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Sampling of continuous-time signals

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Ideal periodic sampling of continuous-time signals

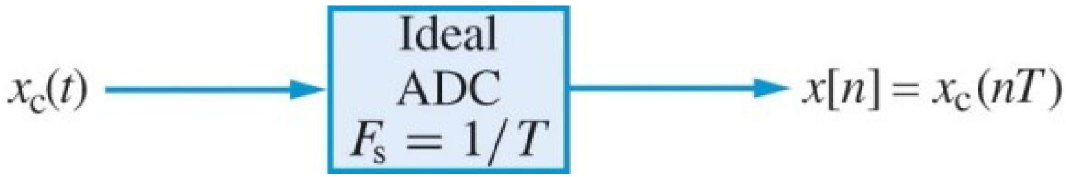
Periodic sampling

Ideal analog-to-digital converter (ADC) or ideal sampler

$$x[n] \triangleq x_c(t)|_{t=nT} = x_c(nT), \quad -\infty < n < \infty$$

T: ***sampling period*** (the fixed time interval between samples)

$Fs = 1/T$: ***sampling frequency*** (when expressed in cycles per second or Hz)
sampling rate (when expressed in samples per second)

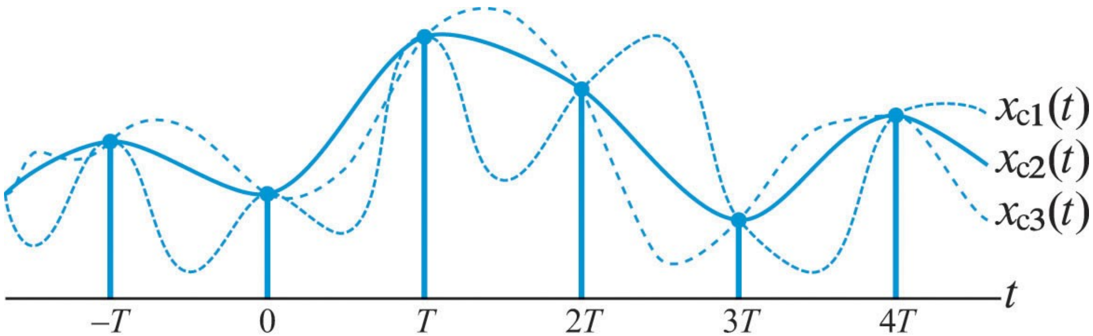


The main difference between an ideal ADC and a practical ADC is the finite number of bits

Ideal periodic sampling of continuous-time signals

Question

Are the samples $x[n]$ sufficient to describe uniquely the original continuous-time signal and, if so, how can $x_c(t)$ be reconstructed from $x[n]$?



Three different continuous-time signals with the same set of sample values, that is, $x[n] = x_{c1}(nT) = x_{c2}(nT) = x_{c3}(nT)$.

- An infinite number of signals can generate the same set of samples.
- In other words, the samples do not tell us anything about the values of the signal between sampling times.
- The only way to determine these values is by putting some constraints on the behavior of the continuous-time signal.
- The answer to these questions lies in the frequency domain, that is, in the relation between the spectra of $x_c(t)$ and $x[n]$.

Ideal periodic sampling of continuous-time signals

Continuous-time

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t} dt, \quad (1)$$

$$x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega)\, e^{j\Omega t} d\Omega. \quad (2)$$

Discrete-time
(at the output of the ideal ADC)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad (3)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \quad (4)$$

Substituting t by $t = nT$ in (2) $\rightarrow$ $x_c(nT) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega)\, e^{j\Omega T n} d\Omega.$ (5)

Comparing (4) to (5) shows that the necessary relationship is

$$\omega = \Omega T = 2\pi F T = 2\pi \frac{F}{F_s} = 2\pi f.$$

Replace the variable ω in (4) by the equivalent variable ΩT,

$$x[n] = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left[T X(e^{j\Omega T}) \right] e^{j(\Omega T) n} d\Omega. \quad (6)$$

- $x[n] = x_c(nT)$
- Replace Ω by θ in (5)
- Express (5) as a sum of integrals over intervals of length $2\pi/T$

$$\Bigg\}\; x_c(nT) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi/T}^{(2k+1)\pi/T} X_c(j\theta)\, e^{j\theta T n} d\theta.$$

Ideal periodic sampling of continuous-time signals

$$x_c(nT) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi/T}^{(2k+1)\pi/T} X_c(j\theta)\, e^{j\theta Tn} d\theta.$$

Change the integration variable from θ to $\Omega = \theta + (2\pi/T)k$

$$x_c(nT) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi/T}^{\pi/T} X_c\left(j\Omega - j\frac{2\pi}{T}k\right) e^{j\Omega Tn} e^{-j2\pi kn} d\Omega.$$

$$\rightarrow \qquad x_c(nT) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left[\sum_{k=-\infty}^{\infty} X_c\left(j\Omega - j\frac{2\pi}{T}k\right) \right] e^{j\Omega Tn} d\Omega. \qquad (7)$$

Comparing (7) to (6) yields the desired relationship between $X(e^{j\Omega T})$ and $X_c(j\Omega)$

$$X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\Omega - j\frac{2\pi}{T}k\right)$$

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)$$

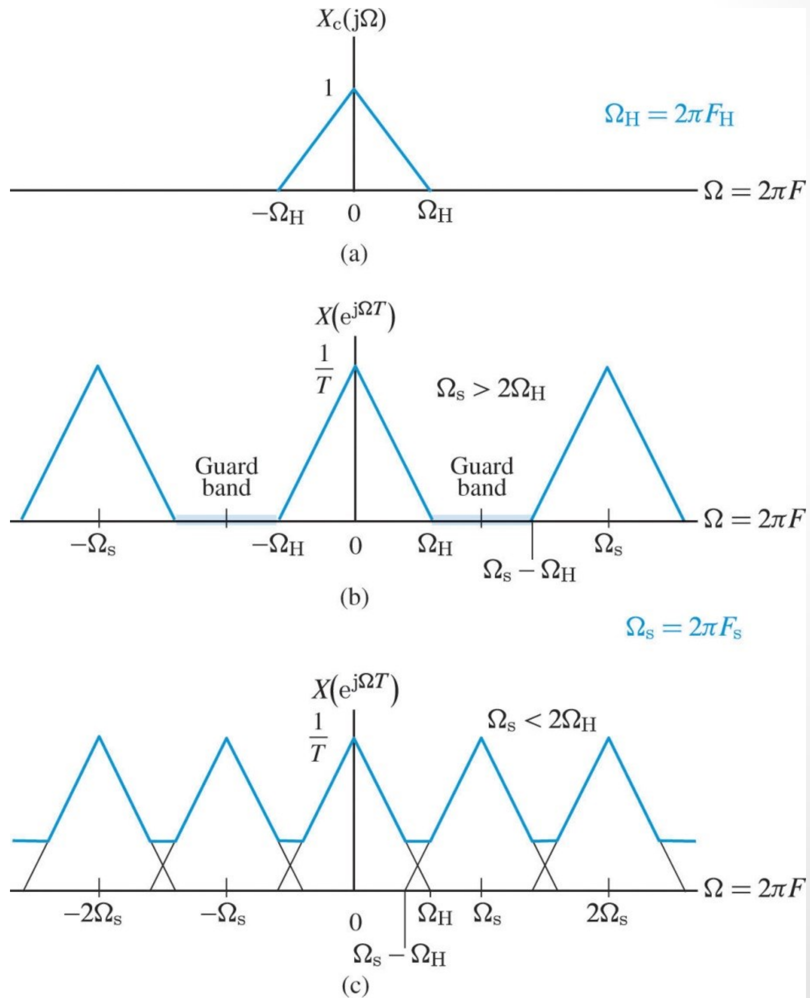
Ideal periodic sampling of continuous-time signals

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)$$

The spectrum of $x[n]$ can be readily sketched if $x_c(t)$ is assumed to be bandlimited, that is, $X_c(j\Omega) = 0$ for $|\,\Omega\,| > \Omega_H = 2\pi F_H$.

Aliasing (aliasing distortion)

The effect of individual terms overlap.



Frequency-domain interpretation of uniform sampling. (a) Spectrum of continuous-time bandlimited signal $x_c(t)$, (b) spectrum of discrete-time signal $x[n] = x_c(nT)$ with $\Omega_s > 2\Omega_H$, and (c) spectrum of $x[n]$, showing aliasing distortion, when $\Omega_s < 2\Omega_H$.

Ideal periodic sampling of continuous-time signals

Two conditions obviously are necessary to prevent overlapping spectral bands:

- the continuous-time signal must be bandlimited, and $$X_c(j\Omega) = 0, \quad |\Omega| > \Omega_H$$
- the sampling frequency must be sufficiently large so that $\Omega_s - \Omega_H > \Omega_H$. $$\Omega_s \geq 2\Omega_H \quad \text{or} \quad T \leq \frac{1}{2F_H}.$$

If the sampled signal is sinusoidal, its spectrum consists of lines and hence $\Omega_s > 2\,\Omega_H$

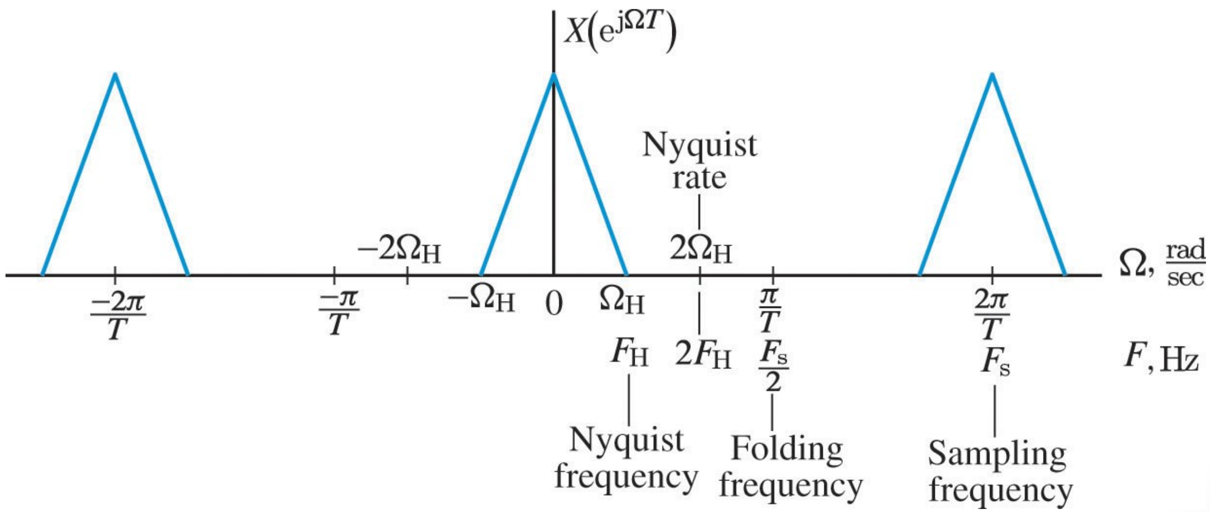
Ideal periodic sampling of continuous-time signals

Terminology in sampling operation

Nyquist frequency: The highest frequency F_H, in Hz, present in a bandlimited signal $x_c(t)$.

Nyquist rate: The minimum sampling frequency required to avoid overlapping bands is $2F_H$.

Folding frequency: The actual highest frequency that the sampled signal $x[n]$ contains is $F_s/2$, in Hz.



Ideal periodic sampling of continuous-time signals

Sampling theorem:

Let $x_c(t)$ be a continuous-time bandlimited signal with Fourier transform

$$X_c(j\Omega) = 0 \quad \text{for} |\Omega| > \Omega_H.$$

Then $x_c(t)$ can be uniquely determined by its samples $x[n] = x_c(nT)$, where $n = 0, \pm1, \pm2, \ldots,$ if the sampling frequency Ω_s satisfies the condition

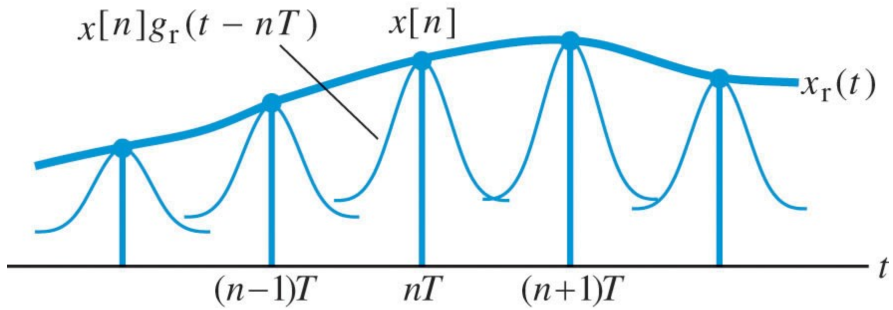
$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_H.$$

Reconstruction of a bandlimited signal from its samples

The objective is to find a formula which yields the values of $x_c(t)$ between samples in terms of the sample values.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] g_r(t - nT),$$

where $x_r(t)$ is the reconstructed signal and $g_r(t)$ is an interpolating reconstruction function.



The interpolation process for reconstruction of a continuous-time signal from its samples.

