Electrical and Computer Engineering Department
Tarbiat Modares University

Z Transform

Foad Ghaderi, PhD

In general, any sequence that passes through a LTI system changes shape, i.e if x[n] is an arbitrary sequence

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k],$$

there is no direct relation between the waveforms of x[n] and y[n]:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Question:
Is there any sequence that retains its shape when it passes through an LTI system?

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Consider the complex exponential sequence:

$$x[n] = z^n, \quad \text{for all } n$$

$z = \mathcal{R}e(z) + j\mathcal{I}m(z)$ is a complex variable defined everywhere on the complex plane.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right) z^n, \quad \text{for all } n.$$

If the summation inside the parentheses converges, the result is a function of z

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}.$$

Then the output sequence is given by

$$y[n] = H(z)z^n, \quad \text{for all } n.$$

The output sequence is the same complex exponential as the input sequence, multiplied by a constant *H(z)* that depends on the value of z. H(z) is called ***System function*** (***transfer function***):

The complex exponential sequences $x[n] = z^n$ are eigenfunctions of LTI systems.

The constant *H(z)*, for a specified value of the complex variable z, is the eigenvalue associated with the eigenfunction z^n.

In contrast to impulse sequences, whose shape changes when they pass through LTI systems, complex exponential sequences *retain* their shape.

If the input to a LTI system can be expressed as a linear combination of complex exponentials

$$x[n] = \sum_k c_k z_k^n, \quad \text{for all } n$$

the output will be

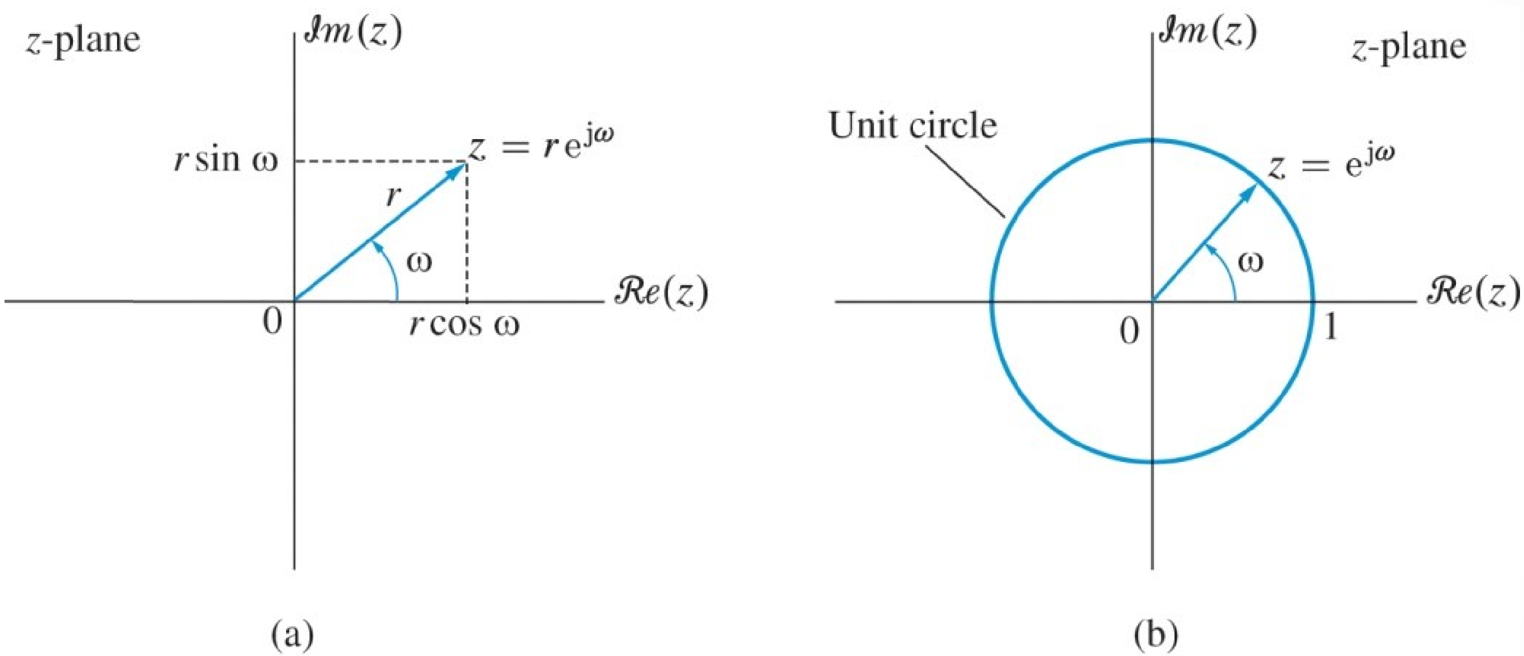
$$y[n] = \sum_k c_k H(z_k) z_k^n, \quad \text{for all } n.$$

The ***z*-transform** of a sequence $x[n]$ is a function $X(z)$ defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n},$$

