DIFFERENTIAL GRADED CATEGORIES

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1. Lecture 1: review of chain complexes

1.1. **Basics.**

Setup 1.1. We will work in the following framework:

- Everything will be over a fixed base *commutative ring* $\mathbf{k}$ (always with unit). If $\mathbf{k} = \mathbb{Z}$, we get chain complexes of abelian group; if $\mathbf{k}$ is a field, we get chain complexes of $\mathbf{k}$-vector spaces.
- We will use *cohomological notation*. This means that we will have increasing indices. Hence, we actually work with *cochain* complexes, but we will simplify terminology and still refer to them as *chain* complexes.
- Notationally, we will sometimes drop parentheses, especially when dealing with differential maps. So, we will write dx instead of $d(x)$.

Definition 1.2. A *chain complex* (sometimes also called just *complex*) is a sequence of $\mathbf{k}$-modules $(V^i i)_{i\in\mathbb{Z}}$ together with morphisms $d^i \colon V^i \to V^{i+1}$ such that

$$d^{i+1} \circ d^i = 0,$$

for all $i \in \mathbb{Z}$. We picture this data as follows:

$$\cdots \to V^i \xrightarrow{d^i} V^{i+1} \xrightarrow{d^{i+1}} V^{i+2} \to \cdots$$

A *morphism* (or *chain map*)

$$f \colon (V^i, d_V^i) \to (W^i, d_W^i)$$

between chain complexes (V^i, d_V^i) and (W^i, d_W^i) is a family of morphisms of $\mathbf{k}$-modules

$$f^i \colon V^i \to W^i,$$

such that $d_W^i \circ f^i = f^{i+1} \circ d_V^i$, for all $i \in \mathbb{Z}$. In other words, the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & V^{i-1} \arrow[r, "d_V^{i-1}"] \arrow[d, "f^{i-1}"] & V^i \arrow[r, "d_V^i"] \arrow[d, "f^i"] & V^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & W^{i-1} \arrow[r, "d_W^{i-1}"] & W^i \arrow[r, "d_W^i"] & W^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 1.3. *Let $V = (V^i, d^i)$ be a chain complex. The identity morphism*

$$1_V \colon V \to V,$$

defined by identities $1_{V^i} \colon V^i \to V^i$ for all $i \in \mathbb{Z}$, is a chain map.

Date: March 8, 2022.

If $f\colon (V^i, d_V^i) \to (W^i, d_W^i)$ and $g\colon (W^i, d_W^i) \to (Z^i, d_Z^i)$ are chain maps, the composition morphism

$$g \circ f\colon (V^i, d_V^i) \to (Z^i, d_Z^i),$$

defined by the compositions $g^i \circ f^i\colon V^i \to Z^i$ for all $i \in \mathbb{Z}$, is also a chain map.

Proof. Exercise 1.1. □

Definition 1.4. Let $f\colon (V^i, d_V^i) \to (W^i, d_W^i)$ be a chain map of complexes. We say that f is an *isomorphism* if there is a (unique) chain map $f^{-1}\colon (W^i, d_W^i) \to (V^i, d_V^i)$ such that $f^{-1} \circ f = 1_V$ and $f \circ f^{-1} = 1_W$.

Isomorphisms are completely understood "componentwise", namely:

Lemma 1.5. *A chain map $f = (f^i)\colon (V^i, d_V^i) \to (W^i, d_W^i)$ is an isomorphism if and only if $f^i\colon V^i \to W^i$ is an isomorphism of* **k***-modules for all $i \in \mathbb{Z}$.*

Proof. If f is an isomorphism, clearly all f^i are isomorphisms by definition. Conversely, assume that all $f^i\colon V^i \to W^i$ are isomorphisms, with inverses $(f^i)^{-1}$. There is a unique way of defining $f^{-1}\colon (W^i, d_W^i) \to (V^i, d_V^i)$, namely:

$$(f^{-1})^i = (f^i)^{-1}.$$

We just need to check that f^{-1} is indeed a chain map. f is a chain map, so by definition

$$d_W^i \circ f^i = f^{i+1} \circ d_V^i$$

for all $i \in \mathbb{Z}$. We may compose with inverses of f^i and f^{i+1}, and obtain:

$$(f^{i+1})^{-1} \circ d_W^i = d_V^i \circ (f^i)^{-1}.$$

This means precisely that f^{-1} is a chain map. □

Remark 1.6. It is useful to identify a chain complex $V = (V^i, d_V^i)$ with the following **k**-module:

$$V = \bigoplus_i V^i,$$

keeping track of the direct sum decomposition over $\mathbb{Z}$ and together with a morphism $d_V\colon V \to V$ (obtained uniquely from the d^i) with the following properties:

$$\begin{aligned} d_V(V^i) &\subseteq V^{i+1} \qquad \text{for all } i \in \mathbb{Z}, \\ d_V \circ d_V &= 0. \end{aligned}$$

Moreover, a chain map $f\colon (V^i, d_V^i) \to (W^i, d_W^i)$ can be identified with a morphism

$$f\colon V = \bigoplus_i V^i \to \bigoplus_i W^i = W,$$

with the following properties:

$$\begin{aligned} f(V^i) &\subseteq W^i, \qquad \text{for all } i \in \mathbb{Z}, \\ d_W \circ f &= f \circ d_V. \end{aligned}$$

As an exercise (cf. Exercise 1.2), we may check that this description of chain complexes and chain maps is indeed equivalent to the one given in Definition 1.2. In what follows, we will go back and forth between these two equivalent definitions without mentioning it.

Remark 1.7. Any k-module M can be viewed as a chain complex concentrated in degree 0, namely:

$$\begin{aligned} M^0 &= M, \\ M^i &= 0, \qquad i \neq 0. \end{aligned}$$

The differentials d^i_M are all zero.

The most important operation that we can perform on a chain complex is taking *cohomology*.

Definition 1.8. Let $V = (V^i, d^i)$ be a chain complex. For all $i \in \mathbb{Z}$, we define:

$$\begin{aligned} Z^i(V) &= \ker(d^i \colon V^i \to V^{i+1}), \\ B^i(V) &= \operatorname{Im}(d^{i-1} \colon V^{i-1} \to V^i). \end{aligned}$$

Both $Z^i(V)$ and $B^i(V)$ are submodules of V^i. They are sometimes called (respectively) *i-cocycles* and *i-coboundaries*, or (again, respectively) *closed degree i elements* and *exact degree i elements*.

The assumption that $d^i \circ d^{i-1} = 0$ ensures that $B^i(V) \subseteq Z^i(V)$. Hence, we may take the quotient:

$$H^i(V) = Z^i(V)/B^i(V).$$

This is called *i-th cohomology* of the chain complex V.

If $v \in Z^i(V)$, we will denote by $[v]$ the image of v in $H^i(V)$ with respect to the natural projection $Z^i(V) \to H^i(V)$.

Notation 1.9. Thanks to the identification described in Remark 1.6, we may and will view $V^i, Z^i(V), B^i(V)$ as submodules of $V = \bigoplus_i V^i$, for all $i \in \mathbb{Z}$. If an element $v \in V$ lies in V^i, we say that it has *degree i* and we write:

$$\deg(v) = |v| = i.$$

Proposition 1.10 (Functoriality)**.** *Let $f \colon V \to W$ be a chain map of complexes. For all $i \in \mathbb{Z}$, f restricts to morphisms*

$$\begin{aligned} Z^i(f) &\colon Z^i(V) \to Z^i(W), \\ B^i(f) &\colon B^i(V) \to B^i(W). \end{aligned}$$

This also induces a morphism

$$H^i(f) \colon H^i(V) \to H^i(W).$$

Moreover, if V is a chain complex, we have

$$\begin{aligned} Z^i(1_V) &= 1_{Z^i(V)} \colon Z^i(V) \to Z^i(V), \\ B^i(1_V) &= 1_{B^i(V)} \colon B^i(V) \to B^i(V), \\ H^i(1_V) &= 1_{H^i(V)} \colon H^i(V) \to H^i(V), \end{aligned}$$

for all $i \in \mathbb{Z}$. If $f \colon V \to W$ and $g \colon W \to X$ are chain maps, we have equalities:

$$\begin{aligned} Z^i(g \circ f) &= Z^i(g) \circ Z^i(f), \\ B^i(g \circ f) &= B^i(g) \circ B^i(f), \\ H^i(g \circ f) &= H^i(g) \circ H^i(f). \end{aligned}$$

Proof. Let $f\colon V \to W$ and $g\colon W \to X$ be chain maps as above. If $x \in Z^i(V)$, namely $dx = 0$, we have $d(f(x)) = f(dx) = f(0) = 0$, so f indeed yields a restricted morphism $Z^i(f)\colon Z^i(V) \to Z^i(W)$. Analogusly, if $x = dx' \in B^i(V)$, then $f(x) = f(dx') = df(x') \in B^i(W)$, so f also restricts to a morphism $B^i(f)\colon B^i(V) \to B^i(W)$. Equalities $Z^i(g \circ f) = Z^i(g) \circ Z^i(f)$ and $B^i(g \circ f) = B^i(g) \circ B^i(f)$, and $Z^i(1_V) = 1_{Z^i(V)}, B^i(1_V) = 1_{B^i(V)}$ follow immediately.

The induced morphism $H^i(f)\colon H^i(V) \to H^i(W)$ is the unique which makes the following diagram commute:

$$\begin{tikzcd}
Z^i(V) \arrow[r, "Z^i(f)"] \arrow[d, "\mathrm{pr}_V"'] & Z^i(W) \arrow[d, "\mathrm{pr}_W"] \\
H^i(V) \arrow[r, dashed, "H^i(f)"'] & H^i(W),
\end{tikzcd}$$

where the vertical morphisms are the canonical projections onto the quotients. By such uniqueness, we easily see that indeed $H^i(1_V) = 1_{H^i(V)}$ and $H^i(g \circ f) = H^i(g) \circ H^i(f)$. $\square$

Remark 1.11. Let V be a chain complex. It is sometimes useful to collect all cohomologies $H^i(V)$ to define

$$H^*(V) = \bigoplus_i H^i(V).$$

This is a *graded module*, namely, a $\mathbf{k}$-module together with a direct sum decomposition over the integers. This can be also described just as the sequence of $\mathbf{k}$-modules $(H^i(V))_{i\in\mathbb{Z}}$ (compare with the case of complexes described in Remark 1.6).