Frequency-domain expression:

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] G_r(j\Omega)\, e^{-j\Omega nT}. \quad \rightarrow \quad X_r(j\Omega) = G_r(j\Omega)\, X(e^{j\Omega T}).$$

Reconstruction of a bandlimited signal from its samples

If we choose $g_r(t)$ so that

$$G_r(j\Omega) \triangleq G_{BL}(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_s/2 \\ 0, & |\Omega| > \Omega_s/2 \end{cases}$$

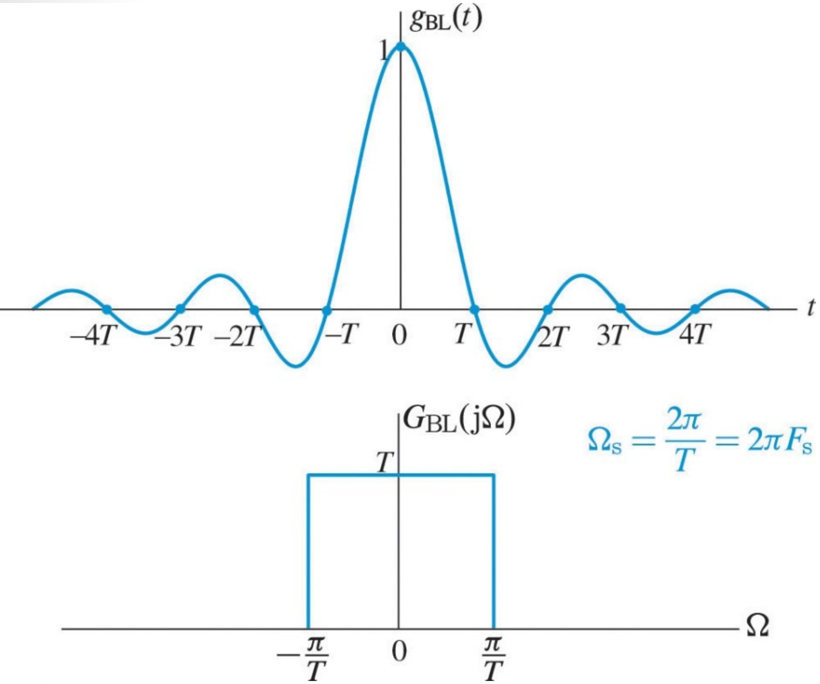
then $X_r(j\Omega) = X_c(j\Omega)$ and therefore $x_r(t) = x_c(t)$.

$$\rightarrow \qquad g_r(t) \triangleq g_{BL}(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

$$\rightarrow \qquad x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left[\pi(t-nT)/T\right]}{\pi(t-nT)/T}.$$

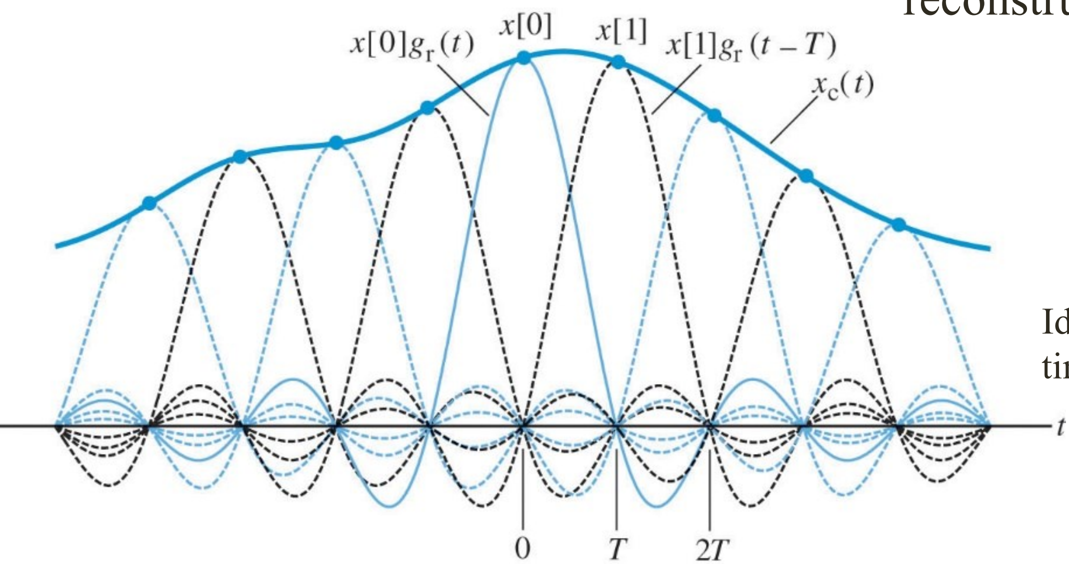
This is called *ideal bandlimited interpolation*, because it provides a perfect reconstruction for all t, if $x_c(t)$ is bandlimited in F_H and $F_s \geq 2F_H$.

Reconstruction of a bandlimited signal from its samples



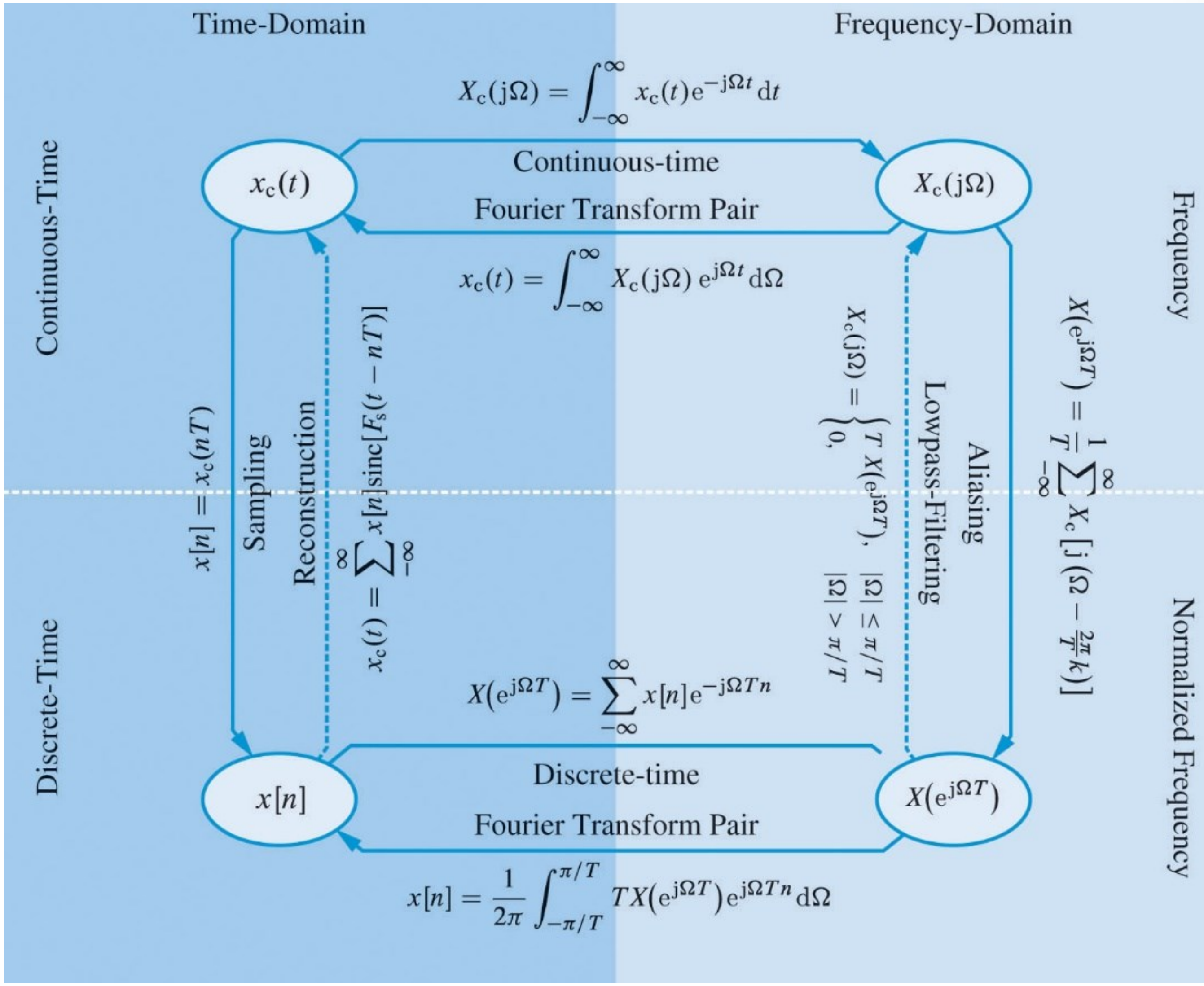
Time- and frequency-domain characteristics of the ideal DAC.

- $g_{BL}(t) = 0$ at all sampling instants $t = nT$, except at $t = 0$ where $g_{BL}(t) = 1$.
- $x_r(nT) = x_c(nT)$ regardless of whether aliasing distortion occurred during sampling.
- If there is no aliasing, the original signal is reconstructed perfectly for all values of time.
- Because $g_{BL}(t)$ has infinite extent, each sample value contributes to the reconstruction $x_r(t)$ of $x_c(t)$ for all values of t.



Ideal bandlimited interpolation in the time domain.

Reconstruction of a bandlimited signal from its samples



Relationships between the spectra of a continuous-time signal $x_c(t)$ and the discrete-time signal $x[n] = x_c(nT)$ obtained by periodic sampling. The dashed paths hold for bandlimited signals sampled at a rate $F_s > 2F_H$.

The effect of undersampling: aliasing

Example: Aliasing in sinusoidal signals

$$x_c(t) = \cos(2\pi F_0 t) = \frac{1}{2}e^{j2\pi F_0 t} + \frac{1}{2}e^{-j2\pi F_0 t}$$

has a discrete spectrum with spectral lines at frequencies $F = \pm F_0$.
We shall study the effect of changing the frequency F_0 while the sampling frequency F_s is kept fixed.

To obtain the spectrum of $x[n] = x_c(nT)$, we replicate (after scaling by $1/T$) the line at $F = F_0$ to $F = kF_s + F_0$ and the line at $F = -F_0$ to $F = kF_s - F_0$, for all k.

The effect of undersampling: aliasing

There are two cases of interest:

- $0 < F_0 < F_s/2$ (there is no aliasing)
- $F_s/2 < F_0 < F_s$ (causes aliasing)

$$x_r(t) = \cos[2\pi(F_s - F_0)t] \neq x_c(t).$$

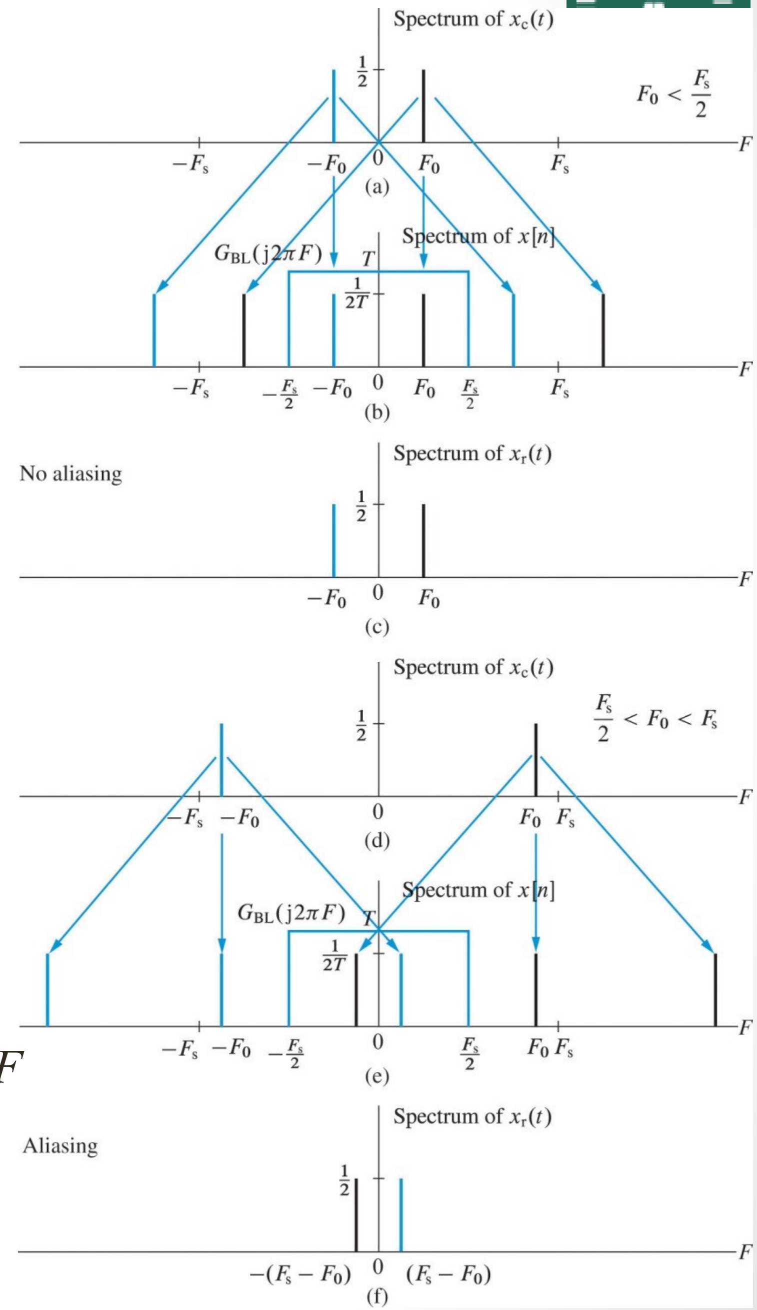
Assume $F_0 = F_s/2 + \Delta F$, where $0 \leq \Delta F \leq F_s/2$

$$\rightarrow \; x_c(t) = \cos(2\pi F_0 t) = \cos[2\pi\,(F_s/2 + \Delta F)\,t].$$

The *apparent frequency* of the reconstructed signal $x_r(t)$ is $F_a = F_s - F_0 = F_s/2 - \Delta F$

$$\rightarrow \; x_r(t) = \cos 2(\pi F_a t) = \cos[2\pi\,(F_s/2 - \Delta F)\,t].$$

A sinusoid of higher frequency $F_0 = F_s/2 + \Delta F$ *appear* as a sinusoid of lower ("apparent") frequency $F_a = F_s/2 - \Delta F$.



