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Z Transform

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In general, any sequence that passes through a LTI system changes shape, i.e if $x[n]$ is an arbitrary sequence

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k],$$

there is no direct relation between the waveforms of $x[n]$ and $y[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k] = \sum_{k=-\infty}^{\infty} h[k]x[n - k]$$

Question:

Is there any sequence that retains its shape when it passes through an LTI system?

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Consider the complex exponential sequence:

$$x[n] = z^n, \quad \text{for all } n$$

$z = \mathcal{R}e(z) + j\mathcal{I}m(z)$ is a complex variable defined everywhere on the complex plane.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k} \right) z^n, \quad \text{for all } n.$$



If the summation inside the parentheses converges, the result is a function of z

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}.$$

Then the output sequence is given by

$$y[n] = H(z)z^n, \quad \text{for all } n.$$

The output sequence is the same complex exponential as the input sequence, multiplied by a constant $H(z)$ that depends on the value of z . $H(z)$ is called

System function (transfer function):

The complex exponential sequences $x[n] = z^n$ are eigenfunctions of LTI systems.

The constant $H(z)$, for a specified value of the complex variable z , is the eigenvalue associated with the eigenfunction z^n .

In contrast to impulse sequences, whose shape changes when they pass through LTI systems, complex exponential sequences *retain* their shape.⁴



If the input to a LTI system can be expressed as a linear combination of complex exponentials

$$x[n] = \sum_k c_k z_k^n, \quad \text{for all } n$$

the output will be

$$y[n] = \sum_k c_k H(z_k) z_k^n, \quad \text{for all } n.$$

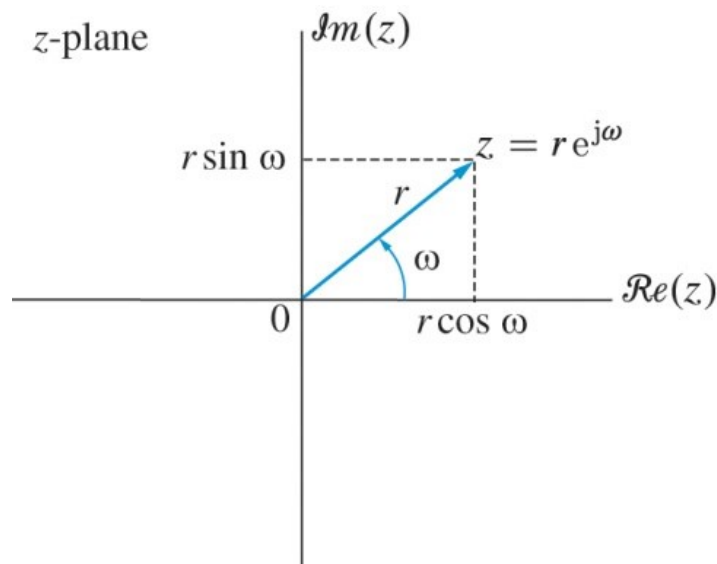
The **z-transform** of a sequence $x[n]$ is a function $X(z)$ defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n},$$

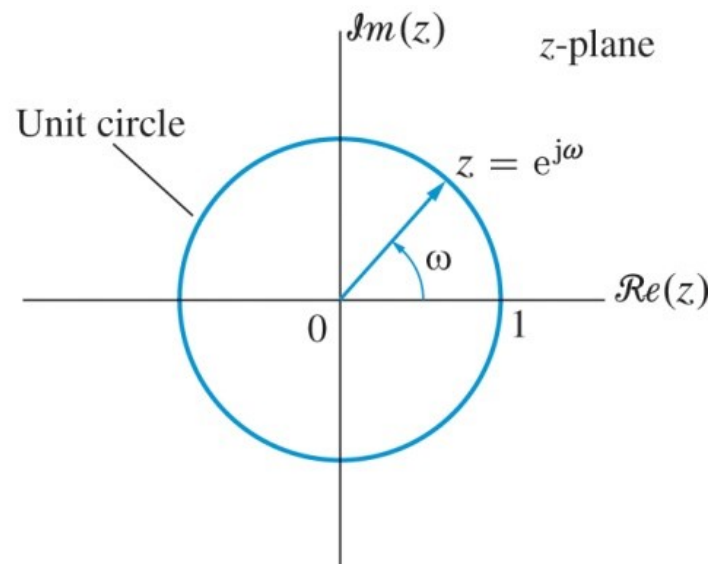
where the independent variable z can represent any complex number.



It is convenient to interpret the z -transform using the correspondence between complex numbers and points in the plane.



(a)

