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Design of FIR filters

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The filter design problem

The term "**filter**" is used for LTI systems that alter their input signals in a prescribed way.

Frequency-selective filters are designed to pass a set of desired frequency components from a mixture of desired and undesired components or to shape the spectrum of the input signal in a desired way.

The **filter design problem** consists of finding a practically realizable filter whose frequency response best approximates the desired ideal magnitude and phase responses within specified tolerances.

The design of **FIR** filters: finding a polynomial frequency response function.
The design of **IIR** filters: a rational approximating function.

The filter design problem

Design of frequency-selective discrete-time filters for practical signal processing applications involves, in general, the following five stages:

1. **Specification:** Specify the desired frequency response function characteristics to address the needs of a specific application.

2. **Approximation:** Approximate the desired frequency response function by the frequency response of a filter with a polynomial or a rational system function. The goal is to meet the specifications with minimum complexity, that is, by using the filter with the lowest number of coefficients.

3. **Quantization:** Quantize the filter coefficients at the required fixed-point arithmetic representation.

4. **Verification:** Check whether the filter satisfies the performance requirements by simulation or testing with real data. If the filter does not satisfy the requirements, return to Stage 2, or reduce the performance requirements and repeat Stage 4.

5. **Implementation:** Implement the system obtained in hardware, software, or both.

The filter design problem

Filter specifications

Ideal filters have to be approximated by practically realizable filters. Practical filters differ from ideal filters in several respects. More specifically, in practical filters

(a) the passband responses are not perfectly flat,

(b) the stopband responses cannot completely reject (eliminate) bands of frequencies, and

(c) the transition between passband and stopband regions takes place over a finite transition band.

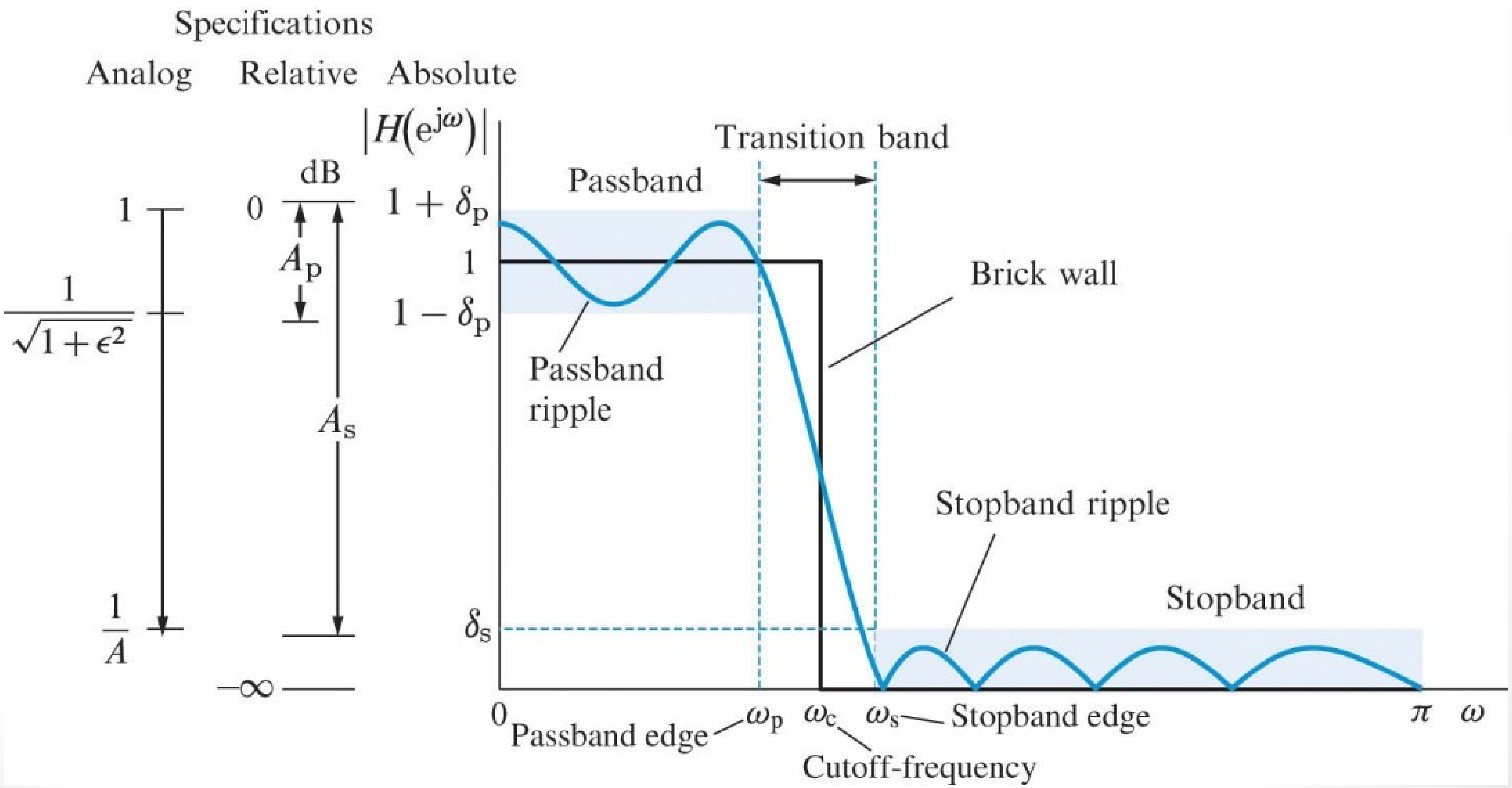
Tolerance diagram: the specifications of practical filters

The filter design problem

ω_p: the passband edge

ω_s: the stopband edge

The phase response is either left completely unspecified or may be required to be linear



Example of tolerance diagram for a lowpass filter.

The filter design problem

Absolute specifications

In the passband: $1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p$, $0 \leq \omega \leq \omega_p$
where δ_p « 1 for a well designed filter

In the stopband: $|H(e^{j\omega})| \leq \delta_s$, $\omega_s \leq \omega \leq \pi$
where δ_s « 1 for a well designed filter

Relative specifications

$$\frac{1-\delta_p}{1+\delta_p} \leq \left|H\left(e^{j\omega}\right)\right| \leq 1, \quad 0 \leq \omega \leq \omega_p \qquad \left|H\left(e^{j\omega}\right)\right| \leq \frac{\delta_s}{1+\delta_p}, \quad \omega_s \leq \omega \leq \pi$$

Passband ripple A_p and the ***stopband attenuation*** A_s are defined in logarithmic units (dB) as follows:

$$A_p \triangleq 20\log_{10}\left(\frac{1+\delta_p}{1-\delta_p}\right), \quad A_s \triangleq 20\log_{10}\left(\frac{1+\delta_p}{\delta_s}\right) \approx -20\log_{10}\left(\delta_s\right)$$

δ_p « 1 $\longrightarrow$

$$-A_p \leq \left|H\left(e^{j\omega}\right)\right|, \text{(in dB)} \leq 0, \qquad 0 \leq \omega \leq \omega_p$$

$$\left|H\left(e^{j\omega}\right)\right|, \text{(in dB)} \leq -A_s. \qquad \omega_s \leq \omega \leq \pi$$

