A Comparison of Numerical Simulation Models For One-Dimensional Infiltration[1]

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ABSTRACT

Six models, employing different ways of discretization of the nonlinear infiltration equation were compared in terms of execution time, accuracy, and programming considerations. All models yielded excellent agreement with water content profiles measured at various times in a sand column. The two explicit models, the θ-based CSMP model and the h-based explicit model, used between 5 and 10 times more computer time than the implicit models. Results obtained with the two models which used the Kirchhoff integral transformation were no better than those obtained with the two h-based implicit models. The implicit schemes with implicit, or explicit evaluation of the hydraulic conductivity and water capacity functions appear to have the widest range of applicability for predicting water movement in soil with both saturated and nonsaturated regions. Excellent agreement was obtained between water content distributions, infiltration rates, and cumulative infiltration volumes calculated with the implicit finite difference model and Philip's quasi-analytical solution.

Additional Index Words: **Fokker-Planck equation, finite differences, CSMP, cumulative infiltration.**

THE THEORY for transient isothermal flow of water into nonswelling unsaturated soil is well understood and has been developed to a large extent in terms of solutions of the nonlinear Fokker-Planck equation. Quasianalytical solutions of this equation in one-dimensional form, subject to fairly restrictive and simple initial and boundary conditions, were developed by Philip (1957a) and more recently by Parlange (1971, 1972b). The predictions appear generally to be in agreement with laboratory observations for situations where the air phase is free to escape in advance of the wetting front.

In the field the description of infiltration is highly complicated since the initial and boundary conditions are usually not constant while the soil characteristics may vary with time and space. In view of this, most efforts have in recent years been concentrated on seeking numerical solutions (Hanks and Bowers, 1962; Rubin, 1966; Whisler and Klute, 1965; Giesel et al. 1972; de Wit and van Keulen, 1972). Freeze (1969) presented a list of papers with finite difference solutions of the one-dimensional infiltration equation. He also listed the boundary and initial conditions used in each paper. This review, and further literature search showed that there exist quite a variety of finite difference solutions employing different forms of the nonlinear Fokker-Planck equation and different ways of discretization. It is the purpose of this paper to compare six of these finite difference schemes in terms of execution time, accuracy, ease of programming, and flexibility. For each of the finite difference schemes a comparison is also made between measured and calculated water content profiles at various times in a sandy soil. Finally, a comparison is made between infiltration profiles as calculated with one of the six tested numerical schemes and as calculated with the quasianalytical solutions of Philip (1957b) and Parlange (1971).

METHODS AND MATERIALS

Theoretical

The partial differential equation used to describe one-dimensional vertical water movement in nonswelling soil is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K(\theta)\left(\frac{\partial h}{\partial z} - 1\right)\right] \qquad [1]$$

where θ is the volumetric water content (cm^3/cm^3); $K(\theta)$ the hydraulic conductivity of the soil (cm/h); z the depth (cm), oriented positively downward; and $h(\theta)$ the soil water pressure (relative to the atmosphere) expressed in cm of water.

With the soil water diffusivity defined as

$$D(\theta) = K(\theta)/C(\theta) \qquad [2]$$

where

$$C(\theta) = d\theta/dh \qquad [3]$$

is the specific water capacity, Eq. [1] can also be written as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) - \frac{\partial K(\theta)}{\partial z}. \qquad [4]$$

Equation (4) is only valid if one assumes the $h(\theta)$ relationship to be unique.

Due to the strong nonlinearity of Eq. [1] and [4], there exists no general analytical solution. However, a specific solution of Eq. [4] was first obtained by Philip (1957b) in the case of infiltration in an homogeneous semiinfinite column satisfying the boundary conditions:

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$$\left.\begin{array}{lll} t<0 & z\geq 0 & \theta=\theta_n \\ t\geq 0 & z=0 & \theta=\theta_o \end{array}\right\} \quad [5a]$$

In a later paper (Philip, 1958), Eq. [1] was solved for the conditions:

$$\left.\begin{array}{lll} t<0 & z\geq 0 & h=h_n \\ t\geq 0 & z=0 & h=h_o \end{array}\right\} \quad [5b]$$

where h_o could take positive values corresponding to an infiltration experiment with submersion. Philip's method led to a solution in the form of a power series in $t^{1/2}$, (Philip, 1955, 1957a, 1957b). Since the series converges only for finite t, the solution becomes unreliable as $t \to \infty$; the t-range of convergence is depending upon the characteristics of soil and the initial and boundary conditions.

By introducing the relation

$$\left(\frac{\partial z}{\partial t}\right)_\theta = -\left(\frac{\partial \theta}{\partial t}\right)_z \cdot \left(\frac{\partial z}{\partial \theta}\right)_t \quad [6]$$

Eq. [4] may be transformed into

$$\frac{\partial z}{\partial t} + \frac{\partial}{\partial \theta}\left[D\frac{\partial \theta}{\partial z}\right] = \frac{dK}{d\theta} \quad [7]$$

which is valid only if the $K(\theta)$ relationship is unique.

In a series of papers Parlange (1971, 1972b) has proposed a quasianalytical solution of Eq. [7] with the boundary conditions given by Eq. [5a] and [5b], as well as with a constant flux condition at the soil surface,

$$\left.\begin{array}{lll} t<0 & z\geq 0 & \theta=\theta_n \\ t\geq 0 & z=0 & q=K-D\dfrac{\partial\theta}{\partial z}=q_o \end{array}\right\} \quad [8]$$

A serious limitation of the use of quasianalytical solutions for practical cases is imposed by the representativity of the initial and boundary conditions ([5a], [5b] and [8]). The soil column is considered to be semiinfinite and with an initial uniform water content, the boundary conditions are constant in time, the flux at the soil surface cannot exceed saturated conductivity—or if this is the case, the Parlange solution is valid until ponding occurs; heterogeneity cannot be taken into account. Most of these conditions are scarcely met in practice. Consequently, numerical solutions without such restrictions were developed; they mostly differ in the way of discretization or in the method of linearization used to solve Eq. [1] or [4].

Of these, a total of six models will be compared in this paper. They are listed in difference form in Appendix 1. Since we are dealing with a flow domain with very simple geometry, it was decided not to consider the finite element method which is up to now mostly used for more complex flow domains (Neuman et al., 1974).

Model 1 uses a continuous system simulation language, CSMP (IBM, 1967), employed by Wierenga and de Wit (1970) to describe heat transfer in soil and subsequently used by Bhuiyan et al., 1971; de Wit and van Keulen, 1972; van der Ploeg, 1974; Dane and Wierenga, 1975; and others for infiltration problems. For the integration in model 1, CSMP provides six fixed-step integration routines and three variable-step integration methods. The availability of these and other preprogrammed subroutines makes this a very simple approach.

