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Transform analysis of LTI systems

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Sinusoidal response of LTI systems

$$x[n] = e^{j\omega n} \xmapsto{\mathcal{H}} y[n] = H(e^{j\omega})e^{j\omega n}, \text{ all } n$$

The frequency response is a complex function that can be expressed in either polar or rectangular form

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})} = H_R(e^{j\omega}) + jH_I(e^{j\omega}).$$

Using the definition above, the linearity property, and the polar notation, we obtain

$$x[n] = Ae^{j(\omega n+\phi)} \xmapsto{\mathcal{H}} y[n] = A|H(e^{j\omega})|e^{j[\omega n+\phi+\angle H(e^{j\omega})]}.$$

Therefore, the response of a stable LTI system to a complex exponential sequence is a complex exponential sequence with the same frequency; only the amplitude and phase are changed by the system.

Sinusoidal response of LTI systems

Suppose that the input is a real sinusoidal sequence

$$x[n] = A_x \cos(\omega n + \phi_x) = \frac{A_x}{2} e^{j\phi_x} e^{j\omega n} + \frac{A_x}{2} e^{-j\phi_x} e^{-j\omega n}.$$

$$x_1[n] = \tfrac{A_x}{2} e^{j\phi_x} e^{j\omega n} \quad \longrightarrow \quad y_1[n] = |H(e^{j\omega})| \frac{A_x}{2} e^{j\phi_x} e^{j[\omega n + \angle H(e^{j\omega})]}.$$

$$x_2[n] = \tfrac{A_x}{2} e^{-j\phi_x} e^{-j\omega n} \quad \longrightarrow \quad y_2[n] = |H(e^{-j\omega})| \frac{A_x}{2} e^{-j\phi_x} e^{j[-\omega n + \angle H(e^{-j\omega})]}.$$

$$\longrightarrow \quad y[n] = \frac{A_x}{2} |H(e^{j\omega})| e^{j[\omega n + \phi_x + \angle H(e^{j\omega})]} + \frac{A_x}{2} |H(e^{-j\omega})| e^{j[-\omega n - \phi_x + \angle H(e^{-j\omega})]}.$$

If we assume that the impulse response $h[n]$ is real-valued, we have $|H(e^{-j\omega})| = |H(e^{j\omega})|$ and $\angle H(e^{-j\omega}) = -\angle H(e^{j\omega})$.

$$\longrightarrow \quad y[n] = A_x |H(e^{j\omega})| \cos\left[\omega n + \phi_x + \angle H(e^{j\omega})\right].$$

$$x[n] = A_x \cos(\omega n + \phi_x) \xmapsto{\mathcal{H}} y[n] = A_y \cos(\omega n + \phi_y),$$

$$A_y = |H(e^{j\omega})| A_x, \quad \phi_y = \angle H(e^{j\omega}) + \phi_x.$$

Conclusion: all an LTI system can do to a sinusoidal input is to scale its amplitude and change its phase; its frequency remains the *same*.

Sinusoidal response of LTI systems

$$x[n] = Ae^{j(\omega n+\phi)} \xmapsto{\mathcal{H}} y[n] = A|H(e^{j\omega})|e^{j[\omega n+\phi+\angle H(e^{j\omega})]}.$$

- *Since* $A_y = |H(e^{j\omega})|A_x$, at frequency ω, the quantity $|H(e^{j\omega})|$ is known as the *magnitude response* or *gain* of the system.

- Since $\varphi_y = \angle H(e^{j\omega}) + \varphi_x$, $\angle H(e^{j\omega})$ is called the *phase response* of the system.

- $H(e^{j\omega})$ is known as the *frequency response* function of the system.

- When $|H(e^{j\omega})|$ is small at a frequency $\omega = \omega_0$, the component at this frequency is essentially removed, that is, "filtered out," from the input signal. For this reason, LTI systems are often called filters.

Sinusoidal response of LTI systems

Example: Illustration of frequency response function

Consider a stable system described by the first-order difference equation

$$y[n] = ay[n-1] + bx[n].\ -1 < a < 1$$

Using $x[n] = e^{j\omega n} \stackrel{\mathcal{H}}{\longmapsto} y[n] = H(e^{j\omega})e^{j\omega n}, \text{ all } n$

$$\longrightarrow \quad H(e^{j\omega})e^{j\omega n} = aH(e^{j\omega})e^{j\omega(n-1)} + be^{j\omega n}. \quad \longrightarrow \quad H(e^{j\omega}) = \frac{b}{1-ae^{-j\omega}}.$$

$$\longrightarrow \quad \left|1-ae^{-j\omega}\right| = \sqrt{(1-a\cos\omega)^2 + (a\sin\omega)^2} = \sqrt{1+a^2-2a\cos\omega},$$

$$\angle(1-ae^{-j\omega}) = \tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right].$$

$$\longrightarrow \quad |H(e^{j\omega})| = \frac{|b|}{\sqrt{1-2a\cos\omega+a^2}}, \qquad \angle H(e^{j\omega}) = \angle b - \tan^{-1}\frac{a\sin\omega}{1-a\cos\omega}.$$

It is customary to choose b so that the maximum of $|H(e^{j\omega})|$ equals one.

If $a > 0$, the denominator of $|H(e^{j\omega})|$ attains its minimum at $\omega = 0$. Therefore, we require that $|H(e^{j0})| = |b|/(1-a) = 1$. This yields $b = \pm(1-a)$.

If $a < 0$, the maximum of $|H(e^{j\omega})|$ occurs at $\omega = \pi$. By requiring that $|H(e^{j\pi})| = |b|/(1+a) = 1$, we obtain $b = \pm(1+a)$.

Both cases can be satisfied by choosing $\boldsymbol{b = 1 - |a|}$.

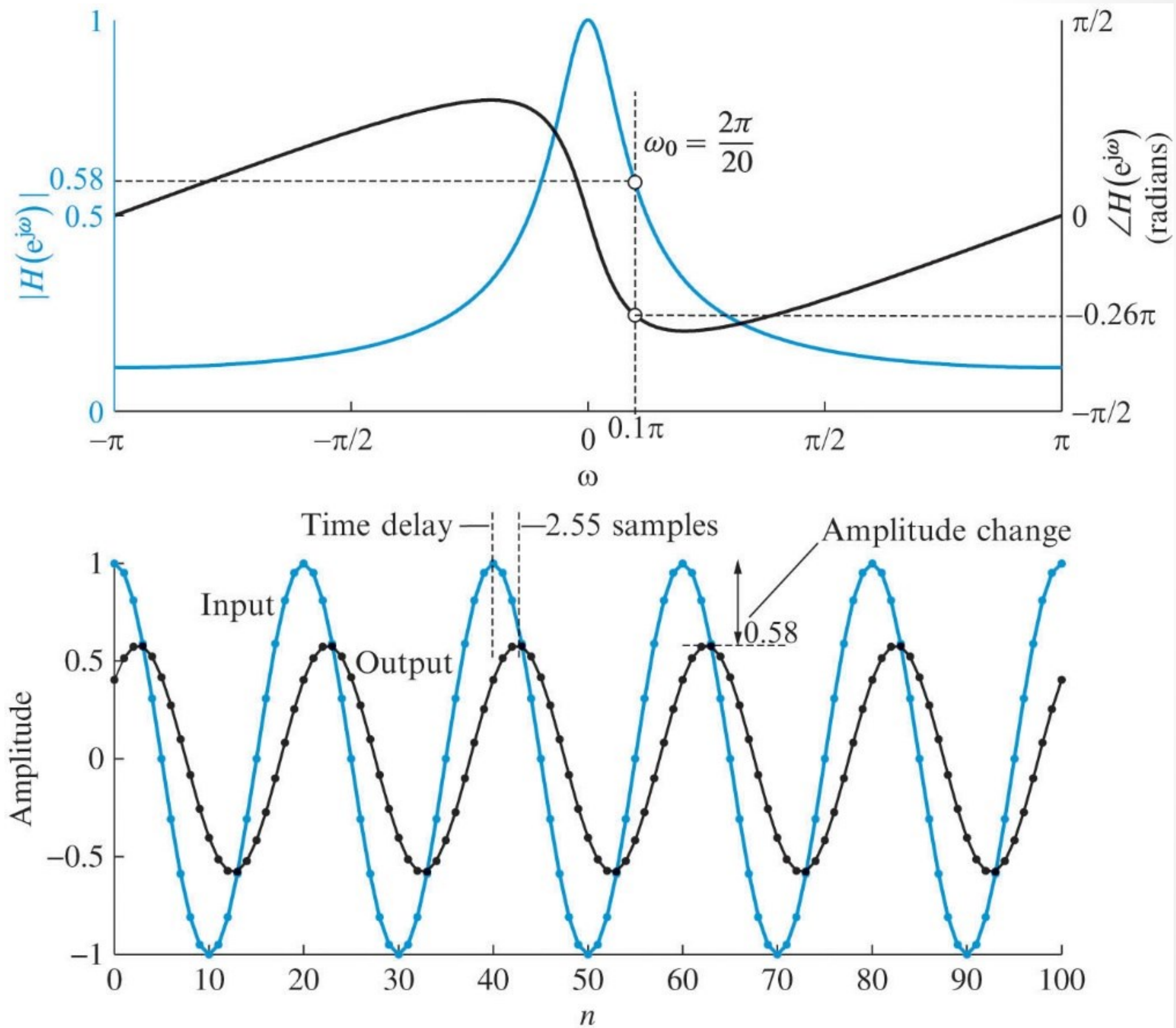
Sinusoidal response of LTI systems

Example: Illustration of frequency response function

$a = 0.8$

The normalized gain at $\omega = 2\pi/20$ is about 0.58 while the phase shift is about -0.26π radians (or $-0.26\pi/\omega = -2.55$ samples).

Example_5.1.py



Magnitude and phase response functions and input–output signals for the LTI system.
The higher frequency suffers more attenuation than the lower frequency (lowpass filter).

Sinusoidal response of LTI systems

Example: Response to a linear FM signal

It is possible to evaluate the magnitude response at several frequencies at once by using an input signal known as a *linear FM pulse*.

$$x(t) = A\cos(\Omega_0 t + \phi) = \mathcal{R}e\left\{A e^{j(\Omega_0 t + \phi)}\right\}.$$

$$\theta(t) = \Omega_0 t + \phi = 2\pi F_0 t + \phi$$

$$F_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = F_0.$$

Suppose now that the phase changes with time according to $\theta(t) = 2\pi F_0 t + \pi\beta t^2$.

$$F_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = F_0 + \beta t.$$

We call $F_i(t)$ the *instantaneous frequency* of $x(t)$.

If $\beta = B/\tau$, the instantaneous frequency of the continuous-time signal

$$x(t) = A\cos(2\pi F_0 t + \pi\beta t^2), \quad 0 \le t \le \tau$$

increases from F_0 to $F_1 = F_0 + B$ Hz at a constant rate. $x(t)$ is also known as *chirp*.