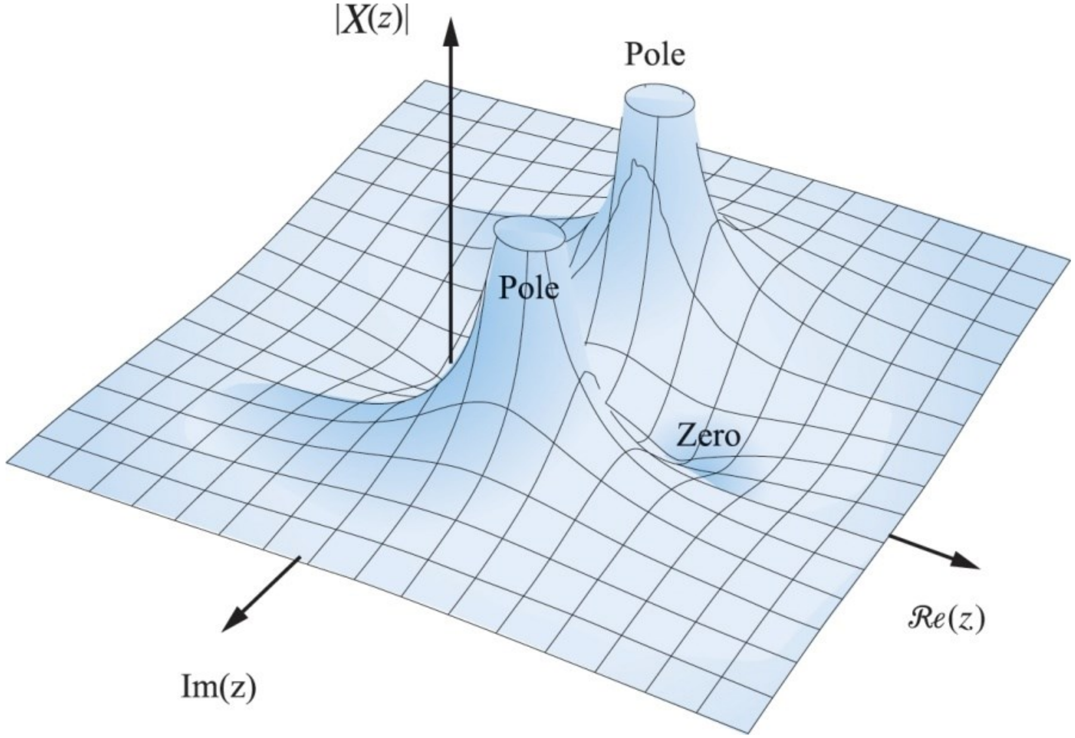
where the independent variable z can represent any complex number.

It is convenient to interpret the z-transform using the correspondence between complex numbers and points in the plane.



(a) A point $z = re^{j\omega}$ in the complex plane can be specified by the distance r from the origin and the angle ω with the positive real axis (polar coordinates) or the rectangular coordinates r cos(ω) and r sin(ω). (b) The unit circle, $|z| = 1$, in the complex plane.

The set of values of z for which the series $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$, converges is known as the *region of convergence* (ROC) of the z-transform. The values of z for which $X(z) = 0$ are called *zeros* of $X(z)$, and the values of z for which $X(z)$ is infinite are known as *poles*.



The magnitude $|X(z)|$ of the z-transform represents a surface in the z-plane. There are two zeros at $z_1 = 0$, $z_2 = 1$ and two poles at $p_{1,2} = 0.9e^{\pm j\pi/4}$.

Example: Unit sample sequence

The z-transform of the unit sample sequence is given by

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^0 = 1. \quad \text{ROC: All } z$$

Example: Square-pulse sequence

The z-transform of the square-pulse sequence

$$x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

is given by

$$X(z) = \sum_{n=0}^{M} 1 z^{-n} = \frac{1 - z^{-(M+1)}}{1 - z^{-1}}. \quad \text{ROC: } z \neq 0$$