If both saturated and nonsaturated regions in the soil profile are of interest, it is better to consider h instead of θ as the independent variable (Philip, 1958). Using the specific water capacity, Eq. [1] is then transformed into

$$C(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\left(\frac{\partial h}{\partial z} - 1\right)\right]. \quad [9]$$

In the saturated zone Eq. [9] becomes Laplace's equation, provided the soil is homogeneous and isotropic. With both saturated and non-saturated regions h varies from positive values in the saturated zone to negative values in the unsaturated zone. Equation [9] can be solved using explicit (model 2) or implicit (model 3) methods. In the explicit method (Staple, 1966) a series of linearized independent equations is solved directly, while in the implicit method (Hanks-Bower, 1962; Whisler-Klute, 1965; Rubin, 1966; Freeze, 1969; Moltz-Remson, 1970; Vauclin et al. 1975; and others) a system of linearized equations has to be solved. For a given grid point at a given time, the values of the coefficients $C(h)$ and $K(h)$ can be expressed either from their values at the preceding time step (explicit linearization, model 3) or from a prediction at

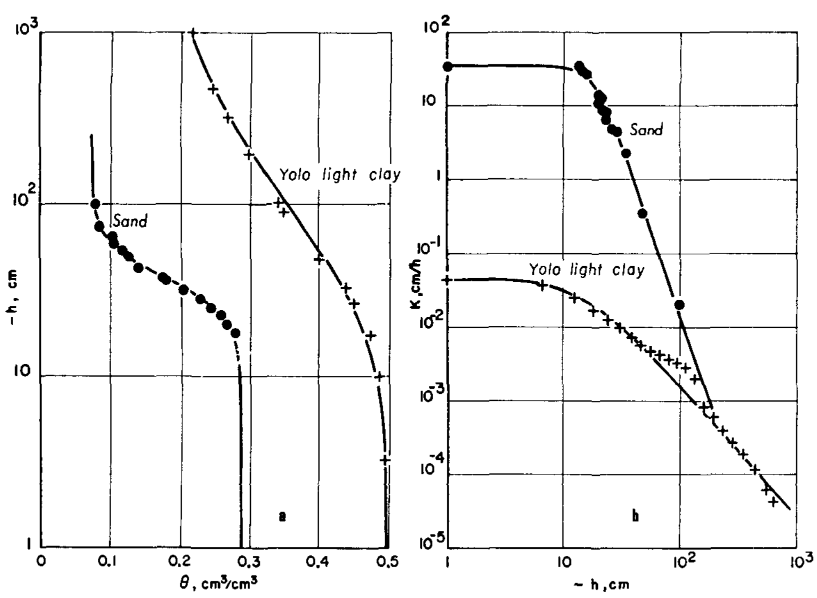

Fig. 1—Relationships between the soil water pressure h, the water content θ and the hydraulic conductivity K for the two soils used in this article. The plain circles are experimental points obtained for the sand; the crosses, experimental points obtained for the Yolo light clay, and the full lines correspond to the functional relationships given by Eq. [12] and [17], respectively.

time ($t + 1/2\ \Delta t$) using a method described by Douglas and Jones, 1963 (implicit linearization, model 4).

In order to overcome stability problems due to the presence of very high water pressure gradients near the wetting front during infiltration in a dried material, the Kirchhoff integral transformation has also been used (Rubin, 1966; Raats and Gardner, 1974; Khanji et al., 1974)

$$U(h) = \int_{h_0}^{h} K(h)dh. \qquad [10]$$

Eq. [9] is then written as

$$F(U)\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial z^2} - G(U)\frac{\partial U}{\partial z} \qquad [11]$$

where

$$F(U) = \frac{C(h)}{K(h)} \quad \text{and} \quad G(U) = \frac{1}{K(h)}\frac{dK(h)}{dh}.$$

Equation [11] can also be solved implicitly with explicit linearization (model 5) or implicit linearization (model 6).

The six different models were tested by comparing water content profiles calculated at given times by each of the models with results obtained from an infiltration experiment carried out in the laboratory. Total computer time on an IBM 360/67, and the mass balance for each method were also compared.

Experimental

A series of infiltration experiments were done in the laboratory using a plexiglass column, 93.5-cm long and 6-cm inside diameter (ID) uniformly packed with sand to a bulk density of 1.66 g/cm^3. The column was equipped with tensiometers at depths of 7, 22, 37, 52, 67, and 82 cm below the soil surface. Each tensiometer had its own pressure transducer. The changes of water content at different depths were obtained by gamma ray attenuation using a source of 100 mCi of Americium-241. Details of the measuring techniques were reported in a previous paper (Vachaud and Thony, 1971). A constant water pressure was maintained at the lower end of the column and a constant flux was imposed at the soil surface (z=0). The hydraulic conductivity-water content relationship of the soil was obtained by analysis of the water-content and water-pressure profiles during transient flow (Watson, 1966; Vachaud and Thony, 1971). The soil-water pressure water content relationship was obtained at each tensiometer depth by correlating tensiometer readings and water content measurements during the experiments. The following analytical expressions, obtained by a least square fit through all data points were chosen for characterizing this soil (Fig. 1):

$$K = K_s\frac{A}{A + |h|^{\beta}},\ K_s = 34 \text{ cm/}h,\ A = 1.175 \times 10^6,\ \beta = 4.74 \qquad [12]$$

$$\theta = \frac{\alpha(\theta_s - \theta_r)}{\alpha + |h|^{\beta}} + \theta_r;\ \theta_s = 0.287,\ \theta_r = 0.075;\ \alpha = 1.611 \times 10^6,\ \beta = 3.96$$

where subscript s refers to *saturation*, i.e. the value of θ for which h = 0, and the subscript r to *residual water content*. The initial and boundary conditions for infiltration of water in the sand were

$$\begin{array}{lll} t < 0 & 0 < z < 70 \text{ cm} & \theta_n = 0.10 \text{ cm}^3/\text{cm}^3, \\ & & (\text{or } h_n = -61.5 \text{ cm}) \\ t \geq 0 & z = 0 & q = 13.69 \text{ cm/}h \qquad [13] \\ t \geq 0 & z \geq 70 \text{ cm} & \theta = \theta_n = 0.10 \text{ cm}^3/\text{cm}^3. \end{array}$$

RESULTS AND DISCUSSION

Numerical vs. Experimental Results

Observed water content profiles during the constant flux infiltration are presented in Fig. 2, together with water con-

