DIFFERENTIAL GRADED CATEGORIES

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1. Lecture 1: review of chain complexes

1.1. Basics.

Setup 1.1. We will work in the following framework:

- Everything will be over a fixed base *commutative ring* $\mathbf{k}$. If $\mathbf{k} = \mathbb{Z}$, we get chain complexes of abelian group; if $\mathbf{k}$ is a field, we get chain complexes of $\mathbf{k}$-vector spaces.
- We will use *cohomological notation*. This means that we will have increasing indices. Hence, we actually work with *cochain* complexes, but we will simplify terminology and still refer to them as *chain* complexes.

Definition 1.2. A *chain complex* (sometimes also called just *complex*) is a sequence of $\mathbf{k}$-modules $(V^i i)_{i\in\mathbb{Z}}$ together with morphisms $d^i \colon V^i \to V^{i+1}$ such that

$$d^{i+1} \circ d^i = 0,$$

for all $i \in \mathbb{Z}$. We picture this data as follows:

$$\cdots \to V^i \xrightarrow{d^i} V^{i+1} \xrightarrow{d^{i+1}} V^{i+2} \to \cdots$$

A *morphism* (or *chain map*)

$$f \colon (V^i, d^i_V) \to (W^i, d^i_W)$$

between chain complexes (V^i, d^i_V) and (W^i, d^i_W) is a family of morphisms of $\mathbf{k}$-modules

$$f^i \colon V^i \to W^i,$$

such that $d^i_W \circ f^i = f^{i+1} \circ d^i_V$, for all $i \in \mathbb{Z}$. In other words, the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & V^{i-1} \arrow[r, "d^{i-1}_V"] \arrow[d, "f^{i-1}"] & V^i \arrow[r, "d^i_V"] \arrow[d, "f^i"] & V^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & W^{i-1} \arrow[r, "d^{i-1}_W"] & W^i \arrow[r, "d^i_W"] & W^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 1.3. *Let $V = (V^i, d^i)$ be a chain complex. The identity morphism*

$$1_V \colon V \to V,$$

defined by identities $1_{V^i} \colon V^i \to V^i$ for all $i \in \mathbb{Z}$, is a chain map.

If $f \colon (V^i, d^i_V) \to (W^i, d^i_W)$ and $g \colon (W^i, d^i_W) \to (Z^i, d^i_Z)$ are chain maps, the composition morphism

$$g \circ f \colon (V^i, d^i_V) \to (Z^i, d^i_Z),$$

defined by the compositions $g^i \circ f^i \colon V^i \to Z^i$ for all $i \in \mathbb{Z}$, is also a chain map.

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Proof. Exercise 1.1. □

Definition 1.4. Let $f\colon (V^i, d_V^i) \to (W^i, d_W^i)$ be a chain map of complexes. We say that f is an *isomorphism* if there is a (unique) chain map $f^{-1}\colon (W^i, d_W^i) \to (V^i, d_V^i)$ such that $f^{-1} \circ f = 1_V$ and $f \circ f^{-1} = 1_W$.

Isomorphisms are completely understood "componentwise", namely:

Lemma 1.5. *A chain map $f = (f^i)\colon (V^i, d_V^i) \to (W^i, d_W^i)$ is an isomorphism if and only if $f^i\colon V^i \to W^i$ is an isomorphism of* **k***-modules for all $i \in \mathbb{Z}$.*

