of control of manager of type 2 x_2 (same):

$$n_{2\to1}(x_2) = \frac{1}{h(1-x_1)} \quad \Rightarrow \quad f(x_2)dx_2 = h(1-x_1)f(x_1)dx_1.$$

- **Intermediary** Span of control of manager of type 3 x_3:

$$n_{3\to2}(x_3) = \frac{1}{h\dfrac{1-x_2}{1-x_1}} \quad \Rightarrow \quad f(x_3)dx_3 = h\frac{1-x_2}{1-x_1}f(x_2)dx_2.$$

Firm with $L = 3$ layers

- **Total** span of control of manager of type 3 x_3:

$$n_{3\to1}(x_3) = n_{3\to2}(x_3)n_{2\to1}(x_2) = \frac{1}{h^2(1-x_2)}.$$

- Previously we had:

$$f(x_2)dx_2 = h(1-x_1)f(x_1)dx_1 \quad \Rightarrow \quad 1-x_2 \sim (1-x_1)^2.$$

- Now we have:

$$f(x_3)dx_3 = h\frac{1-x_2}{1-x_1}f(x_2)dx_2 \quad \Rightarrow 1-x_3 \sim (1-x_2)^{3/2}.$$

- Intuitively, exponent on the matching function gives the tail index of the Pareto, thus:

$$\alpha_3 = \frac{3}{2}.$$

Any L

- First layer always special with:

$$\boxed{m'(x_1)f(m(x_1)) = h(1-x_1)f(x_1).}$$

- Subsequent layers $l \in [2, ..., L-1]$ with conditional probability:

$$\boxed{m'(x_l)f(m(x_l)) = h\frac{1-x_l}{1-m^{-1}(x_l)}f(x_l).}$$

- Matching the more skilled and less skilled:
  - $L-1$ initial conditions.
  - $L-1$ equations for occupational cutoffs.

- Equilibrium number of layers: fixed cost, or indivisibility with a discrete number N of agents:

$$L = \max_L \left\{ L \quad \text{s.t.} \quad 1 - z_L \geq \frac{1}{N} \right\}.$$

Zipf's law for firm sizes

- Total span of control $n(x_L) \equiv n_{L\to 1}(x_L)$ is given by:

$$n(x_L) = n_{L\to L-1}(x_L) * n_{L-1\to L-2}(x_{L-1}) * \dots * n_{2\to 1}(x_2)$$
$$n(x_L) = \prod_{l=1}^{L-1} n_{l+1\to l}(x_l).$$

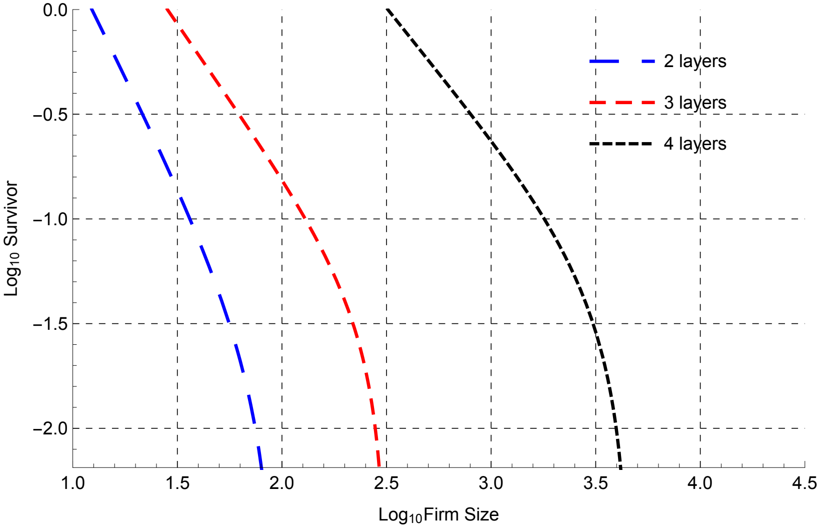
- Generalizing $\alpha_2 = 2$ and $\alpha_3 = 3/2$ by iteration, the tail exponent for $n(x_L)$ is:

$$\boxed{\alpha_L = 1 + \frac{1}{L-1}}.$$

- When $L \to \infty$, **Zipf's law** for firm sizes:

