DIFFERENTIAL GRADED CATEGORIES

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1. Lecture 1: review of chain complexes

1.1. Basics.

Setup 1.1. We will work in the following framework:

- Everything will be over a fixed base *commutative ring* $\mathbf{k}$ (always with unit). If $\mathbf{k} = \mathbb{Z}$, we get chain complexes of abelian group; if $\mathbf{k}$ is a field, we get chain complexes of $\mathbf{k}$-vector spaces.
- We will use *cohomological notation*. This means that we will have increasing indices. Hence, we actually work with *cochain* complexes, but we will simplify terminology and still refer to them as *chain* complexes.
- Notationally, we will sometimes drop parentheses, especially when dealing with differential maps. So, we will write dx instead of $d(x)$.

Definition 1.2. A *chain complex* (sometimes also called just *complex*) is a sequence of $\mathbf{k}$-modules $(V^i)_{i\in\mathbb{Z}}$ together with morphisms $d^i \colon V^i \to V^{i+1}$ such that

$$d^{i+1} \circ d^i = 0,$$

for all $i \in \mathbb{Z}$. We picture this data as follows:

$$\cdots \to V^i \xrightarrow{d^i} V^{i+1} \xrightarrow{d^{i+1}} V^{i+2} \to \cdots$$

A *morphism* (or *chain map*)

$$f \colon (V^i, d^i_V) \to (W^i, d^i_W)$$

between chain complexes (V^i, d^i_V) and (W^i, d^i_W) is a family of morphisms of $\mathbf{k}$-modules

$$f^i \colon V^i \to W^i,$$

such that $d^i_W \circ f^i = f^{i+1} \circ d^i_V$, for all $i \in \mathbb{Z}$. In other words, the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & V^{i-1} \arrow[r, "d^{i-1}_V"] \arrow[d, "f^{i-1}"] & V^i \arrow[r, "d^i_V"] \arrow[d, "f^i"] & V^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & W^{i-1} \arrow[r, "d^{i-1}_W"] & W^i \arrow[r, "d^i_W"] & W^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 1.3. *Let $V = (V^i, d^i)$ be a chain complex. The identity morphism*

$$1_V \colon V \to V,$$

defined by identities $1_{V^i} \colon V^i \to V^i$ for all $i \in \mathbb{Z}$, is a chain map.

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If $f\colon (V^i, d_V^i) \to (W^i, d_W^i)$ and $g\colon (W^i, d_W^i) \to (Z^i, d_Z^i)$ are chain maps, the composition morphism

$$g \circ f\colon (V^i, d_V^i) \to (Z^i, d_Z^i),$$

defined by the compositions $g^i \circ f^i\colon V^i \to Z^i$ for all $i \in \mathbb{Z}$, is also a chain map.

Proof. Exercise 1.1. □

Definition 1.4. Let $f\colon (V^i, d_V^i) \to (W^i, d_W^i)$ be a chain map of complexes. We say that f is an *isomorphism* if there is a (unique) chain map $f^{-1}\colon (W^i, d_W^i) \to (V^i, d_V^i)$ such that $f^{-1} \circ f = 1_V$ and $f \circ f^{-1} = 1_W$.

Isomorphisms are completely understood "componentwise", namely:

Lemma 1.5. *A chain map $f = (f^i)\colon (V^i, d_V^i) \to (W^i, d_W^i)$ is an isomorphism if and only if $f^i\colon V^i \to W^i$ is an isomorphism of* **k***-modules for all $i \in \mathbb{Z}$.*

Proof. If f is an isomorphism, clearly all f^i are isomorphisms by definition. Conversely, assume that all $f^i\colon V^i \to W^i$ are isomorphisms, with inverses $(f^i)^{-1}$. There is a unique way of defining $f^{-1}\colon (W^i, d_W^i) \to (V^i, d_V^i)$, namely:

$$(f^{-1})^i = (f^i)^{-1}.$$

We just need to check that f^{-1} is indeed a chain map. f is a chain map, so by definition

$$d_W^i \circ f^i = f^{i+1} \circ d_V^i$$

for all $i \in \mathbb{Z}$. We may compose with inverses of f^i and f^{i+1}, and obtain:

$$(f^{i+1})^{-1} \circ d_W^i = d_V^i \circ (f^i)^{-1}.$$

This means precisely that f^{-1} is a chain map. □

Remark 1.6. It is useful to identify a chain complex $V = (V^i, d_V^i)$ with the following **k**-module:

$$V = \bigoplus_i V^i,$$

keeping track of the direct sum decomposition over $\mathbb{Z}$ and together with a morphism $d_V\colon V \to V$ (obtained uniquely from the d^i) with the following properties:

$$\begin{aligned} &d_V(V^i) \subseteq V^{i+1} \qquad \text{for all } i \in \mathbb{Z}, \\ &d_V \circ d_V = 0. \end{aligned}$$

Moreover, a chain map $f\colon (V^i, d_V^i) \to (W^i, d_W^i)$ can be identified with a morphism

$$f\colon V = \bigoplus_i V^i \to \bigoplus_i W^i = W,$$

with the following properties:

$$\begin{aligned} &f(V^i) \subseteq W^i, \qquad \text{for all } i \in \mathbb{Z}, \\ &d_W \circ f = f \circ d_V. \end{aligned}$$

As an exercise (cf. Exercise 1.2), we may check that this description of chain complexes and chain maps is indeed equivalent to the one given in Definition 1.2. In what follows, we will go back and forth between these two equivalent definitions without mentioning it.

Remark 1.7. Any k-module M can be viewed as a chain complex concentrated in degree 0, namely:

$$\begin{aligned} M^0 &= M, \\ M^i &= 0, \qquad i \neq 0. \end{aligned}$$

The differentials d^i_M are all zero.

The most important operation that we can perform on a chain complex is taking *cohomology*.

Definition 1.8. Let $V = (V^i, d^i)$ be a chain complex. For all $i \in \mathbb{Z}$, we define:

$$\begin{aligned} Z^i(V) &= \ker(d^i \colon V^i \to V^{i+1}), \\ B^i(V) &= \operatorname{Im}(d^{i-1} \colon V^{i-1} \to V^i). \end{aligned}$$

Both $Z^i(V)$ and $B^i(V)$ are submodules of V^i. They are sometimes called (respectively) *i-cocycles* and *i-coboundaries*, or (again, respectively) *closed degree i elements* and *exact degree i elements*.

The assumption that $d^i \circ d^{i-1} = 0$ ensures that $B^i(V) \subseteq Z^i(V)$. Hence, we may take the quotient:

$$H^i(V) = Z^i(V)/B^i(V).$$

This is called *i-th cohomology* of the chain complex V.

If $v \in Z^i(V)$, we will denote by $[v]$ the image of v in $H^i(V)$ with respect to the natural projection $Z^i(V) \to H^i(V)$.

Notation 1.9. Thanks to the identification described in Remark 1.6, we may and will view $V^i, Z^i(V), B^i(V)$ as submodules of $V = \bigoplus_i V^i$, for all $i \in \mathbb{Z}$. If an element $v \in V$ lies in V^i, we say that it has *degree i* and we write:

$$\deg(v) = |v| = i.$$

Proposition 1.10 (Functoriality)**.** *Let $f \colon V \to W$ be a chain map of complexes. For all $i \in \mathbb{Z}$, f restricts to morphisms*

$$\begin{aligned} Z^i(f) &\colon Z^i(V) \to Z^i(W), \\ B^i(f) &\colon B^i(V) \to B^i(W). \end{aligned}$$

This also induces a morphism

$$H^i(f) \colon H^i(V) \to H^i(W).$$

Moreover, if V is a chain complex, we have

$$\begin{aligned} Z^i(1_V) &= 1_{Z^i(V)} \colon Z^i(V) \to Z^i(V), \\ B^i(1_V) &= 1_{B^i(V)} \colon B^i(V) \to B^i(V), \\ H^i(1_V) &= 1_{H^i(V)} \colon H^i(V) \to H^i(V), \end{aligned}$$

for all $i \in \mathbb{Z}$. If $f \colon V \to W$ and $g \colon W \to X$ are chain maps, we have equalities:

$$\begin{aligned} Z^i(g \circ f) &= Z^i(g) \circ Z^i(f), \\ B^i(g \circ f) &= B^i(g) \circ B^i(f), \\ H^i(g \circ f) &= H^i(g) \circ H^i(f). \end{aligned}$$

Proof. Let $f\colon V \to W$ and $g\colon W \to X$ be chain maps as above. If $x \in Z^i(V)$, namely $dx = 0$, we have $d(f(x)) = f(dx) = f(0) = 0$, so f indeed yields a restricted morphism $Z^i(f)\colon Z^i(V) \to Z^i(W)$. Analogusly, if $x = dx' \in B^i(V)$, then $f(x) = f(dx') = df(x') \in B^i(W)$, so f also restricts to a morphism $B^i(f)\colon B^i(V) \to B^i(W)$. Equalities $Z^i(g \circ f) = Z^i(g) \circ Z^i(f)$ and $B^i(g \circ f) = B^i(g) \circ B^i(f)$, and $Z^i(1_V) = 1_{Z^i(V)}, B^i(1_V) = 1_{B^i(V)}$ follow immediately.

The induced morphism $H^i(f)\colon H^i(V) \to H^i(W)$ is the unique which makes the following diagram commute:

$$\begin{tikzcd} Z^i(V) \arrow[r, "Z^i(f)"] \arrow[d, "\mathrm{pr}_V"'] & Z^i(W) \arrow[d, "\mathrm{pr}_W"] \\ H^i(V) \arrow[r, dashed, "H^i(f)"'] & H^i(W), \end{tikzcd}$$

where the vertical morphisms are the canonical projections onto the quotients. By such uniqueness, we easily see that indeed $H^i(1_V) = 1_{H^i(V)}$ and $H^i(g \circ f) = H^i(g) \circ H^i(f)$. □

Remark 1.11. Let V be a chain complex. It is sometimes useful to collect all cohomologies $H^i(V)$ to define

$$H^*(V) = \bigoplus_i H^i(V).$$

This is a *graded module*, namely, a $\mathbf{k}$-module together with a direct sum decomposition over the integers. This can be also described just as the sequence of $\mathbf{k}$-modules $(H^i(V))_{i\in\mathbb{Z}}$ (compare with the case of complexes described in Remark 1.6).

If $f\colon V \to W$ is a chain map of complexes, the collection $(H^i(f))_{i\in\mathbb{Z}}$ defines a morphism of graded modules

$$H^*(f)\colon H^*(V) \to H^*(W).$$

Clearly, we have compatibility with identities and compositions:

$$\begin{aligned} H^*(1_V) &= 1_{H^*(V)}, \\ H^*(g \circ f) &= H^*(g) \circ H^*(f). \end{aligned}$$

1.2. **Hom and tensor.** If M and N are $\mathbf{k}$-modules, there is a $\mathbf{k}$-module $\mathrm{Hom}(M, N)$ of morphisms $M \to N$, with the obvious elementwise addition and action of $\mathbf{k}$. We may also define the *tensor product* $M \otimes N$. Hom and tensor are connected by the following natural isomorphism:

$$\mathrm{Hom}(V \otimes W, X) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, X)),$$

which is sometimes called the “hom-tensor adjunction”. We are going to discuss this in the framework of chain complexes.

Hom-complexes. If V and W are chain complexes, we denote by

$$\mathrm{Hom}(V, W)$$

the $\mathbf{k}$-module of chain maps $V \to W$, with the obvious elementwise addition and action of $\mathbf{k}$ (see Exercise 1.3). Such $\mathbf{k}$-module can itself be “enhanced” to a chain complex, as follows.

Definition 1.12. Let $V = (V^i, d_V^i)$ and $W = (W^i, d_W^i)$ be chain complexes. We define a chain complex

$$\underline{\mathrm{Hom}}(V, W) = (\underline{\mathrm{Hom}}^p(V, W), d_{\underline{\mathrm{Hom}}})_{p\in\mathbb{Z}}$$

as follows.

- The $\mathbf{k}$-module $\underline{\mathrm{Hom}}^p(V, W)$ is the $\mathbf{k}$-module of *degree p morphisms*, namely, of morphisms $f\colon V \to W$ such that $f(V^i) \subseteq W^{i+p}$ for all $i \in \mathbb{Z}$. Equivalently, they are sequences of morphisms $(f^i\colon V^i \to W^{i+p})_{i\in\mathbb{Z}}$. We don't require any compatibility with the differentials d_V and d_W.
- The differential
$$d^p_{\underline{\mathrm{Hom}}}\colon \underline{\mathrm{Hom}}^p(V, W) \to \underline{\mathrm{Hom}}^{p+1}(V, W)$$
is defined as follows:
$$d^p_{\underline{\mathrm{Hom}}}(f) = d_W \circ f - (-1)^p f \circ d_V.$$

Notationally, we will almost always write d instead of $d_{\underline{\mathrm{Hom}}}$ for the differential of $\underline{\mathrm{Hom}}(V, W)$. We can directly check (Exercise 1.4) that $d^{p+1} \circ d^p = 0$, hence $\underline{\mathrm{Hom}}(V, W)$ is indeed a complex.

Remark 1.13. Let V and W be chain complexes. What is the $\mathbf{k}$-module $Z^0(\underline{\mathrm{Hom}}(V, W))$? By definition, it contains precisely the morphisms $f\colon V \to W$ such that $f(V^i) \subseteq W^i$ for all $i \in \mathbb{Z}$, and $d_W \circ f - f \circ d_V = 0$. This means that
$$Z^0(\underline{\mathrm{Hom}}(V, W)) = \mathrm{Hom}(V, W)$$
is precisely the $\mathbf{k}$-module of chain maps $V \to W$.

What about the zeroth cohomology $H^0(\underline{\mathrm{Hom}}(V, W))$? Its elements are equivalence classes $[f]$, where f is a chain map $V \to W$. By definition, $[f] = [g]$ if and only if $f - g = dh$ for some $h \in \underline{\mathrm{Hom}}^{-1}(V, W)$. Explicitly, this means:
$$f - g = d_W \circ h + h \circ d_V,$$
namely, that f and g are *chain homotopic*. The degree -1 morphism $h\colon V \to W$ is a chain homotopy between f and g. The effort we made to define the hom-complex $\underline{\mathrm{Hom}}(V, W)$ pays off giving us a better framework to treat such chain homotopies.

Tensor products. We are able to generalize the definition of tensor product to chain complexes.

Definition 1.14. Let V and W be chain complexes. We define the *tensor product* $V \otimes W = ((V \otimes W)^p, d^p_{V\otimes W})$ as follows. First:
$$(V \otimes W)^p = \bigoplus_{i+j=p} V^i \otimes W^j,$$
where $V^i \otimes W^j$ is the usual tensor product of $\mathbf{k}$-modules. Moreover, the differential is defined by:
$$d^p_{V\otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^i v \otimes d_W(w),$$
if $v \in V^i$ and $w \in W^j$ with $i+j = p$ (and then "extending by linearity"). With a direct computation (see Exercise 1.5) we can show that $d^{p+1}_{V\otimes W} \circ d^p_{V\otimes W} = 0$ for all $p \in \mathbb{Z}$, hence $V \otimes W$ is indeed a chain complex. We shall often simplify notation and write d instead of $d_{V\otimes W}$ when the context is clear.

The tensor product of complexes behaves nicely. Namely, it is associative, commutative and unital (with unit being $\mathbf{k}$ viewed as a complex concentrated in degree 0, see Remark 1.7). We list all these properties in the following proposition; the proofs are left as an exercise.

Proposition 1.15. *Let V, W, X be chain complexes. There are natural isomorphisms:*

$$(V \otimes W) \otimes X \xrightarrow{\sim} V \otimes (W \otimes X), \quad (v \otimes w) \otimes x \mapsto v \otimes (w \otimes x),$$
$$\mathbf{k} \otimes V \xrightarrow{\sim} V, \quad \lambda \otimes v \mapsto \lambda v,$$
$$V \otimes W \xrightarrow{\sim} W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

Proof. Exercise 1.6. □

Remark 1.16. It is convenient to comment the "commutativity" isomorphism

$$V \otimes W \xrightarrow{\sim} W \otimes V,$$
$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

Assuming that $v \in V^i$ and $w \in W^j$, the tensor $v \otimes w$ is mapped to $(-1)^{ij} w \otimes v$ (and the isomorphism is obtained by extending this by linearity). The occurrence of the sign $(-1)^{ij}$ is an instance of the *Koszul sign rule*.

Warning! What follows are just a few heuristic and informal ideas. To my best understanding, the Koszul sign rule can be informally summarized as follows: *every time we swap two graded symbols a and b, we make the sign* $(-1)^{|a||b|}$ *appear.* This occurs for example in the definition of the differential $d_{V \otimes W}$ of the tensor product $V \otimes W$ (cf. Definition 1.14):

$$d_{V \otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w).$$

The second summand is obtained by "switching" the triple of symbols (d_V, v, w) with the triple (v, d_W, w). The symbols d_V and d_W have degree 1, and $(-1)^{|v|} = (-1)^{1|v|}$ is the sign which correctly appears according to the rule.

We may finally state the "tensor-hom adjunction" for chain complexes.

Proposition 1.17. *Let V, W, X be chain complexes. There are natural isomorphisms of complexes, one inverse to the other:*

$$\Psi \colon \underline{\mathrm{Hom}}(V \otimes W, X) \xrightarrow{\sim} \underline{\mathrm{Hom}}(V, \underline{\mathrm{Hom}}(W, X)), \quad f \mapsto (v \mapsto f_v, f_v(w) = f(v \otimes w)),$$
$$\Phi \colon \underline{\mathrm{Hom}}(V, \underline{\mathrm{Hom}}(W, X)) \xrightarrow{\sim} \underline{\mathrm{Hom}}(V \otimes W, X), \quad g \mapsto (v \otimes w \mapsto g(v)(w)).$$

Proof. The fact that the above morphisms are mutual inverses is clear from the definition. To conclude, we just have to check that they are chain maps. Thanks to Lemma 1.5, we just need to check that the above morphisms preserve the gradings (i.e. they map degree p morphisms to degree p morphisms) and that *one of those* is a chain map. This is straightforward but tedious and not particularly instructive, so we leave it behind. □

1.3. **Quasi-isomorphisms.** Why are complexes interesting and important? A possible answer is that *they essentially bring (linear) algebra to higher dimensions*. They give a unified framework to treat resolutions of modules, derived functors and so on. Cohomology is often the piece of information of a given complex that we really want to retain in many contexts, and this motivates the following definition:

Definition 1.18. Let $V = (V^i, d^i_V)$ and $W = (W^i, d^i_W)$ be chain complexes, and let $f \colon V \to W$ be a chain map. We say that f is a *quasi-isomorphism* if $H^i(f) \colon H^i(V) \to H^i(W)$ is an isomorphism for all $i \in \mathbb{Z}$ (or, equivalently, that the graded morphism $H^*(f)$ is an isomorphism). See Exercise 1.7 for another equivalent definition.

Quasi-isomorphisms are abundant. A typical family of examples is given by projective or injective resolutions, one of which we see in the following example.

Example 1.19. Assume that $\mathbf{k} = \mathbb{Z}$, the integers. We describe a simple free resolution of the abelian group $\mathbb{Z}/2\mathbb{Z}$. This is understood as the following chain map:

$$\begin{tikzcd} V \arrow[d, "f"'] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

The abelian group $\mathbb{Z}/2\mathbb{Z}$ is viewed as a complex concentrated in degree 0, and the morphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is multiplication by 2. The complex V in the upper row is concentrated in degrees -1 and 0. A direct computation shows that $H^{-1}(V) = 0$ and $H^{-1}(\mathbb{Z}/2\mathbb{Z}) = 0$, and that

$$H^0(f) = 1_{\mathbb{Z}/2\mathbb{Z}} \colon H^0(V) \to H^0(\mathbb{Z}/2\mathbb{Z}).$$

We conclude that f is a quasi-isomorphism. This somehow captures the idea that we could replace the module $\mathbb{Z}/2\mathbb{Z}$ (which is torsion) with the complex V, which is made of free abelian groups but has a nontrivial component in degree -1.

There is an important caveat. While we would like to view quasi-isomorphisms as some kind of isomorphism, unfortunately *not all quasi-isomorphisms have inverses*. This can be seen even from the above example:

Remark 1.20. In the setup of the above Example 1.19 we can't find any chain map $g \colon \mathbb{Z}/2\mathbb{Z} \to V$ such that $H^*(g)$ is inverse to $H^*(f)$. The point is that we can't find a nonzero group homomorphism

$$\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z},$$

so there are actually no nonzero chain maps $\mathbb{Z}/2\mathbb{Z} \to V$.

The problem of "inverting quasi-isomorphisms" is highly nontrivial. In the following lectures, we will see – among other things – how we are able to overcome it.

Exercises to Lecture 1.

Exercise 1.1. Prove Lemma 1.3.

Exercise 1.2. Check that the "alternative definition" of chain complexes and chain maps described in Remark 1.6 is equivalent to the one given in Definition 1.2. More explicitly, define bijections

$$\{\text{chain complexes "version 1" } (V^i, d^i)\} \leftrightarrow \{\text{chain complexes "version 2" } (V = \oplus_i V^i, d)\},$$
$$\{\text{chain maps "version 1" } (V^i, d^i) \to (W, d^i_W)\} \leftrightarrow \{\text{chain maps "version 2" } \oplus_i V^i \to \oplus_i W^i\}.$$

Exercise 1.3. Let $f, g \colon V \to W$ be chain maps of complexes. Then, the sum $f + g \colon V \to W$ defined elementwise by

$$(f + g)(v) = f(v) + g(v)$$

is again a chain map. The opposite $-f \colon V \to W$ defined elementwise by

$$(-f)(v) = -(f(v))$$

is also a chain map. Moreover, the zero morphism $0\colon V \to W$ is a chain map. Finally, let $\lambda \in \mathbf{k}$. Then, the morphism $\lambda f\colon V \to W$ defined by

$$(\lambda f)(v) = \lambda f(v)$$

is a chain map.

Exercise 1.4. Check that the differential $d = d_{\underline{\mathrm{Hom}}}$ of the complex $\underline{\mathrm{Hom}}(V, W)$ described in Definition 1.12 actually satisfies $d^{p+1} \circ d^p = 0$ for all $p \in \mathbb{Z}$.

Exercise 1.5. Check that the differential $d_{V\otimes W}$ of the complex $V \otimes W$ described in Definition 1.14 actually satisfies $d^{p+1}_{V\otimes W} \circ d^p_{V\otimes W} = 0$ for all $p \in \mathbb{Z}$.

Exercise 1.6. Prove the claims of Proposition 1.15.

Exercise 1.7. Let $f\colon V \to W$ be a chain map of complexes. Prove that f is a quasi-isomorphism if and only if the following hold:

- Let $y \in W^p$ and $x' \in V^{p+1}$ such that $dy = f(x')$. Then, there is $z \in W^{p-1}$ and $x \in V^p$ such that:

$$\begin{aligned} dx &= x', \\ y - dz &= f(x). \end{aligned}$$

2. Lecture 2: basics on dg-categories

In this lecture we give the definition of the main object of this course, and develop some basics of the theory.

2.1. **Towards dg-categories.** We start by having an even closer look to the *hom-complexes* $\underline{\mathrm{Hom}}(V, W)$ (cf. Definition 1.12).

Example 2.1. Let $f \in \underline{\mathrm{Hom}}^p(V, W)$ and $g \in \underline{\mathrm{Hom}}^q(W, X)$ be respectively a degree p and a degree q morphism between chain complexes. Namely, $f(V^i) \subseteq W^{i+p}$ and $g(W^i) \subseteq W^{i+q}$ for all $i \in \mathbb{Z}$. Can we compose f and g? The answer is yes, of course if we are careful with the components. If $v \in V^i$ is an element of degree i, then $f(v) = f^i(v) \in W^{i+p}$. Moreover, $g(f(v)) = g^{i+p}(f^i(v)) \in X^{i+p+q}$. We conclude that we have indeed a composition $g \circ f$, and this satisfies $(g \circ f)(V^i) \subseteq X^{i+p+q}$ for all $i \in \mathbb{Z}$. It is a degree $p + q$ morphism. We end up with a composition function

$$\begin{aligned} \underline{\mathrm{Hom}}^q(W, X) \times \underline{\mathrm{Hom}}^p(V, W) &\to \underline{\mathrm{Hom}}^{p+q}(V, X), \\ (g, f) &\mapsto g \circ f, \end{aligned}$$

defined for all $p, q \in \mathbb{Z}$. Let us explore its properties.

- The composition is $\mathbf{k}$-bilinear. This is very easy to see, since addition and action of $\mathbf{k}$ on hom complexes is defined elementwise, and we are always dealing with morphisms of $\mathbf{k}$-modules.
- The composition is associative, namely

$$(h \circ g) \circ f = h \circ (g \circ f),$$

if $f \in \underline{\mathrm{Hom}}^p(V, W)$, $g \in \underline{\mathrm{Hom}}^q(W, X)$ and $h \in \underline{\mathrm{Hom}}^r(X, Y)$. This follows from the usual associativity of compositions of morphisms of $\mathbf{k}$-modules.

- The composition is unital. Indeed, for any chain complex V, we know that we have an identity morphism $1_V \colon V \to V$. By the way, $1_V \in \underline{\mathrm{Hom}}^0(V,V)$ and $d(1_V) = 0$. This is also immediate to check.
- What about differentials? Let us compute, for given $f \in \underline{\mathrm{Hom}}^p(V,W)$ and $g \in \underline{\mathrm{Hom}}^q(W,X)$:

$$d(g \circ f) = d_X \circ g \circ f - (-1)^{p+q} g \circ f \circ d_V.$$

Can we perhaps relate this to $d(g)$ and $d(f)$? Let's add and subtract a suitable element, namely $(-1)^q g \circ d_W \circ f$ (beware the sign choice):

$$\begin{aligned} d(g \circ f) &= d_X \circ g \circ f - (-1)^{p+q} g \circ f \circ d_V \\ &= d_X \circ g \circ f - (-1)^q g \circ d_W \circ f + (-1)^q g \circ d_W \circ f - (-1)^{p+q} g \circ f \circ d_V \\ &= (d_X \circ g - (-1)^q f \circ d_W) \circ f + (-1)^q g \circ (d_W \circ f - (-1)^p f \circ d_V) \\ &= d(g) \circ f + (-1)^q g \circ d(f). \end{aligned}$$

Hence, differentials and compositions are indeed compatible. The equation

$$d(g \circ f) = d(g) \circ f + (-1)^q g \circ d(f)$$

is called *graded Leibniz rule*.