Proof. If f is an isomorphism, clearly all f^i are isomorphisms by definition. Conversely, assume that all $f^i\colon V^i \to W^i$ are isomorphisms, with inverses $(f^i)^{-1}$. There is a unique way of defining $f^{-1}\colon (W^i, d_W^i) \to (V^i, d_V^i)$, namely:

$$(f^{-1})^i = (f^i)^{-1}.$$

We just need to check that f^{-1} is indeed a chain map. f is a chain map, so by definition

$$d_W^i \circ f^i = f^{i+1} \circ d_V^i$$

for all $i \in \mathbb{Z}$. We may compose with inverses of f^i and f^{i+1}, and obtain:

$$(f^{i+1})^{-1} \circ d_W^i = d_V^i \circ (f^i)^{-1}.$$

This means precisely that f^{-1} is a chain map. □

Remark 1.6. It is useful to identify a chain complex $V = (V^i, d_V^i)$ with the following **k**-module:

$$V = \bigoplus_i V^i,$$

keeping track of the direct sum decomposition over $\mathbb{Z}$ and together with a morphism $d_V\colon V \to V$ (obtained uniquely from the d^i) with the following properties:

$$\begin{aligned} &d_V(V^i) \subseteq V^{i+1} \qquad \text{for all } i \in \mathbb{Z}, \\ &d_V \circ d_V = 0. \end{aligned}$$

Moreover, a chain map $f\colon (V^i, d_V^i) \to (W^i, d_W^i)$ can be identified with a morphism

$$f\colon V = \bigoplus_i V^i \to \bigoplus_i W^i = W,$$

with the following properties:

$$\begin{aligned} &f(V^i) \subseteq W^i, \qquad \text{for all } i \in \mathbb{Z}, \\ &d_W \circ f = f \circ d_V. \end{aligned}$$

As an exercise (cf. Exercise 1.2), we may check that this description of chain complexes and chain maps is indeed equivalent to the one given in Definition 1.2. In what follows, we will go back and forth between these two equivalent definitions without mentioning it.

Remark 1.7. Any k-module M can be viewed as a chain complex concentrated in degree 0, namely:

$$\begin{aligned} &M^0 = M, \\ &M^i = 0, \qquad i \neq 0. \end{aligned}$$

The differentials d^i_M are all zero.

The most important operation that we can perform on a chain complex is taking *cohomology*.

Definition 1.8. Let $V = (V^i, d^i)$ be a chain complex. For all $i \in \mathbb{Z}$, we define:

$$\begin{aligned} Z^i(V) &= \ker(d^i \colon V^i \to V^{i+1}), \\ B^i(V) &= \operatorname{Im}(d^{i-1} \colon V^{i-1} \to V^i). \end{aligned}$$

Both $Z^i(V)$ and $B^i(V)$ are submodules of V^i. They are sometimes called (respectively) *i-cocycles* and *i-coboundaries*, or (again, respectively) *closed degree i elements* and *exact degree i elements*.

The assumption that $d^i \circ d^{i-1} = 0$ ensures that $B^i(V) \subseteq Z^i(V)$. Hence, we may take the quotient:

$$H^i(V) = Z^i(V)/B^i(V).$$

This is called *i-th cohomology* of the chain complex V.

If $v \in Z^i(V)$, we will denote by $[v]$ the image of v in $H^i(V)$ with respect to the natural projection $Z^i(V) \to H^i(V)$.

Notation 1.9. Thanks to the identification described in Remark 1.6, we may and will view $V^i, Z^i(V), B^i(V)$ as submodules of $V = \bigoplus_i V^i$, for all $i \in \mathbb{Z}$. If an element $v \in V$ lies in V^i, we say that it has *degree i* and we write:

$$\deg(v) = |v| = i.$$

Proposition 1.10 (Functoriality). *Let $f \colon V \to W$ be a chain map of complexes. For all $i \in \mathbb{Z}$, f restricts to morphisms*

$$\begin{aligned} Z^i(f) &\colon Z^i(V) \to Z^i(W), \\ B^i(f) &\colon B^i(V) \to B^i(W). \end{aligned}$$