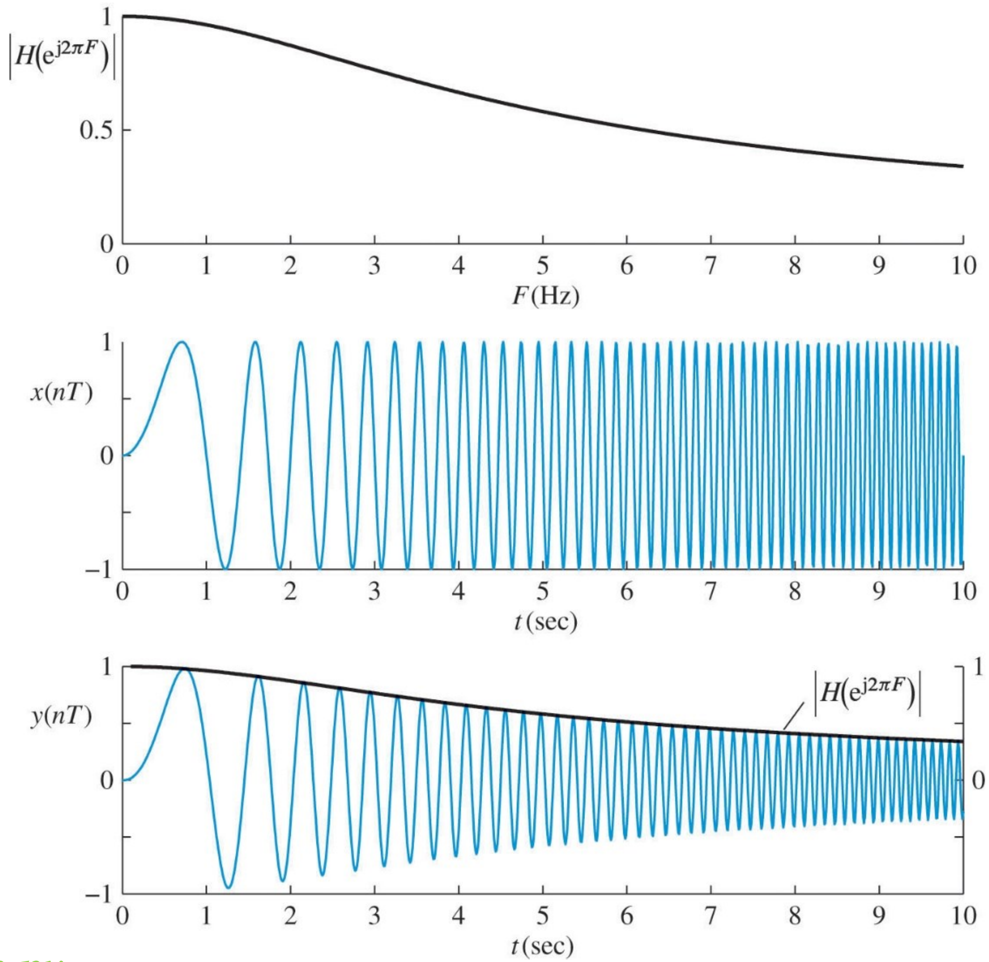
If we set $F_0 = 0$, sample $x(t)$ at a rate of $F_s = 1/T$, choose τ so that $\tau = NT$, and $\beta_d = (B/F_s)/N$, we obtain a discrete-time chirp signal

$$x[n] = x(nT) = A\cos(\pi\beta n^2 T^2) = A\cos(\pi\beta_d n^2), \quad 0 \le n \le N$$

Sinusoidal response of LTI systems

Example: Response to a linear FM signal

Response of the system $y[n] = ay[n-1] + bx[n]$ $-1 < a < 1$ with $a = 0.8$ to a chirp signal $x[n]$ with $A = 1$, $B = 10$ Hz, $\tau = 10$ s, and $F_s = 100$ Hz.



Example_5.2_chirp.py

Evaluation of the magnitude response of an LTI system using a linear FM (chirp) input signal.

Sinusoidal response of LTI systems

Steady-state and transient response

The eigenfunction property $x[n] = e^{j\omega n} \overset{\mathcal{H}}{\longmapsto} y[n] = H(e^{j\omega})e^{j\omega n}, \text{ all } n$
holds if the input sequence $x[n]$ is a complex exponential sequence that exists over the entire interval $-\infty < n < \infty$. However, in practice every input starts at a finite time.
Consider a complex exponential starting at time $n = 0$, that is,

$$x[n] = e^{j\omega n}u[n].$$

The response of a causal system ($h[n] = 0$, $n < 0$) to this input is

$$\begin{aligned} y[n] &= \sum_{k=0}^{n} h[k]x[n-k] = \sum_{k=0}^{n} h[k]e^{j\omega(n-k)} \\ &= \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} \\ &= \underbrace{H(e^{j\omega})e^{j\omega n}}_{y_{ss}[n]} - \underbrace{\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}}_{y_{tr}[n]}. \end{aligned}$$

For an FIR system with $h[n] = 0$ for $n > M$, the transient response vanishes for $n > M$.

$$\lim_{n\to\infty} y[n] = H(e^{j\omega})e^{j\omega n} = y_{ss}[n].$$

Sinusoidal response of LTI systems

Steady-state and transient response

Example

Consider a causal and stable system described by the impulse response $h[n] = 0.8^n u[n]$. We will compute and plot the response $y[n]$ of the system to the input $x[n] = \cos(0.05\pi n)u[n]$.

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{1}{1-0.8z^{-1}}, \quad |z| > 0.8$$

$$X(z) = \mathcal{Z}\{x[n]\} = \frac{1-\cos(0.05\pi)z^{-1}}{1-2\cos(0.05\pi)z^{-1}+z^{-2}}. \quad |z| > 1$$

$$\begin{aligned}
Y(z) = H(z)X(z) &= \frac{1-\cos(0.05\pi)z^{-1}}{(1-0.8z^{-1})[1-2\cos(0.05\pi)z^{-1}+z^{-2}]} \\
&= \frac{\frac{0.8[0.8-\cos(0.05\pi)]}{0.8^2-2(0.8)\cos(0.05\pi)+1}}{1-0.8z^{-1}} + \frac{\frac{e^{j0.05\pi}}{2(e^{j0.05\pi}-0.8)}}{1-e^{j0.05\pi}z^{-1}} + \frac{\frac{e^{-j0.05\pi}}{2(e^{-j0.05\pi}-0.8)}}{1-e^{-j0.05\pi}z^{-1}} \\
&= \frac{-2.5151}{1-0.8z^{-1}} + \frac{\frac{1}{2}H(z)|_{z=e^{j0.05\pi}}}{1-e^{j0.05\pi}z^{-1}} + \frac{\frac{1}{2}H(z)|_{z=e^{-j0.05\pi}}}{1-e^{-j0.05\pi}z^{-1}}, \quad |z| > 1 \\
&= \frac{-2.5151}{1-0.8z^{-1}} + \frac{\frac{1}{2}H(e^{j0.05\pi})}{1-e^{j0.05\pi}z^{-1}} + \frac{\frac{1}{2}H(e^{-j0.05\pi})}{1-e^{-j0.05\pi}z^{-1}}. \quad |z| > 1
\end{aligned}$$

Sinusoidal response of LTI systems

Steady-state and transient response

Example

$$Y(z)=H(z)X(z)=\frac{1-\cos(0.05\pi)z^{-1}}{(1-0.8z^{-1})[1-2\cos(0.05\pi)z^{-1}+z^{-2}]}$$

$$=\frac{\frac{0.8[0.8-\cos(0.05\pi)]}{0.8^2-2(0.8)\cos(0.05\pi)+1}}{1-0.8z^{-1}}+\frac{\frac{e^{j0.05\pi}}{2(e^{j0.05\pi}-0.8)}}{1-e^{j0.05\pi}z^{-1}}+\frac{\frac{e^{-j0.05\pi}}{2(e^{-j0.05\pi}-0.8)}}{1-e^{-j0.05\pi}z^{-1}}$$

$$=\frac{-2.5151}{1-0.8z^{-1}}+\frac{\frac{1}{2}H(z)|_{z=e^{j0.05\pi}}}{1-e^{j0.05\pi}z^{-1}}+\frac{\frac{1}{2}H(z)|_{z=e^{-j0.05\pi}}}{1-e^{-j0.05\pi}z^{-1}},\quad |z|>1$$

$$=\frac{-2.5151}{1-0.8z^{-1}}+\frac{\frac{1}{2}H(e^{j0.05\pi})}{1-e^{j0.05\pi}z^{-1}}+\frac{\frac{1}{2}H(e^{-j0.05\pi})}{1-e^{-j0.05\pi}z^{-1}}.\quad |z|>1$$

$$y[n]=-2.5151(0.8)^n u[n]+\frac{1}{2}H(e^{j0.05\pi})e^{j0.05\pi n}u[n]$$

$$+\frac{1}{2}H(e^{-j0.05\pi})e^{-j0.05\pi n}u[n]$$

$$=-2.5151(0.8)^n u[n]+\mathcal{Re}\left\{|H(e^{j0.05\pi})|e^{j\angle H(e^{j0.05\pi})}\right\}u[n]$$

$$=-2.5151(0.8)^n u[n]+|H(e^{j0.05\pi})|\cos[0.05\pi n+\angle H(e^{j0.05\pi})]u[n]$$

$$=\underbrace{-2.5151(0.8)^n u[n]}_{y_{tr}[n]}+\underbrace{4.0928\cos(0.05\pi n-0.5377)u[n]}_{y_{ss}[n]}.$$

Sinusoidal response of LTI systems

Steady-state and transient response

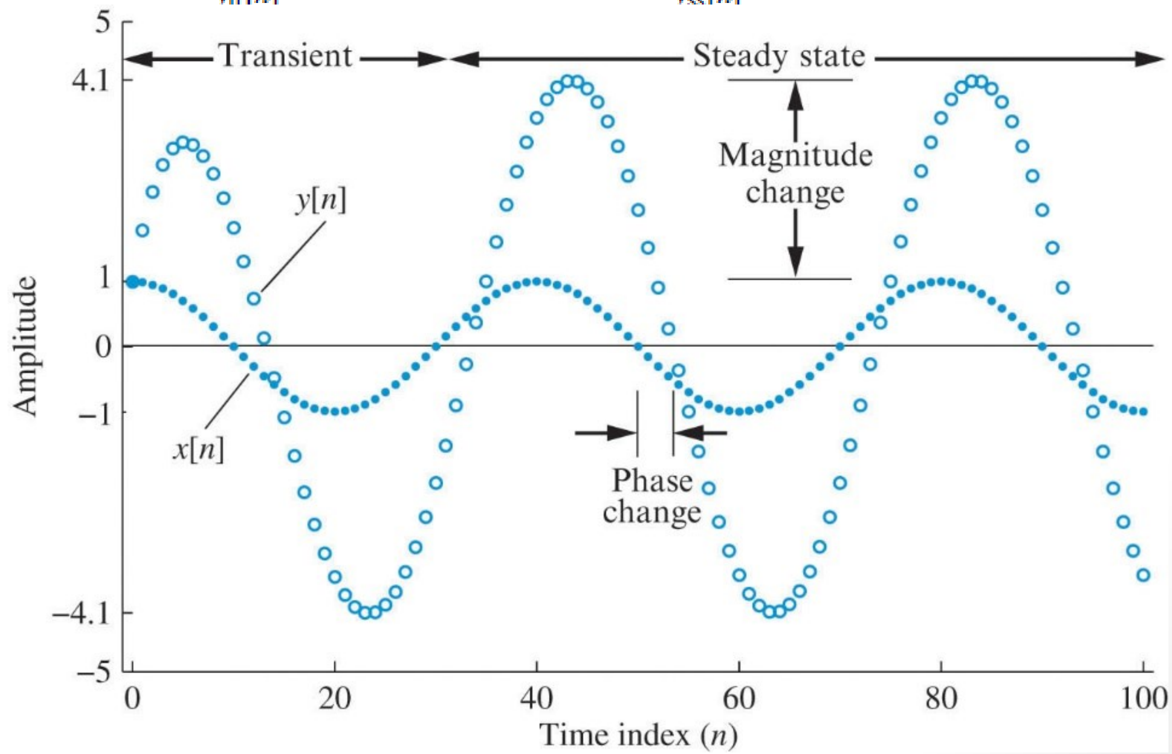
Example

$$y[n] = -2.5151(0.8)^n u[n] + \frac{1}{2}H(e^{j0.05\pi})e^{j0.05\pi n}u[n] + \frac{1}{2}H(e^{-j0.05\pi})e^{-j0.05\pi n}u[n]$$

$$= -2.5151(0.8)^n u[n] + \mathcal{R}e\left\{|H(e^{j0.05\pi})|e^{j\angle H(e^{j0.05\pi})}\right\}u[n]$$

$$= -2.5151(0.8)^n u[n] + |H(e^{j0.05\pi})|\cos[0.05\pi n + \angle H(e^{j0.05\pi})]u[n]$$

$$= \underbrace{-2.5151(0.8)^n u[n]}_{y_{tr}[n]} + \underbrace{4.0928\cos(0.05\pi n - 0.5377)u[n]}_{y_{ss}[n]}.$$



Transient and steady-state responses for sinusoidal excitation $x[n] = \cos(0.05\pi n)$.