The filter design problem

Continuous-time (analog) filter specifications

In practical applications, the passband and stopband edge frequencies are specified in Hz. The values of the normalized frequencies ω_p and ω_s are calculated from the sampling frequency F_s and the edge frequencies F_{pass} and F_{stop} by

$$\omega_p = 2\pi \frac{F_{pass}}{F_s}, \quad \omega_s = 2\pi \frac{F_{stop}}{F_s}.$$

Analog specification

Analog filters are traditionally specified using the quantities ε and A. These quantities are defined by

$$20\log_{10}\left(\sqrt{1+\epsilon^2}\right) = A_p \quad \text{and} \quad 20\log_{10}(A) = A_s,$$

$$\longrightarrow \quad \epsilon = \sqrt{10^{(0.1A_p)} - 1} \quad \text{and} \quad A = 10^{(0.05A_s)}.$$

The filter design problem

Example: Conversion of filter specifications

A lowpass digital filter is specified by the following relative specifications:

$$\omega_p = 0.3\pi,\ A_p = 0.5 \text{ dB};\ \omega_s = 0.5\pi,\ A_s = 40 \text{ dB}.$$

The absolute specifications for the filter are given by

$$A_p = 0.5 = 20\log_{10}\left(\frac{1+\delta_p}{1-\delta_p}\right) \Rightarrow \delta_p = 0.0288,$$

$$A_s = 40 = 20\log_{10}\left(\frac{1+\delta_p}{\delta_s}\right) \Rightarrow \delta_s = 0.0103.$$

The analog filter specifications are given by

$$\epsilon = \sqrt{10^{(0.1A_p)} - 1} = 0.3493 \quad \text{and} \quad A = 10^{(0.05A_s)} = 100.$$

The filter design problem

Filter approximation

Consider an ideal filter with impulse response sequence $h_d[n]$ and frequency response function $H_d(e^{j\omega})$. We wish to find a *practical* filter $H(e^{j\omega})$ which approximates the *desired* filter $H_d(e^{j\omega})$.

The practical filter should be causal, stable, and should have a finite-order rational system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}},$$

The filter design problem

Theorem 1 (Paley–Wiener):
If $h[n]$ has finite energy and $h[n] = 0$ for $n < 0$, then

$$\int_{-\pi}^{\pi} \left| \ln \left| H(e^{j\omega}) \right| \right| d\omega < \infty.$$

Conversely, if $H(e^{j\omega})$ is square integrable and the above integral is finite, then we can obtain a phase response $\angle H(e^{j\omega})$ so that the filter $H(e^{j\omega}) = \left|H(e^{j\omega})\right| \times e^{j\angle H(e^{j\omega})}$ is causal; the solution $\angle H(e^{j\omega})$ is unique if $H(z)$ is minimum phase.

Minimum-phase system: *A causal and stable system with a causal and stable inverse is called a minimum-phase system. It has all poles and zeros inside the unit circle and imparts the minimum phase or group delay to the input signal.*

- An important consequence of this theorem is that the frequency response of a stable and causal system cannot be zero over any finite band of frequencies because, in this case, the integral becomes infinite. Hence, *any stable ideal frequency-selective filter must be noncausal*.

The filter design problem

Given the magnitude response $|H(e^{j\omega})|$ of a causal and stable system, we cannot assign its phase response arbitrarily.

There are two approaches to deal with this problem:

1. Impose constraints on the phase response, for example $\angle H(e^{j\omega}) = -\alpha\omega$, and obtain a filter whose magnitude response satisfies the design specifications.

2. Obtain a filter whose magnitude response satisfies the design specifications irrespective of the resulting phase response.

The filter design problem

Discrete Hilbert transform

A real $h[n]$ can be decomposed into its even and odd parts as follows:

$$h[n] = h_e[n] + h_o[n],$$

where

$$h_e[n] = \tfrac{1}{2}(h[n] + h[-n]),$$

$$h_o[n] = \tfrac{1}{2}(h[n] - h[-n]).$$

If $h[n]$ is causal, it is uniquely defined by its even part

$$h[n] = 2h_e[n]u[n] - h_e[0]\delta[n].$$

If $h[n]$ is absolutely summable, the DTFT of $h[n]$ exists, and can be written as

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}),$$

where $H_R(e^{j\omega})$ is the DTFT of $h_e[n]$.

Thus, if a filter is real, causal, and stable, its frequency response $H(e^{j\omega})$ is uniquely defined by its real part $H_R(e^{j\omega})$.

1. obtain $h_e[n]$ by inverting $H_R(e^{j\omega})$,
2. determine $h[n]$ from $h_e[n]$
3. obtain $H(e^{j\omega})$ from $h[n]$.

This implies a relationship between the real and imaginary parts of $H(e^{j\omega})$,

$$H_I(e^{j\omega}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\omega})\cot\left(\frac{\omega-\theta}{2}\right)d\theta.$$

FIR filters with linear phase

The frequency response of an ideal lowpass filter with linear phase is

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega}, & |\omega| < \omega_c \\ 0. & \omega_c < |\omega| \leq \pi \end{cases} \longleftrightarrow \quad h_{\mathrm{lp}}[n] = \frac{\sin \omega_c (n-\alpha)}{\pi (n-\alpha)}.$$

The impulse response can be obtained by sampling the impulse response $h_{lp}^{c}(t)$ of an ideal continuous-time lowpass filter with cutoff frequency $\Omega_c = \omega_c/T$ at $t = nT$.

$$h_{\mathrm{lp}}[n] = h_{\mathrm{lp}}^{c}(t)|_{t=nT} = \left. \frac{\sin \Omega_c (t - \alpha T)}{\pi (t - \alpha T)} \right|_{t=nT}$$

The function $h_{lp}^{c}(t)$ is symmetric about $t = \alpha T$ for any value of α. However, the sequence $h_{lp}[n] = h_{lp}^{c}(nT)$ may or may not be symmetric dependent on the value of delay α.

