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Fourier representation of signals

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Fourier representation of signals

The goal of Fourier analysis of signals is to break up all signals into summations of sinusoidal components.

Continuous-time sinusoids

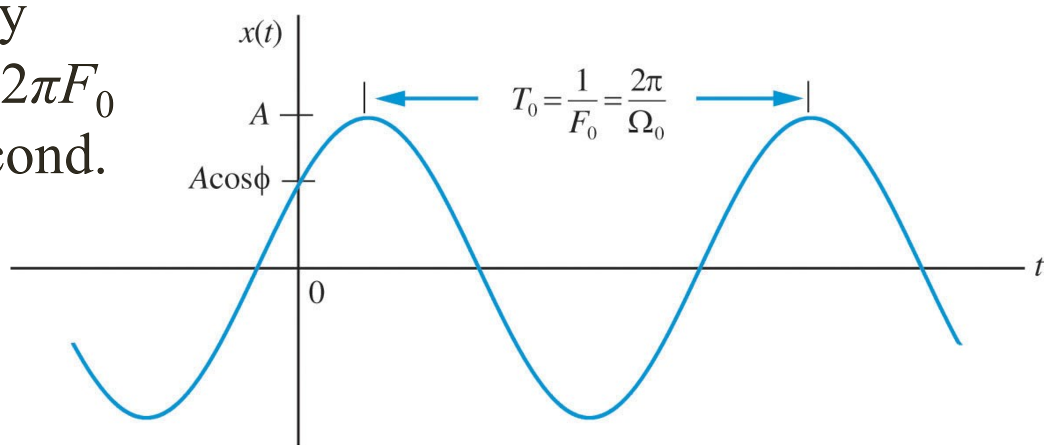
$$x(t) = A\cos(2\pi F_0 t + \theta), -\infty < t < \infty$$

where A is the amplitude, θ is the phase in radians, and F_0 is the frequency (in cycles per second or Hertz (Hz)).

In analytical manipulations it is more convenient to use the *angular* or *radian* frequency

$$\Omega_0 = 2\pi F_0$$

measured in radians per second.



Fourier representation of signals

Euler's identity, $e^{\pm j\varphi} = \cos\varphi \pm j\sin\varphi$, can be used to express the sinusoidal signals as:

$$A\cos(\Omega_0 t + \theta) = A/2\, e^{j\theta} e^{j\Omega_0 t} + A/2\, e^{-j\theta} e^{-j\Omega_0 t}$$

Therefore the properties of the sinusoidal signals can be studied by studying the properties of the complex exponential $x(t) = e^{j\Omega t}$.
The response $y(t)$ of LTI systems to the input $x(t)$ is:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{\mathrm{j}\Omega(t-\tau)}\mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{\mathrm{j}\Omega t}\mathrm{e}^{-\mathrm{j}\Omega\tau}\mathrm{d}\tau = \left(\int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-\mathrm{j}\Omega\tau}\mathrm{d}\tau\right)\mathrm{e}^{\mathrm{j}\Omega t}.$$

Thus, the response to $e^{j\Omega t}$ is of the form

$$y(t) = H(j\Omega)\, e^{j\Omega t}. \qquad -\infty < t < \infty$$

Fourier representation of signals

A set of *harmonically related* complex exponential signals, with fundamental frequency $\Omega_0 = 2\pi/T_0 = 2\pi F_0$, is defined by

$$s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi k F_0 t}. \quad k = 0, \pm 1, \pm 2, \ldots$$

$s_1(t)$ and $s_k(t)$ are called the *fundamental* and the kth *harmonic* of the set, respectively.

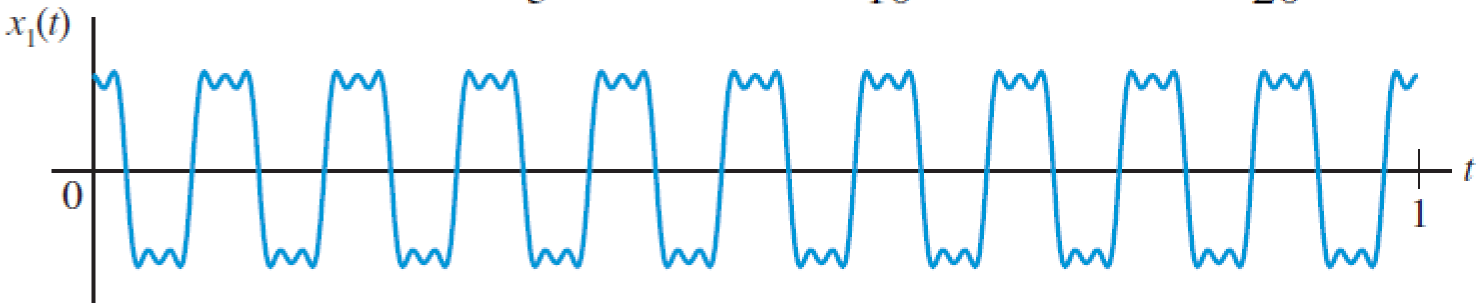
Orthogonality property

$$\int_{T_0} s_k(t) s_m^*(t) \mathrm{d}t = \int_{T_0} e^{jk\Omega_0 t} e^{-jm\Omega_0 t} \mathrm{d}t = \begin{cases} T_0, & k = m \\ 0, & k \neq m \end{cases}$$

Fourier representation of signals

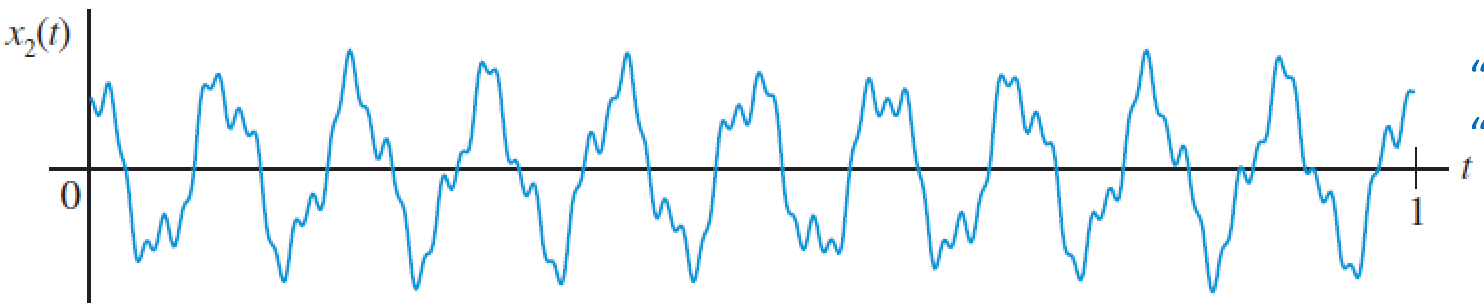
Example: a periodic signal composed of three sinusoids with harmonically related frequencies (F_0 = 10 Hz)

$$x_1(t) = \frac{1}{3}\cos(2\pi F_0 t) - \frac{1}{10}\cos(2\pi 3F_0 t) + \frac{1}{20}\cos(2\pi 5F_0 t),$$



Example: The frequencies of the three sinusoids are *not* harmonically related

$$x_2(t) = \frac{1}{3}\cos(2\pi F_0 t) - \frac{1}{10}\cos\left(2\pi\sqrt{8}F_0 t\right) + \frac{1}{20}\cos\left(2\pi\sqrt{51}F_0 t\right)$$



“almost”-periodic or “quasi”-periodic

Fourier representation of signals

Discrete-time sinusoids

are obtained by sampling the continuous-time sinusoid at equally spaced points $t = nT$

$$x[n] = x(nT) = A\cos(2\pi F_0 nT + \theta) = A\cos\left(2\pi \frac{F_0}{F_s} n + \theta\right)$$

Normalized frequency: $$f \triangleq \frac{F}{F_s} = FT,$$

Normalized angular frequency: $$\omega \triangleq 2\pi f = 2\pi \frac{F}{F_s} = \Omega T,$$

$$x[n] = A\cos(2\pi f_0 n + \theta) = A\cos(\omega_0 n + \theta), \quad -\infty < n < \infty$$

If the input to a discrete-time LTI system is a complex exponential sequence, its output is a complex exponential with the same frequency.

$$x[n] = e^{j\omega n} \xmapsto{\mathcal{H}} y[n] = H(e^{j\omega})e^{j\omega n}, \quad \text{for all } n,$$

Fourier representation of signals

Continuous-time: $x(t) = A\cos(2\pi F t + \theta)$

Unit of

t : *seconds*

2π : *radians per cycle*

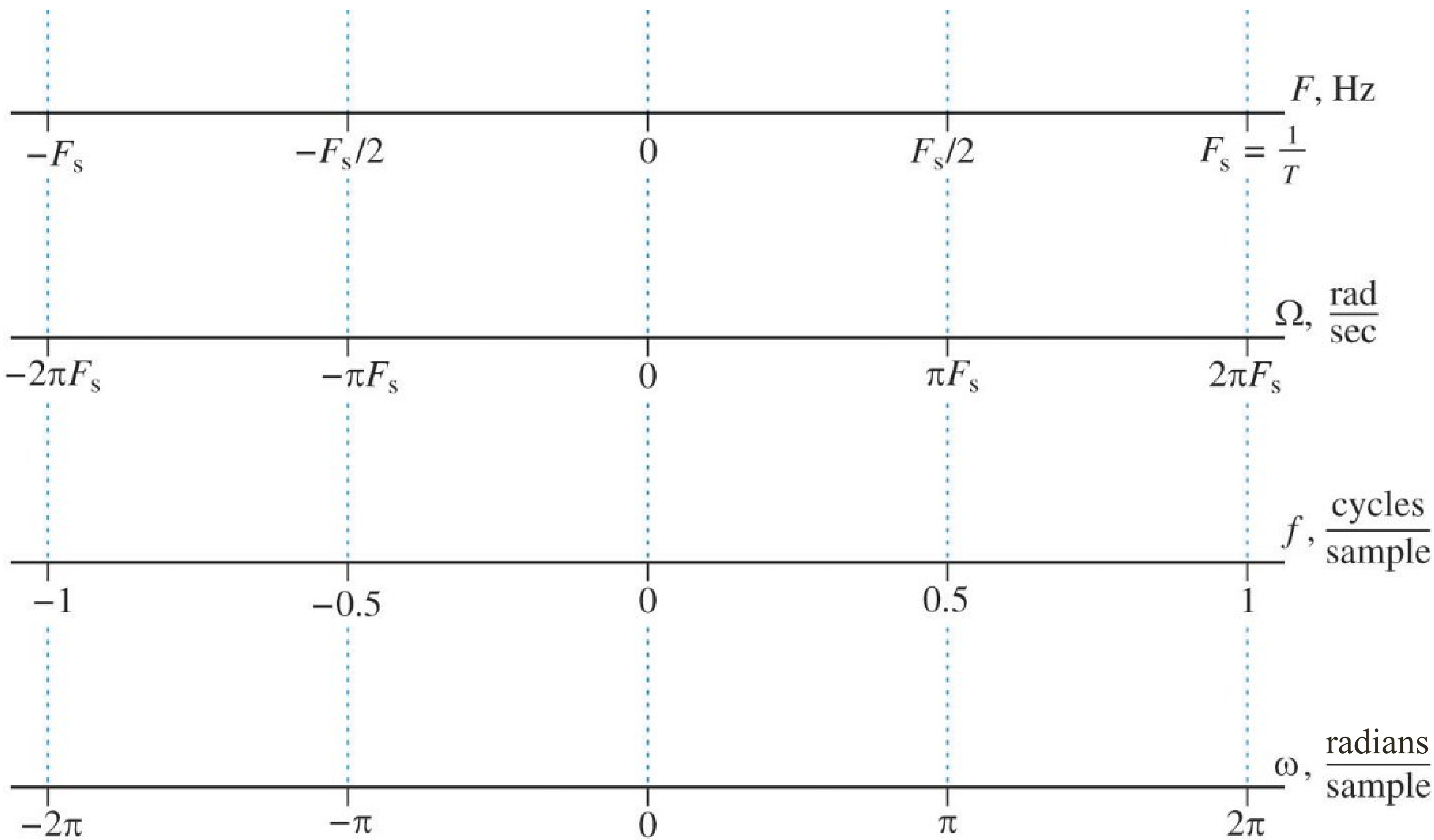
$2\pi F t$: *radians*

→

Unit of

F : *cycles per second* or Hertz (Hz)

$\Omega = 2\pi F$: *radians per second*



Fourier representation of signals

Discrete-time: $x[n] = x(nT) = A\cos(2\pi FnT + \theta)$

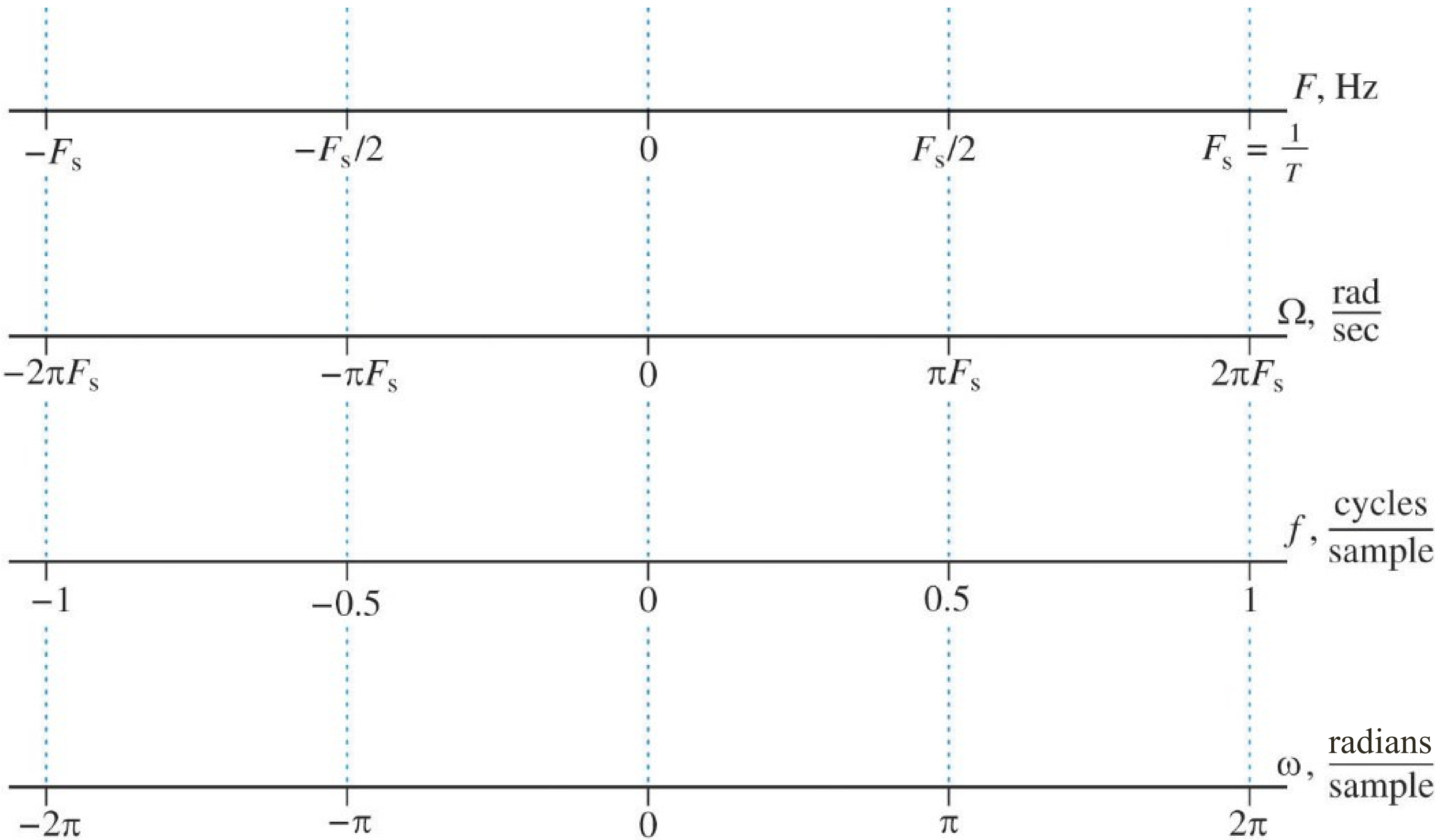
Unit of

$$n : \ \textit{samples}$$
$$2\pi : \textit{radians per cycle}$$
$$2\pi FnT : \textit{radians}$$

$\longrightarrow$

Unit of

$$f : \textit{cycles per sample}$$
$$\omega = 2\pi FT : \textit{radians per sample}$$



Fourier representation of continuous-time signals

Fourier series for continuous-time periodic signals

A periodic signal *x(t)* can be synthesized using a linear combination of the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t},$$

where the coefficients c_k are constants. Each term in the summation is periodic with period $T_0 = 2\pi/\Omega_0$.

What is the relation between the coefficients c_k and the function *x(t)*?

- multiply both sides by $e^{-jm\Omega_0 t}$,
- change the order of integration with summation,
- integrate over a full period,
- simplify the result

$$\rightarrow \qquad c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\Omega_0 t} dt.$$

Fourier representation of continuous-time signals

Fourier series for continuous-time periodic signals

Continuous-Time Fourier Series (CTFS) **pair**

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t} \quad \xleftrightarrow{\text{CTFS}} \quad c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\Omega_0 t} dt.$$

The set of coefficients $\{c_k\}$ are known as the ***Fourier series coefficients***.

$$c_k = |c_k| e^{j\angle c_k}.$$

The plot of $|c_k|$ is known as the ***magnitude spectrum*** of $x(t)$, while the plot of $\angle c_k$ is known as the ***phase spectrum*** of $x(t)$. If c_k is real-valued, we can use a single plot, known as the ***amplitude spectrum***.

The plot of $x(t)$ as a function of time t (waveform) provides a description of the signal in the time-domain.
The plot of c_k as a function of frequency $F = kF_0$ (*spectrum*) constitutes a description of the signal in the frequency-domain.