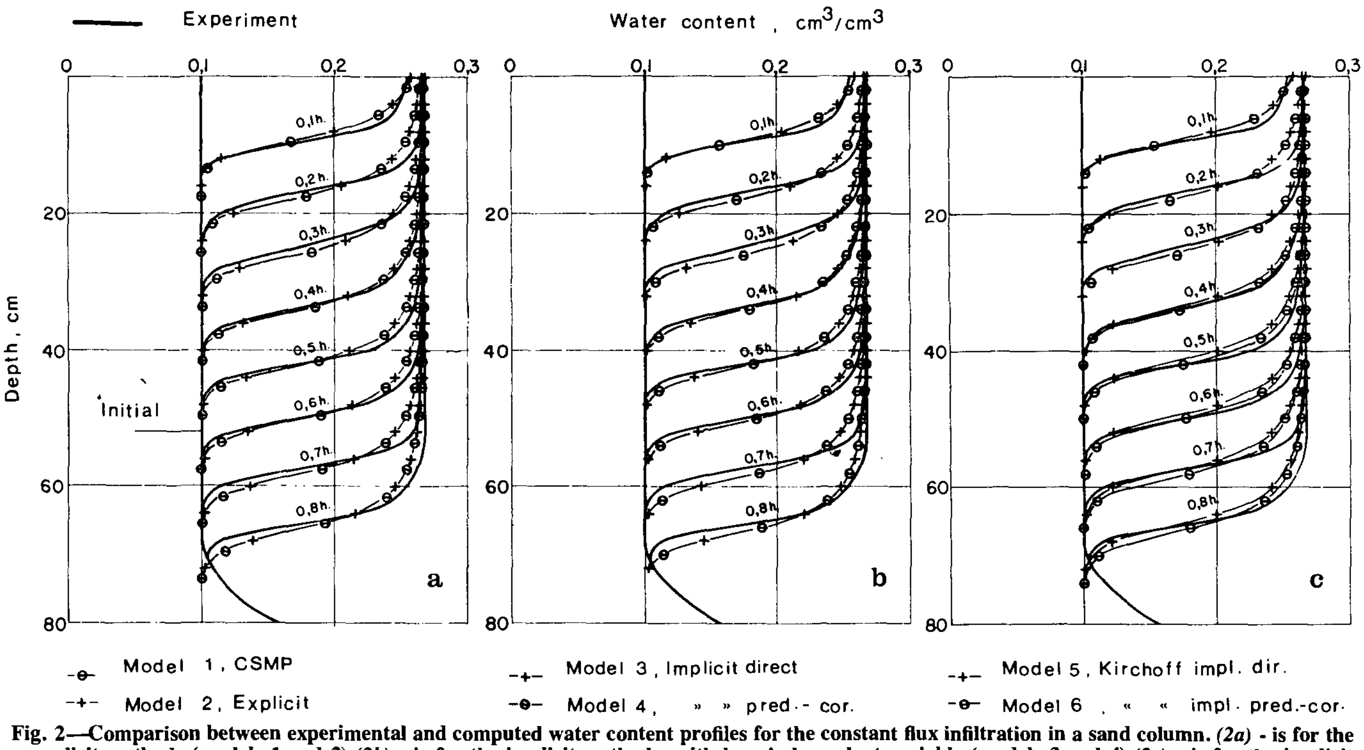

Fig. 2—Comparison between experimental and computed water content profiles for the constant flux infiltration in a sand column. *(2a)* - is for the explicit methods (models 1 and 2) *(2b)* - is for the implicit methods, with *h* as independent variable (models 3 and 4) *(2c)* - is for the implicit methods with *U* as independent variable (models 5 and 6).

Table 1—Comparison between execution times and mass balance for the various models.

Reference	Type of discretisation	Linearization	Time and depth intervals	Execution time	Number of computation nodes	Mass balance %
				sec		
Model 1	C.S.M.P. Eq. [1]	Explicit	$\Delta t = 0.4$ sec	209	313,085	0.21
Model 2	Explicit direct-Eq. [9]	Explicit	$\Delta z = 1$ cm	223	301,172	0.23
Model 3	Implicit,	Explicit		23	24,172	0.41
Model 4	Eq. [9]	Implicit with prediction-correction	$\Delta t = 5$ sec	43	48,304	0.28
Model 5	Implicit,	Explicit	$\Delta z = 1$ cm	29	23,819	1.80
Model 6	Eq. [11]	Implicit with prediction-correction		58	48,010	0.33

tent profiles calculated with the various models presented in Appendix I, using the functional expressions (12) and the boundary conditions (13). In all cases the rate of advance of the water front is particularly well described. Some discrepancies are found between numerical and actual water content profiles in the high water content domain, but because of uncertainties in the $K(\theta)$ and $\theta(h)$ relationships, and in the experimental determination of the water content, it is impossible to determine which is the correct water content distribution. Note that all numerical models yield comparable results, which are not significantly different from the measured water contents.

Table 1 presents data on execution time and the mass balance at $t = 0.8$ h, after 10.952 cm of water had infiltrated into the soil. Execution time is defined as the total time required for execution of the program, including compilation. To reduce computer time, all calculations were started with a uniform soil profile 5 cm deep; the profile depth considered in the program was activated with the advance of the water front till a maximum depth of 70 cm. The criterion used for activating the grid was the same for all programs: a change of water content of 0.002 cm^3/cm^3. Table 1 shows that the explicit methods required 5-10 times more execution time at a comparable mass balance, defined as the percentage difference between the water infiltrated and the water in the column at the end of infiltration, minus the water initially present, and corrected for drainage out of the profile. Except for model 5 all the methods had an excellent mass balance (from 0.2 to 0.4%). The best performance in terms of execution time was obtained with model 3 (implicit solution with explicit linearization of Eq. [9]. Other aspects should, however, be taken into account when considering which method is to be used under given conditions.

The use of Model 1 (CSMP) requires a longer execution time than the implicit models, but, as discussed by Bhuiyan et al., 1970 and van der Ploeg, 1974, it is very easy to program and has the advantage that it contains many preprogrammed routines, including those for data entry and output. The execution time depends upon the integration method and was found to be twice as long when the fourth order Runge Kutta integration method was used, instead of the rectangular method used for Fig. 2. The execution time may be reduced, however, by increasing the layer thickness and adjusting the time step as a function of the soil water diffusivity (Van Keulen and Van Beek, 1971). Defining D_{max} as the maximum value of the soil water diffusivity in the soil profile at time t, the CSMP program with rectangular integration method is stable when

$$\Delta t < (r \cdot (\Delta z)^2 / D_{max}) \tag{14}$$

where Δz is the layer thickness and r an arbitrary chosen coefficient equal to 0.5 for the sand of Fig. 1. The CSMP method is limited by the availability of a large computer with simulation software, and by extensive use of computer time when the water content approaches saturation and Δt becomes very small (D_{max} becomes large). On the other hand, it is a very useful tool for describing rather complex systems such as the movement of solutes through soil (van Genuchten and Wierenga, 1976).

Model 2 (direct explicit) has also the advantage that it is simple and easy to program. However, for reasons of stability the time step should be adjusted with Eq. [14] during each calculation with $r < 0.5$. Staple (1966) used a value of r between 0.15 and 0.3, with a depth interval of 1.5 cm, to compute infiltration and redistribution in a vertical column of loam soil. Since D, is large in wet soil, this method is too slow when the soil is quite wet, notably during the infiltration. Another limitation is that when the soil becomes saturated, the right hand side of Eq. [9] is divided by very small C-values (see Appendix I).