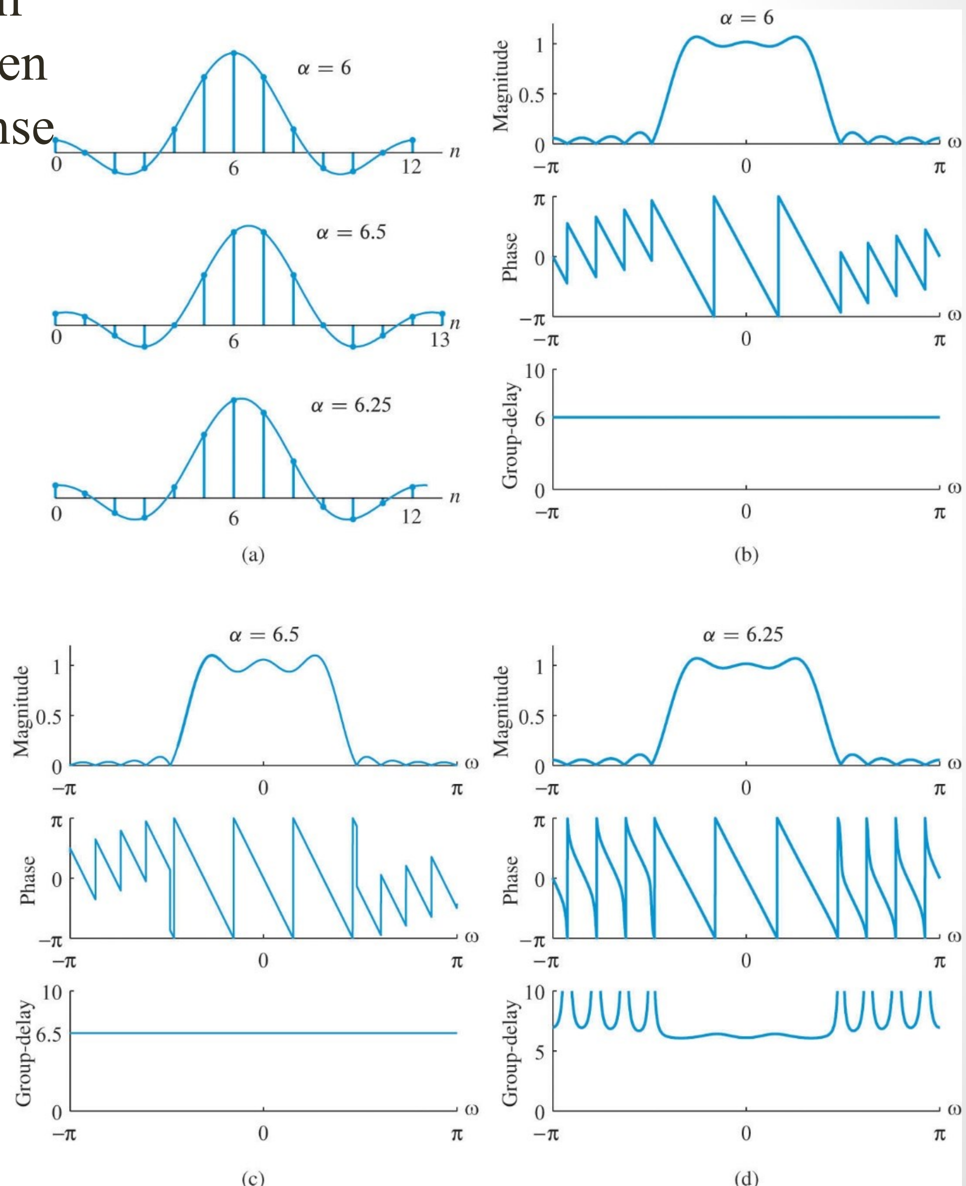
1. The delay α is an integer n_d. In this case, the sequence $h_{\mathrm{lp}}[n]$ is symmetric about its sample at $n = n_d$.
2. The quantity 2α is an integer or α is an integer plus one-half. In this case, the sequence is symmetric about the middle between the samples at $\alpha - 1/2$ and $\alpha + 1/2$.

FIR filters with linear phase

3. The quantity 2α is not an integer. In this case, there is *no* symmetry at all because of the misalignment between the continuous-time impulse response and the sampling grid.

We create a causal FIR filter by setting $h[n] = h_{lp}[n]$ for $0 \leq n \leq M$ and zero elsewhere.

- if $2\alpha = M =$ integer, the impulse $h[n]$ is symmetric about $\alpha = M/2$ and the resulting filters have linear phase.
- The time delay $\alpha = M/2$ is an integer multiple of the sampling interval only when M is even.
- If 2α is *not* an integer, symmetry is lost and the resulting discrete-time filter has a nonlinear phase response and a variable group delay.



FIR filters with linear phase

If for a given value of delay α, we choose a value of $M > 2\alpha$ the symmetry of $h[n]$ is lost and the resulting filter has a nonlinear phase response; thus, we cannot have causal IIR filters with linear phase ($M = \infty$).

Depending on the type of symmetry (even or odd) and whether the filter order M is an even or odd integer, there are four types of FIR filter with linear phase.

If the frequency response is given by

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n},$$

the four cases of interest are specified by the following conditions:

$$h[n] = \pm h[M-n]. \quad M = \text{ even or odd integer}$$

FIR filters with linear phase

Type-I FIR linear-phase filters

A type-I FIR system has a symmetric impulse response with even order M

$$h[n] = h[M-n],\ 0 \leq n \leq M$$

we consider the case $M = 4$.

$$\begin{aligned}
H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\
&= \left(h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[3]e^{-j\omega} + h[4]e^{-j2\omega}\right) e^{-j2\omega} \\
&= \left(h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega}\right) e^{-j2\omega} \\
&= (h[2] + 2h[1]\cos\omega + 2h[0]\cos 2\omega)e^{-j2\omega} \\
&\triangleq (a[0] + a[1]\cos\omega + a[2]\cos 2\omega)e^{-j2\omega}.
\end{aligned}$$

FIR filters with linear phase

Type-I FIR linear-phase filters

In general, the frequency response of a type-I FIR filter can be expressed as

$$H(e^{j\omega}) = \left(\sum_{k=0}^{M/2} a[k] \cos \omega k\right) e^{-j\omega M/2} \triangleq A(e^{j\omega}) e^{-j\omega M/2},$$

where $A(e^{j\omega})$ is a real, even, and periodic function of ω with coefficients given by

$$a[0] = h[M/2], \quad a[k] = 2h[(M/2) - k]. \quad k = 1, 2, \ldots, M/2$$

FIR filters with linear phase

Type-II FIR linear-phase filters

A type-II FIR system has a symmetric impulse response $h[n] = h[M - n]$, $0 \le n \le M$, with odd order M.

For $M = 5$

$$
\begin{aligned}
H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[2]e^{-j3\omega} + h[1]e^{-j4\omega} + h[0]e^{-j5\omega} \\
&= \{2h[2]\cos(\omega/2) + 2h[1]\cos(3\omega/2) + 2h[0]\cos(5\omega/2)\}\, e^{-j(5/2)\omega} \\
&\triangleq \{b[1]\cos(\omega/2) + b[2]\cos(3\omega/2) + b[3]\cos(5\omega/2)\}\, e^{-j(5/2)\omega}.
\end{aligned}
$$

The general expression for the frequency response for a type-II FIR filter is

$$
H(e^{j\omega}) = \left(\sum_{k=1}^{(M+1)/2} b[k] \cos\left[\omega\left(k - \frac{1}{2}\right)\right]\right) e^{-j\omega M/2} \triangleq A(e^{j\omega}) e^{-j\omega M/2},
$$

where the delay $M/2$ is an integer plus one-half and the coefficients of $A(e^{j\omega})$ are

$$
b[k] = 2h[(M+1)/2 - k]. \quad k = 1, 2, \ldots, (M+1)/2
$$

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FIR filters with linear phase

Type-II FIR linear-phase filters

Using $2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$ we can express $A(e^{j\omega})$ in terms of $\cos(\omega k)$. For $M = 5$

$$A(e^{j\omega}) = b[1]\cos(\omega/2) + b[2]\cos(3\omega/2) + b[3]\cos(5\omega/2)$$

$$= \cos\left(\frac{\omega}{2}\right)\{(b[1] - b[2] + b[3]) + 2(b[2] - b[3])\cos\omega + 2b[3]\cos 2\omega\}$$