The above example motivates the following key definition:

Definition 2.2. A *differential graded category* (in short, *dg-category*) $\mathcal{A}$ is the datum of:

- A family of *objects* $\mathrm{Ob}(\mathcal{A})$. We will denote them by $A, B, C, \ldots$. Strictly speaking, $\mathrm{Ob}(\mathcal{A})$ need not be a set but even something "larger". There are possibly serious set-theoretical issues going on in that case but for now and unless otherwise specified we will just adopt a naive point of view on the problem and just forget about them.
- For any pair of objects A and B, a chain complex $\mathcal{A}(A,B)$ (sometimes also stylized as $\underline{\mathrm{Hom}}_{\mathcal{A}}(A,B)$ or even $\underline{\mathrm{Hom}}(A,B)$ if the context does not allow confusion. Elements in $\mathcal{A}(A,B)^p$ are called *degree p morphisms (from A to B)* If $f \in \mathcal{A}(A,B)^p$, we also write $|f| = \deg(f) = p$. Recall from Remark 1.6 that we may view

$$\mathcal{A}(A,B) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}(A,B)^i,$$

and $\mathcal{A}(A,B)^i$ itself as a submodule of $\mathcal{A}(A,B)$.
- Composition morphisms:

$$\begin{aligned} \mathcal{A}(B,C)^q \times \mathcal{A}(A,B)^p &\to \mathcal{A}(A,C)^{p+q}, \\ (g,f) &\mapsto g \circ f, \end{aligned}$$

for $A, B, C \in \mathrm{Ob}(\mathcal{A})$ and $p, q \in \mathbb{Z}$.

These data have the following properties and features:

- The composition morphisms are **k**-bilinear.
- Composition is associative:

$$(h \circ g) \circ f = h \circ (g \circ f),$$

for $f \in \mathcal{A}(A,B)$, $g \in \mathcal{A}(B,C)$ and $h \in \mathcal{A}(C,D)$.

- Composition is unital. Namely, for all $A \in \mathrm{Ob}(\mathcal{A})$, there is a morphism $1_A \in \mathcal{A}(A,A)^0$ such that $d(1_A) = 0$ and for any $f \in \mathcal{A}(A,B)$ we have:
$$f \circ 1_A = f,$$
$$1_B \circ f = f.$$
- The graded Leibniz rule holds. Namely, if $f \in \mathcal{A}(A,B)^p$ and $g \in \mathcal{A}(B,C)^q$, we have:
$$d(g \circ f) = d(g) \circ f + (-1)^q g \circ d(f).$$

Notation 2.3. Elements of the *hom complexes* $\mathcal{A}(A,B)$ will be often described as arrows. Namely, we shall often write $f \colon A \to B$ instead of $f \in \mathcal{A}(A,B)$.

Moreover, we will also ease notation and often write gf instead of $g \circ f$, for composable morphisms $f \colon A \to B$ and $g \colon B \to C$.

Remark 2.4. In the above Definition 2.2, it is not actually necessary to require that identity morphisms 1_A are closed and of degree 0, for it follows from the other properties. Let's try and prove it. Maybe it's a good idea to start by writing
$$1_A = \sum_{n \in \mathbb{Z}} (1_A)_n,$$
where $(1_A)_n \in \mathcal{A}(A,A)^n$ and $(1_A)_n = 0$ for all but a finite number of indices n. From the relation $1_A \circ 1_A = 1_A$ and using bilinearity, we find out:
$$1_A = \sum_{n \in \mathbb{Z}} \Big(\sum_{i+j=n} (1_A)_i \circ (1_A)_j \Big),$$
from which we conclude that for all $n \in \mathbb{Z}$:
$$(1_A)_n = (1_A)_i \circ (1_A)_j.$$
Do we really go anywhere from this? Well, maybe not!

Let's try something different. First, we may observe that *identity morphisms are unique*. Namely: if for any object $A \in \mathrm{Ob}(\mathcal{A})$ we have morphisms $1_A, 1'_A$ such that they behave as identities (according to the above definition), then necessarily $1_A = 1'_A$. Indeed, we have:
$$1_A \circ 1'_A = 1_A = 1'_A.$$
Next, we look again at the decomposition
$$1_A = \sum_{n \in \mathbb{Z}} (1_A)_n.$$
If the degree 0 component $(1_A)_0$ behaves as an identity, then it has to coincide with 1_A by what we said just before. Then, does it really? Let us take any other morphisms $g \colon A \to B$ and $g' \colon B \to A$, of some fixed arbitrary degree p. Let's have a look at the degree p component of the composition $(g \circ 1_A)_p$:
$$(g \circ 1_A)_p = g_p = g,$$
for g is concentrated in degree p and 1_A is the identity. On the other hand:
$$(g \circ 1_A)_p = g_p \circ (1_A)_0 = g \circ (1_A)_0,$$
being careful with degrees. We conclude that indeed
$$g \circ (1_A)_0 = g,$$

and similarly we can prove that

$$(1_A)_0 \circ g' = g'.$$

From this, we indeed conclude that 1_A is a degree 0 morphism.

Finally, what about checking that $d(1_A) = 0$. Now, we can maybe really use the relation

$$1_A \circ 1_A = 1_A,$$

and the graded Leibniz rule:

$$\begin{aligned} d(1_A) &= d(1_A \circ 1_A) = d(1_A) \circ 1_A + 1_A \circ d(1_A), \\ &= d(1_A) + d(1_A), \end{aligned}$$

using that $\deg(1_A) = 0$ as we prove above. We substract $d(1_A)$ in the above relation and finally conclude that $d(1_A) = 0$.

Example 2.5. Example 2.1 on hom complexes tells us precisely that the family of all chain complexes $V, W, \dots$ together with the hom-complexes $\underline{\mathrm{Hom}}(V, W)$ and the compositions as described there is a dg-category. We will denote it as $\mathrm{dgm}(\mathbf{k})$. We will still continue using the notation $\underline{\mathrm{Hom}}(V, W)$ to refer to the hom-complexes in $\mathrm{dgm}(\mathbf{k})$:

$$\mathrm{dgm}(\mathbf{k})(V, W) = \underline{\mathrm{Hom}}(V, W).$$

Remark 2.6. We may use the tensor product of complexes to give a more compact yet equivalent definition of the compositions in a dg-category $\mathcal{A}$. Indeed, we can describe them as chain maps of complexes:

$$\begin{aligned} \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) &\to \mathcal{A}(A, C), \\ g \otimes f &\mapsto g \circ f, \end{aligned}$$

which we require to be associative and unital. $\mathbf{k}$-bilinearity, preservation of degrees and the graded Leibniz rule are encoded in the requirement that it is a chain map defined on the tensor product. See also Exercise 2.1.

You may not (yet) be very familiar with dg-categories, but you may be familiar with *dg-algebras*. We can define them very easily now:

Definition 2.7. A *differential graded algebra* (or *dg-algebra*) A is a dg-category whose family of objects has exactly one element $\star$. We normally identify A with the complex of endomorphisms of the single object $\star$:

$$A = \underline{\mathrm{Hom}}(\star, \star).$$

Hence, a dg-algebra A can be defined just as a chain complex endowed with a composition

$$\begin{aligned} A^q \times A^p &\to A^{p+q}, \\ (b, a) &\mapsto ba, \end{aligned}$$

which is $\mathbf{k}$-bilinear, unital, associative and satisfies the graded Leibniz rule. Recalling Exercise 2.1, we can also define the composition as a chain map

$$A \otimes A \to A.$$

Example 2.8. Let R be a $\mathbf{k}$-algebra. Then, it is a dg-algebra when we view it as a complex concentrated in degree 0, together with its multiplication. In particular, R can be also identified with a dg-category with a single object $\star$ and such that $R = \underline{\mathrm{Hom}}(\star, \star)$.

In particular, we may view $\mathbf{k}$ itself as a dg-algebra, and also a dg-category with a single object.

The above Example 2.8 can be extended to the “several objects” setup. We give the following definition:

Definition 2.9. A **k**-*linear* (or more simply *linear*) *category* $\mathcal{A}$ is the datum of:

- A family of *objects* $\mathrm{Ob}(\mathcal{A})$.
- For any pair of objects A and B, a **k**-module $\mathcal{A}(A, B)$ (sometimes also stylized as $\mathrm{Hom}_{\mathcal{A}}(A, B)$ or even $\mathrm{Hom}(A, B)$ if the context does not allow confusion.
- Composition morphisms:
$$\begin{aligned} \mathcal{A}(B, C) \times \mathcal{A}(A, B) &\to \mathcal{A}(A, C), \\ (g, f) &\mapsto g \circ f, \end{aligned}$$
for $A, B, C \in \mathrm{Ob}(\mathcal{A})$.

These data have the following properties and features:

- The composition morphisms are **k**-bilinear. We may also identify them with **k**-linear morphisms:
$$\begin{aligned} \mathcal{A}(B, C) \times \mathcal{A}(A, B) &\to \mathcal{A}(A, C), \\ g \otimes f &\mapsto g \circ f, \end{aligned}$$
- Composition is associative:
$$(h \circ g) \circ f = h \circ (g \circ f),$$
for $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$ and $h \in \mathcal{A}(C, D)$.
- Composition is unital. Namely, for all $A \in \mathrm{Ob}(\mathcal{A})$, there is a morphism $1_A \in \mathcal{A}(A, A)$ such that for any $f \in \mathcal{A}(A, B)$ we have:
$$\begin{aligned} f \circ 1_A &= f, \\ 1_B \circ f &= f. \end{aligned}$$

Example 2.10. A typical example of a **k**-linear category is given by $\mathrm{Mod}(\mathbf{k})$, the linear category of **k**-modules, defined as follows:

- The family of objects $\mathrm{Ob}(\mathrm{Mod}(\mathbf{k}))$ is given by the **k**-modules $M, N, \dots$
- If M and N are **k**-modules, we have a **k**-module of morphisms (sometimes called **k**-linear morphisms):
$$\mathrm{Mod}(\mathbf{k})(M, N) = \mathrm{Hom}(M, N).$$
- Compositions are given by the usual composition of **k**-linear morphisms.

Example 2.11. Let $\mathcal{A}$ be a **k**-linear category. We may view it as a dg-category, viewing every **k**-module $\mathcal{A}(A, B)$ as a complex concentrated in degree 0. Compositions and units are the obvious ones.

As usual in mathematics, once we have defined a structure we also want to define transformations preserving that structure. This leads to the following definition:

Definition 2.12. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. A *dg-functor* $F \colon \mathcal{A} \to \mathcal{B}$ is the datum of:

- A function $F \colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
- For any pair of objects $A, B \in \mathrm{Ob}(\mathcal{A})$, a chain map of complexes
$$F = F_{A,B} \colon \mathcal{A}(A, B) \to \mathcal{B}(F(A), F(B)).$$

These data are compatible with identities and compositions, as follows:

$$F(1_A) = 1_{F(A)}, \qquad A \in \mathrm{Ob}(\mathcal{A}),$$
$$F(g \circ f) = F(g) \circ F(f), \qquad f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C).$$

Clearly, we can also define the notion of *(**k**-)linear functor* between linear categories, cf. Exercise 2.2.

2.2. **Some examples.** Let us become more familiar with dg-categories and dg-functors.

Example 2.13. Take any dg-category $\mathcal{A}$. What about dg-functors $\mathbf{k} \to \mathcal{A}$, where $\mathbf{k}$ is our base commutative ring viewed as a dg-category (cf. Example 2.8)? Let $F \colon \mathbf{k} \to \mathcal{A}$ be such a dg-functor. Then, F is determined by the following data:

- An object $F(\star) = A \in \mathrm{Ob}(\mathcal{A})$.
- A chain map
$$F \colon \mathbf{k} = \underline{\mathrm{Hom}}(\star, \star) \to \mathcal{A}(A, A),$$
but by $\mathbf{k}$-linearity F is completely determined by $F(1_{\mathbf{k}})$, which in turn is forced to be $1_A \colon A \to A$.

In the end, we see that F is completely determined by the object $F(\star)$. We end up with a bijection:

$$\{\text{dg-functors } \mathbf{k} \to \mathcal{A}\} \to \mathrm{Ob}(\mathcal{A}),$$
$$F \mapsto F(\star).$$

Example 2.14. Let A be a dg-algebra, which we view as a dg-category with a single object $\star$, and $A = \underline{\mathrm{Hom}}(\star, \star)$. Let $F \colon A \to \mathrm{dgm}(\mathbf{k})$ be a dg-functor. F is determined by:

- An object $F(\star) = V \in \mathrm{Ob}(\mathrm{dgm}(\mathbf{k}))$, namely, a chain complex.
- A chain map
$$F \colon A \to \underline{\mathrm{Hom}}(V, V),$$
$$a \mapsto F(a),$$
such that $F(1_A) = 1_V$ and $F(ba) = F(b)F(a)$ for all $a, b \in A$.

Let us adopt the following notation:

$$F(a)(v) = av.$$

We can invoke Proposition 1.17 and get from F an induced chain map of complexes

$$A \otimes V \to V,$$
$$a \otimes v \mapsto F(a)(v) = av$$

with the following additional properties:

$$1_a v = v,$$
$$(ba)v = b(av),$$

for all $v \in V$ and $a, b \in A$. These equalities come from the fact that F is a dg-functor.

Let us unravel this even more. We know (see Exercise 2.1) that we can identify the chain map

$$a \otimes v \mapsto av = F(a)(v)$$

with a family of **k**-bilinear morphisms

$$A^i \times V^j \to V^{i+j},$$

for $i, j \in \mathbb{Z}$. The relation $F(da) = dF(a) \in \underline{\mathrm{Hom}}(V, V)$ translates to the following:

$$(da)v = d(av) - (-1)^{|a|} a(dv),$$

which can be rearranged in the following variant of the graded Leibniz rule:

$$d(av) = (da)v + (-1)^{|a|} a(dv).$$

We could also obtain this by writing explicitly what it means for $A \otimes V \to V$ to be a chain map.

What we have discussed so far is nothing more than the notion of *left A-dg-module*. Hence, we get a bijection:

$$\{\text{dg-functors } A \to \mathrm{dgm}(\mathbf{k})\} \leftrightarrows \{\text{left } A\text{-dg-modules } V, \text{ defined by an action } A \otimes V \to V\}.$$

In the framework of ordinary **k**-algebras and modules, we can obtain an analogous result, cf. Exercise 2.3.

Example 2.15. We give an example of a "small" dg-category $\mathbf{Q}$, which is actually a **k**-linear category which we view as a dg-category (cf. Example 2.11):

- It has three objects $0, 1, 2$.
- The hom **k**-modules are described as follows:

$$\begin{aligned} \mathbf{Q}(i,i) &= \mathbf{k}\langle 1_i \rangle, \qquad i = 0,1,2, \\ \mathbf{Q}(0,1) &= \mathbf{k}\langle e_{01} \rangle, \\ \mathbf{Q}(1,2) &= \mathbf{k}\langle e_{12} \rangle, \\ \mathbf{Q}(0,2) &= 0, \\ \mathbf{Q}(i,j) &= 0, \qquad i > j. \end{aligned}$$

  The notation $\mathbf{k}\langle S \rangle$ simply means *free* **k***-module having S as a basis*. So far, the morphisms 1_i are to be thought as "formal identities". They will become proper identities once we define the compositions suitably.
- Compositions are the "obvious" ones. For example, we may define

$$\mathbf{Q}(1,1) \times \mathbf{Q}(0,1) \to \mathbf{Q}(0,1)$$

  by sending $(1_1, e_{01})$ to e_{01} and then extending by bilinearity. Observe that in particular we have

$$e_{12} \circ e_{01} = 0.$$

The linear category $\mathbf{Q}$ can be pictured as the following diagram:

$$\begin{tikzcd} 0 \arrow[r, "e_{01}"] \arrow[rr, bend left, "0"] & 1 \arrow[r, "e_{12}"] & 2. \end{tikzcd}$$

A key feature of $\mathbf{Q}$ is that it is *not free*, indeed, we have the nontrivial relation $e_{12} \circ e_{01} = 0$. Dg-categories allow us to replace $\mathbf{Q}$ with something which will look like some kind of "free resolution" of $\mathbf{Q}$. To that purpose, we define a dg-category $\mathbf{Q}'$ as follows:

- It has three objects $0, 1, 2$, as $\mathbf{Q}$.

- The hom-complexes are defined as follows:

$$
\begin{aligned}
\mathbf{Q}'(i,i) &= \mathbf{k}\langle 1_i\rangle, \quad |1_i| = 0,\ d(1_i) = 0, \qquad i = 0,1,2,\\
\mathbf{Q}'(0,1) &= \mathbf{k}\langle e_{01}\rangle, \quad |e_{01}| = 0,\ d(e_{01}) = 0,\\
\mathbf{Q}'(1,2) &= \mathbf{k}\langle e_{12}\rangle, \quad |e_{12}| = 0,\ d(e_{12}) = 0,\\
\mathbf{Q}'(0,2) &= \mathbf{k}\langle e_{12}\circ e_{01}, e_{012}\rangle, \quad |e_{12}\circ e_{01}| = 0, |e_{012}| = -1,\ d(e_{012}) = e_{12}\circ e_{01},\\
\mathbf{Q}'(i,j) &= 0, \qquad i > j.
\end{aligned}
$$

  All hom-complexes are made of free $\mathbf{k}$-modules. The only actual differences we have with respect to $\mathbf{Q}$ is that $\mathbf{Q}'(0,2)$ is no longer 0, but it is the following complex:

$$\cdots \to 0 \to \mathbf{k}\langle e_{012}\rangle \xrightarrow{d} \mathbf{k}\langle e_{12}\circ e_{01}\rangle \to 0 \to \cdots,$$

  where e_{012} is of degree -1 and the "formal composition" $e_{12}\circ e_{01}$ is of degree 0. The differential d is actually just the identity map which maps the basis element e_{012} to $e_{12}\circ e_{01}$.
- Compositions are defined in the "obvious" way, again. For instance, the morphism

$$\mathbf{Q}'(1,2)\otimes \mathbf{Q}'(0,1) \to \mathbf{Q}(0,2)$$

  maps $e_{12}\otimes e_{01}$ to the "formal composition" $e_{12}\circ e_{01}$, which then becomes the actual composition in $\mathbf{Q}'$.

We can check that our definition of $\mathbf{Q}'$ actually yields a dg-category. $\mathbf{Q}'$ is essentially built in such a way that the nontrivial relation $e_{12}\circ e_{01} = 0$ is replaced by instead saying that $e_{12}\circ e_{01}$ is a coboundary. We can picture $\mathbf{Q}'$ as the following diagram:

$$\begin{tikzcd} 0 \arrow[r, "e_{01}"] \arrow[rr, bend left, "e_{012}"] & 1 \arrow[r, "e_{12}"] & 2, \end{tikzcd}$$

keeping track of the relation $d(e_{012}) = e_{12}\circ e_{01}$.

We may define a dg-functor $F\colon \mathbf{Q}' \to \mathbf{Q}$ as follows:

- F is the identity on objects: $F(i) = i$ for $i = 0,1,2$.
- On the hom-complexes, F is defined as follows:

$$
\begin{aligned}
F(e_{01}) &= e_{01},\\
F(e_{12}) &= e_{12},\\
F(e_{12}\circ e_{01}) &= 0,\\
F(e_{012}) &= 0,
\end{aligned}
$$

  and also by preserving the identities 1_i and extending by $\mathbf{k}$-linearity. We can check that this yields a well-defined dg-functor. Actually, we did not do give the complete details on the definitions of $\mathbf{Q}, \mathbf{Q}'$ and F. This is the content of Exercise 2.4.

  We leave you with the following question: the chain maps

$$F\colon \mathbf{Q}'(i,j) \to \mathbf{Q}(i,j)$$

  induce, as we know, morphisms in cohomology

$$H^*(F)\colon H^*(\mathbf{Q}'(i,j)) \to H^*(\mathbf{Q}(i,j)).$$

  What can we say about them?

Exercises to Lecture 2.

Exercise 2.1. Let V, W, X be chain complexes. Prove that giving $\mathbf{k}$-bilinear morphisms

$$f_{q,p} \colon V^q \times W^p \to X^{p+q}$$

for $p, q \in \mathbb{Z}$, satisfying

$$d(f_{q,p}(v, w)) = f_{q+1,p}(dv, w) + (-1)^q f_{q,p+1}(v, dw),$$

is the same as giving a chain map of complexes

$$\begin{aligned} &V \otimes W \to X, \\ &v \otimes w \mapsto f_{q,p}(v, w) \quad (v \in V^q \text{ and } w \in W^p). \end{aligned}$$

In particular, prove the claims of Remark 2.6.

Exercise 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbf{k}$-linear categories. Define the notion of a $\mathbf{k}$-linear functor $F \colon \mathcal{A} \to \mathcal{B}$.

Exercise 2.3. Let R be a $\mathbf{k}$-algebra. Classically, we define a *left R-module* as a $\mathbf{k}$-module M together with a $\mathbf{k}$-bilinear action

$$R \times M \to M,$$

or equivalently a $\mathbf{k}$-linear morphism

$$R \otimes M \to M,$$

satisfying the usual compatibilities. View R as a $\mathbf{k}$-linear category with a single object $\star$, analogously to the case of dg-algebras (cf. Definition 2.7). Check that we can establish a bijection:

$$\{\mathbf{k}\text{-linear functors } R \to \mathrm{Mod}(\mathbf{k})\} \leftrightarrows \{\text{left } R\text{-modules } M, \text{ defined by an action } R \otimes M \to M\}.$$

Exercise 2.4. Fill in the details of the constructions of the dg-categories $\mathbf{Q}, \mathbf{Q}'$ and the dg-functor $F \colon \mathbf{Q}' \to \mathbf{Q}$ in Example 2.15.

3. Lecture 3: dg-modules and the Yoneda lemma

We have introduced dg-categories and $\mathbf{k}$-linear categories as "many-object versions" of respectively dg-algebra and $\mathbf{k}$-algebras. Quite informally, we have inclusions:

$$\begin{tikzcd}
\{\mathbf{k}\text{-algebras}\} \arrow[r, hook, "\text{complexes concentrated in degree 0}"] \arrow[d, hook, "\text{single object linear cat.}"'] & \{\text{dg-algebras}\} \arrow[d, hook, "\text{single object dg-cat.}"] \\
\{\mathbf{k}\text{-linear categories}\} \arrow[r, hook, "\text{complexes concentrated in degree 0}"'] & \{\text{dg-categories}\}.
\end{tikzcd}$$

In this lecture, we are going to deal with *modules* over these structures. As a reminder, to enhance familiarity: if $\mathbf{k} = \mathbb{Z}$, the notion of $\mathbf{k}$-algebra is the same as the usual notion of (unital, associative) ring.

3.1. **Modules, dg-modules and morphisms.** We recall again the classical definition of a module over a **k**-algebra. We will start using tensor products.

Definition 3.1. Let R be a **k**-algebra. A *left R-module M* is a **k**-module together with a **k**-linear morphism:

$$\begin{aligned} R \otimes M &\to M, \\ r \otimes m &\mapsto rm, \end{aligned}$$

subject to the relations:

$$\begin{aligned} 1_R m &= m, && m \in M, \\ (rs)m &= r(sm) && r, s \in R,\ m \in M. \end{aligned}$$

A *right R-module N* is a **k**-module together with a **k**-linear morphism:

$$\begin{aligned} M \otimes R &\to M, \\ m \otimes r &\mapsto mr, \end{aligned}$$

subject to the relations:

$$\begin{aligned} m1_R &= m, && m \in M, \\ m(rs) &= (mr)s && r, s \in R,\ m \in M. \end{aligned}$$

Definition 3.2. Let R be a **k**-algebra, and let M and M' be left R-modules. A *morphism $\varphi \colon M \to M'$* is a **k**-linear morphism which preserves the left action of R, namely:

$$\varphi(rm) = r\varphi(m), \qquad r \in R,\ m \in M.$$

Let N and N' be right R-modules. A *morphism $\psi \colon N \to N'$* is a **k**-linear morphism which preserves the right action of R, namely:

$$\psi(mr) = \psi(m)r, \qquad r \in R,\ m \in M.$$

Morphisms of either left or right R-modules form a **k**-module, using elementwise addition and action of **k**. Morphisms of right R-modules $N \to N'$ will be denoted by

$$\operatorname{Hom}_R(N, N') \qquad \text{or} \qquad \operatorname{Hom}_{\mathbf{k}-R}(N, N'),$$

whereas morphisms of left R-modules $M \to M'$ will be denoted by

$$\operatorname{Hom}_{R^{\mathrm{op}}}(M, M') \qquad \text{or} \qquad \operatorname{Hom}_{R-\mathbf{k}}(M, M').$$

So far, view the above as notational tricks, in particular R^{op}. They will be justified later on.

Remark 3.3. Recall thay giving (for instance) a **k**-linear morphism $R \otimes M \to M$ is just the same as giving a **k**-bilinear morphism $R \times M \to M$. This is the actual purpose and key property of the tensor product.

The tensor product allows us to directly generalize the above definitions to the differential graded setting. We start with the "single-object framework":

Definition 3.4. Let R be a dg-algebra. A *left R-dg-module M* is a chain complex of **k**-modules together with a chain map:

$$\begin{aligned} R \otimes M &\to M, \\ r \otimes m &\mapsto rm, \end{aligned}$$

subject to the relations:

$$1_R m = m, \qquad m \in M,$$
$$(rs)m = r(sm) \qquad r, s \in R,\ m \in M.$$

A *right R-dg-module N* is a chain complex of $\mathbf{k}$-modules together with a chain map:

$$M \otimes R \to M,$$
$$m \otimes r \mapsto mr,$$

subject to the relations:

$$m 1_R = m, \qquad m \in M,$$
$$m(rs) = (mr)s \qquad r, s \in R,\ m \in M.$$

Remark 3.5. Giving a chain map $R \otimes M \to M$, as in the case of a left R-dg-module, is the same as giving $\mathbf{k}$-bilinear morphisms

$$R^q \times M^p \to M^{p+q},$$

satisfying a graded Leibniz rule:

$$d(rm) = (dr)m + (-1)^{|r|} r(dm).$$

This holds essentially by the definition of the tensor product of chain complexes.

Morphisms of dg-modules are quite more interesting to define than the "classical" counterparts. Indeed, we will end up with a *complex* of morphisms.

Definition 3.6. Let R be a dg-algebra, and let M and M' be left R-dg-modules. A *degree p morphism* $\varphi \colon M \to M'$ is a $\mathbf{k}$-linear morphism such that $\varphi(M^i) \subseteq M^{i+p}$ for all $i \in \mathbb{Z}$ and it preserves the left action of R, namely:

$$\varphi(rm) = (-1)^{|\varphi||r|} r\varphi(m), \qquad r \in R,\ m \in M,$$

where $|\varphi| = \deg(\varphi) = p$. Beware the sign: it appears because of the swapping of the "graded symbols" φ and r.