$$\boxed{\alpha_{+\infty} = 1}.$$

Many layers

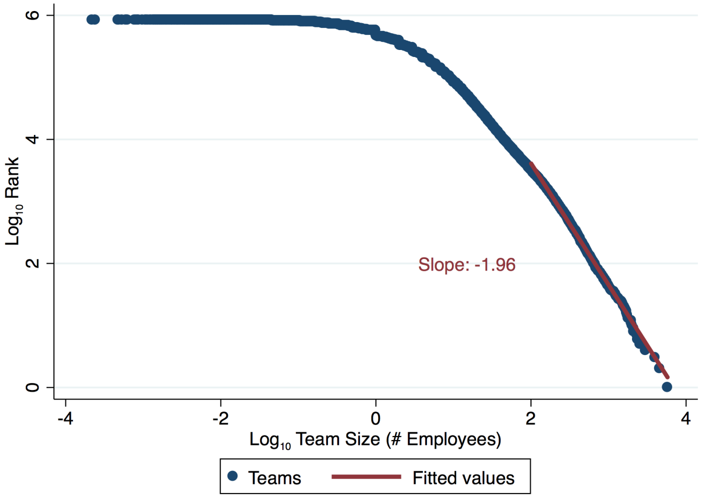

2 layers
3 layers
4 layers
0.0
−0.5
−1.0
−1.5
−2.0
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
Log_{10} Survivor
Log_{10}Firm Size

Higher level of disaggregation

Using French matched employer-employee data, and Caliendo, Monte, Rossi-Hansberg's (JPE, 2015) methodology.

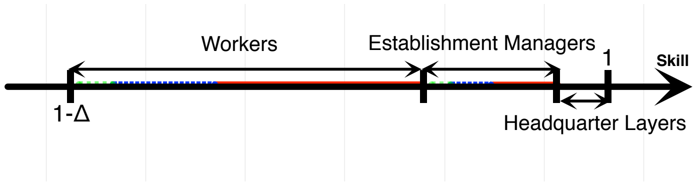
- Use of "PCS-ESE": Profession Catégorie Socioprofessionnelle.
- First Digit Corresponds to one of 6 categories:
  1. **Farmers**
  2. **Self-employed / Owners**: *Plumbers, film directors, CEOs*.
  3. **Senior staff or top management positions**: *CFOs, heads of HRs, purchasing managers.*
  4. **Employees at the supervisor level**: *Quality control technicians, sales supervisors.*
  5. **Clerical, white-collar employees**: *Secretaries, HR or accounting, sales employees.*
  6. **Blue-collar workers**: *Assemblers, machine operators, maintenance workers.*
- Form "teams" in establishments, dividing the # of employees in a layer by the # of employees in the layer above.

French DADS - Distribution of "teams"

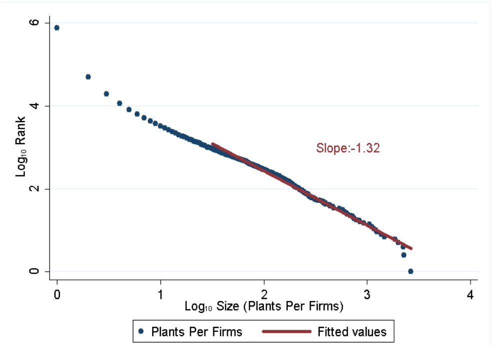


- Data lends support to Zipf's law as **compounding of elementary Pareto (2)** $\neq$ Random Growth .

French DADS - establishments per firms

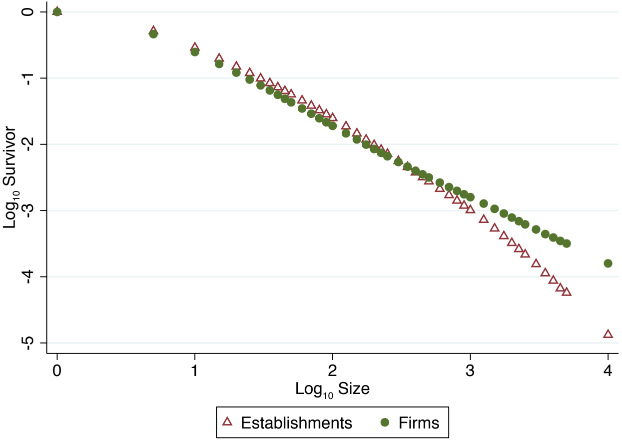


- Pareto on most of the range consistent with the model: the uniform distribution = better approximation locally.