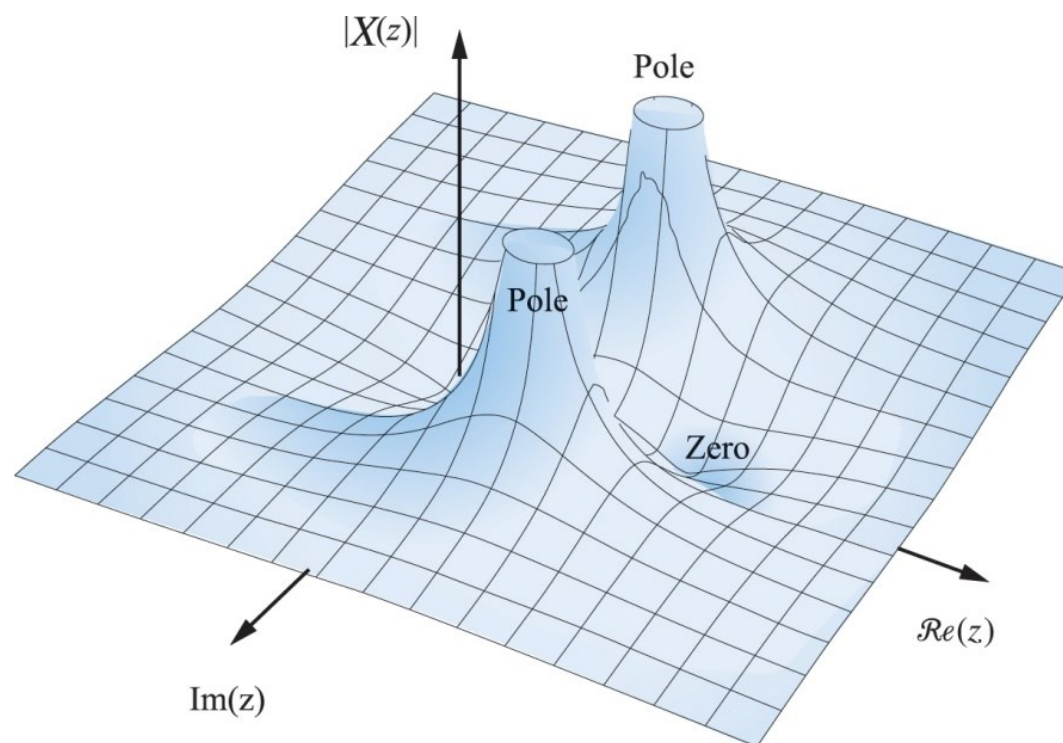


(b)

(a) A point $z = re^{j\omega}$ in the complex plane can be specified by the distance r from the origin and the angle ω with the positive real axis (polar coordinates) or the rectangular coordinates $r \cos(\omega)$ and $r \sin(\omega)$. (b) The unit circle, $|z| = 1$, in the complex plane.



The set of values of z for which the series $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$, converges is known as the *region of convergence* (ROC) of the z -transform. The values of z for which $X(z) = 0$ are called *zeros* of $X(z)$, and the values of z for which $X(z)$ is infinite are known as *poles*.



The magnitude $|X(z)|$ of the z -transform represents a surface in the z -plane. There are two zeros at $z_1 = 0$, $z_2 = 1$ and two poles at $p_{1,2} = 0.9e^{\pm j\pi/4}$.

**Example: Unit sample sequence**

The z -transform of the unit sample sequence is given by

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = z^0 = 1. \quad \text{ROC: All } z$$

Example: Square-pulse sequence

The z -transform of the square-pulse sequence

$$x[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

is given by

$$X(z) = \sum_{n=0}^M 1z^{-n} = \frac{1 - z^{-(M+1)}}{1 - z^{-1}}. \quad \text{ROC: } z \neq 0$$

**Example: Exponential-pulse sequence**

The z -transform of the exponential-pulse sequence

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

is given by

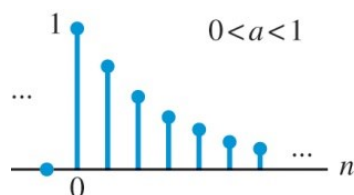
$$X(z) = \sum_{n=0}^M a^n z^{-n} = \sum_{n=0}^M (az^{-1})^n = \frac{1 - a^{M+1}z^{-(M+1)}}{1 - az^{-1}}. \quad \text{ROC: } z \neq 0$$



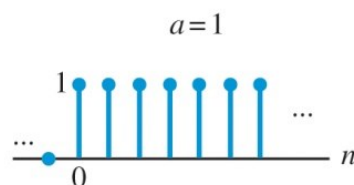
Example: Causal exponential sequence

The z -transform of the causal exponential sequence $x[n] = a^n u[n]$ is given by

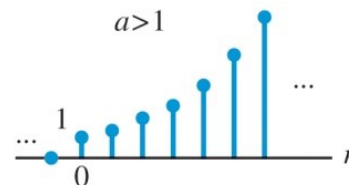
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}. \quad \text{ROC: } |z| > |a|$$



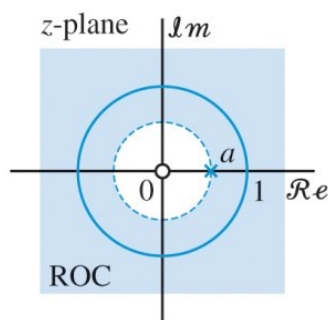
Decaying exponential



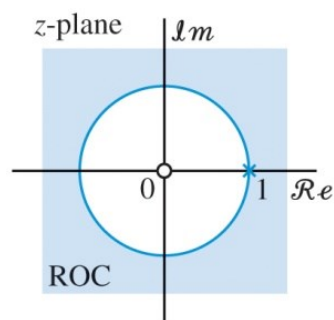
Unit step



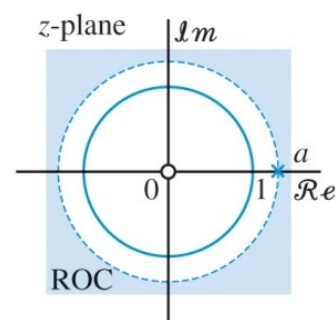
Growing exponential



(a)



(b)



(c)

Pole-zero plot and region of convergence of a causal exponential sequence $x[n] = a^n u[n]$ with (a) decaying amplitude ($0 < a < 1$), (b) fixed amplitude (unit step sequence), and (c) growing amplitude ($a > 1$).