Fourier representation of continuous-time signals

Continuous-Time Fourier Series (CTFS) pair

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t} \quad \xleftrightarrow{\text{CTFS}} \quad c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\Omega_0 t}dt.$$

Parseval's relation

The average power in one period of *x(t)* can be expressed in terms of the Fourier coefficients using Parseval's relation

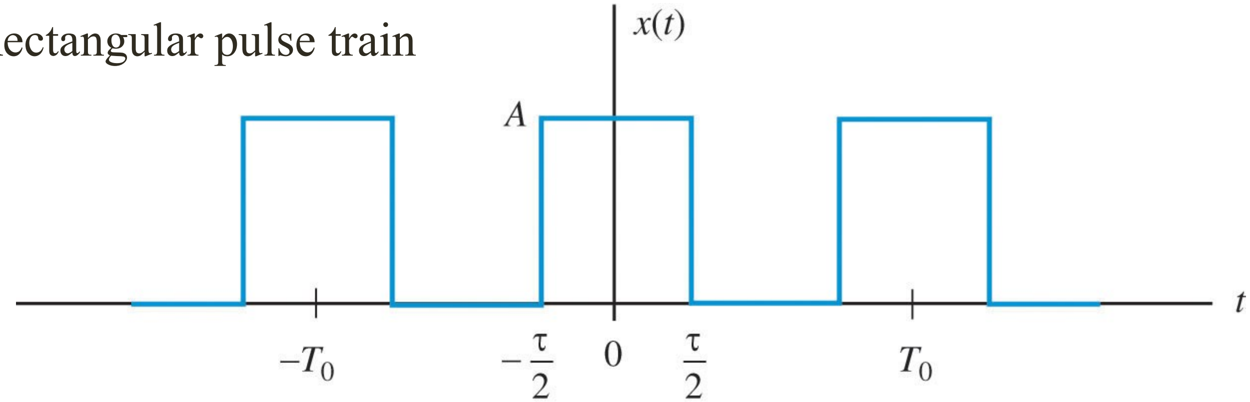
$$P_{\text{av}} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

The graph of $|c_k|^2$ as a function of $F = kF_0$ is known as the ***power spectrum*** of the periodic signal *x(t)*.

Because the power is distributed at a set of discrete frequencies, we say that periodic continuous-time signals have *discrete* or *line* spectra.

Fourier representation of continuous-time signals

Example: Rectangular pulse train



$$c_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t} dt = \frac{A}{T_0}\left[\frac{e^{-j2\pi k F_0 t}}{-j2\pi k F_0}\right]_{-\tau/2}^{\tau/2}$$

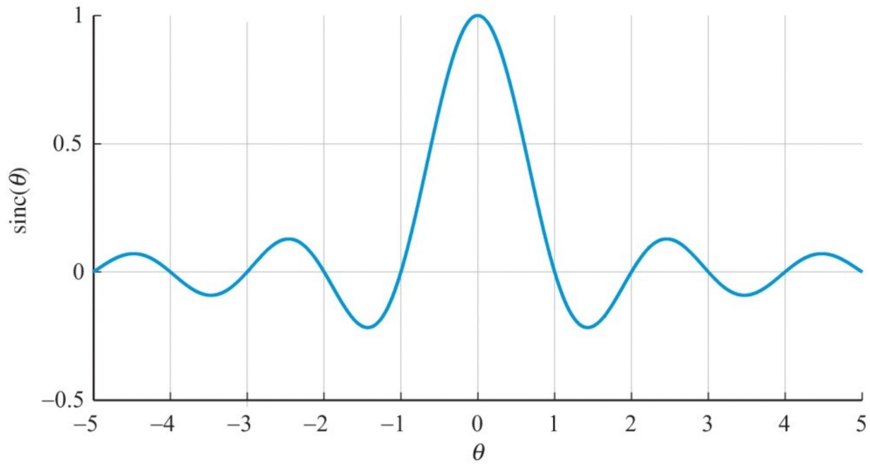
$$= \frac{A}{\pi F_0 k T_0} \frac{e^{j\pi k F_0 \tau} - e^{-j\pi k F_0 \tau}}{2j}$$

$$= \frac{A\tau}{T_0} \frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}. \quad k = 0, \pm 1, \pm 2, \ldots$$

Fourier representation of continuous-time signals

Sinc function

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}.$$



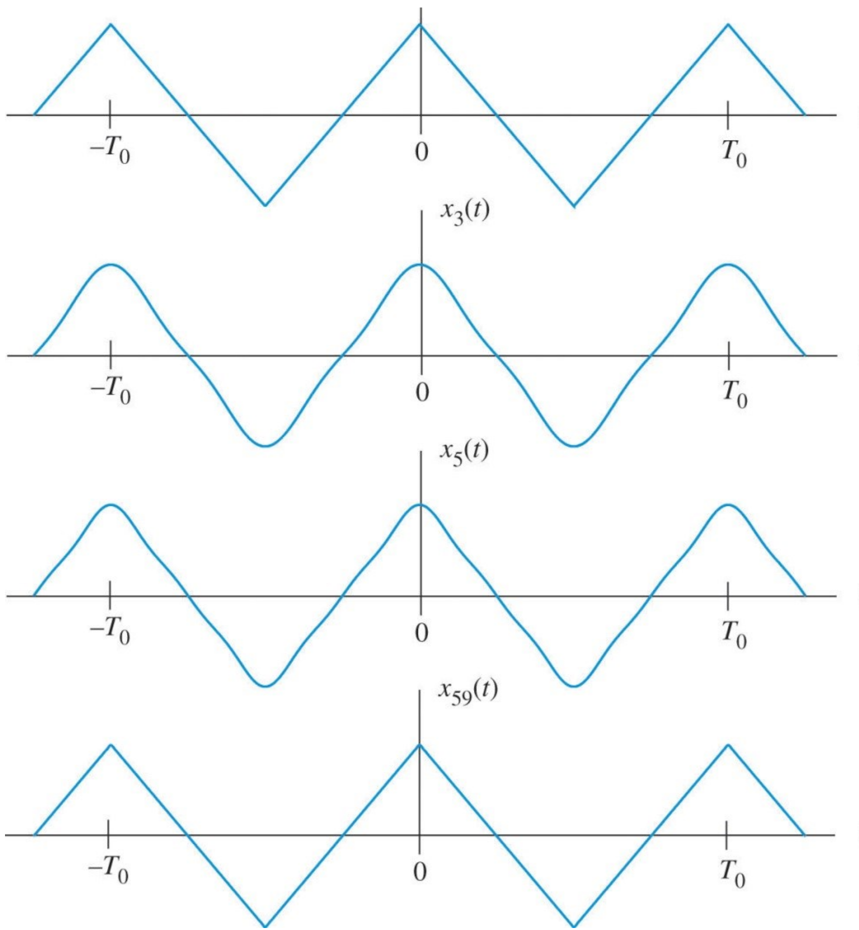
Sinc function properties:

1. The sinc function is an even function of θ, that is, $\mathrm{sinc}(-\theta) = \mathrm{sinc}(\theta)$.
2. $\mathrm{sinc}(\theta) = 0$ when $\sin\theta = 0$, except at $\theta = 0$, where it appears indeterminate. This means that $\mathrm{sinc}(\theta) = 0$ when $\theta = \pm 1, \pm 2, \ldots$
3. It can be shown that $\mathrm{sinc}(0) = 1$.
4. $\mathrm{sinc}(\theta)$ is the product of the periodic function $\sin(\pi\theta)$ with the monotonically decreasing function $1/(\pi\theta)$. Hence, $\mathrm{sinc}(\theta)$ exhibits sinusoidal oscillations of period $\theta=2$ with amplitude decreasing continuously at $1/(\pi\theta)$.

Fourier representation of continuous-time signals

Approximation error

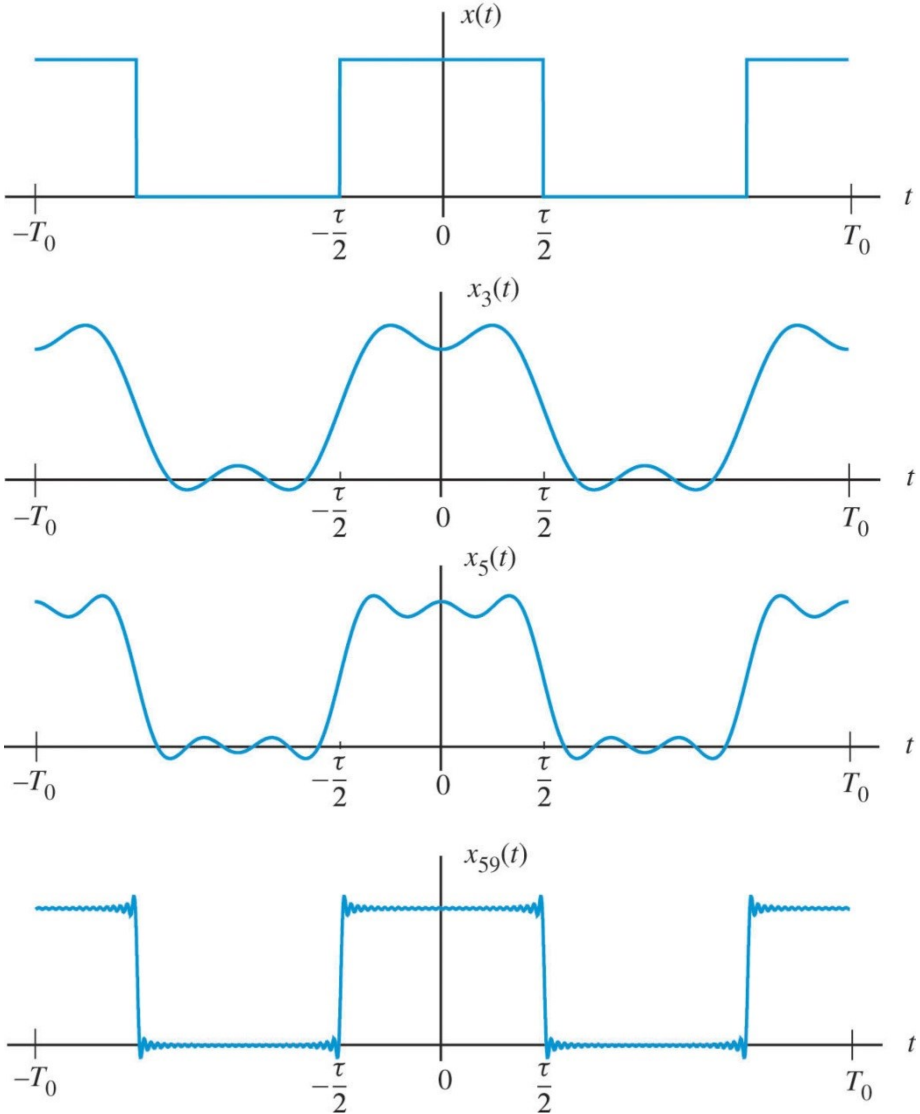
$$e_m(t) = x(t) - x_m(t) = x(t) - \sum_{k=-m}^{m} c_k e^{jk\Omega_0 t}$$



Fourier representation of continuous-time signals

Since the values of the function *x(t)* are *not* defined at $\pm\tau/2$, (points of discontinuity), the Fourier series does not converge at these points.

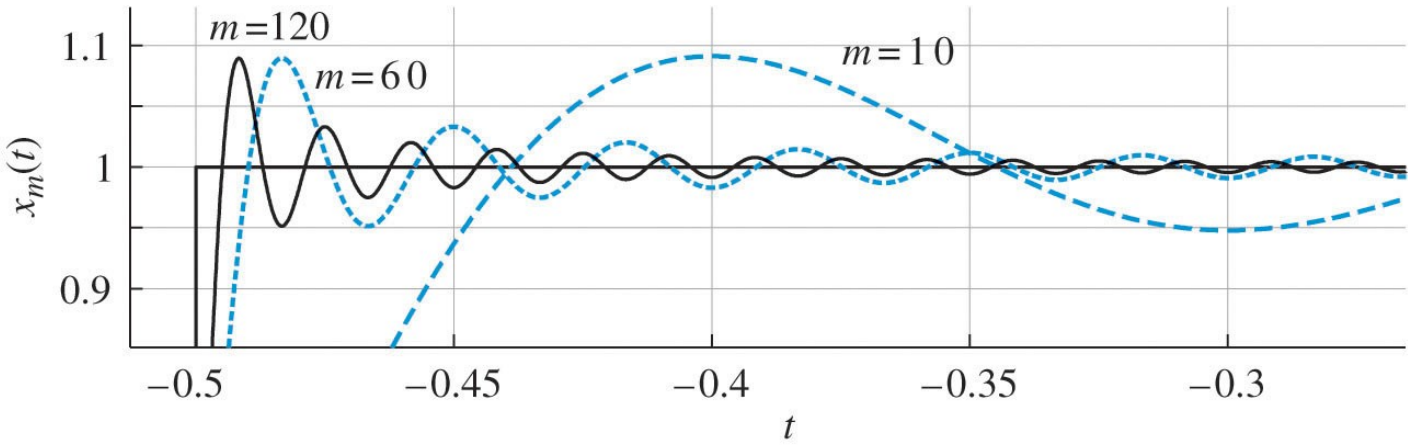
However, the Fourier series handles such situations in a very reasonable way: it converges to the average of the left- and right-hand limits at the jump.



Fourier representation of continuous-time signals

Gibbs phenomenon

The behavior of the Fourier series at the vicinity of points of discontinuity.



The partial sum, even for large values of m, exhibits an oscillatory overshoot with period $T_0/(2m)$ and peak value of about 9 percent of the height of the jump.

As m increases, the ripples are squeezed closer to the discontinuity and the area "under the ripples" decreases; eventually, the area becomes zero as $m \to \infty$. However, the size of the overshoot does not decrease and remains the same for any finite value of m.

Fourier representation of continuous-time signals

From Fourier series to Fourier transform

Any aperiodic signal can be considered as a periodic signal with infinite period. Therefore the concept of Fourier series can be generalized for Fourier representation of aperiodic signals.

Example:

Assume $x(t)$ is defined over one period and is repeated with period T_0

$$x(t) = \begin{cases} A, & |t| < \tau \\ 0, & \tau < |t| < T_0/2 \end{cases}$$

The Fourier coefficients are given by

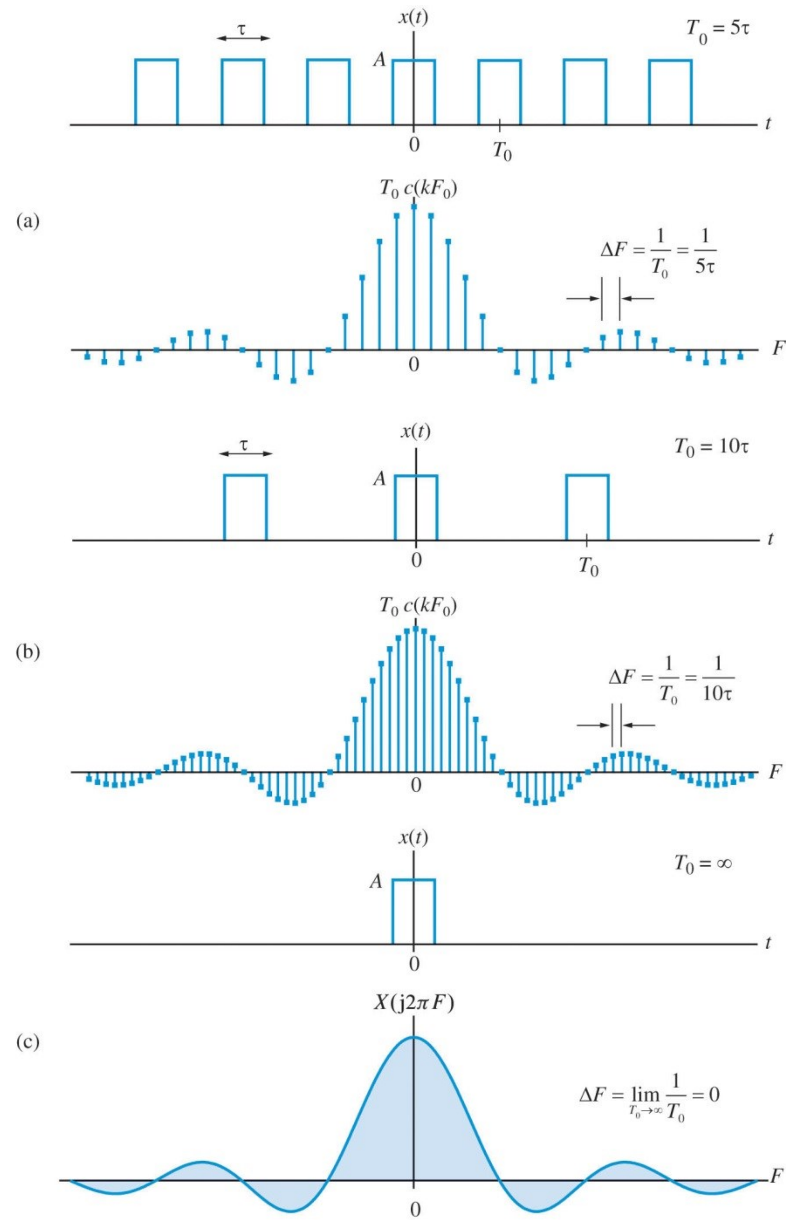
$$c_k = \frac{A\tau}{T_0} \frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau} \triangleq c(kF_0).$$

The size of the coefficients c_k depends on the period T_0, i.e., $c_k \rightarrow 0$ as $T_0 \rightarrow \infty$.

$$c(kF_0)T_0 = A\tau \left. \frac{\sin \pi F \tau}{\pi F \tau} \right|_{F=kF_0}$$

Fourier representation of continuous-time signals

$$c(kF_0)T_0 = A\tau \left.\frac{\sin \pi F\tau}{\pi F\tau}\right|_{F=kF_0}$$



As $T_0 \to \infty$

- **in the time domain**
  the result is an aperiodic signal corresponding to one period of the rectangular pulse train,
- **in the frequency domain**
  the result is a "continuum" of spectral lines.

Fourier representation of continuous-time signals

$x(t)$ is a signal with finite duration, i.e, $x(t) = 0$ for $|t| > \tau/2$.
Repeat $x(t)$ with period $T_0 > \tau$ to create a periodic signal $x_p(t)$. The Fourier series representation of $x_p(t)$ is:

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}, \qquad c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_p(t) e^{-j2\pi k F_0 t} dt.$$

$x(t) = x_p(t)$ for $|t| < T_0/2$ and $x(t) = 0$ for $|t| > T_0/2$

$$\rightarrow \qquad c(kF_0)T_0 = \int_{-\infty}^{\infty} x(t) e^{-j2\pi k F_0 t} dt.$$

$F = kF_0 \qquad \rightarrow \qquad X(j2\pi F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt,$

The function $X(j2\pi F)$ which is basically the envelope of $c_k T_0$, is called the *Fourier transform* or *Fourier integral* of $x(t)$.

$$c_k = \frac{X(j2\pi k F_0)}{T_0} = F_0\, X(j2\pi F)|_{F=kF_0} = X(j2\pi K \Delta F)\Delta F.$$

The Fourier coefficients c_k of a periodic signal $x_p(t)$ are proportional to uniformly spaced samples of the Fourier transform of one period of $x_p(t)$.

Fourier representation of continuous-time signals

Inverse Fourier transform.