The general expression

$$A(e^{j\omega}) = \cos\left(\frac{\omega}{2}\right)\sum_{k=0}^{(M-1)/2}\tilde{b}[k]\cos\omega k,$$

where

$$b[k] = \begin{cases} \frac{1}{2}(\tilde{b}[1] + 2\tilde{b}[0]), & k = 1 \\ \frac{1}{2}(\tilde{b}[k] + \tilde{b}[k-1]), & 2 \le k \le (M-1)/2 \\ \frac{1}{2}\tilde{b}[(M-1)/2]. & k = (M+1)/2 \end{cases}$$

FIR filters with linear phase

Type-III FIR linear-phase filters

A type-III FIR system has an antisymmetric impulse response with even order M

$$h[n] = -h[M-n].\ 0 \le n \le M$$

In this case the delay $M/2$ is an integer and the frequency response is given by

$$H(e^{j\omega}) = \left(\sum_{k=1}^{M/2} c[k] \sin \omega k \right) j e^{-j\omega M/2} \triangleq jA(e^{j\omega}) e^{-j\omega M/2},$$

where

$$c[k] = 2h[M/2-k].\quad k = 1, 2, \ldots, M/2$$

FIR filters with linear phase

Type-III FIR linear-phase filters

Using $2 \sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$ we can show that

$$A(e^{j\omega}) = \sin \omega \sum_{k=0}^{M/2} \tilde{c}[k] \cos \omega k,$$

where

$$c[k] = \begin{cases} \frac{1}{2}(2\tilde{c}[0] - \tilde{c}[1]), & k = 1 \\ \frac{1}{2}(\tilde{c}[k-1] - \tilde{c}[k]), & 2 \le k \le (M/2) - 1 \\ \frac{1}{2}\tilde{c}[(M/2) - 1]. & k = M/2 \end{cases}$$

At $\omega = 0$ and $\omega = \pi$, $A(e^{j\omega}) = 0$. In addition, the factor $j = e^{j\pi/2}$ shows that the frequency response is imaginary. Thus, type-III filters are most suitable for the design of differentiators and Hilbert transformers.

FIR filters with linear phase

Type-IV FIR linear-phase filters

The impulse response is antisymmetric, $h[n] = -h[M-n]$. $0 \le n \le M$, and M is odd

$$H(e^{j\omega}) = \left(\sum_{k=1}^{(M+1)/2} d[k] \sin\left[\omega\left(k - \frac{1}{2}\right)\right]\right) je^{-j\omega M/2} \triangleq jA(e^{j\omega})e^{-j\omega M/2},$$

where

$$d[k] = 2h[(M+1)/2 - k]. \quad k = 1, 2, \ldots, (M+1)/2$$

$$A(e^{j\omega}) = \sin\left(\frac{\omega}{2}\right) \sum_{k=0}^{(M-1)/2} \tilde{d}[k] \cos \omega k,$$

$$d[k] = \begin{cases} \frac{1}{2}(2\tilde{d}[0] - \tilde{d}[1]), & k = 1 \\ \frac{1}{2}(\tilde{d}[k-1] - \tilde{d}[k]), & 2 \le k \le (M-1)/2 \\ \frac{1}{2}\tilde{d}[(M-1)/2]. & k = (M+1)/2 \end{cases}$$

we have $A(e^{j\omega}) = 0$ at $\omega = 0$. this class of filters is most suitable for approximating differentiators and Hilbert transformers.

FIR filters with linear phase

Amplitude response function of FIR filters with linear phase

Frequency response functions of type-I–IV FIR filters with linear phase can all be expressed in the form

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} \triangleq A(e^{j\omega})e^{j\Psi(e^{j\omega})}.$$

The real function $A(e^{j\omega})$ is called *amplitude response*

$$\Psi(e^{j\omega}) \triangleq -\alpha\omega + \beta.$$

Type	$h[k]$	M	$A(e^{j\omega})$	$A(e^{j\omega})$	$\Psi(e^{j\omega})$	Uses
I	even	even	$\sum_{k=0}^{M/2} a[k]\cos\omega k$	even–no restriction	$-\frac{\omega M}{2}$	LP, HP, BP, BS, multiband filters
II	even	odd	$\sum_{k=1}^{\frac{M+1}{2}} b[k]\cos\left[\omega\left(k-\frac{1}{2}\right)\right]$	even $A(e^{j\pi})=0$	$-\frac{\omega M}{2}$	LP, BP
III	odd	even	$\sum_{k=1}^{M/2} c[k]\sin\omega k$	odd $A(e^{j0})=0$ $A(e^{j\pi})=0$	$\frac{\pi}{2}-\frac{\omega M}{2}$	differentiators, Hilbert transformers
IV	odd	odd	$\sum_{k=1}^{\frac{M+1}{2}} d[k]\sin\left[\omega\left(k-\frac{1}{2}\right)\right]$	odd $A(e^{j0})=0$	$\frac{\pi}{2}-\frac{\omega M}{2}$	differentiators, Hilbert transformers

FIR filters with linear phase

Zero locations of FIR filters with linear phase

The symmetry or antisymmetry of the impulse response of FIR systems and its length impose restrictions on the locations of the system function zeros. To investigate the effects of the symmetry condition on the zeros of type-I systems, we note that the system function can be written

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} = \sum_{n=0}^{M} h[M-n]z^{-n}$$

$$= \sum_{k=M}^{0} h[k]z^{k}z^{-M} = z^{-M}H(z^{-1}).$$

if $z_0 = re^{j\theta}$ is a zero of *H(z)*, then $z_0^{-1} = r^{-1}e^{-j\theta}$ is also a zero of *H(z)*. If $h[n]$ is real, then its complex conjugate $z_0^{*} = re^{-j\theta}$ is also a zero of *H(z)*.

Therefore, if $h[n]$ is real, each zero not on the unit circle will be part of a cluster of four conjugate reciprocal zeros of the form

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

Zeros on the unit circle ($r = 1$) and real zeros ($\theta = \pm k\pi$) appear in pairs as

$$(1 - e^{j\theta} z^{-1})(1 - e^{-j\theta} z^{-1}) \text{ or } (1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}),$$

FIR filters with linear phase

Zero locations of FIR filters with linear phase

The same considerations apply for type-II systems, with one important exception: type-II systems *always* have a zero at $z = -1$:

$$H(-1) = (-1)^M H(-1)$$

If M is even, it is a simple identity; if M is odd, we have $H(-1) = -H(-1)$, which implies that $H(-1)$ must be zero.