The effect of undersampling: aliasing

Example: Verification of aliasing in the time domain

We shall demonstrate that sampling the signals

$x_c(t) = \cos(2\pi F_0 t) = \cos[2\pi\,(F_s/2 + \Delta F)\,t]$. and $x_r(t) = \cos 2(\pi F_a t) = \cos[2\pi\,(F_s/2 - \Delta F)\,t]$.

at a sampling frequency $Fs = 1/T$, results in identical sample values.

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$
$\cos(\pi n) = (-1)^n$, for all integer n
$\sin(\pi n) = 0$ for all integer n

$\rightarrow \quad \cos[2\pi\,(F_s/2 \pm \Delta F)\,nT] = (-1)^n \cos(2\pi\,\Delta F\,nT)$

the samples of a sinusoid of frequency $Fs/2 + \Delta F$ are identical to the samples of a sinusoid of frequency $Fs/2 - \Delta F$.

$\rightarrow \quad \sin[2\pi\,(F_s/2 \pm \Delta F)\,nT] = \pm(-1)^n \sin(2\pi\,\Delta F nT)$

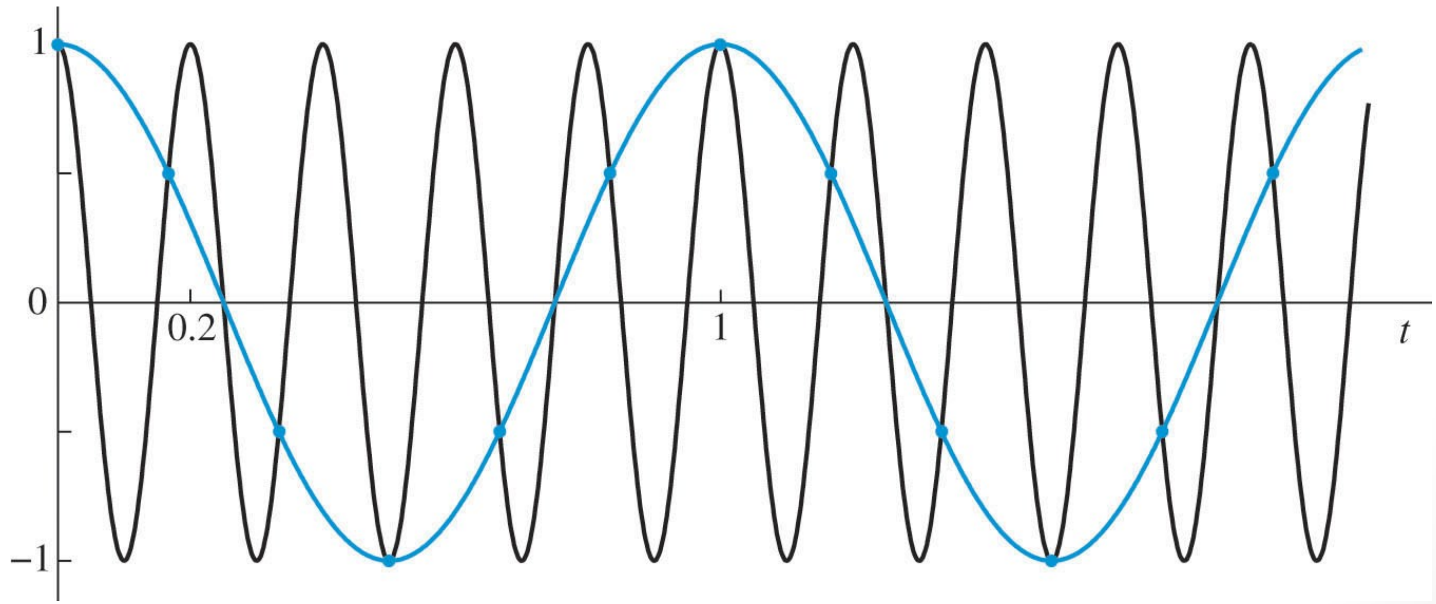
undersampling makes a sine signal of higher frequency $F_0 = F_s/2 + \Delta F$ appear as a sine of lower frequency $F_a = F_s/2 - \Delta F$ and opposite amplitude.

The effect of undersampling: aliasing

Example: Verification of aliasing in the time domain

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$
$\cos(\pi n) = (-1)^n,$ for all integer n
$\sin(\pi n) = 0$ for all integer n

$\rightarrow \quad \cos[2\pi\,(F_s/2 \pm \Delta F)\,nT] = (-1)^n \cos(2\pi\,\Delta F\,nT)$



The signals $x(t) = \cos(2\pi F_1 t)$, $F_1 = 1$ Hz, and $x_2(t) = \cos(2\pi F_2 t)$, $F_2 = 5$ Hz, sampled at a rate $F_s = 6$ Hz generate the same set of samples. The ideal DAC reconstructs the sinusoid whose frequency is in the fundamental range.

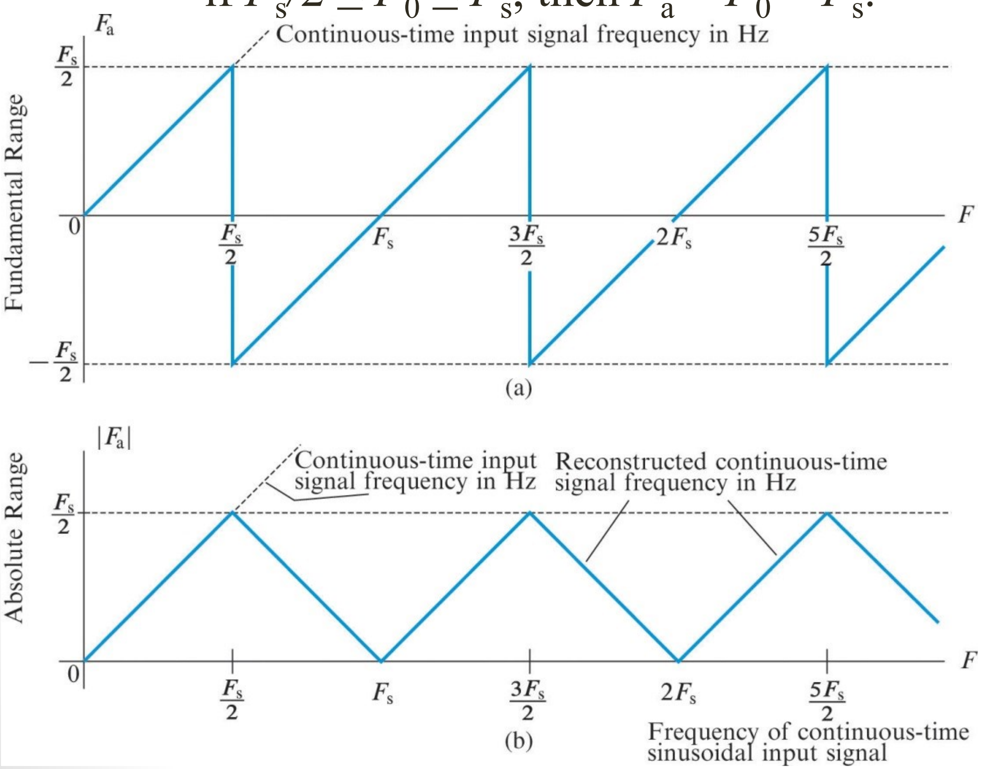
The effect of undersampling: aliasing

Apparent frequency

$$\cos[2\pi(kF_s + F_0)nT] = \cos(2\pi kn + 2\pi F_0 nT) = \cos(2\pi F_0 nT)$$

After sampling and reconstruction at a rate $F_s = 1/T$, a sinusoid of frequency $F > F_s$ appears as a sinusoid of frequency $F_0 = F - kF_s$, where k is chosen such that $0 \le F_0 \le F_s$.

- If $0 \le F_0 \le F_s/2$, the apparent frequency of the reconstructed signal is $F_a = F_0$
- if $F_s/2 \le F_0 \le F_s$, then $F_a = F_0 - F_s$.



The ideal DAC always reconstructs a cosine or sine signal with frequency in the range $-F_s/2 < F_0 \le F_s/2$.

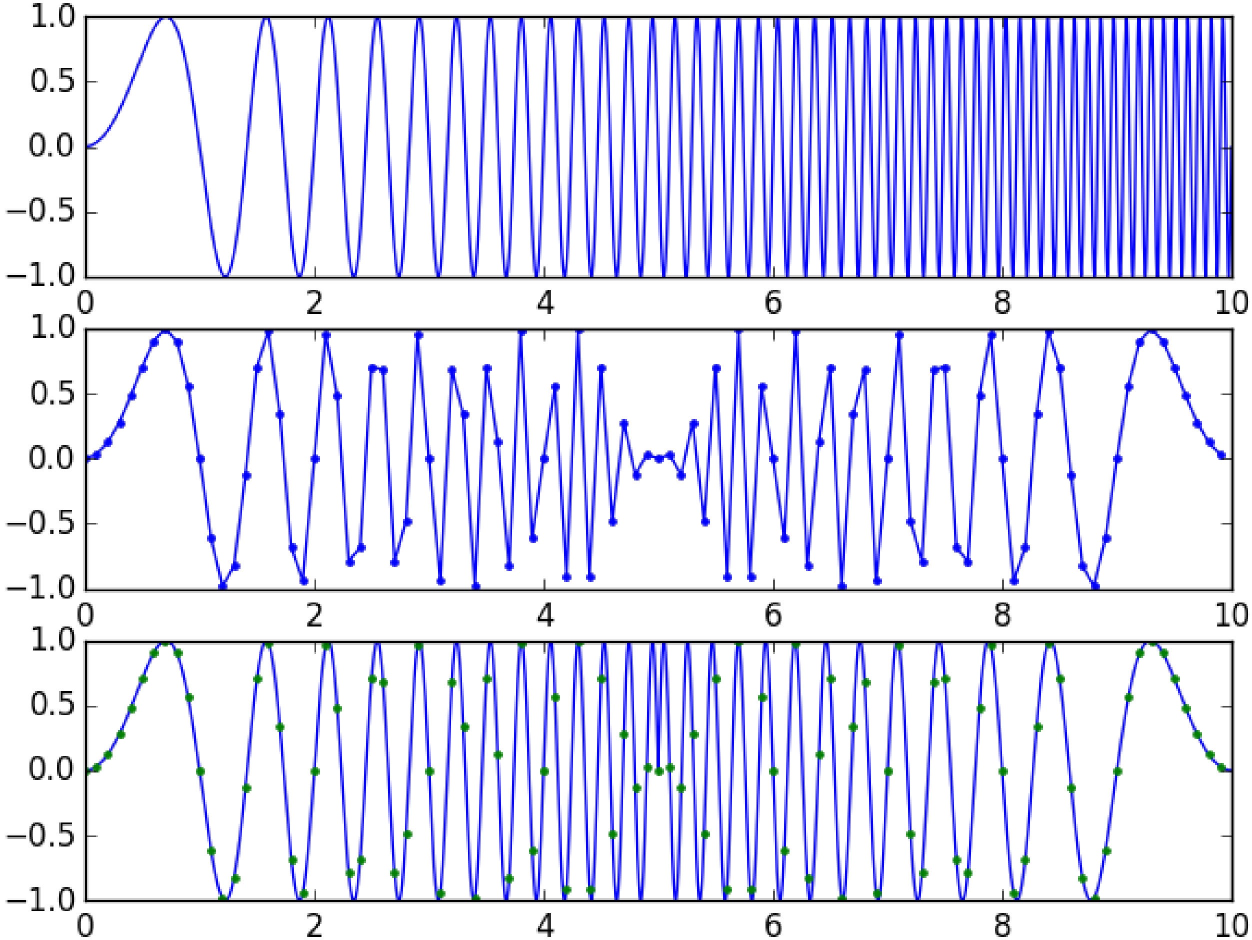
Since $\cos(-2\pi F_a t + \theta) = \cos(2\pi F_a t - \theta)$, the *apparent value* of $-F_a$ is also F_a with a reversal of sign change in phase. This implies that the apparent frequency of any sinusoid lies in the range $0 \le F_0 \le F_s/2$

Apparent frequencies reconstructed by an ideal DAC of a continuous-time sinusoidal signal sampled by an ideal ADC: (a) fundamental range of Fa, and (b) absolute range $|F$a$|$.

The effect of undersampling: aliasing

Sampling a linear FM signal

Consider the linear FM signal $x_c(t) = \sin(\frac{\pi B t^2}{\tau})$, $0 \le t \le \tau$ with $B = 10$ Hz and $\tau = 10$ s. It is applied to a talk-through system with sampling rate of $F_s = B$ Hz to obtain sampled signal $x[n]$ and reconstructed signal $x_r(t)$. Simulate this operation and graph $x_c(t)$, $x[n]$, and $x_r(t)$ in one figure using sub-plots.