Response of LTI systems in the frequency domain

The frequency response leads to a simple relationship between the spectra of input and output signals of LTI systems. The form of this relationship depends on whether the input sequence is periodic or aperiodic.

Response to periodic inputs

Consider a periodic input $x[n] = x[n + N]$ with fundamental period N.

$$x[n] = \sum_{k=0}^{N-1} c_k^{(x)} e^{j\frac{2\pi}{N}kn}.$$

Using the eigenfunction property and the linearity property

$$e^{j\frac{2\pi}{N}kn} \stackrel{\mathcal{H}}{\longmapsto} H(e^{j\frac{2\pi}{N}k})e^{j\frac{2\pi}{N}kn},$$

$$x[n] = \sum_{k=0}^{N-1} c_k^{(x)} e^{j\frac{2\pi}{N}kn} \stackrel{\mathcal{H}}{\longmapsto} \sum_{k=0}^{N-1} c_k^{(x)} H(e^{j\frac{2\pi}{N}k})e^{j\frac{2\pi}{N}kn} = y[n].$$

the output sequence is periodic with Fourier coefficients $c_k^{(y)}$ given by

$$c_k^{(y)} = H(e^{j\frac{2\pi}{N}k})c_k^{(x)}, \quad -\infty < k < \infty.$$

The response of an LTI system to a periodic input sequence is a periodic sequence with the same fundamental period.

Response of LTI systems in the frequency domain

Response to periodic inputs

LTI systems *cannot* alter the frequencies of the input signals; they can only change their amplitude and phase

$$|c_k^{(y)}| = |H(e^{j\frac{2\pi}{N}k})||c_k^{(x)}|,$$

$$\angle c_k^{(y)} = \angle H(e^{j\frac{2\pi}{N}k}) + \angle c_k^{(x)}.$$

Response of LTI systems in the frequency domain

Response to aperiodic inputs

Aperiodic sequences can be expressed as a "continuous" superposition of complex exponentials, using the inverse DTFT, as follows

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = \lim_{\substack{\Delta\omega\to 0\\ k\Delta\omega\to\omega}} \frac{1}{2\pi}\sum_k X(e^{jk\Delta\omega})e^{j(k\Delta\omega)n}\Delta\omega.$$

$$\rightarrow\quad y[n] = \lim_{\substack{\Delta\omega\to 0\\ k\Delta\omega\to\omega}} \frac{1}{2\pi}\sum_k H(e^{jk\Delta\omega})X(e^{jk\Delta\omega})\, e^{j(k\Delta\omega)n}\,\Delta\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})X(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$\rightarrow\qquad Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}).$$

Distortion of signals passing through LTI systems

LTI systems change the relative magnitudes and phases of the frequency components in an input signal in a way dictated by its frequency response function. These changes may be either desirable, that is, the input signal is modified in a useful way, or undesirable, that is, the input signal is subject to distortion.

Distortionless response systems

A system has distortionless response if the input signal $x[n]$ and the output signal $y[n]$ have the same "shape." This is possible if the input and output signals satisfy the condition

$$y[n] = Gx[n - n_d],\ G > 0$$

where G and n_d are constants.

$$\rightarrow \qquad Y(e^{j\omega}) = Ge^{-j\omega n_d} X(e^{j\omega}).$$

$$\rightarrow \qquad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = Ge^{-j\omega n_d}.$$

$$|H(e^{j\omega})| = G,$$

$$\rightarrow$$

$$\angle H(e^{j\omega}) = -\omega n_d.$$

Distortion of signals passing through LTI systems

Distortionless response systems

$$|H(e^{j\omega})| = G,$$

$$\angle H(e^{j\omega}) = -\omega n_d.$$

For a LTI system to have a distortionless response,

- the magnitude response $|H(e^{j\omega})|$ must be a constant,
- the phase response $\angle H(e^{j\omega})$ must be a linear function of ω with slope $-n_d$, where n_d is the delay of the output with respect to the input.
- the phase response should not only be a linear function of frequency, but it should also pass through the origin $\omega = 0$.

Distortion of signals passing through LTI systems

Magnitude distortion

We say that a system introduces *magnitude distortion* if

$$|H(e^{j\omega})| \neq G.$$

Systems without magnitude distortion, are known as *allpass* systems.

Consider the simple test signal

$$x[n] = \cos(\omega_0 n) - \tfrac{1}{3}\cos(3\omega_0 n) + \tfrac{1}{5}\cos(5\omega_0 n),$$

Suppose now that a system $H_i(e^{j\omega})$ with input $x[n]$ produces an output signal $y_i[n]$ given by

$$y_i[n] = c_1 \cos(\omega_0 n + \phi_1) + c_2 \cos(3\omega_0 n + \phi_2) + c_3 \cos(5\omega_0 n + \phi_3).$$

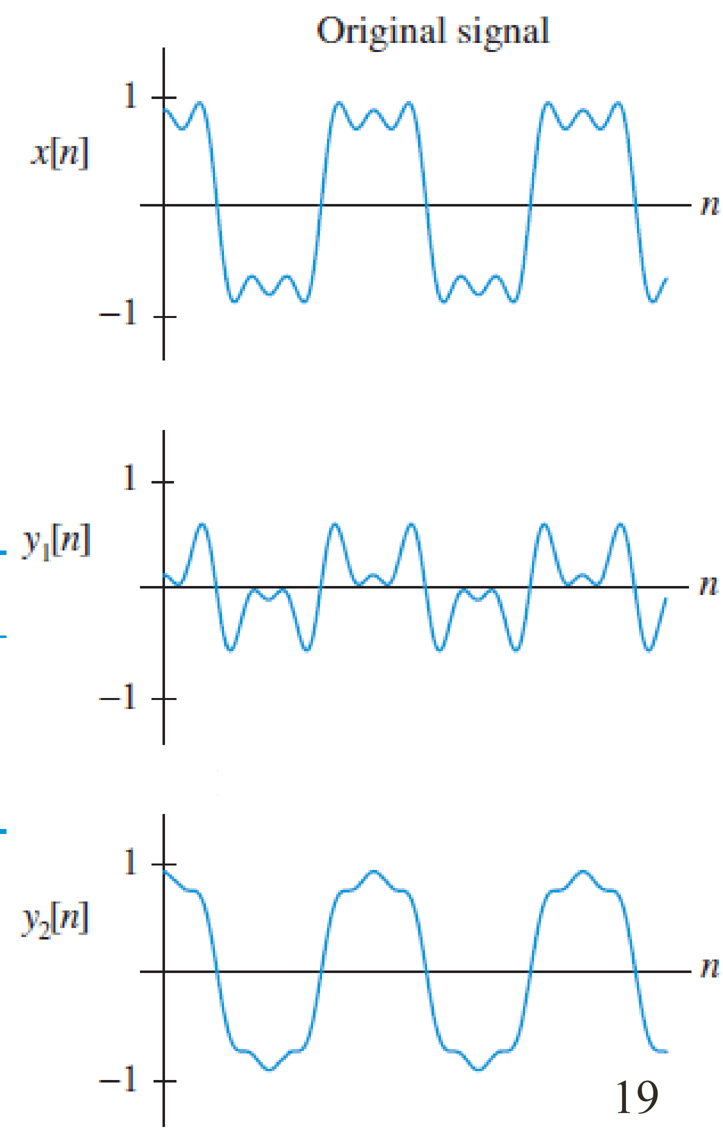
Distortion of signals passing through LTI systems

Magnitude distortion

$$y_i[n] = c_1 \cos(\omega_0 n + \phi_1) + c_2 \cos(3\omega_0 n + \phi_2) + c_3 \cos(5\omega_0 n + \phi_3).$$

$$\omega_0 = 0.004\pi$$

Signal	c_1	c_2	c_3	ϕ_1	ϕ_2	ϕ_3	Amplitude
$x[n]$	1	$-1/3$	$1/5$	0	0	0	original
$y_1[n]$	$1/4$	$-1/3$	$1/5$	0	0	0	
$y_2[n]$	1	$-1/6$	$1/10$	0	0	0	



Distortion of signals passing through LTI systems

Phase or delay distortion

If the phase response is not a linear function of frequency, that is

$$\angle H(e^{j\omega}) \neq -\omega n_d,$$

The phase response $\angle H(e^{j\omega})$ gives the phase shift (in radians) experienced by each sinusoidal component of the input signal. Assume $x[n] = A_x \cos(\omega n + \phi_x)$ and $h[n]$ is real-valued

$$\begin{aligned} y[n] &= A_x |H(e^{j\omega})| \cos[\omega n + \phi_x + \angle H(e^{j\omega})] \\ &= A_x |H(e^{j\omega})| \cos\left\{\omega\left[n + \frac{\phi_x}{\omega} + \frac{\angle H(e^{j\omega})}{\omega}\right]\right\}, \end{aligned}$$

The quantity $\angle H(e^{j\omega})/\omega$ shows the time shift (in number of sampling intervals) experienced by each sinusoidal component of the input signal. Therefore, sometimes it is more meaningful to use the *phase delay* defined by

$$\tau_{pd}(\omega) \triangleq -\frac{\angle H(e^{j\omega})}{\omega}.$$

The phase delay gives the *time delay* experienced by each sinusoidal component of the input signal.