For every signal $x(t)$ that is equal to $x_p(t)$ over exactly one period,

$$x(t) = \begin{cases} x_p(t), & t_0 < t < t_0 + T_0 \\ 0. & \text{otherwise} \end{cases}$$

$$\left.\begin{aligned} x_p(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}, \\ c_k &= X(j2\pi k \Delta F)\,\Delta F \end{aligned}\right\} \rightarrow \quad x_p(t) = \sum_{k=-\infty}^{\infty} X(j2\pi k\Delta F)\, e^{j2\pi k \Delta F t} \Delta F.$$

$$T_0 \rightarrow \infty \quad \rightarrow \quad \begin{cases} x_p(t) \rightarrow x(t) \\ \Delta F \rightarrow 0 \end{cases} \quad \rightarrow \quad x(t) = \int_{-\infty}^{\infty} X(j2\pi F)\, e^{j2\pi F t} dF,$$

Fourier representation of continuous-time signals

Continuous-Time Fourier Transform (CTFT)

Fourier Synthesis Equation | **Fourier Analysis Equation**

$$x(t) = \int_{-\infty}^{\infty} X(\mathrm{j}2\pi F)\, \mathrm{e}^{\mathrm{j}2\pi F t} \mathrm{d}F \xleftrightarrow{\text{CTFT}} X(\mathrm{j}2\pi F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t} \mathrm{d}t,$$

X(j2πF) is the ***spectrum*** of the aperiodic signal *x(t)*.

- **Periodic** signals must have **discrete spectra** with lines at harmonically related frequencies; otherwise they cannot be periodic.
- A **continuous spectrum** results in an **aperiodic** signal because almost all frequencies in a continuous interval are not harmonically related.

Fourier representation of continuous-time signals

Example: Causal exponential signal

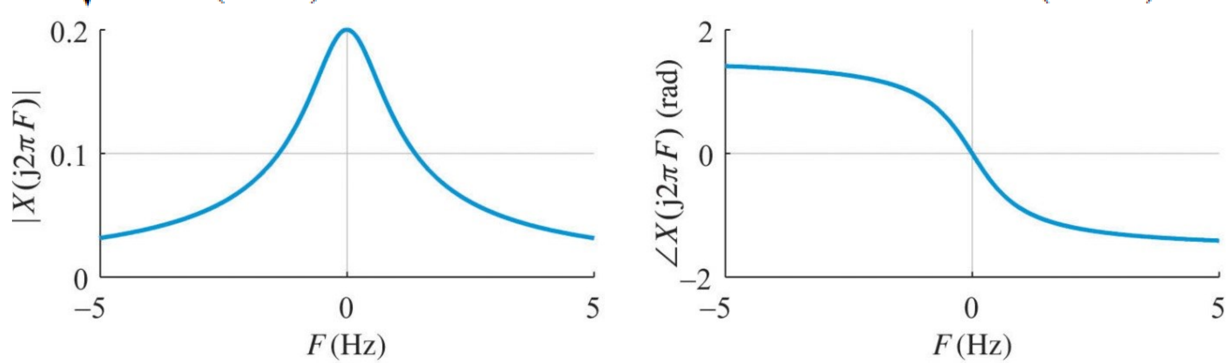
$$x(t) = \begin{cases} e^{-at}, & t > 0 \\ 0. & t < 0 \end{cases}$$

If $a > 0$

$$X(j2\pi F) = \int_0^{\infty} e^{-at} e^{-j2\pi F t} dt = -\frac{1}{a + j2\pi F} e^{-(a+j2\pi F)t} \Big|_0^{\infty}$$

$$\rightarrow \quad X(j2\pi F) = \frac{1}{a + j2\pi F} \quad \text{or} \quad X(j\Omega) = \frac{1}{a + j\Omega}. \quad a > 0$$

$$|X(j2\pi F)| = \frac{1}{\sqrt{a^2 + (2\pi F)^2}}, \qquad \angle X(j2\pi F) = -\tan^{-1}\left(2\pi \frac{F}{a}\right). \quad a > 0$$



Fourier transform of the signal $x(t) = e^{-at}u(t)$ for $a = 5$. (a) Magnitude and (b) phase of $X(j2\pi F)$ in the finite interval $-5 < F < 5$ (Hz).

Fourier representation of continuous-time signals

When the Fourier coefficients are real, we can plot c_k on a single graph. However, for consistency, we plot the magnitude and phase spectra.

To obtain these magnitude and phase spectra, we use the following general conventions:

- Phase angles are always measured with respect to cosine waves. Thus, sine waves have a phase of $-\pi/2$ since $\sin \Omega t = \cos(\Omega t - \pi/2)$.

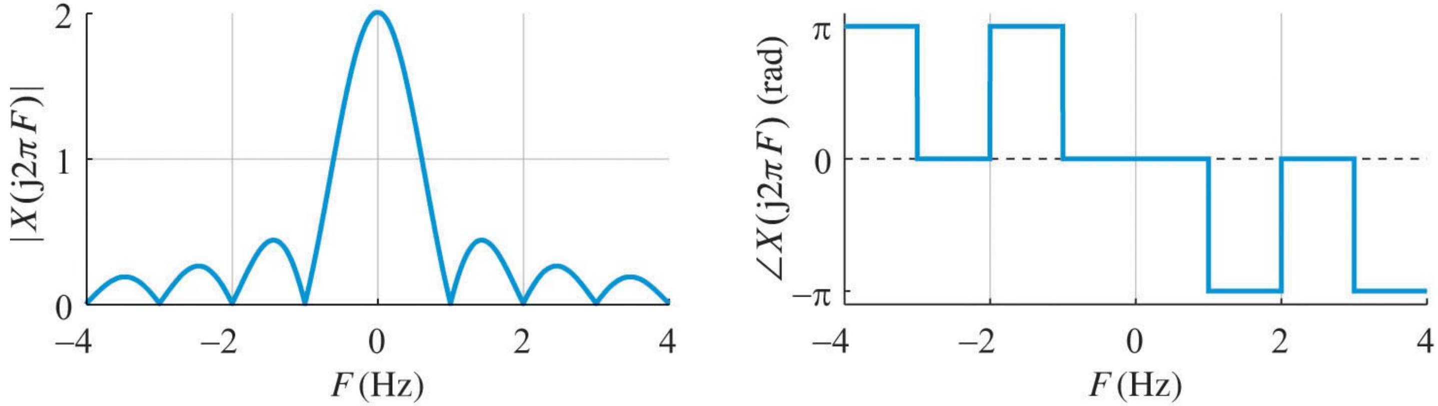
- Magnitude spectra are always positive. Hence, negative signs should be absorbed in the phase using the identity: $-A \cos \Omega t = \cos(\Omega t \pm \pi)$. It does not matter whether we take $+\pi$ or $-\pi$ because $\cos(-\pi) = \cos \pi$. However, we use both $+\pi$ and $-\pi$ to bring out the odd symmetry of the phase.

Fourier representation of continuous-time signals

Example: Rectangular pulse signal

$$x(t) = \begin{cases} A, & |t| < \tau/2 \\ 0. & |t| > \tau/2 \end{cases}$$

$$\rightarrow \qquad X(j2\pi F) = \int_{-\tau/2}^{\tau/2} A e^{-j2\pi F t} dt = A\tau \frac{\sin(\pi F \tau)}{\pi F \tau}.$$



(a) (b)

Fourier representation of continuous-time signals

Example: Multiplying a periodic with an aperiodic signal

Consider an aperiodic signal *x(t)* with Fourier transform *X(j2πF)* and a periodic signal *s(t)* with fundamental frequency F_0 and Fourier coefficients c_k. The product $x_s(t) = x(t)s(t)$ is clearly an aperiodic signal.

$$X_s(\mathrm{j}2\pi F) = \int_{-\infty}^{\infty} x(t)\left[\sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{j}2\pi F_0 k t}\right]\mathrm{e}^{-\mathrm{j}2\pi F t}\mathrm{d}t$$

$$= \sum_{k=-\infty}^{\infty} c_k\left[\int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-\mathrm{j}2\pi(F-kF_0)t}\mathrm{d}t\right].$$

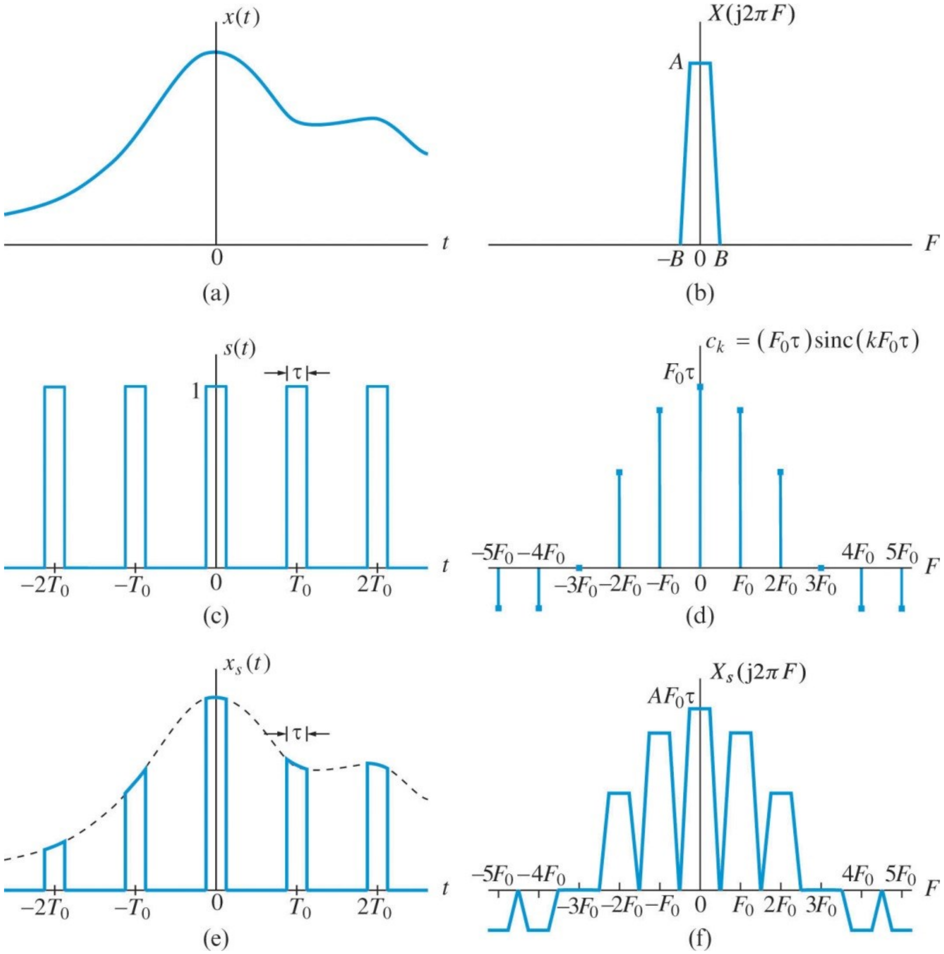
$$X_s(\mathrm{j}2\pi F) = \sum_{k=-\infty}^{\infty} c_k X[\mathrm{j}2\pi(F-kF_0)].$$

The spectrum of $x_s(t)$ is obtained by putting copies of *X(j2πF)*, scaled by c_k, at integer multiples of F_0 and then adding them together.

Fourier representation of continuous-time signals

Example: Multiplying a periodic with an aperiodic signal

If $X(j2\pi F) = 0$ for $|F| > B$ and $F_0 > 2B$, we have $X(j2\pi F) = X_s(j2\pi F)/c_0$ for $|F| < B$, therefore $x(t)$ can be recovered from $x_s(t)$ using the inverse Fourier transform.



Fourier representation of discrete-time signals

Fourier series for discrete-time periodic signals

Consider a linear combination of N harmonically related complex exponentials

$$x[n] = \sum_{k=0}^{N-1} c_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$$

$x[n]$ is periodic with fundamental period N and our objective is to determine the coefficients c_k from the values of the periodic signal $x[n]$

Using orthogonality property:

$$\sum_{n=\langle N\rangle} s_k[n]s_m^*[n] = \sum_{n=\langle N\rangle} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}mn} = \begin{cases} N, & k=m \\ 0, & k\neq m \end{cases}$$

$$s_k[n] = \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}, \quad -\infty < k, n < \infty$$

Fourier representation of discrete-time signals

Fourier series for discrete-time periodic signals

By multiplying both sides of $$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$$

by $e^{-j(2\pi/N)mn}$ and summing from $n = 0$ to $n = N - 1$

$$\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}mn} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}(k-m)n}$$

$$\rightarrow \quad \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}mn} = \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

$$\rightarrow \quad c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Fourier representation of discrete-time signals

Fourier series for discrete-time periodic signals

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

Provides the closed-form expression for obtaining the Fourier series coefficients required by the Fourier series.
This results in the *Discrete-Time Fourier Series (DTFS)* pair:

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn} \quad \xleftrightarrow{\text{DTFS}} \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}.$$

Parseval's relation

$$P_{\text{av}} = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2.$$

The graph of $|c_k|^2$ as a function of $f = k/N$, $\omega = 2\pi k/N$, or simply k, is known as the ***power spectrum*** of the periodic signal $x[n]$.

Fourier representation of discrete-time signals

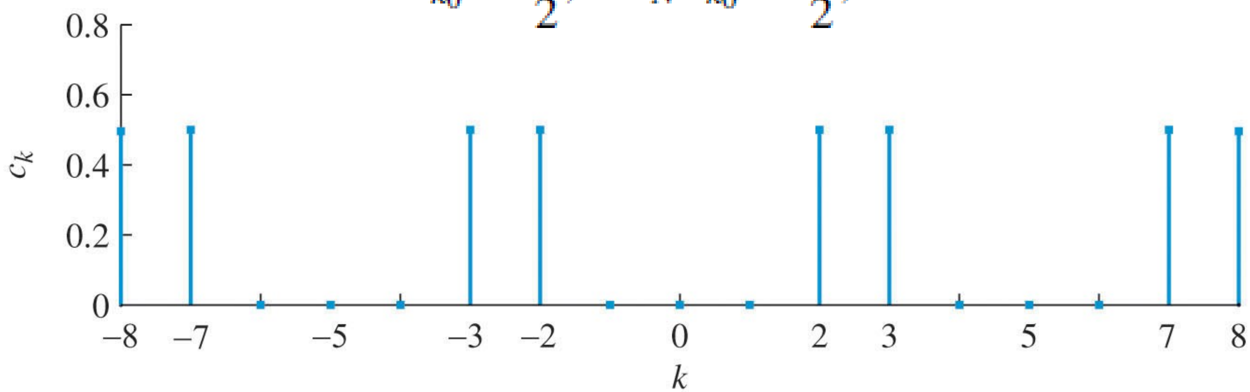
Example: Sinusoidal sequence

$$x[n] = \cos \omega_0 n = \cos 2\pi f_0 n,$$

Suppose that $f_0 = k_0/N$, $0 \le k_0 \le N-1$. Then, $x[n]$ has a DTFS representation.

$$x[n] = \frac{1}{2}e^{j\frac{2\pi}{N}k_0 n} + \frac{1}{2}e^{-j\frac{2\pi}{N}k_0 n} = \frac{1}{2}e^{j\frac{2\pi}{N}k_0 n} + \frac{1}{2}e^{j\frac{2\pi}{N}(N-k_0)n}$$

$$\rightarrow \qquad c_{k_0} = \frac{1}{2}, \quad c_{N-k_0} = \frac{1}{2},$$



Plot of the DTFS of the sinusoidal sequence $x[n] = \cos(2\pi(2/5)n)$. The coefficients outside the fundamental interval $0 \le k \le N-1$ are obtained by periodic repetition.

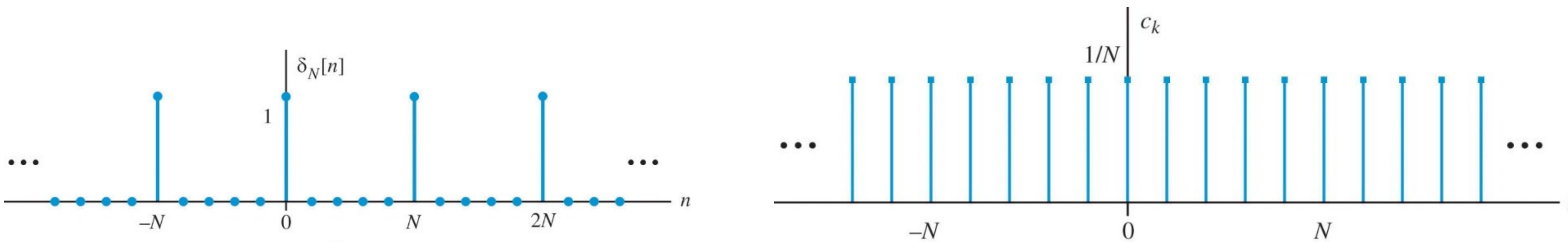
Fourier representation of discrete-time signals

Example: Periodic impulse train

$$\delta_N[n] \triangleq \sum_{\ell=-\infty}^{\infty} \delta[n-\ell N] = \begin{cases} 1, & n = mN,\ m \text{ any integer} \\ 0. & \text{otherwise} \end{cases}$$

Since $\delta_N[n] = \delta[n]$ for $0 \le n \le N-1$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} \delta[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}, \quad \text{all } k \qquad \delta_N[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn}, \quad \text{all } n.$$



(a) (b)

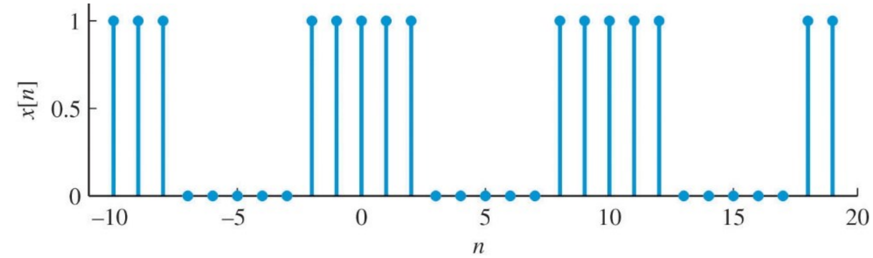
Fourier representation of discrete-time signals

Example: Rectangular pulse train

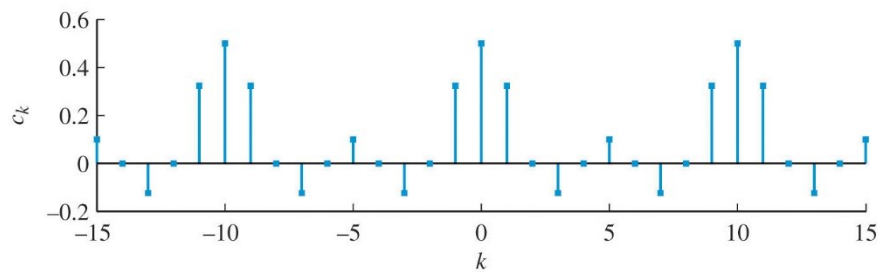
Consider the rectangular pulse train, where $N > 2L+1$.