**Example: Anticausal exponential sequence**

The z -transform of the anticausal exponential sequence

$$y[n] = -b^n u[-n-1] = \begin{cases} 0, & n \geq 0 \\ -b^n, & n < 0 \end{cases}$$

is given by

$$Y(z) = - \sum_{n=-\infty}^{-1} b^n z^{-n} = -b^{-1} z (1 + b^{-1} z + b^{-2} z^2 + \dots)$$

The infinite geometric series inside the parenthesis converges if $|b^{-1}z| < 1$ or $|z| < |b|$.

$$Y(z) = \frac{-bz^{-1}}{1 - b^{-1}z} = \frac{1}{1 - bz^{-1}} = \frac{z}{z - b}. \quad \text{ROC: } |z| < |b|$$

The z -transform function $Y(z)$ has a zero at $z = 0$ and a pole at $p = b$.



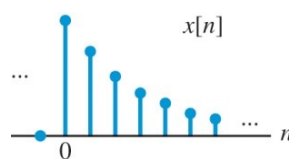
Example: Two-sided exponential sequence

The z-transform of the two-sided exponential sequence

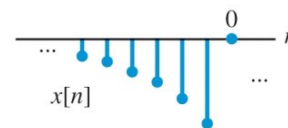
$$x[n] = \begin{cases} a^n, & n \geq 0 \\ -b^n, & n < 0 \end{cases}$$

is given by

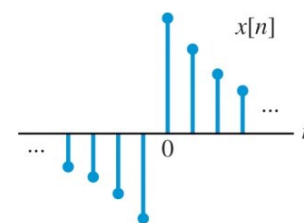
$$X(z) = - \sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$



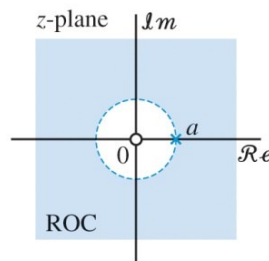
Causal sequence



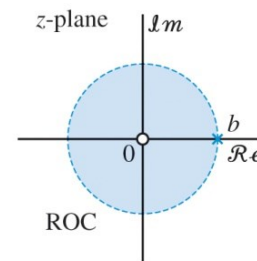
Anticausal sequence



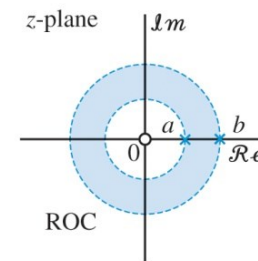
Two-sided sequence



(a)



(b)



(c)

Pole-zero plot and region of convergence for the (a) causal, (b) anticausal, and (c) two-sided exponential sequences

**Example: Exponentially oscillating sequence**

Consider a causal sinusoidal sequence with exponentially varying amplitude:

$$x[n] = r^n (\cos \omega_0 n) u[n], \quad r > 0, \quad 0 \leq \omega_0 < 2\pi$$

Using the identity $\cos \theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$

$$X(z) = \sum_{n=0}^{\infty} r^n (\cos \omega_0 n) z^{-n} = \frac{1}{2} \sum_{n=0}^{\infty} (r e^{j\omega_0} z^{-1})^n + \frac{1}{2} \sum_{n=0}^{\infty} (r e^{-j\omega_0} z^{-1})^n.$$

Since $|e^{\pm j\omega_0}| = 1$, both sums converge if $|r z^{-1}| < 1$, or, equivalently, $|z| > r$.

$$X(z) = \frac{1}{2} \frac{1}{1 - r e^{j\omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - r e^{-j\omega_0} z^{-1}}, \quad \text{ROC: } |z| > r$$

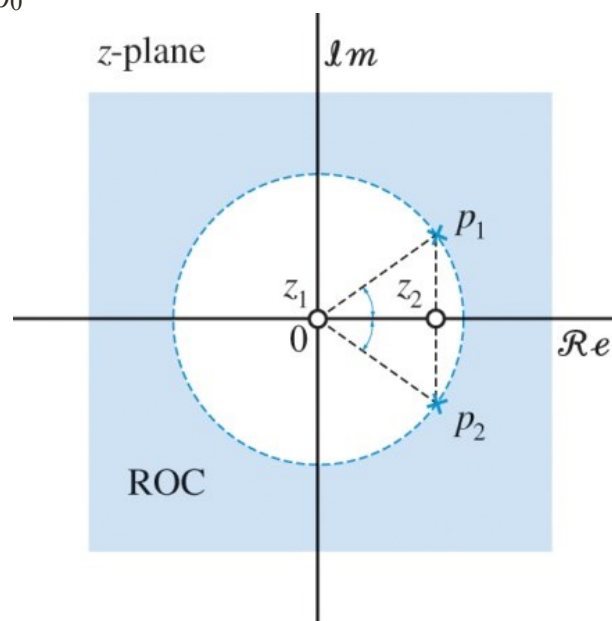
$$X(z) = \frac{1 - (r \cos \omega_0) z^{-1}}{(1 - r e^{j\omega_0} z^{-1})(1 - r e^{-j\omega_0} z^{-1})} = \frac{1 - (r \cos \omega_0) z^{-1}}{1 - 2(r \cos \omega_0) z^{-1} + r^2 z^{-2}}.$$