If $f\colon V \to W$ is a chain map of complexes, the collection $(H^i(f))_{i\in\mathbb{Z}}$ defines a morphism of graded modules

$$H^*(f)\colon H^*(V) \to H^*(W).$$

Clearly, we have compatibility with identities and compositions:

$$H^*(1_V) = 1_{H^*(V)},$$
$$H^*(g \circ f) = H^*(g) \circ H^*(f).$$

1.2. **Hom and tensor.** If M and N are $\mathbf{k}$-modules, there is a $\mathbf{k}$-module $\mathrm{Hom}(M, N)$ of morphisms $M \to N$, with the obvious elementwise addition and action of $\mathbf{k}$. We may also define the *tensor product* $M \otimes N$. Hom and tensor are connected by the following natural isomorphism:

$$\mathrm{Hom}(V \otimes W, X) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, X)),$$

which is sometimes called the "hom-tensor adjunction". We are going to discuss this in the framework of chain complexes.

Hom-complexes. If V and W are chain complexes, we denote by

$$\mathrm{Hom}(V, W)$$

the $\mathbf{k}$-module of chain maps $V \to W$, with the obvious elementwise addition and action of $\mathbf{k}$ (see Exercise 1.3). Such $\mathbf{k}$-module can itself be "enhanced" to a chain complex, as follows.

Definition 1.12. Let $V = (V^i, d^i_V)$ and $W = (W^i, d^i_W)$ be chain complexes. We define a chain complex

$$\underline{\mathrm{Hom}}(V, W) = (\underline{\mathrm{Hom}}^p(V, W), d_{\underline{\mathrm{Hom}}})_{p\in\mathbb{Z}}$$

as follows.

- The $\mathbf{k}$-module $\underline{\mathrm{Hom}}^p(V, W)$ is the $\mathbf{k}$-module of *degree p morphisms*, namely, of morphisms $f\colon V \to W$ such that $f(V^i) \subseteq W^{i+p}$ for all $i \in \mathbb{Z}$. Equivalently, they are sequences of morphisms $(f^i\colon V^i \to W^{i+p})_{i\in\mathbb{Z}}$. We don't require any compatibility with the differentials d_V and d_W.
- The differential
$$d^p_{\underline{\mathrm{Hom}}}\colon \underline{\mathrm{Hom}}^p(V, W) \to \underline{\mathrm{Hom}}^{p+1}(V, W)$$
is defined as follows:
$$d^p_{\underline{\mathrm{Hom}}}(f) = d_W \circ f - (-1)^p f \circ d_V.$$

Notationally, we will almost always write d instead of $d_{\underline{\mathrm{Hom}}}$ for the differential of $\underline{\mathrm{Hom}}(V, W)$. We can directly check (Exercise 1.4) that $d^{p+1} \circ d^p = 0$, hence $\underline{\mathrm{Hom}}(V, W)$ is indeed a complex.

Remark 1.13. Let V and W be chain complexes. What is the $\mathbf{k}$-module $Z^0(\underline{\mathrm{Hom}}(V, W))$? By definition, it contains precisely the morphisms $f\colon V \to W$ such that $f(V^i) \subseteq W^i$ for all $i \in \mathbb{Z}$, and $d_W \circ f - f \circ d_V = 0$. This means that
$$Z^0(\underline{\mathrm{Hom}}(V, W)) = \mathrm{Hom}(V, W)$$
is precisely the $\mathbf{k}$-module of chain maps $V \to W$.

What about the zeroth cohomology $H^0(\underline{\mathrm{Hom}}(V, W))$? Its elements are equivalence classes $[f]$, where f is a chain map $V \to W$. By definition, $[f] = [g]$ if and only if $f - g = dh$ for some $h \in \underline{\mathrm{Hom}}^{-1}(V, W)$. Explicitly, this means:
$$f - g = d_W \circ h + h \circ d_V,$$
namely, that f and g are *chain homotopic*. The degree -1 morphism $h\colon V \to W$ is a chain homotopy between f and g. The effort we made to define the hom-complex $\underline{\mathrm{Hom}}(V, W)$ pays off giving us a better framework to treat such chain homotopies.

Tensor products. We are able to generalize the definition of tensor product to chain complexes.

Definition 1.14. Let V and W be chain complexes. We define the *tensor product* $V \otimes W = ((V \otimes W)^p, d^p_{V\otimes W})$ as follows. First:
$$(V \otimes W)^p = \bigoplus_{i+j=p} V^i \otimes W^j,$$
where $V^i \otimes W^j$ is the usual tensor product of $\mathbf{k}$-modules. Moreover, the differential is defined by:
$$d^p_{V\otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^i v \otimes d_W(w),$$
if $v \in V^i$ and $w \in W^j$ with $i+j = p$ (and then "extending by linearity"). With a direct computation (see Exercise 1.5) we can show that $d^{p+1}_{V\otimes W} \circ d^p_{V\otimes W} = 0$ for all $p \in \mathbb{Z}$, hence $V \otimes W$ is indeed a chain complex. We shall often simplify notation and write d instead of $d_{V\otimes W}$ when the context is clear.

The tensor product of complexes behaves nicely. Namely, it is associative, commutative and unital (with unit being $\mathbf{k}$ viewed as a complex concentrated in degree 0, see Remark 1.7). We list all these properties in the following proposition; the proofs are left as an exercise.

Proposition 1.15. *Let V, W, X be chain complexes. There are natural isomorphisms:*

$$\begin{aligned} &(V \otimes W) \otimes X \xrightarrow{\sim} V \otimes (W \otimes X), \quad (v \otimes w) \otimes x \mapsto v \otimes (w \otimes x), \\ &\mathbf{k} \otimes V \xrightarrow{\sim} V, \quad \lambda \otimes v \mapsto \lambda v, \\ &V \otimes W \xrightarrow{\sim} W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v. \end{aligned}$$

Proof. Exercise 1.6. □

Remark 1.16. It is convenient to comment the "commutativity" isomorphism

$$\begin{aligned} V \otimes W &\xrightarrow{\sim} W \otimes V, \\ v \otimes w &\mapsto (-1)^{|v||w|} w \otimes v. \end{aligned}$$

Assuming that $v \in V^i$ and $w \in W^j$, the tensor $v \otimes w$ is mapped to $(-1)^{ij} w \otimes v$ (and the isomorphism is obtained by extending this by linearity). The occurrence of the sign $(-1)^{ij}$ is an instance of the *Koszul sign rule*.

Warning! What follows are just a few heuristic and informal ideas. To my best understanding, the Koszul sign rule can be informally summarized as follows: *every time we swap two graded symbols a and b, we make the sign* $(-1)^{|a||b|}$ *appear.* This occurs for example in the definition of the differential $d_{V \otimes W}$ of the tensor product $V \otimes W$ (cf. Definition 1.14):

$$d_{V \otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w).$$

The second summand is obtained by "switching" the triple of symbols (d_V, v, w) with the triple (v, d_W, w). The symbols d_V and d_W have degree 1, and $(-1)^{|v|} = (-1)^{1|v|}$ is the sign which correctly appears according to the rule.

We may finally state the "tensor-hom adjunction" for chain complexes.

Proposition 1.17. *Let V, W, X be chain complexes. There are natural isomorphisms of complexes, one inverse to the other:*

$$\begin{aligned} \Psi \colon \underline{\mathrm{Hom}}(V \otimes W, X) &\xrightarrow{\sim} \underline{\mathrm{Hom}}(V, \underline{\mathrm{Hom}}(W, X)), \quad f \mapsto (v \mapsto f_v, f_v(w) = f(v \otimes w)), \\ \Phi \colon \underline{\mathrm{Hom}}(V, \underline{\mathrm{Hom}}(W, X)) &\xrightarrow{\sim} \underline{\mathrm{Hom}}(V \otimes W, X), \quad g \mapsto (v \otimes w \mapsto g(v)(w)). \end{aligned}$$

Proof. The fact that the above morphisms are mutual inverses is clear from the definition. To conclude, we just have to check that they are chain maps. Thanks to Lemma 1.5, we just need to check that the above morphisms preserve the gradings (i.e. they map degree p morphisms to degree p morphisms) and that *one of those* is a chain map. This is straightforward but tedious and not particularly instructive, so we leave it behind. □

1.3. **Quasi-isomorphisms.** Why are complexes interesting and important? A possible answer is that *they essentially bring (linear) algebra to higher dimensions*. They give a unified framework to treat resolutions of modules, derived functors and so on. Cohomology is often the piece of information of a given complex that we really want to retain in many contexts, and this motivates the following definition:

Definition 1.18. Let $V = (V^i, d^i_V)$ and $W = (W^i, d^i_W)$ be chain complexes, and let $f \colon V \to W$ be a chain map. We say that f is a *quasi-isomorphism* if $H^i(f) \colon H^i(V) \to H^i(W)$ is an isomorphism for all $i \in \mathbb{Z}$ (or, equivalently, that the graded morphism $H^*(f)$ is an isomorphism). See Exercise 1.7 for another equivalent definition.

Quasi-isomorphisms are abundant. A typical family of examples is given by projective or injective resolutions, one of which we see in the following example.

Example 1.19. Assume that $\mathbf{k} = \mathbb{Z}$, the integers. We describe a simple free resolution of the abelian group $\mathbb{Z}/2\mathbb{Z}$. This is understood as the following chain map:

$$\begin{tikzcd} V \arrow[d, "f"'] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

The abelian group $\mathbb{Z}/2\mathbb{Z}$ is viewed as a complex concentrated in degree 0, and the morphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is multiplication by 2. The complex V in the upper row is concentrated in degrees -1 and 0. A direct computation shows that $H^{-1}(V) = 0$ and $H^{-1}(\mathbb{Z}/2\mathbb{Z}) = 0$, and that

$$H^0(f) = 1_{\mathbb{Z}/2\mathbb{Z}} \colon H^0(V) \to H^0(\mathbb{Z}/2\mathbb{Z}).$$

We conclude that f is a quasi-isomorphism. This somehow captures the idea that we could replace the module $\mathbb{Z}/2\mathbb{Z}$ (which is torsion) with the complex V, which is made of free abelian groups but has a nontrivial component in degree -1.

There is an important caveat. While we would like to view quasi-isomorphisms as some kind of isomorphism, unfortunately *not all quasi-isomorphisms have inverses*. This can be seen even from the above example:

Remark 1.20. In the setup of the above Example 1.19 we can't find any chain map $g \colon \mathbb{Z}/2\mathbb{Z} \to V$ such that $H^*(g)$ is inverse to $H^*(f)$. The point is that we can't find a nonzero group homomorphism

$$\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z},$$

so there are actually no nonzero chain maps $\mathbb{Z}/2\mathbb{Z} \to V$.

The problem of "inverting quasi-isomorphisms" is highly nontrivial. In the following lectures, we will see – among other things – how we are able to overcome it.

Exercises to Lecture 1.

Exercise 1.1. Prove Lemma 1.3.