This also induces a morphism

$$H^i(f) \colon H^i(V) \to H^i(W).$$

Moreover, if V is a chain complex, we have

$$\begin{aligned} Z^i(1_V) &= 1_{Z^i(V)} \colon Z^i(V) \to Z^i(V), \\ B^i(1_V) &= 1_{B^i(V)} \colon B^i(V) \to B^i(V), \\ H^i(1_V) &= 1_{H^i(V)} \colon H^i(V) \to H^i(V), \end{aligned}$$

for all $i \in \mathbb{Z}$. If $f \colon V \to W$ and $g \colon W \to X$ are chain maps, we have equalities:

$$\begin{aligned} Z^i(g \circ f) &= Z^i(g) \circ Z^i(f), \\ B^i(g \circ f) &= B^i(g) \circ B^i(f), \\ H^i(g \circ f) &= H^i(g) \circ H^i(f). \end{aligned}$$

Proof. Let $f \colon V \to W$ and $g \colon W \to X$ be chain maps as above. If $x \in Z^i(V)$, namely $dx = 0$, we have $d(f(x)) = f(dx) = f(0) = 0$, so f indeed yields a restricted morphism $Z^i(f) \colon Z^i(V) \to Z^i(W)$. Analogusly, if $x = dx' \in B^i(V)$, then $f(x) = f(dx') = df(x') \in B^i(W)$, so f also restricts to a morphism $B^i(f) \colon B^i(V) \to B^i(W)$. Equalities $Z^i(g \circ f) = Z^i(g) \circ Z^i(f)$ and $B^i(g \circ f) = B^i(g) \circ B^i(f)$, and $Z^i(1_V) = 1_{Z^i(V)}, B^i(1_V) = 1_{B^i(V)}$ follow immediately.

The induced morphism $H^i(f)\colon H^i(V) \to H^i(W)$ is the unique which makes the following diagram commute:

$$\begin{tikzcd}
Z^i(V) \arrow[r, "Z^i(f)"] \arrow[d, "\mathrm{pr}_V"'] & Z^i(W) \arrow[d, "\mathrm{pr}_W"] \\
H^i(V) \arrow[r, dashed, "H^i(f)"'] & H^i(W),
\end{tikzcd}$$

where the vertical morphisms are the canonical projections onto the quotients. By such uniqueness, we easily see that indeed $H^i(1_V) = 1_{H^i(V)}$ and $H^i(g \circ f) = H^i(g) \circ H^i(f)$. □

Remark 1.11. Let V be a chain complex. It is sometimes useful to collect all cohomologies $H^i(V)$ to define

$$H^*(V) = \bigoplus_i H^i(V).$$

This is a *graded module*, namely, a $\mathbf{k}$-module together with a direct sum decomposition over the integers. This can be also described just as the sequence of $\mathbf{k}$-modules $(H^i(V))_{i\in\mathbb{Z}}$ (compare with the case of complexes described in Remark 1.6).

If $f\colon V \to W$ is a chain map of complexes, the collection $(H^i(f))_{i\in\mathbb{Z}}$ defines a morphism of graded modules

$$H^*(f)\colon H^*(V) \to H^*(W).$$

Clearly, we have compatibility with identities and compositions:

$$H^*(1_V) = 1_{H^*(V)},$$
$$H^*(g \circ f) = H^*(g) \circ H^*(f).$$

1.2. **Hom and tensor.** If M and N are $\mathbf{k}$-modules, there is a $\mathbf{k}$-module $\mathrm{Hom}(M, N)$ of morphisms $M \to N$, with the obvious elementwise addition and action of $\mathbf{k}$. We may also define the *tensor product* $M \otimes N$. Hom and tensor are connected by the following natural isomorphism:

$$\mathrm{Hom}(V \otimes W, X) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, X)),$$

which is sometimes called the "hom-tensor adjunction". We are going to discuss this in the framework of chain complexes.

Hom-complexes. If V and W are chain complexes, we denote by

$$\mathrm{Hom}(V, W)$$

the $\mathbf{k}$-module of chain maps $V \to W$, with the obvious elementwise addition and action of $\mathbf{k}$ (see Exercise 1.3). Such $\mathbf{k}$-module can itself be "enhanced" to a chain complex, as follows.