The implicit methods (models 3 to 6) generally use much larger time steps than the explicit methods, but their stability conditions have to be determined by trial and error, as they depend upon the degree of nonlinearity of the equations. Also, the programming is more involved than for the explicit method. A tridiagonal system of equations results, which can be solved by direct elimination using Thomas' algorithm (Remson et al., 1971, p. 170). Implicit evaluation of the coefficients at time ($t + 1/2\ \Delta t$) (model 4 and 6, a method described by Douglas and Jones, 1963) requires that the tridiagonal system of equations be solved twice for each time step: first at time ($t + 1/2\ \Delta t$) to obtain values for K and C, then at time ($t + \Delta t$) to evaluate the pressure distribution; on the other hand, the implicit models with explicit evaluation of the coefficient (model 3 and 5) use only about half the computer time, but as a result have a poorer mass balance. The mass balance is still good with Model 3, but very poor with model 5; our feeling is that for this type of soil U (Eq. 10) changes too drastically with time during infiltration for using any explicit evaluation of this integral. The main advantages of using implicit methods are their stability, even for fairly large time steps (5 sec instead of 0.4 sec in our case for the explicit model), and their flexibility for solving flow problems when saturated and unsaturated zones have to be considered simultaneously, since for $C = 0$ one simply has to solve the Laplace's equation. Because the use of the Kirchhoff's transform resulted in a poor mass balance with explicit linearization, and with implicit linearization had comparable accuracy and execution time than the solution of Eq. [9] with an implicit linearization, it appears preferable to use Eq. [9] directly.

The numerical scheme corresponding to Model 3 (implicit discretization of Eq. [9] with explicit linearization of K and C), and referred to hereafter as "h implicit model," was used for further comparison with the quasianalytical solutions of Parlange (1972b) and Philip (1957a, 1957b).

Comparison between the "h-Implicit Model" and Parlange's Solution

Parlange showed that after transformation of Eq. [7], subject to conditions 8 (flux type upper boundary condition), the water content profiles can be estimated by

$$z(\theta, t) = \int_{\theta}^{\theta_1(t)} \frac{D(\beta)}{(q_o - K_n)\left[\dfrac{\beta - \theta_n}{\theta_1(t) - \theta_n}\right] - [K(\beta) - K_n]} d\beta \quad [15]$$

where

K_n is the hydraulic conductivity at the initial water content θ_n

q_o is the value of the flux imposed at $z = 0$ for $t \geq 0$, and

$\theta_1(t)$ is the water content at the soil surface for given time t where

$$t = \int_{\theta_n}^{\theta_1(t)} \frac{D(\alpha)(\alpha - \theta_n)}{[q_o - K(\alpha)][q_o - K_n]} d\alpha. \quad [16]$$

The water content at the soil surface $\theta_1\ (t)$ was calculated with $D(\theta)$ obtained from Eq. [2], [3], and [12], and solving Eq. [16] numerically with Simpson's rule with an integration interval of

$$\Delta\alpha = [\theta_1(t) - \theta_n]/100.$$

Figure 3a presents the water content at the soil surface calculated with this method, as well as the water content at the soil surface calculated with "h-implicit model." The agreement between θ_1 calculated with these two methods is satisfactory, since for $t > 0.05$ hours the difference in surface water contents calculated with the two methods is < 0.0025 cm³/cm³ (relative error $< 10^{-2}$).

Notwithstanding this fair agreement, the water content profiles, obtained by integration of Eq. [15] at times 0.05, 0.1, and 0.2 hour (crosses in Fig. 3b) do not correspond to the profiles obtained by numerical simulation (solid lines in Fig. 3b). Realizing that the method used by Parlange to establish his Eq. [16] fails to satisfy the continuity requirement, another computation of the water content profiles was done by integration of Eq. [15], using for $\theta_1\ (t)$ the values obtained by our numerical solution which are slightly different, but which satisfy the mass balance within 4.10^{-3} (see Table 1). The corresponding profiles (circles in Fig. 3b) now coincide with the values obtained with the "h-implicit model." Thus if θ_1 can be estimated accurately, Parlange's solution and numerical methods yield the same result. Therefore, Parlange proposes a simple iterative procedure (1973), where the time corresponding to any particular value of θ_1 is obtained by measuring the area under the profile and dividing it by q_o. However the time value thus obtained does not correspond with Eq. [16], stressing once more the problem of mass balance requirement. As an example we have calculated the position of the waterfront at 0.05 hours during infiltration into sand at a constant flux of 13.69 cm/hour, having an initial water content of 0.10 cm³/cm³ (Table 2). At $t = 0.05$ hour, θ_1 calculated with the "h-implicit" method is 0.2459 cm³/cm³. With Eq. [15] we obtained $\theta_1 = 0.2506$, a relative difference of only 2%. However, this relatively small difference in surface water content caused a difference in advance of the wetting front near $\theta = 0.16$ cm³/cm³ of 1.45 cm, or a relative difference of 24%.

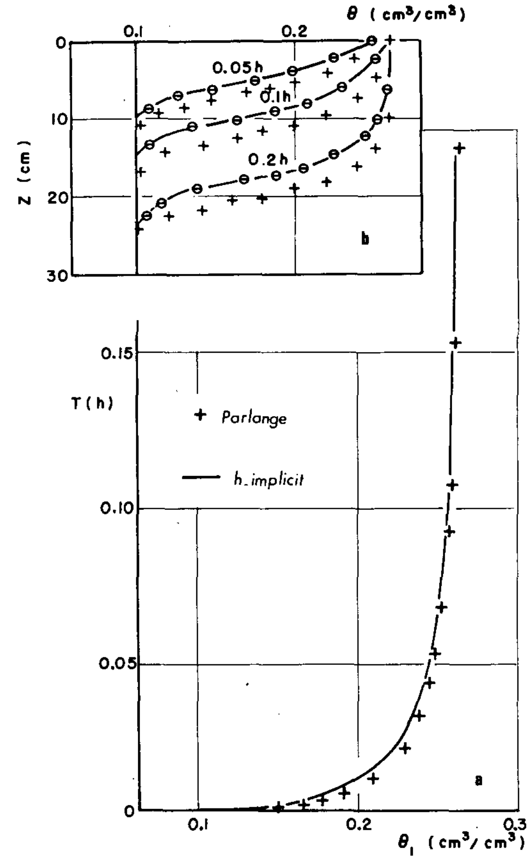

Fig. 3—Comparison between the numerical solution obtained by the h-implicit model (*solid line*) and the quasianalytical solution of Parlange (*crosses*) for the constant flux infiltration. (*3a*) The evolution of the water content θ_1 at the soil surface with time. (*3b*) The water content profiles at time 0.05, 0.1, and 0.2 hour. The crosses correspond to the values computed at time t (Eq. 15) with θ_1 obtained from Eq. [16]; the circles correspond to values computed at the same time, with the same equation, but with θ_1 obtained from the h-implicit model; the solid lines were obtained with model 3.

Comparison between the "h-Implicit" Model and Philip's Solution

In order to test the validity of the numerical model under a different type of boundary condition and for a different soil, numerical results were also compared with the qua-

Table 2—Determination of the water content profile, for the sand, at $t = 0.05$ h, using Eq. [15]. For a given θ, the first depth was obtained with the value of θ_1 from Eq. [16] ($\theta_1 = 0.2506$ cm³/cm³). The second depth was obtained by using as surface water content the values computed with the h-implicit model ($\theta_1 = 0.2549$ cm³/cm³).