Example: Exponential-pulse sequence

The z-transform of the exponential-pulse sequence

$$x[n] = \begin{cases} a^n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

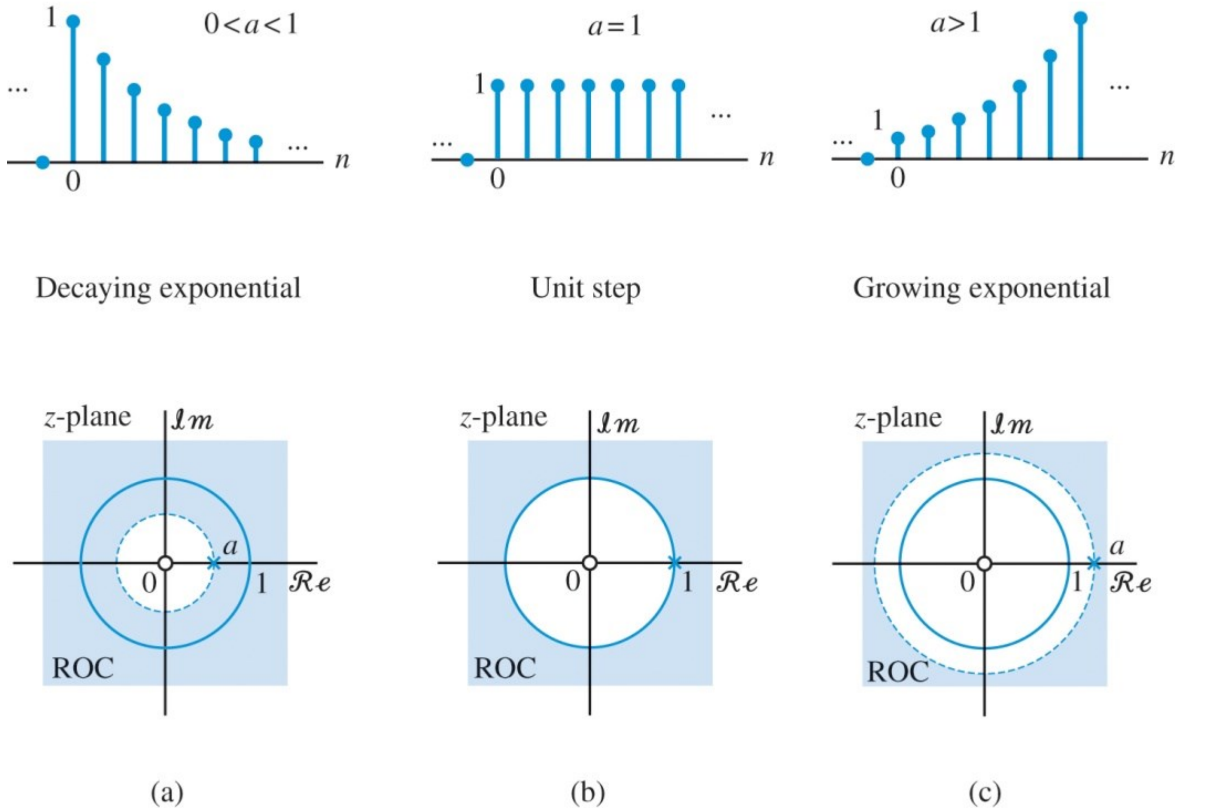
is given by

$$X(z) = \sum_{n=0}^{M} a^n z^{-n} = \sum_{n=0}^{M} (az^{-1})^n = \frac{1 - a^{M+1} z^{-(M+1)}}{1 - az^{-1}}. \quad \text{ROC: } z \neq 0$$

Example: Causal exponential sequence

The z-transform of the causal exponential sequence $x[n] = a^n u[n]$ is given by

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}. \quad \text{ROC: } |z| > |a|$$



Pole-zero plot and region of convergence of a causal exponential sequence $x[n] = a^n u[n]$ with (a) decaying amplitude ($0 < a < 1$), (b) fixed amplitude (unit step sequence), and (c) growing amplitude ($a > 1$).

Example: Anticausal exponential sequence

The z-transform of the anticausal exponential sequence

$$y[n] = -b^n u[-n-1] = \begin{cases} 0, & n \geq 0 \\ -b^n, & n < 0 \end{cases}$$

is given by

$$Y(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} = -b^{-1}z(1 + b^{-1}z + b^{-2}z^2 + \cdots)$$

The infinite geometric series inside the parenthesis converges if $|b^{-1}z| < 1$ or $|z| < |b|$.

$$Y(z) = \frac{-bz^{-1}}{1-b^{-1}z} = \frac{1}{1-bz^{-1}} = \frac{z}{z-b}. \quad \text{ROC: } |z| < |b|$$

The z-transform function $Y(z)$ has a zero at $z = 0$ and a pole at $p = b$.

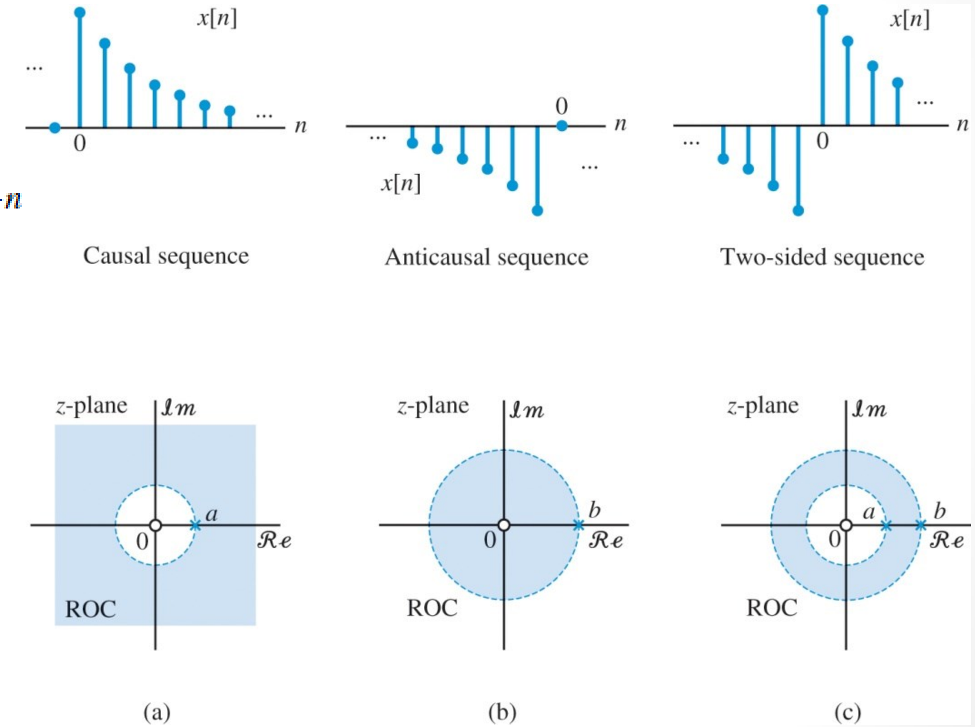
Example: Two-sided exponential sequence

The z-transform of the two-sided exponential sequence

$$x[n] = \begin{cases} a^n, & n \geq 0 \\ -b^n, & n < 0 \end{cases}$$

is given by

$$X(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$



Pole-zero plot and region of convergence for the (a) causal, (b) anticausal, and (c) two-sided exponential sequences