As the input frequency increases from zero to the folding frequency $F_s/2$ the apparent frequency is equal to the input frequency; however, as the input frequency increases linearly from $F_s/2$ to F_s the apparent frequency decreases linearly from $F_s/2$ to zero.

6-6.py

The effect of undersampling: aliasing

Aliasing in nonbandlimited signals

$$x_c(t) = e^{-A|t|} \overset{\text{CTFT}}{\longleftrightarrow} X_c(j\Omega) = \frac{2A}{A^2 + \Omega^2}, \quad A > 0$$

this signal has infinite duration and infinite bandwidth.
Sampling $x_c(t)$ at a rate $F_s = 1/T$

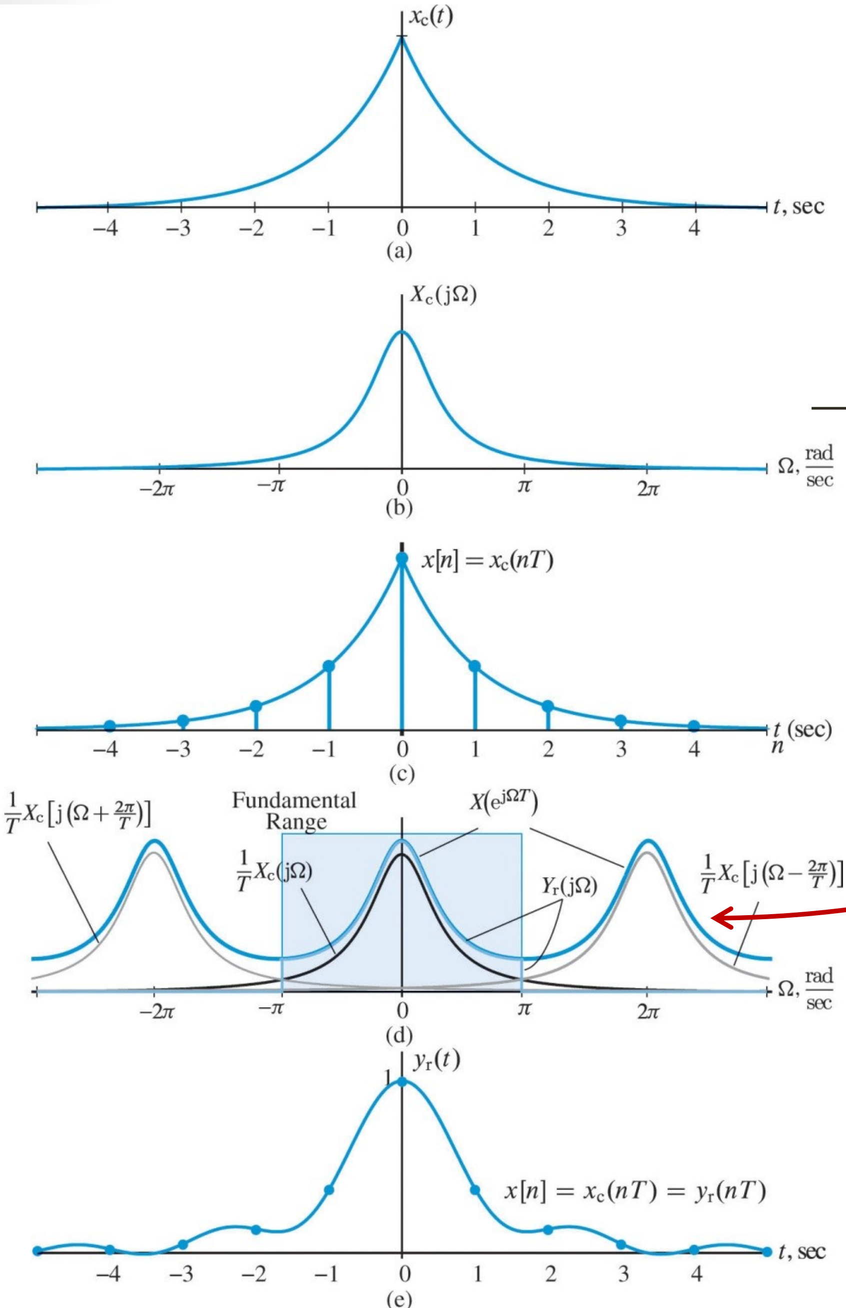
$$x[n] = x_c(nT) = e^{-A|n|T} = (e^{-AT})^{|n|} = a^{|n|}, \quad a \triangleq e^{-AT}$$

$$\rightarrow \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \frac{1-a^2}{1-2a\cos(\omega)+a^2}, \quad \omega = \Omega/F_s$$

$$X(e^{j\Omega T}) = \frac{1-a^2}{1-2a\cos(\Omega T)+a^2}$$

$X(e^{j\omega})$ is periodic in ω with period 2π or periodic in Ω with period $2\pi/T$.

The effect of undersampling: aliasing



The reconstructed signal $y_r(t)$ corresponds to the inverse Fourier transform of $Y_c(j\Omega) = TX(e^{j\Omega T})$ for $|\Omega| < \pi/T$ and $Y_c(j\Omega) = 0$ for $|\Omega| > \pi/T$.

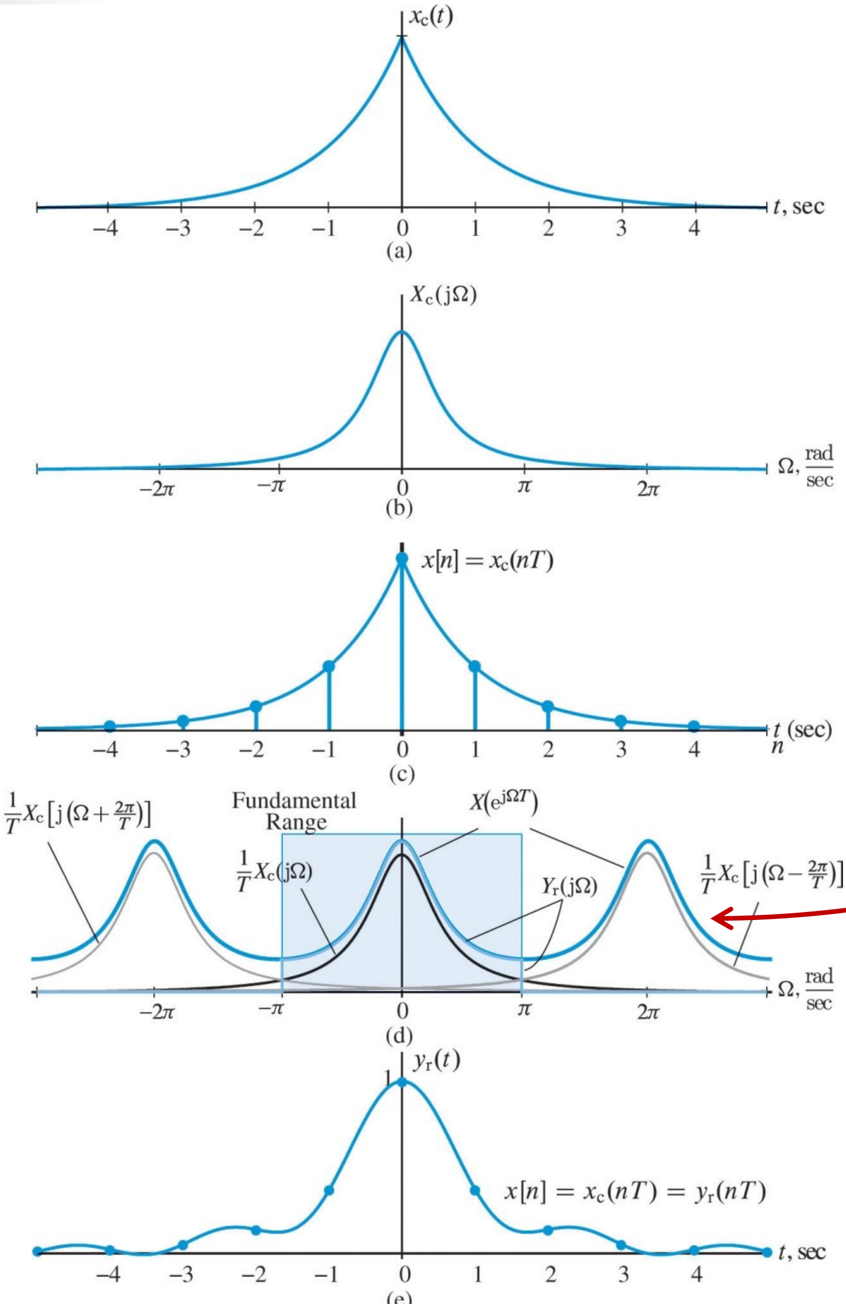
$$\rightarrow \quad y_r(m\Delta t) \approx \sum_{n=N_1}^{N_2} x[n] \frac{\sin[\pi(m\Delta t - nT)/T]}{\pi(m\Delta t - nT)/T}.$$

$$t_1 \leq m\Delta t \leq t_2$$

$X(e^{j\Omega T})$ for $T = 1$ s.

Aliasing effects in sampling and reconstruction of a continuous-time nonbandlimited signal: (a) continuous-time signal $x_c(t)$, (b) spectrum of $x_c(t)$, (c) discrete-time signal $x[n]$ sampled at $T = 1$ s, (d) spectrum of $x[n]$, and (e) bandlimited reconstruction $y_r(t)$. In this case, aliasing distortion is unavoidable.

The effect of undersampling: aliasing



As a result of aliasing, $y_r(t) \neq x_c(t)$ for $t \neq nT$. The strong frequency components of $Y_c(j\Omega)$ close to π/T create oscillations in $y_r(t)$ with an approximate period of $1/(F_s/2) = 2T$.

$X(e^{j\Omega T})$ for $T = 1$ s.

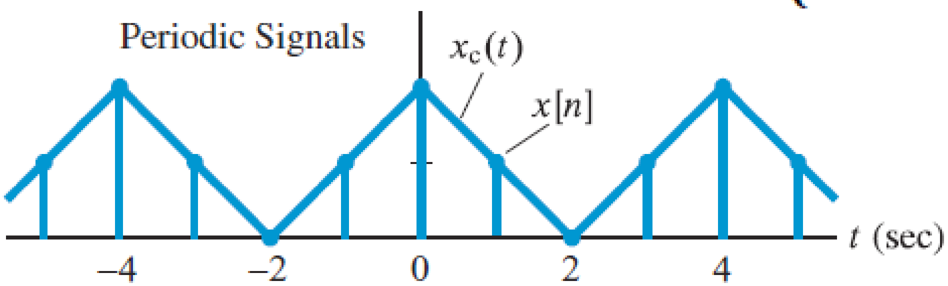
Aliasing effects in sampling and reconstruction of a continuous-time nonbandlimited signal: (a) continuous-time signal $x_c(t)$, (b) spectrum of $x_c(t)$, (c) discrete-time signal $x[n]$ sampled at $T = 1$ s, (d) spectrum of $x[n]$, and (e) bandlimited reconstruction $y_r(t)$. In this case, aliasing distortion is unavoidable.

The effect of undersampling: aliasing

Sampling of a periodic signal

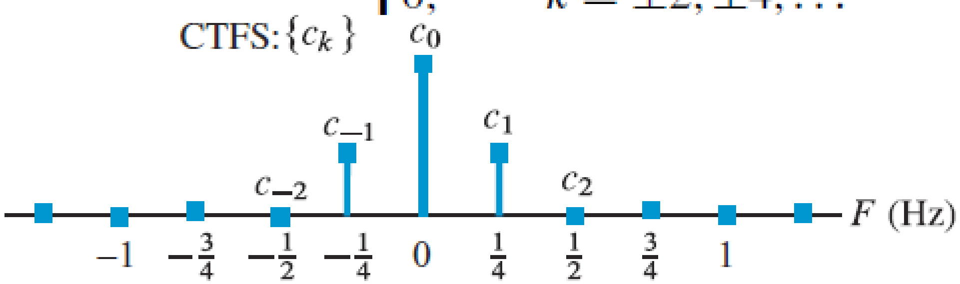
Consider a periodic triangular signal $x_c(t)$ with period $T_0 = 4$, given by

$$x_c(t) = x_c(t+4) = \begin{cases} 2+t & -2 \le t \le 0 \\ 2-t & 0 \le t \le 2 \end{cases}$$



$$\rightarrow \qquad c_k = \frac{1}{T_0}\int_{T_0} x_c(t) e^{-j\frac{2\pi}{T_0}kt} dt = \frac{1}{4}\left[\int_{-2}^{0} t e^{-j\frac{2\pi}{4}kt} dt - \int_{0}^{2} t e^{-j\frac{2\pi}{4}kt} dt\right]$$

$$= \begin{cases} 1, & k = 0, \\ \frac{4}{\pi^2 k^2}, & k = \pm 1, \pm 3, \ldots; \\ 0, & k = \pm 2, \pm 4, \ldots \end{cases} \qquad F_0 = \frac{1}{T_0} = \frac{1}{4} \text{ Hz},$$



Clearly, $x_c(t)$ has an infinite number of harmonics and hence an infinite bandwidth.