Distribution of US firms and establishments. Source: Census bureau.

- Equivalent for the US? Establishment Level.



Firms: $\boxed{1.01}$ **Establishments:** $\boxed{1.33}$

Assignment equation

- Skill prices $w(.)$ decentralizing optimal allocations:

$$w(y) = \max_{x} \frac{y - w(x)}{h(1-x)}.$$

- Envelope condition:

$$w'(y) = \frac{1}{h(1-x)} = n(y) \quad \Rightarrow \quad \boxed{\underbrace{\frac{dw(y(n))}{dn}}_{\Delta\text{Wages}} = \underbrace{n(y(n))}_{\text{Size}} \; \underbrace{y'(n)}_{\Delta\text{Talents}}}.$$

- Comparison:
  - Gabaix, Landier (2008). **Small** differences in talent across managers, **large and Pareto** firm sizes $\Rightarrow$ Large differences in pay.
  - This paper: **Small** differences in talents across **workers and managers** $\Rightarrow$ Large differences in pay. (through endogenous large and Pareto firm sizes)

Integrating truncated Pareto distributions

- Slight difference: Zipf's law is truncated $\Rightarrow$ hypergeometric functions instead of exact Pareto distributions.

- Example:

$$f(x) = \begin{cases} A_1 & \text{if} \quad x \in [1-\Delta_1-\Delta_2, 1-\Delta_2] \\ A_2(\rho+1)(1-x)^\rho & \text{if} \quad x \in [1-\Delta_2, 1] \end{cases}$$

- Comparative statics shown 2 layer case, where this is an hypergeometric function:

$$w(y) = w(z_2) + \int_{z_2}^{y} \frac{du}{h\sqrt{(1-z_2)^2 + \frac{2}{h}\frac{A_2}{A_1}(1-u)^{\rho+1}}}.$$

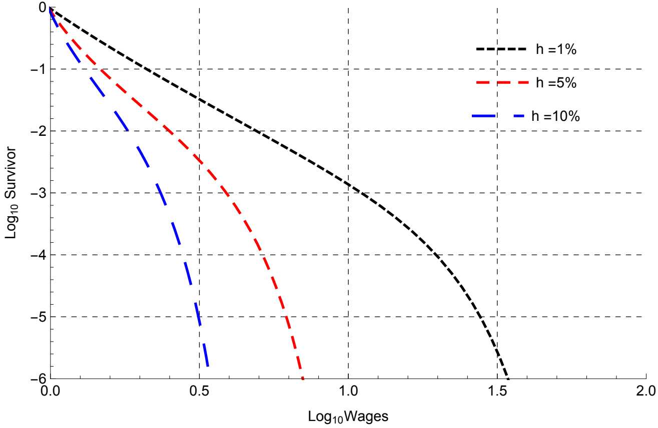
Reduced form VS full model

Apart from positive aspects is a reduced form approach sufficient? Not always.

- **Calculate all wages.** "Trickle-down" effects.
- **Relate change in firm sizes to deep parameters**. Here h and Δ shift the distribution out.
- And: truncation is key for comparative statics of the Pareto distribution:
  - Gabaix and Landier (2008) attribute the 5x increase in CEO compensation to a 5-fold in the scale. h or Δ.
  - Difficulty: $\alpha = -3$ in 1970s to $\alpha = -1.8$ now. In Gabaix and Landier (2008), α is constant.