Exercise 1.2. Check that the "alternative definition" of chain complexes and chain maps described in Remark 1.6 is equivalent to the one given in Definition 1.2. More explicitly, define bijections

$$\{\text{chain complexes "version 1" } (V^i, d^i)\} \leftrightarrow \{\text{chain complexes "version 2" } (V = \oplus_i V^i, d)\},$$

$$\{\text{chain maps "version 1" } (V^i, d^i) \to (W, d^i_W)\} \leftrightarrow \{\text{chain maps "version 2" } \oplus_i V^i \to \oplus_i W^i\}.$$

Exercise 1.3. Let $f, g \colon V \to W$ be chain maps of complexes. Then, the sum $f + g \colon V \to W$ defined elementwise by

$$(f+g)(v) = f(v) + g(v)$$

is again a chain map. The opposite $-f \colon V \to W$ defined elementwise by

$$(-f)(v) = -(f(v))$$

is also a chain map. Moreover, the zero morphism $0\colon V \to W$ is a chain map. Finally, let $\lambda \in \mathbf{k}$. Then, the morphism $\lambda f\colon V \to W$ defined by

$$(\lambda f)(v) = \lambda f(v)$$

is a chain map.

Exercise 1.4. Check that the differential $d = d_{\underline{\mathrm{Hom}}}$ of the complex $\underline{\mathrm{Hom}}(V, W)$ described in Definition 1.12 actually satisfies $d^{p+1} \circ d^p = 0$ for all $p \in \mathbb{Z}$.

Exercise 1.5. Check that the differential $d_{V\otimes W}$ of the complex $V \otimes W$ described in Definition 1.14 actually satisfies $d^{p+1}_{V\otimes W} \circ d^p_{V\otimes W} = 0$ for all $p \in \mathbb{Z}$.

Exercise 1.6. Prove the claims of Proposition 1.15.

Exercise 1.7. Let $f\colon V \to W$ be a chain map of complexes. Prove that f is a quasi-isomorphism if and only if the following hold:

- Let $y \in W^p$ and $x' \in V^{p+1}$ such that $dy = f(x')$. Then, there is $z \in W^{p-1}$ and $x \in V^p$ such that:

$$\begin{aligned} dx &= x', \\ y - dz &= f(x). \end{aligned}$$

2. Lecture 2: basics on dg-categories

In this lecture we give the definition of the main object of this course, and develop some basics of the theory.

2.1. **Towards dg-categories.** We start by having an even closer look to the *hom-complexes* $\underline{\mathrm{Hom}}(V, W)$ (cf. Definition 1.12).

Example 2.1. Let $f \in \underline{\mathrm{Hom}}^p(V, W)$ and $g \in \underline{\mathrm{Hom}}^q(W, X)$ be respectively a degree p and a degree q morphism between chain complexes. Namely, $f(V^i) \subseteq W^{i+p}$ and $g(W^i) \subseteq W^{i+q}$ for all $i \in \mathbb{Z}$. Can we compose f and g? The answer is yes, of course if we are careful with the components. If $v \in V^i$ is an element of degree i, then $f(v) = f^i(v) \in W^{i+p}$. Moreover, $g(f(v)) = g^{i+p}(f^i(v)) \in X^{i+p+q}$. We conclude that we have indeed a composition $g \circ f$, and this satisfies $(g \circ f)(V^i) \subseteq X^{i+p+q}$ for all $i \in \mathbb{Z}$. It is a degree $p + q$ morphism. We end up with a composition function

$$\begin{gathered} \underline{\mathrm{Hom}}^q(W, X) \times \underline{\mathrm{Hom}}^p(V, W) \to \underline{\mathrm{Hom}}^{p+q}(V, X), \\ (g, f) \mapsto g \circ f, \end{gathered}$$

defined for all $p, q \in \mathbb{Z}$. Let us explore its properties.

- The composition is $\mathbf{k}$-bilinear. This is very easy to see, since addition and action of $\mathbf{k}$ on hom complexes is defined elementwise, and we are always dealing with morphisms of $\mathbf{k}$-modules.
- The composition is associative, namely

$$(h \circ g) \circ f = h \circ (g \circ f),$$

if $f \in \underline{\mathrm{Hom}}^p(V, W)$, $g \in \underline{\mathrm{Hom}}^q(W, X)$ and $h \in \underline{\mathrm{Hom}}^r(X, Y)$. This follows from the usual associativity of compositions of morphisms of $\mathbf{k}$-modules.

- The composition is unital. Indeed, for any chain complex V, we know that we have an identity morphism $1_V \colon V \to V$. By the way, $1_V \in \underline{\mathrm{Hom}}^0(V, V)$ and $d(1_V) = 0$. This is also immediate to check.
- What about differentials? Let us compute, for given $f \in \underline{\mathrm{Hom}}^p(V, W)$ and $g \in \underline{\mathrm{Hom}}^q(W, X)$:

$$d(g \circ f) = d_X \circ g \circ f - (-1)^{p+q} g \circ f \circ d_V.$$

Can we perhaps relate this to $d(g)$ and $d(f)$? Let's add and subtract a suitable element, namely $(-1)^q g \circ d_W \circ f$ (beware the sign choice):

$$\begin{aligned} d(g \circ f) &= d_X \circ g \circ f - (-1)^{p+q} g \circ f \circ d_V \\ &= d_X \circ g \circ f - (-1)^q g \circ d_W \circ f + (-1)^q g \circ d_W \circ f - (-1)^{p+q} g \circ f \circ d_V \\ &= (d_X \circ g - (-1)^q f \circ d_W) \circ f + (-1)^q g \circ (d_W \circ f - (-1)^p f \circ d_V) \\ &= d(g) \circ f + (-1)^q g \circ d(f). \end{aligned}$$

Hence, differentials and compositions are indeed compatible. The equation

$$d(g \circ f) = d(g) \circ f + (-1)^q g \circ d(f)$$

is called *graded Leibniz rule*.

The above example motivates the following key definition:

Definition 2.2. A *differential graded category* (in short, *dg-category*) $\mathcal{A}$ is the datum of:

- A family of *objects* $\mathrm{Ob}(\mathcal{A})$. We will denote them by $A, B, C, \dots$. Strictly speaking, $\mathrm{Ob}(\mathcal{A})$ need not be a set but even something "larger". There are possibly serious set-theoretical issues going on in that case but for now and unless otherwise specified we will just adopt a naive point of view on the problem and just forget about them.
- For any pair of objects A and B, a chain complex $\mathcal{A}(A, B)$ (sometimes also stylized as $\underline{\mathrm{Hom}}_{\mathcal{A}}(A, B)$ or even $\underline{\mathrm{Hom}}(A, B)$ if the context does not allow confusion. Elements in $\mathcal{A}(A, B)^p$ are called *degree p morphisms (from A to B)* If $f \in \mathcal{A}(A, B)^p$, we also write $|f| = \deg(f) = p$. Recall from Remark 1.6 that we may view

$$\mathcal{A}(A, B) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}(A, B)^i,$$

  and $\mathcal{A}(A, B)^i$ itself as a submodule of $\mathcal{A}(A, B)$.
- Composition morphisms:

$$\begin{aligned} \mathcal{A}(B, C)^q \times \mathcal{A}(A, B)^p &\to \mathcal{A}(A, C)^{p+q}, \\ (g, f) &\mapsto g \circ f, \end{aligned}$$

  for $A, B, C \in \mathrm{Ob}(\mathcal{A})$ and $p, q \in \mathbb{Z}$.

These data have the following properties and features:

- The composition morphisms are $\mathbf{k}$-bilinear.
- Composition is associative:

$$(h \circ g) \circ f = h \circ (g \circ f),$$

  for $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$ and $h \in \mathcal{A}(C, D)$.

- Composition is unital. Namely, for all $A \in \mathrm{Ob}(\mathcal{A})$, there is a morphism $1_A \in \mathcal{A}(A,A)^0$ such that $d(1_A) = 0$ and for any $f \in \mathcal{A}(A,B)$ we have:
$$f \circ 1_A = f,$$
$$1_B \circ f = f.$$
- The graded Leibniz rule holds. Namely, if $f \in \mathcal{A}(A,B)^p$ and $g \in \mathcal{A}(B,C)^q$, we have:
$$d(g \circ f) = d(g) \circ f + (-1)^q g \circ d(f).$$

Notation 2.3. Elements of the *hom complexes* $\mathcal{A}(A,B)$ will be often described as arrows. Namely, we shall often write $f \colon A \to B$ instead of $f \in \mathcal{A}(A,B)$.

Moreover, we will also ease notation and often write gf instead of $g \circ f$, for composable morphisms $f \colon A \to B$ and $g \colon B \to C$.

Remark 2.4. In the above Definition 2.2, it is not actually necessary to require that identity morphisms 1_A are closed and of degree 0, for it follows from the other properties. Let's try and prove it. Maybe it's a good idea to start by writing
$$1_A = \sum_{n \in \mathbb{Z}} (1_A)_n,$$
where $(1_A)_n \in \mathcal{A}(A,A)^n$ and $(1_A)_n = 0$ for all but a finite number of indices n. From the relation $1_A \circ 1_A = 1_A$ and using bilinearity, we find out:
$$1_A = \sum_{n \in \mathbb{Z}} \Big(\sum_{i+j=n} (1_A)_i \circ (1_A)_j \Big),$$
from which we conclude that for all $n \in \mathbb{Z}$:
$$(1_A)_n = (1_A)_i \circ (1_A)_j.$$
Do we really go anywhere from this? Well, maybe not!

Let's try something different. First, we may observe that *identity morphisms are unique*. Namely: if for any object $A \in \mathrm{Ob}(\mathcal{A})$ we have morphisms $1_A, 1'_A$ such that they behave as identities (according to the above definition), then necessarily $1_A = 1'_A$. Indeed, we have:
$$1_A \circ 1'_A = 1_A = 1'_A.$$
Next, we look again at the decomposition
$$1_A = \sum_{n \in \mathbb{Z}} (1_A)_n.$$
If the degree 0 component $(1_A)_0$ behaves as an identity, then it has to coincide with 1_A by what we said just before. Then, does it really? Let us take any other morphisms $g \colon A \to B$ and $g' \colon B \to A$, of some fixed arbitrary degree p. Let's have a look at the degree p component of the composition $(g \circ 1_A)_p$:
$$(g \circ 1_A)_p = g_p = g,$$
for g is concentrated in degree p and 1_A is the identity. On the other hand:
$$(g \circ 1_A)_p = g_p \circ (1_A)_0 = g \circ (1_A)_0,$$
being careful with degrees. We conclude that indeed
$$g \circ (1_A)_0 = g,$$

and similarly we can prove that

$$(1_A)_0 \circ g' = g'.$$

From this, we indeed conclude that 1_A is a degree 0 morphism.

