Electrical and Computer Engineering Department
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Signals and systems

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Signal processing

Signal processing is a discipline concerned with the **acquisition**, **representation**, **manipulation**, and **transformation** of signals.

Discrete-time signals

A discrete-time signal $x[n]$ is a sequence of numbers defined for every value of the integer variable n.

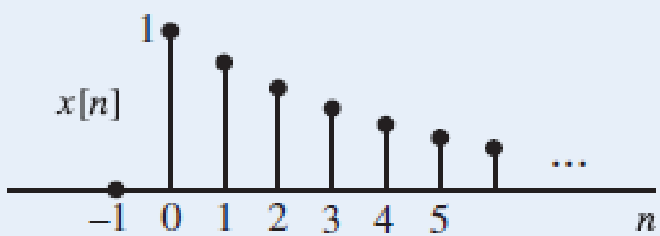
When $x[n]$ is obtained by sampling a continuous-time signal $x(t)$, the interval T between two successive samples is known as the ***sampling period*** or ***sampling interval***.

The quantity $F_s = 1/T$, called **the *sampling frequency*** or ***sampling rate***, equals the number of samples per unit of time.

If T is measured in seconds, the units of F_s are number of samples per second (sampling rate) or Hz (sampling frequency).

Signal representation

Table 2.1 Discrete-time signal representations.

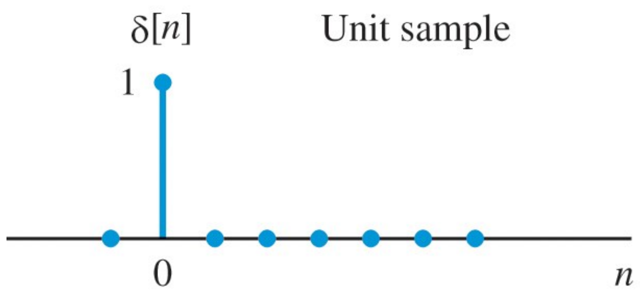
Representation	Example
Functional	$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$
Tabular	$\begin{array}{c\|cccccccc} n & \ldots & -2 & -1 & 0 & 1 & 2 & 3 & \ldots \\ \hline x[n] & \ldots & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \ldots \end{array}$
Sequence	$x[n] = \{ \ \ldots \ 0 \ \underset{\uparrow}{1} \ \frac{1}{2} \ \frac{1}{4} \ \frac{1}{8} \ \ldots \ \}$
Pictorial	

[1] The symbol $\uparrow$ denotes the index $n = 0$; it is omitted when the table starts at $n = 0$.

Elementary discrete-time signals

Unit sample sequence

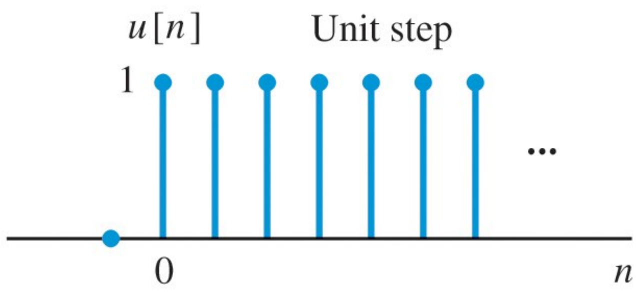
$$\delta[n] \triangleq \begin{cases} 1, & n = 0 \\ 0. & n \neq 0 \end{cases}$$



(a)

Unit step sequence

$$u[n] \triangleq \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



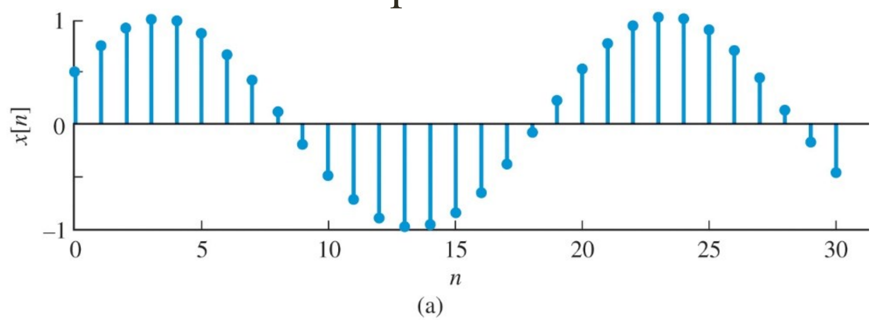
(b)

Elementary discrete-time signals

Sinusoidal sequence

$$x[n] = A\cos(\omega_0 n + \phi), \quad -\infty < n < \infty$$

A (*amplitude*) and φ (*phase*) are real constants. The quantity ω_0 is the *frequency* of the sinusoid and has units of radians per sampling interval.