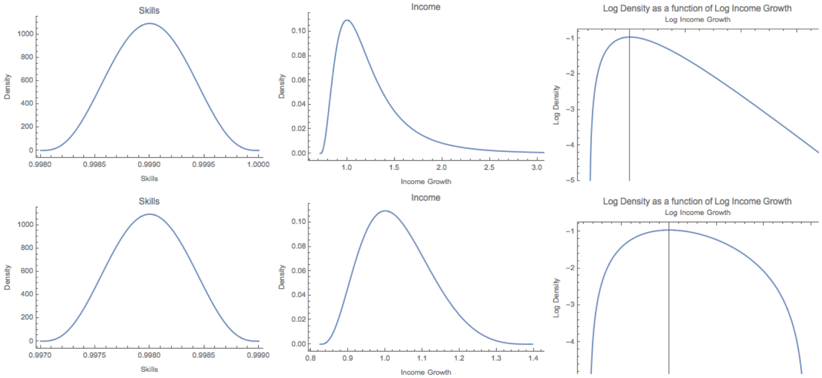
Labor income distribution: effect of a decrease in h (IT?)

- Gabaix, Landier (2008): if skill distribution does not change, Pareto coefficient does not change.
- Not true in this paper when h diminishes (IT?).

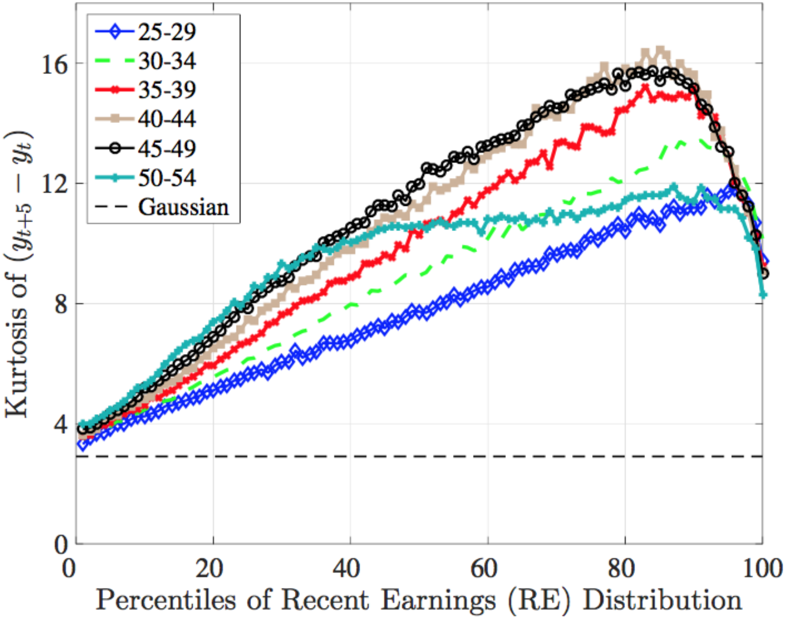


Telling theories of income apart: Dynamics

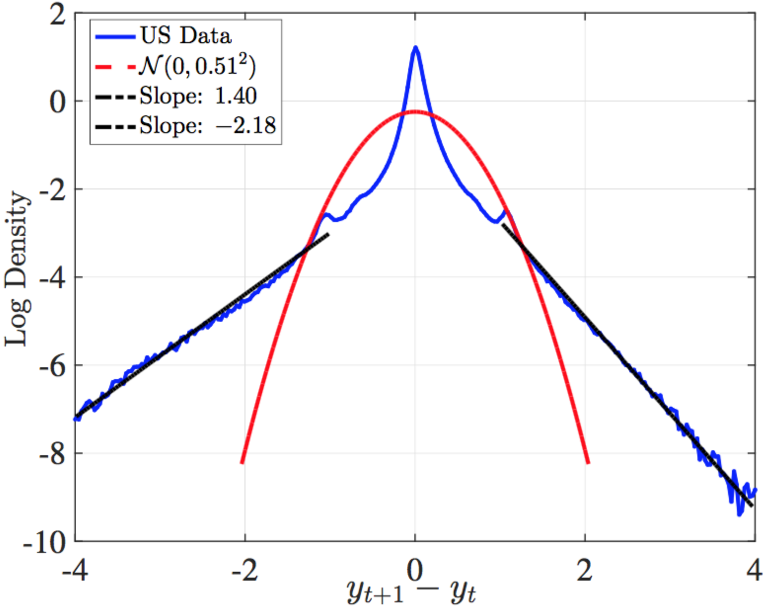
1. Pareto distributed primitives (Lucas (1978), Mirrlees (1971)). But Francis Galton: mental abilities normally distributed.
2. Paretos from random growth models. (Gabaix, Lasry, Lions, Moll (2016))
3. Paretos from production functions. Small shocks to y. Non-linear mapping $w(y)$.



