

# Lecture 2 - The Solow Growth Model

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The first part of this note considers the case of a Solow [1956] growth model with a general, constant returns to scale, production function. The second part of the note looks at a special case of the Solow [1956] growth model for a case of a Cobb and Douglas [1928] production function.

## 1 General Production Function

### 1.1 Assumptions

Solow [1956] starts from a general production function, giving at any point in time output  $Y_t$  as a function of inputs, capital  $K_t$  and labor  $L_t$ :

$$Y_t = F(K_t, L_t).$$

A very important assumption is also **constant returns to scale** with respect to capital and labor, so that for any scaling factor  $a$ :

$$F(aK_t, aL_t) = aF(K_t, L_t).$$

For simplicity, we shall assume from now on that the quantity of labor is fixed with  $L_t = L$ , so that the production function becomes  $Y_t = F(K_t, L)$ . Because of constant returns to scale with respect to capital and labor (and setting  $a = 1/L$  in the previous expression), we have:

$$\frac{Y_t}{L} = F\left(\frac{K_t}{L}, 1\right) = f\left(\frac{K_t}{L}\right)$$

where  $f$  is defined as a function of  $F$  such that

$$f(x) \equiv F(x, 1).$$

An example of such a production function is the Cobb and Douglas [1928] production function, which we started studying in Lecture 1, and which we look at in the next section.

Solow [1956] abstracts from public saving, so that **total saving** at time  $t$  equals **private saving** at time  $t$ , and both are denoted  $S_t$ , which also equals investment  $I_t$  at time  $t$ :

$$S_t = I_t.$$

Saving is assumed to be a constant fraction  $s$  of output  $Y_t$ , and therefore:

$$S_t = sY_t.$$

This constant saving rate may seem a bit ad-hoc; it is. We will investigate more in detail the determinants of saving and consumption behavior in the next lectures. Depreciation of capital is given by a share  $\delta$  (think for example that 8% of the capital stock depreciates each period; the rate of depreciation is much lower for structures, and much higher for computers). The capital stock evolves according to:

$$K_{t+1} = (1 - \delta) K_t + I_t$$

## 1.2 Solution

Replace investment in the previous equation and divide both sides by  $L$ :

$$\frac{K_{t+1}}{L} = (1 - \delta) \frac{K_t}{L} + s \frac{Y_t}{L} \quad \Rightarrow \quad \boxed{\frac{K_{t+1}}{L} - \frac{K_t}{L} = s \frac{Y_t}{L} - \delta \frac{K_t}{L}}$$

The change in the capital stock per person from  $t$  to  $t + 1$  has two components: investment (or saving) and depreciation:

$$\underbrace{\frac{K_{t+1}}{L} - \frac{K_t}{L}}_{\text{Change in capital}} = \underbrace{s f\left(\frac{K_t}{L}\right)}_{\text{Investment}} - \underbrace{\delta \frac{K_t}{L}}_{\text{Depreciation}}.$$

The steady state level of the capital stock  $K^*$  is such that  $K_{t+1} = K_t = K^*$ , and it therefore satisfies:

$$\boxed{s f\left(\frac{K^*}{L}\right) = \delta \frac{K^*}{L}}$$

Note that without further specifying  $f(\cdot)$ , we can't say much more about the value of  $K^*/L$ , we just know it satisfies this implicit equation. The steady-state value of output per worker  $Y^*/L$ , as a function of  $K^*/L$  is given by:

$$\frac{Y^*}{L} = f\left(\frac{K^*}{L}\right)$$

## 1.3 Three cases

There are 3 cases:

1. If capital per worker is relatively low, that is  $K_t/L < K^*/L$ , then investment per worker is larger than depreciation per worker, and therefore from the above equation, capital per worker increases:

$$\frac{K_{t+1}}{L} > \frac{K_t}{L}$$

2. If capital per worker is exactly equal to steady state capital per worker, that is  $K_t/L = K^*/L$ , then investment per worker is equal to depreciation per worker, and therefore from the above equation, capital per worker stays constant:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} = \frac{K^*}{L}$$

3. If capital per worker is relatively high, that is  $K_t/L > K^*/L$ , then depreciation per worker is larger than investment per worker, and therefore, capital per worker declines:

$$\frac{K_{t+1}}{L} < \frac{K_t}{L}.$$

## 2 Cobb and Douglas [1928] production function

### 2.1 Solving for the model

Assume now that the production function is a Cobb and Douglas [1928] production function, so that:

$$F(K, L) = K^\alpha L^{1-\alpha}$$

As we saw during lecture 1,  $\alpha$  should be thought of as roughly equal to  $\alpha = 1/3$ . This implies then that function  $f$  defined above is such that:

$$f(x) = x^\alpha$$

The law of motion for capital is given by:

$$\frac{K_{t+1}}{L} = \frac{K_t}{L} + s \left( \frac{K_t}{L} \right)^\alpha - \delta \frac{K_t}{L}.$$

Given  $L, K_0, \alpha, s, \delta$ , we are able to calculate  $K_1, K_2, \dots$ , as well as  $K_t$  for any  $t$ , by calculating the quantities of capital successively from the formula above.