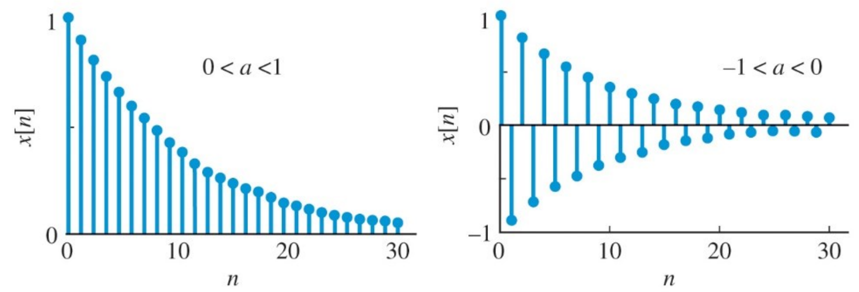


(a)

Exponential sequence

$$x[n] \triangleq Aa^n, \quad -\infty < n < \infty$$

If A and a are real numbers, $x[n]$ is termed as a *real exponential* sequence.



(b)

If both $A = |A|\,e^{j\varphi}$ and $a = \sigma_0 + j\omega_0$ are complex-valued, then we have:

$$x[n] = |A|e^{j\phi}e^{\sigma_0 n + j\omega_0 n} = |A|e^{\sigma_0 n}e^{j(\omega_0 n + \phi)}$$

$$= |A|e^{\sigma_0 n}\cos(\omega_0 n + \phi) + j|A|e^{\sigma_0 n}\sin(\omega_0 n + \phi)$$

Elementary discrete-time signals

Complex sinusoidal sequence

$$x[n] = Ae^{j\omega_0 n} = A\cos(\omega_0 n) + jA\sin(\omega_0 n).$$

One special case of the exponential sequence is when A is real-valued but $a = e^{j\omega_0}$ is complex-valued,

Periodic sequence

A sequence $x[n]$ is called *periodic* if

$$x[n] = x[n+N], \quad \text{all } n$$

The smallest value of N is known as the *fundamental period* or simply *period* of $x[n]$.

Definitions

Causal systems:
A system is called *causal* if the present value of the output does not depend on future values of the input, that is, $y[n_0]$ is determined by the values of $x[n]$ for $n \leq n_0$, only.

Causal:

$$y[n] = 1/3\{x[n] + x[n-1] + x[n-2]\}$$

Noncausal:

$$y[n] = \text{median}\{x[n-1], x[n-2], x[n], x[n+1], x[n+2]\}.$$

Causality implies that if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$; that is, a causal system cannot produce an output before the input is applied.

Definitions

Stable systems:

A system is said to be *stable*, if every bounded input signal results in a bounded output signal, that is $|x[n]| \le M_x < \infty \Rightarrow |y[n]| \le M_y < \infty$.

A signal $x[n]$ is bounded if there exists a positive finite constant M_x such that $|x[n]| \le M_x$ for all n.

Stable:

$$y[n] = 1/3\{x[n] + x[n-1] + x[n-2]\}$$

Unstable:

$$y[n] = \sum_{k=0}^{\infty} x[n-k]$$

Definitions

Linear system:

A system is called linear if and only if for every real or complex constant a_1, a_2 and every input signal $x_1[n]$ and $x_2[n]$

$$H\{a_1x_1[n] + a_2x_2[n]\} = a_1H\{x_1[n]\} + a_2H\{x_2[n]\},$$

for all values of n.

Example:

$$y[n] = x^2[n]$$

Time-invariant systems:

A system is called *time-invariant* or *fixed* if and only if

$$y[n] = H\{x[n]\} \Rightarrow y[n - n_0] = H\{x[n - n_0]\},$$

for every input $x[n]$ and every time shift n_0. That is, a time shift in the input results in a corresponding time shift in the output.

Example:

$y[n] = x[n] \cos \omega_0 n.$

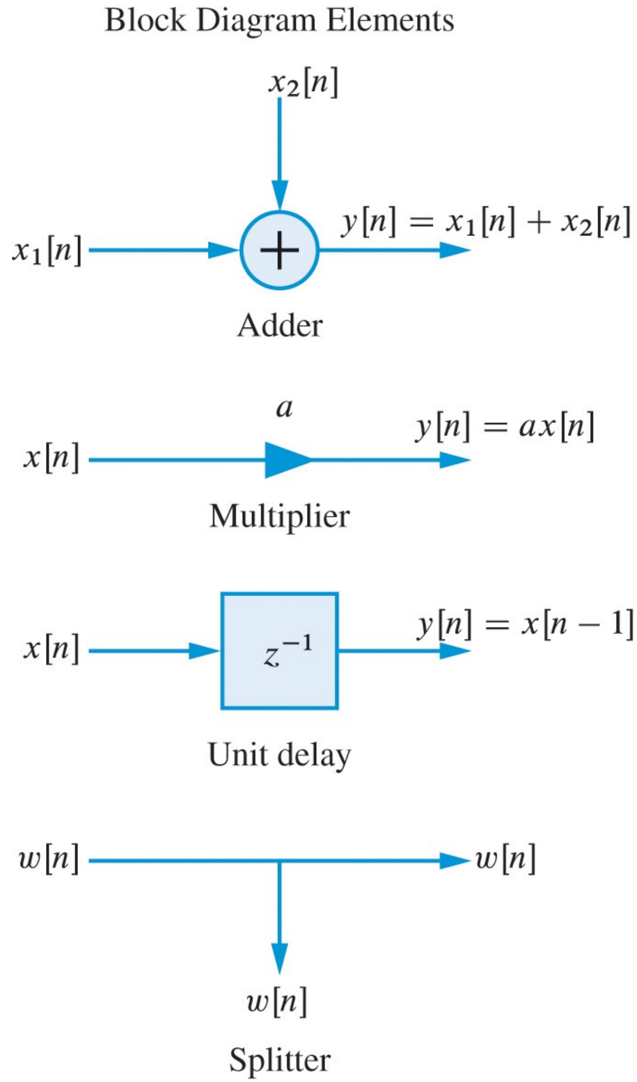
$y[n] = H\{x[n]\} = x[nM]$ (Downsampler)