Finally, what about checking that $d(1_A) = 0$. Now, we can maybe really use the relation

$$1_A \circ 1_A = 1_A,$$

and the graded Leibniz rule:

$$\begin{aligned} d(1_A) &= d(1_A \circ 1_A) = d(1_A) \circ 1_A + 1_A \circ d(1_A), \\ &= d(1_A) + d(1_A), \end{aligned}$$

using that $\deg(1_A) = 0$ as we prove above. We substract $d(1_A)$ in the above relation and finally conclude that $d(1_A) = 0$.

Example 2.5. Example 2.1 on hom complexes tells us precisely that the family of all chain complexes $V, W, \dots$ together with the hom-complexes $\underline{\mathrm{Hom}}(V, W)$ and the compositions as described there is a dg-category. We will denote it as $\mathrm{dgm}(\mathbf{k})$. We will still continue using the notation $\underline{\mathrm{Hom}}(V, W)$ to refer to the hom-complexes in $\mathrm{dgm}(\mathbf{k})$:

$$\mathrm{dgm}(\mathbf{k})(V, W) = \underline{\mathrm{Hom}}(V, W).$$

Remark 2.6. We may use the tensor product of complexes to give a more compact yet equivalent definition of the compositions in a dg-category $\mathcal{A}$. Indeed, we can describe them as chain maps of complexes:

$$\begin{aligned} \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) &\to \mathcal{A}(A, C), \\ g \otimes f &\mapsto g \circ f, \end{aligned}$$

which we require to be associative and unital. $\mathbf{k}$-bilinearity, preservation of degrees and the graded Leibniz rule are encoded in the requirement that it is a chain map defined on the tensor product. See also Exercise 2.1.

You may not (yet) be very familiar with dg-categories, but you may be familiar with *dg-algebras*. We can define them very easily now:

Definition 2.7. A *differential graded algebra* (or *dg-algebra*) A is a dg-category whose family of objects has exactly one element $\star$. We normally identify A with the complex of endomorphisms of the single object $\star$:

$$A = \underline{\mathrm{Hom}}(\star, \star).$$

Hence, a dg-algebra A can be defined just as a chain complex endowed with a composition

$$\begin{aligned} A^q \times A^p &\to A^{p+q}, \\ (b, a) &\mapsto ba, \end{aligned}$$

which is $\mathbf{k}$-bilinear, unital, associative and satisfies the graded Leibniz rule. Recalling Exercise 2.1, we can also define the composition as a chain map

$$A \otimes A \to A.$$

Example 2.8. Let R be a $\mathbf{k}$-algebra. Then, it is a dg-algebra when we view it as a complex concentrated in degree 0, together with its multiplication. In particular, R can be also identified with a dg-category with a single object $\star$ and such that $R = \underline{\mathrm{Hom}}(\star, \star)$.

In particular, we may view $\mathbf{k}$ itself as a dg-algebra, and also a dg-category with a single object.

The above Example 2.8 can be extended to the "several objects" setup. We give the following definition:

Definition 2.9. A $\mathbf{k}$-*linear* (or more simply *linear*) *category* $\mathcal{A}$ is the datum of:

- A family of *objects* $\mathrm{Ob}(\mathcal{A})$.
- For any pair of objects A and B, a $\mathbf{k}$-module $\mathcal{A}(A, B)$ (sometimes also stylized as $\mathrm{Hom}_{\mathcal{A}}(A, B)$ or even $\mathrm{Hom}(A, B)$ if the context does not allow confusion.
- Composition morphisms:
$$\begin{aligned} \mathcal{A}(B, C) \times \mathcal{A}(A, B) &\to \mathcal{A}(A, C), \\ (g, f) &\mapsto g \circ f, \end{aligned}$$
for $A, B, C \in \mathrm{Ob}(\mathcal{A})$.

These data have the following properties and features:

- The composition morphisms are $\mathbf{k}$-bilinear. We may also identify them with $\mathbf{k}$-linear morphisms:
$$\begin{aligned} \mathcal{A}(B, C) \times \mathcal{A}(A, B) &\to \mathcal{A}(A, C), \\ g \otimes f &\mapsto g \circ f, \end{aligned}$$
- Composition is associative:
$$(h \circ g) \circ f = h \circ (g \circ f),$$
for $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$ and $h \in \mathcal{A}(C, D)$.
- Composition is unital. Namely, for all $A \in \mathrm{Ob}(\mathcal{A})$, there is a morphism $1_A \in \mathcal{A}(A, A)$ such that for any $f \in \mathcal{A}(A, B)$ we have:
$$\begin{aligned} f \circ 1_A &= f, \\ 1_B \circ f &= f. \end{aligned}$$

Example 2.10. A typical example of a $\mathbf{k}$-linear category is given by $\mathrm{Mod}(\mathbf{k})$, the linear category of $\mathbf{k}$-modules, defined as follows:

- The family of objects $\mathrm{Ob}(\mathrm{Mod}(\mathbf{k}))$ is given by the $\mathbf{k}$-modules $M, N, \ldots$
- If M and N are $\mathbf{k}$-modules, we have a $\mathbf{k}$-module of morphisms (sometimes called $\mathbf{k}$-linear morphisms):
$$\mathrm{Mod}(\mathbf{k})(M, N) = \mathrm{Hom}(M, N).$$
- Compositions are given by the usual composition of $\mathbf{k}$-linear morphisms.

Example 2.11. Let $\mathcal{A}$ be a $\mathbf{k}$-linear category. We may view it as a dg-category, viewing every $\mathbf{k}$-module $\mathcal{A}(A, B)$ as a complex concentrated in degree 0. Compositions and units are the obvious ones.

As usual in mathematics, once we have defined a structure we also want to define transformations preserving that structure. This leads to the following definition:

Definition 2.12. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. A *dg-functor* $F \colon \mathcal{A} \to \mathcal{B}$ is the datum of:

- A function $F \colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
- For any pair of objects $A, B \in \mathrm{Ob}(\mathcal{A})$, a chain map of complexes
$$F = F_{A,B} \colon \mathcal{A}(A, B) \to \mathcal{B}(F(A), F(B)).$$

These data are compatible with identities and compositions, as follows:

$$F(1_A) = 1_{F(A)}, \qquad A \in \mathrm{Ob}(\mathcal{A}),$$
$$F(g \circ f) = F(g) \circ F(f), \qquad f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C).$$

Clearly, we can also define the notion of *($\mathbf{k}$-)linear functor* between linear categories, cf. Exercise 2.2.

2.2. **Some examples.** Let us become more familiar with dg-categories and dg-functors.

Example 2.13. Take any dg-category $\mathcal{A}$. What about dg-functors $\mathbf{k} \to \mathcal{A}$, where $\mathbf{k}$ is our base commutative ring viewed as a dg-category (cf. Example 2.8)? Let $F : \mathbf{k} \to \mathcal{A}$ be such a dg-functor. Then, F is determined by the following data:

- An object $F(\star) = A \in \mathrm{Ob}(\mathcal{A})$.
- A chain map
$$F : \mathbf{k} = \underline{\mathrm{Hom}}(\star, \star) \to \mathcal{A}(A, A),$$
but by $\mathbf{k}$-linearity F is completely determined by $F(1_{\mathbf{k}})$, which in turn is forced to be $1_A : A \to A$.

In the end, we see that F is completely determined by the object $F(\star)$. We end up with a bijection:

$$\{\text{dg-functors } \mathbf{k} \to \mathcal{A}\} \to \mathrm{Ob}(\mathcal{A}),$$
$$F \mapsto F(\star).$$

Example 2.14. Let A be a dg-algebra, which we view as a dg-category with a single object $\star$, and $A = \underline{\mathrm{Hom}}(\star, \star)$. Let $F : A \to \mathrm{dgm}(\mathbf{k})$ be a dg-functor. F is determined by:

- An object $F(\star) = V \in \mathrm{Ob}(\mathrm{dgm}(\mathbf{k}))$, namely, a chain complex.
- A chain map
$$F : A \to \underline{\mathrm{Hom}}(V, V),$$
$$a \mapsto F(a),$$
such that $F(1_A) = 1_V$ and $F(ba) = F(b)F(a)$ for all $a, b \in A$.

Let us adopt the following notation:

$$F(a)(v) = av.$$

We can invoke Proposition 1.17 and get from F an induced chain map of complexes

$$A \otimes V \to V,$$
$$a \otimes v \mapsto F(a)(v) = av$$

with the following additional properties:

$$1_a v = v,$$
$$(ba)v = b(av),$$

for all $v \in V$ and $a, b \in A$. These equalities come from the fact that F is a dg-functor.

Let us unravel this even more. We know (see Exercise 2.1) that we can identify the chain map

$$a \otimes v \mapsto av = F(a)(v)$$

with a family of **k**-bilinear morphisms

$$A^i \times V^j \to V^{i+j},$$

for $i, j \in \mathbb{Z}$. The relation $F(da) = dF(a) \in \underline{\mathrm{Hom}}(V, V)$ translates to the following:

$$(da)v = d(av) - (-1)^{|a|}a(dv),$$

which can be rearranged in the following variant of the graded Leibniz rule:

$$d(av) = (da)v + (-1)^{|a|}a(dv).$$

We could also obtain this by writing explicitly what it means for $A \otimes V \to V$ to be a chain map.

What we have discussed so far is nothing more than the notion of *left A-dg-module*. Hence, we get a bijection:

$$\{\text{dg-functors } A \to \mathrm{dgm}(\mathbf{k})\} \leftrightarrows \{\text{left } A\text{-dg-modules } V, \text{ defined by an action } A \otimes V \to V\}.$$

In the framework of ordinary **k**-algebras and modules, we can obtain an analogous result, cf. Exercise 2.3.

Example 2.15. We give an example of a "small" dg-category $\mathbf{Q}$, which is actually a **k**-linear category which we view as a dg-category (cf. Example 2.11):

- It has three objects $0, 1, 2$.
- The hom **k**-modules are described as follows:

$$\begin{aligned} \mathbf{Q}(i,i) &= \mathbf{k}\langle 1_i \rangle, \qquad i = 0,1,2, \\ \mathbf{Q}(0,1) &= \mathbf{k}\langle e_{01} \rangle, \\ \mathbf{Q}(1,2) &= \mathbf{k}\langle e_{12} \rangle, \\ \mathbf{Q}(0,2) &= 0, \\ \mathbf{Q}(i,j) &= 0, \qquad i > j. \end{aligned}$$

  The notation $\mathbf{k}\langle S \rangle$ simply means *free* **k***-module having S as a basis*. So far, the morphisms 1_i are to be thought as "formal identities". They will become proper identities once we define the compositions suitably.
- Compositions are the "obvious" ones. For example, we may define

$$\mathbf{Q}(1,1) \times \mathbf{Q}(0,1) \to \mathbf{Q}(0,1)$$

  by sending $(1_1, e_{01})$ to e_{01} and then extending by bilinearity. Observe that in particular we have

$$e_{12} \circ e_{01} = 0.$$

The linear category $\mathbf{Q}$ can be pictured as the following diagram:

$$\begin{tikzcd} 0 \arrow[r, "e_{01}"] \arrow[rr, bend left, "0"] & 1 \arrow[r, "e_{12}"] & 2. \end{tikzcd}$$

A key feature of $\mathbf{Q}$ is that it is *not free*, indeed, we have the nontrivial relation $e_{12} \circ e_{01} = 0$. Dg-categories allow us to replace $\mathbf{Q}$ with something which will look like some kind of "free resolution" of $\mathbf{Q}$. To that purpose, we define a dg-category $\mathbf{Q}'$ as follows:

- It has three objects $0, 1, 2$, as $\mathbf{Q}$.