For systems with an antisymmetric impulse response:

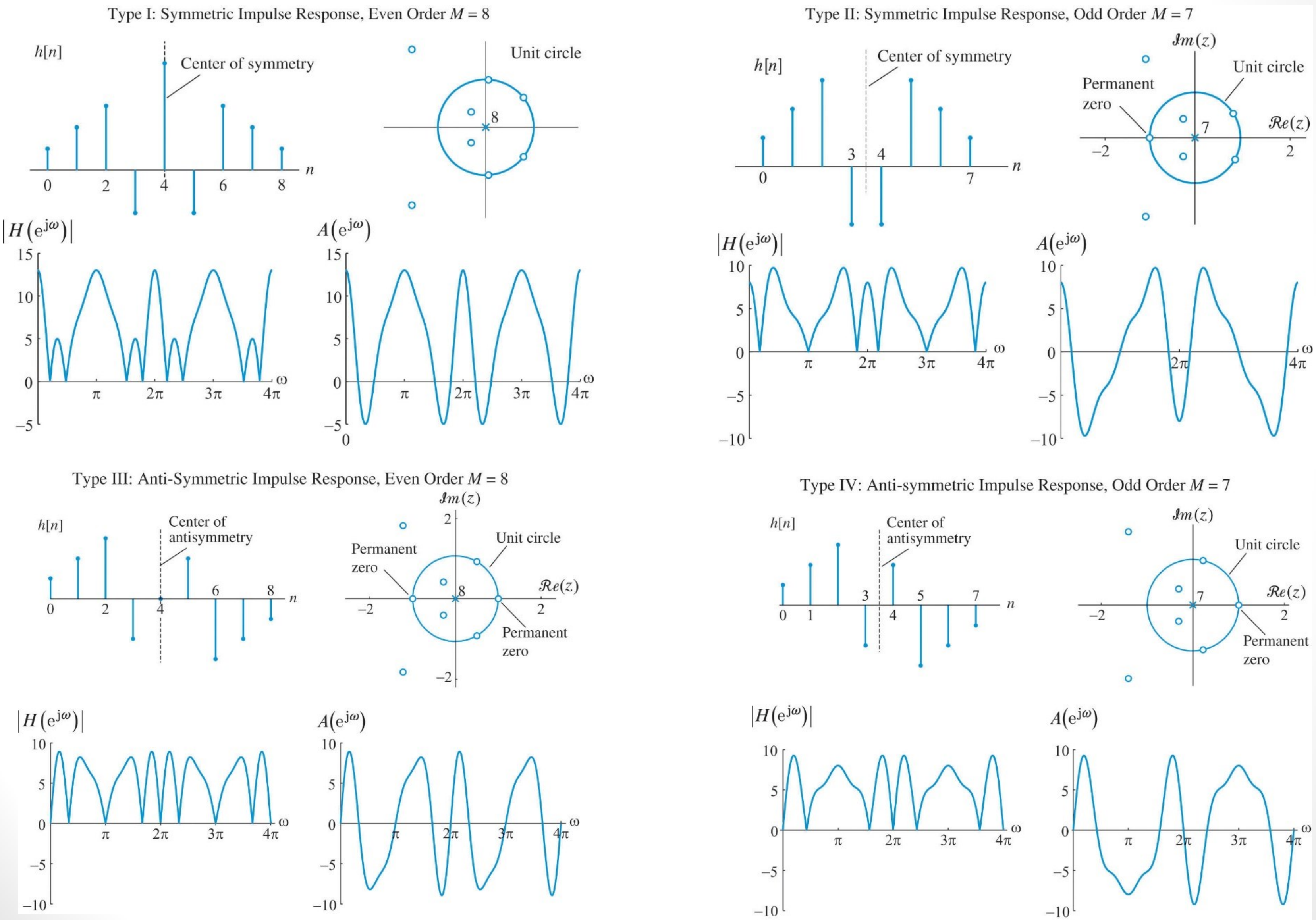
$$H(z) = -z^{-M} H(z^{-1})$$

For $z = 1$, we have $H(1) = -H(1)$; thus, $H(z)$ *must* have a zero at $z = 1$ for any M (even or odd). For $z = -1$, we obtain $H(-1) = -(-1)^M H(-1)$; if M is even, we have $H(-1) = -H(-1)$, which implies that $H(-1) = 0$.

The presence of a zeros at $z = -1$ implies that $H(e^{j\omega})$ has a zero at $\omega = \pi$; thus, a type-III filter cannot be used for the design of highpass filters.

FIR filters with linear phase

Zero locations of FIR filters with linear phase



Design of FIR filters by windowing

Direct truncation of an ideal impulse response

Suppose that we wish to approximate a desired ideal frequency response function

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n},$$

with an FIR filter $h[n]$, $0 \le n \le M$, by minimizing the mean-square error

$$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega.$$

Using Parseval's identity $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega,$

$$\varepsilon^2 = \sum_{n=0}^{M}(h_d[n] - h[n])^2 + \sum_{n=-\infty}^{-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n].$$

The optimum solution is

$$h[n] = \begin{cases} h_d[n], & 0 \le n \le M \\ 0. & \text{otherwise} \end{cases}$$

The best, in the mean-square error sense, FIR approximation to the ideal IIR impulse response $h_d[n]$ is obtained by truncation.

Design of FIR filters by windowing

Frequency domain effects of truncation

we express the truncated impulse response as the product of the desired IIR impulse response and the finite *rectangular window* sequence:

$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0. & \text{otherwise} \end{cases}$$

$$W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} = \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$$

$$\rightarrow \qquad h[n] = h_d[n]w[n].$$

Applying the windowing theorem

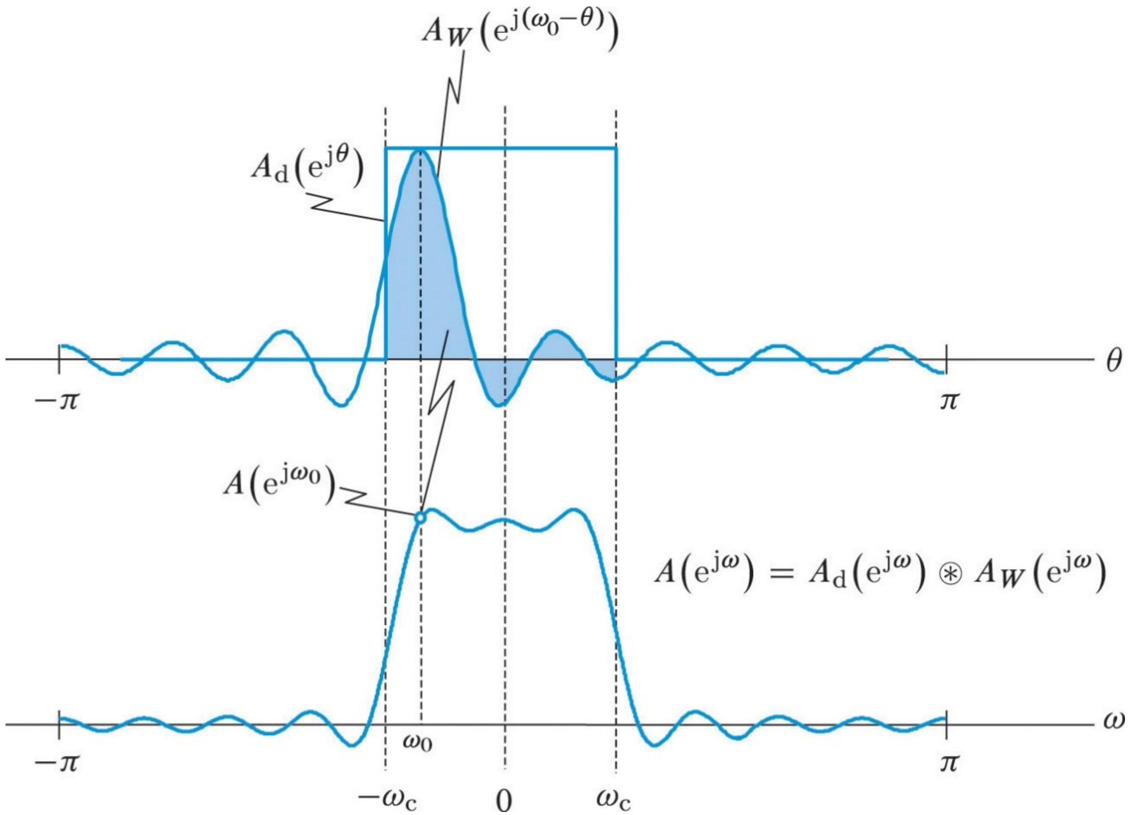
$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta.$$