θ	0.24	0.22	0.20	0.18	0.16	0.14	0.12	0.10
z cm ($\theta_1 = 0.2506$)	2.02	4.05	5.26	6.15	6.93	7.72	8.77	11.9
z cm ($\theta_1 = 0.2459$)	0.91	2.86	4.02	4.82	5.60	6.35	7.36	10

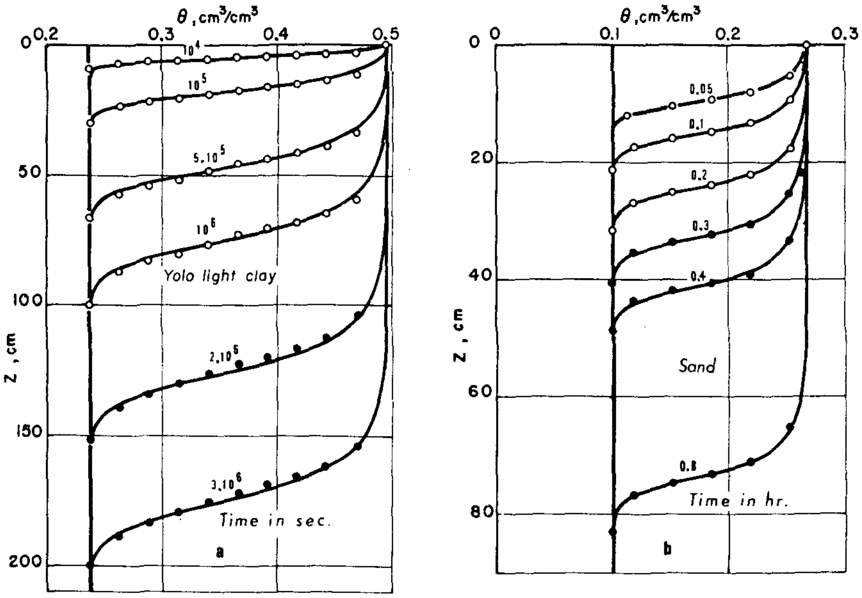

Fig. 4—Comparison between water content profiles obtained by the h-implicit model *(solid line)* and by Philip's solution *(circles)* for constant head infiltration. The open circles were calculated with Eq. [18] and the closed circles with the profile at infinity equation.

sianalytical solution of Philip, which was obtained by solving the Eq. [4] subject to condition of a constant pressure at the soil surface (Eq. [5b]. Comparisons were made with two very different soil materials, e.g., the Yolo light clay used by Philip and first described by Moore (1939) and the sand used in the present experiment.

Infiltration in the Yolo Light Clay

This well-known experiment was used by Philip, 1957, as an example of application of his quasianalytical solution. The soil characteristics are given in Fig. 1 and in Eq. [17], using the same representation as in the previous case. The data points for $\theta(h)$ were taken from Philip, 1969, p. 221; values for $K(\theta)$ were presented by Philip, 1957b, p. 353; points for $K(h)$ were determined from $\theta(h)$ and $K(\theta)$.

$$K = K_s \cdot \frac{A}{A + |h|^{\beta}};\ K_s = 4.428.10^{-2}\ \text{cm}/h,\ A = 124.6,$$
$$B = 1.77$$
$$\theta = \frac{\alpha(\theta_s - \theta_r)}{\alpha + (\ln|h|)^{\beta}} + \theta_r;\ \theta_s = 0.495,\ \theta_r = 0.124,$$
$$\alpha = 739,\ \beta = 4 \text{ for } h < -1 \text{ cm} \qquad [17]$$
$$\theta = \theta_s \text{ for } h \geqslant -1 \text{ cm}$$

The functional relationships represented by Eq. [17] and by solid lines in Fig. 1 describe fairly well the data as tabulated by Philip except near $h = -100$ cm for the $K(h)$ curve. Also the diffusivity values derived from Eq. [17] using Eq. [2] and [3] are somewhat different from those presented by Philip (1957b, p. 353), particularly near θ_s.

Figure 4a compares infiltration profiles obtained by Philip (dot), with infiltration profiles computed with the "h-implicit model" (solid line) using Eq. [9] subject to the conditions:

$$t < 0 \qquad z \geqslant 0 \qquad \theta_n = 0.2376$$
$$t \geqslant 0 \qquad z = 0 \qquad \theta_o = 0.4950$$

The numerical computation was made with a depth interval $\Delta z = 1$ cm, and a time step varying from 40 sec (for short time) to 500 sec (for $t > 5.10^5$ sec). The execution time on an IBM 360-67 was 3 min for simulating 10^6 sec (11.6 days).

The agreement is excellent. Note simply that in the region of the wetting front the Philip's solution is less smooth than the profiles obtained by the finite difference method. In order to compare more accurately each method, numerical data are given in Table 3. For "limited times" Philip's method gives at each time t the depth, $z(t)$ which reaches a given water content, θ_i, according to;

$$z(t, \theta_i) = \phi(\theta_i)t^{1/2} + \chi(\theta_i)t + \psi(\theta_i)t^{3/2} + \omega(\theta_i)t^2 + \cdots + f_n(\theta_i)t^{n/2}. \qquad [18]$$

The numerical model, on the other hand, calculates h (and consequently θ) for a given value of z. As a result, extrapolations are necessary, at a given stage of calculations to compare the results. Since Philip's solution is here considered as the standard, θ was selected as the independent variable in Table 3; at times $t = 10^5$, 10^6, and 3×10^6 sec the numbers in the column labeled Philip were obtained directly from Philip (1957c, Fig. 7, p. 447) and personal communication; the numbers in the column labeled NUMER were obtained by extrapolation from the $\theta(z)$ profiles computed with $z = 1$ cm. The relative error is $< 3\%$ at 10^6 sec, and $< 2\%$ (for any value) at 3.10^6 sec.

In order to test the validity and the stability of the method when saturation occurs, additional calculations were made with a constant positive head at the soil surface as upper boundary condition. This example, corresponding to Eq. [5b] was solved analytically by Philip (1958). Figure 5

Table 3—Comparison between water content profiles determined with the solution of Philip and the *h*-implicit model.

	Infiltration in the Yolo light clay					
	$t = 10^5$ sec, (1.16 days)		$t = 10^6$ sec, (11.6 days)		$t = 3.10^6$ sec, (34.7 days)	
	z	z	z	z	z	z
θ	Philip	Numer	Philip	Numer	Philip	Numer
cm³/cm³	cm					
0.4822	9.1	7.8	52.5	49.1	--	--
0.4563	12.9	11.8	63.4	61.2	157.8	158.1
0.4306	14.5	14.2	66.9	66.6	163.3	164.5
0.4049	15.6	15.8	69.5	70.3	166.8	168.4
0.3791	16.7	17.2	71.9	73.1	169.2	171.9
0.3534	18.0	18.4	74.8	75.6	173.0	174.6
0.3277	19.8	19.5	78.9	78.0	176.0	177.2
0.3019	21.3	20.4	82.3	80.5	179.6	179.9
0.2762	22.7	22.2	85.5	83.5	184.2	182.9
0.2504	24.6	24.4	90.0	88.5	187.1	187.1
0.2401	26.5	26.9	94.6	94.6	199.9	200.0
	Infiltration in the sand					
	$t = 0.1\,h$		$t = 0.2\,h$		$t = 0.8\,h$	
	z	z	z	z	z	z
	Philip	Numer	Philip	Numer	Philip	Numer
0.2523	9.4	9.4	17.7	17.3	65.2	65.7
0.2356	12.0	12.0	20.7	20.5	69.2	69.3
0.2189	13.2	13.3	22.1	22.1	71.1	71.2
0.2021	14.1	14.2	23.1	23.1	72.3	72.3
0.1854	14.8	14.8	23.8	23.9	73.2	73.2
0.1686	15.3	15.4	24.5	24.6	74.0	74.1
0.1519	15.9	15.9	25.2	25.3	74.8	74.8
0.1351	16.5	16.6	25.9	26.0	75.7	75.6
0.1184	17.3	17.5	26.8	26.9	76.8	76.7
0.1016	19.5	19.9	29.5	29.8	78.6	79.8