$y[n] = x[-n]$ (Time-flip)

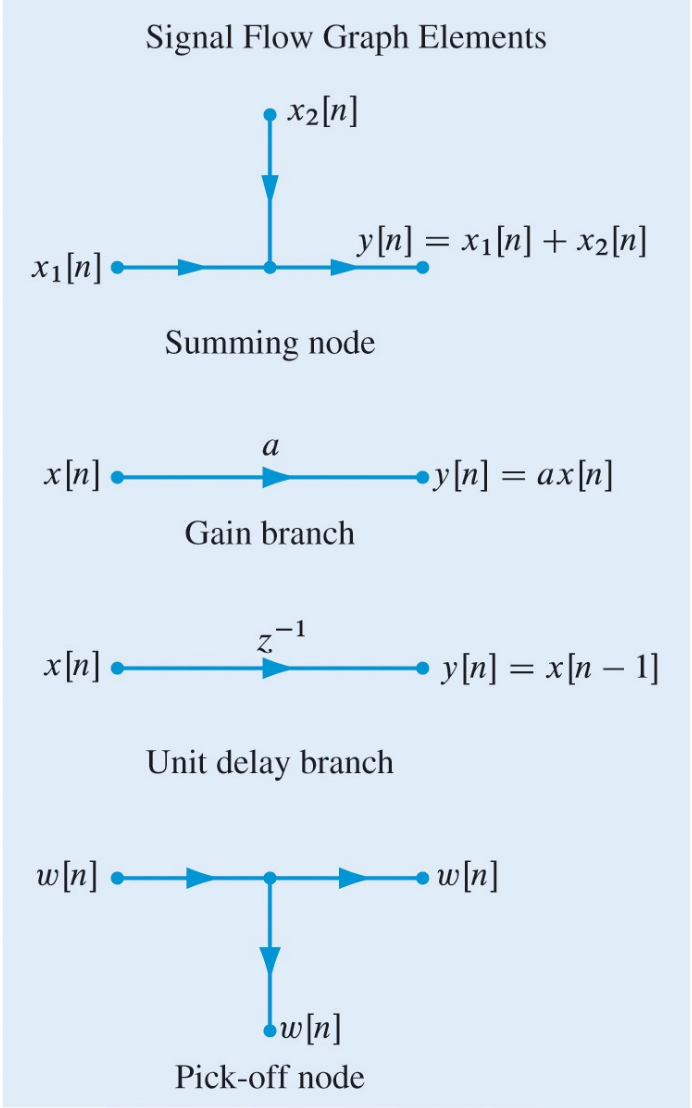
$y[n] = x[n] - x[n-1]$ (First-difference)

System illustrations

Block diagrams

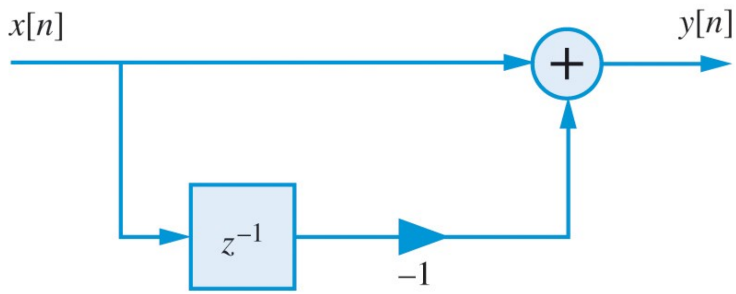


Signal flow graphs

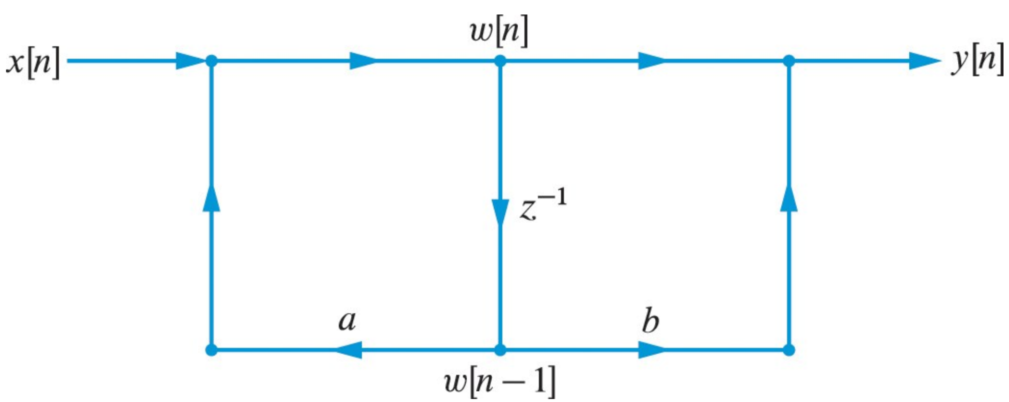


System illustrations

Examples:



$$y[n] = x[n] - x[n-1]$$

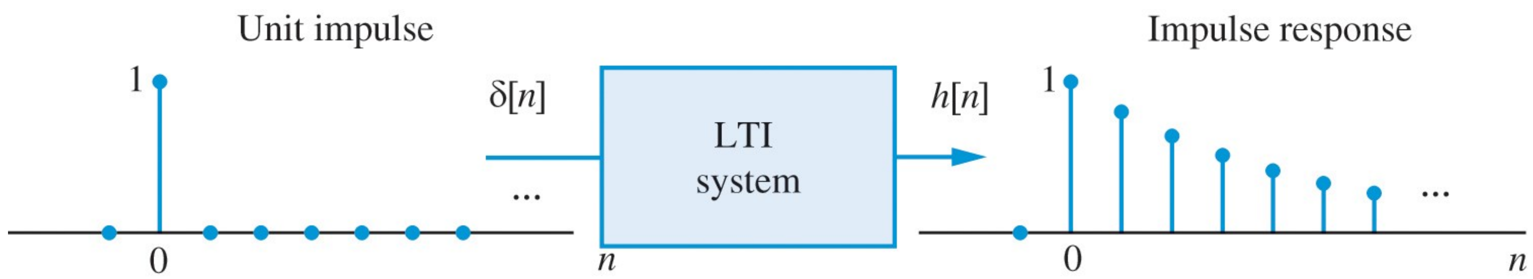


$$w[n] = x[n] + aw[n-1]$$
$$y[n] = w[n] + bw[n-1]$$

$$\rightarrow \quad y[n] = x[n] + bx[n-1] + ay[n-1]$$

Convolution description of linear time-invariant systems

The response of a linear time-invariant (LTI) system to any input can be determined from its response $h[n]$ to the unit sample sequence $\delta[n]$.
The sequence $h[n]$, which is known as *impulse response*, can also be used to infer all properties of a linear time-invariant system.
$h[n]$ can be determined using a formula known as convolution summation.



Convolution description of linear time-invariant systems

We express an input $x[n]$ as a sum of simpler sequences

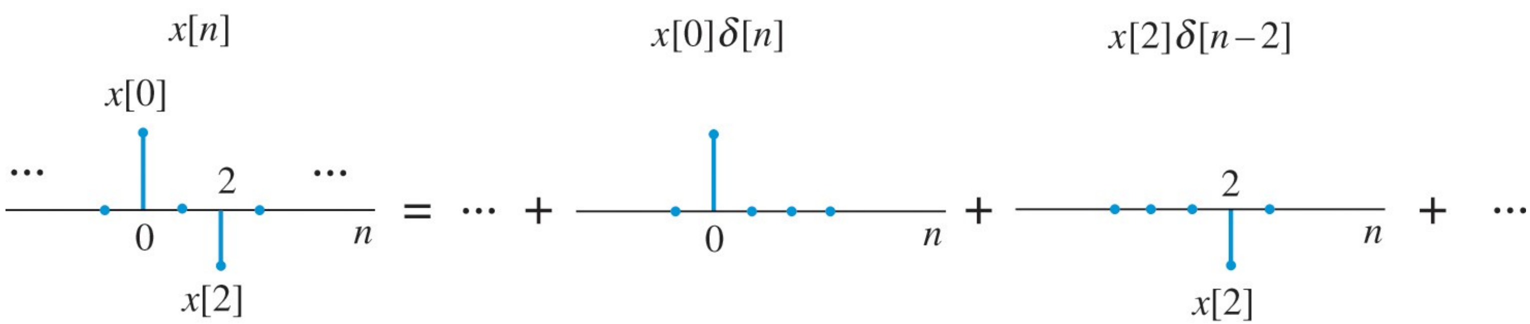
$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \cdots$$

then the response $y[n]$ is given by

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \cdots$$

Signal decomposition into impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \qquad -\infty < n < \infty$$



Convolution sum

Key idea: the output of any LTI system can be determined if we know its impulse response.