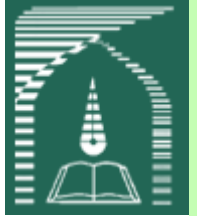
$$X(z) = \frac{1 - (r \cos \omega_0)z^{-1}}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})} = \frac{1 - (r \cos \omega_0)z^{-1}}{1 - 2(r \cos \omega_0)z^{-1} + r^2z^{-2}}.$$

$$X(z) = \frac{z(z - r \cos \omega_0)}{(z - re^{j\omega_0})(z - re^{-j\omega_0})}.$$

$X(z)$ has two zeros at $z_1 = 0$, $z_2 = r \cos \omega_0$ and two complex-conjugate poles at $p_1 = re^{j\omega_0}$, $p_2 = re^{-j\omega_0}$



Pole-zero plot and region of convergence



Some common z-transform pairs

| | | | |
|-----|------------------------------|---|-------------|
| 1. | $\delta[n]$ | 1 | All z |
| 2. | $u[n]$ | $\frac{1}{1 - z^{-1}}$ | $ z > 1$ |
| 3. | $a^n u[n]$ | $\frac{1}{1 - az^{-1}}$ | $ z > a $ |
| 4. | $-a^n u[-n - 1]$ | $\frac{1}{1 - az^{-1}}$ | $ z < a $ |
| 5. | $na^n u[n]$ | $\frac{az^{-1}}{(1 - az^{-1})^2}$ | $ z > a $ |
| 6. | $-na^n u[-n - 1]$ | $\frac{az^{-1}}{(1 - az^{-1})^2}$ | $ z < a $ |
| 7. | $(\cos \omega_0 n) u[n]$ | $\frac{1 - (\cos \omega_0)z^{-1}}{1 - 2(\cos \omega_0)z^{-1} + z^{-2}}$ | $ z > 1$ |
| 8. | $(\sin \omega_0 n) u[n]$ | $\frac{(\sin \omega_0)z^{-1}}{1 - 2(\cos \omega_0)z^{-1} + z^{-2}}$ | $ z > 1$ |
| 9. | $(r^n \cos \omega_0 n) u[n]$ | $\frac{1 - (r \cos \omega_0)z^{-1}}{1 - 2(r \cos \omega_0)z^{-1} + r^2 z^{-2}}$ | $ z > r$ |
| 10. | $(r^n \sin \omega_0 n) u[n]$ | $\frac{(\sin \omega_0)z^{-1}}{1 - 2(r \cos \omega_0)z^{-1} + r^2 z^{-2}}$ | $ z > r$ |



ROC Properties

- The ROC *cannot* include any poles.
- The ROC is a connected (that is, a single contiguous) region.
- For finite duration sequences the ROC is the entire z -plane, with the possible exception of $z = 0$ or $z = \infty$.
- For infinite duration sequences the ROC can have one of the following shapes:

| Type of sequence | ROC |
|-------------------------------------|----------------------------------|
| Right-sided ($x[n] = 0, n < n_0$) | \Rightarrow ROC: $ z > r$ |
| Left-sided ($x[n] = 0, n > n_0$) | \Rightarrow ROC: $ z < r$ |
| Two-sided | \Rightarrow ROC: $a < z < b$ |

- The z -transform of a sequence consists of an algebraic formula and its associated ROC. Thus, to uniquely specify a sequence $x[n]$ we need both $X(z)$ and its ROC.
- The function $X(z)$ is legitimate only for z within its ROC. We stress that $X(z)$ is *not* defined when z is outside the ROC, even if the formula for $X(z)$ yields meaningful results for these values.



The inverse z -transform

The recovery of a sequence $x[n]$ from its z -transform ($X(z)$ and ROC) can be formally done using the formula:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz,$$

Simpler procedures

For most sequences and z -transforms encountered in the analysis of LTI systems the following simple procedure is enough:

- Expansion into a series of terms in the variables z and z^{-1} and picking their coefficients.
- Partial fraction expansion and table look-up.



The inverse z-transform

The z-transform of a linear combination of distinct exponentials (that is, $p_k \neq p_m, k \neq m$) is given by:

$$x[n] = \sum_{k=1}^N A_k (p_k)^n \xleftrightarrow{z} X(z) = \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}},$$

ROC: the intersection of the ROCs of the individual exponential sequences

$$\rightarrow X(z) = \frac{\sum_{k=1}^N A_k \prod_{\substack{m=1 \\ m \neq k}}^N (1 - p_m z^{-1})}{\prod_{k=1}^N (1 - p_k z^{-1})} = \frac{b_0 + b_1 z^{-1} + \dots + b_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}},$$

which is a *proper* rational function because the degree of the numerator is less than the degree of the denominator.