Distortion of signals passing through LTI systems

Phase or delay distortion

$$\tau_{\mathrm{pd}}(\omega) \triangleq -\frac{\angle H(\mathrm{e}^{\mathrm{j}\omega})}{\omega}.$$

Example: the simplest lowpass filter:

$$y[n] = x[n] + x[n\text{-}1]$$

The phase response is $\angle H(\mathrm{e}^{\mathrm{j}\omega}) = -\omega/2$ $\rightarrow$ $\tau_{pd}(\omega) = \frac{1}{2}$ samples

Example: if the input to a filter with frequency response $H(e^{j\omega T}) = G(\omega)e^{j\Theta(\omega)}$ is

$$x(n) = \cos(\omega nT)$$

then the output is

$$\begin{aligned} y(n) &= G(\omega)\cos[\omega nT + \Theta(\omega)] \\ &= G(\omega)\cos\{\omega[nT - P(\omega)]\} \end{aligned}$$

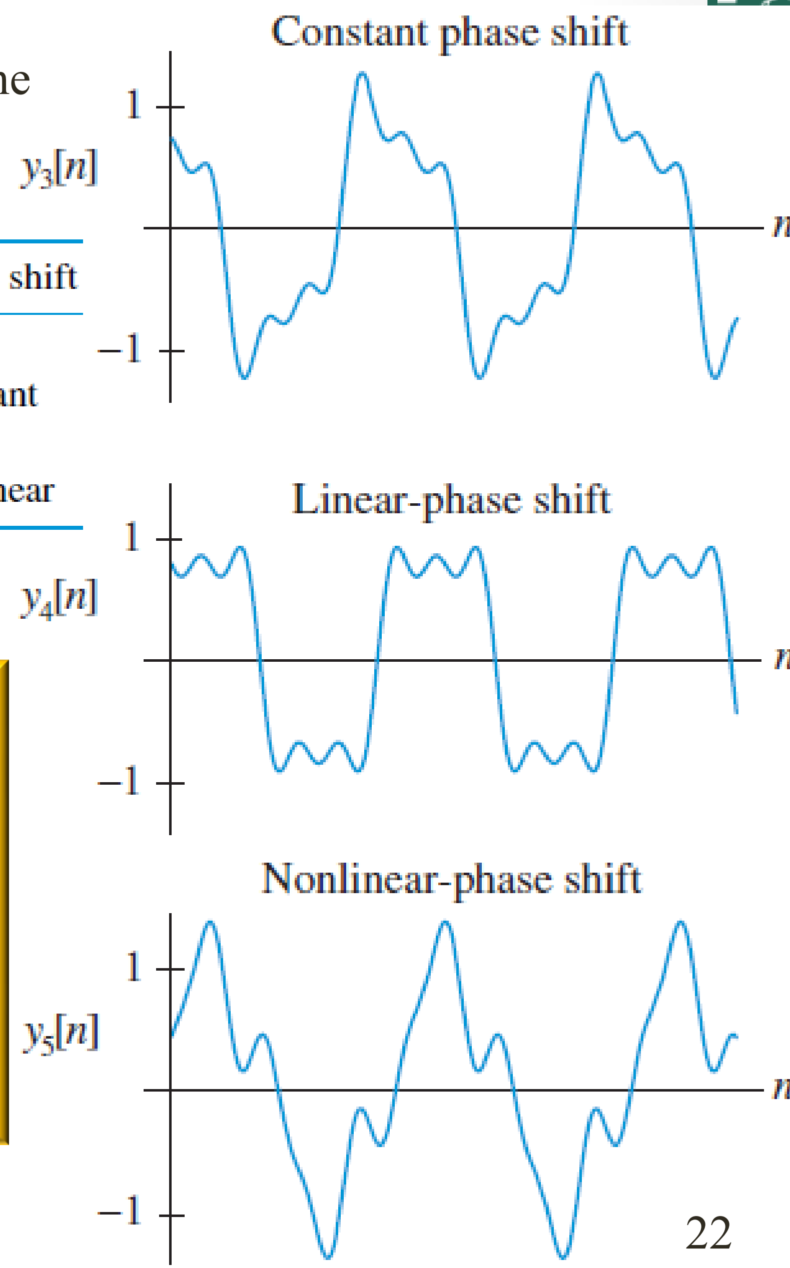
the phase delay expresses the phase response as a time delay in seconds.

Distortion of signals passing through LTI systems

Phase or delay distortion

Consider an allpass system that changes the input signal phase as shown in the list below.

Signal	c_1	c_2	c_3	ϕ_1	ϕ_2	ϕ_3	Phase shift
$x[n]$	1	$-1/3$	$1/5$	0	0	0	zero
$y_3[n]$	1	$-1/3$	$1/5$	$\pi/6$	$\pi/6$	$\pi/6$	constant
$y_4[n]$	1	$-1/3$	$1/5$	$-\pi/4$	$-3\pi/4$	$-5\pi/4$	linear
$y_5[n]$	1	$-1/3$	$1/5$	$-\pi/3$	$\pi/4$	$\pi/7$	nonlinear



Conclusion:
For distortionless transmission it is not enough that the system amplifies (or attenuates) all frequency components equally. All these frequency components must also undergo an identical time delay in order to add up correctly. This demands a constant phase delay, that is, a phase shift proportional to frequency.

Distortion of signals passing through LTI systems

Group delay

A convenient way to check the linearity of phase response is to use the *group delay*

$$\tau_{\mathrm{gd}}(\omega) \triangleq -\frac{\mathrm{d}\Psi(\omega)}{\mathrm{d}\omega}.$$

where ψ(ω) represents the delay.

The derivative in this definition requires that the phase response is a continuous function of frequency

$$\Psi(\omega) = -\int_0^{\omega} \tau_{\mathrm{gd}}(\theta)\mathrm{d}\theta + \Psi(0).$$

Phase responses which are linear in frequency correspond to constant phase delay and constant group delay; both delays are identical, and each may be interpreted as time delay.
If the phase response is nonlinear, then the relative phase of each frequency component is delayed by a different amount resulting in severe shape distortions.

Distortion of signals passing through LTI systems

Group delay

consider a bandpass signal obtained by modulating a lowpass signal such as

$$x[n] = s[n]\cos\omega_c n,$$

where $s[n]$ is a lowpass signal with maximum frequency $\omega_m \ll \omega_c$

If the phase response $\psi(\omega)$ is approximately linear around $\omega = \omega_c$, it can be expressed using a Taylor's series expansion by

$$\Psi(\omega) \approx \Psi(\omega_c) + \left.\frac{d\Psi(\omega)}{d\omega}\right|_{\omega=\omega_c} (\omega - \omega_c)$$

$$= -\tau_{pd}(\omega_c)\omega_c - \tau_{gd}(\omega_c)(\omega - \omega_c),$$

It can be shown that:

$$y[n] \approx \left|H(e^{j\omega_c})\right| s[n - \tau_{gd}(\omega_c)] \cos\{\omega_c[n - \tau_{pd}(\omega_c)]\}.$$

We see that the group delay evaluated at the carrier frequency ω_c is the delay of the envelop $s[n]$ of the input and the phase delay is equal to the delay of the carrier.

The name group delay comes because $\tau_{gd}(\omega_c)$ shows the delay of the "bundle" (group) of frequency components about ω_c.

Distortion of signals passing through LTI systems

Example: Magnitude and group delay distortions

Consider a filter with system function

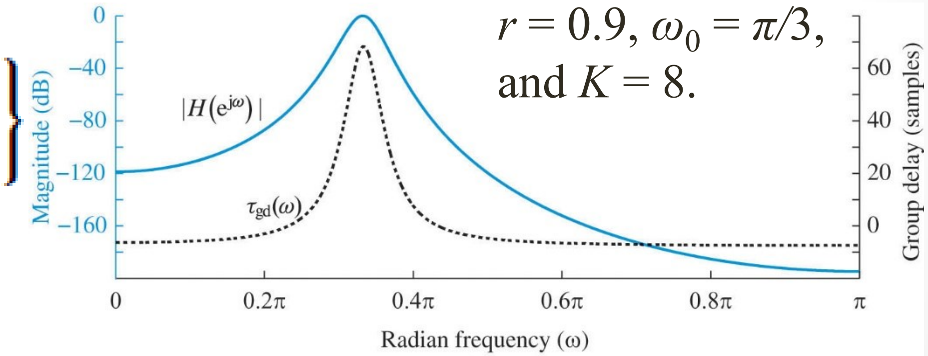
$$H(z) = \frac{b_0}{\left[1 - 2r\cos(\omega_0)\, z^{-1} + r^2 z^{-2}\right]^K}.$$

The input signal $x[n]$ consists of two consecutive narrowband Gaussian pulses followed by a trail of zeros. To create this signal, we first compute $N = 100$ samples of a Gaussian pulse

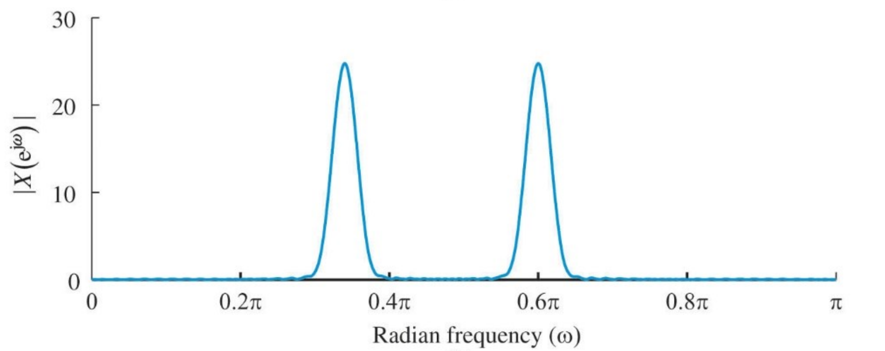
$$s(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}\right\}$$

with $\mu = 0$ and $\sigma = 2$ in the range $-5 \leq t \leq 5$.

$r = 0.9$, $\omega_0 = \pi/3$, and $K = 8$.



(a)



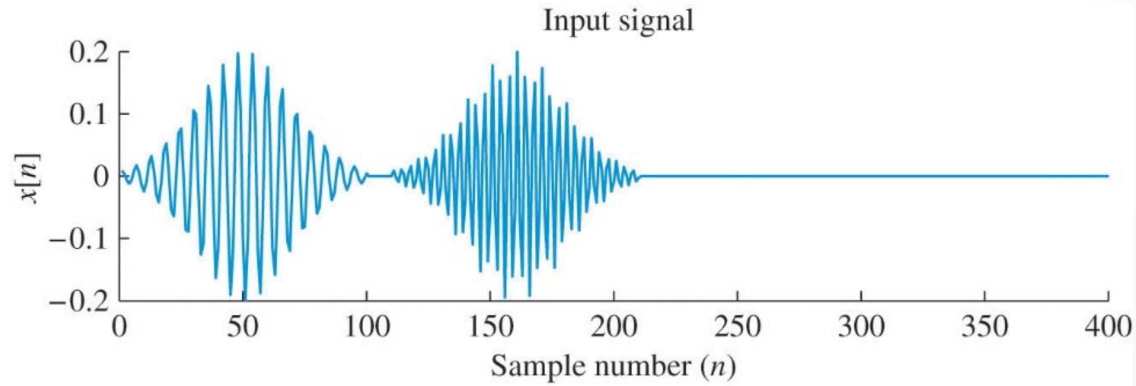
(b)

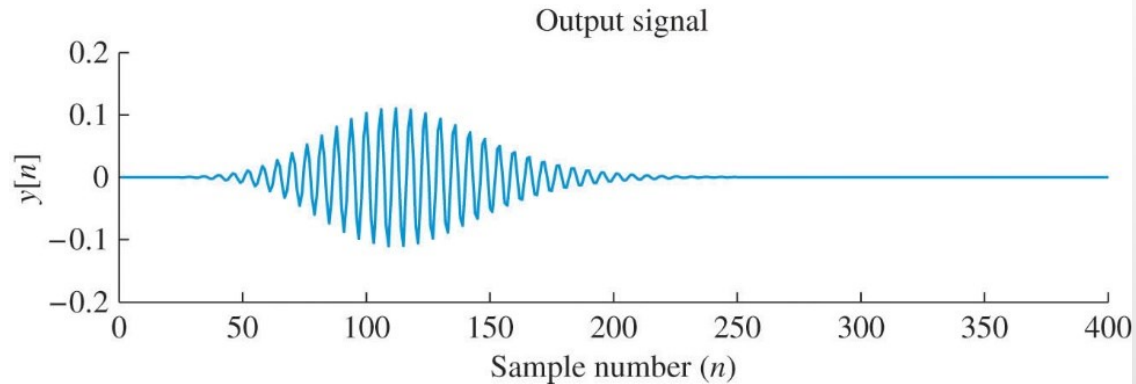
Distortion of signals passing through LTI systems

Example: Magnitude and group delay distortions

The two modulated pulses are generated by $s[n]\cos(\omega_1 n)$ and $s[n]\cos(\omega_2 n)$, where $\omega_1 = 0.34\pi$, $\omega_2 = 0.6\pi$, and for s[n] we have $\mu = 0$, $\sigma = 2$ in the range $-5 \leq t \leq 5$.