Example: Exponentially oscillating sequence

Consider a causal sinusoidal sequence with exponentially varying amplitude:

$$x[n] = r^n (\cos \omega_0 n) u[n]. \quad r > 0,\ 0 \le \omega_0 < 2\pi$$

Using the identity $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$

$$X(z) = \sum_{n=0}^{\infty} r^n (\cos \omega_0 n) z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}(re^{j\omega_0}z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty}(re^{-j\omega_0}z^{-1})^n.$$

Since $|e^{\pm j\omega_0}| = 1$, both sums converge if $|rz^{-1}| < 1$, or, equivalently, $|z| > r$.

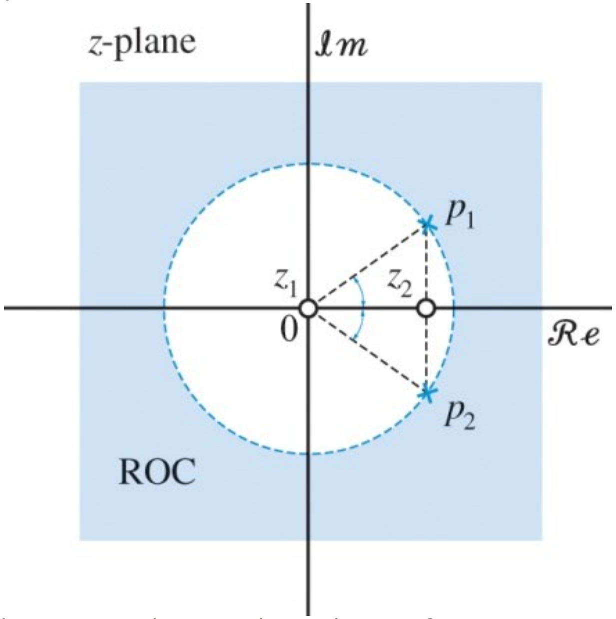
$$X(z) = \frac{1}{2}\frac{1}{1 - re^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1 - re^{-j\omega_0}z^{-1}}, \quad \text{ROC: } |z| > r$$

$$X(z) = \frac{1 - (r\cos\omega_0)z^{-1}}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})} = \frac{1 - (r\cos\omega_0)z^{-1}}{1 - 2(r\cos\omega_0)z^{-1} + r^2 z^{-2}}.$$

$$X(z) = \frac{1-(r\cos\omega_0)z^{-1}}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})} = \frac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}.$$

$$X(z) = \frac{z(z-r\cos\omega_0)}{(z-re^{j\omega_0})(z-re^{-j\omega_0})}.$$

X(z) has two zeros at $z_1 = 0$, $z_2 = r\cos\omega_0$ and two complex-conjugate poles at $p_1 = re^{j\omega_0}$, $p_2 = re^{-j\omega_0}$



Pole-zero plot and region of convergence

Some common z-transform pairs

1.	$\delta[n]$	1	All z
2.	$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4.	$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
5.	$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
6.	$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7.	$(\cos\omega_0 n)u[n]$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
8.	$(\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
9.	$(r^n\cos\omega_0 n)u[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
10.	$(r^n\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

ROC Properties

- The ROC *cannot* include any poles.
- The ROC is a connected (that is, a single contiguous) region.
- For finite duration sequences the ROC is the entire z-plane, with the possible exception of $z = 0$ or $z=\infty$.
- For infinite duration sequences the ROC can have one of the following shapes:

Type of sequence		ROC
Right-sided ($x[n] = 0, n < n_0$)	$\Rightarrow$	ROC: $\|z\| > r$
Left-sided ($x[n] = 0, n > n_0$)	$\Rightarrow$	ROC: $\|z\| < r$
Two-sided	$\Rightarrow$	ROC: $a < \|z\| < b$

- The z-transform of a sequence consists of an algebraic formula and its associated ROC. Thus, to uniquely specify a sequence $x[n]$ we need both $X(z)$ and its ROC.
- The function $X(z)$ is legitimate only for z within its ROC. We stress that $X(z)$ is *not* defined when z is outside the ROC, even if the formula for $X(z)$ yields meaningful results for these values.

The inverse *z*-transform

The recovery of a sequence $x[n]$ from its z-transform ($X(z)$ and ROC) can be formally done using the formula:

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\mathrm{d}z,$$

Simpler procedures

For most sequences and z-transforms encountered in the analysis of LTI systems the following simple procedure is enough:

- Expansion into a series of terms in the variables z and z^{-1} and picking their coefficients.
- Partial fraction expansion and table look-up.