Due to the even symmetry of $x[n]$, it is convenient to compute the Fourier coefficients using the following summation:

$$c_k = \frac{1}{N}\sum_{n=-L}^{L} e^{-j\frac{2\pi}{N}kn}.$$



(a)



(b)

Changing the index of summation,
from n to $m = n + L$,

$$c_k = \frac{1}{N}\sum_{m=0}^{2L} e^{-j\frac{2\pi}{N}k(m-L)} = \frac{1}{N} e^{j\frac{2\pi}{N}kL}\sum_{m=0}^{2L}\left(e^{-j\frac{2\pi}{N}k}\right)^m$$

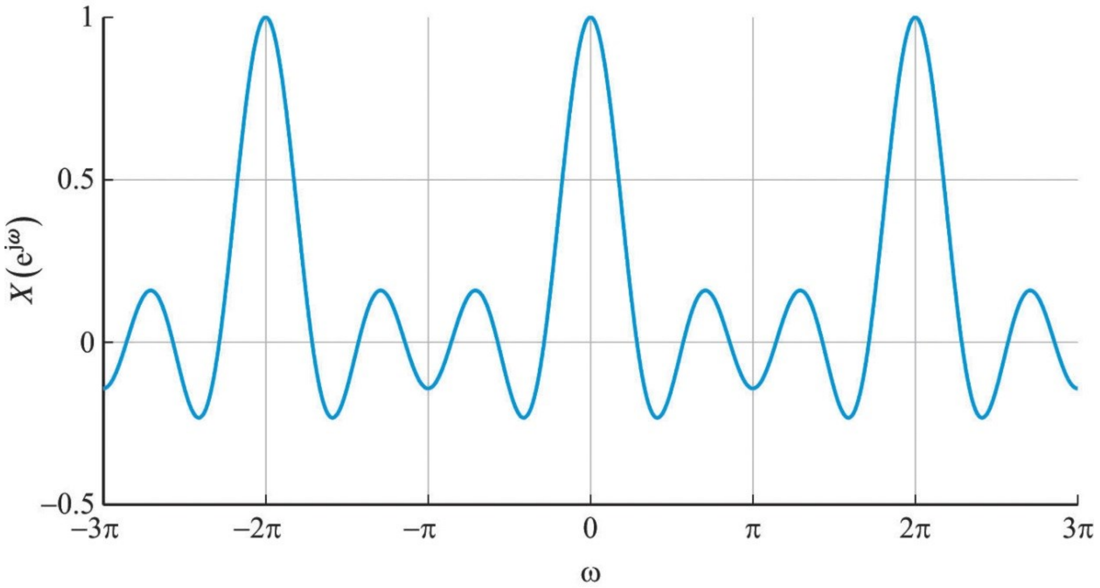
$$(1-e^{-j\theta}) = e^{-j\theta/2}(e^{j\theta/2}-e^{-j\theta/2}) = 2je^{-j\theta/2}\sin(\theta/2).$$

$$c_k = \frac{1}{N} e^{j\frac{2\pi}{N}kL}\left[\frac{1-e^{-j\frac{2\pi}{N}k(2L+1)}}{1-e^{-j\frac{2\pi}{N}k}}\right] = \frac{1}{N}\frac{\sin\left[\frac{2\pi}{N}k\left(L+\frac{1}{2}\right)\right]}{\sin\left(\frac{2\pi}{N}k\frac{1}{2}\right)}, \quad \rightarrow \quad c_k = \begin{cases} \frac{2L+1}{N}, & k = 0, \pm N, \pm 2N, \ldots \\ \frac{1}{N}\frac{\sin\left[\frac{2\pi}{N}k\left(L+\frac{1}{2}\right)\right]}{\sin\left(\frac{2\pi}{N}k\frac{1}{2}\right)}. & \text{otherwise} \end{cases}$$

Fourier representation of discrete-time signals

Dirichlet's function (discrete-time counterpart of the sinc function)

$$D_L(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)},$$



The Dirichlet or "digital sinc" function

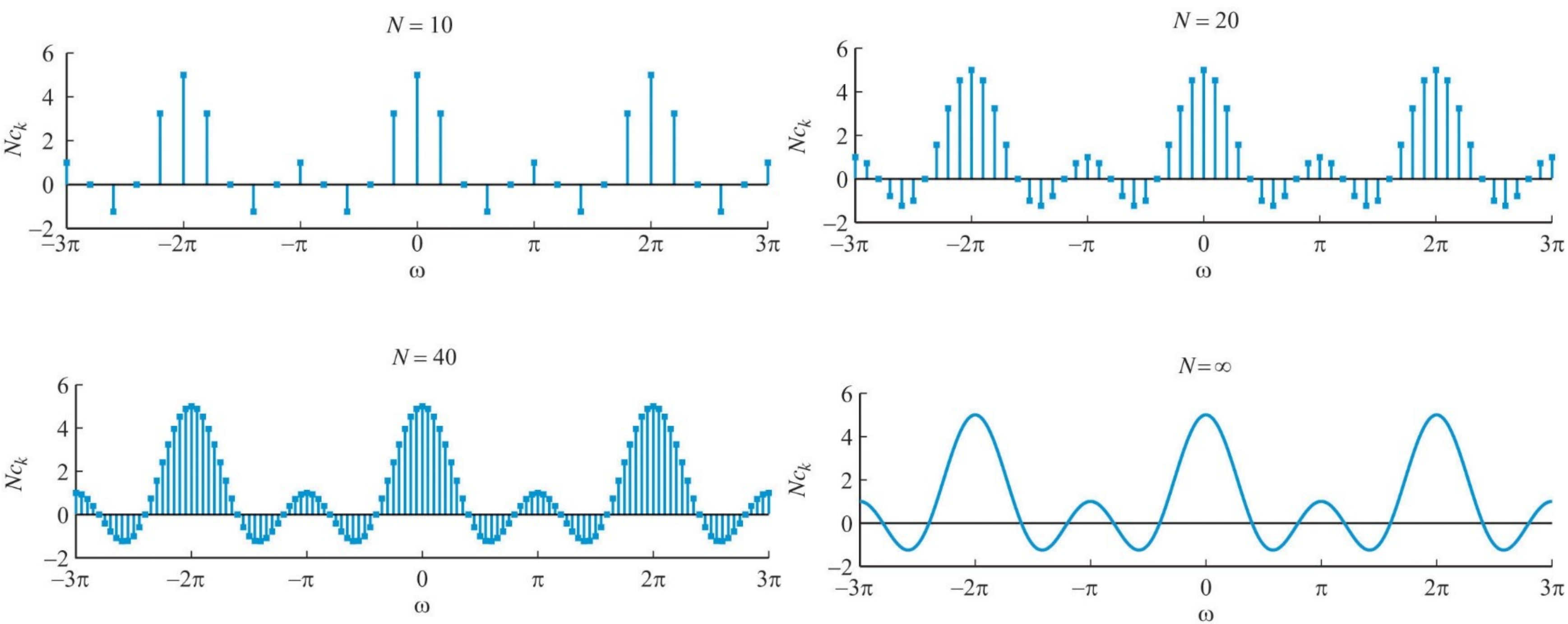
Fourier transforms for discrete-time aperiodic signals

An aperiodic sequence can be viewed as a periodic sequence with infinite period.

Example

Consider the rectangular pulse train $x[n]$ and its DTFS coefficients:

$$c_k = \frac{1}{N}\frac{\sin\left[\frac{2\pi}{N}k\left(L+\frac{1}{2}\right)\right]}{\sin\left(\frac{2\pi}{N}k\frac{1}{2}\right)}.$$



$\omega_k = (2\pi/N)k$ for $N = 10, 20, 40$.

Fourier transforms for discrete-time aperiodic signals

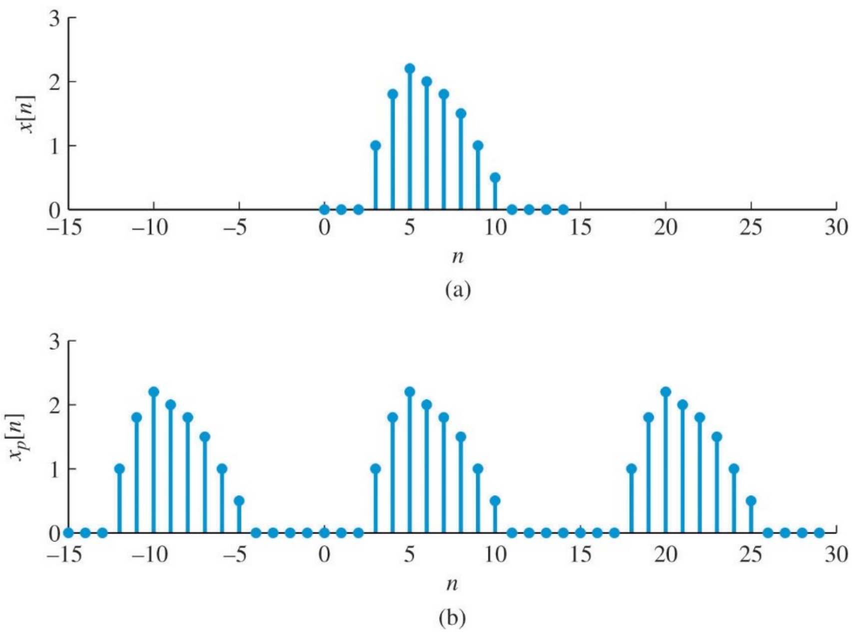
Fourier transforms for discrete-time aperiodic signals

Consider a finite duration sequence $x[n]$, such that $x[n] = 0$ outside the range $-L1 \leq n \leq L2$. we construct a periodic signal $x_p[n]$ by repeating $x[n]$ every $N > L2 + L1 + 1$ samples.

The DTFS of $x_p[n]$ is given by

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn},$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{-j\frac{2\pi}{N}kn}$$



$x_p[n] = x[n]$ for $-L_1 \leq n \leq L_2$ $\quad\rightarrow\quad$ $c_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{2\pi}{N}kn}$

Fourier transforms for discrete-time aperiodic signals

We define the "envelope" function as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The Fourier series coefficients c_k can be obtained by taking equidistant samples of $X(e^{j\omega})$ as follows:

$$c_k = \frac{1}{N} X(e^{j\omega})\Big|_{\omega=k\omega_0}$$

where $\omega_0 = 2\pi/N = \Delta\omega$ is the spacing between successive spectral samples. Using $1/N = \Delta\omega/(2\pi)$, we obtain

$$x_p[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{jk\Delta\omega})e^{j(k\Delta\omega)n}\Delta\omega$$

Fourier transforms for discrete-time aperiodic signals

As $N \to \infty$, $x_p[n] = x[n]$ for any finite n.
As $N \to \infty$, $\Delta\omega \to 0$,
$\omega = k\,\Delta\omega$

$x_p[n] = \dfrac{1}{2\pi} \displaystyle\sum_{k=0}^{N-1} X(e^{jk\Delta\omega}) e^{j(k\Delta\omega)n} \Delta\omega$ passes to the integral of $X(e^{j\omega})e^{j\omega n}$ over the frequency range from 0 to 2π, since $(2\pi/N)(N-1) \to 2\pi$ as $N \to \infty$. Therefore

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Since $X(e^{j\omega})e^{j\omega n}$ is periodic with period 2π, we can use any interval of integration of length 2π.

Fourier transforms for discrete-time aperiodic signals

Discrete-Time Fourier Transform (DTFT) pair

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

The quantities $X(e^{j\omega})$, $|X(e^{j\omega})|$, and $\angle X(e^{j\omega})$ are known as the *spectrum*, *magnitude spectrum*, and *phase spectrum* of the aperiodic sequence $x[n]$, respectively.

Fourier transforms for discrete-time aperiodic signals

Discrete-Time Fourier Transform (DTFT) pair

Fourier Synthesis Equation **Fourier Analysis Equation**

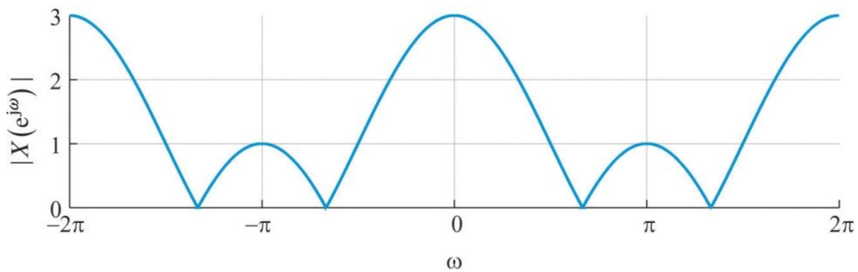
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

Example: Evaluate and plot the magnitude and phase of the DTFT of the sequence $x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$

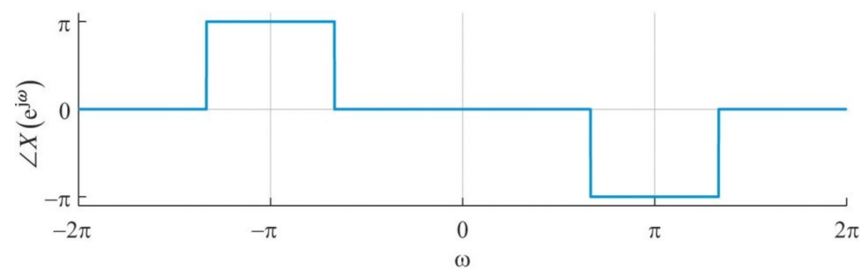
$$X(e^{j\omega}) = \sum_{n=-1}^{1} x[n]e^{-j\omega n} = e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos(\omega)$$

$$|X(e^{j\omega})| = |1 + 2\cos(\omega)|$$

$$\angle X(e^{j\omega}) = \begin{cases} 0, & X(e^{j\omega}) > 0 \\ \pi. & X(e^{j\omega}) < 0 \end{cases}$$



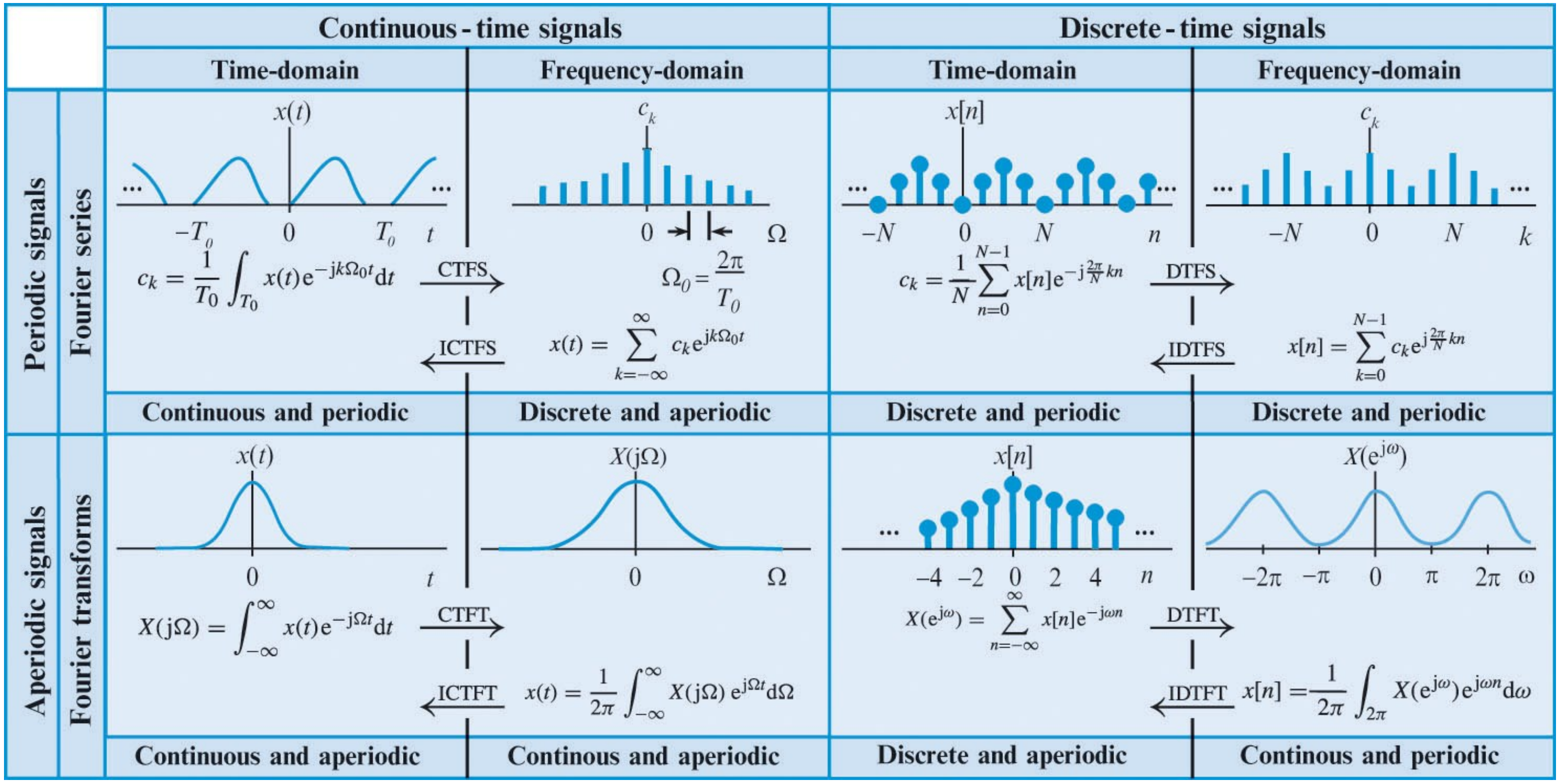
(a)



(b)