- The hom-complexes are defined as follows:

$$\begin{aligned}
\mathbf{Q}'(i,i) &= \mathbf{k}\langle 1_i\rangle, \quad |1_i| = 0,\ d(1_i) = 0, \qquad i = 0,1,2,\\
\mathbf{Q}'(0,1) &= \mathbf{k}\langle e_{01}\rangle, \quad |e_{01}| = 0,\ d(e_{01}) = 0,\\
\mathbf{Q}'(1,2) &= \mathbf{k}\langle e_{12}\rangle, \quad |e_{12}| = 0,\ d(e_{12}) = 0,\\
\mathbf{Q}'(0,2) &= \mathbf{k}\langle e_{12}\circ e_{01}, e_{012}\rangle, \quad |e_{12}\circ e_{01}| = 0, |e_{012}| = -1,\ d(e_{012}) = e_{12}\circ e_{01},\\
\mathbf{Q}'(i,j) &= 0, \qquad i > j.
\end{aligned}$$

  All hom-complexes are made of free $\mathbf{k}$-modules. The only actual differences we have with respect to $\mathbf{Q}$ is that $\mathbf{Q}'(0,2)$ is no longer 0, but it is the following complex:

$$\cdots \to 0 \to \mathbf{k}\langle e_{012}\rangle \xrightarrow{d} \mathbf{k}\langle e_{12}\circ e_{01}\rangle \to 0 \to \cdots,$$

  where e_{012} is of degree -1 and the "formal composition" $e_{12}\circ e_{01}$ is of degree 0. The differential d is actually just the identity map which maps the basis element e_{012} to $e_{12}\circ e_{01}$.
- Compositions are defined in the "obvious" way, again. For instance, the morphism

$$\mathbf{Q}'(1,2)\otimes\mathbf{Q}'(0,1) \to \mathbf{Q}(0,2)$$

  maps $e_{12}\otimes e_{01}$ to the "formal composition" $e_{12}\circ e_{01}$, which then becomes the actual composition in $\mathbf{Q}'$.

We can check that our definition of $\mathbf{Q}'$ actually yields a dg-category. $\mathbf{Q}'$ is essentially built in such a way that the nontrivial relation $e_{12}\circ e_{01} = 0$ is replaced by instead saying that $e_{12}\circ e_{01}$ is a coboundary. We can picture $\mathbf{Q}'$ as the following diagram:

$$\begin{tikzcd} 0 \arrow[r, "e_{01}"] \arrow[rr, bend left, "e_{012}"] & 1 \arrow[r, "e_{12}"] & 2, \end{tikzcd}$$

keeping track of the relation $d(e_{012}) = e_{12}\circ e_{01}$.

We may define a dg-functor $F\colon \mathbf{Q}' \to \mathbf{Q}$ as follows:

- F is the identity on objects: $F(i) = i$ for $i = 0,1,2$.
- On the hom-complexes, F is defined as follows:

$$\begin{aligned}
F(e_{01}) &= e_{01},\\
F(e_{12}) &= e_{12},\\
F(e_{12}\circ e_{01}) &= 0,\\
F(e_{012}) &= 0,
\end{aligned}$$

  and also by preserving the identities 1_i and extending by $\mathbf{k}$-linearity. We can check that this yields a well-defined dg-functor. Actually, we did not do give the complete details on the definitions of $\mathbf{Q}, \mathbf{Q}'$ and F. This is the content of Exercise 2.4.

  We leave you with the following question: the chain maps

$$F\colon \mathbf{Q}'(i,j) \to \mathbf{Q}(i,j)$$

  induce, as we know, morphisms in cohomology

$$H^*(F)\colon H^*(\mathbf{Q}'(i,j)) \to H^*(\mathbf{Q}(i,j)).$$

  What can we say about them?

Exercises to Lecture 2.

Exercise 2.1. Let V, W, X be chain complexes. Prove that giving $\mathbf{k}$-bilinear morphisms

$$f_{q,p}\colon V^q \times W^p \to X^{p+q}$$

for $p, q \in \mathbb{Z}$, satisfying

$$d(f_{q,p}(v,w)) = f_{q+1,p}(dv, w) + (-1)^q f_{q,p+1}(v, dw),$$

is the same as giving a chain map of complexes

$$\begin{aligned} &V \otimes W \to X, \\ &v \otimes w \mapsto f_{q,p}(v,w) \quad (v \in V^q \text{ and } w \in W^p). \end{aligned}$$

In particular, prove the claims of Remark 2.6.

Exercise 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbf{k}$-linear categories. Define the notion of a $\mathbf{k}$-linear functor $F\colon \mathcal{A} \to \mathcal{B}$.

Exercise 2.3. Let R be a $\mathbf{k}$-algebra. Classically, we define a *left R-module* as a $\mathbf{k}$-module M together with a $\mathbf{k}$-bilinear action

$$R \times M \to M,$$

or equivalently a $\mathbf{k}$-linear morphism

$$R \otimes M \to M,$$

satisfying the usual compatibilities. View R as a $\mathbf{k}$-linear category with a single object $\star$, analogously to the case of dg-algebras (cf. Definition 2.7). Check that we can establish a bijection:

$$\{\mathbf{k}\text{-linear functors } R \to \mathrm{Mod}(\mathbf{k})\} \leftrightarrows \{\text{left } R\text{-modules } M, \text{ defined by an action } R \otimes M \to M\}.$$

Exercise 2.4. Fill in the details of the constructions of the dg-categories $\mathbf{Q}, \mathbf{Q}'$ and the dg-functor $F\colon \mathbf{Q}' \to \mathbf{Q}$ in Example 2.15.

3. Lecture 3: dg-modules and the Yoneda lemma

We have introduced dg-categories and $\mathbf{k}$-linear categories as "many-object versions" of respectively dg-algebra and $\mathbf{k}$-algebras. Quite informally, we have inclusions:

$$\begin{tikzcd}
\{\mathbf{k}\text{-algebras}\} \arrow[r, hook, "\text{complexes concentrated in degree 0}"] \arrow[d, hook, "\text{single object linear cat.}"'] & \{\text{dg-algebras}\} \arrow[d, hook, "\text{single object dg-cat.}"] \\
\{\mathbf{k}\text{-linear categories}\} \arrow[r, hook, "\text{complexes concentrated in degree 0}"'] & \{\text{dg-categories}\}.
\end{tikzcd}$$

In this lecture, we are going to deal with *modules* over these structures. As a reminder, to enhance familiarity: if $\mathbf{k} = \mathbb{Z}$, the notion of $\mathbf{k}$-algebra is the same as the usual notion of (unital, associative) ring.

3.1. **Modules, dg-modules and morphisms.** We recall again the classical definition of a module over a **k**-algebra. We will start using tensor products.

Definition 3.1. Let R be a **k**-algebra. A *left R-module M* is a **k**-module together with a **k**-linear morphism:

$$\begin{aligned} R \otimes M &\to M, \\ r \otimes m &\mapsto rm, \end{aligned}$$

subject to the relations:

$$\begin{aligned} 1_R m &= m, && m \in M, \\ (rs)m &= r(sm) && r, s \in R,\ m \in M. \end{aligned}$$

A *right R-module N* is a **k**-module together with a **k**-linear morphism:

$$\begin{aligned} M \otimes R &\to M, \\ m \otimes r &\mapsto mr, \end{aligned}$$

subject to the relations:

$$\begin{aligned} m 1_R &= m, && m \in M, \\ m(rs) &= (mr)s && r, s \in R,\ m \in M. \end{aligned}$$

Definition 3.2. Let R be a **k**-algebra, and let M and M' be left R-modules. A *morphism* $\varphi \colon M \to M'$ is a **k**-linear morphism which preserves the left action of R, namely:

$$\varphi(rm) = r\varphi(m), \qquad r \in R,\ m \in M.$$

Let N and N' be right R-modules. A *morphism* $\psi \colon N \to N'$ is a **k**-linear morphism which preserves the right action of R, namely:

$$\psi(mr) = \psi(m)r, \qquad r \in R,\ m \in M.$$

Morphisms of either left or right R-modules form a **k**-module, using elementwise addition and action of **k**. Morphisms of right R-modules $N \to N'$ will be denoted by

$$\operatorname{Hom}_R(N, N') \qquad \text{or} \qquad \operatorname{Hom}_{\mathbf{k}-R}(N, N'),$$

whereas morphisms of left R-modules $M \to M'$ will be denoted by

$$\operatorname{Hom}_{R^{\mathrm{op}}}(M, M') \qquad \text{or} \qquad \operatorname{Hom}_{R-\mathbf{k}}(M, M').$$

So far, view the above as notational tricks, in particular R^{op}. They will be justified later on.

Remark 3.3. Recall thay giving (for instance) a **k**-linear morphism $R \otimes M \to M$ is just the same as giving a **k**-bilinear morphism $R \times M \to M$. This is the actual purpose and key property of the tensor product.

The tensor product allows us to directly generalize the above definitions to the differential graded setting. We start with the "single-object framework":

Definition 3.4. Let R be a dg-algebra. A *left R-dg-module M* is a chain complex of **k**-modules together with a chain map:

$$\begin{aligned} R \otimes M &\to M, \\ r \otimes m &\mapsto rm, \end{aligned}$$

subject to the relations:

$$1_R m = m, \qquad m \in M,$$
$$(rs)m = r(sm) \qquad r, s \in R,\ m \in M.$$

A *right R-dg-module N* is a chain complex of $\mathbf{k}$-modules together with a chain map:

$$M \otimes R \to M,$$
$$m \otimes r \mapsto mr,$$

subject to the relations:

$$m1_R = m, \qquad m \in M,$$
$$m(rs) = (mr)s \qquad r, s \in R,\ m \in M.$$

Remark 3.5. Giving a chain map $R \otimes M \to M$, as in the case of a left R-dg-module, is the same as giving $\mathbf{k}$-bilinear morphisms

$$R^q \times M^p \to M^{p+q},$$

satisfying a graded Leibniz rule:

$$d(rm) = (dr)m + (-1)^{|r|} r(dm).$$

This holds essentially by the definition of the tensor product of chain complexes.