The inverse z-transform

The z-transform of a linear combination of distinct exponentials (that is, $p_k \neq p_m$, $k \neq m$) is given by:

$$x[n] = \sum_{k=1}^{N} A_k (p_k)^n \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}},$$

ROC: the intersection of the ROCs of the individual exponential sequences

$$\rightarrow \quad X(z) = \frac{\displaystyle\sum_{k=1}^{N} A_k \prod_{\substack{m=1 \\ m \neq k}}^{N} (1 - p_m z^{-1})}{\displaystyle\prod_{k=1}^{N} (1 - p_k z^{-1})} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}},$$

which is a *proper* rational function because the degree of the numerator is less than the degree of the denominator.

The inverse *z*-transform

Example: Real and distinct poles

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

1- Decompose the sequence into partial fractions:

distinct poles $p_1 = 1$ and $p_2 = 0.5$

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 - 0.5z^{-1}}.$$

$$\rightarrow \qquad z + 1 = A_1(z - 0.5) + A_2(z - 1),$$

$$\begin{array}{lcl} z = 1 & \rightarrow & A_1 = 4 \\ z = 0.5 & \rightarrow & A_2 = -3 \end{array}$$

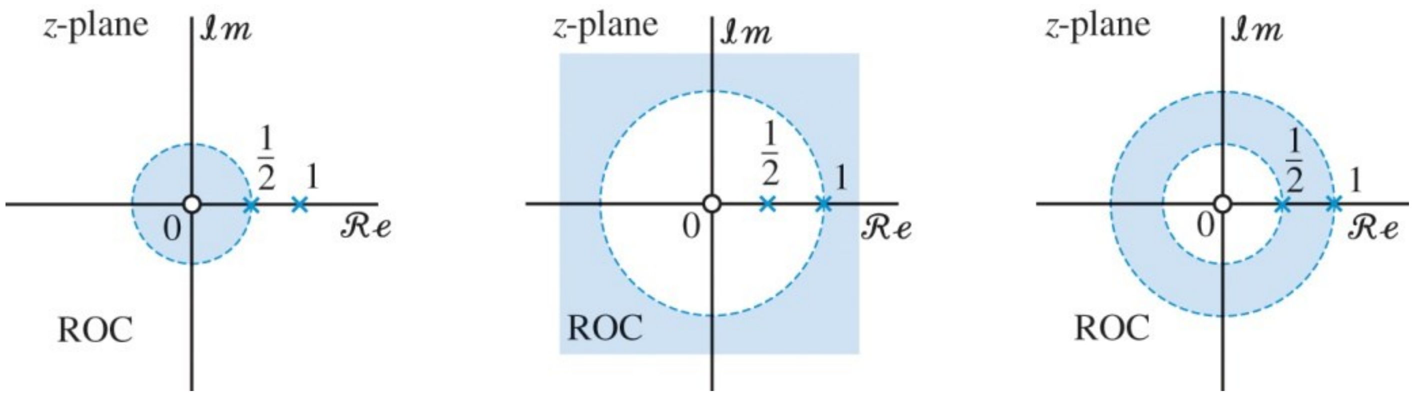
The inverse *z*-transform

Example: Real and distinct poles

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

2- Find the sequences corresponding to the partial fractions,

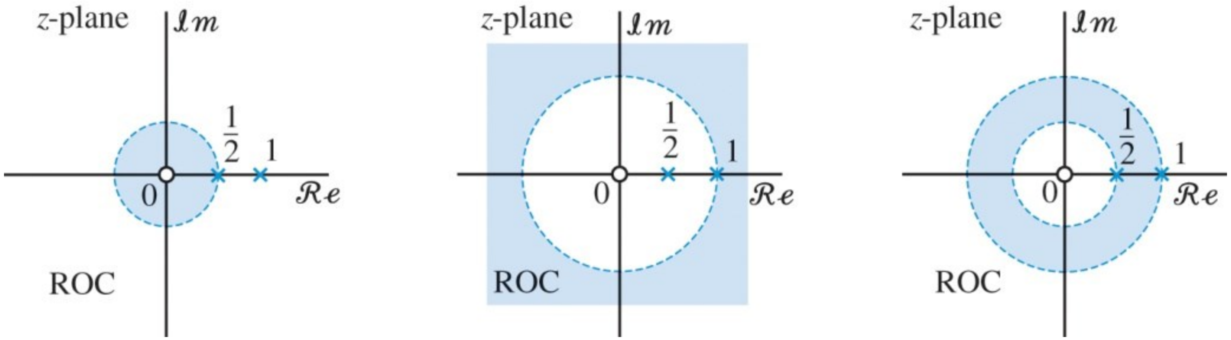
A ROC cannot include any poles, so there are three possible choices for valid ROCs.