The first pulse, which is centered at the passband of the filter, passes through with a group or envelope delay of about 50 samples. The attenuation and smearing of the envelope is due to the magnitude distortion of the filter. We note that the pulse centered at ω_2 is attenuated by more than 100 dB and it does not appear in the output.





Ideal and practical filters

Frequency-selective filters:

Systems that are designed to pass some frequency components without significant distortion while severely or completely eliminating others.

Ideal bandpass filter (BPF) is defined by

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_\ell \le |\omega| \le \omega_u \\ 0, & \text{otherwise} \end{cases}$$

where $n_d \ge 0$ and $0 \le \omega_l \le \omega_u \le \pi$.

- $H(e^{j\omega})$ is periodic with period 2π radians, hence we only specify and plot the frequency response over a single period.
- “Low-frequencies” are located around $\omega = 0$,
- “high-frequencies” are close to $\omega = \pi$ radians.
- The parameters ω_l and ω_u, which specify the end points of the *passband*, are called the lower and upper *cutoff* frequencies.
- The *bandwidth* of the filter, defined as the width of the passband at the positive part of the frequency axis, is given by

$$\omega = \omega_u - \omega_l$$

Ideal and practical filters

Ideal bandpass filter (BPF) is defined by

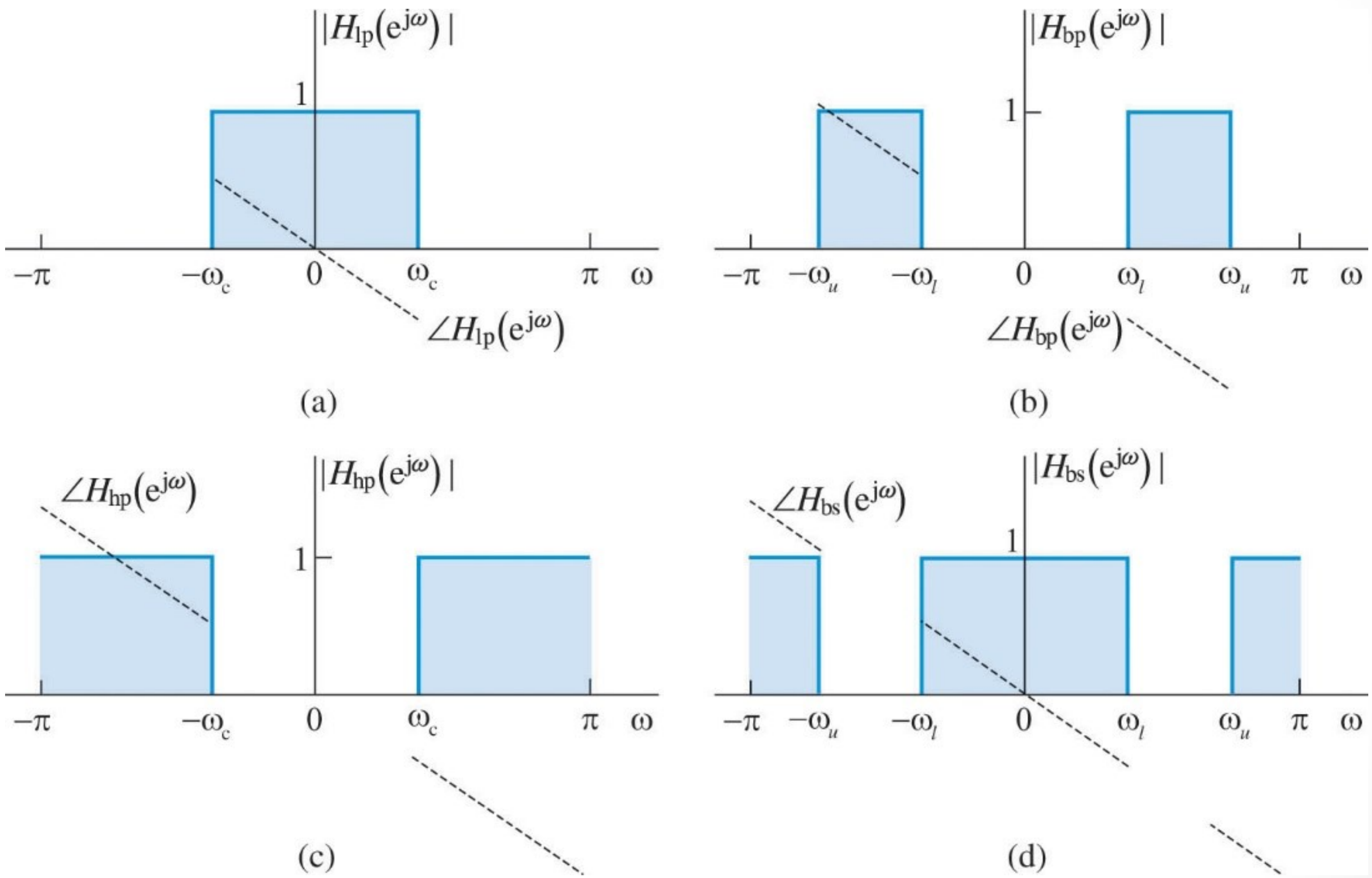
$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_\ell \leq |\omega| \leq \omega_u \\ 0, & \text{otherwise} \end{cases}$$

an ideal *lowpass* filter is defined with $\omega_l = 0$,
an ideal *highpass* filter has $\omega_u = \pi$,
ideal *bandstop* filters have a distortionless response over all frequencies except some *stopband*, $\omega_l \leq |\omega| \leq \omega_u$, where $H(e^{j\omega}) = 0$.

The phase response $\angle H(e^{j\omega})$ is required to be linear only in the passband; there is no need for it to be defined elsewhere because the response of the filter is zero.

Ideal and practical filters

Ideal frequency-selective filters



Ideal frequency-selective filters: (a) lowpass filter, (b) bandpass filter, (c) highpass filter, and (d) bandstop filter.

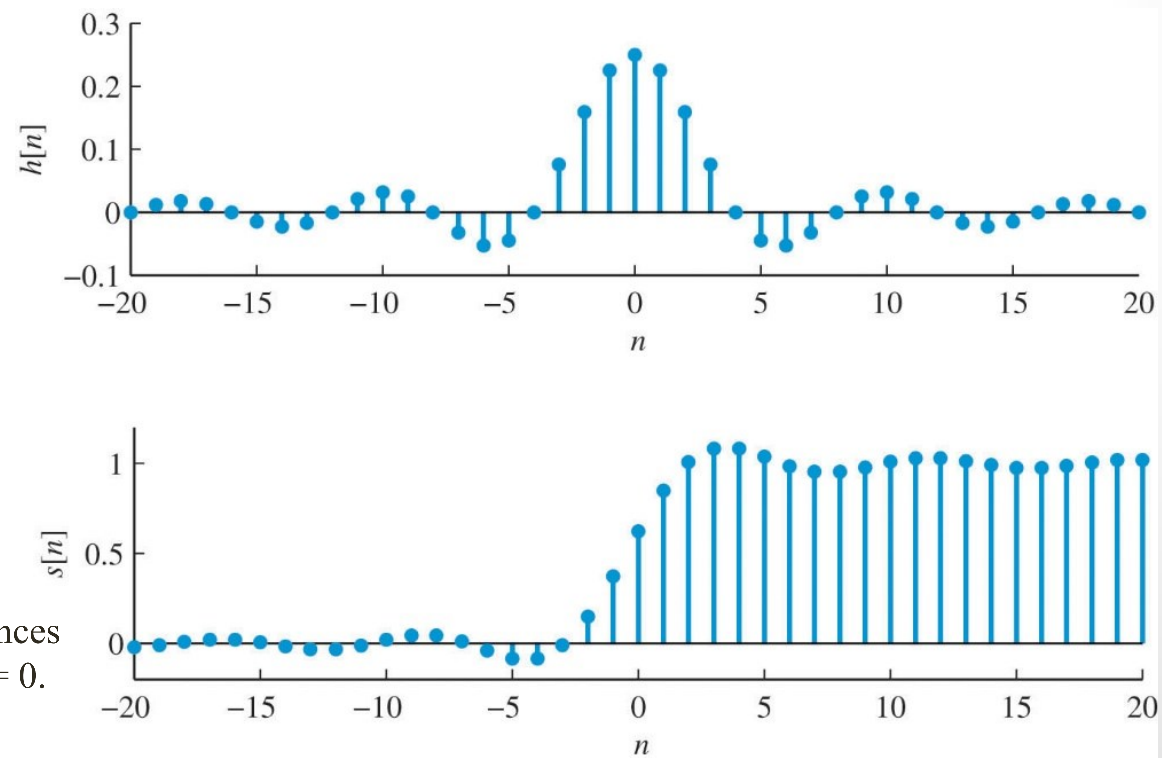
Ideal and practical filters

Consider an ideal lowpass filter with frequency response

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0. & \omega_c < |\omega| \le \pi \end{cases}$$

with the impulse response

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}.$$



Impulse and step response sequences of the ideal lowpass filter for $n_d = 0$.

Ideal and practical filters

- $h_{lp}[n]$ extends from $-\infty$ to ∞; therefore we *cannot* compute the output of the ideal lowpass filter using a convolution sum.
- $h_{lp}[n]$ has a DTFT $H_{lp}(e^{j\omega})$ because it has finite energy. However, it should be noted that $h_{lp}[n]$ is *not* absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} |h_{lp}[n]| = \infty.$$

- Therefore, the ideal lowpass filter is unstable.
- Furthermore, since $r^{-n}h_{lp}[n]$ is not absolutely summable for any value of r, the sequence $h_{lp}[n]$ does *not* have a z-transform.
- Since only systems with a rational system function can be computed recursively, we *cannot* compute the output of the ideal lowpass filter either recursively or nonrecursively.

Conclusion:
The ideal lowpass filter is unstable and practically unrealizable.

Ideal and practical filters

The impulse response of the ideal bandpass filter can be obtained by modulating the impulse response of an ideal lowpass filter with $\omega_c = (\omega_u - \omega_l)/2 = \Delta\omega/2$ using a carrier with frequency $\omega_0 = (\omega_u + \omega_l)/2$:

$$h_{bp}[n] = 2\frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)} \cos \omega_0 n.$$

$H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$ and $H_{bs}(e^{j\omega}) = 1 - H_{bp}(e^{j\omega})$.

$$h_{hp}[n] = \delta[n] - h_{lp}[n],$$
$$h_{bs}[n] = \delta[n] - h_{bp}[n],$$

Ideal filters are used in the early stages of a design process to specify the modules in a signal processing system. However, since they are not realizable in practice, they must be approximated by *practical* or *nonideal* filters. This is usually done by minimizing some approximation error between the nonideal filter and a prototype ideal filter.