Design of FIR filters by windowing

Frequency domain effects of truncation

Using the amplitude response $A_d(e^{j\omega})$ of the ideal lowpass filter for $\alpha = M/2$ and the amplitude function $A_w(e^{j\omega})$ of the rectangular window:

$$A_d(e^{j\omega}) \triangleq \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}, \quad A_w(e^{j\omega}) \triangleq \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}$$



Design of FIR filters by windowing

Smoothing the frequency response using fixed windows

Using nonrectangular windows to obtain a less abrupt truncation of the impulse response reduces the height of the ripples at the expense of a wider transition band:

Rectangular

$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2, \quad M \text{ even} \\ 2 - 2n/M, & M/2 < n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hann

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hamming

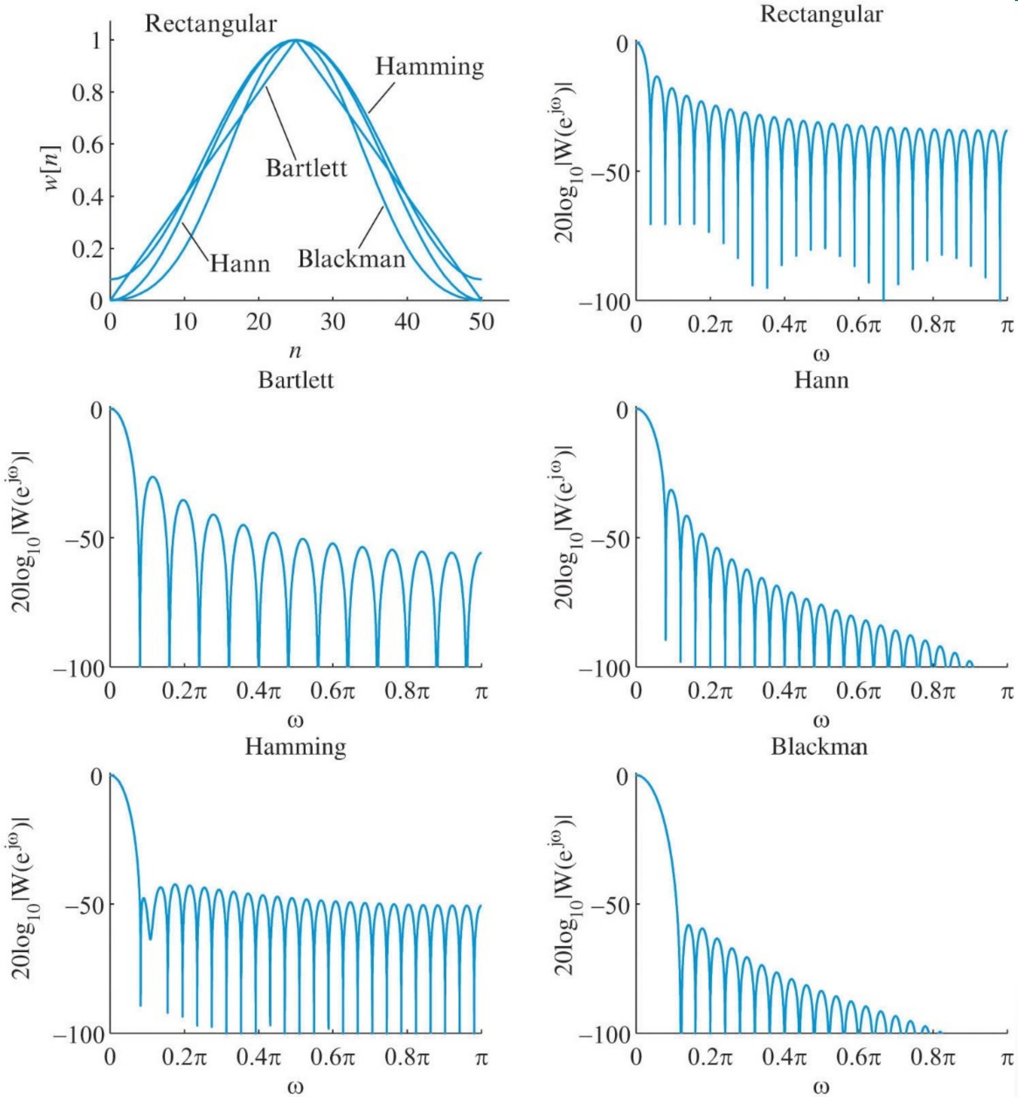
$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Blackman

$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M \\ 0. & \text{otherwise} \end{cases}$$

Design of FIR filters by windowing

Smoothing the frequency response using fixed windows



Time-domain and frequency-domain characteristics of some commonly used windows for M=50.

The filter design problem

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Table 10.3 Properties of commonly used windows ($L = M + 1$).

Window name	Side lobe level (dB)	Approx. $\Delta\omega$	Exact $\Delta\omega$	$\delta_p \approx \delta_s$	A_p (dB)	A_s (dB)
Rectangular	-13	$4\pi/L$	$1.8\pi/L$	0.09	0.75	21
Bartlett	-25	$8\pi/L$	$6.1\pi/L$	0.05	0.45	26
Hann	-31	$8\pi/L$	$6.2\pi/L$	0.0063	0.055	44
Hamming	-41	$8\pi/L$	$6.6\pi/L$	0.0022	0.019	53
Blackman	-57	$12\pi/L$	$11\pi/L$	0.0002	0.0017	74