The inverse z-transform

Example: Real and distinct poles

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$



- $|z| > 1$, both fractions are the z-transform of causal sequences. Hence

$$x[n] = 4u[n] - 3\left(\frac{1}{2}\right)^n u[n]. \quad \text{(causal)}$$

- $0.5 < |z| < 1$, this can be obtained as the intersection of ROC: $|z| < 1$ and ROC:$|z| > 0.5$.

$$x[n] = -4u[-n-1] - 3\left(\frac{1}{2}\right)^n u[n]. \quad \text{(two-sided)}$$

- $|z| < 0.5$, both fractions are the z-transform of anticausal sequences.

$$x[n] = -4u[-n-1] + 3\left(\frac{1}{2}\right)^n u[-n-1]. \quad \text{(anticausal)}$$

The inverse *z*-transform

Example: Complex conjugate distinct poles

Consider a causal sequence $x[n]$ with z-transform

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

$$z^2 - z + 0.5 = 0 \quad \rightarrow p_1 = \frac{1}{2}(1 + \mathrm{j}) = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{j}\pi/4} \quad \text{and} \quad p_2 = \frac{1}{2}(1 - \mathrm{j}) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{j}\pi/4}$$

$$p1 \neq p2 \quad \rightarrow \quad X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}} = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

$z + 1 = A_1(z - p_2) + A_2(z - p_1)$, setting $z = p_1$ and $z = p_2$

$$A_1 = \frac{1}{2} - \mathrm{j}\frac{3}{2} = \frac{\sqrt{10}}{2}\mathrm{e}^{-\mathrm{j}71.56^\circ} \quad \text{and} \quad A_2 = \frac{1}{2} + \mathrm{j}\frac{3}{2} = \frac{\sqrt{10}}{2}\mathrm{e}^{\mathrm{j}71.56^\circ}$$

Since $x[n]$ is causal $\quad \rightarrow \quad x[n] = A_1(p_1)^n u[n] + A_1^*(p_1^*)^n u[n]$

$$A_1 = A\mathrm{e}^{\mathrm{j}\theta}, p_1 = r\mathrm{e}^{\mathrm{j}\omega_0} \qquad r = 1/\sqrt{2}, \omega_0 = \pi/4, A = \sqrt{10}/2, \text{ and } \theta = -71.56^\circ$$

$$x[n] = Ar^n\left(\mathrm{e}^{\mathrm{j}\omega_0 n}\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\omega_0 n}\mathrm{e}^{-\mathrm{j}\theta}\right)u[n] = 2Ar^n\cos(\omega_0 n + \theta)u[n]$$

The inverse *z*-transform

If we have a rational function with distinct poles

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}},$$

the complete partial fraction expansion takes the form

$$X(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}, \qquad A_k = (1 - p_k z^{-1}) X(z)|_{z=p_k},$$

Example Partial fraction expansion (using **scipy.signal.residuez**)

$$X(z) = \frac{6 - 10z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = 1 + \frac{2}{1 - z^{-1}} + \frac{3}{1 - 2z^{-1}}$$

```
import scipy.signal as sig
b = [6, -10, 2]
a = [1, -3, 2]
[B, A, C] = sig.residuez(b,a)
```

$$x[n] = \delta[n] + (2 + 3 \times 2^n) u[n], \qquad \text{ROC: } |z| > 2$$

$$x[n] = \delta[n] - (2 + 3 \times 2^n) u[-n-1], \qquad \text{ROC: } |z| < 1$$

$$x[n] = \delta[n] + 2u[n] - 3 \times 2^n u[-n-1]. \qquad \text{ROC: } 1 < |z| < 2$$

Properties of the z-transform

Linearity

$$a_1 x_1[n] + a_2 x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Time shifting

$$x[n-k] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k} X(z). \quad \text{ROC} = R_x (\text{except } z = 0 \text{ or } z = \infty)$$

Example

$$x[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0. & \text{otherwise} \end{cases}$$

The ROC is the entire z-plane, except $z = 0$.

$$X(z) = \sum_{n=0}^{N-1} 1 z^{-n} = 1 + z^{-1} + \cdots + z^{-(N-1)} = \begin{cases} N, & z = 1 \\ \dfrac{1 - z^{-N}}{1 - z^{-1}}, & z \ne 1. \end{cases}$$

Alternative approach:

$$x[n] = u[n] - u[n-N].$$

$$X(z) = U(z) - z^{-N} U(z) = (1 - z^{-N}) U(z) = \frac{1 - z^{-N}}{1 - z^{-1}}.$$

Properties of the z-transform

Convolution of sequences

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z). \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Multiplication by an exponential sequence

$$a^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/a). \quad \text{ROC} = |a|R_x$$

Since *a* and *z* take complex values, the result is scaling (expansion or shrinking) and rotation of the *z*-plane, the ROC, and the pole-zero pattern.

Differentiation of the z-transform

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{\mathrm{d}X(z)}{\mathrm{d}z}, \quad \text{ROC} = R_x$$

Both sequences have the same ROC.

Example Second-order pole

$$x[n] = na^n u[n] = n(a^n u[n]), \rightarrow X(z) = -z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

Properties of the *z*-transform

Conjugation of a complex sequence

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*). \quad \text{ROC} = R_x$$

Time reversal

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z). \quad \text{ROC} = \frac{1}{R_x}$$

Initial-value theorem

If $x[n]$ is a causal sequence, that is, $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z),$$

Summary of the z-transform properties

Table 3.2 Some z-transform properties.