shows a comparison between water content profiles calculated with Philip's (1959) solution and with the "*h*-implicit method" assuming a constant positive head of water on the soil surface of $h_o = +25$ cm. The depths of the saturated zone where $h = 0$, at times $t = 10^4$ and 2.10^5 sec, were 3.8 and 20 cm as calculated by the analytical method (Philip, 1958; p. 283, Fig. 4) and 3.5 and 20.5 cm, respectively, as calculated by the numerical method. Again the agreement is excellent. The numerical model was stable, and water contents and fluxes in the saturated zone ($h \geq 0$) were constant to the sixth decimal.

Infiltration in the Sand

The same comparison as for the Yolo light clay was made for the sand material. The results are presented in Fig. [4b]. Using the functional relations given in Eq. [12] for characterizing the hydraulic properties of this soil, the water content profiles subject to the conditions

$$\begin{array}{lll} t < 0 & z \geqslant 0 & \theta_n = 0.10\ \text{cm}^3/\text{cm}^3 \\ t \geqslant 0 & z = 0 & \theta_0 = 0.267\ \text{cm}^3/\text{cm}^3 \\ & & (\text{or } h_o = -20.73\ \text{cm}) \end{array} \qquad [19]$$

were determined with the "*h*-implicit model" ($\Delta z = 1$ cm, $\Delta t = 5$ sec; total execution time was 30 sec for simulating 0.8 hours), and quasianalytically with the solution of Philip.

The prediction of the water content profiles using Philip's method is only valid within the domain of convergence of the series (Eq. [18]). To calculate the time for which the series would converge, Philip (1959, p. 250) introduces a characteristic time of infiltration, t_{grav}, as:

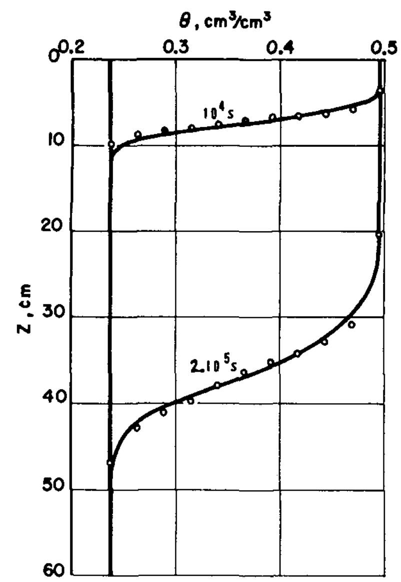

Fig. 5—Comparison between water content profiles obtained by the *h*-implicit model *(solid line)* and by Philip's solution *(circles)* for infiltration in Yolo light clay with a constant positive head $h_0 = +25$ cm at the soil surface.

$$t_{\text{grav}} = [S/(K_o - K_n)]^2 \qquad [20]$$

where K_o is the hydraulic conductivity corresponding with θ_o and S is the sorptivity, defined as

$$S = \int_{\theta_n}^{\theta_o} \phi d\theta.$$

For the sand material, S was found to be 5.441 cm/$h^{1/2}$, and the characteristic time was $t_{\text{grav}} = 0.16\ h$. Consequently for $t \leq 0.2$ hour the water content profile could be calculated with Eq. [18]. In our calculations the series was limited to four terms. To use more terms of the series would according to Philip, "extend the range of accurate results only by small amounts, quite disproportionate to the extra labor involved." For $t \geq 0.3\ h$, the profiles were calculated by an approximation of the "infinite" profile, as proposed by Philip (1957c, p. 444). The agreement between infiltration profiles calculated with Philip's method and the "*h*-implicit method is very good (see Fig. 4b and Table 3). It appears, from Fig. 4a, that the power series solution (Eq. 18) and the asymptotic solution of the profile at infinity do overlap, as they should. Philip assumed that the *t*-range of convergence for the expansion of Eq. [18] was of the same order of magnitude as t_{grav}, defined in Eq. [20].

Avoiding any problems of continuity, Parlange (1971) developed a solution for Eq. [7] subject to the condition of a constant water content at the soil surface (condition 5a). The flux at the soil surface, denoted as $-C(t)$ by Parlange (1971; Eq. [9], p. 171), was sought as part of the required solution, and obtained by a series of approximations. However, without entering into the details of his solution, we experienced the same problems as with the constant flux solution, e.g., the sensitivity of the $C(t)$ relationship made it impossible to predict the position of the water content profiles in accordance with Philip's solution. The profiles

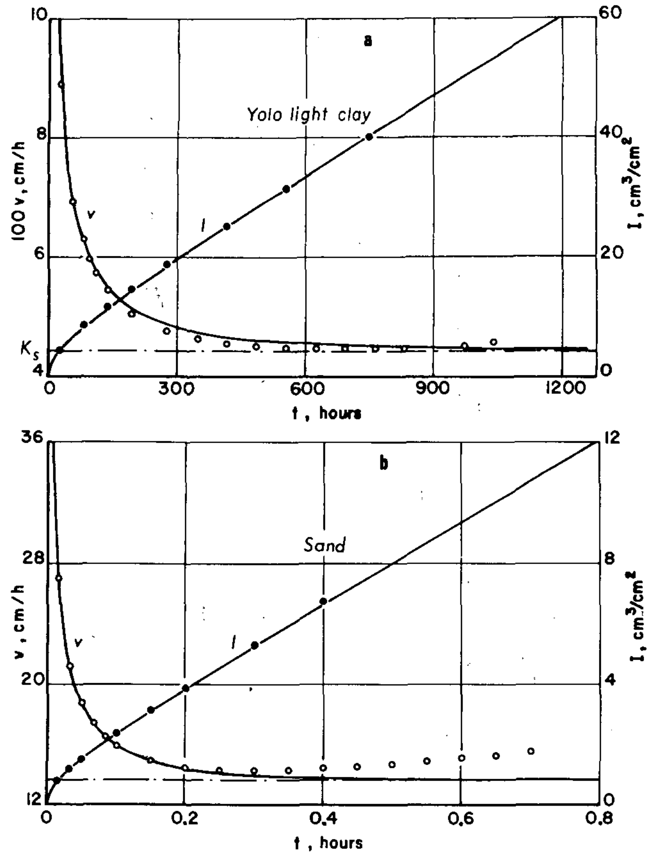

Fig. 6—Comparison between infiltration rate and cumulative infiltration obtained by the *h*-implicit model *(solid line)* and by Philip's solution *(circles)*.

determined by different numbers of the approximation sequences oscillated around Philip's profiles; this result is similar to the previous observations of Knight and Philip (1974). An iterative procedure analog to the one developed for the constant flux problem can be applied for this constant head problem, where time values corresponding to arbitrary C-values are calculated from the integral $\int_0^t - C \; dt$ measured as the area under the profile. However the $C(t)$ relationship obtained this way doesn't correspond with the relation used by Parlange (1971, Eq. [12], p. 171), emphasizing once more the problem of continuity requirement.