Definition 1.12. Let $V = (V^i, d_V^i)$ and $W = (W^i, d_W^i)$ be chain complexes. We define a chain complex

$$\underline{\mathrm{Hom}}(V, W) = (\underline{\mathrm{Hom}}^p(V, W), d_{\underline{\mathrm{Hom}}})_{p\in\mathbb{Z}}$$

as follows.

- The $\mathbf{k}$-module $\underline{\mathrm{Hom}}^p(V, W)$ is the $\mathbf{k}$-module of *degree p morphisms*, namely, of morphisms $f\colon V \to W$ such that $f(V^i) \subseteq W^{i+p}$ for all $i \in \mathbb{Z}$. Equivalently, they are sequences of morphisms $(f^i\colon V^i \to W^{i+p})_{i\in\mathbb{Z}}$. We don't require any compatibility with the differentials d_V and d_W.

- The differential
$$d^p_{\underline{\mathrm{Hom}}} \colon \underline{\mathrm{Hom}}^p(V,W) \to \underline{\mathrm{Hom}}^{p+1}(V,W)$$
is defined as follows:
$$d^p_{\underline{\mathrm{Hom}}}(f) = d_W \circ f - (-1)^p f \circ d_V.$$

Notationally, we will almost always write d instead of $d_{\underline{\mathrm{Hom}}}$ for the differential of $\underline{\mathrm{Hom}}(V,W)$. We can directly check (Exercise 1.4) that $d^{p+1} \circ d^p = 0$, hence $\underline{\mathrm{Hom}}(V,W)$ is indeed a complex.

Remark 1.13. Let V and W be chain complexes. What is the $\mathbf{k}$-module $Z^0(\underline{\mathrm{Hom}}(V,W))$? By definition, it contains precisely the morphisms $f \colon V \to W$ such that $f(V^i) \subseteq W^i$ for all $i \in \mathbb{Z}$, and $d_W \circ f - f \circ d_V = 0$. This means that
$$Z^0(\underline{\mathrm{Hom}}(V,W)) = \mathrm{Hom}(V,W)$$
is precisely the $\mathbf{k}$-module of chain maps $V \to W$.

What about the zeroth cohomology $H^0(\underline{\mathrm{Hom}}(V,W))$? Its elements are equivalence classes $[f]$, where f is a chain map $V \to W$. By definition, $[f] = [g]$ if and only if $f - g = dh$ for some $h \in \underline{\mathrm{Hom}}^{-1}(V,W)$. Explicitly, this means:
$$f - g = d_W \circ h + h \circ d_V,$$
namely, that f and g are *chain homotopic*. The degree -1 morphism $h \colon V \to W$ is a chain homotopy between f and g. The effort we made to define the hom-complex $\underline{\mathrm{Hom}}(V,W)$ pays off giving us a better framework to treat such chain homotopies.

Tensor products. We are able to generalize the definition of tensor product to chain complexes.

Definition 1.14. Let V and W be chain complexes. We define the *tensor product* $V \otimes W = ((V \otimes W)^p, d^p_{V\otimes W})$ as follows. First:
$$(V \otimes W)^p = \bigoplus_{i+j=p} V^i \otimes W^j,$$
where $V^i \otimes W^j$ is the usual tensor product of $\mathbf{k}$-modules. Moreover, the differential is defined by:
$$d^p_{V\otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^i v \otimes d_W(w),$$
if $v \in V^i$ and $w \in W^j$ with $i+j = p$ (and then "extending by linearity"). With a direct computation (see Exercise 1.5) we can show that $d^{p+1}_{V\otimes W} \circ d^p_{V\otimes W} = 0$ for all $p \in \mathbb{Z}$, hence $V \otimes W$ is indeed a chain complex. We shall often simplify notation and write d instead of $d_{V\otimes W}$ when the context is clear.

The tensor product of complexes behaves nicely. Namely, it is associative, commutative and unital (with unit being $\mathbf{k}$ viewed as a complex concentrated in degree 0, see Remark 1.7). We list all these properties in the following proposition; the proofs are left as an exercise.