The inverse z-transform

Example: Real and distinct poles

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

1- Decompose the sequence into partial fractions:

distinct poles $p_1 = 1$ and $p_2 = 0.5$

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 - 0.5z^{-1}}.$$

$$\rightarrow z + 1 = A_1(z - 0.5) + A_2(z - 1),$$

$$z = 1 \quad \rightarrow \quad A_1 = 4$$

$$z = 0.5 \quad \rightarrow \quad A_2 = -3$$



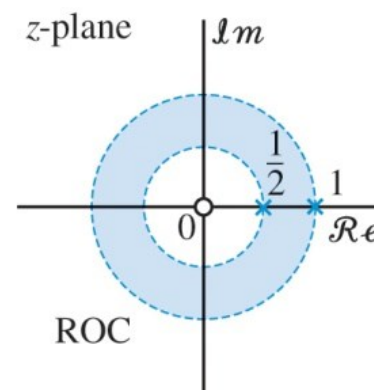
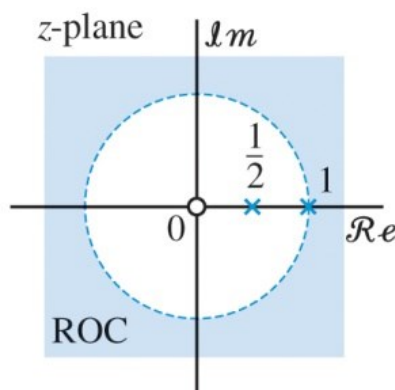
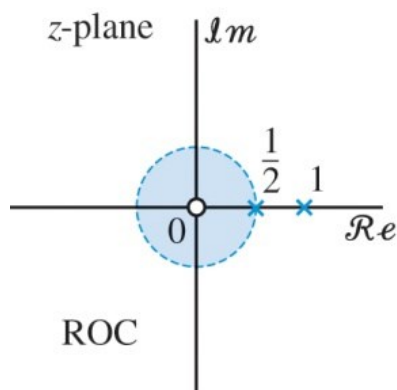
The inverse z-transform

Example: Real and distinct poles

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

2- Find the sequences corresponding to the partial fractions,

A ROC cannot include any poles, so there are three possible choices for valid ROCs.

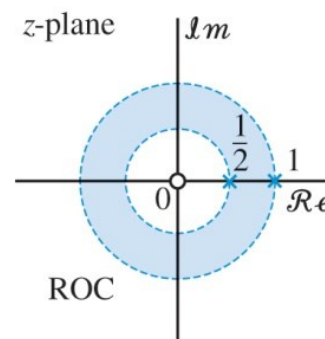
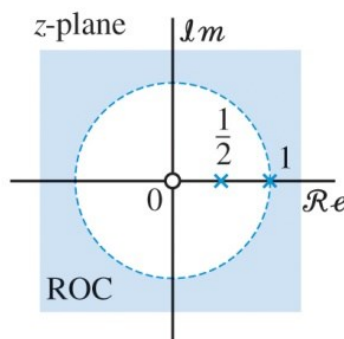
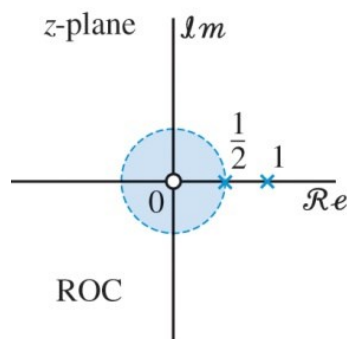




The inverse z-transform

Example: Real and distinct poles

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$



- $|z| > 1$, both fractions are the z-transform of causal sequences. Hence

$$x[n] = 4u[n] - 3\left(\frac{1}{2}\right)^n u[n]. \quad (\text{causal})$$

- $0.5 < |z| < 1$, this can be obtained as the intersection of ROC: $|z| < 1$ and ROC: $|z| > 0.5$.

$$x[n] = -4u[-n-1] - 3\left(\frac{1}{2}\right)^n u[n]. \quad (\text{two-sided})$$

- $|z| < 0.5$, both fractions are the z-transform of anticausal sequences.

$$x[n] = -4u[-n-1] + 3\left(\frac{1}{2}\right)^n u[-n-1]. \quad (\text{anticausal})$$



The inverse z-transform

Example: Complex conjugate distinct poles

Consider a causal sequence $x[n]$ with z-transform

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

$$z^2 - z + 0.5 = 0 \quad \rightarrow \quad p_1 = \frac{1}{2}(1 + j) = \frac{1}{\sqrt{2}}e^{j\pi/4} \quad \text{and} \quad p_2 = \frac{1}{2}(1 - j) = \frac{1}{\sqrt{2}}e^{-j\pi/4}$$

$$p_1 \neq p_2 \quad \rightarrow \quad X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}} = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

$z + 1 = A_1(z - p_2) + A_2(z - p_1)$, setting $z = p_1$ and $z = p_2$

$$A_1 = \frac{1}{2} - j\frac{3}{2} = \frac{\sqrt{10}}{2}e^{-j71.56^\circ} \quad \text{and} \quad A_2 = \frac{1}{2} + j\frac{3}{2} = \frac{\sqrt{10}}{2}e^{j71.56^\circ}$$

$$\text{Since } x[n] \text{ is causal} \quad \rightarrow \quad x[n] = A_1(p_1)^n u[n] + A_1^*(p_1^*)^n u[n]$$

$$A_1 = Ae^{j\theta}, p_1 = re^{j\omega_0} \quad r = 1/\sqrt{2}, \omega_0 = \pi/4, A = \sqrt{10}/2, \text{ and } \theta = -71.56^\circ$$

$$x[n] = Ar^n \left(e^{j\omega_0 n} e^{j\theta} + e^{-j\omega_0 n} e^{-j\theta} \right) u[n] = 2Ar^n \cos(\omega_0 n + \theta) u[n]$$



The inverse z-transform

If we have a rational function with distinct poles

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}},$$

the complete partial fraction expansion takes the form

$$X(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}}, \quad A_k = (1 - p_k z^{-1})X(z)|_{z=p_k},$$