Fourier transforms for discrete-time aperiodic signals

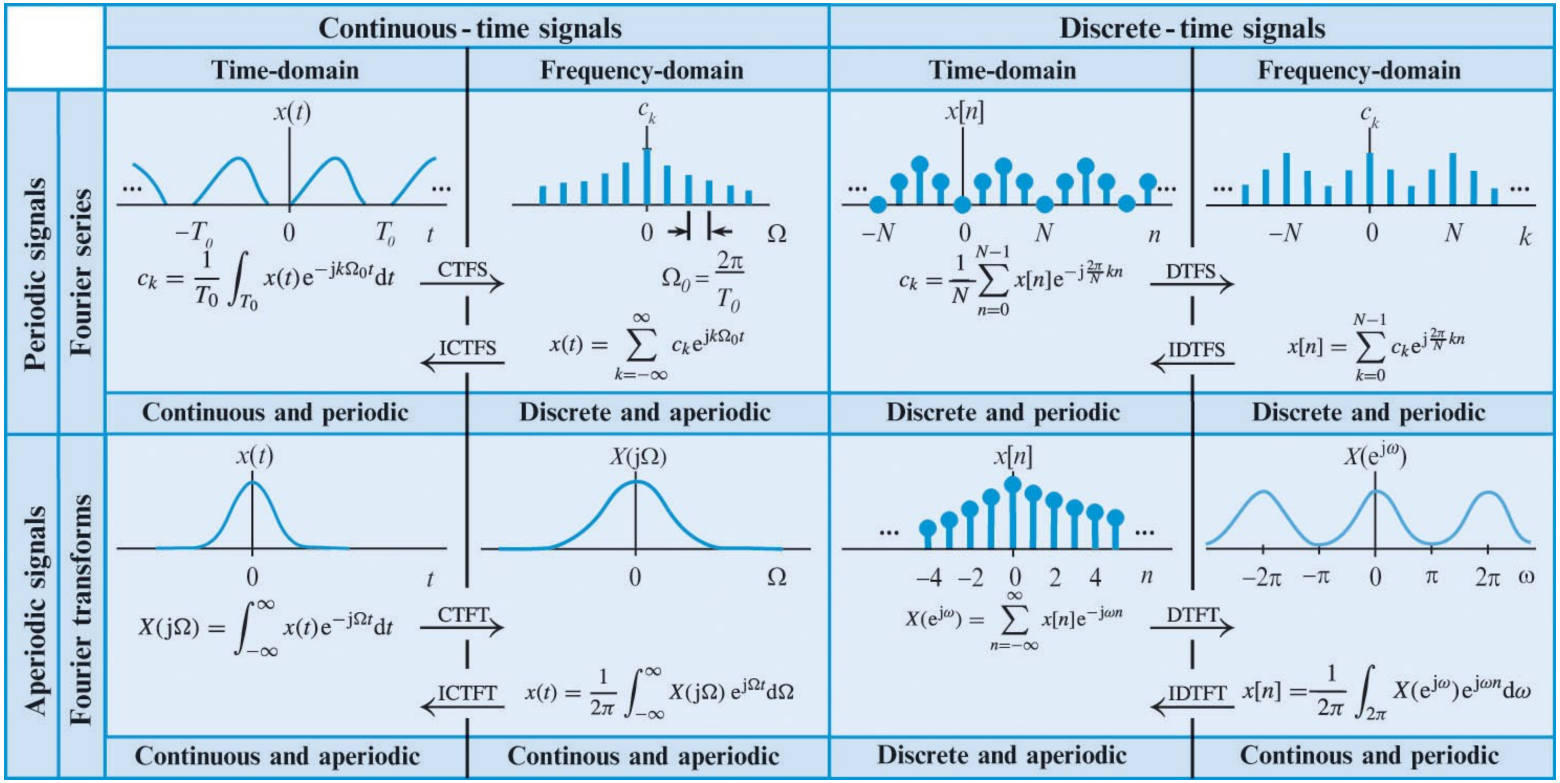
Summary of Fourier representation of signals.



Continuous-time periodic signals are represented by an infinite Fourier series of harmonically related complex exponentials. Therefore, the spectrum exists only at $F = 0, \pm F_0, \pm 2F_0, \ldots$, that is, at discrete values of F. The spacing between the lines of this discrete or line spectrum is $F_0 = 1/T_0$, that is the reciprocal of the fundamental period.

Fourier transforms for discrete-time aperiodic signals

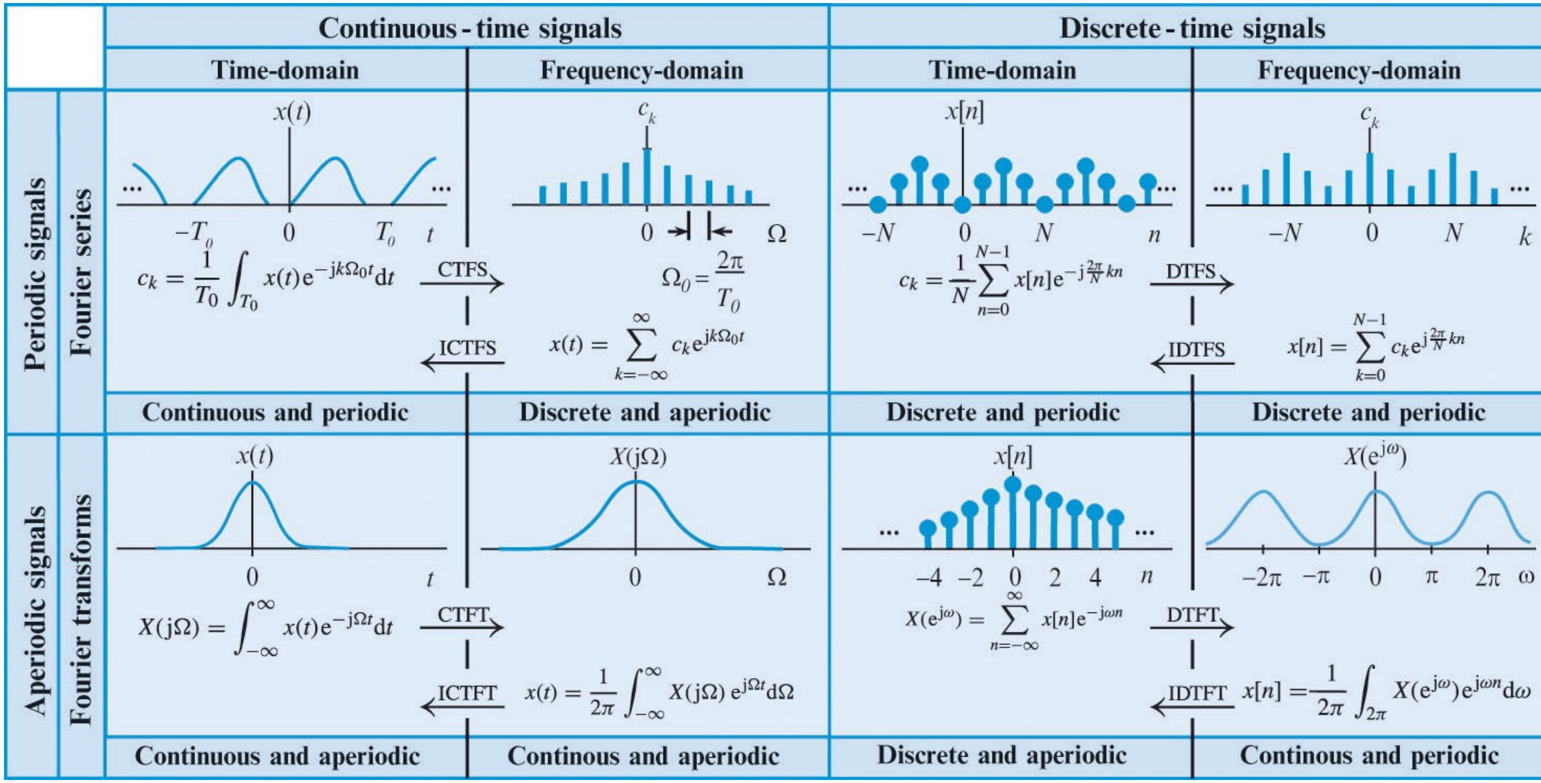
Summary of Fourier representation of signals.



Continuous-time aperiodic signals are represented by a Fourier integral of complex exponentials over the entire frequency axis. Therefore, the spectrum exists for all F, $-\infty < F < \infty$. Knowledge of $X(j2\pi F)$ for $-\infty < F < \infty$ is needed to represent $x(t)$ for $-\infty < t < \infty$.

Fourier transforms for discrete-time aperiodic signals

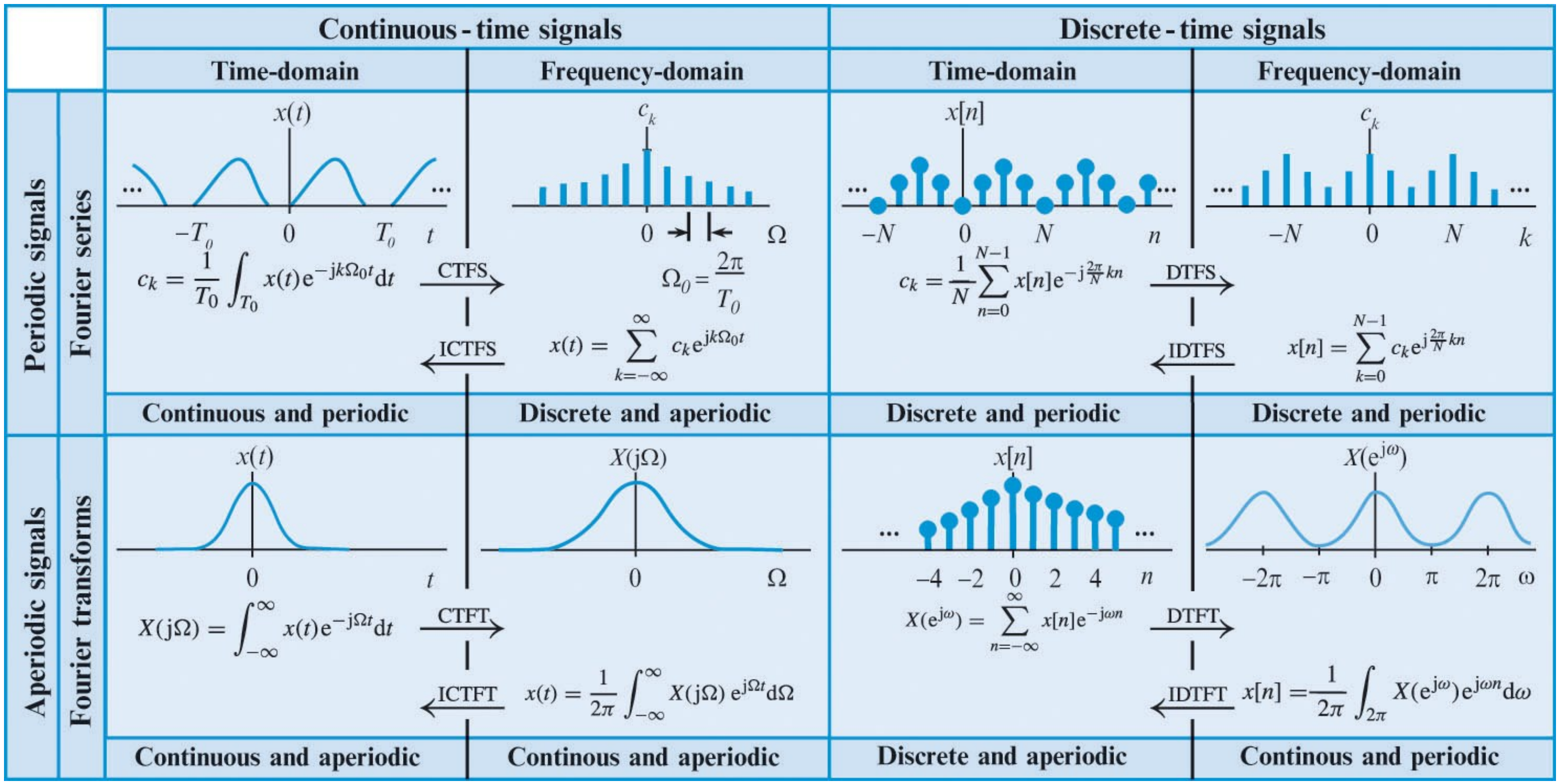
Summary of Fourier representation of signals.



Discrete-time periodic signals are represented by a finite Fourier series of harmonically related complex exponentials. The spacing between the lines of the resulting discrete spectrum is $\omega = 2\pi/N$, where N is the fundamental period. The DTFS coefficients of a periodic signal are periodic and the analysis equation involves a finite sum over a range of 2π.

Fourier transforms for discrete-time aperiodic signals

Summary of Fourier representation of signals.



Discrete-time aperiodic signals are represented by a Fourier integral of complex exponentials over any frequency range of length 2π radians. Knowledge of the periodic DTFT function $X(e^{j\omega})$ over any interval of length 2π is needed to recover $x[n]$ for $-\infty < n < \infty$.

Fourier transforms for discrete-time aperiodic signals

Summary of Fourier representation of signals.

Discrete-time complex exponentials that differ in frequency by a multiple of 2π are identical. This has the following implications:

- low-frequencies, corresponding to slowly-varying signals, are around the points $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$
- High-frequencies, corresponding to rapidly-varying signals, are around the points $\omega = \pm\pi, \pm 3\pi, \ldots$

Periodicity with “period” α in one domain implies discretization with “spacing” of $1/\alpha$ in the other domain, and vice versa.

Fourier transforms for discrete-time aperiodic signals

Summary of Fourier representation of signals.

Bandlimited signals Signals whose frequency components are zero or "small" outside a finite interval $0 \le B1 \le |F| \le B2 < \infty$ are said to be *bandlimited*. The quantity $B = B2 - B1$ is known as the *bandwidth* of the signal. For discrete-time signals we should also have the condition $B2 < Fs/2$. Depending on the values of $B1$ and $B2$, we distinguish the following types of signal:

Type	Continuous-time	Discrete-time
Lowpass	$0 \le \|F\| \le B < \infty$	$0 \le \|F\| \le B < F_s/2$
Bandpass	$0 < B_1 \le \|F\| \le B_2 < \infty$	$0 < B_1 \le \|F\| \le B_2 < F_s/2$
Highpass	$0 < B \le \|F\|$	$0 < B \le \|F\| \le F_s/2$

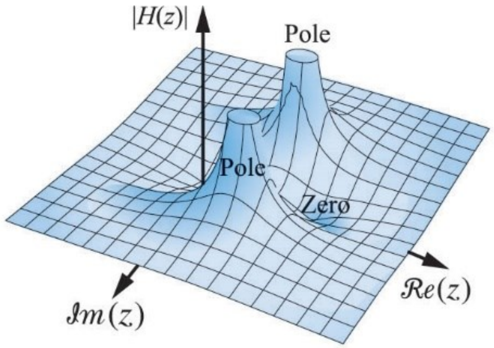
Properties of the discrete-time Fourier transform

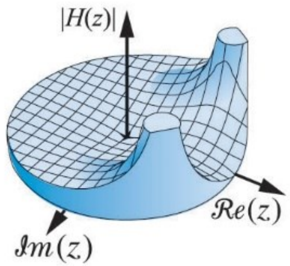
Relationship to the z -transform and periodicity

$$X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}),$$

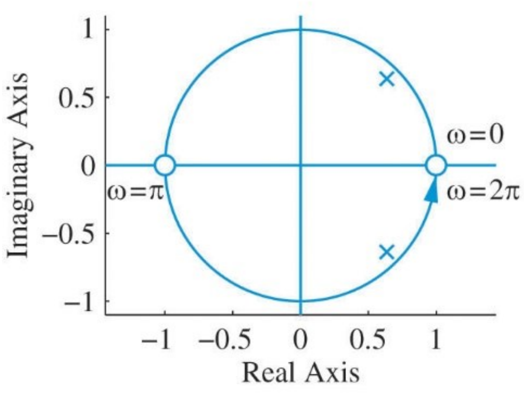
that is, the z-transform reduces to the Fourier transform.

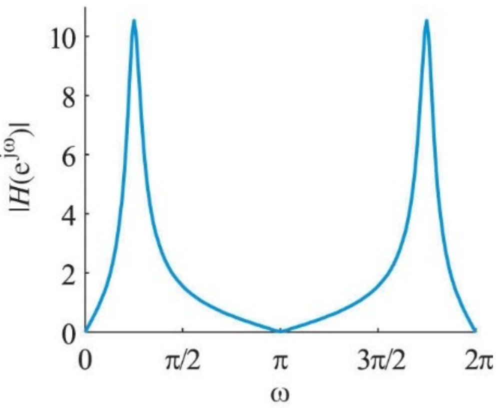
The magnitude of DTFT is obtained by intersecting the surface $|H(z)|$ with a vertical cylinder of radius one, centered at $z = 0$.





$$H(z) = \frac{1 - z^{-2}}{1 - 0.9\sqrt{2}z^{-1} + 0.81z^{-2}}.$$





The relationship between the z-transform and the DTFT for a sequence with two complex-conjugate poles at $z = 0.9e^{j\pm\pi/4}$ and two zeros at $z = \pm 1$.

Properties of the discrete-time Fourier transform

Symmetry properties

Suppose that both the signal $x[n]$ and its DTFT $X(e^{j\omega})$ are complex-valued functions, i.e.,

$$x[n] = x_R[n] + jx_I[n],$$

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

$$\left.\begin{aligned} e^{-j\omega} &= \cos\omega - j\sin\omega \\ X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \end{aligned}\right\} \longrightarrow \begin{aligned} X_R(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \{x_R[n]\cos(\omega n) + x_I[n]\sin(\omega)n\}, \\ X_I(e^{j\omega}) &= -\sum_{n=-\infty}^{\infty} \{x_R[n]\sin(\omega n) - x_I[n]\cos(\omega n)\}. \end{aligned}$$

$$\left.\begin{aligned} e^{j\omega} &= \cos\omega + j\sin\omega \\ x[n] &= \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{aligned}\right\} \rightarrow \begin{aligned} x_R[n] &= \frac{1}{2\pi}\int_{2\pi} \left[X_R(e^{j\omega})\cos(\omega n) - X_I(e^{j\omega})\sin(\omega n)\right]d\omega, \\ x_I[n] &= \frac{1}{2\pi}\int_{2\pi} \left[X_R(e^{j\omega})\sin(\omega n) + X_I(e^{j\omega})\cos(\omega n)\right]d\omega. \end{aligned}$$

Properties of the discrete-time Fourier transform

Symmetry properties

Real signals

$$X_R(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n) \qquad X_I(e^{j\omega}) = -\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)$$

$$\rightarrow \quad \begin{aligned} X_R(e^{-j\omega}) &= X_R(e^{j\omega}), && \text{(even symmetry)} \\ X_I(e^{-j\omega}) &= -X_I(e^{j\omega}), && \text{(odd symmetry)} \end{aligned}$$

or in other words:

$$X^*(e^{j\omega}) = X(e^{-j\omega}). \qquad \text{(Hermitian symmetry)}$$

The magnitude and phase of the DTFT

$$|X(e^{j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})} \qquad \angle X(e^{j\omega}) = \tan^{-1}\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}$$

Properties of the discrete-time Fourier transform

Symmetry properties

Real and even signals If $x[n]$ is real and even, that is, $x[-n] = x[n]$, then $x[n]\cos(\omega n)$ is an even and $x[n]\sin(\omega n)$ is an odd function of n.

$$X_R(e^{j\omega}) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n), \qquad \text{(even symmetry)}$$

$$X_I(e^{j\omega}) = 0,$$

$$x[n] = \frac{1}{\pi}\int_0^{\pi} X_R(e^{j\omega})\cos(\omega n)\,d\omega. \qquad \text{(even symmetry)}$$

Real and odd signals If $x[n]$ is real and odd, that is, $x[-n] = -x[n]$, then $x[n]\cos(\omega n)$ is an odd function and $x[n]\sin(\omega n)$ is an even function of n.

$$X_R(e^{j\omega}) = 0,$$

$$X_I(e^{j\omega}) = -2\sum_{n=1}^{\infty} x[n]\sin(\omega n), \qquad \text{(odd symmetry)}$$

$$x[n] = -\frac{1}{\pi}\int_0^{\pi} X_I(e^{j\omega})\sin(\omega n)\,d\omega. \qquad \text{(odd symmetry)}$$

Properties of the discrete-time Fourier transform

Symmetry properties

Special cases of the DTFT for real signals.