Consider an LTI system and denote by $h_k[n] = h[n-k]$ its response to the basic signal $\delta[n-k]$. Then, from the superposition property for a linear system, the response $y[n]$ to the input $x[n]$ is the same linear combination of the basic responses $h_k[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \quad -\infty < n < \infty$$

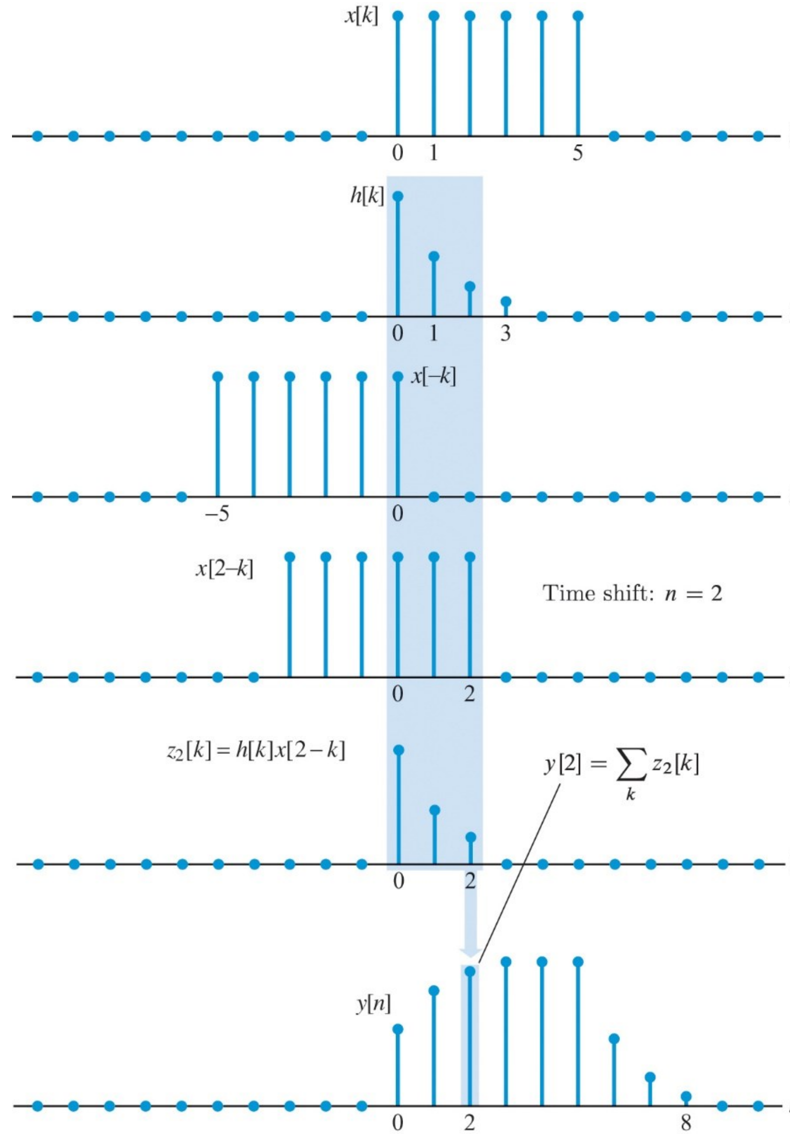
This is called the *convolution sum* or simply *convolution,* and is denoted using the notation $y[n] = x[n] * h[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Convolution sum

Computation of convolution:

1. Change the index of the sequences $h[n]$, $x[n]$ from n to k and plot them as a function of k.
2. Flip (or fold) the sequence $x[k]$ about $k = 0$ to obtain the sequence $x[-k]$.
3. Shift the flipped sequence $x[-k]$ by n samples to the right, if $n > 0$, or to the left, if $n < 0$.
4. Multiply the sequences $h[k]$ and $x[n-k]$ to obtain the sequence $z_n[k] = h[k]x[n-k]$.
5. Sum all nonzero samples of $z_n[k]$ to determine the output sample at the given value of the shift n.
6. Repeat steps $3 - 5$ for all desired values of n.



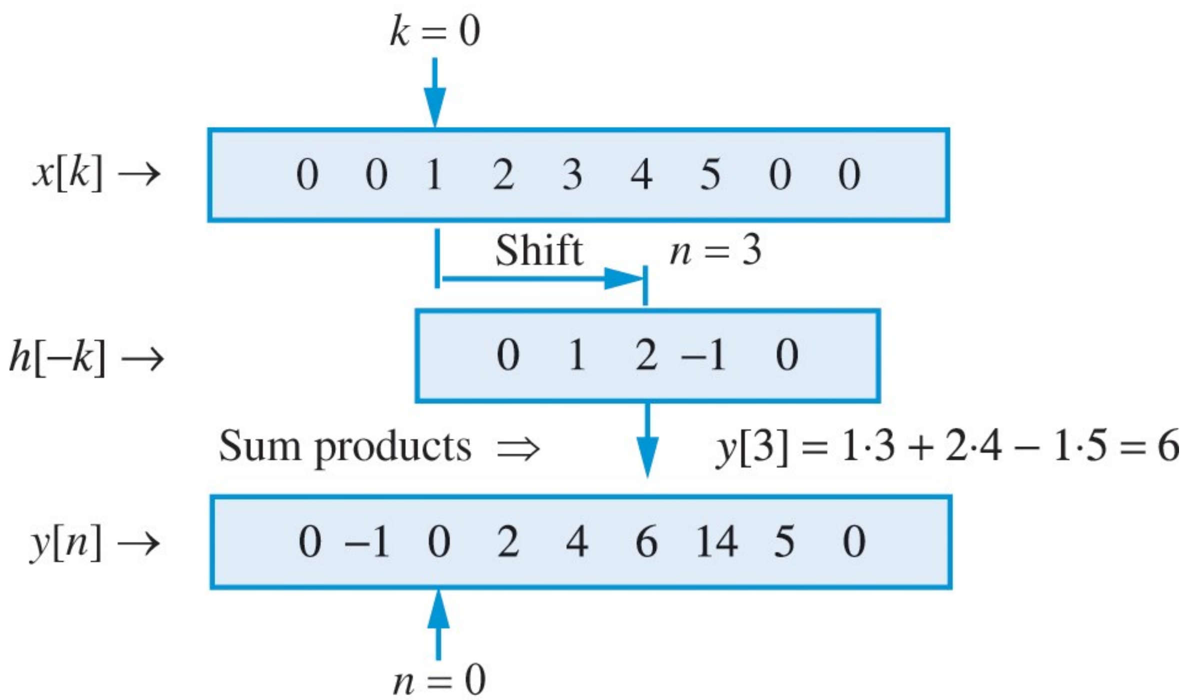
Visit: http://www.fit.vutbr.cz/study/courses/ISS/public/demos/conv/