	Property	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1.	Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
2.	Time shifting	$x[n-k]$	$z^{-k}X(z)$	R_x except $z=0$ or ∞
3.	Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\|a\|R_x$
4.	Differentation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5.	Conjugation	$x^*[n]$	$X^*(z^*)$	R_x
6.	Real-part	$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	At least R_x
7.	Imaginary part	$\mathrm{Im}\{x[n]\}$	$\frac{1}{2}[X(z)-X^*(z^*)]$	At least R_x
8.	Folding	$x[-n]$	$X(1/z)$	$1/R_x$
9.	Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
10.	Initial-value theorem	$x[n]=0$ for $n<0$	$x[0]=\lim_{z\to\infty} X(z)$	

System function of LTI systems

Every LTI system can be completely characterized in the time domain by its impulse response $h[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k], \quad \rightarrow \quad Y(z) = H(z)X(z),$$

Since there is a unique relation between h[n] and H(z) many properties of the system can be inferred from H(z) and its ROC.

Example

Determine the response of a system with impulse response $h[n] = a^n u[n]$, $|a| < 1$ to the input $x[n] = u[n]$ using the convolution theorem.

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \quad |z| > |a| \qquad X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}. \quad |z| > 1$$

$$\rightarrow \quad Y(z) = \frac{1}{(1-az^{-1})(1-z^{-1})}. \quad |z| > \max\{|a|, 1\} = 1.$$

Determining the output sequence *y*[*n*]

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right). \quad |z| > 1$$

$$\rightarrow \quad y[n] = \frac{1}{1-a}(u[n] - a^{n+1}u[n]) = \frac{1-a^{n+1}}{1-a}u[n],$$

System function of LTI systems

Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n < 0$. Therefore, for causal systems the power series H(z) does not include any positive powers of z and its ROC extends outside of a circle for some radius r, that is, $|z| > r$.

A system function H(z) with the ROC that is the exterior of a circle, extending to infinity, is a necessary condition for a discrete-time LTI system to be causal but not a sufficient one.

Stability

For a LTI system to be stable, $\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$

This is equivalent to the condition $|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty,$ for $|z| = 1$

A LTI system is stable if and only if the ROC of the system function H(z) includes the unit circle $|z| = 1$.

System function of LTI systems

Causal and stable system

An LTI system with rational H(z) is both causal and stable if and only if all poles of H(z) are inside the unit circle and its ROC is on the exterior of a circle, extending to infinity.

Stability

A causal LTI with a rational system function is stable if and only if all poles of H(z) are inside the unit circle in the z-plane. The zeros can be anywhere.

Linear constant-coefficient difference equations

Consider a linear constant-coefficient difference equation

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

Using the properties of linearity and time shifting

$$\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z).$$

Therefore the system function is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}},$$

Linear constant-coefficient difference equations

Example

Find the difference equation corresponding to the system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{6 - 10z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}}.$$

$$\rightarrow \qquad (1 - 3z^{-1} + 2z^{-2})Y(z) = (6 - 10z^{-1} + 2z^{-2})X(z).$$

$$\rightarrow \quad y[n] - 3y[n-1] + 2y[n-2] = 6x[n] - 10x[n-1] + 2x[n-2].$$

Linear constant-coefficient difference equations

Poles and zeros

if $z_1, z_2, \ldots, z_M$ and $p_1, p_2, \ldots, p_N$ are the roots of the numerator and the denominator polynomial, respectively

$$B(z) = b_0 z^{-M}\left(z^M + \frac{b_1}{b_0} z^{M-1} + \cdots + \frac{b_M}{b_0}\right)$$

$$= b_0 z^{-M}(z - z_1)\ldots(z - z_M).$$

$$A(z) = z^{-N}\left(z^N + a_1 z^{N-1} + \cdots + a_N\right)$$

$$= z^{-N}(z - p_1)\ldots(z - p_N).$$

$$\rightarrow \qquad H(z) = \frac{B(z)}{A(z)} = b_0 \frac{z^{-M}}{z^{-N}} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})},$$

where b_0 is a constant gain term.
Each of the factors $(1 - z_k z^{-1})$ contributes a zero at $z = z_k$ and a pole at $z = 0$.
Each of the factors $(1 - p_k z^{-1})$ contributes a pole at $z = p_k$ and a zero at $z = 0$.
Poles and zeros at the origin are not counted.

Linear constant-coefficient difference equations

Impulse response

Any rational function of z^{-1} with distinct poles can be expressed in the form

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}},$$

If we assume that the system is causal, then the ROC is the exterior of a circle starting at the outermost pole, and the impulse response is

$$h[n] = \sum_{k=0}^{M-N} C_k \delta[n-k] + \sum_{k=1}^{N} A_k (p_k)^n u[n].$$

Linear constant-coefficient difference equations

System classifications

LTI systems can be classified into different classes:

Length of impulse response

Infinite Impulse Response (IIR)

If at least one nonzero pole of $H(z)$ is not canceled by a zero, there will a term of the form $A_k(p_k)^n u[n]$. In this case $h[n]$ has infinite duration.

Finite Impulse Response (FIR)

If $N = 0$, the system function becomes a polynomial.

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Linear constant-coefficient difference equations

System classifications

LTI systems can be classified into different classes:

Feedback in implementation

Recursive systems

If $N \geq 1$ the output of the system is fed back into the input.

Nonrecursive systems

If $N = 0$ the output is a linear combination of the present and previous inputs, only.

Poles and zeros

If $a_k = 0$, for $k = 1, \dots, N$, the system has M zeros (*all-zero* systems).
If $b_k = 0$, for $k = 1, \dots, M$, the system has N poles (*all-pole* systems).