Morphisms of dg-modules are quite more interesting to define than the "classical" counterparts. Indeed, we will end up with a *complex* of morphisms.

Definition 3.6. Let R be a dg-algebra, and let M and M' be left R-dg-modules. A *degree p morphism* $\varphi \colon M \to M'$ is a $\mathbf{k}$-linear morphism such that $\varphi(M^i) \subseteq M^{i+p}$ for all $i \in \mathbb{Z}$ and it preserves the left action of R, namely:

$$\varphi(rm) = (-1)^{|\varphi||r|} r\varphi(m), \qquad r \in R,\ m \in M,$$

where $|\varphi| = \deg(\varphi) = p$. Beware the sign: it appears because of the swapping of the "graded symbols" φ and r.

Let N and N' be right R-dg-modules. A *degree p morphism* $\psi \colon N \to N'$ is a $\mathbf{k}$-linear morphism such that $\psi(N^i) \subseteq N^{i+p}$ for all $i \in \mathbb{Z}$, and it preserves the right action of R, namely:

$$\psi(mr) = \psi(m)r, \qquad r \in R,\ m \in N.$$

We have no additional signs here, because we did not swap any graded symbol!

If $\varphi \colon M \to M'$ is a degree $p = |\varphi|$ morphism of either left or right R-dg-modules, we can define its *differential*

$$d\varphi = d_{M'} \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_M.$$

We can directly check (using that φ is a morphisms of dg-modules and the suitable graded Leibniz rules) that $d\varphi$ is indeed a degree $p+1$ morphism of (either left or right) dg-modules $M \to M'$, namely, that is compatible with the action of R. This is quite tedious and left as an exercise (see Exercise 3.1). The formula for $d\varphi$ is the same as the one we already encountered when defining the complex of morphisms between chain complexes (cf. Definition 1.12), so we already know that $d \circ d = 0$. We hence end up with a chain complex.

The complex of morphisms of right R-dg-modules $N \to N'$ will be denoted by

$$\underline{\mathrm{Hom}}_R(N, N') \qquad \text{or} \qquad \underline{\mathrm{Hom}}_{\mathbf{k}-R}(N, N'),$$

whereas the complex of morphisms of left R-modules $M \to M'$ will be denoted by

$$\underline{\mathrm{Hom}}_{R^{\mathrm{op}}}(M, M') \qquad \text{or} \qquad \underline{\mathrm{Hom}}_{R-\mathbf{k}}(M, M').$$

Again, view the above as notational tricks, in particular R^{op}.

Finally, we turn to the "many-object" framework. We will directly deal with *dg-modules over dg-categories*. Watch carefully how this more general setup is dealt with; recalling the notion of *presheaf* can be useful.

Definition 3.7. Let $\mathcal{A}$ be a dg-category. A *left $\mathcal{A}$-dg-module M* is given by the following data:

- A function $A \mapsto M(A)$, where $A \in \mathrm{Ob}(\mathcal{A})$ and $M(A)$ is a chain complex. Essentially, it is a family of chain complexes parametrized by the objects of $\mathcal{A}$.
- For objects $A, B \in \mathrm{Ob}(\mathcal{A})$, chain maps:

$$\begin{aligned} \mathcal{A}(A, B) \otimes M(A) &\to M(B), \\ f \otimes m &\mapsto fm, \end{aligned}$$

subject to the relations:

$$\begin{aligned} 1_A m &= m, \qquad m \in M(A), \qquad A \in \mathrm{Ob}(\mathcal{A}), \\ (gf)m &= g(fm) \qquad g \in \mathcal{A}(B, C), f \in \mathcal{A}(A, B),\ m \in M(A). \end{aligned}$$

A *right $\mathcal{A}$-dg-module N* is given by the following data:

- A function $A \mapsto N(A)$, where $A \in \mathrm{Ob}(\mathcal{A})$ and $N(A)$ is a chain complex. Again, it is a family of chain complexes parametrized by the objects of $\mathcal{A}$.
- For objects $A, B \in \mathrm{Ob}(\mathcal{A})$, chain maps:

$$\begin{aligned} N(B) \otimes \mathcal{A}(A, B) &\to N(A), \\ m \otimes f &\mapsto mf, \end{aligned}$$

subject to the relations:

$$\begin{aligned} m 1_A &= m, \qquad m \in N(A), \qquad A \in \mathrm{Ob}(\mathcal{A}), \\ m(gf) &= (mg)f \qquad g \in \mathcal{A}(B, C), f \in \mathcal{A}(A, B),\ m \in N(C). \end{aligned}$$

Remark 3.8. Giving a chain map $\mathcal{A}(A, B) \otimes M(A) \to M(A)$, as in the case of a left $\mathcal{A}$-dg-module, is the same as giving $\mathbf{k}$-bilinear morphisms

$$\mathcal{A}(A, B)^q \times M(A)^p \to M(B)^{p+q},$$

satisfying a graded Leibniz rule:

$$d_{M(B)}(fm) = (df)m + (-1)^{|f|} f(d_{M(A)} m).$$

Again, this holds essentially by the definition of the tensor product of chain complexes.

We now define morphisms of dg-modules over dg-categories. As in the case of dg-algebras, we will end up with a complex of morphisms. We have to take care of the presence of possibly many objects, but this is not too difficult.

Definition 3.9. Let $\mathcal{A}$ be a dg-category, and let M and M' be left $\mathcal{A}$-dg-modules. A *degree p morphism $\varphi \colon M \to M'$* is a family of $\mathbf{k}$-linear morphisms

$$\varphi_A \colon M(A) \to M'(A),$$

parametrized by $A \in \mathrm{Ob}(\mathcal{A})$ (namely, a function $A \mapsto \varphi_A$), such that:

- $\varphi_A(M(A)^i) \subseteq M(A)^{i+p}$ for all $i \in \mathbb{Z}$, for all $A \in \mathrm{Ob}(\mathcal{A})$.
- The left action of $\mathcal{A}$ is preserved, namely, for any $f \in \mathcal{A}(A, B)$ and $m \in M(A)$ we have:

$$\varphi_B(fm) = (-1)^{|\varphi||f|} f\varphi_A(m),$$

where $|\varphi| = p$. Beware the sign! It appears because of the swapping of the "graded symbols" φ and f.

Let N and N' be right $\mathcal{A}$-dg-modules. A *degree p morphism* $\psi\colon N \to N'$ is a family of $\mathbf{k}$-linear morphisms

$$\psi_A \colon N(A) \to N'(A),$$

parametrized by $A \in \mathrm{Ob}(\mathcal{A})$ (namely, a function $A \mapsto \psi_A$), such that:

- $\psi_A(N(A)^i) \subseteq N(A)^{i+p}$ for all $i \in \mathbb{Z}$, for all $A \in \mathrm{Ob}(\mathcal{A})$.
- The right action of $\mathcal{A}$ is preserved, namely, for any $f \in \mathcal{A}(A, B)$ and $m \in M(B)$ we have:

$$\varphi_A(mf) = \varphi_B(m)f,$$

We have no additional sign here, because we did not swap any graded symbol!

If $\varphi\colon M \to M'$ is a degree $p = |\varphi|$ morphism of either left or right R-dg-modules, we can define its *differential* as the parametrized family $((d\varphi)_A : A \in \mathrm{Ob}(\mathcal{A}))$, where

$$(d\varphi)_A = d(\varphi_A),$$

where $d(\varphi_A)$ is the now familiar differential of morphisms between twisted complexes:

$$d(\varphi_A) = d_{M'(A)} \circ \varphi_A - (-1)^{|\varphi|}\varphi_A d_{M(A)}.$$

As in Definition 3.6, we can directly check that $d\varphi$ is indeed a degree $p+1$ morphism of (either left or right) dg-modules $M \to M'$, namely, that is compatible with the action of $\mathcal{A}$. The identity $d \circ d = 0$ clearly holds. We hence end up with a chain complex.

The complex of morphisms of right $\mathcal{A}$-dg-modules $N \to N'$ will be denoted by

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(N, N') \qquad \text{or} \qquad \underline{\mathrm{Hom}}_{\mathbf{k}-\mathcal{A}}(N, N'),$$

whereas the complex of morphisms of left R-modules $M \to M'$ will be denoted by

$$\underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{op}}}(M, M') \qquad \text{or} \qquad \underline{\mathrm{Hom}}_{\mathcal{A}-\mathbf{k}}(M, M').$$

Soon we will give a meaningful definition of the gadget $\mathcal{A}^{\mathrm{op}}$.

Remark 3.10. You can see that there are not many differences between the above definition and the one of dg-modules over a dg-algebras: one has just to be careful with having a parametrized family of morphisms ($\varphi_A : A \in \mathrm{Ob}(\mathcal{A})$ instead of just one map. On the other hand, you can see that specializing to dg-categories with a single object (indeed, dg-algebras) one gets back the "old" definition.

3.2. **Represented dg-modules and the dg-Yoneda lemma.** If R is a $\mathbf{k}$-algebra, there is a very special example of a both right and left R-module: that is, R itself where the action is just induced by the product

$$R \otimes R \to R.$$

Something similar happens with a dg-algebra. In the case of dg-categories, one has to be careful with having a family of objects. Products of elements will become compositions of morphisms:

Definition 3.11. Let $\mathcal{A}$ be a dg-category, and let $A \in \mathrm{Ob}(\mathcal{A})$. We define a right dg-module $\mathcal{A}(-, A) = h_A$ as follows:

$$h_A(B) = \mathcal{A}(B, A), \qquad B \in \mathrm{Ob}(\mathcal{A}),$$

with actions

$$h_A(B') \otimes \mathcal{A}(B, B') \to h_A(B)$$

given just by the compositions

$$\mathcal{A}(B', A) \otimes \mathcal{A}(B, B') \to \mathcal{A}(B, A).$$

Verifying that this indeed yields an $\mathcal{A}$-dg-module is straightforward.

Analogously, we define a left $\mathcal{A}$-dg-module $\mathcal{A}(A, -) = h^A$ as:

$$h^A(B) = \mathcal{A}(A, B), \qquad B \in \mathrm{Ob}(\mathcal{A}),$$

with actions

$$\mathcal{A}(B, B') \otimes h^A(B) \to h^A(B')$$

given by just compositions.

The right $\mathcal{A}$-dg-module $\mathcal{A}(-, A)$ is said to be *represented by* A; the left $\mathcal{A}$-dg-module $\mathcal{A}(A, -)$ is said to be *corepresented by* A.