Convolution sum

Example:

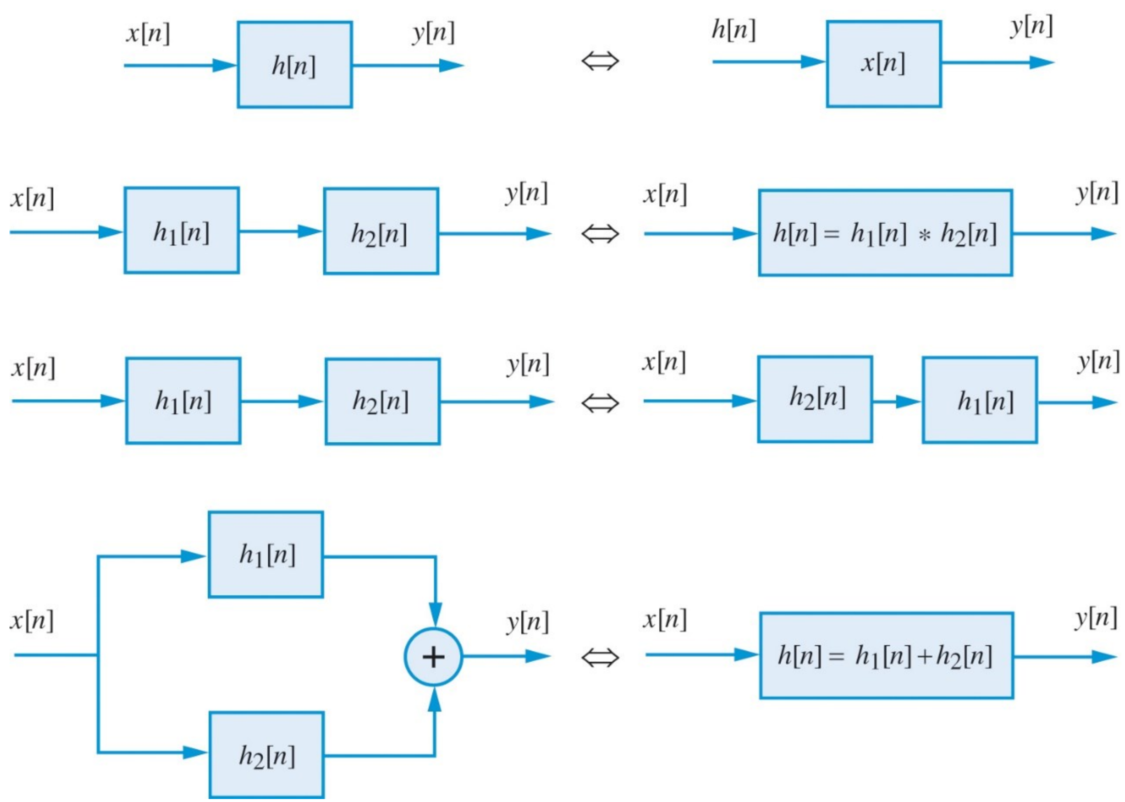
$$x[n] = \{\underset{\uparrow}{1}\ 2\ 3\ 4\ 5\}, \quad h[n] = \{-1\ \underset{\uparrow}{2}\ 1\}.$$

(x[0] =1, h[0] = 2)



Properties of convolution

Property	Formula
Identity	$x[n] * \delta[n] = x[n]$
Delay	$x[n] * \delta[n-n_0] = x[n-n_0]$
Commutative	$x[n] * h[n] = h[n] * x[n]$
Associative	$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$
Distributive	$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$



Causality and stability

A linear time-invariant system with impulse response $h[n]$ is causal if

$$h[n] = 0 \text{ for } n < 0.$$

A linear time-invariant system with impulse response $h[n]$ is stable, in the bounded-input bounded-output sense, if and only if the impulse response is absolutely summable, that is, if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Properties of convolution

Causality and stability

Example

Consider the system with impulse response $h[n] = ba^n u[n]$.

$$S_h = \sum_{k=-\infty}^{\infty} |h[k]| = |b| \sum_{k=0}^{\infty} |a|^n.$$

If $|a| < 1$, the sum converges to $|b|/(1 - |a|)$ where we have used the sum of geometric series formula. Therefore, the system is stable only when $|a| < 1$

Response to simple test sequences

Assume we know the impulse response $h[n]$, of a system.