Signal	Fourier transform
Real and even	real and even
Real and odd	imaginary and odd
Imaginary and even	imaginary and even
Imaginary and odd	real and odd

Properties of the discrete-time Fourier transform

Example: Causal exponential sequence

Consider the sequence $x[n] = a^n u[n]$. For $|a| < 1$, the sequence is absolutely summable, that is

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty$$

Therefore

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}. \quad \text{if } |ae^{-j\omega}| < 1 \text{ or } |a| < 1$$

If $x[n]$ is real $(-1 < a < 1)$, using the properties of complex numbers, we obtain

$$X_R(e^{j\omega}) = \frac{1-a\cos(\omega)}{1-2a\cos(\omega)+a^2} = X_R(e^{-j\omega}),$$

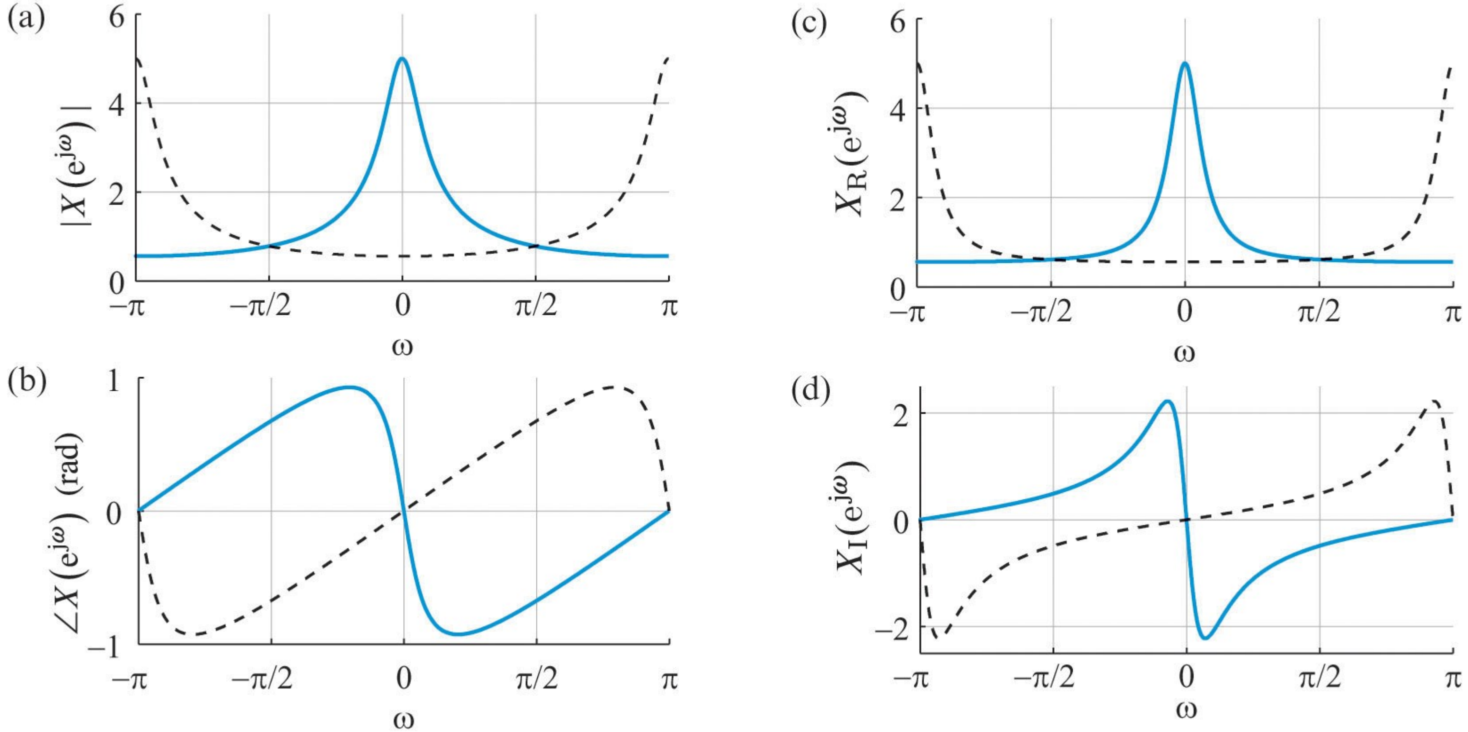
$$X_I(e^{j\omega}) = \frac{-a\sin(\omega)}{1-2a\cos(\omega)+a^2} = -X_I(e^{-j\omega}),$$

$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1-2a\cos(\omega)+a^2}} = \left|X(e^{-j\omega})\right|,$$

$$\angle X(e^{j\omega}) = \tan^{-1}\frac{-a\sin(\omega)}{1-a\cos(\omega)} = -\angle X(e^{-j\omega}).$$

Properties of the discrete-time Fourier transform

Example: Causal exponential sequence



Plots of the magnitude (a), phase (b), real part (c), and imaginary part (d) of the DTFT for the sequence $x[n] = a^n u[n]$. The solid lines correspond to a lowpass sequence $(a = 0.8)$ and the dashed lines to a highpass sequence $(a = -0.8)$.

Properties of the discrete-time Fourier transform

Example: Ideal lowpass sequence

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0. & \omega_c < |\omega| < \pi \end{cases}$$

The sequence $x[n]$ can be obtained using the synthesis formula

$$x[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi\, jn}\, e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{\sin(\omega_c n)}{\pi n}. \quad n \neq 0$$

For $n = 0$ we obtain $x[0] = 0/0$, which is undefined. However, if we use the definition directly, we obtain

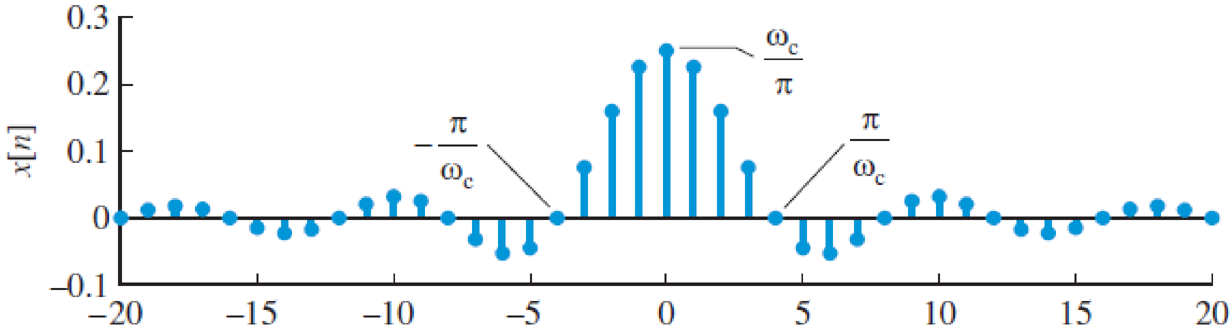
$$x[0] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} \quad \longrightarrow \quad x[n] = \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} = \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty$$

with the understanding that at $n = 0$, $x[n] = \omega_c/\pi$.

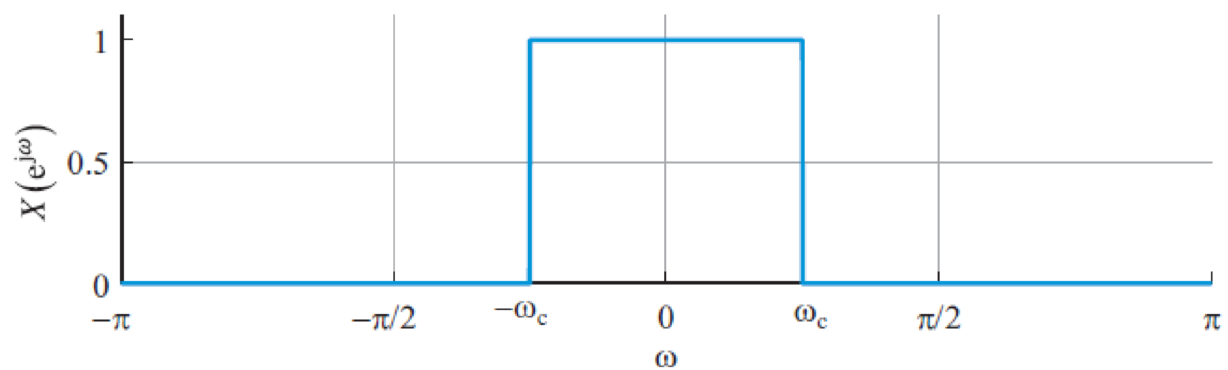
Properties of the discrete-time Fourier transform

Example: Ideal lowpass sequence

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0. & \omega_c < |\omega| < \pi \end{cases}$$



(a)



(b)

Properties of the discrete-time Fourier transform

Symmetry properties of the DTFT

Sequence $x[n]$	Transform $X(e^{j\omega})$
	Complex signals
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x_R[n]$	$X_e(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
$jx_I[n]$	$X_o(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$
$x_e[n] \triangleq \frac{1}{2}(x[n] + x^*[-n])$	$X_R(e^{j\omega})$
$x_o[n] \triangleq \frac{1}{2}(x[n] - x^*[-n])$	$jX_I(e^{j\omega})$
	Real signals
Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ $\|X(e^{j\omega})\| = \|X(e^{-j\omega})\|$ $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
$x_e[n] = \frac{1}{2}(x[n] + x[-n])$ Even part of $x[n]$	$X_R(e^{j\omega})$ real part of $X(e^{j\omega})$ (even)
$x_o[n] = \frac{1}{2}(x[n] - x[-n])$ Odd part of $x[n]$	$jX_I(e^{j\omega})$ imaginary part of $X(e^{j\omega})$ (odd)

Properties of the discrete-time Fourier transform

Operational properties of the DTFT

Since the DTFT is a special case of the *z*-transform, all properties of the *z*-transform translate into similar properties for the Fourier transform.

Linearity

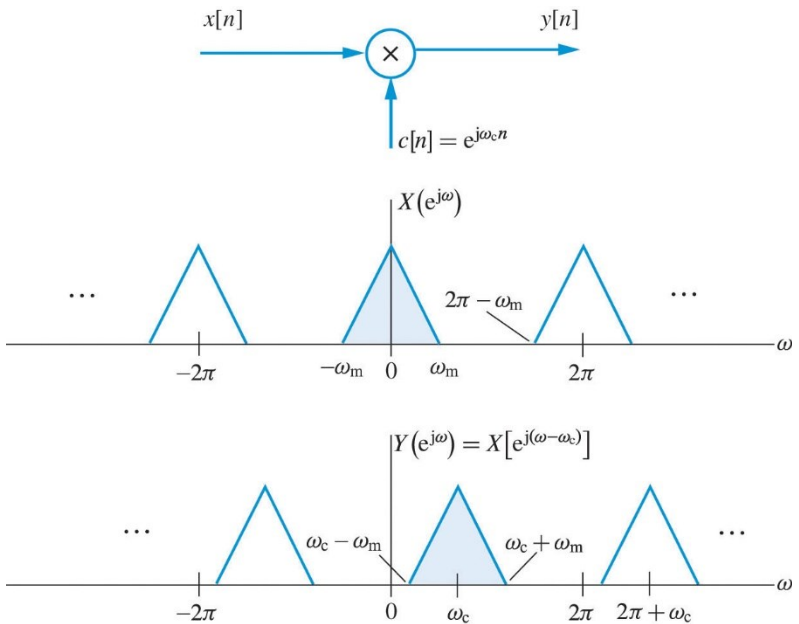
$$a_1x_1[n] + a_2x_2[n] \xleftrightarrow{\text{DTFT}} a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega}),$$

Time shifting

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \quad \longrightarrow \quad x[n-k] \xleftrightarrow{\text{DTFT}} e^{-j\omega k}X(e^{j\omega}).$$

Frequency shifting (*amplitude modulation with exponential carrier* $c[n] = e^{j\omega_c n}$)

$$e^{j\omega_c n}x[n] \xleftrightarrow{\text{DTFT}} X(e^{j[\omega-\omega_c]}).$$

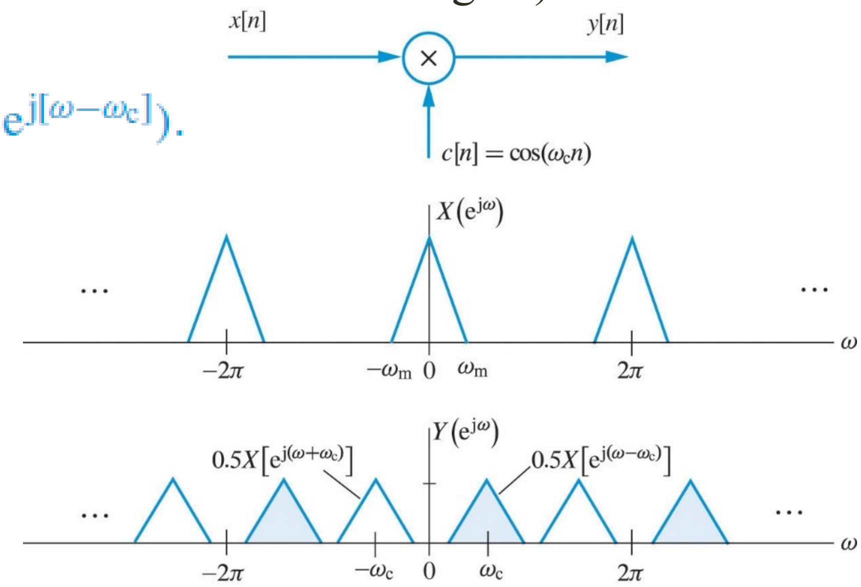


Properties of the discrete-time Fourier transform

Operational properties of the DTFT

Modulation ((amplitude) modulation with a sinusoidal carrier signal)

$$x[n]\cos(\omega_c n) \xleftrightarrow{\text{DTFT}} \frac{1}{2}X(e^{j[\omega+\omega_c]}) + \frac{1}{2}X(e^{j[\omega-\omega_c]}).$$



Differentiation in frequency

$$nx[n] \xleftrightarrow{\text{DTFT}} -j\frac{dX(e^{j\omega})}{d\omega},$$

Time reversal

$$x[-n] \xleftrightarrow{\text{DTFT}} X(e^{-j\omega}).$$

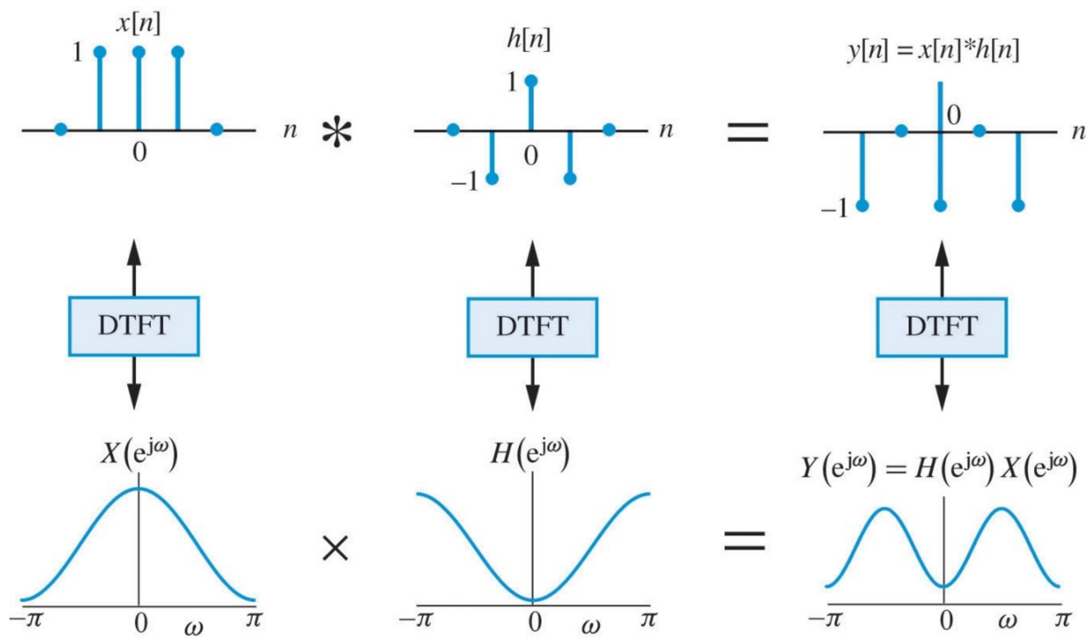
Properties of the discrete-time Fourier transform
Operational properties of the DTFT

Conjugation of a complex sequence

$$x^*[n] \stackrel{\mathrm{DTFT}}{\longleftrightarrow} X^*(e^{-j\omega}).$$

Convolution of sequences

$$y[n] = x[n] * h[n] \stackrel{\mathrm{DTFT}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$



Properties of the discrete-time Fourier transform

Operational properties of the DTFT

Multiplication of sequences (windowing theorem)

$$s[n] = x[n]w[n] \xleftrightarrow{\text{DTFT}} S(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})W[e^{j(\omega-\theta)}]d\theta.$$

This integral, which can be evaluated over any interval of length 2π, generates what is known as the periodic convolution of $X(e^{j\omega})$ and $W(e^{j\omega})$.

Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega.$$

$$\begin{aligned}
\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega &= \frac{1}{2\pi}\int_{2\pi}\left[\sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n}\right]X_2^*(e^{j\omega})d\omega \\
&= \sum_{n=-\infty}^{\infty} x_1[n]\left[\frac{1}{2\pi}\int_{2\pi} X_2^*(e^{j\omega})e^{-j\omega n}d\omega\right] \\
&= \sum_{n=-\infty}^{\infty} x_1[n]\left[\frac{1}{2\pi}\int_{2\pi} X_2^*(e^{-j\omega})e^{j\omega n}d\omega\right] \\
&= \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]. \quad \text{(using (4.148))}
\end{aligned}$$

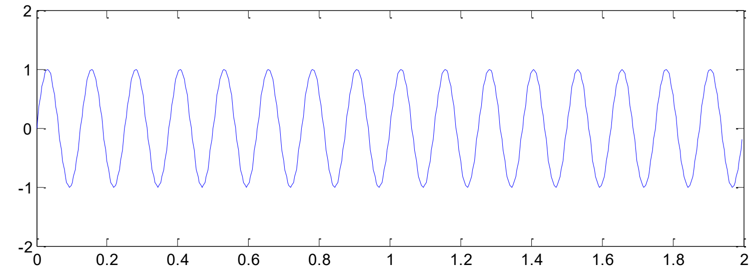
Properties of the discrete-time Fourier transform

Operational properties of the DTFT

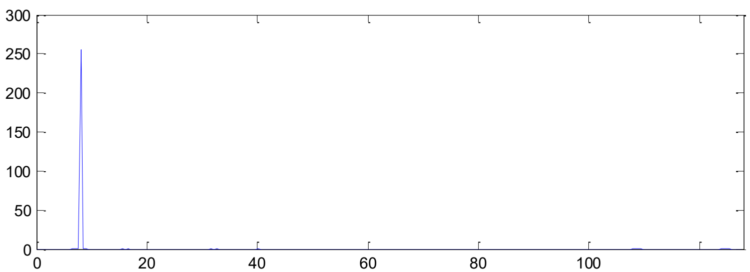
	Property	Sequence	Transform
		$x[n]$	$\mathcal{F}\{x[n]\}$
1.	Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$
2.	Time shifting	$x[n-k]$	$e^{-jk\omega} X(e^{j\omega})$
3.	Frequency shifting	$e^{j\omega_0 n} x[n]$	$X[e^{j(\omega-\omega_0)}]$
4.	Modulation	$x[n]\cos\omega_0 n$	$\frac{1}{2}X[e^{j(\omega+\omega_0)}] + \frac{1}{2}X[e^{j(\omega-\omega_0)}]$
5.	Folding	$x[-n]$	$X(e^{-j\omega})$
6.	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
7.	Differentiation	$nx[n]$	$-j\dfrac{dX(e^{j\omega})}{d\omega}$
8.	Convolution	$x[n] * h[n]$	$X(e^{j\omega})H(e^{j\omega})$
9.	Windowing	$x[n]w[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})W[e^{j(\omega-\theta)}]d\theta$
10.	Parseval's theorem	$\displaystyle\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega$	
11.	Parseval's relation	$\displaystyle\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

Fourier analysis applications

Spectral analysis of stationary signals

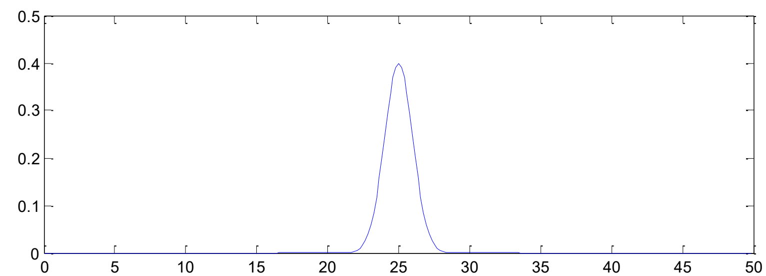


Sine wave

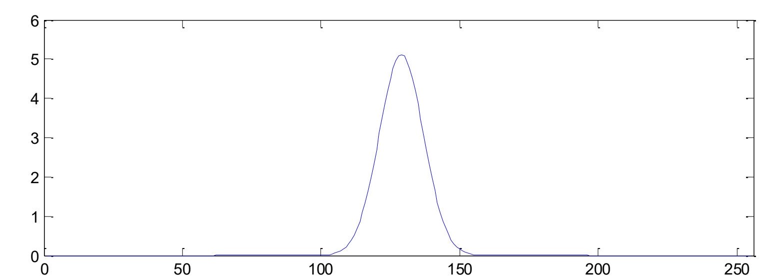


Delta function

Gaussian

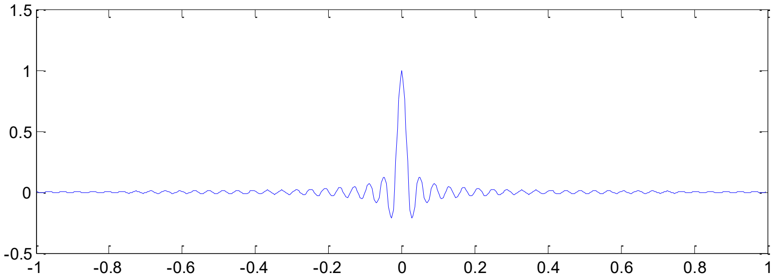


Gaussian

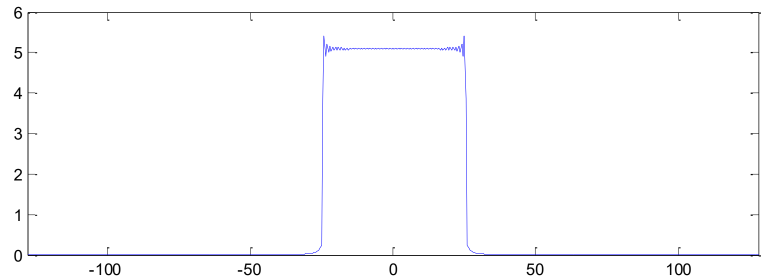


Fourier analysis applications

Spectral analysis of stationary signals

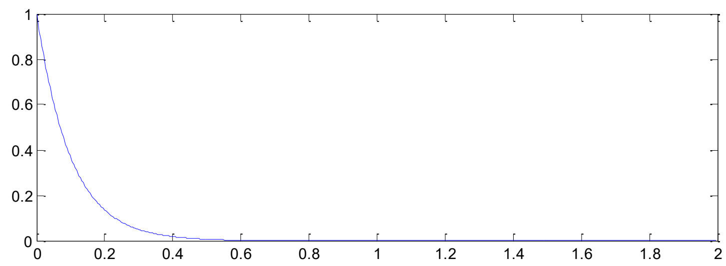


Sinc function

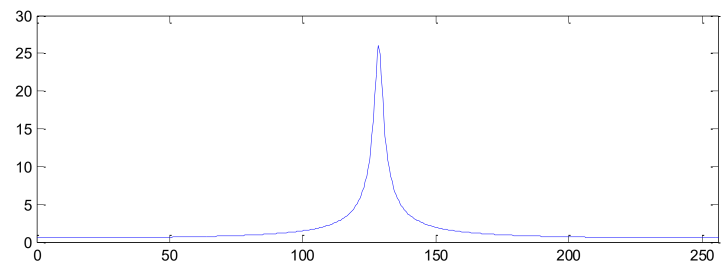


Square wave

Exponential



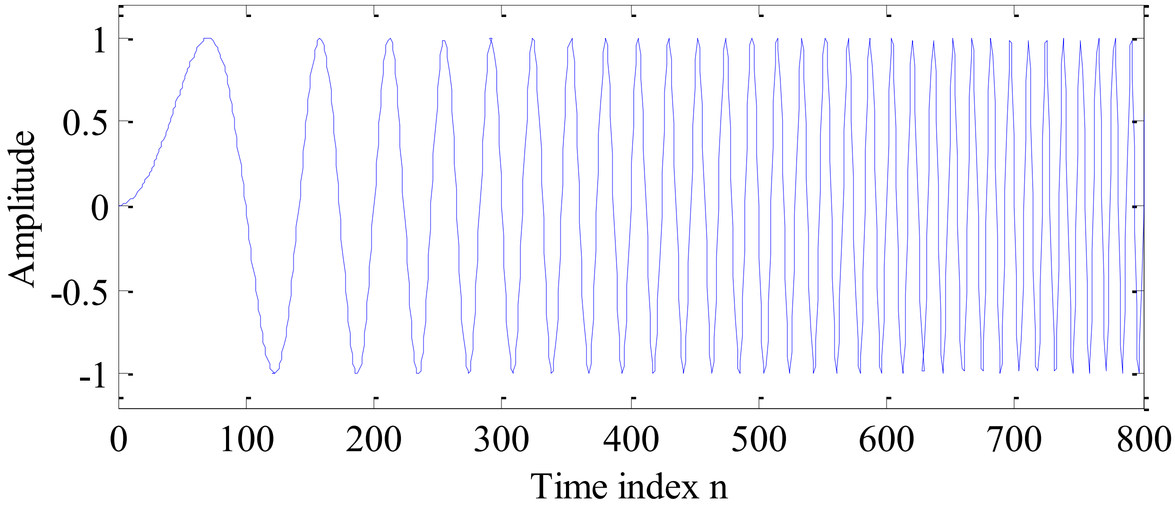
Lorentzian



Fourier analysis applications

Spectral analysis of nonstationary signals

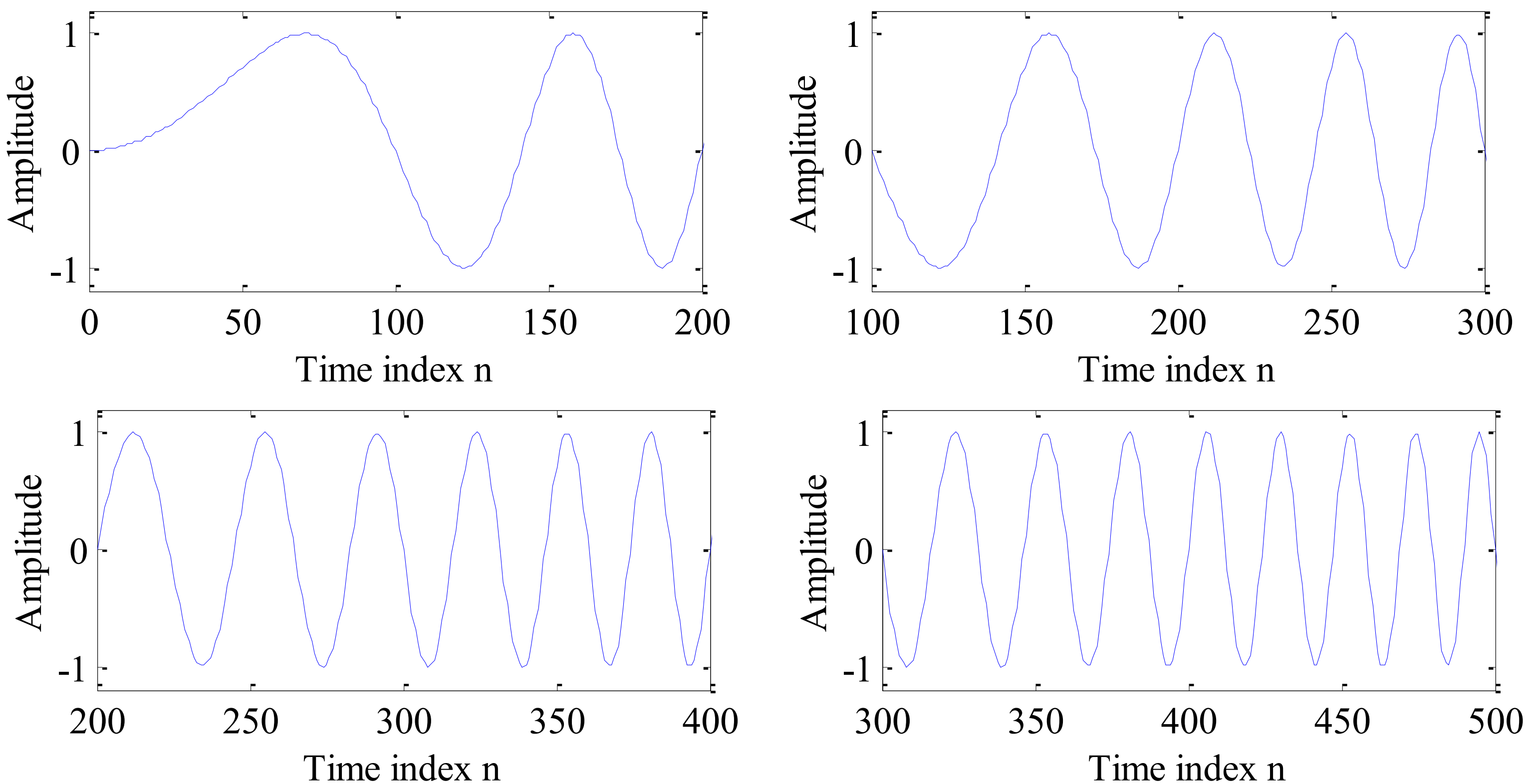
- An example of a time-varying signal is the chirp signal $x[n] = A\cos(\omega_o n^2)$ and shown below for $\omega_o = 10\pi \times 10^{-5}$



- The instantaneous frequency of $x[n]$ is $2\omega_o n$

Fourier analysis applications

Spectral analysis of nonstationary signals

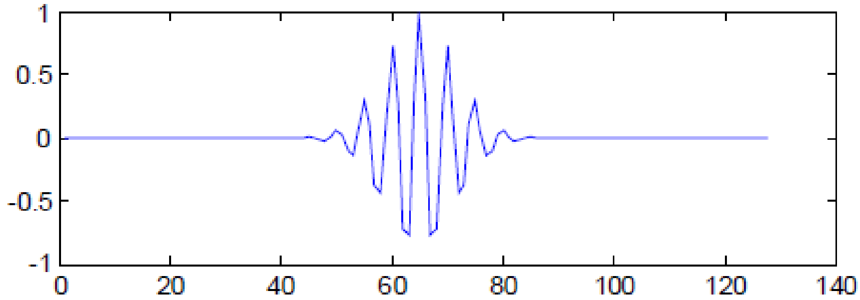


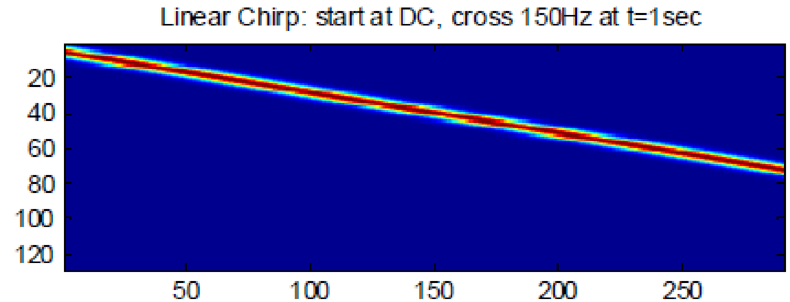
Four segments of the chirp signal as seen through a stationary length-200 rectangular window

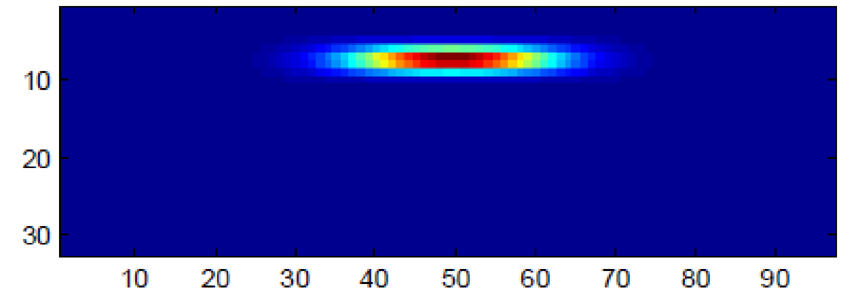
Fourier analysis applications

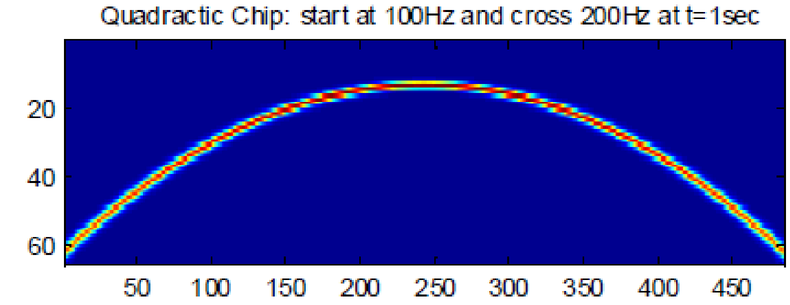
Spectral analysis of nonstationary signals

Short-Time Fourier Transform







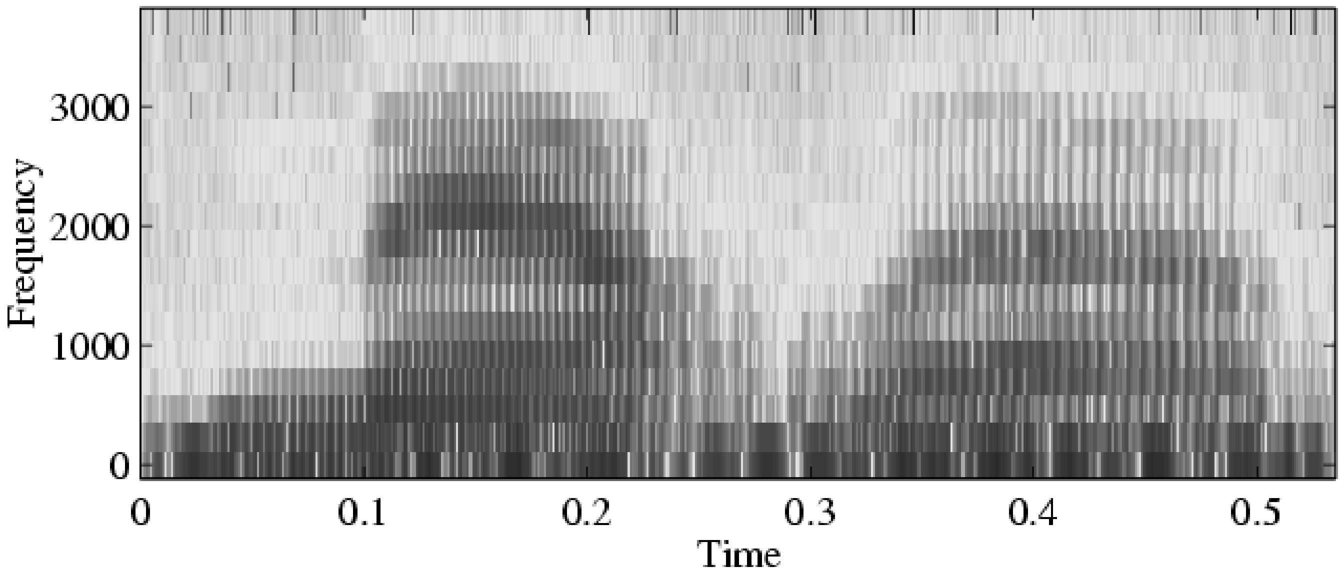


Digital Signal Processing, Fall 2019

Fourier analysis applications

Spectral analysis of nonstationary signals

Short-Time Fourier Transform



spectrogram of speech signal

Computational Fourier analysis

The basic premise of Fourier analysis is that any signal can be expressed as a linear superposition, that is, a sum or integral of sinusoidal signals. The exact mathematical form of the representation depends on whether the signal is continuous-time or discrete-time and whether it is periodic or aperiodic.