Let N and N' be right R-dg-modules. A *degree p morphism* $\psi \colon N \to N'$ is a $\mathbf{k}$-linear morphism such that $\psi(N^i) \subseteq N^{i+p}$ for all $i \in \mathbb{Z}$, and it preserves the right action of R, namely:

$$\psi(mr) = \psi(m)r, \qquad r \in R,\ m \in N.$$

We have no additional signs here, because we did not swap any graded symbol!

If $\varphi \colon M \to M'$ is a degree $p = |\varphi|$ morphism of either left or right R-dg-modules, we can define its *differential*

$$d\varphi = d_{M'} \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_M.$$

We can directly check (using that φ is a morphisms of dg-modules and the suitable graded Leibniz rules) that $d\varphi$ is indeed a degree $p+1$ morphism of (either left or right) dg-modules $M \to M'$, namely, that is compatible with the action of R. This is quite tedious and left as an exercise (see Exercise 3.1). The formula for $d\varphi$ is the same as the one we already encountered when defining the complex of morphisms between chain complexes (cf. Definition 1.12), so we already know that $d \circ d = 0$. We hence end up with a chain complex.

The complex of morphisms of right R-dg-modules $N \to N'$ will be denoted by

$$\underline{\mathrm{Hom}}_R(N, N') \qquad \text{or} \qquad \underline{\mathrm{Hom}}_{\mathbf{k}-R}(N, N'),$$

whereas the complex of morphisms of left R-modules $M \to M'$ will be denoted by

$$\underline{\mathrm{Hom}}_{R^{\mathrm{op}}}(M, M') \qquad \text{or} \qquad \underline{\mathrm{Hom}}_{R-\mathbf{k}}(M, M').$$

Again, view the above as notational tricks, in particular R^{op}.

Finally, we turn to the "many-object" framework. We will directly deal with *dg-modules over dg-categories*. Watch carefully how this more general setup is dealt with; recalling the notion of *presheaf* can be useful.

Definition 3.7. Let $\mathcal{A}$ be a dg-category. A *left $\mathcal{A}$-dg-module M* is given by the following data:

- A function $A \mapsto M(A)$, where $A \in \mathrm{Ob}(\mathcal{A})$ and $M(A)$ is a chain complex. Essentially, it is a family of chain complexes parametrized by the objects of $\mathcal{A}$.
- For objects $A, B \in \mathrm{Ob}(\mathcal{A})$, chain maps:

$$\begin{aligned} \mathcal{A}(A, B) \otimes M(A) &\to M(B), \\ f \otimes m &\mapsto fm, \end{aligned}$$

subject to the relations:

$$\begin{aligned} 1_A m &= m, \qquad m \in M(A), \qquad A \in \mathrm{Ob}(\mathcal{A}), \\ (gf)m &= g(fm) \qquad g \in \mathcal{A}(B, C), f \in \mathcal{A}(A, B),\ m \in M(A). \end{aligned}$$

A *right $\mathcal{A}$-dg-module N* is given by the following data:

- A function $A \mapsto N(A)$, where $A \in \mathrm{Ob}(\mathcal{A})$ and $N(A)$ is a chain complex. Again, it is a family of chain complexes parametrized by the objects of $\mathcal{A}$.
- For objects $A, B \in \mathrm{Ob}(\mathcal{A})$, chain maps:

$$\begin{aligned} N(B) \otimes \mathcal{A}(A, B) &\to N(A), \\ m \otimes f &\mapsto mf, \end{aligned}$$

subject to the relations:

$$\begin{aligned} m 1_A &= m, \qquad m \in N(A), \qquad A \in \mathrm{Ob}(\mathcal{A}), \\ m(gf) &= (mg)f \qquad g \in \mathcal{A}(B, C), f \in \mathcal{A}(A, B),\ m \in N(C). \end{aligned}$$

Remark 3.8. Giving a chain map $\mathcal{A}(A, B) \otimes M(A) \to M(A)$, as in the case of a left $\mathcal{A}$-dg-module, is the same as giving $\mathbf{k}$-bilinear morphisms

$$\mathcal{A}(A, B)^q \times M(A)^p \to M(B)^{p+q},$$

satisfying a graded Leibniz rule:

$$d_{M(B)}(fm) = (df)m + (-1)^{|f|} f(d_{M(A)}m).$$

Again, this holds essentially by the definition of the tensor product of chain complexes.

We now define morphisms of dg-modules over dg-categories. As in the case of dg-algebras, we will end up with a complex of morphisms. We have to take care of the presence of possibly many objects, but this is not too difficult.

Definition 3.9. Let $\mathcal{A}$ be a dg-category, and let M and M' be left $\mathcal{A}$-dg-modules. A *degree p morphism $\varphi \colon M \to M'$* is a family of $\mathbf{k}$-linear morphisms

$$\varphi_A \colon M(A) \to M'(A),$$

parametrized by $A \in \mathrm{Ob}(\mathcal{A})$ (namely, a function $A \mapsto \varphi_A$), such that:

- $\varphi_A(M(A)^i) \subseteq M(A)^{i+p}$ for all $i \in \mathbb{Z}$, for all $A \in \mathrm{Ob}(\mathcal{A})$.
- The left action of $\mathcal{A}$ is preserved, namely, for any $f \in \mathcal{A}(A, B)$ and $m \in M(A)$ we have:

$$\varphi_B(fm) = (-1)^{|\varphi||f|} f\varphi_A(m),$$

  where $|\varphi| = p$. Beware the sign! It appears because of the swapping of the "graded symbols" φ and f.

Let N and N' be right $\mathcal{A}$-dg-modules. A *degree p morphism* $\psi\colon N \to N'$ is a family of $\mathbf{k}$-linear morphisms

$$\psi_A\colon N(A) \to N'(A),$$

parametrized by $A \in \mathrm{Ob}(\mathcal{A})$ (namely, a function $A \mapsto \psi_A$), such that:

- $\psi_A(N(A)^i) \subseteq N(A)^{i+p}$ for all $i \in \mathbb{Z}$, for all $A \in \mathrm{Ob}(\mathcal{A})$.
- The right action of $\mathcal{A}$ is preserved, namely, for any $f \in \mathcal{A}(A, B)$ and $m \in M(B)$ we have:

$$\varphi_A(mf) = \varphi_B(m)f,$$

  We have no additional sign here, because we did not swap any graded symbol!

If $\varphi\colon M \to M'$ is a degree $p = |\varphi|$ morphism of either left or right R-dg-modules, we can define its *differential* as the parametrized family $((d\varphi)_A : A \in \mathrm{Ob}(\mathcal{A}))$, where

$$(d\varphi)_A = d(\varphi_A),$$

where $d(\varphi_A)$ is the now familiar differential of morphisms between twisted complexes:

$$d(\varphi_A) = d_{M'(A)} \circ \varphi_A - (-1)^{|\varphi|}\varphi_A d_{M(A)}.$$

As in Definition 3.6, we can directly check that $d\varphi$ is indeed a degree $p+1$ morphism of (either left or right) dg-modules $M \to M'$, namely, that is compatible with the action of $\mathcal{A}$. The identity $d \circ d = 0$ clearly holds. We hence end up with a chain complex.

The complex of morphisms of right $\mathcal{A}$-dg-modules $N \to N'$ will be denoted by

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(N, N') \qquad \text{or} \qquad \underline{\mathrm{Hom}}_{\mathbf{k}-\mathcal{A}}(N, N'),$$

whereas the complex of morphisms of left R-modules $M \to M'$ will be denoted by

$$\underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{op}}}(M, M') \qquad \text{or} \qquad \underline{\mathrm{Hom}}_{\mathcal{A}-\mathbf{k}}(M, M').$$

Soon we will give a meaningful definition of the gadget $\mathcal{A}^{\mathrm{op}}$.

Remark 3.10. You can see that there are not many differences between the above definition and the one of dg-modules over a dg-algebras: one has just to be careful with having a parametrized family of morphisms $(\varphi_A : A \in \mathrm{Ob}(\mathcal{A})$ instead of just one map. On the other hand, you can see that specializing to dg-categories with a single object (indeed, dg-algebras) one gets back the "old" definition.

3.2. **Represented dg-modules and the dg-Yoneda lemma.** If R is a $\mathbf{k}$-algebra, there is a very special example of a both right and left R-module: that is, R itself where the action is just induced by the product

$$R \otimes R \to R.$$

Something similar happens with a dg-algebra. In the case of dg-categories, one has to be careful with having a family of objects. Products of elements will become compositions of morphisms:

Definition 3.11. Let $\mathcal{A}$ be a dg-category, and let $A \in \mathrm{Ob}(\mathcal{A})$. We define a right dg-module $\mathcal{A}(-, A) = h_A$ as follows:

$$h_A(B) = \mathcal{A}(B, A), \qquad B \in \mathrm{Ob}(\mathcal{A}),$$

with actions

$$h_A(B') \otimes \mathcal{A}(B, B') \to h_A(B)$$

given just by the compositions

$$\mathcal{A}(B', A) \otimes \mathcal{A}(B, B') \to \mathcal{A}(B, A).$$

Verifying that this indeed yields an $\mathcal{A}$-dg-module is straightforward.

Analogously, we define a left $\mathcal{A}$-dg-module $\mathcal{A}(A, -) = h^A$ as:

$$h^A(B) = \mathcal{A}(A, B), \qquad B \in \mathrm{Ob}(\mathcal{A}),$$

with actions

$$\mathcal{A}(B, B') \otimes h^A(B) \to h^A(B')$$

given by just compositions.

The right $\mathcal{A}$-dg-module $\mathcal{A}(-, A)$ is said to be *represented by* A; the left $\mathcal{A}$-dg-module $\mathcal{A}(A, -)$ is said to be *corepresented by* A.

We now state a result which should be quite straightforward to prove, but whose relevance cannot be underestimated, even if this will maybe not be apparent right now.

Theorem 3.12 (Dg-categorical Yoneda lemma)**.** *Let $\mathcal{A}$ be a dg-category, and let $A \in \mathrm{Ob}(\mathcal{A})$ be an object. Let M be a right $\mathcal{A}$-dg-module. We have an isomorphism of complexes:*

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M) &\xrightarrow{\sim} M(A), \\ \varphi &\mapsto \varphi_A(1_A), \end{aligned}$$

where $\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M)$ is the hom complex of morphisms between $\mathcal{A}(-, A)$ and M, cf. Definition 3.9.

The proof of this crucial result is left to you, reader. Exercises from Exercise 3.2 to Exercise 3.6 should give you a reasonable path to achieve that. I believe that everything revolving around the Yoneda lemma should be worked out directly by you, until you become "fluent" with the result. In my experience, this has happened after years and it is still happening – so, take your time!

Exercises to Lecture 3.

Exercise 3.1. Prove that the differential of a degree p morphism of left or right R-dg-modules (as in Definition 3.6) is a degree $p + 1$ morphism of R-dg-modules.

Exercise 3.2 (Yoneda for $\mathbf{k}$-modules). Let M be a $\mathbf{k}$-module. Find an isomorphism of $\mathbf{k}$-modules

$$\mathrm{Hom}(\mathbf{k}, M) \cong M.$$

Exercise 3.3 (Yoneda for modules over a $\mathbf{k}$-algebra). Let R be a $\mathbf{k}$-algebra, and let M be a right R-module. Find an isomorphism of $\mathbf{k}$-modules:

$$\mathrm{Hom}_R(R, M) \cong M,$$

where we view R as a right R-module with the canonical action $R \otimes R \to R$ given by the product.

Exercise 3.4 (Yoneda for dg-modules over a dg-algebra). Let R be a dg-algebra, and let M be a right dg-module. Find an isomorphism of complexes of $\mathbf{k}$-modules:

$$\underline{\mathrm{Hom}}_R(R, M) \cong M,$$

where we view R as a right R-dg-module with the canonical action $R \otimes R \to R$ given by the product.

Exercise 3.5. Prove the dg-Yoneda lemma (Theorem 3.12).

Exercise 3.6 (Yoneda for left dg-modules). State and prove a variant of the dg-Yoneda lemma (Theorem 3.12) for *left* dg-modules over a dg-category.

4. Lecture 4: more on dg-modules and dg-functors

4.1. **Dg-modules as dg-functors.** Let $\mathcal{A}$ be a dg-category. The definition of left $\mathcal{A}$-dg-modules we gave (cf. Definition 3.7) revolves around the actions

$$\mathcal{A}(A, B) \otimes M(A) \xrightarrow{m_{A,B}} M(B),$$

which we call $m_{A,B}$ for our current purposes. $m_{A,B}$ is a chain map of complexes; thanks to the tensor-hom adjunction isomorphism

$$\underline{\mathrm{Hom}}(\mathcal{A}(A, B) \otimes M(A), M(B)) \cong \underline{\mathrm{Hom}}(\mathcal{A}(A, B), \underline{\mathrm{Hom}}(M(A), M(B))),$$

described in Proposition 1.17, we may actually identify $m_{A,B}$ with a chain map

$$M_{A,B} \colon \mathcal{A}(A, B) \to \underline{\mathrm{Hom}}(M(A), M(B)),$$
$$M_{A,B}(f)(x) = m_{A,B}(f \otimes x) = fx.$$

What properties do the *chain maps* $M_{A,B}$ satisfy? For the sake of simplicity, let us drop the notational reference to the objects. If $f \in \mathcal{A}(A, B)$ and $g \in \mathcal{A}(B, C)$, we have:

$$M(gf)(x) = (gf)(x) = g(fx) = M(g)(fx) = M(g)(M(f)(x)),$$

namely

$$\boxed{M(gf) = M(g)M(f).}$$

Notice that this last relation does not explicitly mention any element $x \in M(A)$. Moreover, if $1_A \colon A \to A$ is an identity morphism, we have:

$$M(1_A)(x) = 1_A x = x = 1_{M(A)}(x),$$

namely

$$\boxed{M(1_A) = 1_{M(A)}.}$$

The above boxed relations, together with the fact that $f \mapsto M(f)$ is a chain map, immediately shows that we get a *dg-functor*:

$$M \colon \mathcal{A} \to \mathrm{dgm}(\mathbf{k}),$$

where $\mathrm{dgm}(\mathbf{k})$ is the dg-category of chain complexes, cf. Example 2.5.

Let us have a look at the complex of morphisms of (left) dg-modules. If $\varphi \colon M \to M'$ is a morphism of left $\mathcal{A}$-dg-modules, we have by definition (for $x \in M(A)$ and $f \in \mathcal{A}(A, B)$):

$$\varphi_B(fx) = (-1)^{|\varphi||f|} f \varphi_A(x),$$

where $|f|$ is the degree of f and $|\varphi|$ is the degree of φ. Viewing M and M' as dg-functors, we may replace fx with $M(f)(x)$ and $f\varphi_A(x)$ with $M'(f)(\varphi_A(x))$ and get the following relation:

$$\varphi_B(M(f)(x)) = (-1)^{|\varphi||f|} M'(f)(\varphi_A(x)).$$

We may drop the explicit reference to the element x and just write:

$$\boxed{\varphi_B \circ M(f) = (-1)^{|\varphi||f|} M'(f) \circ \varphi_A.}$$

This relation can be pictured by saying that the diagram

$$\begin{tikzcd} M(A) \arrow[r, "\varphi_A"] \arrow[d, "M(f)"'] & M'(A) \arrow[d, "M'(f)"] \\ M(B) \arrow[r, "\varphi_B"] & M'(B) \end{tikzcd}$$

is commutative up to the sign $(-1)^{|\varphi||f|}$, for all $f \colon A \to B$. Finally, recall that differentials of morphisms of dg-modules were defined objectwise:

$$\boxed{(d\varphi)_A = d(\varphi_A).}$$

The fact that the above boxed relations do not depend of elements of the complexes $M(A)$ allow us to vastly generalize, from dg-modules to any dg-functor.

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. We define the *dg-category of dg-functors* $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ as follows:

- Objects are dg-functors $F \colon \mathcal{A} \to \mathcal{B}$.
- Let $F, G \colon \mathcal{A} \to \mathcal{B}$ be dg-functors. A *degree p morphism* $\varphi \colon F \to G$ is given by a family $A \mapsto \varphi_A$ of degree p morphisms

  $$\varphi_A \in \mathcal{B}(F(A), G(A))^p,$$

  such that

  $$\varphi_B \circ F(f) = (-1)^{|\varphi||f|} G(f) \circ \varphi_A$$

  (here $|\varphi| = p$), namely, the following diagram is commutative

  $$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\varphi_B"] & G(B) \end{tikzcd}$$

  up to the sign $(-1)^{|\varphi||f|}$.

  The differential $d\varphi$ of a degree p morphism $\varphi \colon F \to G$ is defined objectwise:

  $$(d\varphi)_A = d(\varphi_A).$$

  Morphisms of dg-functors will be also called *dg-natural transformations*; the hom-complex $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})(F, G)$ will be also denoted by $\mathrm{Nat}_{\mathrm{dg}}(F, G)$.

Remark 4.2. Let us show that, in the above definition, $d\varphi$ is indeed a natural transformation. We compute, using the graded Leibniz rule:

$$\begin{aligned} d(\varphi_B) \circ F(f) &= d(\varphi_B \circ F(f)) - (-1)^{|\varphi|}\varphi_B \circ dF(f) \\ &= (-1)^{|\varphi||f|} d(G(f) \circ \varphi_A) - (-1)^{|\varphi|}\varphi_B \circ F(df) \\ &= (-1)^{|\varphi||f|} d(G(f) \circ \varphi_A) - (-1)^{|\varphi|}(-1)^{|\varphi|(|f|+1)} dG(f) \circ \varphi_A \\ &= (-1)^{|\varphi||f|} d(G(f) \circ \varphi_A) - (-1)^{|\varphi||f|} dG(f) \circ \varphi_A \\ &= (-1)^{|\varphi||f|} (d(G(f) \circ \varphi_A - dG(f) \circ \varphi_A) \\ &= (-1)^{|\varphi||f|}(-1)^{|f|} G(f) d\varphi_A \\ &= (-1)^{(|\varphi|+1)|f|} G(f) d\varphi_A. \end{aligned}$$

4.2. **The case of right dg-modules, the opposite dg-category and "official" dg-modules.** Let $\mathcal{A}$ be a dg-category. We viewed (actually identified) left $\mathcal{A}$-dg-modules with dg-functors $\mathcal{A} \to \mathrm{dgm}(\mathbf{k})$. What about *right* $\mathcal{A}$-dg-modules? The situation is a little bit trickier. Let N be a right $\mathcal{A}$-dg-module, with actions

$$\begin{aligned} N(B) \otimes \mathcal{A}(A, B) &\xrightarrow{m} N(A), \\ x \otimes f &\mapsto m(x \otimes f) = xf. \end{aligned}$$

We first apply the symmetry isomorphism (cf. Proposition 1.15):

$$\begin{aligned} N(B) \otimes \mathcal{A}(A, B) &\xrightarrow{\sim} \mathcal{A}(A, B) \otimes N(B), \\ x \otimes f &\mapsto (-1)^{|x||f|} f \otimes x. \end{aligned}$$

Then, we apply the tensor-hom adjunction isomorphism (Proposition 1.17):

$$\underline{\mathrm{Hom}}(\mathcal{A}(A, B) \otimes N(B), N(A)) \cong \underline{\mathrm{Hom}}(\mathcal{A}(A, B), \underline{\mathrm{Hom}}(N(B), N(A))).$$

In the end, the action $m\colon N(B) \otimes \mathcal{A}(A, B) \to N(A)$ is identified with the chain map:

$$\begin{aligned} N = N_{A,B} \colon \mathcal{A}(A, B) &\to \underline{\mathrm{Hom}}(N(B), N(A)), \\ f &\mapsto N(f)(x) = (-1)^{|f||x|} xf. \end{aligned}$$

What properties do the chain maps $N = N_{A,B}$ satisfy? Let us compute, for composable morphisms $f\colon A \to B$ and $g\colon B \to C$:

$$\begin{aligned} N(gf)(x) &= (-1)^{|x|(|g|+|f|)} x(gf) \\ &= (-1)^{|x|(|g|+|f|)} (xg)f \\ &= (-1)^{|x||g|}(-1)^{|f||g|} N(f)(xg) \\ &= (-1)^{|f||g|} N(f)(N(g)(x)), \end{aligned}$$

hence

$$\boxed{N(gf) = (-1)^{|f||g|} N(f)N(g).}$$

Moreover

$$N(1_A)(x) = x1_A = x = 1_{N(A)}(x),$$

hence

$$\boxed{N(1_A) = 1_{N(A)}.}$$

We see that N is not a dg-functor from $\mathcal{A}$ to $\mathrm{dgm}(\mathbf{k})$, but something that we can temporarily call a *contravariant dg-functor*.

What about the complex of morphisms $\psi\colon N \to N'$ of right $\mathcal{A}$-dg-modules? By definition, for $x \in N(B)$ and $f\colon A \to B$:

$$\psi_A(xf) = \psi_B(x)f.$$

Identifying N and N' with contravariant dg-functors as above, we get:

$$(-1)^{|f||x|}\psi_A(N(f)(x)) = (-1)^{(|x|+|\psi|)|f|}N'(f)(\psi_B(x)),$$

from which we obtain

$$\boxed{\psi_A \circ N(f) = (-1)^{|\psi||f|}N'(f) \circ \psi_B.}$$

Hence, natural transformations of contravariant dg-functors look precisely the same as the natural transformations of ordinary dg-functors. Finally, we know that the differential of ψ is defined objectwise:

$$\boxed{(d\psi)_A = d(\psi_A).}$$

Can we make such contravariant dg-functors into ordinary dg-functors? The answer is yes, but we first need to introduce a new concept in order to formally switch left and right compositions and actions:

Definition 4.3. Let $\mathcal{A}$ be a dg-category. We define the *opposite dg-category* $\mathcal{A}^{\mathrm{op}}$ as follows:

- Objects of $\mathcal{A}^{\mathrm{op}}$ are the same as the objects of $\mathcal{A}$: $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$.
- Hom-complexes in $\mathcal{A}^{\mathrm{op}}$ are the same as the ones of $\mathcal{A}$, but "with arrows reversed":
$$\mathcal{A}^{\mathrm{op}}(A, B) = \mathcal{A}(B, A).$$
Notationally, we will put an "op" decoration on every morphism of $\mathcal{A}$ when we want to view it as a morphism in $\mathcal{A}^{\mathrm{op}}$. Namely, a morphism $f^{\mathrm{op}}\colon A \to B$ in $\mathcal{A}^{\mathrm{op}}$ corresponds to a morphism $f\colon B \to A$ in $\mathcal{A}$.
- Compositions are induced directly from $\mathcal{A}$, but beware the Koszul sign rule:
$$\begin{aligned}&\mathcal{A}^{\mathrm{op}}(B, C) \otimes \mathcal{A}^{\mathrm{op}}(A, B) \to \mathcal{A}^{\mathrm{op}}(A, C),\\ &\qquad\qquad g^{\mathrm{op}} \otimes f^{\mathrm{op}} \mapsto g^{\mathrm{op}} \circ f^{\mathrm{op}} = (-1)^{|f||g|}(f \circ g)^{\mathrm{op}}.\end{aligned}$$

Observe that we trivially have:

$$(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}.$$

Remark 4.4. What is a dg-functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$? It is given by a map on objects $A \mapsto F(A)$ and chain maps

$$F\colon \mathcal{A}^{\mathrm{op}}(B, A) = \mathcal{A}(A, B) \to \mathcal{B}(F(B), F(A)),$$

compatible with identities and compositions, in particular:

$$F(g^{\mathrm{op}} f^{\mathrm{op}}) = F(g^{\mathrm{op}})F(f^{\mathrm{op}}),$$

namely:

$$(-1)^{|f||g|}F((fg)^{\mathrm{op}}) = F(g^{\mathrm{op}})F(f^{\mathrm{op}}),$$

hence, if we drop the "op" decorations we indeed obtain a contravariant dg-functor, as introduced in the special case where we take $\mathcal{B} = \mathrm{dgm}(\mathbf{k})$.

With this language and recalling the above discussion, we may finally give a concise definition of left and right dg-modules, together with their dg-categories!

Definition 4.5. Let $\mathcal{A}$ be a dg-category. We define the *dg-category of right $\mathcal{A}$-dg-modules* as:

$$\operatorname{dgm}(\mathcal{A}) = \operatorname{Fun}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}}, \operatorname{dgm}\mathbf{k}),$$

and the dg-category of *left $\mathcal{A}$-dg-modules* just as the dg-category of right $\mathcal{A}^{\mathrm{op}}$-dg-modules:

$$\operatorname{dgm}(\mathcal{A}^{\mathrm{op}}) = \operatorname{Fun}_{\mathrm{dg}}(\mathcal{A}, \operatorname{dgm}(\mathbf{k})),$$

recalling that $(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}$. These are dg-categories of dg-functors, with morphisms being dg-natural transformations.

Remark 4.6. The above discussion essentially shows that this "new" definition of dg-modules is essentially the same as the "old" given in terms of left or right actions. For our convenience, we recall how we can pass from dg-functors to left or right actions, and vice-versa.

If N is a right $\mathcal{A}$-dg-module, we have:

$$xf = (-1)^{|f||x|}N(f)(x),$$

for $f \in \mathcal{A}(A, B)$ and $x \in N(B)$. If M is a left $\mathcal{A}$-dg-module, we write:

$$fx = M(f)(x),$$

for $x \in M(A)$ and $f \in \mathcal{A}(A, B)$.

Moreover, under this identification, dg-natural transformations of dg-modules become the dg-module morphisms as previously defined in Definition 3.9.

It is worth mentioning that what we've done so far for dg-categories and dg-modules can be also done for $\mathbf{k}$-linear categories and modules over those, just by replacing $\operatorname{dgm}(\mathbf{k})$ with the $\mathbf{k}$-linear category of $\mathbf{k}$-modules $\operatorname{Mod}(\mathbf{k})$ and making the necessary adjustments. This is actually easier, because we can make the same formal steps but in an easier setting (no sign issues!). We sum up what we need in the following definition:

Definition 4.7. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbf{k}$-linear categories. We denote by $\operatorname{Fun}(\mathcal{A}, \mathcal{B})$ (or sometimes $\operatorname{Fun}_{\mathbf{k}}(\mathcal{A}, \mathcal{B})$) the $\mathbf{k}$-linear category of $\mathbf{k}$-linear functors. Its morphisms are given by the $\mathbf{k}$-module $\operatorname{Nat}(F, G) = \operatorname{Nat}_{\mathbf{k}}(F, G)$ of *natural transformations* $\varphi \colon F \to G$, defined by families $A \mapsto \varphi_A$ such that $G(f) \circ \varphi_A = \varphi_B \circ F(f)$ for all $f \colon A \to B$.