The effect of undersampling: aliasing

The DTFS coefficients of $x[n]$ are given by

We sample $x_c(t)$ at a rate of $F_s = 1/T = N/T_0 = 1$ Hz with $T = 1$ and $N = 4$ to obtain the periodic sequence $x[n] = \{2, 1, 0, 1\}$, $0 \le n \le 3$ with period $N = 4$

$$\tilde{c}_0 = \cdots + c_{-4} + c_0 + c_4 + \cdots = \sum c_{0-4\ell} = \cdots + 0 + 1 + 0 + \cdots = 1$$

$$\tilde{c}_1 = \cdots + c_{-3} + c_1 + c_5 + \cdots = \sum c_{1-4\ell} = \sum_{\ell} \frac{4}{\pi^2(1-4\ell)^2} = \frac{1}{2}$$

$$\tilde{c}_2 = \cdots + c_{-2} + c_2 + c_6 + \cdots = \sum c_{2-4\ell} = 0$$

$$\tilde{c}_3 = \cdots + c_{-1} + c_3 + c_7 + \cdots = \sum_{\ell} c_{3-4\ell} = \sum_{\ell} \frac{4}{\pi^2(3-4\ell)^2} = \frac{1}{2}$$

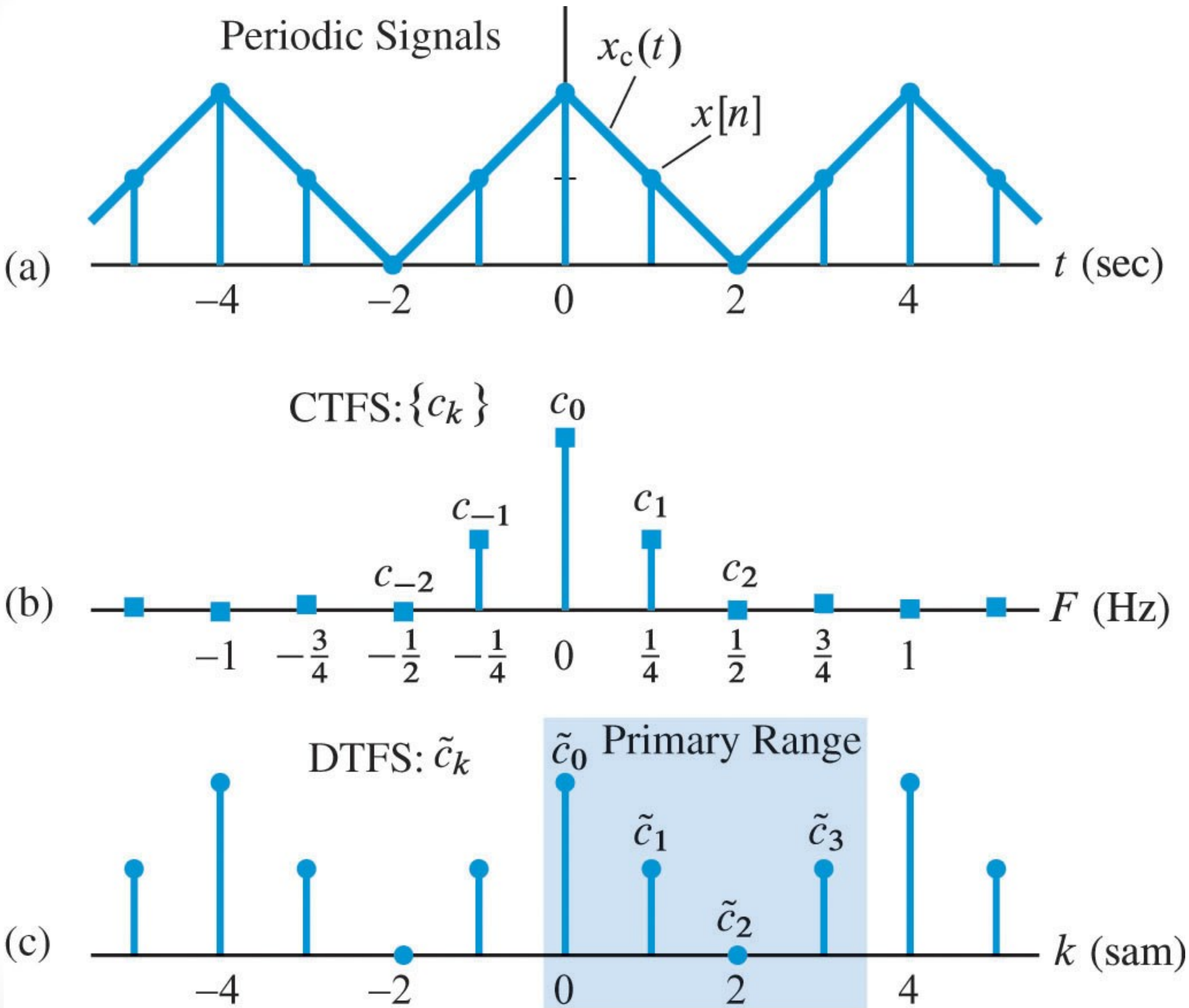
Or

$$\tilde{c}_k = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{4}\left[2 + e^{-j\frac{\pi}{2}k} + e^{-j3\frac{\pi}{2}k}\right]$$

$$= \frac{1}{2}\left[1 + \cos(\pi k/2)\right] = \begin{cases} 1, & k = 0 \\ \frac{1}{2}, & k = \pm 1, \pm 3, \ldots \\ 0. & k = \pm 2, \pm 4, \ldots \end{cases}$$

The effect of undersampling: aliasing

Sampling of a periodic signal



Sampling of a periodic signal: (a) periodic signal $x_c(t)$ and its samples $x[n]$, (b) CTFS coefficients $\{c_k\}$, and (c) DTFS coefficients $\tilde{c}_k$

The effect of undersampling: aliasing

Sampling of a periodic signal

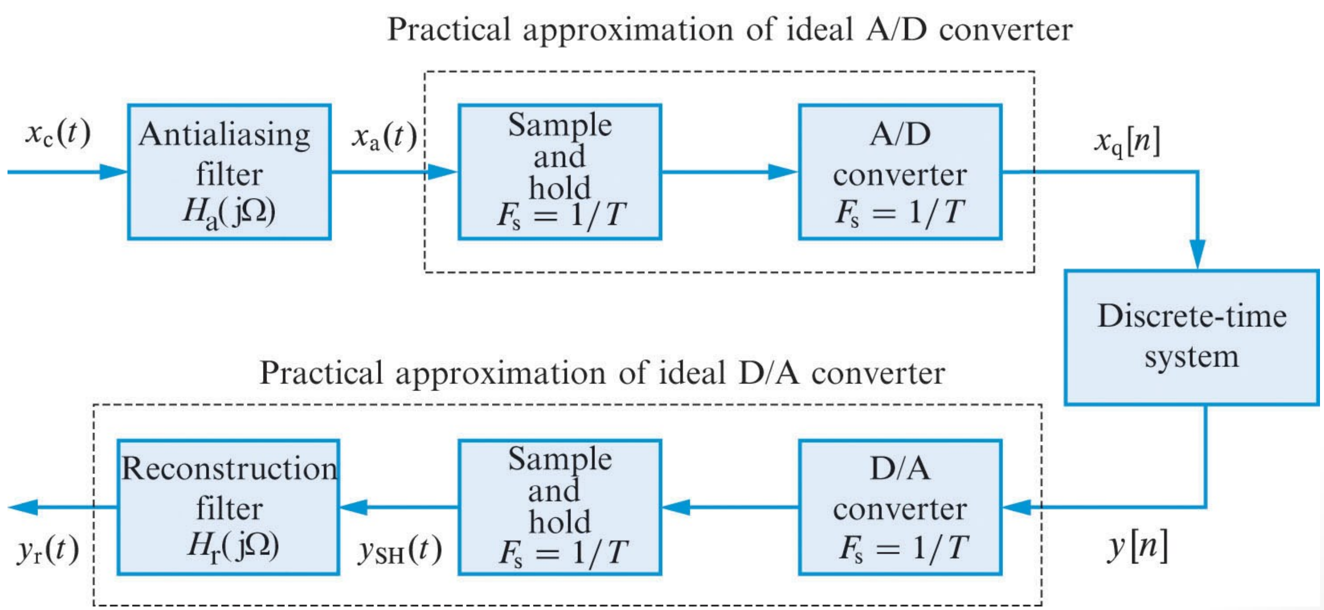
A careful sampling of a periodic signal produces a periodic sequence whose DTFS is an aliased version of the corresponding CTFS coefficients given by

$$\tilde{c}_k = \sum_{\ell=-\infty}^{\infty} c_{k-\ell N}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Practical sampling and reconstruction

Differences between practical sampling and reconstruction and ideal sampling and reconstruction:

- All practical continuous-time signals are timelimited, that is, they have finite duration; therefore they are not, and cannot be, strictly bandlimited.
- In practice, the sampled values $x[n]$ can only be described by a finite number of bits; that is, the values of $x_c(nT)$ should be quantized.
- The ideal DAC is practically unrealizable because the interpolation kernel $g_{BL}(t) = \sin(\pi t/T)/(\pi t/T)$ has infinite duration.



A realistic model for digital processing of continuous-time signals.

Practical sampling and reconstruction

Analog-to-digital conversion:

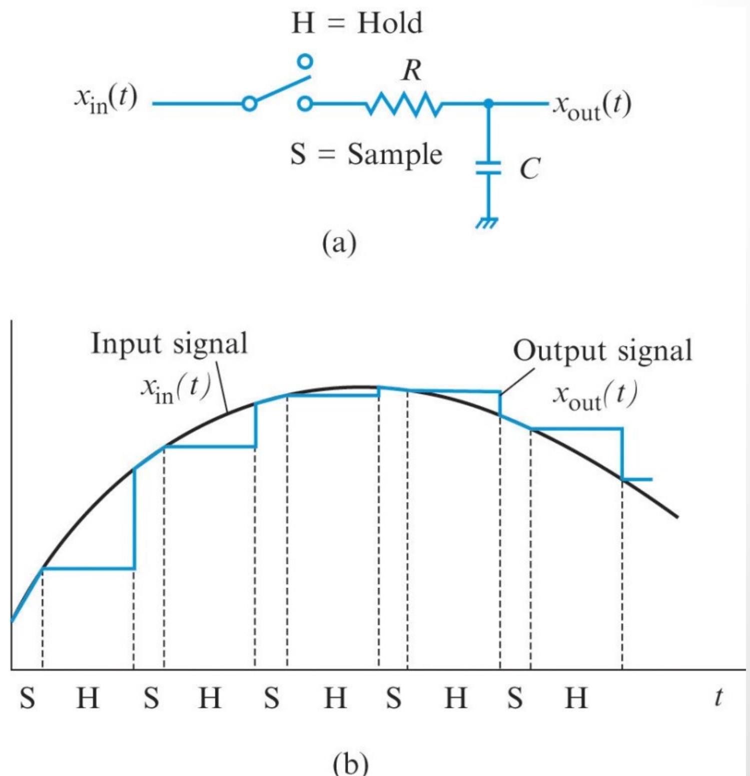
- an analog lowpass filter
- a sample-and-hold circuit
- an A/D converter

Lowpass antialiasing filter

To bandlimit the input signal to the folding frequency without introducing excessive linear or nonlinear distortion, and without generating excessive noise.

Sample-and-hold (S/H) circuit

When an analog voltage is connected directly to the input of an ADC, the conversion process can be adversely affected if the analog voltage is changing during the conversion time. The quality of the conversion process can be improved by using a S/H circuit to hold the analog voltage constant while the A/D conversion is taking place.

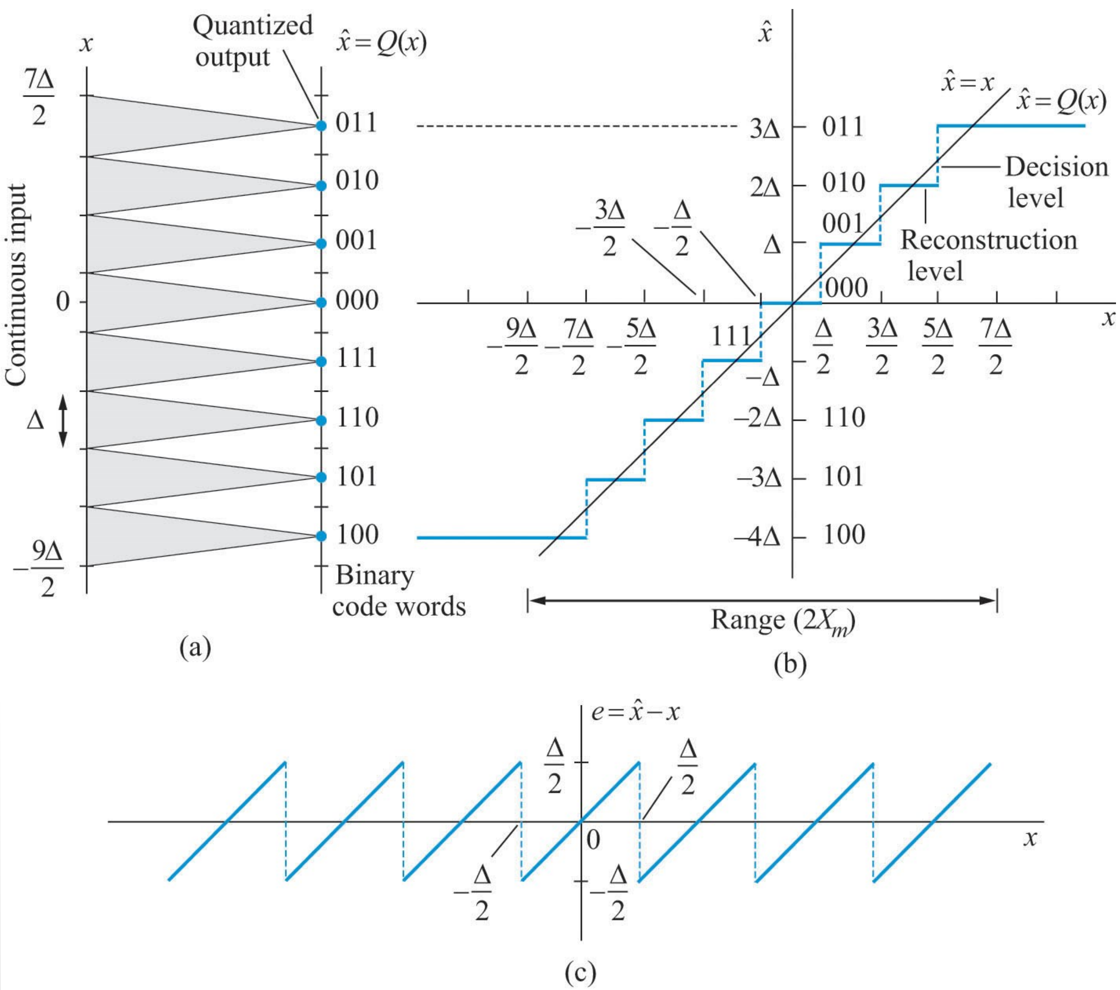


Practical sampling and reconstruction

Analog-to-digital conversion:

A/D converter

The ADC is a physical device that converts the voltage or current value at its input into a binary word, which is the numerical representation of a quantized value closest to the input value.