Example Partial fraction expansion (using `scipy.signal.residuez`)

$$X(z) = \frac{6 - 10z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = 1 + \frac{2}{1 - z^{-1}} + \frac{3}{1 - 2z^{-1}}$$

```
import scipy.signal as sig
```

```
b = [6, -10, 2]
```

```
a = [1, -3, 2]
```

```
[B, A, C] = sig.residuez(b,a)
```

$$x[n] = \delta[n] + (2 + 3 \times 2^n)u[n], \quad \text{ROC: } |z| > 2$$

$$x[n] = \delta[n] - (2 + 3 \times 2^n)u[-n - 1], \quad \text{ROC: } |z| < 1$$

$$x[n] = \delta[n] + 2u[n] - 3 \times 2^n u[-n - 1], \quad \text{ROC: } 1 < |z| < 2$$



Properties of the z-transform

Linearity

$$a_1x_1[n] + a_2x_2[n] \xleftrightarrow{\mathcal{Z}} a_1X_1(z) + a_2X_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Time shifting

$$x[n - k] \xleftrightarrow{\mathcal{Z}} z^{-k}X(z). \quad \text{ROC} = R_x(\text{except } z = 0 \text{ or } z = \infty)$$

Example

$$x[n] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

The ROC is the entire z-plane, except $z = 0$.

$$X(z) = \sum_{n=0}^{N-1} 1z^{-n} = 1 + z^{-1} + \dots + z^{-(N-1)} = \begin{cases} N, & z = 1 \\ \frac{1 - z^{-N}}{1 - z^{-1}}, & z \neq 1. \end{cases}$$

Alternative approach:

$$x[n] = u[n] - u[n - N].$$

$$X(z) = U(z) - z^{-N}U(z) = (1 - z^{-N})U(z) = \frac{1 - z^{-N}}{1 - z^{-1}}.$$



Properties of the z -transform

Convolution of sequences

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z). \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Multiplication by an exponential sequence

$$a^n x[n] \xleftrightarrow{Z} X(z/a). \quad \text{ROC} = |a|R_x$$

Since a and z take complex values, the result is scaling (expansion or shrinking) and rotation of the z -plane, the ROC, and the pole-zero pattern.

Differentiation of the z -transform

$$nx[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}, \quad \text{ROC} = R_x$$

Both sequences have the same ROC.

Example Second-order pole

$$x[n] = na^n u[n] = n(a^n u[n]), \rightarrow X(z) = -z \frac{d}{dz} \left(\frac{1}{1 - az^{-1}} \right) = \frac{az^{-1}}{(1 - az^{-1})^2}, \quad |z| > |a|$$



Properties of the z -transform

Conjugation of a complex sequence

$$x^*[n] \xleftrightarrow{\mathcal{Z}} X^*(z^*). \quad \text{ROC} = R_x$$

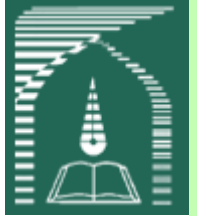
Time reversal

$$x[-n] \xleftrightarrow{\mathcal{Z}} X(1/z). \quad \text{ROC} = \frac{1}{R_x}$$

Initial-value theorem

If $x[n]$ is a causal sequence, that is, $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z \rightarrow \infty} X(z),$$



Summary of the z-transform properties

Table 3.2 Some z-transform properties.

| | Property | Sequence | Transform | ROC |
|-----|-----------------------|-------------------------|---|----------------------------------|
| | | $x[n]$ | $X(z)$ | R_x |
| | | $x_1[n]$ | $X_1(z)$ | R_{x_1} |
| | | $x_2[n]$ | $X_2(z)$ | R_{x_2} |
| 1. | Linearity | $a_1x_1[n] + a_2x_2[n]$ | $a_1X_1(z) + a_2X_2(z)$ | At least $R_{x_1} \cap R_{x_2}$ |
| 2. | Time shifting | $x[n - k]$ | $z^{-k}X(z)$ | R_x except $z = 0$ or ∞ |
| 3. | Scaling | $a^n x[n]$ | $X(a^{-1}z)$ | $ a R_x$ |
| 4. | Differentiation | $nx[n]$ | $-z \frac{dX(z)}{dz}$ | R_x |
| 5. | Conjugation | $x^*[n]$ | $X^*(z^*)$ | R_x |
| 6. | Real-part | $\text{Re}\{x[n]\}$ | $\frac{1}{2}[X(z) + X^*(z^*)]$ | At least R_x |
| 7. | Imaginary part | $\text{Im}\{x[n]\}$ | $\frac{1}{2j}[X(z) - X^*(z^*)]$ | At least R_x |
| 8. | Folding | $x[-n]$ | $X(1/z)$ | $1/R_x$ |
| 9. | Convolution | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | At least $R_{x_1} \cap R_{x_2}$ |
| 10. | Initial-value theorem | $x[n] = 0$ for $n < 0$ | $x[0] = \lim_{z \rightarrow \infty} X(z)$ | |



System function of LTI systems

Every LTI system can be completely characterized in the time domain by its impulse response $h[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k], \quad \rightarrow \quad Y(z) = H(z)X(z),$$

Since there is a unique relation between $h[n]$ and $H(z)$ many properties of the system can be inferred from $H(z)$ and its ROC.

Example

Determine the response of a system with impulse response $h[n] = a^n u[n]$, $|a| < 1$ to the input $x[n] = u[n]$ using the convolution theorem.