We define the $\mathbf{k}$-linear categories of (respectively) right and left $\mathcal{A}$-modules as:

$$\operatorname{Mod}(\mathcal{A}) = \operatorname{Fun}(\mathcal{A}^{\mathrm{op}}, \operatorname{Mod}(\mathbf{k})),$$
$$\operatorname{Mod}(\mathcal{A}^{\mathrm{op}}) = \operatorname{Fun}(\mathcal{A}, \operatorname{Mod}(\mathbf{k})),$$

where $\mathcal{A}^{\mathrm{op}}$ is the opposite $\mathbf{k}$-linear category.

Remark 4.8. Warning! For a given $\mathbf{k}$-linear category, we have the $\mathbf{k}$-linear category of (right) $\mathcal{A}$-modules $\operatorname{Mod}(\mathcal{A})$, but we can also view $\mathcal{A}$ as a dg-category concentrated in degree 0 and consider the dg-category of (right) $\mathcal{A}$-dg-modules $\operatorname{dgm}(\mathcal{A})$. These are two very different things, so don't be confused. Indeed, even when $\mathcal{A} = \mathbf{k}$, we get $\operatorname{Mod}(\mathbf{k})$ ($\mathbf{k}$-modules) on one side and $\operatorname{dgm}(\mathbf{k})$ (complexes of $\mathbf{k}$-modules) on the other.

This identification between "dg-modules defined as dg-functors" and "dg-modules defined with left/right actions" may seem quite exotic, but it is just an instance of the classical identification of "modules over algebras" with "representations of algebras", which we actually obtain from what we have done by specializing to single-object dg-categories (namely, dg-algebras) or single object linear categories (namely, ordinary algebras).

Exercises to Lecture 4.

Exercise 4.1. View $\mathbf{k}$ as a dg-category with a single object $\star$ and concentrated in degree 0 with zero differentials. Let $\mathcal{A}$ be any dg-category. Define a dg-functor

$$\mathrm{Fun}_{\mathrm{dg}}(\mathbf{k}, \mathcal{A}) \to \mathcal{A}$$

which is a bijection on objects and an isomorphism of complexes between the hom-complexes – namely, it is an isomorphism of dg-categories.

Exercise 4.2. Let A be a $\mathbf{k}$-algebra. A *representation* of A is a $\mathbf{k}$-module M together with a morphism

$$A \to \mathrm{Hom}(M, M)$$

of $\mathbf{k}$-algebras, where $\mathrm{Hom}(M, M)$ has the $\mathbf{k}$-algebra structure given by composition of endomorphisms $M \to M$. Check that this is actually the same as giving a $\mathbf{k}$-*linear functor*

$$A \to \mathrm{Mod}(k),$$

viewing A as a $\mathbf{k}$-linear category with a single object. In turn, this is the same as giving a left A-module structure on M by means of an action

$$A \otimes M \to M.$$

Exercise 4.3. Let R be a dg-algebra. Consider the complex $\underline{\mathrm{Hom}}_{\mathbf{k}}(R, R)$ of endomorphisms $R \to R$, and endow it with a structure of $\mathbf{k}$-dg-algebra with the composition of endormorphisms. Find an injective morphism of dg-algebras:

$$R \to \underline{\mathrm{Hom}}_{\mathbf{k}}(R, R).$$

Hint: By the Yoneda lemma, we know that $R \cong \underline{\mathrm{Hom}}_{R-\mathbf{k}}(R, R)$, the complex of left R-module morphisms $R \to R$, endowing R with the obvious structure of left R-module over itself.

Exercise 4.4. *(A bit harder!)* Let $\mathcal{A}$ be a dg-category and let $M \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{dgm}(\mathbf{k})$ be a right $\mathcal{A}$-dg-module. Consider the Yoneda isomorphisms ($A \in \mathrm{Ob}(\mathcal{A})$):

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M) \xrightarrow{\sim} M(A).$$

Prove that we can define a right $\mathcal{A}$-dg-module

$$\begin{aligned} \widetilde{M} \colon \mathcal{A}^{\mathrm{op}} &\to \mathrm{dgm}(\mathbf{k}), \\ A &\mapsto \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M) \end{aligned}$$

(describe it on morphisms!) such that the Yoneda isomorphism is "promoted" to a natural transformation

$$\widetilde{M} \to M$$

which in particular yields isomorphisms of complexes $\widetilde{M}(A) \to M(A)$.

5. Lecture 5: the Yoneda embedding and constructions on dg-modules

The dg-category $\operatorname{dgm}(\mathcal{A})$ of (right) dg-modules over a dg-category $\mathcal{A}$ is a big "playground". In this lecture:

- We will show how $\mathcal{A}$ can actually be embedded in $\operatorname{dgm}(\mathcal{A})$, which can be henceforth thought of as some kind of "extension" of $\mathcal{A}$ itself.
- We will see how essentially all constructions we make on chain complexes (for example, direct sums and products) can be also made on $\mathcal{A}$-dg-modules, just objectwise.
- We will have our first encounter with shifts and mapping cones, which will be a key notion in our journey to the derived category.

It is important to build intuition and familiarity on the notion of dg-modules. If we view them as dg-functors $\mathcal{A}^{\mathrm{op}} \to \operatorname{dgm}(\mathbf{k})$, we can think of them as "representations of $\mathcal{A}$". For instance, take $\mathcal{A}$ to be the dg-category $\mathbf{Q}$ or $\mathbf{Q}'$ of Example 2.11. What would a $\mathbf{Q}$-dg-module be? Or a $\mathbf{Q}'$-dg-module?

5.1. **The Yoneda embedding.** Let $\mathcal{A}$ be a dg-category, and let $A \in \operatorname{Ob}(\mathcal{A})$ be an object of $\mathcal{A}$. For any right $\mathcal{A}$-dg-module $M \colon \mathcal{A}^{\mathrm{op}} \to \operatorname{dgm}(\mathbf{k})$, the Yoneda lemma yields an isomorphism of complexes:

$$\begin{aligned} \operatorname{Nat}_{\mathrm{dg}}(\mathcal{A}(-,A), M) &\xrightarrow{\sim} M(A), \\ \varphi &\mapsto \varphi_A(1_A), \end{aligned}$$

with inverse given by

$$\begin{aligned} M(A) &\to \operatorname{Nat}_{\mathrm{dg}}(\mathcal{A}(-,A), M), \\ x &\mapsto (\varphi_x)_B(f) = xf = (-1)^{|x||f|} M(f)(x). \end{aligned}$$

$\mathcal{A}(-,A) \colon \mathcal{A}^{\mathrm{op}} \to \operatorname{dgm}(\mathbf{k})$ is the right $\mathcal{A}$-dg-module represented by A. How is it defined, when we view it as a dg-functor?

- On objects, we have $\mathcal{A}(-,A)(B) = \mathcal{A}(B,A)$.
- If $f \in \mathcal{A}(B,B')$, then
$$\begin{aligned} \mathcal{A}(-,A)(f) = \mathcal{A}(f,A) = f^* &\colon \mathcal{A}(B',A) \to \mathcal{A}(B,A), \\ g \mapsto \mathcal{A}(f,A)(g) &= f^*(g) = (-1)^{|f||g|} g \circ f. \end{aligned}$$

Is there perhaps a way to make the mapping

$$A \mapsto \mathcal{A}(-,A)$$

itself a dg-functor $\mathcal{A} \to \operatorname{dgm}(\mathcal{A})$? The answer is *yes*. Let us take a morphism $f \in \mathcal{A}(A,A')$ and try to define a morphism $\mathcal{A}(-,A) \to \mathcal{A}(-,A')$. We can use again compositions and define:

$$\begin{aligned} &\mathcal{A}(-,f) = f_* \colon \mathcal{A}(-,A) \to \mathcal{A}(-,A'), \\ &(f_*)_B \colon \mathcal{A}(B,A) \to \mathcal{A}(B,A'), \\ &(f_*)_B(g) = f \circ g. \end{aligned}$$

We can show:

Proposition 5.1. *There is a dg-functor*

$$h \colon \mathcal{A} \to \operatorname{dgm}(\mathcal{A}),$$

defined as follows:

- $h(A) = h_A = \mathcal{A}(-, A)$ *on objects.*
- *If* $f\colon A \to B$ *is a morphism in* $\mathcal{A}$*, we set*

$$h(f) = \mathcal{A}(-, f) = f_*\colon \mathcal{A}(-, A) \to \mathcal{A}(-, A'),$$
$$(f_*)_B(g) = f \circ g.$$

It has the property that for any pair of objects $A, A' \in \mathrm{Ob}(\mathcal{A})$*, the chain map*

$$\mathcal{A}(A, A') \to \mathrm{Nat}_{\mathrm{dg}}(\mathcal{A}(-, A), \mathcal{A}(-, A')),$$
$$f \mapsto \mathcal{A}(-, f)$$

is an isomorphism of complexes.

The dg-functor $h\colon \mathcal{A} \to \mathrm{dgm}(\mathcal{A})$ *is called the* Yoneda embedding.

Proof. Proving that $h\colon \mathcal{A} \to \mathrm{dgm}(\mathcal{A})$ is a dg-functor is straightforward and a bit tedious (just check everything), so we will not do that and it is left as an exercise. More interesting is to see that

$$\mathcal{A}(A, A') \to \mathrm{Nat}_{\mathrm{dg}}(\mathcal{A}(-, A), \mathcal{A}(-, A'))$$

is actually an isomorphism of complexes. One can also check it directly, but a more careful inspection shows that this morphism is just the inverse of

$$\mathrm{Nat}_{\mathrm{dg}}(\mathcal{A}(-, A), \mathcal{A}(-, A')) \to \mathcal{A}(A, A'),$$
$$\varphi \mapsto \varphi_A(1_A).$$

So, we already know from the Yoneda lemma that it is indeed an isomorphism! □

The Yoneda embedding is an example of a *fully faithful dg-functor*:

Definition 5.2. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a dg-functor. We say that F is *fully faithful* if for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{A})$, the chain map

$$F\colon \mathcal{A}(A, B) \to \mathcal{B}(F(A), F(B))$$

is an isomorphism of complexes.

Fully faithful dg-functors can be thought as "embeddings". Indeed, we can prove:

Lemma 5.3. *Let* $F\colon \mathcal{A} \to \mathcal{B}$ *be a fully faithful dg-functor. Then,* F *is* essentially injective*, namely: if there are closed degree* 0 *morphisms* $g\colon F(A) \to F(B)$ *and* $g'\colon F(B) \to F(A)$ *such that* $g \circ g' = 1_{F(B)}$ *and* $g' \circ g = 1_{F(A)}$*, then we can find closed degree* 0 *morphisms* $f\colon A \to B$ *and* $f'\colon B \to A$ *such that* $f' \circ f = 1_A$ *and* $f \circ f' = 1_B$*.*

Proof. By hypothesis, we can find unique $f\colon A \to B$ and $f'\colon B \to A$, which are closed degree 0 morphisms, such that $g = F(f)$ and $g' = F(f')$. Then, from the relations:

$$F(f') \circ F(f) = F(f' \circ f) = F(1_A), \qquad F(f) \circ F(f') = F(f \circ f') = F(1_B)$$

we conclude that $f' \circ f = 1_A$ and $f \circ f' = 1_B$, as claimed. □

Remark 5.4. In a dg-category $\mathcal{A}$, we are certainly tempted to call *isomorphism* a morphism $f\colon A \to B$ which is of degree 0 and closed ($df = 0$) and admits a closed degree 0 inverse $f'\colon B \to A$, such that $f' \circ f = 1_A$ and $f \circ f' = 1_B$. We will come back on this notion later on, when we also deal with the homotopy category $H^0(\mathcal{A})$. If the reader is really interested, they can try and figure out themselves even now basic facts about isomorphisms (are inverses unique, for instance?) and even how to define $H^0(\mathcal{A})$ – see Exercises 5.2 and 5.3.

A common example of fully faithful dg-functors are obtained by choosing a subfamily of objects in some given dg-category:

Definition 5.5. Let $\mathcal{A}$ be a dg-category. Let $\mathrm{Ob}(\mathcal{B})$ be a subfamily of objects of $\mathcal{A}$, namely, $\mathrm{Ob}(\mathcal{B}) \subseteq \mathrm{Ob}(\mathcal{A})$. The *full dg-subcategory* $\mathcal{B}$ *of* $\mathcal{A}$ *spanned by* $\mathrm{Ob}(\mathcal{B})$ is the dg-category defined as follows:

- Objects of $\mathcal{B}$ are given by the subfamily $\mathrm{Ob}(\mathcal{B})$ we have already specified.
- For any pair of objects $B, B' \in \mathrm{Ob}(\mathcal{B})$, we set
$$\mathcal{B}(B, B') = \mathcal{A}(B, B'),$$
with compositions and units induced directly from $\mathcal{A}$.

It is immediate to see that $\mathcal{B}$ is a dg-category, and moreover there is a fully faithful "inclusion" dg-functor
$$\mathcal{B} \hookrightarrow \mathcal{A}.$$

5.2. **Constructions on dg-modules.** Let $\mathcal{A}$ be a dg-category. The dg-category of (right) $\mathcal{A}$-dg-modules $\mathrm{dgm}(\mathcal{A}) = \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}}, \mathrm{dgm}(\mathbf{k}))$ is a big playground where we can make a lot of mathematics. There is a basic philosophy: *essentially every construction that we can make on chain complexes, we can also make on* $\mathcal{A}$*-dg-modules by reasoning objectwise*. Let us see this principle in action with a few basic examples:

Definition 5.6 (Direct product)**.** Let $\{M_i : i \in I\}$ be a family of objects in $\mathrm{dgm}(\mathcal{A})$. We define the *direct product* $\prod_{i\in I} M_i$ of this family by:
$$\left(\prod_{i\in I} M_i\right)(A) = \prod_{i\in I} M_i(A),$$
for $A \in \mathrm{Ob}(\mathcal{A})$, where the latter product is the usual product of complexes. If $f \colon A \to B$ is a morphism in $\mathcal{A}$, we set:
$$\left(\prod_{i\in I} M_i\right)(f) = \prod_{i\in I} M_i(f) \colon \prod_{i\in I} M_i(B) \to \prod_{i\in I} M_i(A).$$
We can check that $\prod_{i\in I} M_i$ yields indeed a dg-functor $\mathcal{A}^{\mathrm{op}} \to \mathrm{dgm}(\mathbf{k})$ (namely, a "contravariant dg-functor" from $\mathcal{A}$ to $\mathrm{dgm}(\mathbf{k})$). More concretely and using the right action notation, we can simply define:
$$(x_i)_{i\in I} f = (x_i f)_{i\in I},$$
for a given element $(x_i)_{i\in I} \in \prod_{i\in I} M_i(B)$ and $f \colon A \to B$.

Definition 5.7 (Direct sum)**.** Let $\{M_i : i \in I\}$ be a family of objects in $\mathrm{dgm}(\mathcal{A})$. We define the *direct sum* $\bigoplus i \in I M_i$ of this family by:
$$\left(\bigoplus_{i\in I} M_i\right)(A) = \bigoplus_{i\in I} M_i(A),$$

for $A \in \mathrm{Ob}(\mathcal{A})$, where the latter direct sum is the usual direct sum of complexes. If $f\colon A \to B$ is a morphism in $\mathcal{A}$, we set:

$$\left(\bigoplus_{i\in I} M_i\right)(f) = \bigoplus_{i\in I} M_i(f)\colon \bigoplus_{i\in I} M_i(B) \to \bigoplus_{i\in I} M_i(A).$$

We can check that $\bigoplus_{i\in I} M_i$ yields indeed a dg-functor $\mathcal{A}^{\mathrm{op}} \to \mathrm{dgm}(\mathbf{k})$ (namely, a "contravariant dg-functor" from $\mathcal{A}$ to $\mathrm{dgm}(\mathbf{k})$). More concretely and using the right action notation, we can simply define:

$$(x_i)_{i\in I} f = (x_i f)_{i\in I},$$

for a given element $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i(B)$ (which is a function $i \mapsto x_i$ such that $x_i = 0$ for i in the complement of a finite subset of I) and $f\colon A \to B$.

Definition 5.8 (Tensor product over $\mathbf{k}$)**.** Let M and N be objects in $\mathrm{dgm}(\mathcal{A})$. We define the *tensor product (over* $\mathbf{k}$*)* $M \otimes N$ as follows:

$$(M \otimes N)(A) = M(A) \otimes N(A),$$

for $A \in \mathrm{Ob}(\mathcal{A})$, where $M(A) \otimes N(A)$ is the tensor product of complexes of $\mathbf{k}$-modules. If $f\colon A \to B$ is a morphism in $\mathcal{A}$, we set:

$$(M \otimes N)(f) = M(f) \otimes N(f)\colon M(B) \otimes N(B) \to M(A) \otimes N(A).$$

Be careful with the Koszul sign rule:

$$M(f) \otimes N(f)(x \otimes y) = (-1)^{|x||f|} M(f)(x) \otimes N(f)(y).$$

Using the right action notation (recall Remark 4.6), we actually find:

$$(x \otimes y)f = (-1)^{|y||f|} xf \otimes yf.$$

We can check that $M \otimes N$ is indeed a right $\mathcal{A}$-dg-module.

Shifts and cones. We introduce two important construction which can be performed on complexes, and then also on dg-module over dg-categories. The first is the easiest one, and it is *shifting*:

Definition 5.9. Let V be a complex of $\mathbf{k}$-modules, and let $n \in \mathbb{Z}$. The *n-shift* $V[n]$ is the complex defined by:

$$V[n]^p = V^{n+p}, \qquad d_{V[n]} = (-1)^n d_V.$$

For example, $V[1]$ means essentially "shifting V by one to the left" and $V[-1]$ means "shifting V by one to the right". Shifting changes the grading of the given complex; just beware of the signs appearing on differentials. It is now immediate to extend this notion to dg-modules over dg-categories:

Definition 5.10. Let $\mathcal{A}$ be a dg-category, let $M \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a right $\mathcal{A}$-dg-module and let $n \in \mathbb{Z}$. We define the *n-shift* $M[n]$ as follows:

$$M[n](A) = M(A)[n],$$

for $A \in \mathrm{Ob}(\mathcal{A})$, where $M(A)[n]$ is the n-shift of the chain complex $M(A)$. If $f\colon A \to B$, we set:

$$M[n](f) = M(f)[n]\colon M(B)[n] \to M(A)[n],$$

where $M(f)[n]$ is just $M(f)$ but keeping track of the shifted gradings, with some care for signs: its p-th component $M(f)[n]^p$ is just $(-1)^{n|f|}M(f)^{n+p}$, the $(n+p)$-th component of $M(f)$. Even more explicitly, if $x \in M(B)[n]^p = M(B)^{n+p}$, then

$$M(f)[n](x) = (-1)^{n|f|}M(f)(x) \in M(A)[n]^{p+|f|} = M(A)^{n+p+|f|}.$$

We can check that $M[n]$ is indeed a (right) $\mathcal{A}$-dg-module.

More involved is the notion of *mapping cone*. It can be thought as a homological variant of both kernels and cokernels. The definition is as follows for chain complexes:

Definition 5.11. Let $f\colon V \to W$ be a chain map of chain complexes. The *mapping cone* $\mathrm{C}(f)$ of f is the complex defined as follows:

- $\mathrm{C}(f)^p = V^{p+1} \oplus W^p$.
- The differential $d^p_{\mathrm{C}(f)}\colon \mathrm{C}(f)^p \to \mathrm{C}(f)^{p+1}$ is given in matrix notation as follows:

$$d^p_{\mathrm{C}(f)} = \begin{pmatrix} -d_V^{p+1} & 0 \\ f^{p+1} & d^p_W \end{pmatrix}.$$

This definition can be directly generalized to dg-modules over dg-categories:

Definition 5.12. Let $\mathcal{A}$ be a dg-category and let $\varphi\colon M \to N$ be a closed degree 0 morphism of (right) $\mathcal{A}$-dg-modules. In particular, $\varphi_A\colon M(A) \to N(A)$ is a chain map of complexes for all $A \in \mathrm{Ob}(\mathcal{A})$. We define the *mapping cone* $\mathrm{C}(\varphi)$ of φ as follows:

$$\mathrm{C}(\varphi)(A) = \mathrm{C}(\varphi_A),$$

for all $A \in \mathrm{Ob}(\mathcal{A})$, where $\mathrm{C}(\varphi_A)$ is the mapping cone of the chain map of complexes φ_A. If $f\colon A \to B$ is a morphism of degree p in $\mathcal{A}$, we define

$$\mathrm{C}(\varphi)(f)\colon \mathrm{C}(\varphi_B) \to \mathrm{C}(\varphi_A)$$

as follows:

$$\mathrm{C}(\varphi)(f)^n\colon M(B)^{n+1} \oplus N(B)^n \to M(A)^{n+p+1} \oplus N(A)^{n+p},$$
$$\mathrm{C}(\varphi)(f)^n = \begin{pmatrix} (-1)^{|f|}M(f)^{n+1} & 0 \\ 0 & N(f)^n \end{pmatrix}.$$

Using the right action notation, we have:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f = \begin{pmatrix} x_1 f \\ x_2 f \end{pmatrix}.$$

If the element $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ has degree n, it means that $|x_1| = n+1$ and $|x_2| = n$. We can check that $\mathrm{C}(\varphi)$ is indeed a right $\mathcal{A}$-dg-module: it is quite tedious but straightforward. To that purpose, the right action notation might be easier to work with. Sign conventions can be confusing: don't worry about them during your first reading.

Example 5.13. To better understand the mapping cone construction, let us look at an easy setup. Let $f\colon M \to N$ be a morphism of $\mathbf{k}$-modules, which we view as a closed degree 0 morphism of complexes concentrated in degree 0. What is the mapping cone of f? By definition, the only nonzero components are:

$$\mathrm{C}(f)^{-1} = M,$$
$$\mathrm{C}(f)^0 = N,$$

and the differential can be identified with the morphism f. Namely, $\mathrm{C}(f)$ is the complex:

$$\cdots 0 \to M \xrightarrow{f} N \to 0 \to \cdots ,$$

where M is in degree -1 and N is in degree 0.

What is the cohomology of this complex? Is it very easy to compute: the only nonzero components are:

$$H^{-1}(\mathrm{C}(f)) = \ker(f),$$
$$H^{0}(\mathrm{C}(f)) = N/\operatorname{Im}(f) = \operatorname{coker}(f).$$

Hence, the cohomology of $\mathrm{C}(f)$ contains the informations of both the kernel and cokernel of f.

Exercises to Lecture 5.

Exercise 5.1. Check that the definition of $h\colon \mathcal{A} \to \mathrm{dgm}(\mathcal{A})$ in Proposition 5.1 actually yields a dg-functor.

Exercise 5.2. Let $\mathcal{A}$ be a dg-category, or a $\mathbf{k}$-linear category, and let $f\colon A \to B$ be a closed degree 0 morphism in $\mathcal{A}$ (if $\mathcal{A}$ is $\mathbf{k}$-linear, just a morphism). We say that f is an *isomorphism* if there is a closed degree 0 morphism (just a morphism if $\mathcal{A}$ is $\mathbf{k}$-linear) $f'\colon B \to A$ such that $f' \circ f = 1_A$ and $f \circ f' = 1_B$.

- Prove that if $f\colon A \to B$ is an isomorphism, then there is a *unique* $f'\colon B \to A$ such that $f' \circ f = 1_A$ and $f \circ f' = 1_B$. This will be called the *inverse of f* and denoted by f^{-1}.
- In the case that $\mathcal{A}$ is a dg-category, prove that in the definition above we need not require f' to be a closed degree 0 morphism. Namely, f is an isomorphism if and only there is a morphism $f'\colon B \to A$ (which, in general, will be an element of $\mathcal{A}(B, A) = \oplus_i \mathcal{A}(B, A)^i$) such that $f' \circ f = 1_A$ and $f \circ f' = 1_B$.

Exercise 5.3. Let $\mathcal{A}$ be a dg-category. Attempt to define a $\mathbf{k}$-linear category $H^0(\mathcal{A})$ before we do that in a following lecture. *Hint:* we want to take zeroth cohomology; of course we can't (in general) do that on objects, but for any pair of objects A, B we have a complex $\mathcal{A}(A, B)$ and we can take cohomology. What about compositions then? And why restrict to zeroth cohomology?

If $f\colon A \to B$ is an isomorphism in $\mathcal{A}$ (see Exercise 5.2), does it perhaps induce (in what sense?) an isomorphism in $H^0(\mathcal{A})$?

Exercise 5.4. Let $\mathcal{A}$ be a dg-category and let $M, N \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$. Find an isomorphism of complexes

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N[k]) \cong \underline{\mathrm{Hom}}_{\mathcal{A}}(M, N)[k],$$

for $k \in \mathbb{Z}$. Recall that $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N)$ is the complex of morphisms of $\mathcal{A}$-dg-modules $M \to N$.

Exercise 5.5. Denote by S^n the complex of $\mathbf{k}$-modules having just the commutative ring $\mathbf{k}$ in degree n:

$$\cdots \to 0 \to \mathbf{k} \to 0 \to \cdots$$

Let $\mathcal{A}$ be a dg-category and let $M \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a right $\mathcal{A}$-dg-module. Define a new object $S^n \otimes M$ in $\mathrm{dgm}(\mathcal{A})$ by:

$$A \mapsto S^n \otimes M(A).$$

If $f\colon A \to B$ we set

$$S^n \otimes M(f)\colon S^n \otimes M(B) \to S^n \otimes M(A),$$
$$\lambda \otimes x \mapsto (-1)^{n|f|}\lambda \otimes M(f)(x),$$

taking care of the Koszul sign rule.

Prove that $S^n \otimes M$ is isomorphic (cf. Exercise 5.2) to the n-shift $M[n]$.

6. Lecture 6: towards the derived category

A crucial operation which can be performed on complex is taking cohomology. This can be extended to dg-categories themselves, and to dg-modules over dg-categories. In this lecture:

- To any dg-category $\mathcal{A}$ we will associate a **k**-linear category $H^0(\mathcal{A})$ obtained by taking zeroth cohomology on hom complexes. We will also define cohomology of dg-modules over a dg-category.
- We will see how mapping cones yield useful long exact sequences in cohomology.
- We will define *h-projective* dg-modules, which are a far-reaching generalization of projective modules. This will allow us to introduce h-projective resolutions and, finally, the derived category of a dg-category.