The quantization operation allocates intervals to a number of discrete levels, that is, quantization is a many-to-one mapping. (a) Allocation of levels in a 3-bit quantizer which rounds x/Δ to the closest integer. Input-output (b) and quantization error (c) transfer function of a uniform rounding quantizer.

Practical sampling and reconstruction

Analog-to-digital conversion:

python functions for quantization:

numpy.round(x):
quantizes x to the nearest integer

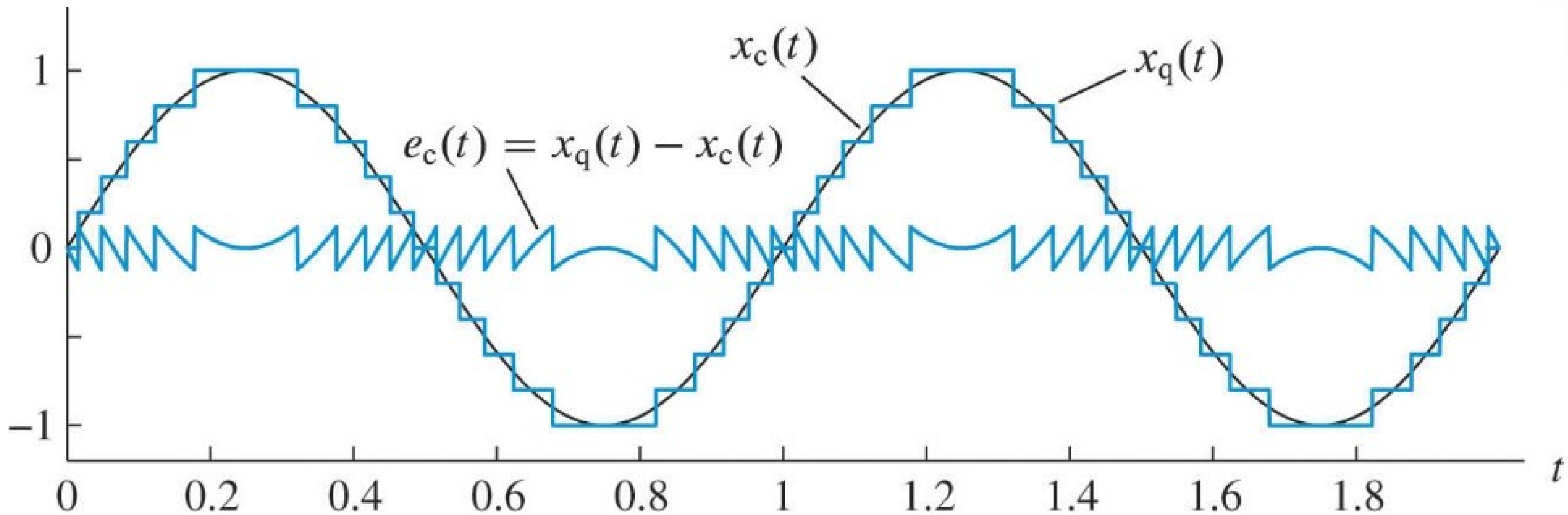
numpy.fix(x):
quantizes x to the nearest integer towards 0

numpy.ceil(x):
quantizes x to the nearest integer towards ∞

numpy.floor(x):
quantizes x to the nearest integer towards $-\infty$

Practical sampling and reconstruction

Analog-to-digital conversion:



Quantization error resulting from the quantization of a continuous-time sinusoidal signal using a rounding quantizer with $\Delta = 0.2$.

Practical sampling and reconstruction

Digital-to-analog conversion

We saw that how a bandlimited signal can be reconstructed from a sequence of samples using the ideal DAC described by

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] g_r(t - nT)$$

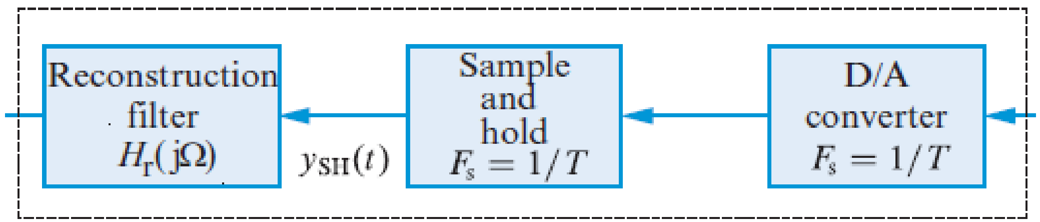
Since $g_{BL}(t) \neq 0$ for $t < 0$ and $\int |g_{BL}(t)| dt = \infty$, the ideal DAC is a noncausal and unstable system; hence, it is not practically realizable.

The main objective of a DAC is to "fill-in" the signal values between the sampling times $t = nT$, a process known as interpolation.

Practical sampling and reconstruction

Digital-to-analog conversion

Practical approximation of ideal D/A converter



In practice, the conversion from a digital signal to analog typically is implemented with the three devices:

1. **DAC**

   generates an analog voltage at its output which is determined by the binary word at its input.

2. **Sample and hold amplifier**

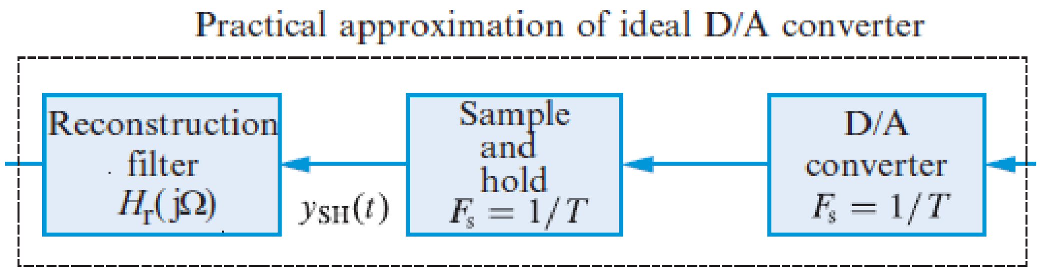
   prevents the internal switching glitches in the DAC from appearing at the output analog signal. This is done by holding the output voltage of the DAC constant for one sampling period; the result is a staircase continuous time signal. The output of the S/H is given by

$$x_{SH}(t) = \sum_{n=-\infty}^{\infty} x_q[n] g_{SH}(t - nT)$$

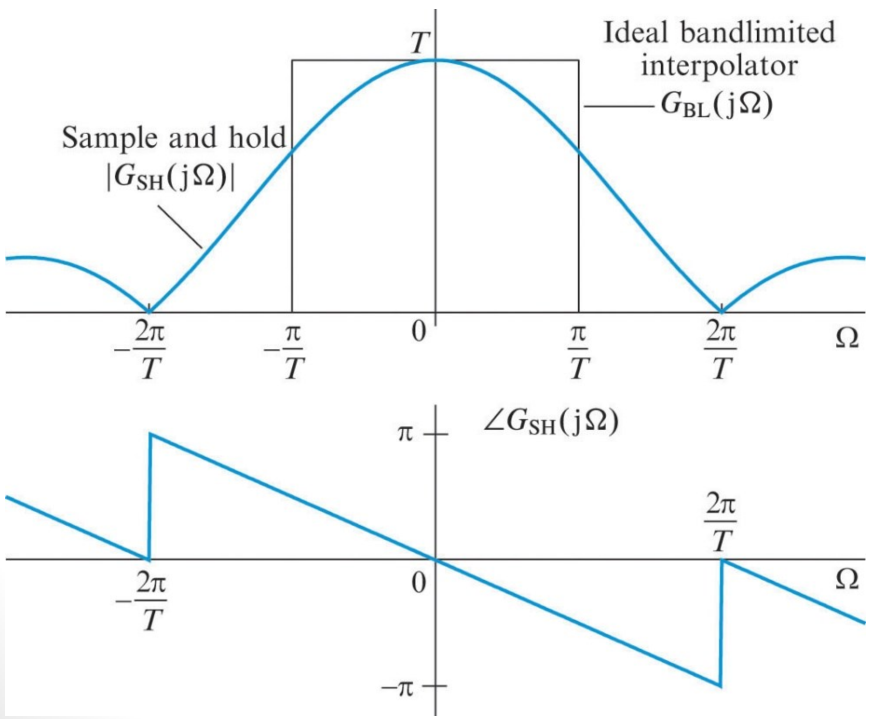
$$g_{SH}(t) = \begin{cases} 1, & 0 \le t \le T \\ 0. & \text{otherwise} \end{cases} \xleftrightarrow{\text{CTFT}} G_{SH}(j\Omega) = \frac{2\sin(\Omega T/2)}{\Omega} e^{-j\Omega T/2}$$

Practical sampling and reconstruction

Digital-to-analog conversion



The S/H circuit, unlike the ideal DAC, does *not* completely eliminate the replicated spectral images introduced by the sampling process; moreover, it introduces an amplitude distortion (known as *droop*) in the Nyquist band $|Fs| < F_s/2$.

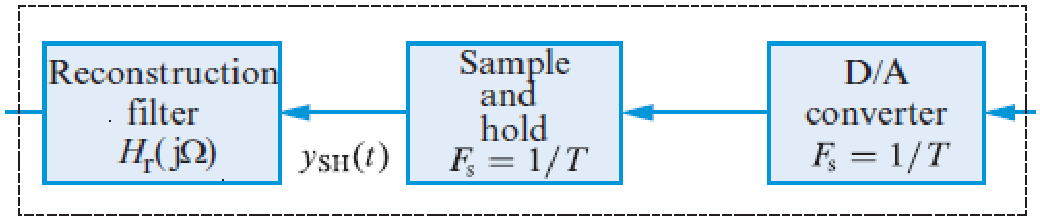


Frequency domain characteristics of the S/H system.

Practical sampling and reconstruction

Digital-to-analog conversion

Practical approximation of ideal D/A converter



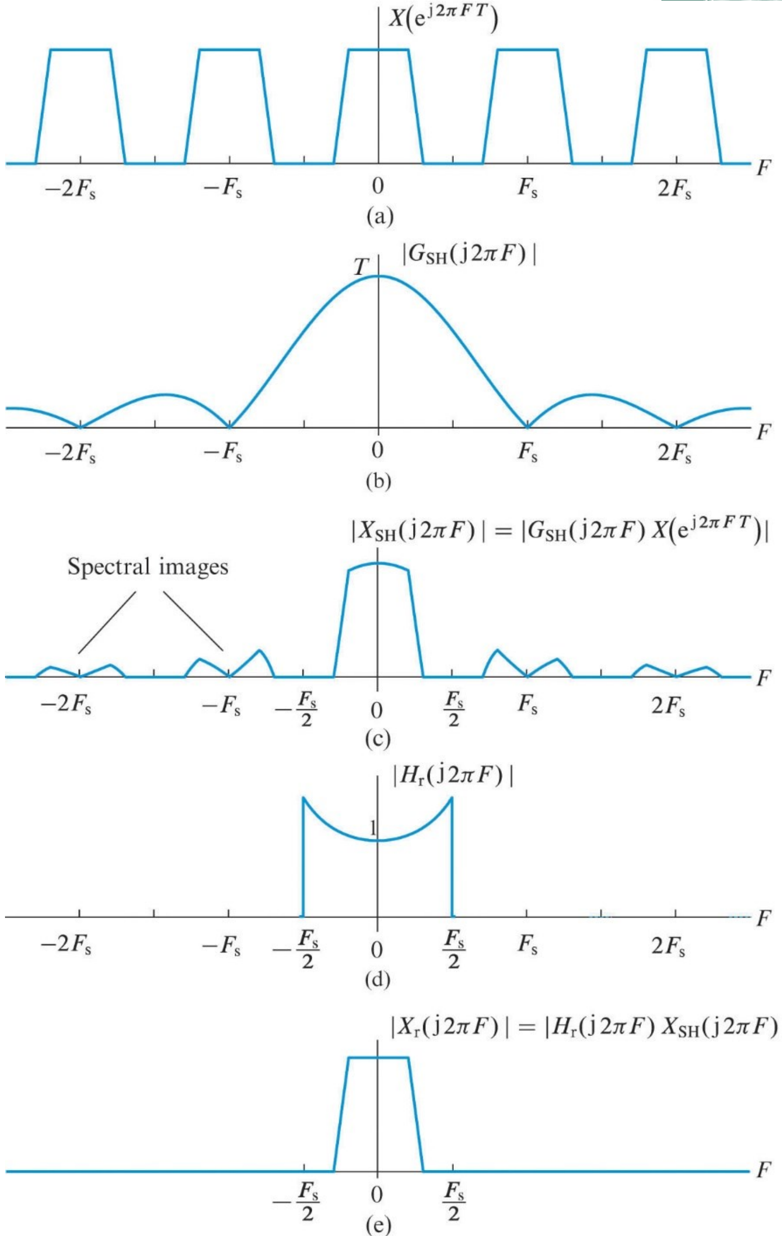
3. **Reconstruction filter**
   To compensate for the effects of the S/H circuit we use an analog lowpass post-filter $H_r(j\Omega)$ such that $G_{SH}(F)H_r(F) = G_{BL}(F)$

$$H_r(j\Omega) = \begin{cases} \dfrac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0. & \text{otherwise} \end{cases}$$

Practical sampling and reconstruction

Digital-to-analog conversion

$$H_r(j\Omega) = \begin{cases} \dfrac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0. & \text{otherwise} \end{cases}$$



Practical sampling and reconstruction

Example: Practical reconstruction of sinusoidal signals

A sinusoidal signal $x_c(t) = \cos 2\pi F_0 t$ with $F_0 = 0.025$ Hz is sampled at a rate of $F_s = 1/T = 0.2$ Hz. The result is the discrete-time signal $x[n] = x_c(nT) = \cos(2\pi f_0 n)$ with $f_0 = 1/8$.