Ideal and practical filters

We can obtain a causal FIR filter by truncating the impulse response of the ideal lowpass filter as follows

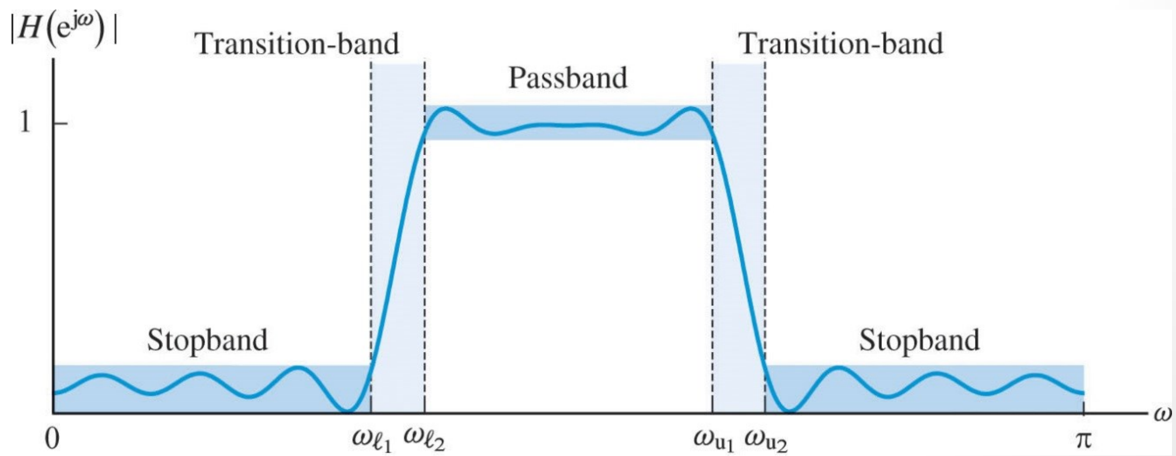
$$\hat{h}_{\mathrm{lp}}[n] = \begin{cases} \dfrac{\sin \omega_{\mathrm{c}}(n - n_{\mathrm{d}})}{\pi (n - n_{\mathrm{d}})}, & 0 \le n \le M - 1 \\ 0. & \text{otherwise} \end{cases}$$

As the delay n_{d} and the length M of $\hat{h}_{\mathrm{lp}}[n]$ increase, the resulting filter $\widehat{H}_{\mathrm{lp}}(e^{j\omega})$ will be a better approximation of the ideal lowpass filter.

***Transition* bands**: the filter neither passes nor rejects the input.

A good filter should have

- only a small ripple in the passband,
- high attenuation in the stopband, and
- very narrow transition bands.



Typical characteristics of a practical bandpass filter.

Frequency response for rational system functions

All LTI systems of practical interest are described by a difference equation of the form

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k],$$

and a rational system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}.$$

$$\rightarrow \qquad H(e^{j\omega}) = \left.\frac{B(z)}{A(z)}\right|_{z=e^{j\omega}} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}},$$

in terms of poles and zeros
$\rightarrow$

$$H(e^{j\omega}) = b_0 \left.\frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}\right|_{z=e^{j\omega}} = b_0 \frac{\prod_{k=1}^{M} (1 - z_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})},$$

Frequency response for rational system functions

$$H(e^{j\omega}) = b_0 \left. \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} \right|_{z=e^{j\omega}} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega})},$$

The magnitude, phase, and group-delay responses are given by

$$|H(e^{j\omega})| = |b_0| \prod_{k=1}^{M} \left|1 - z_k e^{-j\omega}\right| \Big/ \prod_{k=1}^{N} \left|1 - p_k e^{-j\omega}\right|,$$

$$\angle H(e^{j\omega}) = \angle b_0 + \sum_{k=1}^{M} \angle(1 - z_k e^{-j\omega}) - \sum_{k=1}^{N} \angle(1 - p_k e^{-j\omega}),$$

$$\tau_{gd}(\omega) = -\sum_{k=1}^{M} \frac{d}{d\omega}\left[\angle(1 - z_k e^{-j\omega})\right] + \sum_{k=1}^{N} \frac{d}{d\omega}\left[\angle(1 - p_k e^{-j\omega})\right]$$

Frequency response for rational system functions

Expressing the zeros and poles in polar notation as $z_k = q_k\, e^{j\theta k}$ and $p_k = r_k\, e^{j\varphi k}$

$$|H(e^{j\omega})| = |b_0| \left[\frac{\prod_{k=1}^{M} \sqrt{1 + q_k^2 - 2q_k \cos(\omega - \theta_k)}}{\prod_{k=1}^{N} \sqrt{1 + r_k^2 - 2r_k \cos(\omega - \phi_k)}} \right],$$

$$\angle H(e^{j\omega}) = \angle b_0 + \sum_{k=1}^{M} \tan^{-1}\left(\frac{q_k \sin(\omega - \theta_k)}{1 - q_k \cos(\omega - \theta_k)} \right)$$

$$- \sum_{k=1}^{N} \tan^{-1}\left(\frac{r_k \sin(\omega - \phi_k)}{1 - r_k \cos(\omega - \phi_k)} \right),$$

$$\tau_{\mathrm{gd}}(\omega) = \sum_{k=1}^{N} \frac{r_k^2 - r_k \cos(\omega - \phi_k)}{1 + r_k^2 - 2r_k \cos(\omega - \phi_k)} - \sum_{k=1}^{M} \frac{q_k^2 - q_k \cos(\omega - \theta_k)}{1 + q_k^2 - 2q_k \cos(\omega - \theta_k)}.$$

These equations explicitly show the influence of each individual pole or zero on the magnitude, phase, and group-delay responses of the system.

Frequency response for rational system functions

Computation of group delay

$$H(e^{j\omega}) = H_R(\omega) + jH_I(\omega) = G(\omega)e^{j\Psi(\omega)},$$

$$\tilde{H}(\omega) \triangleq \ln H(e^{j\omega}) = \ln G(\omega) + j\Psi(\omega).$$

Differentiating both sides

$$\tilde{H}'(\omega) = \frac{H'(e^{j\omega})}{H(e^{j\omega})} = \frac{G'(\omega)}{G(\omega)} + j\Psi'(\omega),$$

$$\tau_{gd}(\omega) = -\Psi'(\omega) = -\mathcal{I}m\{\tilde{H}'(\omega)\} = -\mathcal{I}m\left\{\frac{H'(e^{j\omega})}{H(e^{j\omega})}\right\}$$

The derivative of $H(e^{j\omega})$ is determined as the DTFT of the sequence $nh[n]$,

$$H_n(e^{j\omega}) = \mathrm{DTFT}\{nh[n]\} = jH'(e^{j\omega})$$

$$\tau_{gd}(\omega) = -\mathcal{I}m\left\{\frac{H'(e^{j\omega})}{H(e^{j\omega})}\right\} = \mathcal{I}m\left\{j\frac{H_n(e^{j\omega})}{H(e^{j\omega})}\right\} = \mathcal{R}e\left\{\frac{H_n(e^{j\omega})}{H(e^{j\omega})}\right\}$$

Frequency response for rational system functions

Example:

$$H(z) = \frac{1 + 1.655z^{-1} + 1.655z^{-2} + z^{-3}}{1 - 1.57z^{-1} + 1.264z^{-2} - 0.4z^{-3}},$$

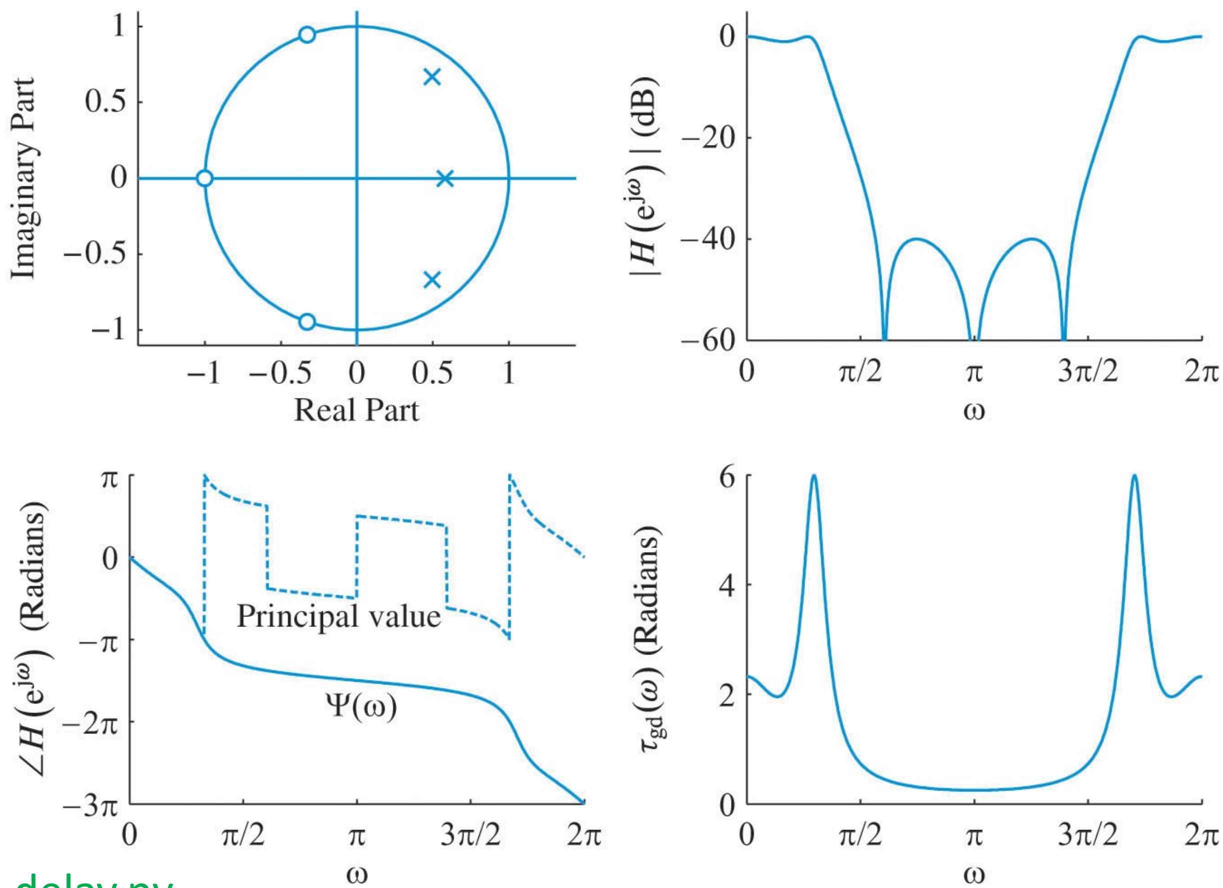


Fig5.15_group_delay.py

Pole-zero plot, magnitude response, phase response (principal value and continuous phase functions), and group delay of the system.

Dependence of frequency response on poles and zeros

There is a strong dependence of the shape of the frequency response on the location of poles and zeros of the system.
We can use this dependence

(a) to obtain a simple and intuitive procedure for determining quickly the magnitude and phase response, and
(b) to gain physical insight into the filtering characteristics of LTI systems.