	Direct transform (spectral analysis)	Inverse transform (signal reconstruction)	Exact computation
DTFS	$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$ finite summation	$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$ finite summation	yes
DTFT	$\tilde{X}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ infinite summation	$x[n] = \frac{1}{2\pi}\int_0^{2\pi} \tilde{X}(e^{j\Omega})e^{j\Omega}d\omega$ integration	no
CTFS	$c_k = \frac{1}{T_0}\int_0^{T_0} \tilde{x}_c(t)e^{-jk\Omega_0 t}dt$ integration	$\tilde{x}_c(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$ infinite summation	no
CTFT	$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt$ integration	$x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega)e^{j\Omega t}d\Omega$ integration	no

Computational Fourier analysis

The DTFT of a sequence $x[n]$ is given by

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

~ sign is used to emphasize that a sequence or function is periodic.

There are two major issues:

1. Evaluation of the infinite summation would be impossible.

   We use the following approximation which will be reasonable if the values of $x[n]$ outside the interval $0 \le n \le N-1$ are either zero or negligibly small.

$$\tilde{X}(e^{j\omega}) \approx \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \triangleq \tilde{X}_N(e^{j\omega})$$

2. The function $\tilde{X}_N(e^{j\omega})$ an only be computed at a finite set of frequencies $0 \le \omega_k < 2\pi$, $0 \le k \le K-1$.

Computational Fourier analysis

The DTFT of a sequence $x[n]$ is given by

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}.$$

~ sign is used to emphasize that a sequence or function is periodic.

2. We define the quantities

$$X[k] \triangleq \tilde{X}_N(e^{j\omega_k}), \quad k = 0, 1, \ldots, K-1$$

$$\rightarrow \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[K-1] \end{bmatrix} = \begin{bmatrix} e^{j\omega_0 0} & e^{j\omega_0 1} & \ldots & e^{j\omega_0 (N-1)} \\ e^{j\omega_1 0} & e^{j\omega_1 1} & \ldots & e^{j\omega_1 (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j\omega_{K-1} 0} & e^{j\omega_{K-1} 1} & \ldots & e^{j\omega_{K-1} (N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Or in matrix form $\qquad \boldsymbol{X} = \boldsymbol{W}\boldsymbol{x}$

Computational Fourier analysis

The DTFT of a sequence $x[n]$ is given by

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

$\tilde{}$ sign is used to emphasize that a sequence or function is periodic.

If $K = N$ we have a linear system of N equations with N unknowns. If the $N \times N$ matrix $\boldsymbol{W}$ is nonsingular, its inverse exists, and the solution is formally expressed as

$$\boldsymbol{x} = \boldsymbol{W}^{-1}\boldsymbol{X}.$$

This can be simplified if we use N equally spaced frequencies:

$$\omega_k = \frac{2\pi}{N}k. \quad k = 0, 1, \ldots, N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{K}kn}. \quad k = 0, 1, \ldots, N-1$$

$$\rightarrow$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{K}kn}, \quad n = 0, 1, \ldots, N-1$$

Computational Fourier analysis
Symmetry properties of the DTFT

The set of equations:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{K}kn}. \quad k = 0, 1, \ldots, N-1 \qquad (1)$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{K}kn}, \quad n = 0, 1, \ldots, N-1 \qquad (2)$$

provide the basis for computational Fourier analysis. Given a set of samples $x[n]$, $0 \leq n \leq N-1$, we can use (1) to compute a set of coefficients $X[k]$, $0 \leq k \leq N-1$. The N signal samples can always be exactly recovered from the N coefficients using (2).

Computational Fourier analysis

However, the meaning or interpretation of the coefficients depends on the "origin" of the N signal samples:

- If $x[n]$ has finite-length N, that is, $x[n] = 0$ outside the range $0 \leq n \leq N - 1$, then we have
$$X[k] = \tilde{X}\left(e^{j\frac{2\pi}{N}k}\right).$$
- If $x[n]$ has finite-length $L > N$ or infinite length, then we have
$$X[k] = \tilde{X}_N\left(e^{j\frac{2\pi}{N}k}\right).$$
- If $x[n]$, $0 \leq n \leq N - 1$ is a period from a periodic sequence, then
$$X[k] = N\tilde{c}_k.$$

Equations (1) and (2) can be used to compute, either exactly or approximately, all Fourier decompositions (DTFS, CTFS, DTFT, CTFT). This pair is known as *Discrete Fourier Transform* (DFT).

The Discrete Fourier Transform (DFT)

Analysis equation **Synthesis equation**

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \xleftrightarrow[N]{\text{DFT}} x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

where the complex quantity W_N, known as the *twiddle factor*, is defined by

$$W_N \triangleq e^{-j\frac{2\pi}{N}}$$

The Discrete Fourier Transform (DFT)

Direct computation of the Discrete Fourier Transform

```
function Xdft=dftdirect(x)
% Direct computation of the DFT
N=length(x); Q=2*pi/N;
for k=1:N
    S=0;
    for n=1:N
        W(k,n)=exp(-j*Q*(k-1)*(n-1));
        S=S+W(k,n)*x(n);
    end
    Xdft(k)=S;
end
```

The computational complexity of the direct DFT algorithm $O(N^2)$.

It is desirable to develop a procedure for calculating DFT. The procedure has to be

- Fast,
- Accurate,
- Simple.

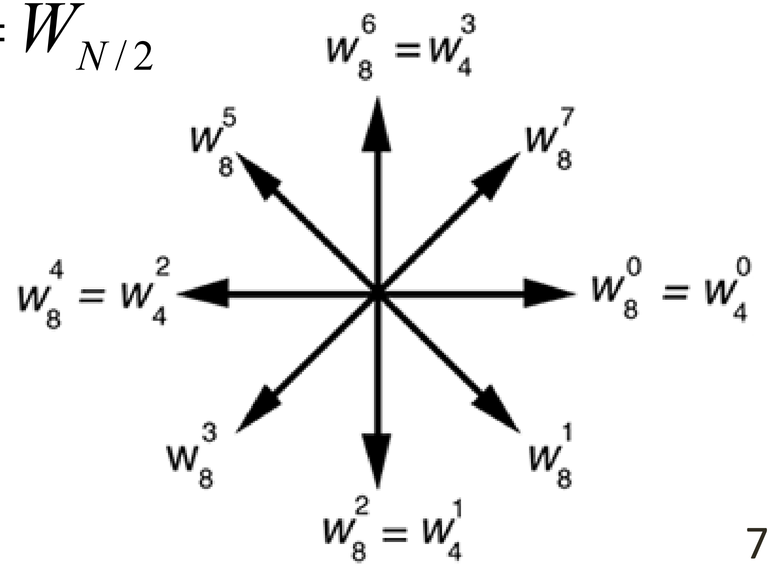
The Discrete Fourier Transform (DFT)

Decimation-in-time fast Fourier transform (FFT) algorithms

Introduced by Cooley and Tukey (1965)

The following properties of $W_N \triangleq e^{-j\frac{2\pi}{N}}$ are exploited:

- Symmetry property $\quad W_N^{k+N/2} = -W_N^k = e^{j\pi} W_N^k$
- Periodicity property $\quad W_N^{k+N} = W_N^k$
- Recursion property $\quad W_N^2 = W_{N/2}$



The Discrete Fourier Transform (DFT)

Decimation-in-time fast Fourier transform (FFT) algorithms

Assumption: N is an even integer

Split the N-point DFT summation into two $N/2$ -point summations: one sum over the even indexed points of $x[n]$ and another sum over the odd-indexed points of $x[n]$.

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, 1, \ldots, N-1 \\
&= \sum_{m=0}^{\frac{N}{2}-1} x[2m]\, W_N^{k(2m)} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1]\, W_N^{k(2m+1)} \\
&= \sum_{m=0}^{\frac{N}{2}-1} x[2m]\, W_N^{k(2m)} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} x[2m+1]\, W_N^{k(2m)}
\end{aligned}
$$

The Discrete Fourier Transform (DFT)

Decimation-in-time fast Fourier transform (FFT) algorithms

We define the following sequences:

$$a[n] \triangleq x[2n], \qquad n = 0, 1, \ldots, \frac{N}{2} - 1$$

$$b[n] \triangleq x[2n+1]. \qquad n = 0, 1, \ldots, \frac{N}{2} - 1$$

Using the identity $W_N^2 = W_{\frac{N}{2}}$

$$X[k] = A[k] + W_N^k B[k], \quad k = 0, 1, \ldots, N-1$$

where

$$A[k] \triangleq \sum_{m=0}^{\frac{N}{2}-1} a[m]\, W_{\frac{N}{2}}^{km}, \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

$$B[k] \triangleq \sum_{m=0}^{\frac{N}{2}-1} b[m]\, W_{\frac{N}{2}}^{km}. \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

The Discrete Fourier Transform (DFT)

Decimation-in-time fast Fourier transform (FFT) algorithms

Using the periodicity property of DFT, and since the $N/2$-point transforms $A[k]$ and $B[k]$ are implicitly periodic with period $N/2$,we have:

$$X\left[k+\frac{N}{2}\right]=A\left[k+\frac{N}{2}\right]+W_N^{k+\frac{N}{2}}B\left[k+\frac{N}{2}\right]$$

$$=A[k]-W_N^kB[k]$$

Therefore, the N-point DFT $X[k]$ can be calculated from the $N/2$-point DFTs $A[k]$ and $B[k]$ using the following merging formulas, which are called **butterfly operations** :

$$X[k]=A[k]+W_N^kB[k],\quad k=0,1,\ldots,\frac{N}{2}-1$$

$$X\left[k+\frac{N}{2}\right]=A[k]-W_N^kB[k].\quad k=0,1,\ldots,\frac{N}{2}-1$$

The Discrete Fourier Transform (DFT)

Decimation-in-time fast Fourier transform (FFT) algorithms

The "even-odd" decomposition of the input sequence can be applied recursively (divide-and-conquer) until we reach the point where the DFT lengths are equal to two:

$$X[0] = x[0] + W_2^0 x[1] = x[0] + x[1],$$

$$X[1] = x[0] + W_2^1 x[1] = x[0] - x[1].$$

Example: Decimation-in-time FFT for $N = 8$

$$X[k] = \text{DFT}_8\{x[0], x[1], x[2], x[3], x[4], x[5], x[6], x[7]\},\ 0 \le k \le 7$$

Using the divide-and-conquer approach:

$$A[k] = \text{DFT}_4\{x[0], x[2], x[4], x[6]\}, \qquad 0 \le k \le 3$$

$$B[k] = \text{DFT}_4\{x[1], x[3], x[5], x[7]\}. \qquad 0 \le k \le 3$$

The Discrete Fourier Transform (DFT)

Decimation-in-time fast Fourier transform (FFT) algorithms

Next, to calculate $A[k]$ and $B[k]$ we need the following two-point transforms

$$\begin{aligned} C[k] &= \mathrm{DFT}_2\{x[0], x[4]\}, & k &= 0, 1 \\ D[k] &= \mathrm{DFT}_2\{x[2], x[6]\}, & k &= 0, 1 \\ E[k] &= \mathrm{DFT}_2\{x[1], x[5]\}, & k &= 0, 1 \\ F[k] &= \mathrm{DFT}_2\{x[3], x[7]\}. & k &= 0, 1 \end{aligned}$$

Assuming that the required twiddle factors have already been computed and stored, the major computational effort is to merge $C[k]$ with $D[k]$, $E[k]$ with $F[k]$, and $A[k]$ with $B[k]$ for $N = 4$ and $N = 8$.

$$A[k] = C[k] + W_8^{2k} D[k], \quad k = 0, 1$$

$$A[k+2] = C[k] - W_8^{2k} D[k], \quad k = 0, 1$$

$$B[k] = E[k] + W_8^{2k} F[k], \quad k = 0, 1$$

$$B[k+2] = E[k] - W_8^{2k} F[k], \quad k = 0, 1$$

The Discrete Fourier Transform (DFT)

Decimation-in-time fast Fourier transform (FFT) algorithms

Finally, we merge the four-point DFTs using:

$$X[k] = A[k] + W_8^k B[k], \qquad k = 0, 1, \ldots, 3$$

$$X[k+4] = A[k] - W_8^k B[k]. \qquad k = 0, 1, \ldots, 3$$

Note that we have used the identity $W_4^k = W_8^{2k}$ so that the twiddle factors for all merging equations correspond to $N = 8$.

The Discrete Fourier Transform (DFT)

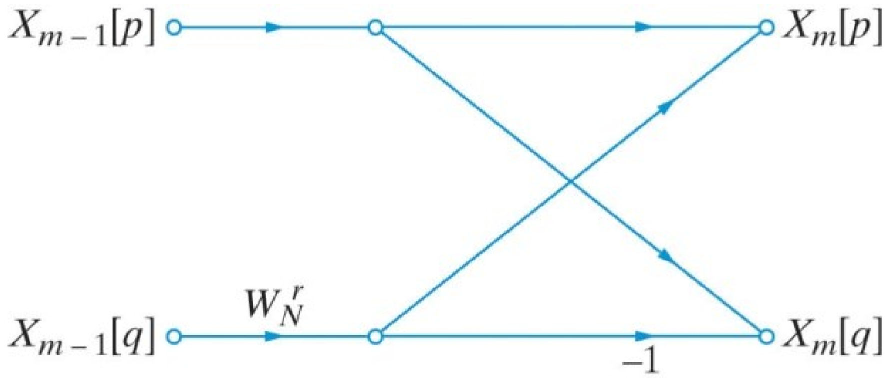
Decimation-in-time FFT algorithms

Butterfly computation

A careful inspection of the merging formulas shows that they all share the same form:

$$X_m[p] = X_{m-1}[p] + W_N^r X_{m-1}[q],$$

$$X_m[q] = X_{m-1}[p] - W_N^r X_{m-1}[q].$$



Flow graph of the butterfly operation for computation of the decimation-in-time FFT algorithm.

The Discrete Fourier Transform (DFT)

Decimation-in-time FFT algorithms

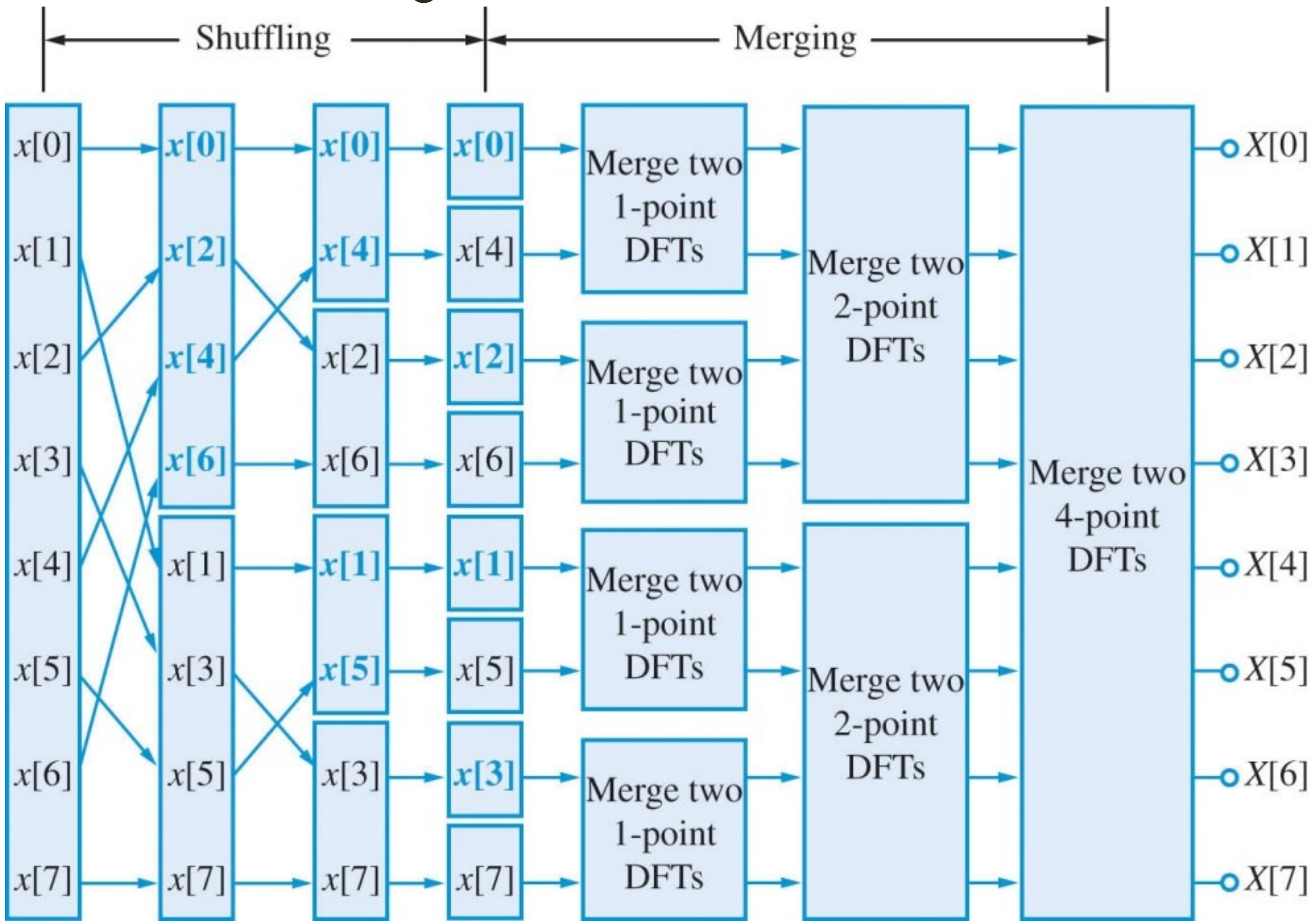
The method takes place in two phases:

Shuffling
The input sequence is successively decomposed into even and odd parts until we end-up with sub-sequences of length two.

Merging
The butterfly operations are used to combine DFTs of length 1 into DFTs of length 2, DFTs of length 2 into DFTs of length 4, and so on, until the final N-point DFT $X[k]$ is formed from two $N/2$-point DFTs.

The Discrete Fourier Transform (DFT)

Decimation-in-time FFT algorithms



The shuffling and merging operations required for recursive computation of the 8-point DFT using the decimation-in-time FFT algorithm.

The Discrete Fourier Transform (DFT)

- Image = 200x320 matrix of values
- Compress by keeping largest 2.5% of FFT components
- Similar idea used by jpeg