6.1. **Cohomology of dg-categories and dg-modules.** A dg-category $\mathcal{A}$ has objects and complexes of morphisms between pairs of objects. We can take cohomology of those complexes, and find a **k**-linear category.

Definition 6.1. Let $\mathcal{A}$ be a dg-category. We define a **k**-linear category $H^0(\mathcal{A})$, sometimes called the *homotopy category* of $\mathcal{A}$, or the *zeroth cohomology* of $\mathcal{A}$, as follows:

- Objects are the same as the objects of $\mathcal{A}$: $\mathrm{Ob}(H^0(\mathcal{A})) = \mathrm{Ob}(\mathcal{A})$.
- We define
$$H^0(\mathcal{A})(A, B) = H^0(\mathcal{A}(A, B)),$$
namely, the hom **k**-modules of $H^0(\mathcal{A})$ are obtained by taking zeroth cohomology of the hom complexes of $\mathcal{A}$.
- Compositions are induced from $\mathcal{A}$ by passing to zeroth cohomology:
$$H^0(\mathcal{A})(B, C) \otimes H^0(\mathcal{A})(A, B) \to H^0(\mathcal{A})(A, C),$$
$$[g] \otimes [f] \mapsto [g \circ f]$$
The identity morphism of an object A in $H^0(\mathcal{A})$ is given by the cohomology class $[1_A]$ of the identify of A in $\mathcal{A}$.

Using the graded Leibniz rule, we can check that the compositions are well defined (if f and g are closed and of degree 0, the same is true for $g \circ f$; the class $[g \circ f]$ does not depend on the choice of representatives of the classes $[f]$ and $[g]$), and $H^0(\mathcal{A})$ is indeed a **k**-linear category – see Exercise 6.1.

We may take H^0 also on dg-functors:

Definition 6.2. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a dg-functor between dg-categories. We define a **k**-linear functor:

$$H^0(F)\colon H^0(\mathcal{A}) \to H^0(\mathcal{B})$$

as follows:

- On objects, it is the same as F, namely: $H^0(F)(A) = F(A)$ for all $A \in \mathrm{Ob}(\mathcal{A})$.
- On morphisms, we take zeroth cohomology:
$$\begin{aligned} H^0(F) = H^0(F)_{A,B} = H^0(F_{A,B})\colon H^0(\mathcal{A}(A,B)) &\to H^0(\mathcal{B}(F(A),F(B))), \\ [f] &\mapsto [F(f)]. \end{aligned}$$

Remark 6.3. You might wonder: why do we take zeroth cohomology H^0 and not H^1, H^2 and so on? The reason is quite trivial: composing degree 0 morphisms yield a degree 0 morphism, but composing degree k morphisms for $k \neq 0$ yields a degree $2k$ morphism, and $k \neq 2k$.

What we can do is to introduce a new notion of *graded category*, namely, something which is similar to a dg-category but *without differentials*: morphisms between object form a *graded module*. Then, to any dg-category $\mathcal{A}$ we can associate its *graded cohomology* $H^*(\mathcal{A})$: its objects are the same of $\mathcal{A}$, and morphisms are given by graded cohomologies $H^*(\mathcal{A}(A,B))$ (see also Remark 1.11). Compositions are induced from $\mathcal{A}$. Notice that units $[1_A]$ will be degree 0 morphisms in $H^0(\mathcal{A})$. Moreover, if $F\colon \mathcal{A} \to \mathcal{B}$ is a dg-functor, we also obtain an induced *graded functor* $H^*(F)\colon H^*(\mathcal{A}) \to H^*(\mathcal{B})$. If you want to fill in the details of this construction, do Exercise 6.2

Example 6.4. Let us take the dg-category $\mathrm{dgm}(\mathbf{k})$ of chain complexes. What is $H^0(\mathrm{dgm}(\mathbf{k}))$?
- Objects are chain complexes $V, W, \dots$
- Morphisms from V to W are given by $H^0(\underline{\mathrm{Hom}}(V,W))$. An element $[f] \in H^0(\underline{\mathrm{Hom}}(V,W))$ are *chain homotopy classes of chain maps*.

The abstract machinery automatically yields the result that compositions of chain maps are compatible with chain homotopies. The category $H^0(\mathrm{dgm}(\mathbf{k}))$ is also called the *homotopy category of complexes*.

We now define cohomology for dg-modules over a dg-category. We follow the usual philosophy and define it objectwise:

Definition 6.5. Let $\mathcal{A}$ be a dg-category, and let $M \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a (right) $\mathcal{A}$-dg-module. The *n-th cohomology* $H^n(M)$ of M is a right $H^0(\mathcal{A})$-module (namely, an object of $\mathrm{Mod}(H^0(\mathcal{A})) = \mathrm{Fun}(H^0(\mathcal{A})^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k}))$), defined as follows:
- On objects, we define: $H^n(M)(A) = H^n(M(A))$, taking the n-th cohomology of the complex $M(A)$.
- Let $[f]\colon A \to B$ be a morphism in $H^0(\mathcal{A})$, namely, a cohomology class of some closed degree 0 morphism $f\colon A \to B$ in $\mathcal{A}$. We define
$$\begin{aligned} H^n(M)([f])\colon H^n((M(B)) &\to H^n(M(A)), \\ [x] &\mapsto [M(f)(x)] = [xf]. \end{aligned}$$

Using the graded Leibniz rule for the dg-module M, we can check that $H^n(M)([f])$ is well defined and depends only on the cohomology class of f. There are no signs in the definition because $|f| = 0$.

Remark 6.6. Recall the notion of *shift* of a dg-module (Definition 5.10). We can immediately see that:
$$H^n(M) = H^0(M[n]).$$

Cohomology of dg-modules can itself be upgraded to a suitable ($\mathbf{k}$-linear) functor. This requires a little notational care:

Proposition 6.7. *Let $\mathcal{A}$ be a dg-category and let $n \in \mathbb{Z}$. There is a* $\mathbf{k}$*-linear functor*

$$\begin{aligned} H^n(-) \colon H^0(\mathrm{dgm}(\mathcal{A})) &\to \mathrm{Mod}(H^0(\mathcal{A})), \\ M &\mapsto H^n(M), \end{aligned}$$

which is defined as follows on morphisms. Take $[\varphi] \colon M \to N$ to be a cohomology class of a closed degree 0 *morphism $\varphi \colon M \to N$ in* $\mathrm{dgm}(\mathcal{A})$*. We map this to a morphism*

$$\begin{gathered} H^n(\varphi) \colon H^n(M) \to H^n(N), \\ H^n(\varphi)_A = H^n(\varphi_A) \colon H^n(M(A)) \to H^n(N(A)), \end{gathered}$$

where $H^n(\varphi_A)$ is the morphism induced on cohomology by the chain map $\varphi_A \colon M(A) \to N(A)$. We can check that everything is well defined and $H^n(-)$ is indeed a $\mathbf{k}$*-linear functor.*

6.2. **Pretriangles and cohomological long exact sequences.** Let $\mathcal{A}$ be a dg-category. Recall the definition of *mapping cone* $\mathrm{C}(\varphi)$ of a closed degree 0 morphism $\varphi \colon M \to N$ in $\mathrm{dgm}(\mathcal{A})$ (Definition 5.12). There are natural morphisms:

$$\begin{aligned} j &\colon N \to \mathrm{C}(\varphi), \\ p &\colon \mathrm{C}(\varphi) \to M[1], \end{aligned}$$

defined respectively as the "inclusion of the second summand" and the "projection on the first summand":

$$\begin{aligned} j^n_A &\colon N^n \to \mathrm{C}(\varphi)^n = M^{n+1} \oplus N^n, \\ p^n_A &\colon \mathrm{C}(\varphi)^n = M^{n+1} \oplus N^n \to M^{n+1} = M[1]^n, \end{aligned}$$

for $A \in \mathrm{Ob}(\mathcal{A})$. We can check that j and p are closed degree 0 morphisms. To check this, it is convenient to use matrix notation. Namely, write:

$$j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad p = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then, for example, we can compute:

$$(d j_A)^n = \begin{pmatrix} -d_M^{n+1} & 0 \\ f^{n+1} & d_N^n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} d_N^n = 0,$$

for $A \in \mathrm{Ob}(\mathcal{A})$. The computation for dp is similar.

Remark 6.8. We have also degree 0 morphisms

$$\begin{aligned} i &\colon M \to \mathrm{C}(\varphi), \\ s &\colon \mathrm{C}(\varphi) \to N, \end{aligned}$$

defined respectively as the "inclusion of the first summand" and the "projection on the second summand". They are *not* closed morphisms. As an exercise, try to compute their differentials (Exercise 6.4)!

Definition 6.9. With the morphisms j and p, we can draw a sequence:

$$M \xrightarrow{\varphi} N \xrightarrow{j} \mathrm{C}(\varphi) \xrightarrow{p} M[1].$$

Such a sequence is called the *pretriangle associated to* φ, for reasons which will be clearer later on.

Remark 6.10. There is also a variant of the pretriangle associated to φ, which looks like:

$$N[-1] \xrightarrow{j} \mathrm{C}(\varphi)[-1] \xrightarrow{p} M \xrightarrow{\varphi} N.$$

We will sometimes call this the *rotated pretriangle associated to* φ. The closed degree 0 morphisms j and p above denote again (respectively) the natural inclusion of the second summand and the projection on the first summand, just taking into account the shift.

Pretriangles are in a sense a generalization of short exact sequences of complexes. Indeed, an important and very useful result tells that for any pretriangle there is an induced long exact sequence in cohomology. We will not prove that, but we will use it extensively:

Black Box 6.11. *Let $\mathcal{A}$ be a dg-category. Let $\varphi\colon M \to N$ be a closed degree 0 morphism in* $\mathrm{dgm}(\mathcal{A})$*, and let*

$$M \xrightarrow{\varphi} N \xrightarrow{j} \mathrm{C}(\varphi) \xrightarrow{p} M[1]$$

be the associated pretriangle. Then, there is a long exact sequence in cohomology:

$$\cdots \to H^n(M) \xrightarrow{H^n(\varphi)} H^n(N) \xrightarrow{H^n(j)} H^n(\mathrm{C}(\varphi)) \xrightarrow{H^n(p)} H^{n+1}(M) \xrightarrow{H^{n+1}(f)} H^{n+1}(N) \to \cdots$$

Analogously, let

$$N[-1] \xrightarrow{j} \mathrm{C}(\varphi)[-1] \xrightarrow{p} M \xrightarrow{\varphi} N$$

be the associated rotated pretriangle. Then, there is a long exact sequence in cohomology (essentially the same as the one above):

$$\cdots \to H^{n-1}(N) \xrightarrow{H^{n-1}(j)} H^{n-1}(\mathrm{C}(\varphi)) \xrightarrow{H^{n-1}(p)} H^n(M) \xrightarrow{H^n(\varphi)} H^n(M) \xrightarrow{H^n(j)} H^n(\mathrm{C}(\varphi)) \to \cdots$$

Later on, we will concentrate more deeply on these pretriangles. For instance, we will be able to define them inside more arbitrary dg-categories and not only for dg-modules, and we will see that they are preserved by dg-functors. This last property actually implies the following result, which for now we take without proof:

Black Box 6.12. *Let $\mathcal{A}$ be a dg-category, let $X \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$, and let*

$$M \xrightarrow{\varphi} N \to \mathrm{C}(\varphi) \to M[1]$$

be a pretriangle. Then, taking $\underline{\mathrm{Hom}}_{\mathcal{A}}(X,-)$ we get an induced pretriangle in $\mathrm{dgm}(\mathbf{k})$*:*

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(X,M) \xrightarrow{\varphi_*} \underline{\mathrm{Hom}}_{\mathcal{A}}(X,N) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(X,\mathrm{C}(\varphi)) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(X,M[1]), \qquad (*1)$$

in the sense that we may identify

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(X,M[1]) \cong \underline{\mathrm{Hom}}_{\mathcal{A}}(X,M)[1],$$
$$\underline{\mathrm{Hom}}_{\mathcal{A}}(X,\mathrm{C}(\varphi)) \cong \mathrm{C}(\underline{\mathrm{Hom}}_{\mathcal{A}}(X,M) \xrightarrow{\varphi_*} \underline{\mathrm{Hom}}_{\mathcal{A}}(X,N)).$$

Dually, taking $\underline{\mathrm{Hom}}_{\mathcal{A}}(-,X)$ we get an induced (rotated) pretriangle in $\mathrm{dgm}(\mathbf{k})$*:*

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M[1],X) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi),X) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(N,X) \xrightarrow{\varphi^*} \underline{\mathrm{Hom}}_{\mathcal{A}}(M,X), \qquad (*2)$$

in the sense that we may identify

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M[1],X) \cong \underline{\mathrm{Hom}}_{\mathcal{A}}(M,X)[-1],$$
$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi),X) \cong \mathrm{C}(\underline{\mathrm{Hom}}_{\mathcal{A}}(N,X) \xrightarrow{\varphi^*} \underline{\mathrm{Hom}}_{\mathcal{A}}(M,X))[-1].$$

Both pretriangles $(*1)$ *and* $(*2)$ *actually yield pretriangles in the dg-category of respectively right or left* $\mathrm{dgm}(\mathcal{A})$*-dg-modules:*

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(-, M) \xrightarrow{\varphi_*} \underline{\mathrm{Hom}}_{\mathcal{A}}(-, N) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(-, \mathrm{C}(\varphi)) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(-, M[1]),$$

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M[1], -) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), -) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(N, -) \xrightarrow{\varphi^*} \underline{\mathrm{Hom}}_{\mathcal{A}}(M, -).$$

6.3. **h-projectives and the derived category.** With the usual "objectwise" philosophy, we can define quasi-isomorphisms of dg-modules:

Definition 6.13. Let $\mathcal{A}$ be a dg-category and let $\varphi \colon M \to N$ be a closed degree 0 morphism in $\mathrm{dgm}(\mathcal{A})$. We say that φ is a *quasi-isomorphism* if $\varphi_A \colon M(A) \to N(A)$ is a quasi-isomorphism of complexes for any object $A \in \mathrm{Ob}(\mathcal{A})$, namely, if $H^n(\varphi_A) \colon H^n(M(A)) \to H^n(N(A))$ is an isomorphism for all $n \in \mathbb{Z}$.

Remark 6.14. Being a quasi-isomorphism actually depends only on the cohomology class of the given closed degree 0 morphism. More precisely: assume that $\varphi \colon M \to N$ is a quasi-isomorphism, and that $[\varphi] = [\varphi']$ in $H^0(\mathrm{dgm}(\mathcal{A})$. Then, φ' is also a quasi-isomorphism. This follows from the fact that taking n-th cohomology $H^n(-)$ is a $\mathbf{k}$-linear functor:

$$H^n(-) \colon H^0(\mathrm{dgm}(\mathcal{A})) \to \mathrm{Mod}(H^0(\mathcal{A})),$$

cf. Proposition 6.7.

Hence, we may also say "$[\varphi]$ is a quasi-isomorphism", referring to the cohomology class $[\varphi]$ in $H^0(\mathrm{dgm}(\mathcal{A}))$.

Another easy definition is that of acyclic dg-modules:

Definition 6.15. Let $\mathcal{A}$ be a dg-category and let $M \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a right $\mathcal{A}$-dg-module. We say that M is *acyclic* if it has zero cohomology, namely:

$$H^n(M(A)) \cong 0,$$

for all $n \in \mathbb{Z}$ and all $A \in \mathrm{Ob}(\mathcal{A})$.

Remark 6.16. Saying that M is acyclic as in the above definition is the same as saying that the zero morphism

$$0 \to M$$

is a quasi-isomorphism. Moreover, it is immediate to see that

$$\mathrm{C}(0 \to X) = X,$$

hence we obtain a pretriangle

$$0 \to X \xrightarrow{1_X} X \to 0.$$

Quasi-isomorphisms have a nice characterization in terms of their mapping cones:

Lemma 6.17. *Let* $\mathcal{A}$ *be a dg-category, and let* $\varphi \colon M \to N$ *be a closed degree* 0 *morphism in* $\mathrm{dgm}(\mathcal{A})$*. Then,* φ *is a quasi-isomorphism if and only if its cone* $\mathrm{C}(\varphi)$ *is acyclic.*

Proof. Take the pretriangle associated to φ:

$$M \xrightarrow{\varphi} N \to \mathrm{C}(\varphi) \to M[1]$$

and then the long exact sequence in cohomology (cf. Black Box 6.11):

$$\cdots \to H^n(M) \xrightarrow{H^n(\varphi)} H^n(N) \xrightarrow{H^n(j)} H^n(\mathrm{C}(\varphi)) \xrightarrow{H^n(p)} H^{n+1}(M) \xrightarrow{H^{n+1}(f)} H^{n+1}(N) \to \cdots$$

Using that, we conclude immediately. □

We may finally define the crucial notion of h-projective dg-modules:

Definition 6.18. Let $\mathcal{A}$ be a dg-category. Let $P \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a right $\mathcal{A}$-dg-module. We say that P is *h-projective* if for any acyclic dg-module $X \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$, the complex

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(P, X)$$

is also acyclic.

We can immediately prove an easy characterization:

Proposition 6.19. *Let $\mathcal{A}$ be a dg-category and let $P \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a right $\mathcal{A}$-dg-module. Then, P is h-projective if and only if for any quasi-isomorphism $\varphi\colon M \to N$, the induced chain map*

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(-, \varphi) = \varphi_* \colon \underline{\mathrm{Hom}}_{\mathcal{A}}(P, M) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(P, N)$$

is also a quasi-isomorphism.

Proof. Assume that P is h-projective, and let $\varphi\colon M \to N$ be a quasi-isomorphism. By Lemma 6.17, we know that $\mathrm{C}(\varphi)$ is acyclic. Moreover, using Black Box 6.12, we have a pretriangle:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(P, M) \xrightarrow{\varphi_*} \underline{\mathrm{Hom}}_{\mathcal{A}}(P, N) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(P, \mathrm{C}(\varphi)) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(X, M[1]).$$

By assumption, $\underline{\mathrm{Hom}}_{\mathcal{A}}(P, \mathrm{C}(\varphi))$ is acyclic; hence, by Lemma 6.17, φ_* is a quasi-isomorphism.

Let us prove the other implication. Let X be any acyclic dg-module in $\mathrm{dgm}(\mathcal{A})$. Recalling Remark 6.16, this means that the zero morphism

$$0 \to X$$

is a quasi-isomorphism. Moreover, we have a pretriangle:

$$0 \to X \xrightarrow{1_X} X \to 0.$$

Applying $\underline{\mathrm{Hom}}_{\mathcal{A}}(P, -)$ we get the following pretriangle:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(P, 0) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(P, X) \xrightarrow{\mathrm{id}} \underline{\mathrm{Hom}}_{\mathcal{A}}(P, X) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(P, 0).$$

By assumption, $\underline{\mathrm{Hom}}_{\mathcal{A}}(P, 0) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(P, X)$ is a quasi-isomorphism, hence $\underline{\mathrm{Hom}}_{\mathcal{A}}(P, X)$ is acyclic, thanks to Lemma 6.17. □

The above characterization tells us that an h-projective dg-module P "does not see the difference" between quasi-isomorphic dg-modules, when taking $\mathrm{Hom}_{\mathcal{A}}(P, -)$. Recalling Example 1.19, we know that *in general, quasi-isomorphisms do not have inverses*. We will see later on that restricting to h-projective dg-modules actually solves this issue. Finally, we end this lecture with the definition of the derived category. This will be further motivated later on.

Definition 6.20. Let $\mathcal{A}$ be a dg-category. The *derived dg-category* $\mathsf{D}_{\mathrm{dg}}(\mathcal{A})$ of $\mathcal{A}$ is the dg-category defined as follows:

- Its objects are the h-projective right $\mathcal{A}$-dg-modules.

- For any pair of h-projective dg-modules, the hom complex is simply given by
$$\mathsf{D}_{\mathrm{dg}}(\mathcal{A})(P, Q) = \underline{\mathrm{Hom}}_{\mathcal{A}}(P, Q),$$
with compositions and identities induced directly from $\mathrm{dgm}(\mathcal{A})$.

In other words, $\mathsf{D}_{\mathrm{dg}}(\mathcal{A})$ is the full dg-subcategory of $\mathrm{dgm}(\mathcal{A})$ spanned by the h-projective dg-modules, cf Definition 5.5.

The *derived category* $\mathsf{D}(\mathcal{A})$ of $\mathcal{A}$ is defined by
$$\mathsf{D}(\mathcal{A}) = H^0(\mathsf{D}_{\mathrm{dg}}(\mathcal{A})).$$

Exercises to Lecture 6.

Exercise 6.1. Let $\mathcal{A}$ be a dg-category. Check that the compositions in the zeroth cohomology $H^0(\mathcal{A})$ (cf. Definition 6.1) are well defined.

Exercise 6.2. Refer to Remark 6.3. Define the notion of *graded category*, and for a given dg-category $\mathcal{A}$, define its graded cohomology category $H^*(\mathcal{A})$. Moreover, define the notion of *graded functor* between graded categories, and for a dg-functor $F\colon \mathcal{A} \to \mathcal{B}$ define the induced graded functor $H^*(F)\colon H^*(\mathcal{A}) \to H^*(\mathcal{A})$.

Exercise 6.3. Prove that the cohomology $\mathbf{k}$-linear functors $H^k(-)\colon H^0(\mathrm{dgm}(\mathcal{A})) \to \mathrm{Mod}(H^0(\mathcal{A}))$ (see Proposition 6.7) are indeed well defined.

Exercise 6.4. Refer to Remark 6.8. Compute di and ds.

7. Lecture 7: a digression on isomorphisms in dg-categories

Before concentrating on the derived category, we need to better understand the concept of *isomorphism* inside a dg-category. It turns out that we actually have at least two notions: isomorphisms and *homotopy equivalences*. We will concentrate on these notions in the specific case of dg-modules, also comparing them with the quasi-isomorphisms.

7.1. **Isomorphisms in dg-categories.**

Inside a dg-category we have two possible notions of isomorphism. One involves closed degree 0 morphisms in $\mathcal{A}$, the other involves their cohomology classes in $H^0(\mathcal{A})$:

Definition 7.1. Let $\mathcal{A}$ be a dg-category, and let $f\colon A \to B$ be a closed degree 0 morphism in $\mathcal{A}$. We say that f is an *isomorphism* if there exists a closed degree 0 morphism $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. We say that f is a *homotopy equivalence* if there exists a closed degree 0 morphism $g\colon B \to A$ such that $[g \circ f] = [1_A]$ and $[f \circ g] = [1_B]$ in $H^0(\mathcal{A})$. A homotopy equivalence f is also called just an *isomorphism in* $H^0(\mathcal{A})$ (more precisely, we should say that $[f]$ is an isomorphism in $H^0(\mathcal{A})$).

Let A, B be objects in $\mathcal{A}$. We say that A and B are *isomorphic* – and write $A \cong B$ – if there is an isomorphism $A \to B$. We say that A and B are *homotopy equivalent* (or *isomorphic in* $H^0(\mathcal{A})$) – and write $A \approx B$ (or $A \cong B$ *in* $H^0(\mathcal{A})$) – if there is a homotopy equivalence $A \to B$.

Remark 7.2. We can show, with a standard , that if $f\colon A \to B$ is an isomorphism in $\mathcal{A}$ (respectively an isomorphism in $H^0(\mathcal{A})$) then the inverse morphism $g\colon B \to A$ is uniquely determined (respectively, its cohomology class is uniquely determined in $H^0(\mathcal{A})$). We will write $f^{-1}\colon B \to A$ for the inverse of f in $\mathcal{A}$, and $[f]^{-1}\colon B \to A$ for the inverse of $[f]$ in $H^0(\mathcal{A})$.

Moreover, we can show that both isomorphisms relations introduced above on $\mathrm{Ob}(\mathcal{A})$ are equivalence relations. More explicitly, we have:

- For any object $A \in \mathrm{Ob}(\mathcal{A})$, the identity morphism 1_A is an isomorphism in $\mathcal{A}$, and its class $[1_A]$ is an isomorphism in $H^0(\mathcal{A})$.
- If $f: A \to B$ is an isomorphism, then its inverse $f^{-1}: B \to A$ is an isomorphism. If $[f]$ is an isomorphism in $H^0(\mathcal{A})$, then its inverse $[f]^{-1}$ is an isomorphism in $H^0(\mathcal{A})$.
- If $f: A \to B$ and $f': B \to C$ are isomorphisms in $\mathcal{A}$, then its composition $f' \circ f: A \to C$ is an isomorphism in $\mathcal{A}$. An analogous conclusion holds for isomorphisms in $H^0(\mathcal{A})$.

Isomorphisms in $\mathcal{A}$ and $H^0(\mathcal{A})$ also satisfy the so-called "two-out-of-three" property, see Exercise 7.1.

Finally, it is clear that if f is an isomorphism in $\mathcal{A}$, it is also an isomorphism in $H^0(\mathcal{A})$ (namely, a homotopy equivalence).

The Yoneda embedding yields a useful characterization of isomorphisms and homotopy equivalences:

Lemma 7.3. *Let $\mathcal{A}$ be a dg-category, and let $f: A \to B$ be a closed degree 0 morphism. Then, f is an isomorphism if and only if the induced morphism via the Yoneda embedding $\mathcal{A} \to \mathrm{dgm}(\mathcal{A})$:*

$$\mathcal{A}(-, f): \mathcal{A}(-, A) \to \mathcal{A}(-, B),$$

is an isomorphism in $\mathrm{dgm}(\mathcal{A})$.

Analogously, f is a homotopy equivalence if and only if the induced morphism

$$\mathcal{A}(-, f): \mathcal{A}(-, A) \to \mathcal{A}(-, B)$$

is a homotopy equivalence in $\mathrm{dgm}(\mathcal{A})$.

The proof follows from a more general result:

Proposition 7.4. *Let $F: \mathcal{A} \to \mathcal{B}$ be a dg-functor, and let $f: A \to B$ be a closed degree 0 morphism.*

If f is an isomorphism (respectively a homotopy equivalence), then $F(f)$ is an isomorphism (respectively a homotopy equivalence).

If in addition F is fully faithful (Definition 5.2), f is an isomorphism (respectively a homotopy equivalence) if and only if $F(f): F(A) \to F(B)$ is an isomorphism (respectively a homotopy equivalence).

Proof. Left to the reader! (see Exercise 7.3). □

7.2. **Isomorphisms in the dg-category of dg-modules.** We now investigate isomorphisms and homotopy equivalences in the specific case of dg-modules.