- **Step response** (the response to the unit step sequence):

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k],$$

  Alternatively, the impulse response is the first-difference of the step response, that is, $h[n] = s[n] - s[n-1]$.

- $\boldsymbol{x[n] = a^n}$, $-\infty < n < \infty$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]a^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]a^{-k}\right) a^n = H(a)a^n$$

Response to simple test sequences

Assume we know the impulse response $h[n]$, of a system.

- Important special case happen when $\boldsymbol{a} = \mathbf{e}^{\mathbf{j}\omega}$

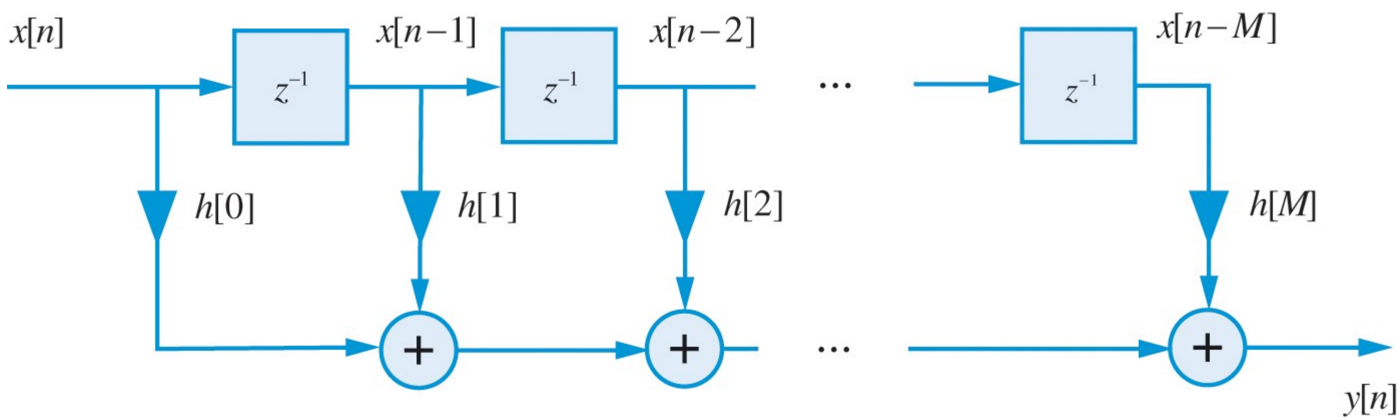
$$y[n] = \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n} = H(e^{j\omega}) e^{j\omega n}.$$

The quantity $H(e^{j\omega})$ *is* known as a frequency response function.

Type of response	Input sequence		Output sequence
Impulse	$x[n] = \delta[n]$	$\xmapsto{\mathcal{H}}$	$y[n] = h[n]$
Step	$x[n] = u[n]$	$\xmapsto{\mathcal{H}}$	$y[n] = s[n] = \sum_{k=-\infty}^{n} h[k]$
Exponential	$x[n] = a^n$, all n	$\xmapsto{\mathcal{H}}$	$y[n] = H(a)a^n$, all n
Complex sinusoidal	$x[n] = e^{j\omega n}$, all n	$\xmapsto{\mathcal{H}}$	$y[n] = H(e^{j\omega})e^{j\omega n}$, all n

$H(a) = \sum_{-\infty}^{\infty} h[n]a^{-n}$

Convolution FIR examples



Block diagram representation of an FIR system.

FIR spatial filters

Digital image:

the sampled version of pictures over a rectangular grid represented by a 2D discrete-space signal $x[m, n]$, $[m, n] \in \{(0, M-1) \times (0, N-1)\}$.

Pixel:

each sample of the digital image is a picture element.

Spatial FIR filters are very popular and useful in the processing of digital images to implement visual effects like noise filtering, edge detection, etc.