At each sampling interval, the DAC takes as input a digital word and generates a voltage equal to $x[n]$; this value is held constant for T seconds by the S/H circuit.

The output $x_{SH}(t)$ of the S/H is given by $X_{SH}(j2\pi F) = G_{SH}(j2\pi F)X(e^{j2\pi FT})$.

The S/H scales the magnitude of each input frequency component by the sinc function and introduces a time delay of $T/2$ seconds.

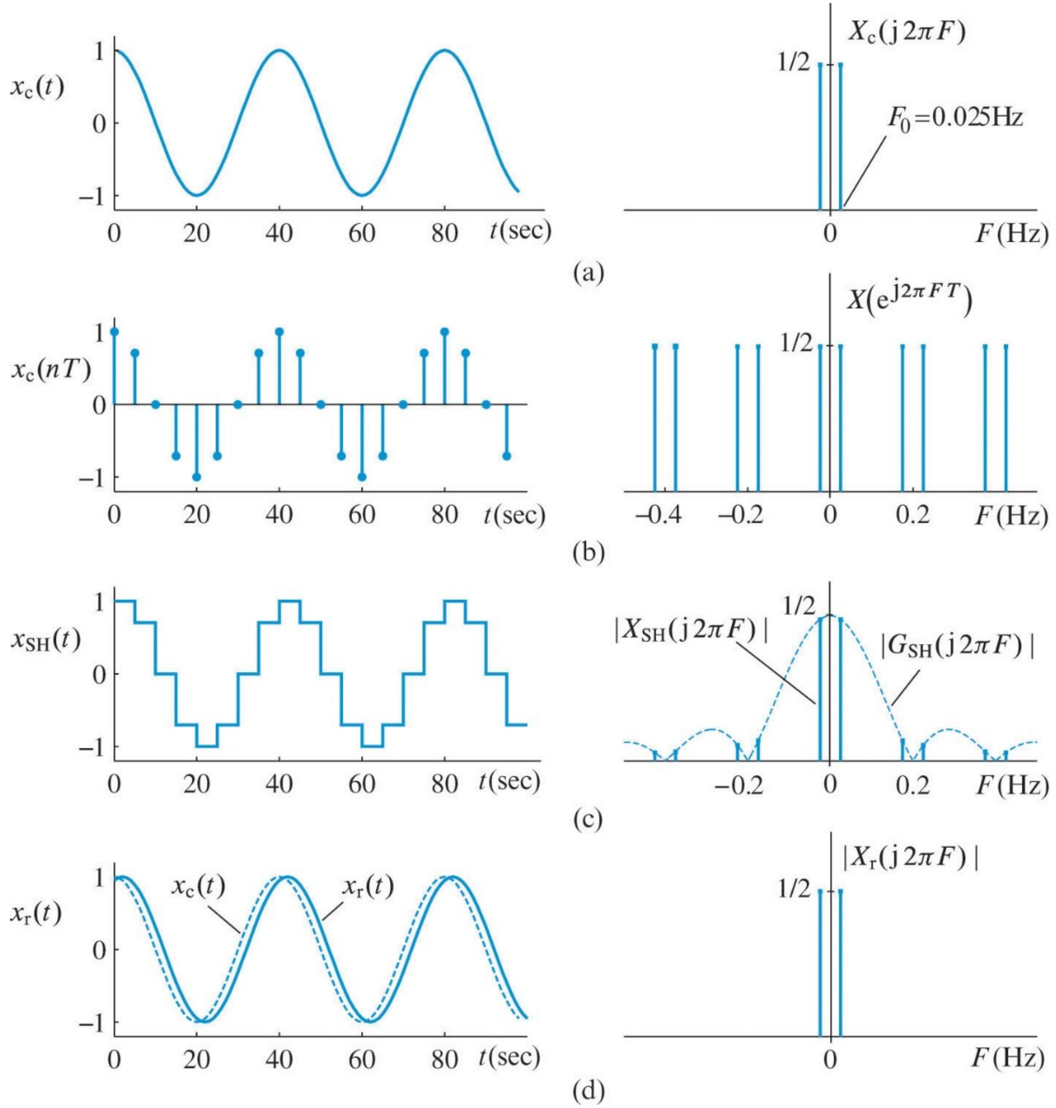
The reconstruction filter removes all frequency components outside the Nyquist interval, compensates for the droop distortion, and scales the input amplitude by T. Therefore, the final reconstructed signal is given by

$$x_r(t) = \frac{1}{2}e^{-jT/2}e^{j2\pi F_0 t} + \frac{1}{2}e^{jT/2}e^{-j2\pi F_0 t} = \cos(2\pi F_0 t - T/2)$$

which is identical with the input signal delayed by $T/2$ seconds.

Practical sampling and reconstruction

Example: Practical reconstruction of sinusoidal signals



Frequency domain characteristics of S/H reconstruction. $F_s = 0.2$ Hz.

Image sampling and reconstruction

In digital image processing a physical image $s_c(x, y)$ is sampled on a rectangular grid of points (mx, ny), *i.e.*,

$$s[m, n] = s_c(m\Delta x, n\Delta y),$$

where (x,y) is the spacing of the grid.

After processing, we obtain another array of numbers $v[m, n]$, which should be used to reconstruct a continuous image $v_r(x, y)$ for viewing.

The two-dimensional, continuous Fourier transform pair

$$S_c(F_x, F_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_c(x, y)e^{-j2\pi(xF_x+yF_y)}dxdy,$$

$$s_c(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_c(F_x, F_y)e^{j2\pi(xF_x+yF_y)}dF_xdF_y$$

where F_x and F_y are the *spatial* frequency variables in units of cycles per unit of distance.

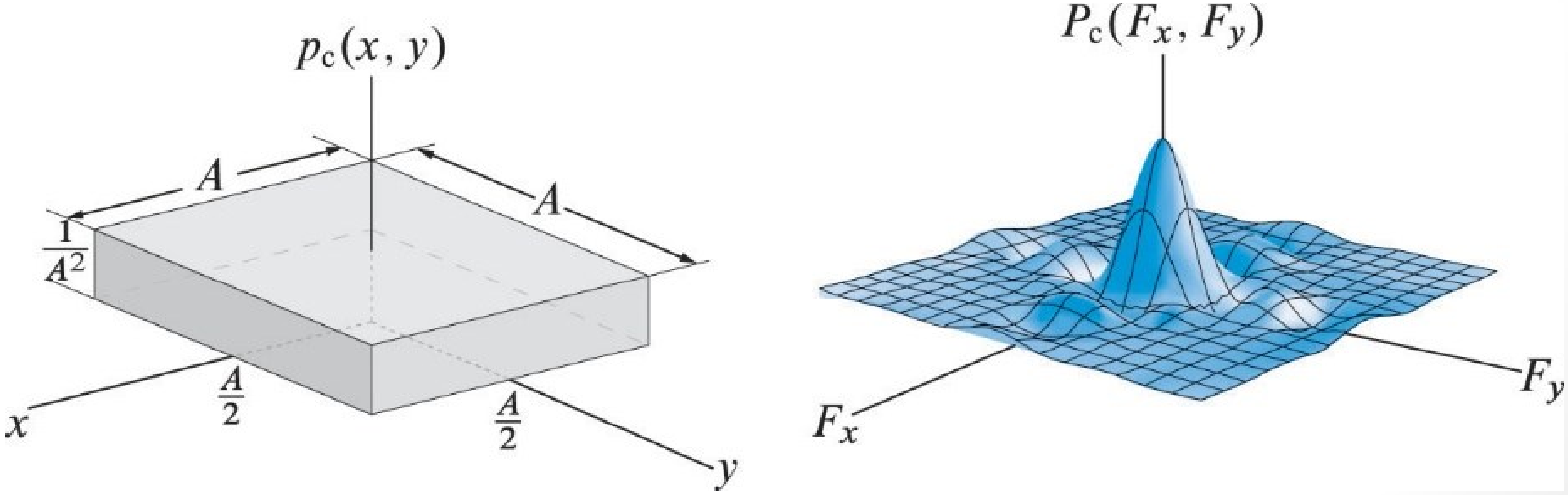
Image sampling and reconstruction

Example: the Fourier transform of the square pulse image

$$p_c(x,y) = \begin{cases} 1/A^2, & |x| < A/2, |y| < A/2 \\ 0, & \text{otherwise} \end{cases}$$

is given by

$$P_c(F_x, F_y) = \frac{\sin(\pi F_x A)}{\pi F_x A} \times \frac{\sin(\pi F_y A)}{\pi F_y A}$$



A 2-D rectangular function and a section of its spectrum about the origin.

Image sampling and reconstruction

2-D sampling theorem

Similar to the 1-D case, the Fourier transform of the sampled image, $s[m, n]$, is given by

$$\tilde{S}(F_x, F_y) \triangleq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n] e^{j2\pi(m\Delta x F_x + n\Delta y F_y)}$$

$$= \frac{1}{\Delta x \Delta y} \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} S_c(F_x - kF_{s_x}, F_y - \ell F_{s_y})$$

where $F_{sx} = 1/\Delta x$ and $F_{sy} = 1/\Delta y$ are the spatial sampling frequencies.

The spectrum of the sampled image is obtained by infinitely repeating the spectrum of the original image over the frequency plane in a rectangular grid with spacing (F_{sx}, F_{sy}).

If the function $s_c(x, y)$ is band-limited, that is

$$S_c(F_x, F_y) = 0 \quad \text{for } |F_x| > B_x \text{ and } |F_y| > B_y,$$

and the spatial sampling frequencies satisfy the conditions

$$F_{s_x} \geq 2B_x \quad \text{and} \quad F_{s_y} \geq 2B_y,$$

then there is *no* spectrum overlap.