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \quad |z| > |a| \quad X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$
$$\rightarrow Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}, \quad |z| > \max\{|a|, 1\} = 1.$$

Determining the output sequence $y[n]$

$$Y(z) = \frac{1}{1 - a} \left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}} \right), \quad |z| > 1$$
$$\rightarrow y[n] = \frac{1}{1 - a} (u[n] - a^{n+1} u[n]) = \frac{1 - a^{n+1}}{1 - a} u[n],$$



System function of LTI systems

Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n < 0$. Therefore, for causal systems the power series $H(z)$ does not include any positive powers of z and its ROC extends outside of a circle for some radius r , that is, $|z| > r$.

A system function $H(z)$ with the ROC that is the exterior of a circle, extending to infinity, is a necessary condition for a discrete-time LTI system to be causal but not a sufficient one.

Stability

For a LTI system to be stable,
$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

This is equivalent to the condition $|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$, for $|z| = 1$

A LTI system is stable if and only if the ROC of the system function $H(z)$ includes the unit circle $|z| = 1$.



System function of LTI systems

Causal and stable system

An LTI system with rational $H(z)$ is both causal and stable if and only if all poles of $H(z)$ are inside the unit circle and its ROC is on the exterior of a circle, extending to infinity.

Stability

A causal LTI with a rational system function is stable if and only if all poles of $H(z)$ are inside the unit circle in the z -plane. The zeros can be anywhere.



Linear constant-coefficient difference equations

Consider a linear constant-coefficient difference equation

$$y[n] + \sum_{k=1}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k].$$

Using the properties of linearity and time shifting

$$\left(1 + \sum_{k=1}^N a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^M b_k z^{-k}\right) X(z).$$

Therefore the system function is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}},$$



Linear constant-coefficient difference equations

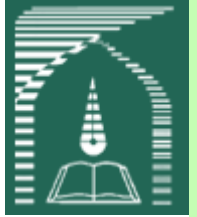
Example

Find the difference equation corresponding to the system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{6 - 10z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}}.$$

$$\rightarrow (1 - 3z^{-1} + 2z^{-2})Y(z) = (6 - 10z^{-1} + 2z^{-2})X(z).$$

$$\rightarrow y[n] - 3y[n - 1] + 2y[n - 2] = 6x[n] - 10x[n - 1] + 2x[n - 2].$$



Linear constant-coefficient difference equations

Poles and zeros

if z_1, z_2, \dots, z_M and p_1, p_2, \dots, p_N are the roots of the numerator and the denominator polynomial, respectively

$$\begin{aligned} B(z) &= b_0 z^{-M} \left(z^M + \frac{b_1}{b_0} z^{M-1} + \dots + \frac{b_M}{b_0} \right) \\ &= b_0 z^{-M} (z - z_1) \dots (z - z_M). \end{aligned}$$

$$\begin{aligned} A(z) &= z^{-N} \left(z^N + a_1 z^{N-1} + \dots + a_N \right) \\ &= z^{-N} (z - p_1) \dots (z - p_N). \end{aligned}$$

$$\rightarrow H(z) = \frac{B(z)}{A(z)} = b_0 \frac{z^{-M}}{z^{-N}} \frac{\prod_{k=1}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)} = b_0 \frac{\prod_{k=1}^M (1 - z_k z^{-1})}{\prod_{k=1}^N (1 - p_k z^{-1})},$$

where b_0 is a constant gain term.

Each of the factors $(1 - z_k z^{-1})$ contributes a zero at $z = z_k$ and a pole at $z = 0$.

Each of the factors $(1 - p_k z^{-1})$ contributes a pole at $z = p_k$ and a zero at $z = 0$.

Poles and zeros at the origin are not counted.



Linear constant-coefficient difference equations

Impulse response

Any rational function of z^{-1} with distinct poles can be expressed in the form

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}},$$

If we assume that the system is causal, then the ROC is the exterior of a circle starting at the outermost pole, and the impulse response is

$$h[n] = \sum_{k=0}^{M-N} C_k \delta[n - k] + \sum_{k=1}^N A_k (p_k)^n u[n].$$



Linear constant-coefficient difference equations

System classifications

LTI systems can be classified into different classes:

Length of impulse response

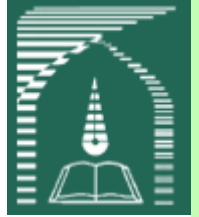
Infinite Impulse Response (IIR)

If at least one nonzero pole of $H(z)$ is not canceled by a zero, there will be a term of the form $A_k(p_k)^n u[n]$. In this case $h[n]$ has infinite duration.

Finite Impulse Response (FIR)

If $N = 0$, the system function becomes a polynomial.

$$h[n] = \sum_{k=0}^M b_k \delta[n - k] = \begin{cases} b_n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$



Linear constant-coefficient difference equations

System classifications

LTI systems can be classified into different classes:

Feedback in implementation

Recursive systems

If $N \geq 1$ the output of the system is fed back into the input.

Nonrecursive systems

If $N = 0$ the output is a linear combination of the present and previous inputs, only.

Poles and zeros

If $a_k = 0$, for $k = 1, \dots, N$, the system has M zeros (*all-zero* systems).

If $b_k = 0$, for $k = 1, \dots, M$, the system has N poles (*all-pole* systems).



Connections between pole-zero locations and time-domain behavior

The roots of a polynomial with real coefficients either must be real or must occur in complex conjugate pairs.