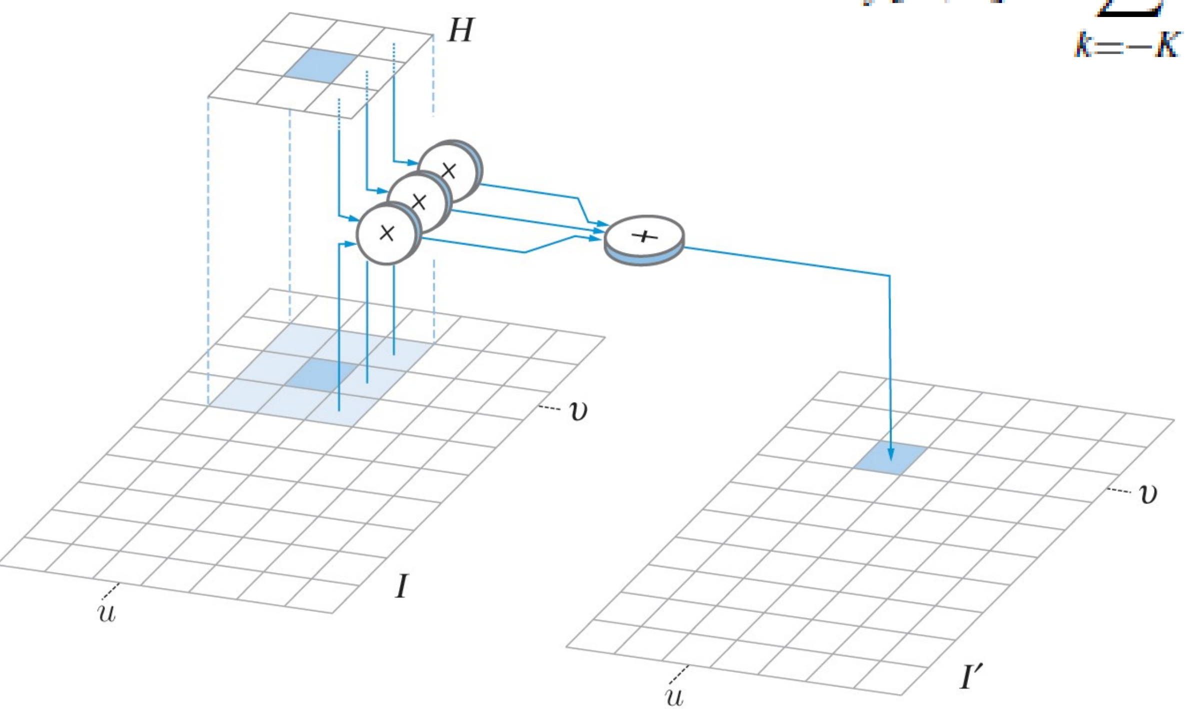
Images have sharp edges when the local intensity rises or drops sharply and have blurred or fuzzy perception when local intensity is smooth.

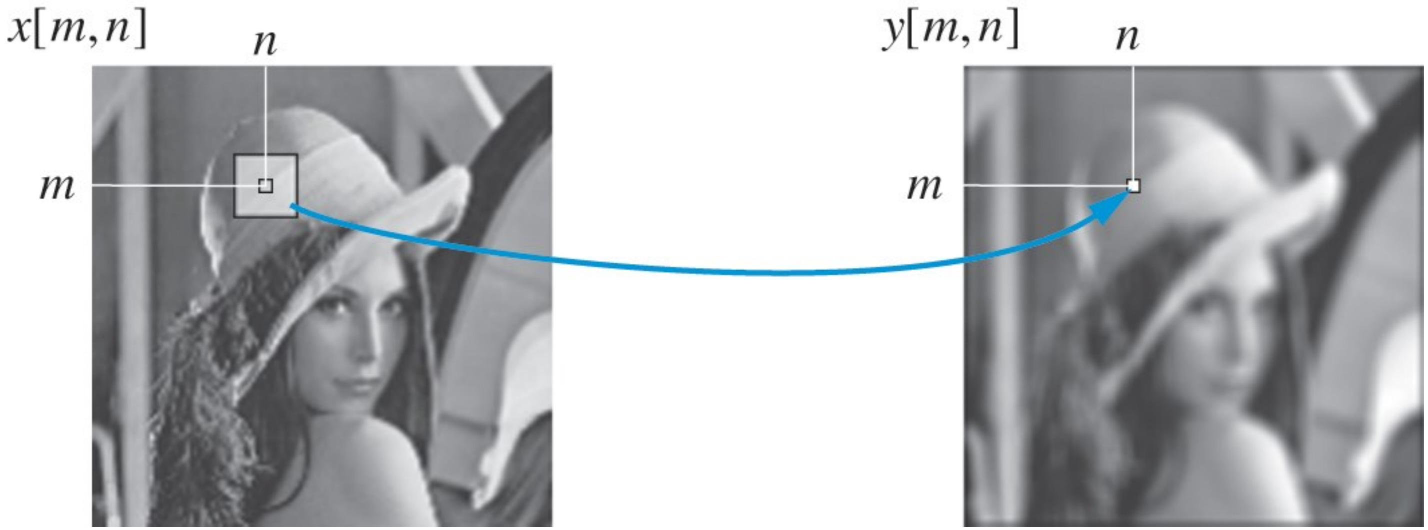
A simple smoothing operation involves replacing each pixel by its average over a local region.

$$h[m,n] = \begin{cases} \frac{1}{9}, & -1 \le m,n \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \rightarrow \quad y[m,n] = \sum_{k=-1}^{1} \sum_{\ell=-1}^{1} h[k,\ell] x[m-k, n-\ell],$$

$$y[m,n] = \sum_{k=-1}^{1} \sum_{\ell=-1}^{1} \left(\frac{1}{9}\right) x[m-k, n-\ell].$$

$$\rightarrow \quad y[m,n] = \sum_{k=-K}^{K} \sum_{\ell=-L}^{L} h[k,\ell] x[m-k, n-\ell].$$




x[m,n]
n
m
y[m,n]
n
m