First, we check that isomorphisms of dg-modules are understood termwise:

Lemma 7.5. *Let $\mathcal{A}$ be a dg-category, and let $\varphi: M \to N$ be a closed degree 0 morphism in* $\mathrm{dgm}(\mathcal{A})$. *Then, φ is an isomorphism if and only if $\varphi_A: M(A) \to N(A)$ is an isomorphism of complexes, for all $A \in \mathrm{Ob}(\mathcal{A})$.*

Proof. Suppose that φ is an isomorphism. Then, we have a (unique) closed degree 0 morphism $\varphi^{-1}: N \to M$ such that $\varphi^{-1} \circ \varphi = 1_M$ and $\varphi \circ \varphi^{-1} = 1_N$. In particular, $\varphi_A^{-1} \circ \varphi_A = 1_{M(A)}$ and $\varphi_A \circ \varphi_A^{-1} = 1_{N(A)}$ for all $A \in \mathrm{Ob}(\mathcal{A})$, namely, φ_A is an isomorphism of complexes for all $A \in \mathrm{Ob}(\mathcal{A})$.

Conversely, suppose that φ_A is an isomorphism for all $A \in \mathrm{Ob}(\mathcal{A})$. Let $\psi_A: N(A) \to M(A)$ be the inverse of φ_A. We just need to show that the family $A \mapsto \psi_A$ is actually a closed degree 0 morphism of dg-modules $N \to M$, namely, a natural transformation $N \to M : \mathcal{A}^{\mathrm{op}} \to \mathrm{dgm}(\mathbf{k})$. We need to check that:

$$\psi_A \circ N(f) = M(f) \circ \psi_B,$$

for any morphism $f : A \to B$. This easily follows by composing on the left with φ_A and on the right with φ_B, and the fact that φ is a natural transformation. Finally, degrees and differentials for morphisms of dg-modules are described objectwise, so we conclude that ψ is indeed a closed degree 0 morphism $N \to M$. □

Remark 7.6. A question may arise: is it true that a closed degree 0 morphism $\varphi : M \to N$ is a homotopy equivalence if and only if $\varphi_A : M(A) \to N(A)$ is a homotopy equivalence in $\mathrm{dgm}(\mathbf{k})$, for all $A \in \mathrm{Ob}(\mathcal{A})$?

If $\varphi : M \to N$ is a homotopy equivalence, then its "homotopy inverse" $[\psi] : N \to M$ yields homotopy inverses $[\psi_A] : N(A) \to M(A)$ to $[\varphi_A]$. When trying to prove the converse statement, there is an issue: we can certainly choose closed degree 0 morphisms $\psi_A : N(A) \to M(A)$ such that $[\psi_A \circ \varphi_A] = [1_{M(A)}]$ and $[\varphi_A \circ \psi_A] = 1_{N(A)}$ for all $A \in \mathrm{Ob}(\mathcal{A})$, but how can we prove that $A \mapsto \psi_A$ yields a natural transformation $\psi : N \to M$? We don't have the assumptions to check the relation

$$\psi_A \circ N(f) = M(f) \circ \psi_B,$$

for $f : A \to B$. We will be able to prove equality of cohomology classes, even with the additional assumption that $f : A \to B$ is a closed degree 0 morphism:

$$[\psi_A \circ N(f)] = [M(f) \circ \psi_B],$$

but this certainly not enough! Not everything works when we take cohomology!

We already encountered the notion of quasi-isomorphism between dg-modules (Definition 6.13). It turns out that it is the "weakest" form of isomorphism for dg-modules.

Lemma 7.7. *Let $\mathcal{A}$ be a dg-category, and let $\varphi : M \to N$ be a homotopy equivalence in* $\mathrm{dgm}(\mathcal{A})$. *Then, it is a quasi-isomorphism.*

Proof. Let $\psi : N \to M$ be a "homotopy inverse" of φ, namely, we have:

$$[\psi] \circ [\varphi] = [1_M],$$
$$[\varphi] \circ [\psi] = [1_N].$$

We may take (degree $k \in \mathbb{Z}$) cohomology, which is a well-defined $\mathbf{k}$-linear functor $H^0(\mathrm{dgm}(\mathcal{A})) \to \mathrm{Mod}(H^0(\mathcal{A}))$ (cf. Proposition 6.7):

$$H^k(\psi) \circ H^k(\varphi) = H^k(1_M) = 1_{H^k(M)},$$
$$H^k(\varphi) \circ H^k(\psi) = H^k(1_N) = 1_{H^k(N)}.$$

We conclude that $H^k(\varphi) : H^k(M) \to H^k(N)$ is an isomorphism for all $k \in \mathbb{Z}$, which is what we wanted. □

We can characterize homotopy equivalences in the dg-category $\mathrm{dgm}(\mathcal{A})$ of right $\mathcal{A}$-dg-modules in terms of a suitable "vanishing" of their mapping cones. We first need an easy lemma:

Lemma 7.8. *Let $\mathcal{A}$ be a dg-category, and let $M \in$* $\mathrm{dgm}(\mathcal{A})$ *be a right $\mathcal{A}$-dg-module. Then, the following are equivalent:*

(1) $M \cong 0$ *(namely, M is isomorphic to* 0*), respectively* $M \approx 0$ *(namely, M is homotopy equivalent to* 0*).*
(2) $1_M = 0$*, respectively* $[1_M] = [0]$.
(3) $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, M) \cong 0$*, respectively* $H^*(\underline{\mathrm{Hom}}_{\mathcal{A}}(M, M)) \cong 0$.
(4) *For any* $N \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$*, we have* $\underline{\mathrm{Hom}}_{\mathcal{A}}(N, M) \cong 0$*, respectively* $H^*(\underline{\mathrm{Hom}}_{\mathcal{A}}(N, M)) \cong 0$.
(5) *For any* $N \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$*, we have* $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N) \cong 0$*, respectively* $H^*(\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N)) \cong 0$.

Proof. (1) $\Rightarrow$ (2). Assume that $M \cong 0$. This means that the (unique) zero morphism $0 \colon 0 \to M$ is an isomorphism. In particular, its inverse is $0 \colon M \to 0$ and the composition $0 \colon M \to 0 \to M$ is necessarily 1_M. By applying the same argument to cohomology classes in $H^0(\mathrm{dgm}(\mathcal{A}))$, we conclude that $M \approx 0$ implies $[1_M] = [0]$.

(2) $\Rightarrow$ (1). Conversely, assume that $1_M = 0$. The composition $0 \to M \to 0$ is the identity on the zero dg-module, and the composition $M \to 0 \to M$ is the zero morphism, which by assumption is 1_M. This means that $0 \to M$ and $M \to 0$ are inverse to each other, and $M \cong 0$. With a similar argument with cohomology classes in $H^0(\mathrm{dgm}(\mathcal{A}))$ we can prove that $[1_M] = [0]$ implies $M \approx 0$.

(3) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Let $f \colon M \to M$ be any element in $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, M)$. Then $f = f \circ 1_M = 0$ if $1_M = 0$. Similarly, let $[f] \in H^p(\underline{\mathrm{Hom}}_{\mathcal{A}}(M, M)$. Then, $[f] = [f \circ 1_M] = [0]$ if $[1_M] = [0]$, with a direct application of the graded Leibniz rule.

(2) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (5) are proven with similar arguments as (2) $\Leftrightarrow$ (3). □

Proposition 7.9. *Let $\mathcal{A}$ be a dg-category. Let $\varphi \colon M \to N$ be a closed degree* 0 *morphism in* $\mathrm{dgm}(\mathcal{A})$*. Then, φ is a homotopy equivalence if and only if its mapping cone is homotopy equivalent to* 0*:* $\mathrm{C}(\varphi) \approx 0$.

Proof. Suppose that φ is a homotopy equivalence. We have a pretriangle in the dg-category of right $\mathrm{dgm}(\mathcal{A})$-dg-modules:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(-, M) \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{A}}(-,\varphi)} \underline{\mathrm{Hom}}_{\mathcal{A}}(-, N) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(-, \mathrm{C}(\varphi)) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(-, M[1]).$$

By Lemma 7.3, we know that $\underline{\mathrm{Hom}}_{\mathcal{A}}(-, \varphi)$ is a homotopy equivalence. In particular (Lemma 7.7) it is a quasi-isomorphism. This means (Lemma 6.17) that its cone $\underline{\mathrm{Hom}}_{\mathcal{A}}(-, \mathrm{C}(\varphi))$ is acyclic. In particular, we have that

$$H^*(\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), \mathrm{C}(\varphi))) \cong 0.$$

Using Lemma 7.8, we conclude that $\mathrm{C}(\varphi)$ is homotopy equivalent to 0.

Conversely, suppose that $\mathrm{C}(\varphi) \approx 0$. Consider the following pretriangle in $\mathrm{dgm}(\mathbf{k})$:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(N, M) \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{A}}(N,\varphi)} \underline{\mathrm{Hom}}_{\mathcal{A}}(N, N) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(N, \mathrm{C}(\varphi)) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(N, M[1]).$$

We take the induced long exact sequence in cohomology and use that $H^*(\underline{\mathrm{Hom}}_{\mathcal{A}}(N, \mathrm{C}(\varphi))) \cong 0$ by assumption (and recalling Lemma 7.8), finding an exact sequence:

$$H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(N, M)) \xrightarrow{H^0(\mathrm{Hom}_{\mathcal{A}}(N,\varphi))} H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(N, N)) \to 0.$$

This means that $H^0(\mathrm{Hom}_{\mathcal{A}}(N, \varphi))$ is surjective. Hence, we can find a morphism $[\psi] \colon N \to M$ in $H^0(\mathrm{dgm}(\mathcal{A}))$ such that

$$[\varphi] \circ [\psi] = [1_N].$$

On the other hand, we have a (rotated) pretriangle:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M[1],-) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi),-) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(N,-) \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{A}}(\varphi,-)} \underline{\mathrm{Hom}}_{\mathcal{A}}(M,-),$$

obtained by applying the "contravariant Yoneda dg-functor" $X \mapsto \underline{\mathrm{Hom}}_{\mathcal{A}}(X,-)$. Taking the long exact sequence in cohomology and using that $H^*(\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi),M)) \cong 0$, we find that the morphism

$$H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(N,M) =\xrightarrow{H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(\varphi,-))} H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(M,M))$$

is surjective. In particular, we can find a morphism $[\psi'] \colon N \to M$ in $H^0(\mathrm{dgm}(\mathcal{A}))$ such that

$$[\psi'] \circ [\varphi] = [1_M].$$

We found both a left and a right inverse to $[\varphi]$ in $H^0(\mathrm{dgm}(\mathcal{A}))$. With an abstract general argument (see Exercise 7.4), we conclude that actually $[\psi] = [\psi']$ is the inverse of $[\varphi]$, which is then an isomorphism in $H^0(\mathrm{dgm}(\mathcal{A}))$ as we wanted. □

Exercises to Lecture 7.

Exercise 7.1. Let $\mathcal{A}$ be a dg-category. Let $f \colon A \to B$ and $g \colon B \to C$ be closed degree 0 morphisms. Assume that two out of $f, g, g \circ f$ is an isomorphism (respectively an isomorphism in $H^0(\mathcal{A})$). Then, this is also true for the third one.

Exercise 7.2. Let $\mathcal{A}$ be a dg-category. Let $\varphi \colon M \to N$ and $\psi \colon N \to P$ be closed degree 0 morphisms in $\mathrm{dgm}(\mathcal{A})$. Assume that two out of $\varphi, \psi, \psi \circ \varphi$ is a quasi-isomorphism. Then, the third is also a quasi-isomorphism.

Exercise 7.3. Prove Proposition 7.4.

Exercise 7.4. Let $\mathcal{A}$ be a dg-category, and let $f \colon A \to B$ be a closed degree 0 morphism. Assume that it has both a left and a right inverse, namely: there exist closed degree 0 morphisms $g, g' \colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g' = 1_B$. Prove that $g = g'$ and it is the inverse f^{-1} of f, which is then an isomorphism.

Prove the analogous claim for homotopy equivalences: if there exist closed degree 0 morphisms $g, g' \colon B \to A$ such that $[g] \circ [f] = [1_A]$ and $[f] \circ [g'] = [1_B]$ in $H^0(\mathcal{A})$, then $[g] = [g']$ and it is the homotopy inverse $[f]^{-1}$ of $[f]$ in $H^0(\mathcal{A})$. In particular, f is a homotopy equivalence.

8. Lecture 8: resolutions

The derived category of a dg-category contains the cohomological information of dg-modules over that dg-category. In this lecture, we will develop some results on h-projective dg-modules and we will see concrete applications of h-projective resolutions.

- We will discuss closure properties of the family of the h-projective dg-modules. In particular, we will see that represented dg-modules are h-projective, and that h-projectives are closed under arbitrary direct sums, shifts and mapping cones.
- We will discuss h-projective resolutions and their properties, in particular their uniqueness.
- We will se how we can define the derived hom (and, taking cohomology, the Ext groups) using h-projective resolutions.

8.1. **Properties of h-projective dg-modules.** We use the tools of §7 to prove some closure properties of h-projective dg-modules – that is, closure properties of the derived dg-category.

Proposition 8.1. *Let $\mathcal{A}$ be a dg-category, and let $A \in \mathrm{Ob}(\mathcal{A})$. Then, the represented right $\mathcal{A}$-dg-module $\mathcal{A}(-, A)$ is h-projective.*

Proof. Let X be an acyclic right $\mathcal{A}$-dg-module. By the Yoneda lemma, we have an isomorphism of complexes:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), X) \cong X(A),$$

and we conclude immediately. □

Proposition 8.2. *H-projective dg-module are closed under arbitrary direct sums, shifts and mapping cones. More precisely, fix a dg-category $\mathcal{A}$:*

1. *If $\{M_i : i \in I\}$ is a family of h-projective (right) $\mathcal{A}$-dg-modules, then its direct sum $\bigoplus_{i \in I} M_i$ is also h-projective.*
2. *If M is an h-projective (right) $\mathcal{A}$-dg-modules, then the shift $M[k]$ is also h-projective.*
3. *If $\varphi \colon M \to N$ is a closed degree 0 morphism between h-projective (right) $\mathcal{A}$-dg-modules, its mapping cone $\mathrm{C}(\varphi)$ is also h-projective.*

Proof. (1). Let X be an acyclic right $\mathcal{A}$-dg-module. Then, we have an isomorphism of complexes:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\bigoplus_{i \in I} M_i, X) \xrightarrow{\sim} \prod_{i \in I} \underline{\mathrm{Hom}}_{\mathcal{A}}(M_i, X),$$
$$\varphi \mapsto (\varphi_{|M_i})_{i \in I}.$$

The complex $\underline{\mathrm{Hom}}_{\mathcal{A}}(M_i, X)$ is acylic by hypothesis, hence the same is true for $\prod_{i \in I} \underline{\mathrm{Hom}}_{\mathcal{A}}(M_i, X)$.

(2). Let X be an acyclic right $\mathcal{A}$-dg-module. We have an isomorphism of complexes:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M[k], X) \cong \underline{\mathrm{Hom}}_{\mathcal{A}}(M, X)[-k],$$

which we can prove directly recalling how shifts of complexes and dg-modules are defined. Since $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, X)$ is acyclic, its shift $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, X)[-k]$ is also acyclic.

(3). Let X be an acyclic right $\mathcal{A}$-dg-module. We have a (rotated) pretriangle:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M[1], X) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), X) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(N, X) \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{A}}(\varphi, -)} \underline{\mathrm{Hom}}_{\mathcal{A}}(M, X).$$

Both $\underline{\mathrm{Hom}}_{\mathcal{A}}(N, X)$ and $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, X)$ are acyclic by assumption, so $\varphi^* = \underline{\mathrm{Hom}}_{\mathcal{A}}(\varphi, -)$ is necessarily a quasi-isomorphism. We conclude that $\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), X)$ is acyclic, as we wanted. □

We discussed more than once the impossibility to "invert quasi-isomorphisms", see for instance the old Example 1.19. It turns out that if we restrict to h-projective dg-modules, this problem is solved:

Proposition 8.3. *Let $\mathcal{A}$ be a dg-category, and let $\varphi \colon M \to N$ be a quasi-isomorphism between h-projective right $\mathcal{A}$-dg-modules. Then, φ is a homotopy equivalence.*

Proof. Thanks to Proposition 7.9, it is enough to show that $\mathrm{C}(\varphi) \approx 0$ (namely, the mapping cone is homotopy equivalent to 0). By the previous Proposition 8.2, we know that $\mathrm{C}(\varphi)$ is also h-projective. We have a pretriangle:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), M) \xrightarrow{\varphi_*} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), N) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), \mathrm{C}(\varphi)) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), M[1]).$$

By Proposition 6.19, we deduce that φ_* is a quasi-isomorphism. Hence, its cone $\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), \mathrm{C}(\varphi))$ is acyclic, namely $H^*(\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{C}(\varphi), \mathrm{C}(\varphi)))$. Thanks to Lemma 7.8, we conclude that $\mathrm{C}(\varphi) \approx 0$, as we wanted. □

8.2. **h-projective resolutions.** The good news is that the dg-category of dg-modules "has enough derived projectives", in the following sense:

Black Box 8.4. *Let $\mathcal{A}$ be a dg-category, and let $M \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a right $\mathcal{A}$-dg-module. Then, There is an h-projective dg-module $Q(M)$ and a quasi isomorphism*

$$q_M \colon Q(M) \to M,$$

which we call an h-projective resolution*.*

Remark 8.5. Assume that we have *two* h-projective resolutions $q_M \colon Q(M) \to M$ and $q'_M \colon Q'(M) \to M$. We end up with a diagram:

$$Q(M) \xrightarrow{q_M} M \xleftarrow{q'_M} Q'(M),$$

where both morphisms are quasi-isomorphisms. With the tools we already have, we can prove that there is actually a homotopy equivalence $Q(M) \xrightarrow{\approx} Q'(M)$, uniquely determined in $H^0(\mathrm{dgm}(\mathcal{A}))$, with the property that the diagram

$$\begin{tikzcd} Q(M) \arrow[r, "\approx"] \arrow[d, "q_M"'] & Q'(M) \arrow[dl, "q'_M"] \\ M & \end{tikzcd}$$

is commutative in $H^0(\mathrm{dgm}(\mathcal{A}))$. In other words, *h-projective resolutions are essentially unique.*

To prove this, consider the pretriangle

$$Q'(M) \xrightarrow{q'_M} M \to \mathrm{C}(q'_M) \to Q'(M)[1],$$

and the induced pretriangle obtained applying $\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), -)$:

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), Q'(M)) \xrightarrow{(q'_M)_*} \underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), M) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), \mathrm{C}(q'_M)) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), Q'(M)[1]).$$

We may take the long exact sequence induced in cohomology, in particular the following sequence

$$\begin{aligned} H^{-1}(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), \mathrm{C}(q'_M)) &\to H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), Q'(M))) \\ &\xrightarrow{[q'_M]_*} H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), M)) \to H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), \mathrm{C}(q'_M))) \end{aligned}$$

is exact. Since q'_M is a quasi-isomorphism, the cone $\mathrm{C}(q'_M)$ is acyclic. Moreover, since $Q(M)$ is h-projective, we have that

$$H^k(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), \mathrm{C}(q'_M))) \cong 0,$$

and in particular by exactness we conclude that

$$H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), Q'(M))) \xrightarrow{[q'_M]_*} H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), M))$$

is an isomorphism. So, taking the cohomology class $[q_M] \in H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), M))$, we find a unique morphism $[\alpha]\colon Q(M) \to Q'(M)$ in $H^0(\mathrm{dgm}(\mathcal{A}))$ such that

$$[q'_M][\alpha] = [q_M].$$

Next, recalling that cohomology of dg-modules is a $\mathbf{k}$-linear functor (cf. Proposition 6.7), we find out:

$$H^k(q'_M)H^k(\alpha) = H^k(q_M), \qquad k \in \mathbb{Z}.$$

Since $H^k(q'_M)$ and $H^k(q_M)$ are isomorphisms, the same is true for $H^k(\alpha)$. We conclude that α is a quasi-isomorphism. Being a quasi-isomorphism between h-projective dg-modules, it is a homotopy equivalence (Proposition 8.3).

For the purpose of this course, we will not prove the general existence of h-projective resolutions. Still, we will see how to construct them in a few select cases.

For instance, we can already see how the "classical" projective resolutions of modules actually yield h-projective resolutions. First, we have:

Lemma 8.6. *Let R be a $\mathbf{k}$-algebra, and let P be a (right) R-module. If P is projective, then it is h-projective as an object in* $\mathrm{dgm}(R)$*.*

Proof. We know that P is a direct summand of a free R-module, namely, there is an R-module P' such that:

$$R^{\oplus I} = P \oplus P',$$

for some set I. Next, let $X \in \mathrm{Ob}(\mathrm{dgm}(R))$ be acyclic. Then:

$$H^*(\underline{\mathrm{Hom}}_R(R^{\oplus I}, X)) \cong H^*(\underline{\mathrm{Hom}}_R(P, X)) \oplus H^*(\underline{\mathrm{Hom}}_R(P', X)).$$

We know that $R^{\oplus I}$ is h-projective by an application of Proposition 8.1, hence

$$H^*(\underline{\mathrm{Hom}}_R(R^{\oplus I}, X)) \cong 0.$$

We conclude that its summands are necessarily 0, in particular $H^*(\underline{\mathrm{Hom}}_R(P, X) \cong 0$, as we wanted. □

What is a projective resolution of an R-module M? It is nothing but a complex $P^\bullet$ of projective module together with a quasi-isomorphism

$$P^\bullet \to M$$

in $\mathrm{dgm}(R)$.

We now prove that projective resolutions yield h-projective resolutions. For simplicity, we will give a proof in the case of *finite* resolutions.

Proposition 8.7. *Let R be a $\mathbf{k}$-algebra, and let $P^\bullet$ be a (bounded above) complex of projectives:*

$$P^\bullet = \quad \cdots \to P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \to \cdots \to P^0 \to 0 \to \cdots$$

Then, $P^\bullet$ is h-projective as an object in $\mathrm{dgm}(R)$*.*

Proof. We give the proof under the additional assumption that $P^\bullet$ is bounded, namely, there is $N \in \mathbb{N}$ such that $P^{-i} \cong 0$ for all $i > N$.

We argue by induction . If $P^\bullet = P^0$ is concentrated in degree 0, then it is h-projective by the above Lemma 8.6.

Next, assume that all bounded complexes of projectives concentrated in degrees $[-n, 0]$ are h-projective, and assume that our $P^\bullet$ is concentrated in degrees $[-n-1, 0]$. We consider the following chain map:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P^{-n-1} \arrow[r] \arrow[d, "d^{-n-1}"] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[d] \\
\cdots \arrow[r] & 0 \arrow[r] & P^{-n} \arrow[r, "d^{-n}"] & P^{-n+1} \arrow[r] & \cdots \arrow[r] & P^0.
\end{tikzcd}$$

By the inductive hypothesis, both rows are h-projective when viewed as objects in $\mathrm{dgm}(R)$. By direct inspection, we see that the cone of the above chain map is precisely the complex $P^\bullet$. We now conclude applying Proposition 8.2.

The proof for general (unbounded) projective resolutions is a bit more complicated and involves viewing $P^\bullet$ as a "(co)limit" of its bounded truncations. □

Remark 8.8. It is worth appreciating that, using the "abstract" uniqueness of h-projective resolutions (Remark 8.5), we immediately deduce that two projective resolutions of an R-module are homotopy equivalent, and that homotopy equivalence is uniquely determined in $H^0(\mathrm{dgm}(R))$.

8.3. **Derived homs and Ext.** Using the derived category and h-projective resolutions, we can compute Ext groups. We start by observing that, in order to compute the hom $\mathbf{k}$-modules in the derived category $\mathsf{D}(\mathcal{A})$ of a dg-category $\mathcal{A}$, it is enough to take the h-projective resolution of the first variable:

Lemma 8.9. *Let $\mathcal{A}$ be a dg-category, and let $M, N \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$. Let $Q(M) \to M$ and $q_N \colon Q(N) \to N$ be h-projective resolutions. Then, the induced morphism*

$$\mathsf{D}_{\mathrm{dg}}(\mathcal{A})(Q(M), Q(N)) \xrightarrow{(q_N)_*} \underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), N)$$

is a quasi-isomorphism. In particular, we have isomorphisms:

$$\mathsf{D}(\mathcal{A})(Q(M), Q(N)[k]) \cong H^k(\mathsf{D}_{\mathrm{dg}}(\mathcal{A})(Q(M), Q(N))) \cong H^k(\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), N)).$$

for all $k \in \mathbb{Z}$.

Proof. $Q(M)$ is h-projective, and $\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), -)$ preserves quasi-isomorphisms by Proposition 6.19. Moreover, it is easy to see that in general

$$H^0(\underline{\mathrm{Hom}}_{\mathcal{A}}(X, Y[k]) \cong H^k(\underline{\mathrm{Hom}}_{\mathcal{A}}(X, Y). \qquad \square$$

Remark 8.10. In setup of the above Lemma 8.9, we may want to define a *derived hom* as follows:

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N) = \underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), N),$$

where $Q(M) \to M$ is an h-projective resolution of M.

Of course, we would like to prove that this definition is somehow independent from the choice of h-projective resolution. To do so, let $Q'(M) \to M$ be another such resolution. We know from Remark 8.5 that we have a homotopy equivalence $Q(M) \to Q'(M)$, uniquely determined in $H^0(\mathrm{dgm}(\mathcal{A}))$. Applying the dg-functor $\underline{\mathrm{Hom}}_{\mathcal{A}}(-, N)$ and recalling that dg-functors preserve homotopy equivalences, we find a homotopy equivalence

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(Q(M), N) \approx \underline{\mathrm{Hom}}_{\mathcal{A}}(Q'(M), N).$$

Hence, $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N)$ is well-defined at least in $H^0(\mathrm{dgm}(\mathbf{k}))$.

We can now define the Ext groups.

Definition 8.11. Let $\mathcal{A}$ be a dg-category, and let $M, N \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$, and let $k \in \mathbb{Z}$. We set:

$$\mathrm{Ext}^k_{\mathcal{A}}(M,N) = H^k(\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{A}}(M,N)).$$

Example 8.12. Let us compute the very well-known $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})$ using the tools we developed.