Geometrical evaluation of $H(e^{j\omega})$ from poles and zeros

We can write

$$H(e^{j\omega}) = b_0 \left[\frac{\prod_{k=1}^{M} (1 - z_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})} \right] = b_0 e^{j\omega(N-M)} \left[\frac{\prod_{k=1}^{M} (e^{j\omega} - z_k)}{\prod_{k=1}^{N} (e^{j\omega} - p_k)} \right]$$

Dependence of frequency response on poles and zeros

The complex number $(e^{j\omega} - z_k)$ can be written in polar form as follows:

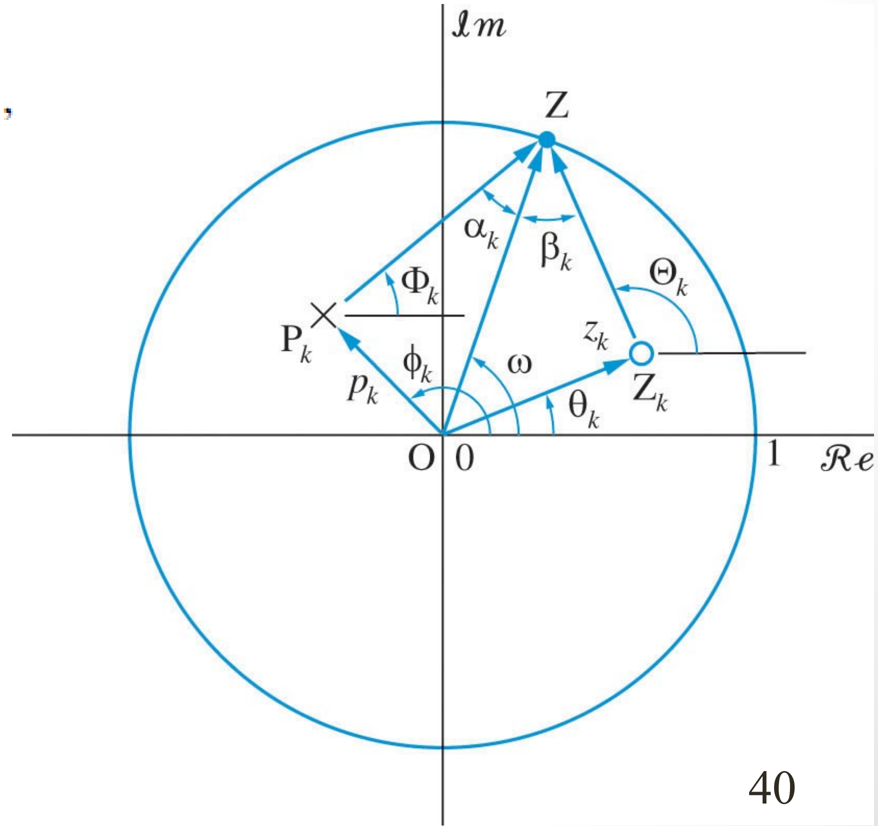
$$\left(e^{j\omega} - z_k\right) = \overrightarrow{Z_k Z} = Q_k e^{j\Theta_k}$$

where Q_k is the distance from the zero z_k to the point $e^{j\omega}$, and Θ_k is the angle of the vector $\overrightarrow{Z_k Z}$ with the (horizontal) positive real axis.

Similarly, the factor $(e^{j\omega} - p_k)$ is a complex number that can be expressed as

$$\left(e^{j\omega} - p_k\right) = \overrightarrow{P_k Z} = R_k e^{j\Phi_k},$$

where R_k is the distance from the pole p_k to the point $e^{j\omega}$ and k the angle of $\overrightarrow{P_k Z}$ with the positive real axis.



Dependence of frequency response on poles and zeros

$$H(e^{j\omega}) = |b_0| \frac{\prod_{k=1}^{M} Q_k(\omega)}{\prod_{k=1}^{N} R_k(\omega)} \times \exp\left[\angle b_0 + \omega(N-M) + \sum_{k=1}^{M} \Theta_k(\omega) - \sum_{k=1}^{N} \Phi_k(\omega)\right]$$

$$b_0 < 0 \quad \rightarrow \quad \angle b_0 = \pi \text{ rads}$$
$$b_0 \geq 0 \quad \rightarrow \quad \angle b_0 = 0$$

The magnitude and phase responses are easily obtained

$$|H(e^{j\omega})| = |b_0| \frac{\prod_{k=1}^{M} Q_k(\omega)}{\prod_{k=1}^{N} R_k(\omega)},$$

$$\angle H(e^{j\omega}) = \angle b_0 + \omega(N-M) + \sum_{k=1}^{M} \Theta_k(\omega) - \sum_{k=1}^{N} \Phi_k(\omega),$$

where ω is the angle of the point $z = e^{j\omega}$ with the positive real axis and

$Q_k(\omega)$ = distance of kth zero from $z = e^{j\omega}$,
$R_k(\omega)$ = distance of kth pole from $z = e^{j\omega}$,
$\Theta_k(\omega)$ = angle of the vector $\overrightarrow{Z_k Z}$ with the positive real axis,
$\Phi_k(\omega)$ = angle of the vector $\overrightarrow{P_k Z}$ with the positive real axis.

Digital Signal Processing, Fall 2019

Dependence of frequency response on poles and zeros

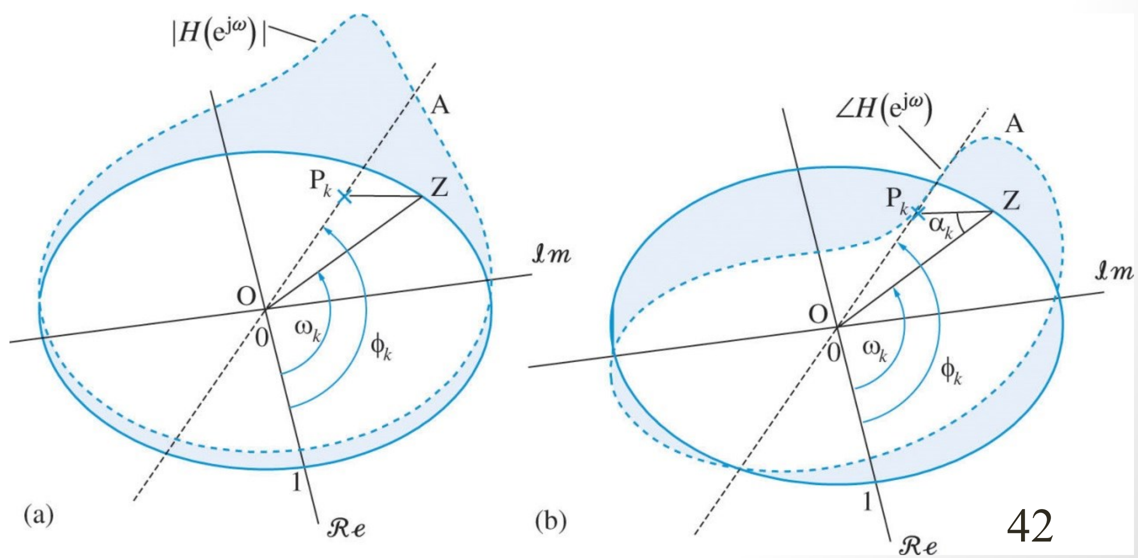
Significance of poles and zeros

Gain enhancement by a pole

Consider a pole $p_k = r_k\, e^{j\varphi k}$. connect the pole (point P_k) to the tip of vector $z = e^{j\omega}$ (point Z on the unit circle). If the length of this line is $R_k(\omega)$, then

$$|H(e^{j\omega})| = \frac{\kappa}{\overline{(P_k Z)}} = \frac{\kappa}{R_k(\omega)},$$

where overbar denotes the length of a vector. The exact value of constant κ is not important at this point.



Dependence of frequency response on poles and zeros

Significance of poles and zeros

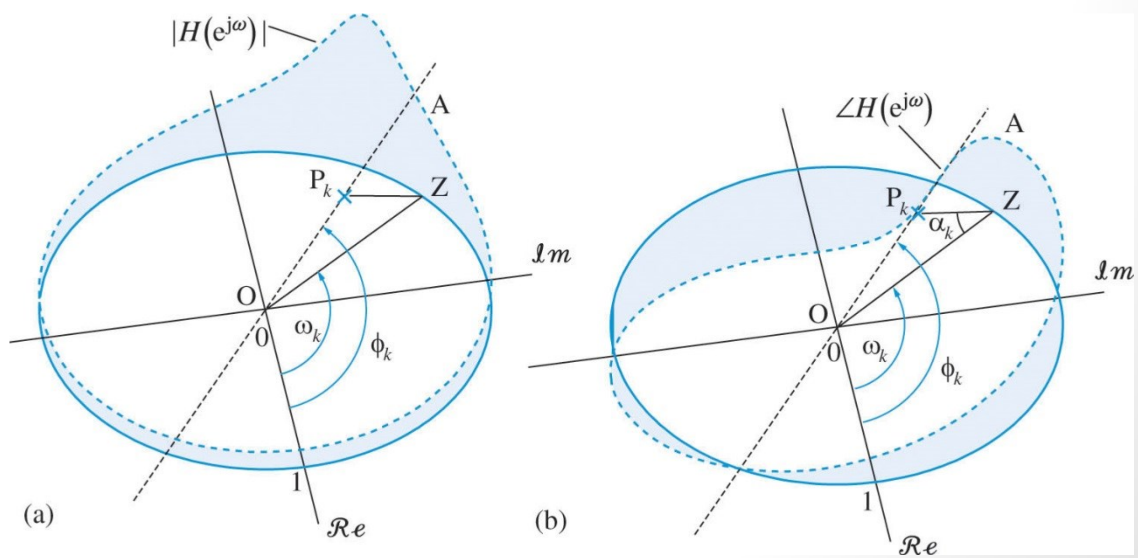
Gain enhancement by a pole

The line segment P_kZ takes its minimum value $1 - r_k$ at $\omega = \varphi_k$, and its maximum value $1 + r_k$ at $\omega = \varphi_k + \pi$.

Therefore, the length P_kZ increases progressively as ω increases from φ_k to $\varphi_k + \pi$ and then decreases continuously until ω approaches the value φ_k.

Then, $|H(e^{j\omega})|$ decreases as ω goes from φ_k to $\varphi_k + \pi$ and then progressively increases as ω moves closer to φ_k.

We conclude that a pole $p_k = r_k\, e^{j\varphi k}$ results in a frequency-selective response that enhances the gain around $\omega = \varphi_k$ (angle of the pole) and attenuates the gain as we move away from φ_k.