Infiltration Rate and Cumulative Infiltration

Finally for both soils a comparison was made between infiltration rate and cumulative inflow obtained by either technique. The results are presented in Fig. 6a (Yolo light clay) and Fig. 6b (sand). For the Yolo light clay, the agreement between the two solutions is very good as long as the series solution of Philip is valid ($t < 600$ hours). The small discrepancy between calculated infiltration rates between 300 and 600 hours can be explained by uncertainties in the $D(\theta)$ relationship. After $t = 600$ hours, the numerical solution of the infiltration rate tended towards its physical limit, K_s. However, the quasianalytical solution started to diverge at this point.

For the sand, the agreement between infiltration rate and cumulative inflow calculated with the two procedures is excellent until $t = 0.29$ hour. After this time the series expansion of Philip diverges, and the asymptotic solution has to be used. This confirms that the t-range of convergence is indeed of the same order of magnitude as t_{grav}, as was stated before. The reason for the relatively short time span over which Eq. [18] is valid for the sand is that the influence of gravity is very important for the sand, but small, compared to capillary forces, for the Yolo light clay. This is further shown in Table 4 where the integrals of the different terms of Eq. [18] are given for both soils. Whereas Eq. [18] could, for most purposes, be reduced to two terms in the case of Yolo light clay, the relative magnitude of the four coefficients ϕ, χ, ψ and ω is such that even a four-term series approximation of Eq. [18] becomes inadequate, after 5-cm water has infiltrated into the soil.

Table 4—Values of the integrals $\int_{\theta_0}^{\theta_n} f_n d\theta$ of Philip's series expansion for Yolo light clay loam and the sand.

	$\int_{\theta_0}^{\theta_n} f_n \, d\theta$	
f_n	Yolo light clay loam	Sand
ϕ (cm·$h^{-1/2}$)	0.7524	5.411
χ (cm·h^{-1})	$1.627 \cdot 10^{-2}$	5.149
ψ (cm·$h^{-3/2}$)	$3.034 \cdot 10^{-4}$	3.643
ω (cm·h^{-2})	$1.161 \cdot 10^{-6}$	1.764

Conclusion

The close agreement between observed water content profiles and those computed with six different numerical schemes, as well as the close agreement among these six schemes, indicate that numerical models are a reliable tool for predicting infiltration of water into soil. For the conditions for which the comparison was made, the explicit methods used between 5 to 10 times more computer time than the implicit methods. Results obtained with the numerical schemes which used the Kirchhoff integral transformation were no better than those obtained with the implicit model with implicit linearization of the hydraulic conductivity and water capacity functions. Considering computer time and stability problems, it appears that the implicit finite difference approximation with implicit or explicit evaluation of the hydraulic conductivity and water capacity functions has the widest range of applicability for predicting water movement in soil with both saturated and nonsaturated regions. For specific cases explicit models may be preferred, mainly because they are easy to program.

The excellent agreement between water content distributions, infiltration rates and cumulative infiltration volumes obtained with the implicit finite difference approximation and Philip's quasianalytical solution shows that numerical solutions can yield very accurate results at moderate costs in terms of computer time. In a future paper, a systematic study will be made of the different equations used for describing infiltration of water into soil. These equations will be tested with the implicit finite difference model used in this paper.

APPENDIX I

Discretization Schemes Used for the Various Models

θ is the water content cm³/cm³,
h is the soil water pressure – cm of water – $h(\theta)$ is unique,
K is the soil hydraulic conductivity – cm/hour ($K(\theta)$ or $K(h)$),
C is the specific water capacity (cm⁻¹) ($C(\theta)$ or $C(h)$),
j refers to time, and
i refers to depth

Model 1—Eq. [1]—Explicit scheme solved with C.S.M.P.

$$\theta_i^{j+1} = \theta_i^j + \frac{1}{\Delta z}\int_t^{t+dt} \cdot \left[K^j_{i+1/2}\left(\frac{h^j_{i+1} - h^j_i}{\Delta z} - 1\right) - K^j_{i-1/2}\left(\frac{h^j_i - h^j_{i-1}}{\Delta z} - 1\right)\right] dt$$

Model 2—Eq. [5]—Explicit scheme solved directly

$$h_i^{j+1} = h_i^j + \frac{\Delta t}{C_i^j \Delta z}\left[K^j_{i+1/2}\left(\frac{h^j_{i+1} - h^j_i}{\Delta z} - 1\right) - K^j_{i-1/2}\left(\frac{h^j_i - h^j_{i-1}}{\Delta z} - 1\right)\right]$$

Model 3—Eq. [5]—Implicit scheme with explicit linearization

$$C_i^j \cdot \frac{h_i^{j+1} - h_i^j}{\Delta t} = \frac{1}{\Delta z}\left[K^j_{i+1/2}\left(\frac{h^{j+1}_{i+1} - h^{j+1}_i}{\Delta z} - 1\right) - K^j_{i-1/2}\left(\frac{h^{j+1}_i - h^{j+1}_{i-1}}{\Delta z} - 1\right)\right]$$

where

$$K^j_{i+1/2} = \frac{K^j_{i+1} + K^j_i}{2}\ ;\ K^j_{i-1/2} = \frac{K^j_i + K^j_{i-1}}{2}.$$

Model 4—Eq. [6]—Implicit scheme with implicit linearization (prediction-correction).

Prediction (estimation of C_i^j and K_i^j)

$$\frac{2C_i^j}{K_i^j}\frac{h_i^{j+1/2} - h_i^j}{\Delta t} = \Delta_z^2 h_i^{j+1/2} + \frac{1}{K_i^j}\cdot \Delta_z K_i^j (\Delta_z h_i^j - 1)$$

Correction (estimation of h_i^j)

$$\frac{C_i^{j+1/2}}{K_i^{j+1/2}}\frac{h_i^{j+1} - h_i^j}{\Delta t} = \frac{1}{2}\Delta_z^2 (h_i^{j+1} + h_i^j) + \frac{1}{K_i^{j+1/2}}\Delta_z K_i^{j+1/2}(\Delta_z h_i^{j+1/2} - 1)$$

where

$$\Delta_z^2 h_i^j = \frac{h^j_{i+1} - 2h^j_i + h^j_{i-1}}{(\Delta z)^2}\ ;\ \Delta_z h_i^j = \frac{h^j_{i+1} - h^j_{i-1}}{2\cdot\Delta z}$$

$$\Delta_z K_i^j = \frac{K^j_{i+1} - K^j_{i-1}}{2\cdot\Delta z}$$

Model 5—Eq. [8]—Implicit scheme with explicit linearization

$$F_i^j \frac{U_i^{j+1} - U_i^j}{\Delta t} = \Delta_z^2 U_i^{j+1} - G_i^j \Delta_z U_i^{j+1}$$

Δ_z^2 and Δ_z have the same meaning as for model 4.

