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Design of IIR filters

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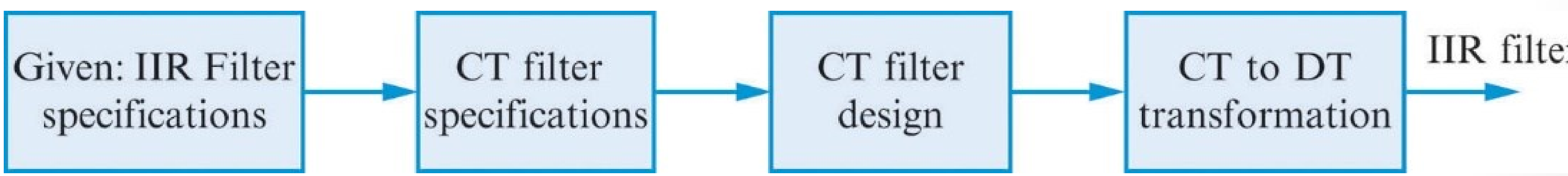
IIR filter design

The system function of a causal, stable, and realizable IIR discrete-time filter can be represented in terms of impulse response, difference-equation coefficients, or zero-pole locations and is, respectively, given by the formulas

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

The objective in IIR filter design is to determine the coefficients so that its frequency response $H(e^{j\omega})$ approximates an ideal desired response $H_d(e^{j\omega})$ according to some criterion of performance.



Procedure for designing IIR filters from continuous-time filters

IIR filter design

FIR filters

Advantages:

- can have exactly linear phase,
- are always stable,
- have design methods that are generally linear in filter parameters,
- can have great flexibility in choosing their frequency response,
- can be realized efficiently in hardware,
- and have finite-duration transients (or start-up responses).

Disadvantages:

- FIR filters often require a much higher filter order than IIR filters to achieve a given level of performance,
- the delay in the output response is often much greater than for an equal performance IIR filter,
- and the design methods often are iterative in nature requiring computer-aided techniques.

IIR filter design

Causal IIR filters *cannot* have linear phase.

- In FIR filter design we avoided this problem by showing that a magnitude response with linear phase can be expressed by an equivalent real-valued amplitude response function.
- This is *not* possible for IIR filters. Since the phase response of IIR filters is generally highly nonlinear, we should always examine the group delay response to see how much frequency dispersal we have within the passband.

Even though FIR filters enjoy many advantages, for most applications, IIR filters are desirable due to their lower order and hence lower cost compared to FIR filters, but if linear-phase response is of paramount interest then FIR filters are preferable.

IIR filter design

Design of continuous-time lowpass filters

For a continuous-time filter with real coefficients we have

$$|H_c(j\Omega)|^2 = H_c(s)H_c(-s)|_{s=j\Omega}$$

A typical pair of factors, like $(s - s_k)(-s - s_k) = s^2_k - s^2$, evaluated at $s = j\Omega$ becomes $(s^2_k + \Omega^2)$. Hence, the magnitude-squared function can always be written as:

$$|H_c(j\Omega)|^2 = G^2 \frac{(\Omega^2 + \zeta_1^2)(\Omega^2 + \zeta_2^2)\cdots(\Omega^2 + \zeta_M^2)}{(\Omega^2 + s_1^2)(\Omega^2 + s_2^2)\cdots(\Omega^2 + s_N^2)}$$