Guvenen, Karahan, Ozkan, Song (2016)

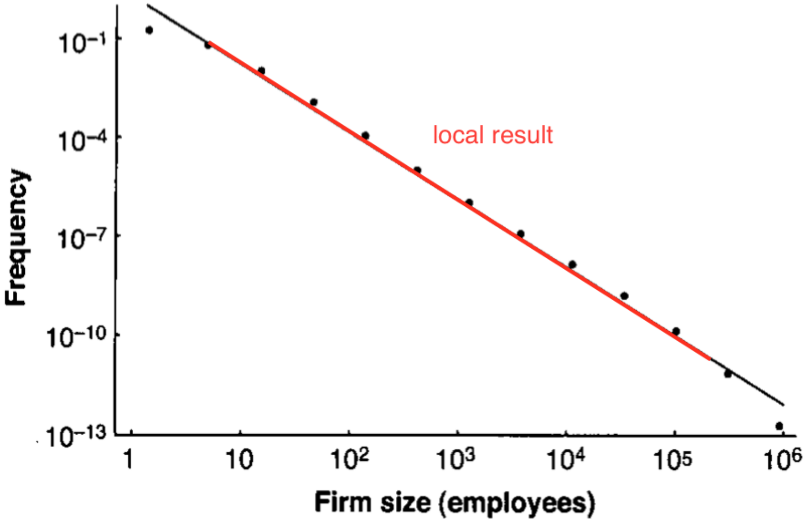


Guvenen, Karahan, Ozkan, Song (2016)



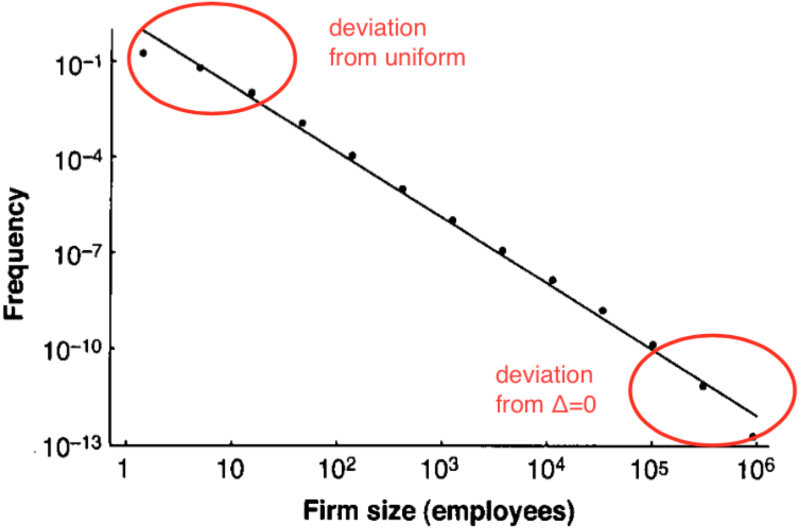
Conclusion: coming back to Axtell (2001)

What does not matter for heterogeneity under a power law production function



Conclusion: coming back to Axtell (2001)

What matters for heterogeneity under a power law production function



Conclusion

- **Main takeaways**:
  - **Maths**:
    - U is Uniform $(0,\Delta) \Rightarrow 1/U$ is Pareto $(1, 1/\Delta)$.
    - X goes through the origin $\Rightarrow 1/X$ has a Pareto tail.
  - Stylized model accounts for **Pareto firm size and labor income distribution**, regardless of the ability distribution.
  - New intuition for why firm sizes and labor incomes are so heterogenous **despite small observable differences**: "power law change of variable near the origin".
  - Endogenous "economics of superstars".
- **Future work**:
  - **Other microfoundations** for power-law production functions.
  - **In applied work**, potential alternative to:
    - <u>Optimal taxation</u>: Pareto distributed skills.
    - <u>Trade</u>: Pareto distributed firm productivities.
    - <u>Misallocation</u>: Pareto distributed manager/firm productivities.