We now state a result which should be quite straightforward to prove, but whose relevance cannot be underestimated, even if this will maybe not be apparent right now.

Theorem 3.12 (Dg-categorical Yoneda lemma)**.** *Let $\mathcal{A}$ be a dg-category, and let $A \in \mathrm{Ob}(\mathcal{A})$ be an object. Let M be a right $\mathcal{A}$-dg-module. We have an isomorphism of complexes:*

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M) &\xrightarrow{\sim} M(A), \\ \varphi &\mapsto \varphi_A(1_A), \end{aligned}$$

where $\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M)$ is the hom complex of morphisms between $\mathcal{A}(-, A)$ and M, cf. Definition 3.9.

The proof of this crucial result is left to you, reader. Exercises from Exercise 3.2 to Exercise 3.6 should give you a reasonable path to achieve that. I believe that everything revolving around the Yoneda lemma should be worked out directly by you, until you become "fluent" with the result. In my experience, this has happened after years and it is still happening – so, take your time!

Exercises to Lecture 3.

Exercise 3.1. Prove that the differential of a degree p morphism of left or right R-dg-modules (as in Definition 3.6) is a degree $p + 1$ morphism of R-dg-modules.

Exercise 3.2 (Yoneda for $\mathbf{k}$-modules). Let M be a $\mathbf{k}$-module. Find an isomorphism of $\mathbf{k}$-modules

$$\mathrm{Hom}(\mathbf{k}, M) \cong M.$$

Exercise 3.3 (Yoneda for modules over a $\mathbf{k}$-algebra). Let R be a $\mathbf{k}$-algebra, and let M be a right R-module. Find an isomorphism of $\mathbf{k}$-modules:

$$\mathrm{Hom}_R(R, M) \cong M,$$

where we view R as a right R-module with the canonical action $R \otimes R \to R$ given by the product.

Exercise 3.4 (Yoneda for dg-modules over a dg-algebra). Let R be a dg-algebra, and let M be a right dg-module. Find an isomorphism of complexes of $\mathbf{k}$-modules:

$$\underline{\mathrm{Hom}}_R(R, M) \cong M,$$

where we view R as a right R-dg-module with the canonical action $R \otimes R \to R$ given by the product.

Exercise 3.5. Prove the dg-Yoneda lemma (Theorem 3.12).

Exercise 3.6 (Yoneda for left dg-modules). State and prove a variant of the dg-Yoneda lemma (Theorem 3.12) for *left* dg-modules over a dg-category.

4. Lecture 4: more on dg-modules and dg-functors

4.1. Dg-modules as dg-functors. Let $\mathcal{A}$ be a dg-category. The definition of left $\mathcal{A}$-dg-modules we gave (cf. Definition 3.7) revolves around the actions

$$\mathcal{A}(A, B) \otimes M(A) \xrightarrow{m_{A,B}} M(B),$$

which we call $m_{A,B}$ for our current purposes. $m_{A,B}$ is a chain map of complexes; thanks to the tensor-hom adjunction isomorphism

$$\underline{\mathrm{Hom}}(\mathcal{A}(A, B) \otimes M(A), M(B)) \cong \underline{\mathrm{Hom}}(\mathcal{A}(A, B), \underline{\mathrm{Hom}}(M(A), M(B))),$$

described in Proposition 1.17, we may actually identify $m_{A,B}$ with a chain map

$$M_{A,B} \colon \mathcal{A}(A, B) \to \underline{\mathrm{Hom}}(M(A), M(B)),$$
$$M_{A,B}(f)(x) = m_{A,B}(f \otimes x) = fx.$$

What properties do the *chain maps* $M_{A,B}$ satisfy? For the sake of simplicity, let us drop the notational reference to the objects. If $f \in \mathcal{A}(A, B)$ and $g \in \mathcal{A}(B, C)$, we have:

$$M(gf)(x) = (gf)(x) = g(fx) = M(g)(fx) = M(g)(M(f)(x)),$$

namely

$$\boxed{M(gf) = M(g)M(f).}$$

Notice that this last relation does not explicitly mention any element $x \in M(A)$. Moreover, if $1_A \colon A \to A$ is an identity morphism, we have:

$$M(1_A)(x) = 1_A x = x = 1_{M(A)}(x),$$

namely

$$\boxed{M(1_A) = 1_{M(A)}.}$$

The above boxed relations, together with the fact that $f \mapsto M(f)$ is a chain map, immediately shows that we get a *dg-functor*:

$$M \colon \mathcal{A} \to \mathrm{dgm}(\mathbf{k}),$$

where $\mathrm{dgm}(\mathbf{k})$ is the dg-category of chain complexes, cf. Example 2.5.

Let us have a look at the complex of morphisms of (left) dg-modules. If $\varphi \colon M \to M'$ is a morphism of left $\mathcal{A}$-dg-modules, we have by definition (for $x \in M(A)$ and $f \in \mathcal{A}(A, B)$):

$$\varphi_B(fx) = (-1)^{|\varphi||f|} f \varphi_A(x),$$

where $|f|$ is the degree of f and $|\varphi|$ is the degree of φ. Viewing M and M' as dg-functors, we may replace fx with $M(f)(x)$ and $f\varphi_A(x)$ with $M'(f)(\varphi_A(x))$ and get the following relation:

$$\varphi_B(M(f)(x)) = (-1)^{|\varphi||f|}M'(f)(\varphi_A(x)).$$

We may drop the explicit reference to the element x and just write:

$$\boxed{\varphi_B \circ M(f) = (-1)^{|\varphi||f|}M'(f) \circ \varphi_A.}$$

This relation can be pictured by saying that the diagram

$$\begin{tikzcd} M(A) \arrow[r, "\varphi_A"] \arrow[d, "M(f)"'] & M'(A) \arrow[d, "M'(f)"] \\ M(B) \arrow[r, "\varphi_B"] & M'(B) \end{tikzcd}$$

is commutative up to the sign $(-1)^{|\varphi||f|}$, for all $f\colon A \to B$. Finally, recall that differentials of morphisms of dg-modules were defined objectwise:

$$\boxed{(d\varphi)_A = d(\varphi_A).}$$

The fact that the above boxed relations do not depend of elements of the complexes $M(A)$ allow us to vastly generalize, from dg-modules to any dg-functor.

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. We define the *dg-category of dg-functors* $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ as follows:

- Objects are dg-functors $F\colon \mathcal{A} \to \mathcal{B}$.
- Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be dg-functors. A *degree p morphism* $\varphi\colon F \to G$ is given by a family $A \mapsto \varphi_A$ of degree p morphisms

  $$\varphi_A \in \mathcal{B}(F(A), G(A))^p,$$

  such that

  $$\varphi_B \circ F(f) = (-1)^{|\varphi||f|}G(f) \circ \varphi_A$$

  (here $|\varphi| = p$), namely, the following diagram is commutative

  $$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\varphi_B"] & G(B) \end{tikzcd}$$

  up to the sign $(-1)^{|\varphi||f|}$.

  The differential $d\varphi$ of a degree p morphism $\varphi\colon F \to G$ is defined objectwise:

  $$(d\varphi)_A = d(\varphi_A).$$

  Morphisms of dg-functors will be also called *dg-natural transformations*; the hom-complex $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})(F, G)$ will be also denoted by $\mathrm{Nat}_{\mathrm{dg}}(F, G)$.

Remark 4.2. Let us show that, in the above definition, $d\varphi$ is indeed a natural transformation. We compute, using the graded Leibniz rule:

$$\begin{aligned} d(\varphi_B) \circ F(f) &= d(\varphi_B \circ F(f)) - (-1)^{|\varphi|}\varphi_B \circ dF(f) \\ &= (-1)^{|\varphi||f|} d(G(f) \circ \varphi_A) - (-1)^{|\varphi|}\varphi_B \circ F(df) \\ &= (-1)^{|\varphi||f|} d(G(f) \circ \varphi_A) - (-1)^{|\varphi|}(-1)^{|\varphi|(|f|+1)} dG(f) \circ \varphi_A \\ &= (-1)^{|\varphi||f|} d(G(f) \circ \varphi_A) - (-1)^{|\varphi||f|} dG(f) \circ \varphi_A \\ &= (-1)^{|\varphi||f|} (d(G(f) \circ \varphi_A - dG(f) \circ \varphi_A) \\ &= (-1)^{|\varphi||f|}(-1)^{|f|} G(f) d\varphi_A \\ &= (-1)^{(|\varphi|+1)|f|} G(f) d\varphi_A. \end{aligned}$$

4.2. **The case of right dg-modules, the opposite dg-category and "official" dg-modules.** Let $\mathcal{A}$ be a dg-category. We viewed (actually identified) left $\mathcal{A}$-dg-modules with dg-functors $\mathcal{A} \to \mathrm{dgm}(\mathbf{k})$. What about *right* $\mathcal{A}$-dg-modules? The situation is a little bit trickier. Let N be a right $\mathcal{A}$-dg-module, with actions

$$\begin{aligned} N(B) \otimes \mathcal{A}(A, B) &\xrightarrow{m} N(A), \\ x \otimes f &\mapsto m(x \otimes f) = xf. \end{aligned}$$

We first apply the symmetry isomorphism (cf. Proposition 1.15):

$$\begin{aligned} N(B) \otimes \mathcal{A}(A, B) &\xrightarrow{\sim} \mathcal{A}(A, B) \otimes N(B), \\ x \otimes f &\mapsto (-1)^{|x||f|} f \otimes x. \end{aligned}$$

Then, we apply the tensor-hom adjunction isomorphism (Proposition 1.17):

$$\underline{\mathrm{Hom}}(\mathcal{A}(A, B) \otimes N(B), N(A)) \cong \underline{\mathrm{Hom}}(\mathcal{A}(A, B), \underline{\mathrm{Hom}}(N(B), N(A))).$$

In the end, the action $m\colon N(B) \otimes \mathcal{A}(A, B) \to N(A)$ is identified with the chain map:

$$\begin{aligned} N = N_{A,B} \colon \mathcal{A}(A, B) &\to \underline{\mathrm{Hom}}(N(B), N(A)), \\ f &\mapsto N(f)(x) = (-1)^{|f||x|} xf. \end{aligned}$$

What properties do the chain maps $N = N_{A,B}$ satisfy? Let us compute, for composable morphisms $f\colon A \to B$ and $g\colon B \to C$:

$$\begin{aligned} N(gf)(x) &= (-1)^{|x|(|g|+|f|)} x(gf) \\ &= (-1)^{|x|(|g|+|f|)} (xg)f \\ &= (-1)^{|x||g|}(-1)^{|f||g|} N(f)(xg) \\ &= (-1)^{|f||g|} N(f)(N(g)(x)), \end{aligned}$$

hence

$$\boxed{N(gf) = (-1)^{|f||g|} N(f)N(g).}$$

Moreover

$$N(1_A)(x) = x1_A = x = 1_{N(A)}(x),$$

hence

$$\boxed{N(1_A) = 1_{N(A)}.}$$

We see that N is not a dg-functor from $\mathcal{A}$ to $\mathrm{dgm}(\mathbf{k})$, but something that we can temporarily call a *contravariant dg-functor*.