Model 6—Eq. [8]—Implicit scheme with implicit linearization (prediction-correction)

Prediction (estimation of F_i^j and G_i^j)

$$2F_i^j \frac{U_i^{j+1/2} - U_i^j}{\Delta t} = \Delta_z^2 U_i^{j+1/2} - G_i^j \Delta_z U_i^j$$

Correction (estimation of U_i^j)

$$F_i^{j+1/2} \frac{U_i^{j+1} - U_i^j}{\Delta t} = \frac{1}{2}\Delta_z^2 (U_i^{j+1} + U_i^j) - G_i^{j+1/2}\Delta_z U_i^{j+1/2}$$

LITERATURE CITED

1. Bhuiyan, S. I., E. A. Hiler, C. H. M. van Bavel, and A. R. Aston. 1971. Dynamic simulation of vertical infiltration into unsaturated soils. Water Resour. Res. 7:1597–1606.
2. Dane, J. H., and P. Wierenga. 1975. Effect of hysteresis on the prediction of infiltration, redistribution and drainage of water in a layered soil. J. Hydrol. 25:229–242.
3. DeWit, C. T., and H. van Keulen. 1972. Simulation of transport processes in soils. Cent. Agric. Publ. Doc. Pudoc., Wageningen, The Netherlands.
4. Douglas, J. J., and B. F. Jones. 1963. On predictor-corrector method for non linear parabolic differential equations. J. Siam 11:195–204.
5. Freeze, R. A. 1969. The mechanism of natural groundwater recharge and discharge 1. One-dimensional vertical, unsteady, unsaturated flow above a recharging, or discharging groundwater flow system. Water Resour. Res. 5:153–171.
6. Giesel, W., M. Renger, and O. Strebel. 1972. Numerical treatment of the unsaturated water flow equation-comparison of experimental and computed results. Water Resour. Res. 9:174–177.
7. Hanks, R., and S. A. Bowers. 1962. Numerical solution of the moisture flow equation into layered soils. Soil Sci. Soc. Am. Proc. 26:530–534.
8. Khanji, D., M. Vauclin and G. Vachaud. 1974. Infiltration non permanente et bidimensionnelle dans une tranche de sol non saturee. Analyse numerique et resultats experimentaux. C. R. Acad. Sci. (Paris) 278:381–384.
9. Knight, J. H., and J. R. Philip. 1974. On solving the unsaturated flow equation. Soil Sci. 116:407–416.
10. Molz, F. J., and I. Remson. 1970. Extraction term models of soil moisture use by transpiring plants. Water Resour. Res. 6:1346–1356.
11. Moore, R. E. 1939. Water conduction from shallow water tables. Hilgardia 12:383–426.
12. Neuman, S. P., R. A. Feddes, and E. Bresler. 1975. Finite element analysis of two-dimensional flow in soils considering water uptake by roots: I. Theory. Soil Sci. Soc. Am. Proc. 39:224–237.
13. Parlange, J. -Y. 1971. Theory of water-movement in soils: 2. One-dimensional infiltration. Soil Sci. 111:170–174.
14. Parlange, J. -Y. 1972a. Theory of water movement in soils: 6. Effect of water depth over soil. Soil Sci. 113:308–312.
15. Parlange, J. -Y. 1972b. Theory of water movement in soils: 8. One-dimensional infiltration with constant flux at the surface. Soil Sci. 114:1–4.
16. Parlange, J. -Y. 1973. Vertical infiltration into a layered soil. Soil Sci. Soc. Am. Proc. 37:673–676.
17. Philip, J. R. 1955. Numerical solution of equations of the diffusion type with diffusivity concentration-dependent I. Trans. Faraday Soc. 51:885–892.
18. Philip, J. R. 1957a. Numerical solution of equations of the diffusion type with diffusivity concentration dependent II. Aus. J. Phys. 10:29–42.
19. Philip, J. R. 1957b. The theory of infiltration 1. Soil Sci. 83:345–357.
20. Philip, J. R. 1957c. The theory of infiltration 2. Soil Sci. 83:435–448.
21. Philip, J. R. 1958. The theory of infiltration 6. Soil Sci. 85:278–286.
22. Philip, J. R. 1969. Theory of infiltration. Adv. Hydrosci. 5:215–305. Academic Press, N.Y.
23. Raats, P. A. C., and W. R. Gardner. 1974. Movement of water in the unsaturated zone near a watertable. *In* Jan Van Schilfgaarde (ed.) Drainage for agriculture: Agronomy 17:311–405. Am. Soc. Agron., Madison, Wis.
24. Remson, I., G. M. Hornberger, and R. D. Molz. 1971. Numerical

methods in subsurface hydrology. John Wiley, New York.

25. Rubin, J. 1966. Numerical analysis of bonded rainfall infiltration. p. 440–451. *In* P. E. Rijtema (ed.) Proc. Wageningen Symposium IASH 82 (1968, UNESCO, Paris)

26. Staple, W. J. 1966. Infiltration and redistribution of water in vertical columns of loam. Soil Sci. Soc. Am. Proc. 33:645–651.

27. Vachaud, G., and J. L. Thony. 1971. Hysteresis during infiltration and redistribution in a soil column at different initial water contents. Water Resour. Res. 7:111–127.

28. Van der Ploeg, R. R. 1974. Simulation of moisture transfer in soils: one-dimensional infiltration. Soil Sci. 118:349–357.

29. Van Genuchten, M. Th., and P. J. Wierenga. 1976. Numerical solution for convective dispersion with intra-aggregate diffusion and non-linear adsorption in biosystems. p. 275–292. *In* G. C. Vansteenkiste (ed.) System simulation in water resources. North-Holland Publishing Co., Amsterdam.

30. Van Keulen H., and G. E. M. Van Beek. 1971. Water movement in layered soils. A. Simulation model. Neth J. Agric. Sci. 19:138–153.

31. Vauclin M., G. Vachaud, and D. Khanji. 1975. Two-dimensional numerical analysis of transient water transfer in saturated-unsaturated soils. p. 299–323. *In* G. C. Vansteenkiste (ed.) Modeling and simulation of water resources systems. North-Holland Publishing Co., Amsterdam.

32. Watson, K. K. 1966. An instantaneous profile method for determining the hydraulic conductivity of unsaturated porous material. Water Resour. Res. 2:709–715.

33. Wierenga, P. J., and C. T. de Wit. 1970. Simulation of heat transfer in soils. Soil Sci. Soc. Am. Proc. 34:845–848.

34. Whisler, F. D., and A. Klute. 1965. The numerical analysis of infiltration considering hysteresis into a vertical soil column at equilibrium under gravity. Soil Sci. Soc. Am. Proc. 29:489–494.