Image sampling and reconstruction

The spectrum of the original image can be recovered by multiplying $\tilde{S}(Fx, Fy)$ with the reconstruction filter

$$G_r(F_x, F_y) = \begin{cases} \Delta x \Delta y, & |F_x| \le F_{s_x}/2 \text{ and } |F_y| \le F_{s_y}/2 \\ 0. & \text{otherwise} \end{cases}$$

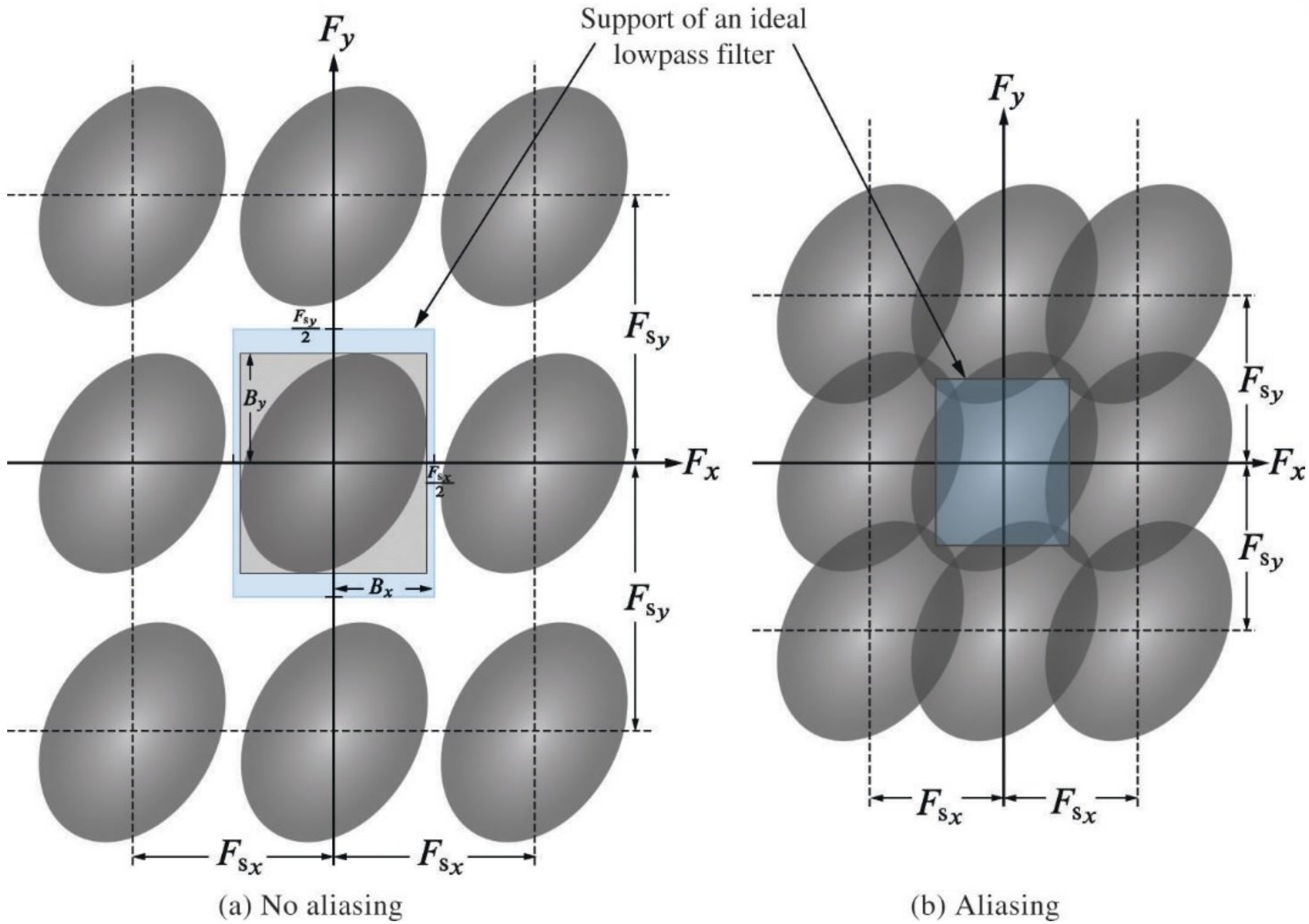
The sampling period must be equal to or smaller than one-half the period of the finest detail within the image.

If equality holds, the image is said to be **sampled at its Nyquist rate**;

If F_{sx} and F_{sy} are greater than required by the Nyquist criterion, the image is called **oversampled**;

If the opposite case holds, the image is **undersampled**.

Image sampling and reconstruction



2D Sampling in the frequency domain: (a) $F_{sx} > 2B_x$, $F_{sy} > 2B_y$ (no aliasing); (b) $F_{sx} < 2B_x$, $F_{sy} < 2B_y$ (aliasing).

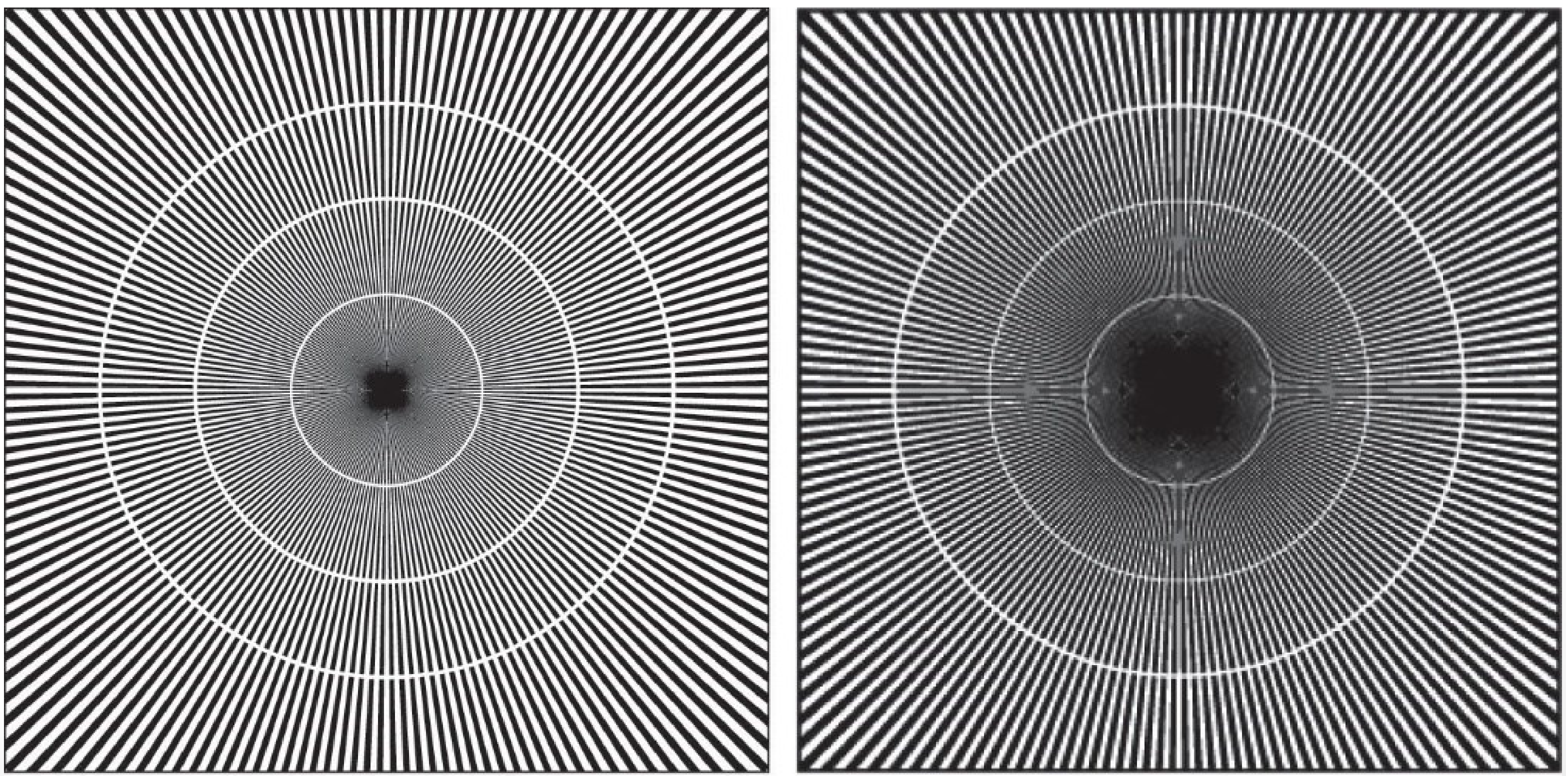
Image sampling and reconstruction

Visual effects of sampling

For real-world images, most prominent aliased frequency components are near folding frequencies in each dimension (that is, half the sampling frequencies), which then results in a beat pattern effect. These aliasing effects are known as *Moiré patterns*.

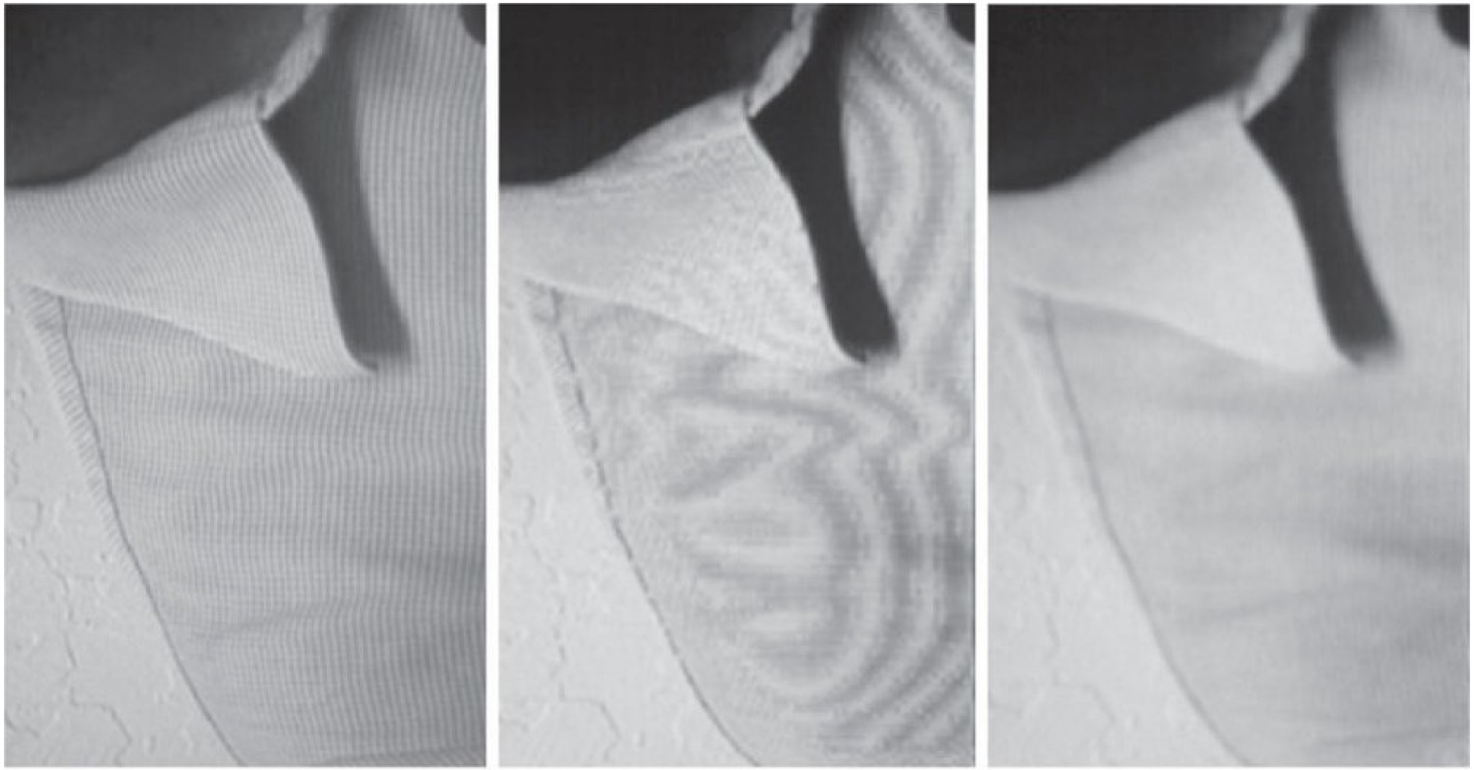
Antialiasing filtering has to be done at the front-end, before the image is sampled.

There is a trade-off between sampled image resolution and aliasing error.



(a) (b)

Visual effects of sampling



(a) (b) (c)

Aliasing in resampled images (digital aliasing):

- **(a)** original image,
- **(b)** resampled without pre-filtering (resampled to 50% of its original size by deleting every other row and column without pre-smoothing), and
- **(c)** resampled with pre-filtering (the same image after first smoothing it using a simple averaging filter prior to resampling to 50% of its original size.).

Image sampling and reconstruction

ADSP Book, P.352, Ex.47

This problem uses a 100 × 300 image file containing letters "DSP" and is available at the book website as dsp.png file. Access this file in MATLAB and store it as a variable x_c.

(a) Sample x_c by taking every 10th pixel horizontally and vertically to create a sampled image x of size 10×30. Rescale x to 100×300 to obtain x_s. Display images x_c, x_s, and x and comment on their appearance.

(b) First blur the image x_c using a 5 × 5 averaging filter to obtain filtered image y_c, then sample it by taking every 10th pixel horizontally and vertically to create a sampled image y of size 10 × 30, and finally rescale y to 100 × 300 to obtain y_s. Display images y_c, y_s, and y. Compare their appearance with those in (a) and comment on the antialiasing effects on font display.

(c) Experiment on font-antialiasing in (b) above with various blurring filter types and sizes and comment on your observations.

DSP_image.py

Image sampling and reconstruction

Ideal reconstruction

To achieve perfect image reconstruction in a digital image processing system, it is necessary to bandlimit the image to be sampled, spatially sample the image at the Nyquist or higher rate, and properly interpolate the image samples.

$$s_r(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m, n] g_r(x - m\Delta x, y - n\Delta y),$$

where $g_r(x, y)$ is a 2-D interpolation function.

$$\longrightarrow \quad S_r(F_x, F_y) = G_r(F_x, F_y)\tilde{S}(F_x, F_y)$$

$$G_r(F_x, F_y) = \begin{cases} \Delta x \Delta y, & |F_x| \le F_{s_x}/2 \text{ and } |F_y| \le F_{s_y}/2 \\ 0. & \text{otherwise} \end{cases}$$

$$\longrightarrow \quad g_r(x, y) = \frac{\sin(\pi F_{s_x} x)}{\pi F_{s_x} x} \times \frac{\sin(\pi F_{s_y} y)}{\pi F_{s_y} y}$$

Image sampling and reconstruction

Practical reconstruction

$$S_r(F_x, F_y) = \frac{1}{\Delta x \Delta y} G_r(F_x, F_y) \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} S_c(F_x - kF_{s_x}, F_y - \ell F_{s_y})$$

For example, displaying a digital image in a cathode ray tube display involves two steps:

First, the sample values are converted into a continuous video signal using a sample-and-hold circuit.

The resulting staircase signal drives the display spot, which has a 2-D Gaussian shape; thus, the display-spot acts as an impulse response of an anti-imaging reconstruction filter.

Image sampling and reconstruction

Practical reconstruction