What about the complex of morphisms $\psi\colon N \to N'$ of right $\mathcal{A}$-dg-modules? By definition, for $x \in N(B)$ and $f\colon A \to B$:
$$\psi_A(xf) = \psi_B(x)f.$$
Identifying N and N' with contravariant dg-functors as above, we get:
$$(-1)^{|f||x|}\psi_A(N(f)(x)) = (-1)^{(|x|+|\psi|)|f|}N'(f)(\psi_B(x)),$$
from which we obtain
$$\boxed{\psi_A \circ N(f) = (-1)^{|\psi||f|}N'(f) \circ \psi_B.}$$
Hence, natural transformations of contravariant dg-functors look precisely the same as the natural transformations of ordinary dg-functors. Finally, we know that the differential of ψ is defined objectwise:
$$\boxed{(d\psi)_A = d(\psi_A).}$$

Can we make such contravariant dg-functors into ordinary dg-functors? The answer is yes, but we first need to introduce a new concept in order to formally switch left and right compositions and actions:

Definition 4.3. Let $\mathcal{A}$ be a dg-category. We define the *opposite dg-category* $\mathcal{A}^{\mathrm{op}}$ as follows:

- Objects of $\mathcal{A}^{\mathrm{op}}$ are the same as the objects of $\mathcal{A}$: $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$.
- Hom-complexes in $\mathcal{A}^{\mathrm{op}}$ are the same as the ones of $\mathcal{A}$, but "with arrows reversed":
$$\mathcal{A}^{\mathrm{op}}(A,B) = \mathcal{A}(B,A).$$
Notationally, we will put an "op" decoration on every morphism of $\mathcal{A}$ when we want to view it as a morphism in $\mathcal{A}^{\mathrm{op}}$. Namely, a morphism $f^{\mathrm{op}}\colon A \to B$ in $\mathcal{A}^{\mathrm{op}}$ corresponds to a morphism $f\colon B \to A$ in $\mathcal{A}$.
- Compositions are induced directly from $\mathcal{A}$, but beware the Koszul sign rule:
$$\begin{aligned}\mathcal{A}^{\mathrm{op}}(B,C) \otimes \mathcal{A}^{\mathrm{op}}(A,B) &\to \mathcal{A}^{\mathrm{op}}(A,C),\\ g^{\mathrm{op}} \otimes f^{\mathrm{op}} &\mapsto g^{\mathrm{op}} \circ f^{\mathrm{op}} = (-1)^{|f||g|}(f \circ g)^{\mathrm{op}}.\end{aligned}$$

Observe that we trivially have:
$$(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}.$$

Remark 4.4. What is a dg-functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$? It is given by a map on objects $A \mapsto F(A)$ and chain maps
$$F\colon \mathcal{A}^{\mathrm{op}}(B,A) = \mathcal{A}(A,B) \to \mathcal{B}(F(B),F(A)),$$
compatible with identities and compositions, in particular:
$$F(g^{\mathrm{op}}f^{\mathrm{op}}) = F(g^{\mathrm{op}})F(f^{\mathrm{op}}),$$
namely:
$$(-1)^{|f||g|}F((fg)^{\mathrm{op}}) = F(g^{\mathrm{op}})F(f^{\mathrm{op}}),$$
hence, if we drop the "op" decorations we indeed obtain a contravariant dg-functor, as introduced in the special case where we take $\mathcal{B} = \mathrm{dgm}(\mathbf{k})$.

With this language and recalling the above discussion, we may finally give a concise definition of left and right dg-modules, together with their dg-categories!

Definition 4.5. Let $\mathcal{A}$ be a dg-category. We define the *dg-category of right $\mathcal{A}$-dg-modules* as:

$$\operatorname{dgm}(\mathcal{A}) = \operatorname{Fun}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}}, \operatorname{dgm}\mathbf{k}),$$

and the dg-category of *left $\mathcal{A}$-dg-modules* just as the dg-category of right $\mathcal{A}^{\mathrm{op}}$-dg-modules:

$$\operatorname{dgm}(\mathcal{A}^{\mathrm{op}}) = \operatorname{Fun}_{\mathrm{dg}}(\mathcal{A}, \operatorname{dgm}(\mathbf{k})),$$

recalling that $(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}$. These are dg-categories of dg-functors, with morphisms being dg-natural transformations.

Remark 4.6. The above discussion essentially shows that this "new" definition of dg-modules is essentially the same as the "old" given in terms of left or right actions. For our convenience, we recall how we can pass from dg-functors to left or right actions, and vice-versa.

If N is a right $\mathcal{A}$-dg-module, we have:

$$xf = (-1)^{|f||x|} N(f)(x),$$

for $f \in \mathcal{A}(A, B)$ and $x \in N(B)$. If M is a left $\mathcal{A}$-dg-module, we write:

$$fx = M(f)(x),$$

for $x \in M(A)$ and $f \in \mathcal{A}(A, B)$.

Moreover, under this identification, dg-natural transformations of dg-modules become the dg-module morphisms as previously defined in Definition 3.9.

It is worth mentioning that what we've done so far for dg-categories and dg-modules can be also done for **k**-linear categories and modules over those, just by replacing $\operatorname{dgm}(\mathbf{k})$ with the **k**-linear category of **k**-modules $\operatorname{Mod}(\mathbf{k})$ and making the necessary adjustments. This is actually easier, because we can make the same formal steps but in an easier setting (no sign issues!). We sum up what we need in the following definition:

Definition 4.7. Let $\mathcal{A}$ and $\mathcal{B}$ be **k**-linear categories. We denote by $\operatorname{Fun}(\mathcal{A}, \mathcal{B})$ (or sometimes $\operatorname{Fun}_{\mathbf{k}}(\mathcal{A}, \mathcal{B})$) the **k**-linear category of **k**-linear functors. Its morphisms are given by the **k**-module $\operatorname{Nat}(F, G) = \operatorname{Nat}_{\mathbf{k}}(F, G)$ of *natural transformations* $\varphi\colon F \to G$, defined by families $A \mapsto \varphi_A$ such that $G(f) \circ \varphi_A = \varphi_B \circ F(f)$ for all $f\colon A \to B$.

We define the **k**-linear categories of (respectively) right and left $\mathcal{A}$-modules as:

$$\operatorname{Mod}(\mathcal{A}) = \operatorname{Fun}(\mathcal{A}^{\mathrm{op}}, \operatorname{Mod}(\mathbf{k})),$$
$$\operatorname{Mod}(\mathcal{A}^{\mathrm{op}}) = \operatorname{Fun}(\mathcal{A}, \operatorname{Mod}(\mathbf{k})),$$

where $\mathcal{A}^{\mathrm{op}}$ is the opposite **k**-linear category.

Remark 4.8. Warning! For a given **k**-linear category, we have the **k**-linear category of (right) $\mathcal{A}$-modules $\operatorname{Mod}(\mathcal{A})$, but we can also view $\mathcal{A}$ as a dg-category concentrated in degree 0 and consider the dg-category of (right) $\mathcal{A}$-dg-modules $\operatorname{dgm}(\mathcal{A})$. These are two very different things, so don't be confused. Indeed, even when $\mathcal{A} = \mathbf{k}$, we get $\operatorname{Mod}(\mathbf{k})$ (**k**-modules) on one side and $\operatorname{dgm}(\mathbf{k})$ (complexes of **k**-modules) on the other.

This identification between "dg-modules defined as dg-functors" and "dg-modules defined with left/right actions" may seem quite exotic, but it is just an instance of the classical identification of "modules over algebras" with "representations of algebras", which we actually obtain from what we have done by specializing to single-object dg-categories (namely, dg-algebras) or single object linear categories (namely, ordinary algebras).

Exercises to Lecture 4

Exercise 4.1. View $\mathbf{k}$ as a dg-category with a single object $\star$ and concentrated in degree 0 with zero differentials. Let $\mathcal{A}$ be any dg-category. Define a dg-functor

$$\mathrm{Fun}_{\mathrm{dg}}(\mathbf{k}, \mathcal{A}) \to \mathcal{A}$$

which is a bijection on objects and an isomorphism of complexes between the hom-complexes – namely, it is an isomorphism of dg-categories.

Exercise 4.2. Let A be a $\mathbf{k}$-algebra. A *representation* of A is a $\mathbf{k}$-module M together with a morphism

$$A \to \mathrm{Hom}(M, M)$$

of $\mathbf{k}$-algebras, where $\mathrm{Hom}(M, M)$ has the $\mathbf{k}$-algebra structure given by composition of endomorphisms $M \to M$. Check that this is actually the same as giving a $\mathbf{k}$-*linear functor*

$$A \to \mathrm{Mod}(k),$$

viewing A as a $\mathbf{k}$-linear category with a single object. In turn, this is the same as giving a left A-module structure on M by means of an action

$$A \otimes M \to M.$$

Exercise 4.3. Let R be a dg-algebra. Consider the complex $\underline{\mathrm{Hom}}_{\mathbf{k}}(R, R)$ of endomorphisms $R \to R$, and endow it with a structure of $\mathbf{k}$-dg-algebra with the composition of endormorphisms. Find an injective morphism of dg-algebras:

$$R \to \underline{\mathrm{Hom}}_{\mathbf{k}}(R, R).$$

Hint: By the Yoneda lemma, we know that $R \cong \underline{\mathrm{Hom}}_{R-\mathbf{k}}(R, R)$, the complex of left R-module morphisms $R \to R$, endowing R with the obvious structure of left R-module over itself.

Exercise 4.4. *(A bit harder!)* Let $\mathcal{A}$ be a dg-category and let $M \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{dgm}(\mathbf{k})$ be a right $\mathcal{A}$-dg-module. Consider the Yoneda isomorphisms ($A \in \mathrm{Ob}(\mathcal{A})$):

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M) \xrightarrow{\sim} M(A).$$

Prove that we can define a right $\mathcal{A}$-dg-module

$$\widetilde{M} \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{dgm}(\mathbf{k}),$$
$$A \mapsto \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M)$$

(describe it on morphisms!) such that the Yoneda isomorphism is "promoted" to a natural transformation

$$\widetilde{M} \to M$$

which in particular yields isomorphisms of complexes $\widetilde{M}(A) \to M(A)$.