If you do so, you will notice that  $K_t$  converges to a steady state value  $K^*$ . However, you do not need to perform an infinity of operations to get at this  $K^*$ . Instead, you can see that capital per worker in steady-state  $K^*/L$  solves:

$$s \left( \frac{K^*}{L} \right)^\alpha = \delta \frac{K^*}{L} \Rightarrow \boxed{\frac{K^*}{L} = \left( \frac{s}{\delta} \right)^{\frac{1}{1-\alpha}}}$$

What was an implicit equation in the previous section can now be solved for explicitly. The steady-state level of output per worker is then:

$$\frac{Y^*}{L} = \left( \frac{s}{\delta} \right)^{\frac{\alpha}{1-\alpha}}$$

We are finally able to compute the capital to output ratio  $K^*/Y^*$  from the Solow growth model:

$$\frac{K^*}{Y^*} = \frac{K^*/L}{Y^*/L} = \left( \frac{s}{\delta} \right)^{\frac{1}{1-\alpha}} \left( \frac{s}{\delta} \right)^{-\frac{\alpha}{1-\alpha}} = \frac{s}{\delta}.$$

Alternatively, you may obtain this expression much more simply by equating saving  $sY^*$  to investment  $\delta K^*$  at the steady state:

$$sY^* = \delta K^* \Rightarrow \boxed{\frac{K^*}{Y^*} = \frac{s}{\delta}}.$$

## 2.2 Golden Rule

Most economists believe that policymakers should not care so much about GDP per person, but rather about consumption per person (however, some people hold a different view – we shall talk about that later). The intuition is simple: if an economy was to produce many goods which were only used for investment purposes (which would be the case if  $s = 1$ ), then people in this economy would be starving, even though it was actually producing a lot. Investment, ultimately, should serve to increase future consumption.

The **Golden Rule level of capital accumulation** is such that the level of steady-state consumption per capita is maximized. The steady-state consumption per capita is given by:

$$\frac{C^*}{L} = (1-s)\frac{Y^*}{L} = (1-s)\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} = \frac{(1-s)s^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}$$

Maximizing this steady state consumption with respect to the saving rate  $s$  consists in finding the maximum of that function with respect to  $s$ :

$$\frac{d(C^*/L)}{ds} = 0 \quad \Rightarrow \quad \frac{d[(1-s)s^{\frac{\alpha}{1-\alpha}}]}{ds} = 0$$

Note that the  $1/\delta^{\alpha/(1-\alpha)}$  is just a constant which does not change anything to the maximization. If you are not convinced, then you may also compute the derivative with respect to the whole  $C^*/L$  expression. This gives:

$$\begin{aligned} -s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)s^{\frac{\alpha}{1-\alpha}-1} &= 0 \quad \Rightarrow \quad \frac{\alpha}{1-\alpha} \frac{1-s}{s} = 1 \\ \Rightarrow \quad \alpha - \alpha s &= s - \alpha s \quad \Rightarrow \quad \boxed{s = \alpha}. \end{aligned}$$

Therefore, the saving rate corresponding to the Golden Rule level of capital accumulation is equal to  $\alpha$  (again, taking  $\alpha$  to be equal to roughly 1/3, this would suggest that an economy would optimally need to save about a third of its production every year).

The Golden Rule level of capital accumulation is then such that capital at the steady-state is given as a function of the exogenous parameters by:

$$\frac{K^*}{L} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} \quad \Rightarrow \quad K^* = L \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$$

The level of GDP corresponding to this Golden rule level is:

$$Y^* = L \left(\frac{\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

## References

- Charles W. Cobb and Paul H. Douglas. A Theory of Production. *The American Economic Review*, 18(1): 139–165, 1928. ISSN 0002-8282. URL <http://www.jstor.org/stable/1811556>.
- Robert M. Solow. A Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics*, 70(1):65–94, 1956. ISSN 0033-5533. doi: 10.2307/1884513. URL <http://www.jstor.org/stable/1884513>.