Proposition 1.15. *Let V, W, X be chain complexes. There are natural isomorphisms:*
$$(V \otimes W) \otimes X \xrightarrow{\sim} V \otimes (W \otimes X), \quad (v \otimes w) \otimes x \mapsto v \otimes (w \otimes x),$$
$$\mathbf{k} \otimes V \xrightarrow{\sim} V, \quad \lambda \otimes v \mapsto \lambda v,$$
$$V \otimes W \xrightarrow{\sim} W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

Proof. Exercise 1.6. □

Remark 1.16. It is convenient to comment the "commutativity" isomorphism

$$\begin{aligned} V \otimes W &\xrightarrow{\sim} W \otimes V, \\ v \otimes w &\mapsto (-1)^{|v||w|} w \otimes v. \end{aligned}$$

Assuming that $v \in V^i$ and $w \in W^j$, the tensor $v \otimes w$ is mapped to $(-1)^{ij} w \otimes v$ (and the isomorphism is obtained by extending this by linearity). The occurrence of the sign $(-1)^{ij}$ is an instance of the *Koszul sign rule*.

Warning! What follows are just a few heuristic and informal ideas. To my best understanding, the Koszul sign rule can be informally summarized as follows: *every time we swap two graded symbols a and b, we make the sign* $(-1)^{|a||b|}$ *appear.* This occurs for example in the definition of the differential $d_{V\otimes W}$ of the tensor product $V \otimes W$ (cf. Definition 1.14):

$$d_{V\otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w).$$

The second summand is obtained by "switching" the triple of symbols (d_V, v, w) with the triple (v, d_W, w). The symbols d_V and d_W have degree 1, and $(-1)^{|v|} = (-1)^{1|v|}$ is the sign which correctly appears according to the rule.

We may finally state the "tensor-hom adjunction" for chain complexes.

Proposition 1.17. *Let V, W, X be chain complexes. There are natural isomorphisms of complexes, one inverse to the other:*

$$\begin{aligned} \Psi \colon \underline{\mathrm{Hom}}(V \otimes W, X) &\xrightarrow{\sim} \underline{\mathrm{Hom}}(V, \underline{\mathrm{Hom}}(W, X)), \quad f \mapsto (v \mapsto f_v, f_v(w) = f(v \otimes w)), \\ \Phi \colon \underline{\mathrm{Hom}}(V, \underline{\mathrm{Hom}}(W, X)) &\xrightarrow{\sim} \underline{\mathrm{Hom}}(V \otimes W, X), \quad g \mapsto (v \otimes w \mapsto g(v)(w)). \end{aligned}$$

Proof. The fact that the above morphisms are mutual inverses is clear from the definition. To conclude, we just have to check that they are chain maps. Thanks to Lemma 1.5, we just need to check that the above morphisms preserve the gradings (i.e. they map degree p morphisms to degree p morphisms) and that *one of those* is a chain map. This is straightforward but tedious and not particularly instructive, so we leave it behind. □

1.3. **Quasi-isomorphisms.**

Why are complexes interesting and important? A possible answer is that *they essentially bring (linear) algebra to higher dimensions*. They give a unified framework to treat resolutions of modules, derived functors and so on. Cohomology is often the piece of information of a given complex that we really want to retain in many contexts, and this motivates the following definition:

Definition 1.18. Let $V = (V^i, d^i_V)$ and $W = (W^i, d^i_W)$ be chain complexes, and let $f \colon V \to W$ be a chain map. We say that f is a *quasi-isomorphism* if $H^i(f) \colon H^i(V) \to H^i(W)$ is an isomorphism for all $i \in \mathbb{Z}$ (or, equivalently, that the graded morphism $H^*(f)$ is an isomorphism). See Exercise 1.7 for another equivalent definition.

Quasi-isomorphisms are abundant. A typical family of examples is given by projective or injective resolutions, one of which we see in the following example.