Connections between pole-zero locations and time-domain behavior

The roots of a polynomial with real coefficients either must be real or must occur in complex conjugate pairs.

Any system with a rational *H(z)* is equivalent to a parallel combination of an FIR system, K_1 first-order systems with real poles, and K_2 second-order systems with complex conjugate poles, where $N = K_1 + 2K_2$:

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{K_1} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}.$$

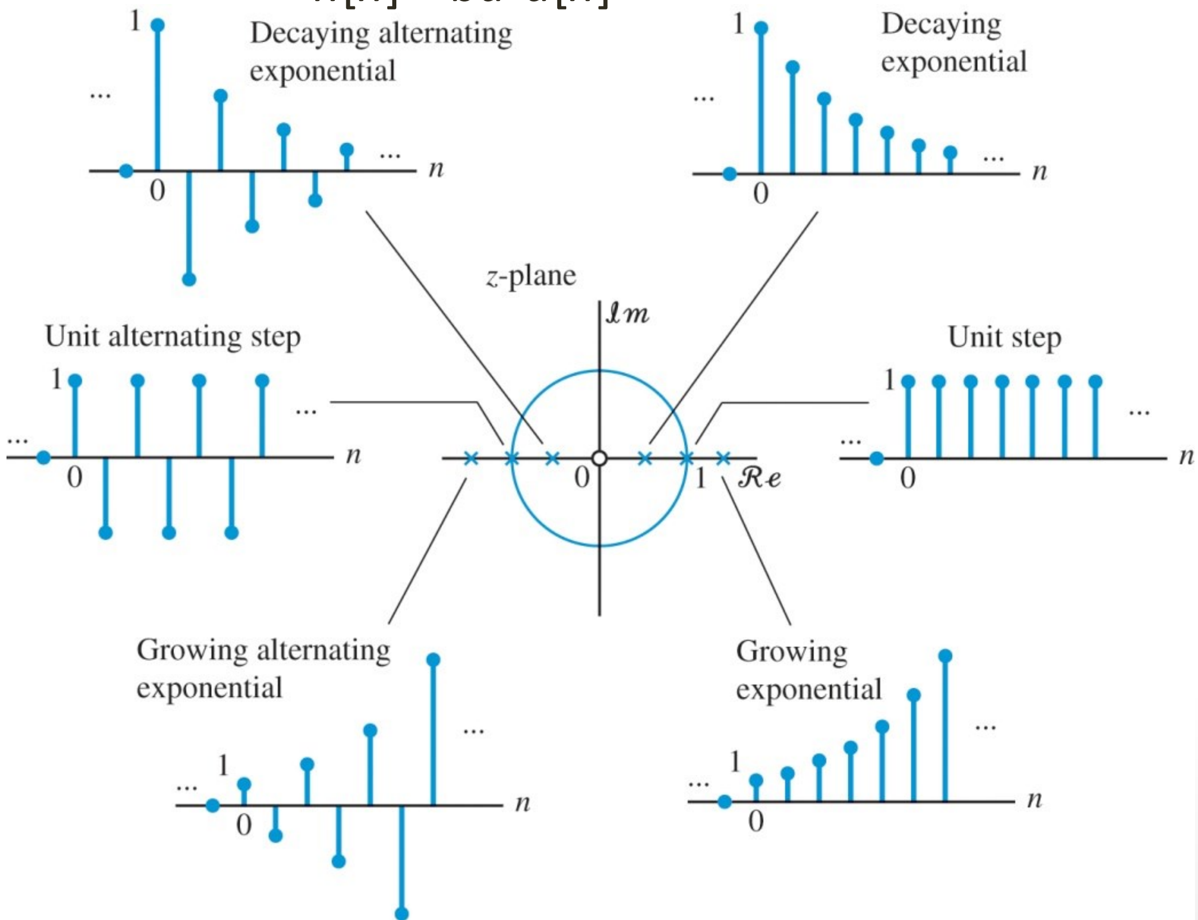
Connections between pole-zero locations and time-domain behavior

First-order systems

$$H(z) = \frac{b}{1 - az^{-1}}, \quad a, b \text{ real}$$

Assuming a causal system, the impulse response is given by the following real exponential sequence:

$$h[n] = ba^n u[n]$$



Connections between pole-zero locations and time-domain behavior

Second-order systems

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z(b_0 z + b_1)}{z^2 + a_1 z + a_2}.$$

Zeros:

$z_1 = 0$ and $z_2 = -b_1/b_0$

Poles:

$$p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}.$$

Poles	Condition
Real and distinct	$a_1^2 > 4a_2$
Real and equal	$a_1^2 = 4a_2$
Complex conjugate	$a_1^2 < 4a_2$

The impulse response of a causal system with a pair of complex conjugate poles

$$\begin{aligned} h[n] &= A p^n u[n] + A^*(p^*)^n u[n] \\ &= |A| e^{j\theta} r^n e^{j\omega_0 n} u[n] + |A| e^{-j\theta} r^n e^{-j\omega_0 n} u[n] \\ &= |A| r^n \left[e^{j(\omega_0 n + \theta)} + e^{-j(\omega_0 n + \theta)} \right] u[n] \\ &= 2|A| r^n \cos(\omega_0 n + \theta) u[n], \end{aligned}$$

Connections between pole-zero locations and time-domain behavior