Dependence of frequency response on poles and zeros

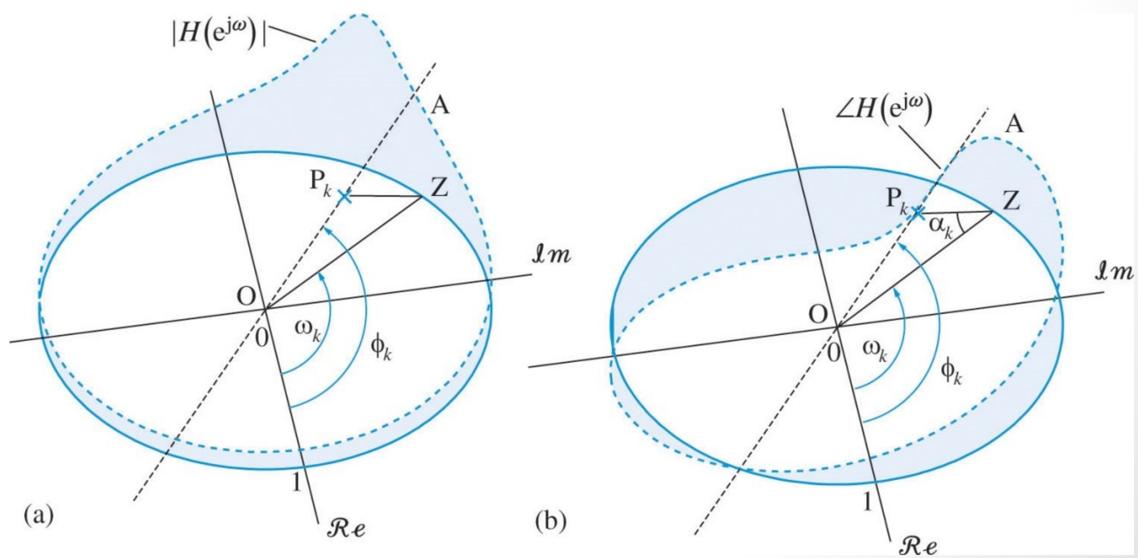
Significance of poles and zeros

Gain enhancement by a pole

The dynamic range of the magnitude response

$$\frac{|H(e^{j\omega})|_{\max}}{|H(e^{j\omega})|_{\min}} = \frac{1+r_k}{1-r_k}$$

increases as the pole is moved closer to the unit circle. As a result, the peak of $H(e^{j\omega})$ at $\omega = \varphi_k$ becomes sharper as the pole approaches the unit circle. The maximum gain $|H(\varphi_k)|$ goes to infinity as the pole moves on the unit circle. However, this should be avoided, because causal LTI systems with poles on or outside the unit circle are unstable.



Dependence of frequency response on poles and zeros

Significance of poles and zeros

Gain enhancement by a pole

$$\angle H(e^{j\omega}) = \angle b_0 + \omega(N-M) + \sum_{k=1}^{M} \Theta_k(\omega) - \sum_{k=1}^{N} \Phi_k(\omega), \longrightarrow \angle H(e^{j\omega}) = \omega - \Phi_k(\omega)$$

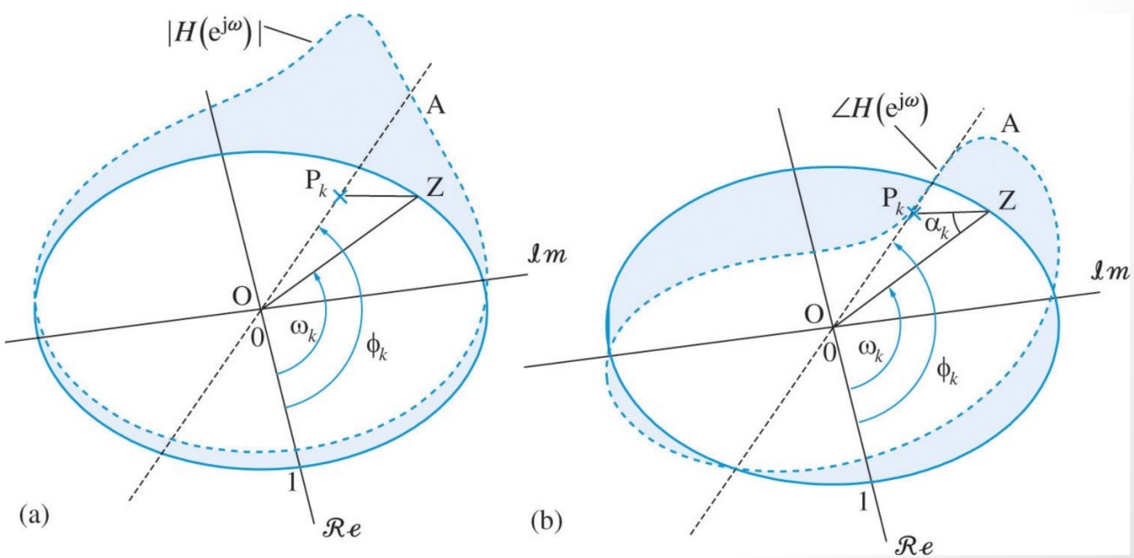
The angles of the triangle OP_kZ sum to π

$$\longrightarrow \quad \alpha_k + (\phi_k - \omega) + (\Phi_k + \pi - \phi_k) = \pi$$

$$\longrightarrow \quad \omega - \Phi_k = \alpha_k$$

$$\longrightarrow \quad \angle H(e^{j\omega}) = \omega - \Phi_k(\omega) = \alpha_k.$$

It is not easy to obtain a reasonably accurate shape for the phase response with this approach.



Dependence of frequency response on poles and zeros

Significance of poles and zeros

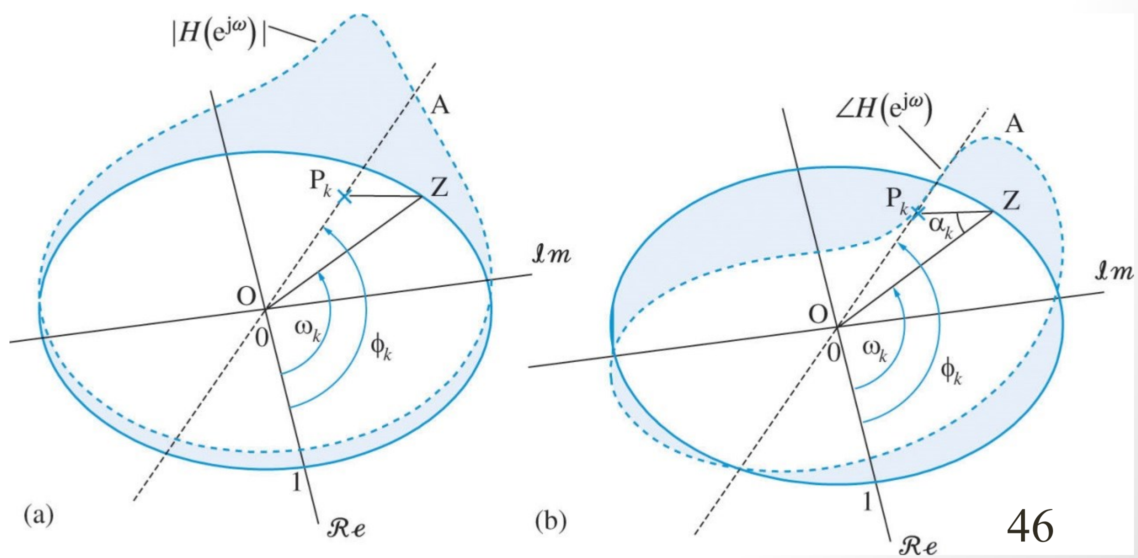
In general, we have

$$|H(e^{j\omega})| = \frac{\kappa}{(\overline{P_1 Z_1})} \neq \frac{\kappa}{(\overline{P_1 Z_2})} = \left|H(e^{-j\omega})\right|$$

$$\angle H(e^{j\omega}) = \alpha_1 \neq \alpha_2 = -\angle H(e^{-j\omega})$$

Two ways to enforce the symmetry

- If the pole P_k moves on the real axis we have $\overline{P_1 Z_1} = \overline{P_1 Z_2}$ and $\alpha_1 = \alpha_2$. Therefore $|H(e^{j\omega})| = |H(e^{-j\omega})|$ (even) and $\angle H(e^{j\omega}) = -\angle H(e^{-j\omega})$ (odd).



Dependence of frequency response on poles and zeros

Significance of poles and zeros

In general, we have

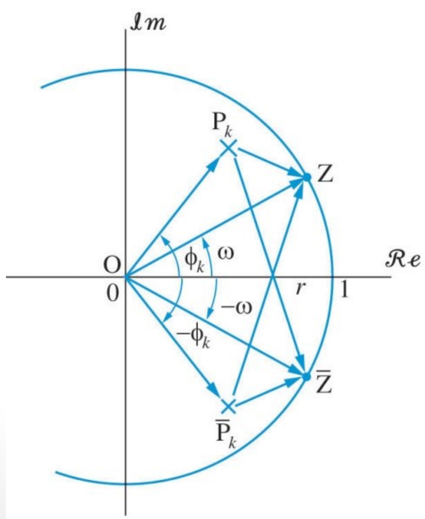
$$|H(e^{j\omega})| = \frac{\kappa}{(\overline{P_1Z_1})} \neq \frac{\kappa}{(\overline{P_1Z_2})} = \left|H(e^{-j\omega})\right|$$

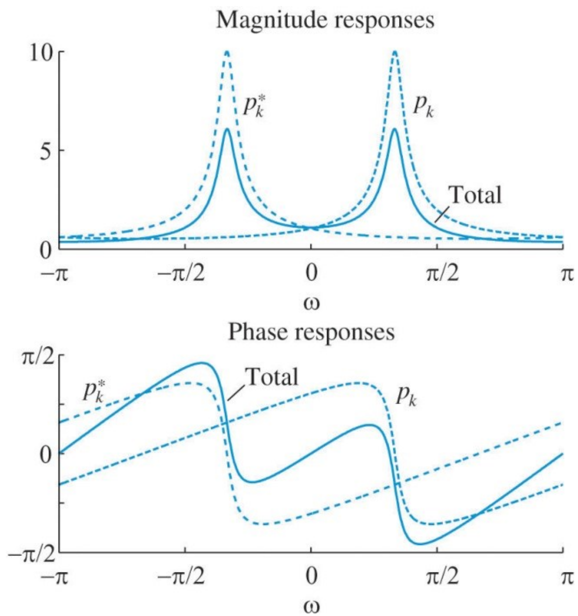
$$\angle H(e^{j\omega}) = \alpha_1 \neq \alpha_2 = -\angle H(e^{-j\omega})$$

Two ways to enforce the symmetry

- If the pole P_k moves on the real axis we have $\overline{P_1Z_1} = \overline{P_1Z_2}$ and $\alpha_1 = \alpha_2$. Therefore $|H(e^{j\omega})| = |H(e^{-j\omega})|$ (even) and $\angle H(e^{j\omega}) = -\angle H(e^{-j\omega})$ (odd).
- To place a complex conjugate pole at $p_k^* = r_k e^{-j\phi_k}$

$$\rightarrow \quad |H(e^{j\omega})| = \frac{\kappa}{(\overline{P_1Z})(\overline{P_2Z})} = \frac{\kappa}{\left(\overline{P_1\bar{Z}}\right)\left(\overline{P_2\bar{Z}}\right)} = \left|H(e^{-j\omega})\right|$$





Design of simple filters by pole-zero placement

The strong dependence between the locations of poles and zeros and the frequency response can be used to design simple filters with desired frequency response.

Guidelines:

- To suppress a frequency component at $\omega = \omega_0$, we should place a zero at angle $\theta = \omega_0$ on the unit circle.
- To enhance or amplify a frequency component at $\omega = \omega_0$, we should place a pole at angle $\varphi = \omega_0$ close but inside the unit circle.
- Complex poles or zeros should appear in complex conjugate pairs to assure that the system has real coefficients.
- Poles or zeros at the origin do not influence the magnitude response because their distance from any point on the unit circle is unity. However, a pole (or zero) at the origin adds (or subtracts) a linear phase of ω rads to the phase response. We often introduce poles and zeros at $z = 0$ to assure that $N = M$.