Any system with a rational $H(z)$ is equivalent to a parallel combination of an FIR system, K_1 first-order systems with real poles, and K_2 second-order systems with complex conjugate poles, where $N = K_1 + 2K_2$:

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{K_1} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}.$$



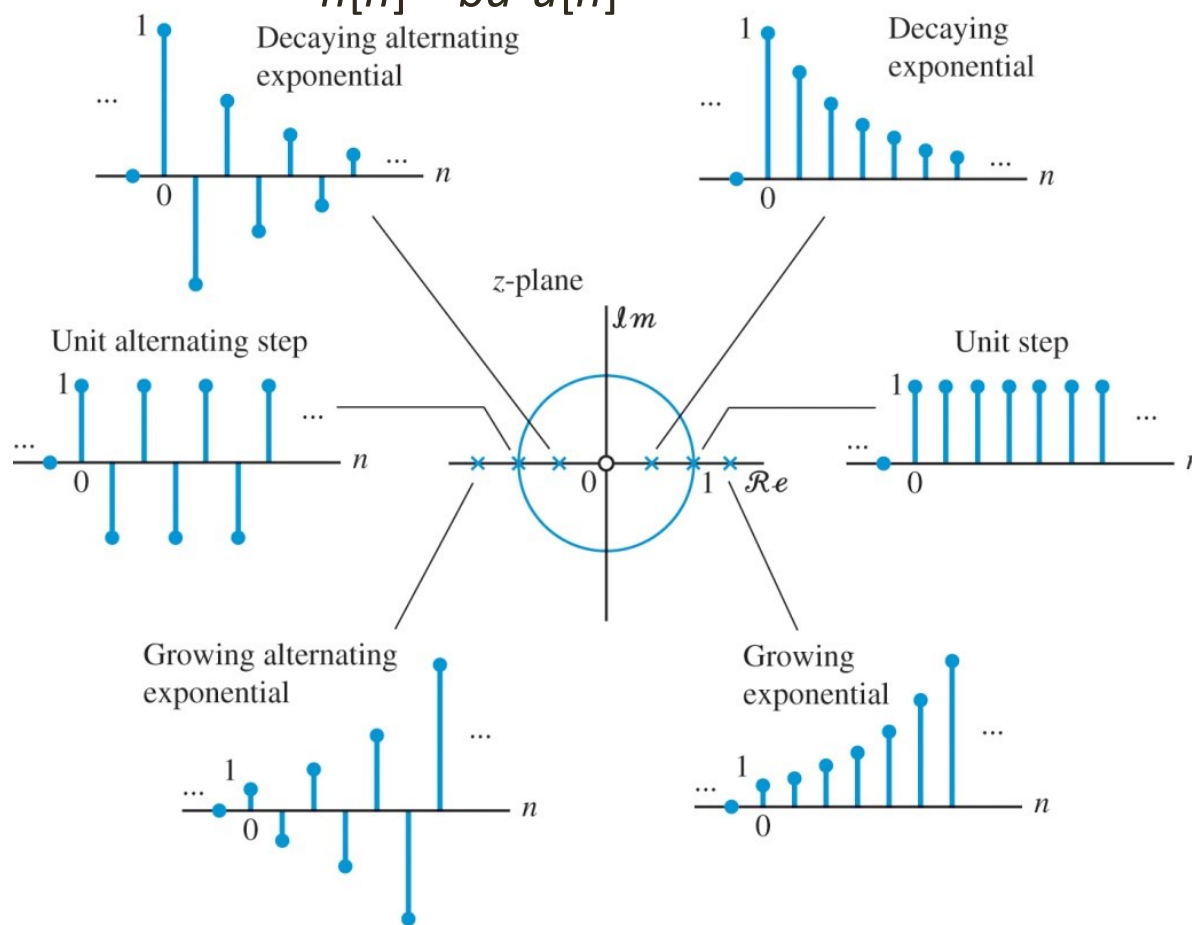
Connections between pole-zero locations and time-domain behavior

First-order systems

$$H(z) = \frac{b}{1 - az^{-1}} \quad a, b \text{ real}$$

Assuming a causal system, the impulse response is given by the following real exponential sequence:

$$h[n] = ba^n u[n]$$





Connections between pole-zero locations and time-domain behavior

Second-order systems

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z(b_0 z + b_1)}{z^2 + a_1 z + a_2}.$$

Zeros:

$$z_1 = 0 \text{ and } z_2 = -b_1/b_0$$

Poles:

$$p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}.$$

| Poles | Condition |
|-------------------|----------------|
| Real and distinct | $a_1^2 > 4a_2$ |
| Real and equal | $a_1^2 = 4a_2$ |
| Complex conjugate | $a_1^2 < 4a_2$ |

The impulse response of a causal system with a pair of complex conjugate poles

$$\begin{aligned} h[n] &= A p^n u[n] + A^* (p^*)^n u[n] \\ &= |A| e^{j\theta} r^n e^{j\omega_0 n} u[n] + |A| e^{-j\theta} r^n e^{-j\omega_0 n} u[n] \\ &= |A| r^n \left[e^{j(\omega_0 n + \theta)} + e^{-j(\omega_0 n + \theta)} \right] u[n] \\ &= 2|A| r^n \cos(\omega_0 n + \theta) u[n], \end{aligned}$$



Connections between pole-zero locations and time-domain behavior