First, we look for an h-projective resolution of $\mathbb{Z}/2\mathbb{Z}$. Recalling Example 1.19 and Proposition 8.7, we know that the complex

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z},$$

concentrated in degrees -1 and 0, is an h-projective resolution of $\mathbb{Z}/2\mathbb{Z}$. Moreover, such complex is by definition the mapping cone $\mathrm{C}(2)$ of the morphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$. This observation is useful, since we then have a pretriangle in $\mathrm{dgm}(\mathbb{Z})$:

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathrm{C}(2) \to \mathbb{Z}[1].$$

Applying $\underline{\mathrm{Hom}}_{\mathbb{Z}}(-,\mathbb{Z})$, we get a (rotated) pretriangle in $\mathrm{dgm}(\mathbb{Z})$:

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[1],\mathbb{Z}) \to \underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathrm{C}(2),\mathbb{Z}) \to \underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \xrightarrow{2^*} \underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}).$$

Using the Yoneda lemma, we may identify the morphism 2^* with

$$2 \colon \mathbb{Z} \to \mathbb{Z}.$$

Moreover, we have

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[1],\mathbb{Z}) \cong \underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})[-1] \cong \mathbb{Z}[-1].$$

Taking the long exact sequence in cohomology, we easily find out that:

$$\begin{aligned}\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) &= H^0(\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathrm{C}(2),\mathbb{Z})) \cong 0,\\ \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) &= H^1(\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathrm{C}(2),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},\end{aligned}$$

and the other Ext groups are zero.

9. Lecture 9: tensor products

We now deal with different kinds of tensor products.

- We introduce the tensor product of dg-categories, and also the derived tensor product by taking resolutions of dg-categories.
- We define *dg-bimodules*. We also see how they generalize dg-functors.
- We discuss the tensor product of dg-bimodules and the derived tensor product.
- We specialize to the computation of the classical Tor groups.

This lecture can feel a bit "heavy". I advise the reader not to focus on the technical aspects of resolutions, and just assume the slogan: *whenever we need compatibility with some suitable "weak isomorphisms", we take resolutions; it can be bothersome, but one way or another it always works*.

9.1. **(Derived) tensor products of dg-categories.** We are going to define tensor products of dg-categories and their "derived" versions. In order to do so, we also need some general preliminaries.

9.1.1. *Quasi-equivalences and resolutions.* We first introduce the "dg-categorical counterpart" of quasi-isomorphisms.

Definition 9.1. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a dg-functor between dg-categories. We say that F is a *quasi-equivalence* if:

- For any pair of objects $A, B \in \mathrm{Ob}(\mathcal{A})$, the chain map
$$F = F_{A,B}\colon \mathcal{A}(A,B) \to \mathcal{B}(F(A), F(B))$$
is a quasi-isomorphism.
- $H^0(F)$ is essentially surjective. Namely, for any $B \in \mathrm{Ob}(\mathcal{B})$ there is an object $A \in \mathrm{Ob}(\mathcal{A})$ such that $B \approx F(A)$.

Example 9.2. Recall the dg-categories $\mathbf{Q}$ and $\mathbf{Q}'$ of Example 2.15. The dg-functor $F\colon \mathbf{Q}' \to \mathbf{Q}$ we constructed there is an example of quasi-equivalence.

We have seen how to take resolutions of dg-modules and why this is useful. An interesting observation is that we can also resolve dg-categories themselves.

Black Box 9.3. *We say that a dg-category $\mathcal{A}$ is* h-projective *if its hom complexes $\mathcal{A}(A,B)$ are h-projective as objects of* $\mathrm{dgm}(\mathbf{k})$.

For any dg-category $\mathcal{A}$, there is an h-projective dg-category $Q(\mathcal{A})$ together with a quasi-equivalence $Q(\mathcal{A}) \to \mathcal{A}$, which is called a h-projective resolution *of $\mathcal{A}$. Moreover, the dg-functor $Q(\mathcal{A}) \to \mathcal{A}$ can be chosen to be the identity on objects and surjective on morphisms.*

Remark 9.4. If $\mathbf{k}$ is a field, we can prove that any object in $\mathrm{dgm}(\mathbf{k})$ is h-projective (Exercise 9.1). Hence, any dg-category over a field is h-projective.

9.1.2. *Tensor products of dg-categories.* The tensor product of a pair of dg-categories is defined in a very straightforward way (just be careful with the Koszul sign rule).

Definition 9.5. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ is the dg-category defined as follows:

- Objects of $\mathcal{A} \otimes \mathcal{B}$ are pairs (A, B) where $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{B})$. Namely, $\mathrm{Ob}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ as sets.
- If (A, B) and (A', B') are two objects in $\mathcal{A} \otimes \mathcal{B}$, we define:
$$(\mathcal{A} \otimes \mathcal{B})((A,B),(A',B')) = \mathcal{A}(A,A') \otimes \mathcal{B}(B,B').$$
- Compositions are described as follows (notice the Koszul sign rule):
$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|}(f' \circ f) \otimes (g' \circ g),$$
if $f \otimes g\colon (A,B) \to (A',B')$ and $f' \otimes g'\colon (A',B') \to (A'',B'')$. Notice that, by definition, the degree of $f \otimes g$ is the sum of the degrees of f and g: $|f \otimes g| = |f| + |g|$.
- For any object $(A,B) \in \mathrm{Ob}(\mathcal{A} \otimes \mathcal{B})$, the identity morphism of (A,B) is given by $1_A \otimes 1_B$.

We can check that $\mathcal{A} \otimes \mathcal{B}$ is indeed a dg-category.

The tensor product has the typical properties you would expect:

Lemma 9.6. *Let $\mathcal{A}$, $\mathcal{B}$ be dg-categories. Then, the tensor product is symmetric, namely, there is an isomorphism (i.e. an invertible dg-functor):*

$$\begin{aligned} \mathcal{A} \otimes \mathcal{B} &\cong \mathcal{B} \otimes \mathcal{A}, \\ (A, B) &\mapsto (B, A), \\ f \otimes g &\mapsto (-1)^{|f||g|} g \otimes f. \end{aligned}$$

Moreover, the tensor product is well-behaved with respect to opposites:

$$(\mathcal{A} \otimes \mathcal{B})^{\mathrm{op}} \cong \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}^{\mathrm{op}}.$$

Furthermore, the base ring **k**, *viewed as a dg-category, is the unit of the tensor product:*

$$\mathbf{k} \otimes \mathcal{A} \cong \mathcal{A}.$$

Proof. Left to the reader! (Exercise 9.2). □

The issue with the tensor product is that it will not, in general, preserve quasi-equivalences, in the sense that if $U \colon \mathcal{A} \to \mathcal{A}'$ is a quasi-equivalence, the induced dg-functor

$$\begin{aligned} U \otimes 1 \colon \mathcal{A} \otimes \mathcal{B} &\to \mathcal{A}' \otimes \mathcal{B}, \\ (A, B) &\mapsto (U(A), B), \\ f \otimes g &\mapsto U(f) \otimes g, \end{aligned}$$

need not be a quasi-equivalence. Fortunately, this is actually true if we are tensoring with an h-projective dg-category.

Black Box 9.7. *Let V be an h-projective object in* $\mathrm{dgm}(\mathbf{k})$. *Then, the dg-functor*

$$V \otimes - \colon \ \mathrm{dgm}(\mathbf{k}) \to \mathrm{dgm}(\mathbf{k})$$

preserves acyclic objects and, equivalently, quasi-isomorphisms. The same is true for the dg-functor $- \otimes V$.

Lemma 9.8. *Let $\mathcal{B}$ be an h-projective dg-category, and let $U \colon \mathcal{A} \to \mathcal{A}'$ be a quasi-equivalence. Then, the induced dg-functor*

$$\begin{aligned} U \otimes 1 \colon \mathcal{A} \otimes \mathcal{B} &\to \mathcal{A}' \otimes \mathcal{B}, \\ (A, B) &\mapsto (U(A), B), \\ f \otimes g &\mapsto U(f) \otimes g, \end{aligned}$$

is a quasi-equivalence. The same conclusion is true for the induced dg-functor

$$1 \otimes U \colon \mathcal{B} \otimes \mathcal{A} \to \mathcal{B} \otimes \mathcal{A}'.$$

Proof. Left to the reader! (Exercise 9.3). □

Using h-projective resolutions, we can define a "derived tensor product" of dg-categories which is preserved by quasi-equivalences:

Definition 9.9. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. We define the *derived tensor product* $\mathcal{A} \overset{\mathbb{L}}{\otimes} \mathcal{B}$ as

$$\mathcal{A} \overset{\mathbb{L}}{\otimes} \mathcal{B} = Q(\mathcal{A}) \otimes \mathcal{B} \approx \mathcal{A} \otimes Q(\mathcal{B}),$$

where $Q(\mathcal{A}) \to \mathcal{A}$ and $Q(\mathcal{B}) \to \mathcal{B}$ are h-projective resolutions.

Remark 9.10. We've written "$\mathcal{A} \overset{\mathbb{L}}{\otimes} \mathcal{B} = Q(\mathcal{A}) \otimes \mathcal{B} \approx \mathcal{A} \otimes Q(\mathcal{B})$" in the above definition, which does not look like it makes much sense. It actually does, but an explanation is needed. Since we know that tensoring with h-projective dg-categories preserves quasi-equivalences (in both variables), we obtain indeed a diagram of quasi-equivalences:

$$Q(\mathcal{A}) \otimes \mathcal{B} \leftarrow Q(\mathcal{A}) \otimes Q(\mathcal{B}) \rightarrow \mathcal{A} \otimes Q(\mathcal{B}).$$

Hence, we may conclude that the derived tensor product is well-defined "up to quasi-equivalence".

9.2. **Dg-bimodules.** Differential graded (dg) bimodules are a generalization of dg-modules. Essentially, they are gadgets with both a left and a right compatible action of two possibly different dg-categories. Using the tensor product of dg-categories, we may define them very quickly:

Definition 9.11. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. The dg-category of $\mathcal{A}$-$\mathcal{B}$-*dg-bimodules* is

$$\mathrm{dgm}(\mathcal{A}, \mathcal{B}) = \mathrm{Fun}_{\mathrm{dg}}(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A}, \mathrm{dgm}(\mathbf{k})).$$

In other words, a $\mathcal{A}$-$\mathcal{B}$-dg-bimodule is a right $\mathcal{B} \otimes \mathcal{A}^{\mathrm{op}}$-dg-module, or equivalently a left $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A}$-dg-module.

The hom complexes in $\mathrm{dgm}(\mathcal{A}, \mathcal{B})$ will often be denoted by $\underline{\mathrm{Hom}}_{\mathcal{A}-\mathcal{B}}(-, =)$.

Remark 9.12. It is worth elaborating a little on the definition.

(1) First, notice that a dg-bimodule $F \in \mathrm{dgm}(\mathcal{A}, \mathcal{B})$ yields a family of left $\mathcal{A}$-dg-modules and of right $\mathcal{B}$-dg-modules. Namely, for any object $A \in \mathrm{Ob}(\mathcal{A})$, we have:

$$\begin{aligned} F(-, A) \colon \mathcal{B}^{\mathrm{op}} &\to \mathrm{dgm}(\mathbf{k}), \\ B &\mapsto F(B, A), \\ (f \colon B \to B') &\mapsto F(f \otimes 1_A) \colon F(B', A) \to F(B, A). \end{aligned}$$

Similarly, for any object $B \in \mathrm{Ob}(\mathcal{B})$, we have:

$$\begin{aligned} F(B, -) \colon \mathcal{A} &\to \mathrm{dgm}(\mathbf{k}), \\ A &\mapsto F(B, A), \\ (g \colon A \to A') &\mapsto F(1_B \otimes g) \colon F(B, A) \to F(B, A'). \end{aligned}$$

Notice that, notationally, the first variable of F is "contravariant" and the second variable is "covariant".

(2) Notice that, for $f \colon B \to B'$ in $\mathcal{B}$ (corresponding to $f^{\mathrm{op}} \colon B' \to B$ in $\mathcal{B}^{\mathrm{op}}$) and for $g \colon A \to A'$ in $\mathcal{A}$, we have:

$$f \otimes g = (-1)^{|f||g|}(1_B \otimes g) \circ (f \otimes 1_A) = (f \otimes 1_{A'}) \circ (1_{B'} \otimes g).$$

From this, we see that to give an $\mathcal{A}$-$\mathcal{B}$-dg-bimodule F is the same as giving a family of left $\mathcal{A}$-dg-modules $B \mapsto F^B$ (for $B \in \mathrm{Ob}(\mathcal{B})$) and a family of right $\mathcal{B}$-dg-modules $A \mapsto F_A$, subject to the compatibilities

$$\begin{aligned} F^B(A) &= F_A(B), \\ (-1)^{|f||g|} F^B(g) \circ F_A(f) &= F_{A'}(f) \circ F_{B'}(g), \end{aligned}$$

for $f \in \mathcal{B}(B, B')$ and $g \in \mathcal{A}(A, A')$. Indeed, we can reconstruct F by

$$\begin{aligned} F(B, A) &= F^B(A) = F_A(B), \\ F(f \otimes g) &= -1)^{|f||g|} F^B(g) \circ F_A(f) = F_{A'}(f) \circ F_{B'}(g), \end{aligned}$$

and in particular we have $F(B,-) = F^B$ and $F(-,A) = F_A$.

(3) Recalling what we did for dg-modules (see Lecture 3 and Lecture 4), we can adopt a "left-right action notation" which describes a given $\mathcal{A}$-$\mathcal{B}$-dg-bimodule F. Namely, we may write:

$$gxf = (-1)^{|f|(|x|+|g|)} F(f \otimes g)(x).$$

Using the observations of the above point (2), we can also say that a $\mathcal{A}$-$\mathcal{B}$-dg-bimodule F is a family of chain complexes $(B,A) \mapsto F(B,A)$, together with both a left action

$$\begin{gathered}\mathcal{A}(A,A') \otimes F(B,A) \to F(B,A'),\\ g \otimes x \mapsto gx,\end{gathered}$$

and a right action

$$\begin{gathered}F(B',A) \otimes \mathcal{B}(B,B') \to F(B,A),\\ x \otimes f \mapsto xf,\end{gathered}$$

subject to the usual relations, and moreover with the compatibility:

$$(gx)f = g(xf),$$

which then allows to set $gxf = (gx)f = g(xf)$.

Remark 9.13. Recalling that $\mathbf{k}$ satisfies $\mathbf{k} \otimes \mathcal{A} \cong \mathcal{A}$ for any dg-category $\mathcal{A}$, we may identify left $\mathcal{A}$-dg-modules with $\mathcal{A}$-$\mathbf{k}$-dg-bimodules, and right $\mathcal{A}$-dg-modules with $\mathbf{k}$-$\mathcal{A}$-dg-bimodules. In both cases, we have a "trivial" right or left action of $\mathbf{k}$.

Example 9.14. Let $\mathcal{A}$ be a dg-category. There is a very important $\mathcal{A}$-$\mathcal{A}$-dg-bimodule, which is called the *diagonal bimodule* and denoted by $h_{\mathcal{A}}$ or sometimes by $\mathcal{A}$ itself. Recalling point (3) of the above Remark 9.12, we can define it as follows. We set:

$$h_{\mathcal{A}}(A,A') = \mathcal{A}(A,A').$$

the complex of morphisms from A to A'. Then, left and right actions are given by compositions, and the compatibility between such actions is just associativity:

$$(gh)f = g(hf).$$

Viewing $h_{\mathcal{A}}$ as a dg-functor $h_{\mathcal{A}} \colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathrm{dgm}(\mathbf{k})$, we have:

$$\begin{aligned}&h_{\mathcal{A}}(A,A') = \mathcal{A}(A,A'),\\ &h_{\mathcal{A}}(f \otimes g) \colon \mathcal{A}(A,A') \to \mathcal{A}(B,B'),\\ &h_{\mathcal{A}}(f \otimes g)(h) = (-1)^{|f|(|h|+|g|)} f^*(g_*(h)) = (-1)^{|f|(|h|+|g|)} ghf,\end{aligned}$$

for $f \colon B \to A$ and $g \colon A' \to B'$.

9.3. **(Derived) tensor product of bimodules.** We may now define a very general version of tensor product.

Definition 9.15. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be dg-categories. Let F be an $\mathcal{A}$-$\mathcal{B}$-dg-bimodule, and let G be a $\mathcal{B}$-$\mathcal{C}$-dg-bimodule. We may define the *tensor product of F and G over* $\mathcal{B}$ as the $\mathcal{A}$-$\mathcal{C}$-dg-bimodule denoted by $F \otimes_{\mathcal{B}} G$ and defined by:

$$(F \otimes_{\mathcal{B}} G)(C, A) = \operatorname{coker}\left(\bigoplus_{B,B' \in \mathrm{Ob}(\mathcal{B})} F(B, A) \otimes \mathcal{B}(B', B) \otimes G(C, B) \to \bigoplus_{B' \in \mathrm{Ob}(\mathcal{B})} F(B', A) \otimes G(C, B') \right)$$
$$x \otimes f \otimes y \mapsto xf \otimes y - x \otimes fy,$$

with the left action of $\mathcal{A}$ and the right action of $\mathcal{C}$ induced respectively from F and G.

The above definition may look rather obscure, but it is actually what you would like a tensor product to do. Let us write

$$x \otimes_{\mathcal{B}} y$$

for an element in $(F \otimes_{\mathcal{B}} G)(C, A)$, which comes from $x \otimes y \in F(B', A) \otimes G(C, B')$, for some $B' \in \mathrm{Ob}(\mathcal{B})$. The left action of $\mathcal{A}$ looks like

$$g(x \otimes_{\mathcal{B}} y) = gx \otimes_{\mathcal{B}} y,$$

whereas the right action of $\mathcal{C}$ looks like

$$(x \otimes_{\mathcal{B}} y)f = x \otimes_{\mathcal{B}} yf.$$

Finally, we have that

$$xh \otimes_{\mathcal{B}} y = x \otimes_{\mathcal{B}} hy,$$

for any morphism h in $\mathcal{B}$ with suitable domain and codomain. The above equations are actually what the tensor $F \otimes_{\mathcal{B}} G$ is all about.

Remark 9.16. It is worth noticing that the tensor product is dg-functorial. More precisely, if $F \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}, \mathcal{B})$, we have a dg-functor

$$F \otimes_{\mathcal{B}} - \colon\ \mathrm{dgm}(\mathcal{B}, \mathcal{C}) \to \mathrm{dgm}(\mathcal{A}, \mathcal{C});$$

if $G \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{B}, \mathcal{C})$, we have a dg-functor

$$- \otimes_{\mathcal{B}} G \colon\ \mathrm{dgm}(\mathcal{A}, \mathcal{B}) \to \mathrm{dgm}(\mathcal{A}, \mathcal{C}).$$

How do you define such dg-functors on the complexes of morphisms? The answer is: in the unique sensible way!

Remark 9.17. Dg-bimodules are actually a generalization of dg-functors. Indeed, if $F \colon \mathcal{A} \to \mathcal{B}$ is a dg-functor, we may associate to it a $\mathcal{A}$-$\mathcal{B}$-dg-bimodule h_F defined by

$$h_F(B, A) = \mathcal{B}(B, F(A)),$$

with the obvious left and right actions. Moreover, the mapping $F \mapsto h_F$ yields a fully faithful dg-functor

$$\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B}) \hookrightarrow \mathrm{dgm}(\mathcal{A}, \mathcal{B}).$$

By definition, the dg-category of dg-bimodules is a particular example of a dg-category of dg-modules. Hence, we can perform on dg-bimodules all the constructions we do with dg-modules. In particular, we have a notion of quasi-isomorphism.

The above tensor product of dg-bimodules need not preserve quasi-isomorphisms, but we can use h-projective resolutions so that we obtain a "derived" version which does. To do so, it will be useful to also take h-projective resolutions of the dg-categories themselves.

Black Box 9.18. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be dg-categories, and assume that $\mathcal{B}$ is h-projective. If F is an h-projective $\mathcal{A}$-$\mathcal{B}$-dg-bimodule and G is an h-projective $\mathcal{B}$-$\mathcal{C}$-dg-bimodule, then $F(-,A)$ is an h-projective right $\mathcal{B}$-dg-module for all $A \in \mathrm{Ob}(\mathcal{A})$, and $G(C,-)$ is an h-projective left $\mathcal{B}$-dg-bimodule for all $C \in \mathrm{Ob}(\mathcal{C})$.*

Black Box 9.19. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be dg-categories. Let F be an $\mathcal{A}$-$\mathcal{B}$-dg-bimodule and let G be a $\mathcal{B}$-$\mathcal{C}$-dg-bimodule. If $F(-,A)$ is an h-projective right $\mathcal{B}$-dg-bimodule for all $A \in \mathrm{Ob}(\mathcal{A})$, then*

$$F \otimes_{\mathcal{B}} - \colon \; \mathrm{dgm}(\mathcal{B},\mathcal{C}) \to \mathrm{dgm}(\mathcal{A},\mathcal{C})$$

preserves quasi-isomorphisms (or, equivalently, acyclic dg-bimodules). Analogously, if $G(C,-)$ is an h-projective left $\mathcal{B}$-dg-bimodule for all $C \in \mathrm{Ob}(\mathcal{C})$, then

$$- \otimes_{\mathcal{B}} G \colon \; \mathrm{dgm}(\mathcal{A},\mathcal{B}) \to \mathrm{dgm}(\mathcal{A},\mathcal{C})$$

preserves quasi-isomorphisms (or, equivalently, acyclic dg-bimodules).

We can now finally define the derived tensor product in this very broad generality.

Definition 9.20. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be dg-categories, with $\mathcal{B}$ being h-projective. Let $F \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A},\mathcal{B}))$ and $G \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{B},\mathcal{C}))$. We define the *derived tensor product over* $\mathcal{B}$

$$F \overset{\mathbb{L}}{\otimes}_{\mathcal{B}} G$$

as

$$F \overset{\mathbb{L}}{\otimes}_{\mathcal{B}} G = Q(F) \otimes_{\mathcal{B}} G \approx F \otimes_{\mathcal{B}} Q(G),$$

where $Q(F) \to F$ is an h-projective resolution of F and $Q(G) \to G$ is an h-projective resolution of G. Thanks to Black Box 9.18 and Black Box 9.19, we know that $Q(F) \otimes_{\mathcal{B}} -$ and $- \otimes_{\mathcal{B}} Q(G)$ preserve quasi-isomorphisms.

Analogously to what happens with the tensor product of dg-categories (see Remark 9.10), the derived tensor product of bimodules is well defined "up to quasi-isomorphism":

$$Q(F) \otimes_{\mathcal{B}} G \xleftarrow{\approx} Q(F) \otimes_{\mathcal{B}} Q(G) \xrightarrow{\approx} F \otimes_{\mathcal{B}} Q(G).$$

Moreover, a similar argument as in Remark 8.10 shows that the definition does not depend (up to quasi-isomorphism) on the choice of resolutions $Q(F) \to F$ or $Q(G) \to G$.

Definition 9.21. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be dg-categories, with $\mathcal{B}$ being h-projective. Let $F \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A},\mathcal{B}))$ and $G \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{B},\mathcal{C}))$. We set:

$$\mathrm{Tor}_i^{\mathcal{B}}(F,G) = H^{-i}(F \overset{\mathbb{L}}{\otimes}_{\mathcal{B}} G).$$

Remark 9.22. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be dg-categories, and assume that $\mathcal{B}$ is h-projective. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a dg-functor. The dg-bimodule h_F of the above Remark 9.17 has the property that $h_F(-,A)$ is the represented right $\mathcal{B}$-dg-module $\mathcal{B}(-,F(A))$, for all $A \in \mathrm{Ob}(\mathcal{A})$.

In the dg-category $\mathrm{dgm}(\mathcal{B})$ we have quasi-isomorphisms, so we can perhaps give a new definition as follows. We say that an $\mathcal{A}$-$\mathcal{B}$-dg-bimodule F is a *quasi-functor* from $\mathcal{A}$ to $\mathcal{B}$ (and write $F \colon \mathcal{A} \to \mathcal{B}$) if, for any $A \in \mathrm{Ob}(\mathcal{A})$, there is a quasi-isomorphism of right $\mathcal{B}$-dg-modules

$$\mathcal{B}(-,\Phi_F(A)) \xrightarrow{\approx} F(-,A),$$

for a suitable $\Phi_F(A) \in \mathrm{Ob}(\mathcal{B})$.

The notion of quasi-functor is actually very important in the *homotopy theory of dg-categories*. If $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ are quasi-functors, the derived tensor product

$$F \overset{\mathbb{L}}{\otimes}_{\mathcal{B}} G$$

works as a composition of F and G.

Computing Tor. We now apply the very general machinery we have developed in the previous parts in order to compute an example of Tor groups.

Remark 9.23. Let us view our base commutative ring $\mathbf{k}$ as a dg-category with a single object. Then, as a dg-category, it is h-projective. Indeed, its only hom complex is $\mathbf{k}$ itself, which is obviously h-projective as a complex of $\mathbf{k}$-modules.

Example 9.24. Let us take $\mathbf{k} = \mathbb{Z}$, the integers. We know from Example 8.12 that the complex

$$\mathrm{C}(2) = \quad (\mathbb{Z} \xrightarrow{2} \mathbb{Z})$$

is an h-projective resolution of $\mathbb{Z}/2\mathbb{Z}$. We conclude that

$$\mathbb{Z}/2\mathbb{Z} \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \mathrm{C}(2) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

Taking $- \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is a dg-functor, so it preserves pretriangles. We hence get a pretriangle:

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2\otimes 1_{\mathbb{Z}/2\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathrm{C}(2) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

Notice that we may identify $2 \otimes 1_{\mathbb{Z}/2\mathbb{Z}}$ with

$$2 = 0\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}.$$

Taking the long exact sequence in cohomology, we easily conclude that

$$\begin{aligned}
&\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \\
&\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \\
&\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong 0, \qquad i \neq 0, 1.
\end{aligned}$$

Exercises to Lecture 9

Exercise 9.1. *(Not so easy)* Assume that $\mathbf{k}$ is a field. Prove that any object in $\mathrm{dgm}(\mathbf{k})$ is h-projective.

Exercise 9.2. Prove Lemma 9.6.

Exercise 9.3. Prove Lemma 9.8.

10. Lecture 10: pretriangulated dg-categories

We have already encountered *mapping cone* of dg-modules, which fit in particular sequences calld *pretriangles*. We are going to deal with these concepts more systematically and see how we can interpret them in *any* dg-category. For this, we will learn the fundamental concept of *representability*.

- We will see how many constructions we do on dg-modules can be "internalized" in any dg-category, using representability.
- We will discuss *pretriangulated dg-categories*, which are particular dg-categories having (internal) shifts and mapping cones.

10.1. **Representability.** We are going to discuss a key notion in dg-category theory. It is a general procedure which allows to make constructions *inside* a given dg-category.

Definition 10.1. Let $\mathcal{A}$ be a dg-category, and let $M \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a right $\mathcal{A}$-dg-module. A *representation* (respectively *homotopy representation* or *quasi-representation*) of M is a pair (A, φ), where $A \in \mathrm{Ob}(\mathcal{A})$ and φ is an isomorphism (respectively a homotopy equivalence or a quasi-isomorphism)

$$\varphi \colon \mathcal{A}(-, A) \xrightarrow{\sim} M.$$

We say that M is *representable* (respectively *homotopy representable* or *quasi-representable*) if it admits a representation (respectively a homotopy representation or a quasi-representation).