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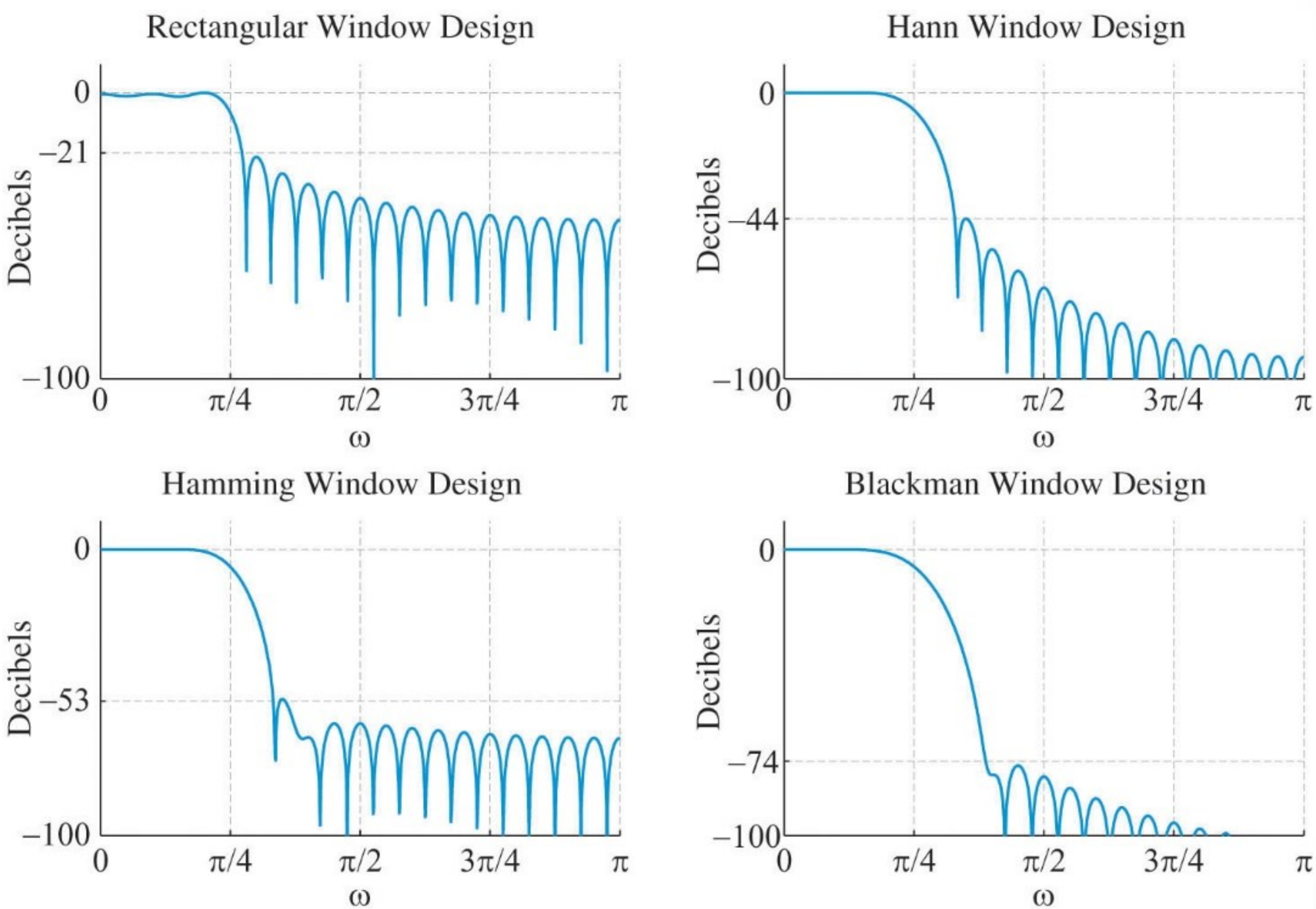

Magnitude responses in dB for 40th-order FIR lowpass filters designed using rectangular, Hann, Hamming, and Blackman windows with cutoff frequency $\omega_c = \pi/4$.

The filter design problem

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FIR filter design using fixed windows (For lowpass filters):

1. Given the design specifications $\{\omega_p, \omega_s, A_p, A_s\}$, determine the ripples δ_p and δ_s and set $\delta = \min\{\delta_p, \delta_s\}$.
2. Determine the cutoff frequency of the ideal lowpass prototype by $\omega_c = (\omega_p + \omega_s)/2$.
3. Determine the design parameters $A = -20 \log_{10} \delta$ and $\Delta\omega = \omega_s - \omega_p$.
4. Choose the window function that provides the smallest stopband attenuation greater than A. For this window function, determine the required value of $M = L-1$ by selecting the corresponding value of ω from the column labeled "exact $\Delta\omega$". If M is odd, we may increase it by one to have a flexible type-I filter.
5. Determine the impulse response of the ideal lowpass filter by

$$h_d[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}.$$

6. Compute the impulse response $h[n] = h_d[n]w[n]$ using the chosen window.
7. Check whether the designed filter satisfies the prescribed specifications; if not, increase the order M and go back to step 5.

The filter design problem

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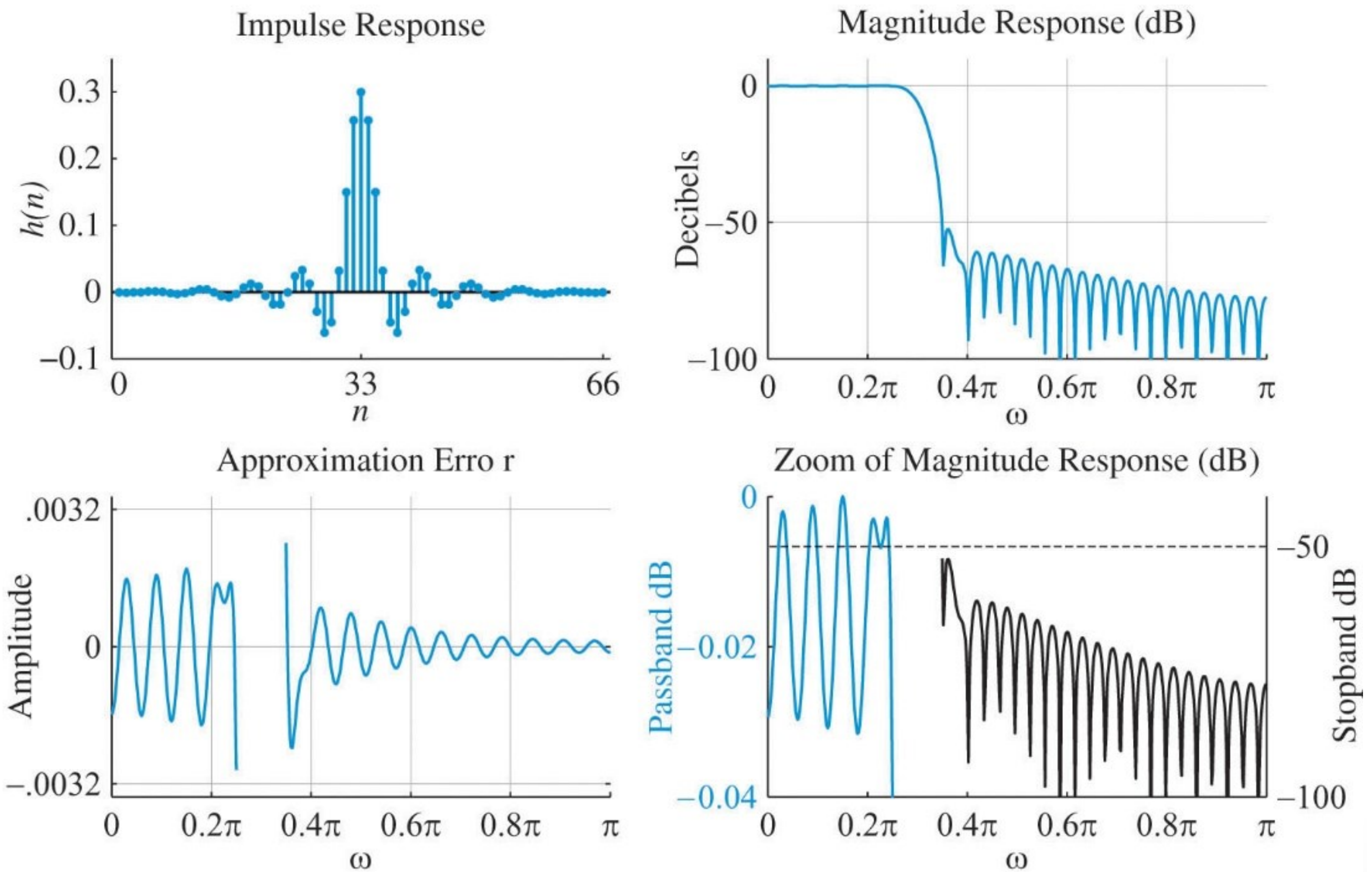
Example

Let us design a lowpass linear-phase FIR filter to satisfy the following specifications: $\omega_p = 0.25\pi$, $\omega_s = 0.35\pi$, $A_p = 0.1$ dB, $A_s = 50$ dB.