Example 1.19. Assume that $\mathbf{k} = \mathbb{Z}$, the integers. We describe a simple free resolution of the abelian group $\mathbb{Z}/2\mathbb{Z}$. This is understood as the following chain map:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

The abelian group $\mathbb{Z}/2\mathbb{Z}$ is viewed as a complex concentrated in degree 0, and the morphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is multiplication by 2. Hence, the complex V in the upper row has $\mathbb{Z}$ in degrees -1 and 0. A direct computation shows that $H^{-1}(V) = 0$ and $H^{-1}(\mathbb{Z}/2\mathbb{Z}) = 0$, and that

$$H^0(f) = 1_{\mathbb{Z}/2\mathbb{Z}} \colon H^0(V) \to H^0(\mathbb{Z}/2\mathbb{Z}).$$

We conclude that f is a quasi-isomorphism. This somehow captures the idea that we could replace the module $\mathbb{Z}/2\mathbb{Z}$ (which is torsion) with the complex V, which is made of free abelian groups but has a nontrivial component in degree -1.

There is an important caveat. While we would like to view quasi-isomorphisms as some kind of isomorphism, unfortunately *not all quasi-isomorphisms have inverses*. This can be seen even from the above example:

Remark 1.20. In the setup of the above Example 1.19 we can't find any chain map $g \colon \mathbb{Z}/2\mathbb{Z} \to V$ such that $H^*(g)$ is inverse to $H^*(f)$. The point is that we can't find a nonzero group homomorphism

$$\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z},$$

so there actually no nonzero chain maps $\mathbb{Z}/2\mathbb{Z} \to V$.

The problem of "inverting quasi-isomorphisms" is highly nontrivial. In the following lectures, we will see – among other things – how we are able to overcome it.

Exercises.

Exercise 1.1. Prove Lemma 1.3.

Exercise 1.2. Check that the "alternative definition" of chain complexes and chain maps described in Remark 1.6 is equivalent to the one given in Definition 1.2. More explicitly, define bijections

$$\{\text{chain complexes "version 1" } (V^i, d^i)\} \leftrightarrow \{\text{chain complexes "version 2" } (V = \oplus_i V^i, d)\},$$

$$\{\text{chain maps "version 1" } (V^i, d^i) \to (W, d^i_W)\} \leftrightarrow \{\text{chain maps "version 2" } \oplus_i V^i \to \oplus_i W^i\}.$$

Exercise 1.3. Let $f, g \colon V \to W$ be chain maps of complexes. Then, the sum $f + g \colon V \to W$ defined elementwise by

$$(f + g)(v) = f(v) + g(v)$$

is again a chain map. The opposite $-f \colon V \to W$ defined elementwise by

$$(-f)(v) = -(f(v))$$

is also a chain map. Moreover, the zero morphism $0 \colon V \to W$ is a chain map. Finally, let $\lambda \in \mathbf{k}$. Then, the morphism $\lambda f \colon V \to W$ defined by

$$(\lambda f)(v) = \lambda f(v)$$

is a chain map.

Exercise 1.4. Check that the differential $d = d_{\underline{\mathrm{Hom}}}$ of the complex $\underline{\mathrm{Hom}}(V, W)$ described in Definition 1.12 actually satisfies $d^{p+1} \circ d^p = 0$ for all $p \in \mathbb{Z}$.

Exercise 1.5. Check that the differential $d_{V \otimes W}$ of the complex $V \otimes W$ described in Definition 1.14 actually satisfies $d^{p+1}_{V \otimes W} \circ d^p_{V \otimes W} = 0$ for all $p \in \mathbb{Z}$.

Exercise 1.6. Prove the claims of Proposition 1.15.

Exercise 1.7. Let $f \colon V \to W$ be a chain map of complexes. Prove that f is a quasi-isomorphism if and only if the following hold:

- Let $y \in W^p$ and $x' \in V^{p+1}$ such that $dy = f(x')$. Then, there is $z \in W^{p-1}$ and $x \in V^p$ such that:

$$\begin{aligned} dx &= x', \\ y - dz &= f(x). \end{aligned}$$