Remark 10.2. Clearly, homotopy representations are quasi-representations. If M is h-projective, the converse is also true thanks to Proposition 8.1 and Proposition 8.3. We will actually concentrate more on representability and quasi-representability.

Lemma 10.3. *let $\mathcal{A}$ be a dg-category and let $M \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$ be a right $\mathcal{A}$-dg-module.*

Representations are uniquely determined up to unique isomorphism. Namely, let (A, φ) and (A', φ') be representations of M. Then, there is a unique isomorphism $\alpha \colon A \to A'$ such that the following diagram is commutative in $\mathrm{dgm}(\mathcal{A})$*:*

$$\begin{tikzcd} \mathcal{A}(-, A) \arrow[r, "\varphi"] \arrow[d, "f_*"'] & M \\ \mathcal{A}(-, A'). \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

Analogously, quasi-representations are uniquely determined up to essentially unique homotopy equivalence. Namely, let (A, φ) and (A', φ') be quasi-representations of M. Then, there is a homotopy equivalence $\alpha \colon A \to A'$, uniquely determined in $H^0(\mathcal{A})$, such that the following diagram is commutative in $H^0(\mathrm{dgm}(\mathcal{A}))$:

$$\begin{tikzcd} \mathcal{A}(-, A) \arrow[r, "\varphi"] \arrow[d, "f_*"'] & M \\ \mathcal{A}(-, A'). \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

Proof. Let us first deal with the (easier) case of representations. Composing $(\varphi')^{-1} \circ \varphi$ we obtain an isomorphism of dg-modules $\mathcal{A}(-, A) \to \mathcal{A}(-, A')$. We know that the Yoneda embedding $\mathcal{A} \hookrightarrow \mathrm{dgm}(\mathcal{A})$ is fully faithful, hence we can find a unique $f \colon A \to A'$, necessarily an isomorphism, such that

$$f_* = (\varphi')^{-1} \circ \varphi.$$

This is precisely what we wanted.

Let us deal with the (slightly more difficult) case of quasi-representations. The dg-module $\mathcal{A}(-, A)$ is h-projective, hence the induced morphism

$$\varphi'_* \colon \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), \mathcal{A}(-, A')) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M)$$

is a quasi-isomorphism (cf. Proposition 6.19). In particular, it induces an isomorphism in zeroth cohomology and from this we find a unique morphism $[\alpha] \colon \mathcal{A}(-, A) \to \mathcal{A}(-, A')$ in $H^0(\mathrm{dgm}(\mathcal{A}))$ such that $[\varphi' \circ \alpha] = [\varphi]$. Taking cohomologies, we see that α is a quasi-isomorphism; being a quasi-isomorphism between h-projective dg-modules, it is a homotopy equivalence. By faithfulness of the zeroth cohomology $H^0(\mathcal{A}) \hookrightarrow H^0(\mathrm{dgm}(\mathcal{A}))$ of the Yoneda embedding, we find a unique $[f]\colon A \to A'$ in $H^0(\mathcal{A})$ such that $[\alpha] = [f_*]$, and $[f]$ is an isomorphism in $H^0(\mathcal{A})$. This is precisely what we wanted. □

Remark 10.4. Let us recall the Yoneda lemma. We have an isomorphism of complexes:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M) &\to M(A), \\ \psi &\mapsto \psi_A(1_A), \end{aligned}$$

which also induces an isomorphism in cohomology:

$$\begin{aligned} H^i(\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(-, A), M)) &\to H^i(M(A)), \\ [\psi] &\mapsto [\psi_A(1_A)]. \end{aligned}$$

Let us assume that M has a representation (A, φ). Thanks to the Yoneda lemma, φ is completely determined by $x = \varphi_A(1_A) \in Z^0(M(A))$, and it can be identified with the "action of x":

$$x_*\colon \mathcal{A}(-, A) \to M, \qquad f \mapsto xf.$$

The assumption that φ is an isomorphism is understood as follows: for any object $B \in \mathrm{Ob}(\mathcal{A})$ and for any $y \in M(B)$, there is a unique morphism $f\colon B \to A$ in $\mathcal{A}$ such that $\varphi_B(f) = y$. We can express this just in terms of $x \in Z^0(M(A))$, since

$$\varphi_B(f) = \varphi_B(1_A \circ f) = xf = M(f)(x).$$

Namely, we can alternatively define a representation of M as a pair (A, x) where $x \in M(A)$ and: *for any object $B \in \mathrm{Ob}(\mathcal{A})$ and for any $y \in M(B)$, there is a unique morphism $f\colon B \to A$ in $\mathcal{A}$ such that $y = xf = M(f)(x)$.* In some sense, this means that $y \in M(B)$ "factors uniquely through x". This is also called the *universal property* of x.

Now, if (A, φ) is just a quasi-representation of M, we get isomorphisms in cohomology:

$$H^i(\varphi)\colon H^i(\mathcal{A}(-, A)) \to H^i(M).$$

Using the Yoneda lemma, we can hence define a quasi-representation of M as a pair (A, x) where $x \in M(A)$ and: *for any object $B \in \mathrm{Ob}(\mathcal{A})$ and for any $[y] \in H^i(M(B))$, there is a unique morphism $[f]\colon B \to A$ of degree i in the graded category $H^*(\mathcal{A})$ such that $[y] = [xf]$.*

Representability is the key to do constructions *inside* a given dg-category $\mathcal{A}$. The trick is more or less always the same. Essentially:

(1) We want to perform some construction involving objects of $\mathcal{A}$.
(2) We perform that construction with the represented dg-modules inside $\mathrm{dgm}(\mathcal{A})$.
(3) We go back to $\mathcal{A}$ requiring representability.

There are some caveats (for instance, we will sometimes want to work with $\mathcal{A}^{\mathrm{op}}$ instead of $\mathcal{A}$), but that is the general philosophy. Let us see it in action!

Definition 10.5. Let $\mathcal{A}$ be a dg-category, and let $A, B \in \mathrm{Ob}(\mathcal{A})$.

A *(strict) direct sum* of A and B is an object $A \oplus B$ in $\mathcal{A}$ together with an isomorphism

$$\mathcal{A}(-, A \oplus B) \xrightarrow{\sim} \mathcal{A}(-, A) \oplus \mathcal{A}(-, B) \tag{$*$}$$

in $\operatorname{dgm}(\mathcal{A})$. Equivalently (cf. Remark 10.4), it is an object $A \oplus B$ together with closed degree 0 morphisms $p_A \colon A \oplus B \to A$ and $p_B \colon A \oplus B \to B$ satisfying the following universal property: for any pair of morphisms $f \colon Z \to A$ and $g \colon Z \to B$, there is a unique morphism

$$\begin{pmatrix} f \\ g \end{pmatrix} \colon Z \to A \oplus B$$

such that

$$p_A \circ \begin{pmatrix} f \\ g \end{pmatrix} = f, \qquad p_B \circ \begin{pmatrix} f \\ g \end{pmatrix} = g.$$

In particular, the isomorphism $(*)$ can be described as:

$$\begin{pmatrix} (p_A)_* \\ (p_B)_* \end{pmatrix} \colon \mathcal{A}(-, A \oplus B) \xrightarrow{\sim} \mathcal{A}(-, A) \oplus \mathcal{A}(-, B), \qquad h \mapsto \begin{pmatrix} p_A \circ h \\ p_B \circ h \end{pmatrix}.$$

A *homotopy direct sum* of A and B is an object $A \oplus B$ in $\mathcal{A}$ together with a quasi-isomorphism

$$\mathcal{A}(-, A \oplus B) \xrightarrow{\approx} \mathcal{A}(-, A) \oplus \mathcal{A}(-, B)$$

in $\operatorname{dgm}(\mathcal{A})$.

The direct sum of two objects in a dg-category is quite special in the sense that it automatically enjoys a "dual" representability property. We will deal with the case of strict direct sums:

Lemma 10.6. *Let $\mathcal{A}$ be a dg-category, let $A, B \in \operatorname{Ob}(\mathcal{A})$ and let $(A \oplus B, p_A, p_B)$ be a (strict) direct sum of A and B in $\mathcal{A}$.*

Define

$$j_A \colon A \to A \oplus B, \qquad j_B \colon B \to A \oplus B$$

by means of the universal property, respectively as the unique (closed, degree 0) morphisms such that

$$p_A \circ j_A = 1_A, \qquad p_B \circ j_A = 0,$$

and

$$p_A \circ j_B = 0, \qquad p_B \circ j_B = 1_B.$$

Then, there is an isomorphism

$$\begin{pmatrix} (j_A)^* \\ (j_B)^* \end{pmatrix} \colon \mathcal{A}(A \oplus B, -) \xrightarrow{\sim} \mathcal{A}(A, -) \oplus \mathcal{A}(B, -), \qquad h \mapsto \begin{pmatrix} h \circ j_A \\ h \circ j_B \end{pmatrix}, \tag{$\star$}$$

in $\operatorname{dgm}(\mathcal{A}^{\mathrm{op}})$.

Proof. We start with an observation. Take any morphism $x \colon Z \to A \oplus B$. By the universal property of $(A \oplus B, p_A, p_B)$, we know that x is completely determined by $x_A \colon Z \to A$ and $x_B \colon Z \to B$ such that $p_A \circ x = x_A$ and $p_B \circ x = x_B$. By the way, we see by uniqueness and by definition of j_A and j_B that actually:

$$x = j_A \circ x_A + j_B \circ x_B.$$

Let us now show that the chain map $(\star)$ is an isomorphism. First, it is injective. Let $h \colon A \oplus B \to Z$ be a morphism such that $h \circ j_A = 0$ and $h \circ j_B = 0$. Then, for any $x \colon Z \to A \oplus B$, we have:

$$h \circ x = h \circ j_A \circ x_A + h \circ j_B \circ x_B = 0,$$

from which we conclude that $h = 0$.

Let us then prove surjectivity. Let $y_A\colon A \to Z$ and $y_B\colon B \to Z$ be morphisms in $\mathcal{A}$. We want to find a morphism $h\colon Z \to A \oplus B$ such that $h \circ j_A = y_A$ and $h \circ j_B = y_B$. We may set:

$$h = y_A \circ p_A + y_B \circ p_B,$$

and this will work. □

10.2. **Shifts and cones.** We may use representability to define shifts and cones *inside* a given dg-category.

Definition 10.7. Let $\mathcal{A}$ be a dg-category. We say that $\mathcal{A}$ is *strongly pretriangulated* if:

- It has a (strict) zero object $0 \in \mathrm{Ob}(\mathcal{A})$, such that
$$\mathcal{A}(0, A) \cong \mathcal{A}(A, 0) \cong 0,$$
for all $A \in \mathrm{Ob}(\mathcal{A})$.
- For any $A \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbb{Z}$, there is an object $A[n] \in \mathrm{Ob}(\mathcal{A})$ called *n-shift* together with an isomorphism
$$\mathcal{A}(-, A[n]) \xrightarrow{\sim} \mathcal{A}(-, A)[n],$$
where $A(-, A)[n]$ is the n-shift of the dg-module $\mathcal{A}(-, A)$.
  Observe that degree p morphisms $X \to A[n]$ correspond to degree $n + p$ morphisms $X \to A$.
- For any closed degree 0 morphism $f\colon A \to B$ in $\mathcal{A}$, there is an object $\mathrm{C}(f)$ in $\mathrm{Ob}(\mathcal{A})$ (called *cone of* f) together with an isomorphism
$$\mathcal{A}(-, \mathrm{C}(f)) \xrightarrow{\sim} \mathrm{C}(\mathcal{A}(-, A) \xrightarrow{f_*} \mathcal{A}(-, B)),$$
where $\mathrm{C}(f_*)$ is the mapping cone taken in $\mathrm{dgm}(\mathcal{A})$.

Example 10.8. Let $\mathcal{A}$ be a dg-category. Then, the dg-category $\mathrm{dgm}(\mathcal{A})$ of right $\mathcal{A}$-dg-modules is strongly pretriangulated. One has to prove that we have isomorphisms

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(-, M[n]) \cong \underline{\mathrm{Hom}}_{\mathcal{A}}(-, M)[n],$$
$$\underline{\mathrm{Hom}}_{\mathcal{A}}(-, \mathrm{C}(f)) \cong \mathrm{C}(\underline{\mathrm{Hom}}_{\mathcal{A}}(-, M) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(-, N)),$$

if $M, N \in \mathrm{Ob}(\mathrm{dgm}(\mathcal{A}))$, $f\colon M \to N$ is a closed degree 0 morphism and $n \in \mathbb{Z}$. The details are left as an exercise (Exercise 10.1).

Example 10.9. Let $\mathcal{A}$ be a dg-category. The derived dg-category $\mathsf{D}_{\mathrm{dg}}(\mathcal{A})$ is strongly pretriangulated. This follows directly from the fact that it is closed under cones and shifts in $\mathrm{dgm}(\mathcal{A})$ (Proposition 8.2).

Let us elaborate on the notion of "internal" shifts and cones as described in Definition 10.7. Using the Yoneda lemma as explained in §10.1, we can describe them with suitable universal properties:

Lemma 10.10. *Let $\mathcal{A}$ be a dg-category.*

(1) *Let $A \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbb{Z}$. A n-shift of A is an object $A[n]$ in $\mathcal{A}$ together with a closed degree n morphism*
$$1_{A,n}\colon A[n] \to A,$$
such that for any degree $n+k$ morphism $f\colon B \to A$, there is a unique degree k morphism $g\colon B \to A[n]$ such that
$$f = 1_{A,n} \circ g.$$

(2) *Let* $f: A \to B$ *be a closed degree* 0 *morphism in* $\mathcal{A}$. *A cone of* f *is an object* $\mathrm{C}(f)$ *in* $\mathcal{A}$ *together with a degree* 1 *morphism* $p: \mathrm{C}(f) \to A$ *and a degree* 0 *morphism* $s: \mathrm{C}(f) \to B$, *such that*

$$dp = 0, \qquad ds = -f \circ p$$

and such that for any pair of morphisms $f': X \to A$ *and* $f'': X \to B$ *with* $|f'| = n + 1$ *and* $|f''| = n$, *there is a unique morphism* $g: X \to \mathrm{C}(f)$ *of degree* n *such that*

$$p \circ g = f, \qquad s \circ g = f'.$$

Proof. We only need to use the Yoneda lemma in order to express representability (see Remark 10.4), and how shifts and mapping cones are defined for dg-modules. □

Being strongly pretriangulated is an "auto-dual" property:

Proposition 10.11. *Let* $\mathcal{A}$ *be a dg-category.* $\mathcal{A}$ *is strongly pretriangulated if and only if* $\mathcal{A}^{\mathrm{op}}$ *is strongly pretriangulated.*

Proof. Since $(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}$, it is enough to prove that $\mathcal{A}^{\mathrm{op}}$ is strongly pretriangulated, assuming that $\mathcal{A}$ is. As usual, morphisms in $\mathcal{A}^{\mathrm{op}}$ will have a "op" decoration, and its absence will mean that we are dealing with a morphism in $\mathcal{A}$.

Let us deal with shifts. If $A \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbb{Z}$, we claim that the n-shift in $\mathcal{A}^{\mathrm{op}}$ is represented by $A[-n]$ (taken in $\mathcal{A}$). We need to find an isomorphism

$$\mathcal{A}(A[-n], -) \xrightarrow{\sim} \mathcal{A}(A, -)[n]$$

in $\mathrm{dgm}(\mathcal{A}^{\mathrm{op}})$. Equivalently, we may look for a closed degree n morphism $1'_{A,n}: A \to A[-n]$, such that for any degree $n+k$ morphism $f: A \to B$ there is a unique degree k morphism $g: A[-n] \to B$ such that

$$f = g \circ 1'_{A,n}.$$

To find both $1'_{A,n}$ and g, we use the fully faithfulness of the Yoneda embedding $\mathcal{A} \hookrightarrow \mathrm{dgm}(\mathcal{A})$. We first look for a closed degree n morphism

$$\mathcal{A}(-, A) \to \mathcal{A}(-, A[-n]) \cong \mathcal{A}(-, A)[-n],$$

which corresponds to a degree 0 morphism

$$\mathcal{A}(-, A) \to \mathcal{A}(-, A),$$

which we may take to be the identity 1_A. Essentially, $1'_{A,n}$ is the identity morphism of A, but viewed as a degree n morphism $A \to A[-n]$. Next, we fix $f: A \to B$ of degree $n + k$. We look for a degree k morphism

$$g_*: \mathcal{A}(-, A[-n]) \cong \mathcal{A}(-, A)[-n] \to \mathcal{A}(-, B)$$

with the desired properties. Yet, such degree k morphisms corresponds to a degree $n + k$ morphism $\mathcal{A}(-, A) \to \mathcal{A}(-, B)$, which is necessarily f_*. Essentially, g is f itself but after suitable shifting. This may look confusing but it is actually quite straightforward.

Let us deal with cones. Let $f: A \to B$ be a closed degree 0 morphism, corresponding to $f^{\mathrm{op}}: B \to A$ in $\mathcal{A}^{\mathrm{op}}$. We claim that the cone of f^{op} is represented by $\mathrm{C}(f)[-1]$ (taken in $\mathcal{A}$). We need to find an isomorphism

$$\mathcal{A}(\mathrm{C}(f)[-1], -) \xrightarrow{\sim} \mathrm{C}(f^*),$$

where $f^* = \mathcal{A}(f,-)\colon \mathcal{A}(B,-) \to \mathcal{A}(A,-)$. Using that $\mathcal{A}(\mathrm{C}(f)[-1],-) \cong \mathcal{A}(\mathrm{C}(f),-)[1]$ thanks to the previous part of the proof, we can equivalently look for an isomorphism

$$\mathcal{A}(\mathrm{C}(f),-) \xrightarrow{\sim} \mathrm{C}(f^*)[-1].$$

Using the Yoneda lemma and unwinding the definitions, this is the same as looking for a degree 0 morphism $j\colon B \to \mathrm{C}(f)$ and a degree -1 morphism $i\colon A \to \mathrm{C}(f)$, such that

$$dj = 0, \qquad di = jf,$$

such that for any pair of morphisms $f'\colon B \to X$ and $f''\colon A \to X$, with $|f'| = n$ and $|f''| = n-1$, there is a unique morphism $g\colon \mathrm{C}(f) \to X$ of degree n such that

$$g \circ j = f', \qquad g \circ i = f''.$$

Using the faithfulness of the Yoneda embedding $\mathcal{A} \hookrightarrow \mathrm{dgm}(\mathcal{A})$, we look for morphisms

$$j_*\colon \mathcal{A}(-,B) \to \mathcal{A}(-,\mathrm{C}(f)) = \mathrm{C}(f_*),$$
$$i_*\colon \mathcal{A}(-,A) \to \mathcal{A}(-,\mathrm{C}(f)) = \mathrm{C}(f_*).$$

Recalling the explicit description of the cone of morphisms of dg-modules, we take j_* to be the inclusion into the second summand $\mathcal{A}(-,B)$ and i_* to be the inclusion into the first summand $\mathcal{A}(-,A)[1]$. We can check they yield morphisms $j\colon B \to \mathrm{C}(f)$ and $i\colon A \to \mathrm{C}(f)$ with the required differentials. Next, let $f'\colon A \to X$ and $f''\colon B \to X$ be given as above. We want a unique morphism

$$g_*\colon \mathcal{A}(-,\mathrm{C}(f)) \to \mathcal{A}(-,X)$$

of degree n, such that $g_* j_* = f'_*$ and $g_* i_* = f''_*$. We may identify

$$\mathcal{A}(-,\mathrm{C}(f)) \cong \mathrm{C}(f_*),$$

and again using that

$$\mathrm{C}(f_*)^p = \mathcal{A}(-,A)^{p+1} \oplus \mathcal{A}(-,B)^p,$$

we are forced to define the morphism $\mathrm{C}(f_*) \to \mathcal{A}(-,X)$ as the unique one having f'_* and f''_* as components. The fully faithfulness of the Yoneda embedding does the rest. $\square$

From the proof of the above result, we obtain *pretriangles* inside any strongly pretriangulated dg-category.

Definition 10.12. Let $\mathcal{A}$ be a strongly pretriangulated dg-category. A *pretriangle* is a sequence of the form

$$A \xrightarrow{f} B \xrightarrow{j} \mathrm{C}(f) \xrightarrow{p} A[1],$$

where f is a closed degree 0 morphism, and the morphisms j and p are closed degree 0 morphisms induced respectively by the inclusion into the second summand and the projection onto the first summand, after using the Yoneda embedding and identifying

$$\mathcal{A}(-,\mathrm{C}(f)) \cong \mathrm{C}(f_*).$$

The morphisms p and j are (up to shifts) the same as the ones described in the universal property of $\mathrm{C}(f)$ and its "dual"

$$\mathcal{A}(\mathrm{C}(f),-) \cong \mathrm{C}(f^*)[-1]$$

as described in the proof of Proposition 10.11. In particular, we also obtain degree a degree -1 morphism $i\colon A \to \mathrm{C}(f)$ and a degree -1 morphism $s\colon \mathrm{C}(f) \to B[1]$ such that

$$di = j \circ f, \qquad ds = f[1] \circ p.$$

Sometimes we will call "rotated pretriangles" the pretriangles in $\mathcal{A}^{\mathrm{op}}$:

$$B[-1] \to \mathrm{C}(f)[-1] \to A \to B.$$

Remark 10.13. Let $f\colon A \to B$ be a closed degree 0 morphism in a dg-category $\mathcal{A}$. Let us concentrate on the morphisms i, j, p, s appearing in the above Definition 10.12. Upon suitable shifting, let us view them as degree 0 morphisms:

$$B \underset{s}{\overset{j}{\rightleftarrows}} \mathrm{C}(f) \underset{p}{\overset{i}{\leftrightarrows}} A[1].$$

If we apply the Yoneda embedding $\mathcal{A} \hookrightarrow \mathrm{dgm}(\mathcal{A})$, they become the natural inclusions and projections to and from the cone $\mathrm{C}(f_*)$. From this, we see that they satisfy the following equations:

$$p \circ i = 1_{A[1]}, \quad p \circ j = 0, \quad s \circ j = 1_B, \quad s \circ i = 0, \quad i \circ p + j \circ s = 1_{\mathrm{C}(f)}.$$

This implies that $\mathrm{C}(f)$ is a direct sum of $A[1]$ and B *in the underlying graded category of* $\mathcal{A}$.

We end this lecture by showing that shifts and cones (and hence pretriangles) are preserved, up to isomorphisms, by dg-functors.

Proposition 10.14. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a dg-functor. Recall Lemma 10.10.*

(1) *Let $A \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbb{Z}$. Let $A[n]$, together with the closed degree n morphism*

$$1_{A,n}\colon A[n] \to A,$$

be an n-shift of A. Then $F(A[n])$, together with the morphism

$$F(1_{A,n})\colon F(A[n]) \to F(A),$$

is an n-shift of $F(A)$. In particular, we have an isomorphism

$$F(A[n]) \cong F(A)[n].$$

(2) *Let $f\colon A \to B$ be a closed degree 0 morphism in $\mathcal{A}$. Let $\mathrm{C}(f)$, together with the morphisms*

$$p\colon \mathrm{C}(f) \to A, \qquad s\colon \mathrm{C}(f) \to B,$$

be a mapping cone of f. Then $F(\mathrm{C}(f))$, together with the morphisms

$$F(p)\colon F(\mathrm{C}(f)) \to F(A), \qquad s\colon F(\mathrm{C}(f)) \to F(B),$$

is a mapping cone of $F(f)$. In particular, we have an isomorphism

$$F(\mathrm{C}(f)) \cong \mathrm{C}(F(f)).$$

Proof. (1) The morphism $1_{A,n}$ yields by definition a closed degree n invertible morphism

$$(1_{A,n})_*\colon \mathcal{A}(-, A[n]) \to \mathcal{A}(-, A).$$

This has a closed degree $-n$ inverse, which is (Yoneda embedding) induced by a closed degree $-n$ morphism

$$1'_{A,-n}\colon A \to A[n],$$

such that
$$1'_{A,-n} \circ 1_{A,n} = 1_{A[n]}, \qquad 1_{A,n} \circ 1'_{A,-n} = 1_A.$$

Let us prove that $(F(A[n]), F(1_{A,n}))$ satisfies the suitable universal property. Let $f\colon B \to F(A)$ be a degree $n+k$ morphism. We look for a unique $g\colon B \to F(A[n])$ of degree k such that $f = F(1_{A,n}) \circ g$. It is immediate to see that necessarily
$$g = F(1'_{A,-n}) \circ f.$$

(2) since F is a dg-functor, the morphisms $F(p)$ and $F(s)$ are respectively of degree 1 and 0 and satisfy
$$dF(p) = 0, \qquad dF(s) = -F(f) \circ F(p).$$
We now show the universal property. Let $f'\colon X \to F(A)$ and $f''\colon X \to B$ with $|f'| = n+1$ and $|f''| = n$. We use the observations in the above Remark 10.13, and we view $F(p)$ and $F(s)$ as degree 0 morphisms, upon shifting objects suitably (we will need to rearrange the degrees of f' and f'', too). If we set
$$g = F(i) \circ f' + F(j) \circ f'',$$
we see that
$$F(p) \circ g = f', \qquad F(s) \circ g = f'',$$
and g is also uniquely determined. Indeed, if g' is such that $F(p) \circ g' = 0$ and $F(s) \circ g' = 0$, then we conclude:
$$g' = F(i)F(p)g' + F(j)F(s)g' = 0.$$
□

Exercises to Lecture 10

Exercise 10.1. Describe explicitly the isomorphisms of Example 10.8. *Hint:* use the Yoneda lemma as in §10.1 and prove suitable universal properties.

Exercise 10.2. Let $\mathcal{A}$ be a dg-category. Prove that if $\mathcal{A}$ is strongly pretriangulated, then for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{A})$, the direct sum $A \oplus B$ exists in $\mathcal{A}$, in the sense that there is an isomorphism
$$\mathcal{A}(-, A \oplus B) \xrightarrow{\sim} \mathcal{A}(-, A) \oplus \mathcal{A}(-, B).$$
Hint: take the cone of a suitable morphism.

Exercise 10.3. Let $\mathcal{A}$ be a strongly pretriangulated dg-category, and let $A \in \mathrm{Ob}(\mathcal{A})$. Describe the shift $A[1]$ as the cone of a suitable morphism. *Hint:* use a zero object.