1. Obtain: $\delta_p = 0.0058$ and $\delta_s = 0.0032$
2. δ is set to 0.0032 or $A = A_s = 50$ dB.
3. Set the ideal lowpass filter cutoff frequency $\omega_c = (\omega_p+\omega_s)/2 = 0.3\pi$
4. Compute transition bandwidth $\omega = \omega_s - \omega_p = 0.1\pi$.
5. Choose a Hamming window since it provides at least 53 dB attenuation which is greater than $A = 50$ dB.
6. Using the transition bandwidth $\omega \approx 6.6\pi/L$, the minimum window length is $L = 66$.
7. Choose $L = 67$ or $M = 66$ to obtain a type-I filter.

The filter design problem

Smoothing the frequency response using fixed windows



Impulse, approximation error, and magnitude response plots of the designed filter using a Hamming window to satisfy specifications: $\omega_p = 0.25\pi$, $\omega_s = 0.35\pi$, $A_p = 0.1$ dB, and $A_s = 50$ dB.

FIR_design_windowing.py

The filter design problem

Chebyshev polynomials and minimax approximation

Design of FIR filters by minimizing the mean square error does not preclude the possibility of large errors at individual frequencies. However, in most applications it is important that the error be small at all frequencies within the range of interest. This can be achieved by minimizing the maximum absolute error.

Equiripple optimum Chebyshev FIR filter design

If the actual filter has amplitude response $A(e^{j\omega})$, we define the approximation error by:

$$E(\omega) \triangleq W(\omega)\left[A_d(e^{j\omega}) - A(e^{j\omega})\right]$$

The design objective is to find the coefficients of a type I–IV FIR filter that minimize the weighted Chebyshev error, defined by

$$\|E(\omega)\|_\infty = \max_{\omega \in \mathcal{B}} |E(\omega)|,$$

where B is a union of disjoint closed subsets (corresponding to frequency bands) of $0 \le \omega \le \pi$.

The filter design problem

Chebyshev polynomials and minimax approximation

The amplitude response function $A(e^{j\omega})$ can be written as

$$A(e^{j\omega}) = Q(e^{j\omega})P(e^{j\omega}),$$

$$Q(e^{j\omega}) = \begin{cases} 1, & \text{Type I} \\ \cos(\omega/2), & \text{Type II} \\ \sin(\omega), & \text{Type III} \\ \sin(\omega/2), & \text{Type IV} \end{cases}, \quad P(e^{j\omega}) = \sum_{k=0}^{R} p[k]\cos(\omega k),$$

and $R = M/2$, if M is even and $R = (M-1)/2$, if M is odd.

$$\rightarrow \qquad E(\omega) = W(\omega)Q(e^{j\omega})\left[\frac{A_d(e^{j\omega})}{Q(e^{j\omega})} - P(e^{j\omega})\right]$$

$$\triangleq \bar{W}(\omega)\left[\bar{A}_d(e^{j\omega}) - P(e^{j\omega})\right]$$

where

$$\bar{W}(\omega) \triangleq W(\omega)Q(e^{j\omega}),\ \bar{A}_d(e^{j\omega}) \triangleq \frac{A_d(e^{j\omega})}{Q(e^{j\omega})}.$$

The filter design problem

Chebyshev polynomials and minimax approximation

Chebyshev approximation problem

Given the filter order M, determine the coefficients of $P(e^{j\omega})$ that minimize the maximum absolute value of $E(\omega)$ over the frequency bands of interest, that is, choose $P(e^{j\omega})$ so that

$$\|E(\omega)\|_\infty = \max_{\omega \in \mathcal{B}} |\bar{W}(\omega)[\bar{A}_d(e^{j\omega}) - P(e^{j\omega})]|$$

is minimum.

The amplitude response of a type I FIR filter is given by

$$P(e^{j\omega}) = A(e^{j\omega}) = \sum_{k=0}^{R} a[k]\cos(\omega k) = \sum_{k=0}^{R} a_k\{\cos(\omega)\}^k$$

where $R = M/2$, $h[R] = a[0]$, and $h[k] = a[R-k]/2$, $k = 1, 2, \ldots, R$.

The filter design problem

Chebyshev polynomials and minimax approximation

The Remez exchange algorithm

If the extremal frequencies ω_i, $i = 1, 2, \ldots, R+2$ for the optimum filter were known, we could find the coefficients $a[k]$, $k = 0, 1, \ldots, R$ and the corresponding error δ by solving the set of linear equations:

$$E(\omega_i) = W(\omega_i)\left[A_{\mathrm{d}}(\mathrm{e}^{\mathrm{j}\omega_i}) - A(\mathrm{e}^{\mathrm{j}\omega_i})\right] = (-1)^{i+1}\delta,\ 1 \leq i \leq R+2,$$

or equivalently

$$\begin{bmatrix} 1 & \cos(\omega_1) & \ldots & \cos(R\omega_1) & \frac{1}{W(\omega_1)} \\ 1 & \cos(\omega_2) & \ldots & \cos(R\omega_2) & \frac{-1}{W(\omega_2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{R+1}) & \ldots & \cos(R\omega_{R+1}) & \frac{(-1)^{R+2}}{W(\omega_{R+1})} \\ 1 & \cos(\omega_{R+2}) & \ldots & \cos(R\omega_{R+2}) & \frac{(-1)^{R+3}}{W(\omega_{R+2})} \end{bmatrix} \begin{bmatrix} a[0] \\ a[1] \\ \vdots \\ a[R] \\ \delta \end{bmatrix} = \begin{bmatrix} A_{\mathrm{d}}(\mathrm{e}^{\mathrm{j}\omega_1}) \\ A_{\mathrm{d}}(\mathrm{e}^{\mathrm{j}\omega_2}) \\ \vdots \\ A_{\mathrm{d}}(\mathrm{e}^{\mathrm{j}\omega_{R+1}}) \\ A_{\mathrm{d}}(\mathrm{e}^{\mathrm{j}\omega_{R+2}}) \end{bmatrix}$$

The filter design problem

Chebyshev polynomials and minimax approximation

scipy.signal.remez

Calculate the minimax optimal filter using the Remez exchange algorithm.

scipy.signal.remez(*numtaps*, *bands*, *desired*, *weight=None*, *Hz=1*, *type='bandpass'*, *maxiter=25*, *grid_density=16***)**

FIR_chebyshev_Remez.py