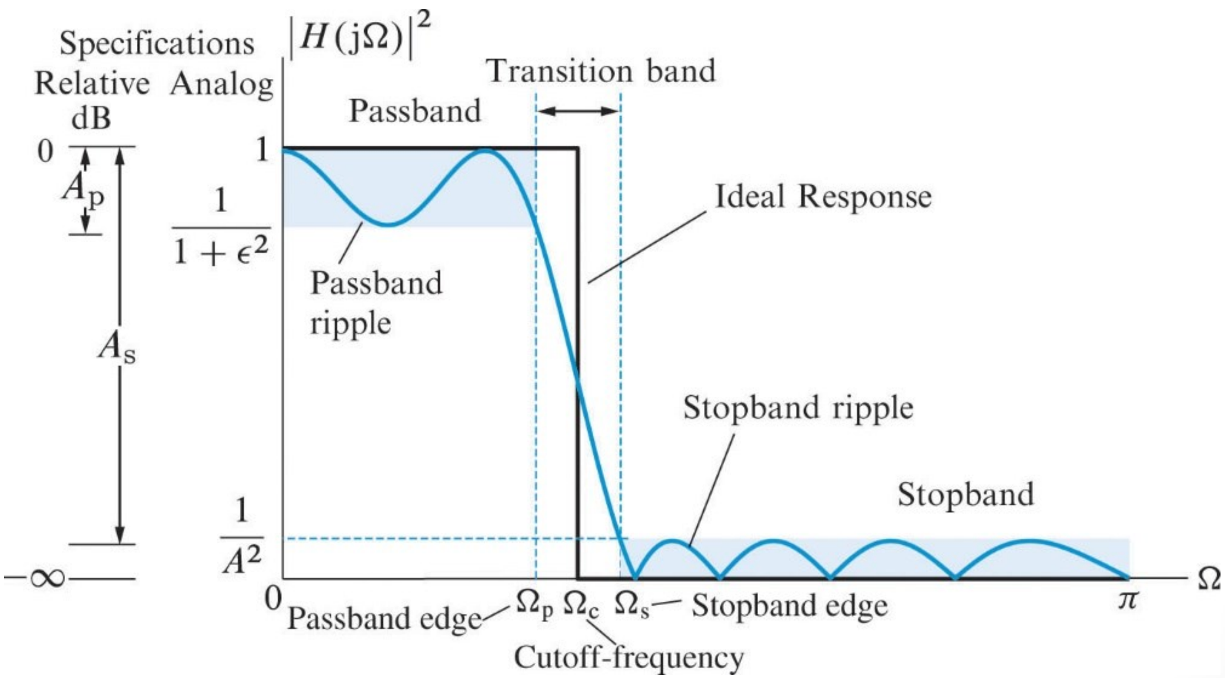
- Since the coefficients of $H_c(s)$ are real, its poles and zeros are either real or they appear in complex conjugate pairs.
- A term like $(\Omega^2 + s^2_k)$ is real when s_k is real; if $s_1 = re^{j\theta}$ and $s_2 = re^{-j\theta}$ are two complex conjugate poles, we have $(\Omega^2+s^2_1)(\Omega^2+s^2_2) = (\Omega^2-r^2)^2 \geq 0$ for all Ω.
- Thus, $|H_c(j\Omega)|^2$ is a positive and real rational function of Ω^2.
- Design techniques for continuous-time filters use $|H_c(j\Omega)|^2$ because it is **real**, **differentiable**, and a **rational function of Ω^2**; the function $|H_c(j\Omega)|$ is real, but it lacks the other two properties.
- Because of the causality and stability requirements we can specify either the magnitude response or the phase response, but *not* both.

IIR filter design

Analog lowpass filter specifications

The objective is to approximate the magnitude-squared response of an ideal lowpass filter

$$|H_d(j\Omega)|^2 = \begin{cases} 1, & 0 \le |\Omega| \le \Omega_c \\ 0. & |\Omega| > \Omega_c \end{cases}$$



Magnitude-squared specifications for lowpass analog filter.

IIR filter design

System function from magnitude-squared response

The classical approximation techniques use a function of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1 + V^2(\Omega)},$$

where $V^2(\Omega) \ll 1$ for $|\Omega| \leq \Omega_c$ and $V^2(\Omega) \gg 1$ for $|\Omega| > \Omega_c$. Different choices for $V(\Omega)$ lead to different design techniques.

The problem is now reduced to obtaining a causal and stable system $H_c(s)$ from the magnitude-squared function. Since $H_c(s)$ has real coefficients, the poles and zeros of $H_c(-s)H_c(s)$ are symmetrically located with respect to both the real and the imaginary axes (quadrantal symmetry).

Therefore, we can obtain a causal and stable system by choosing the poles on the left-half plane; the zeros can be anywhere.

However, we typically choose the zeros on the left-half plane, which results in a minimum-phase system.

IIR filter design

The Butterworth approximation

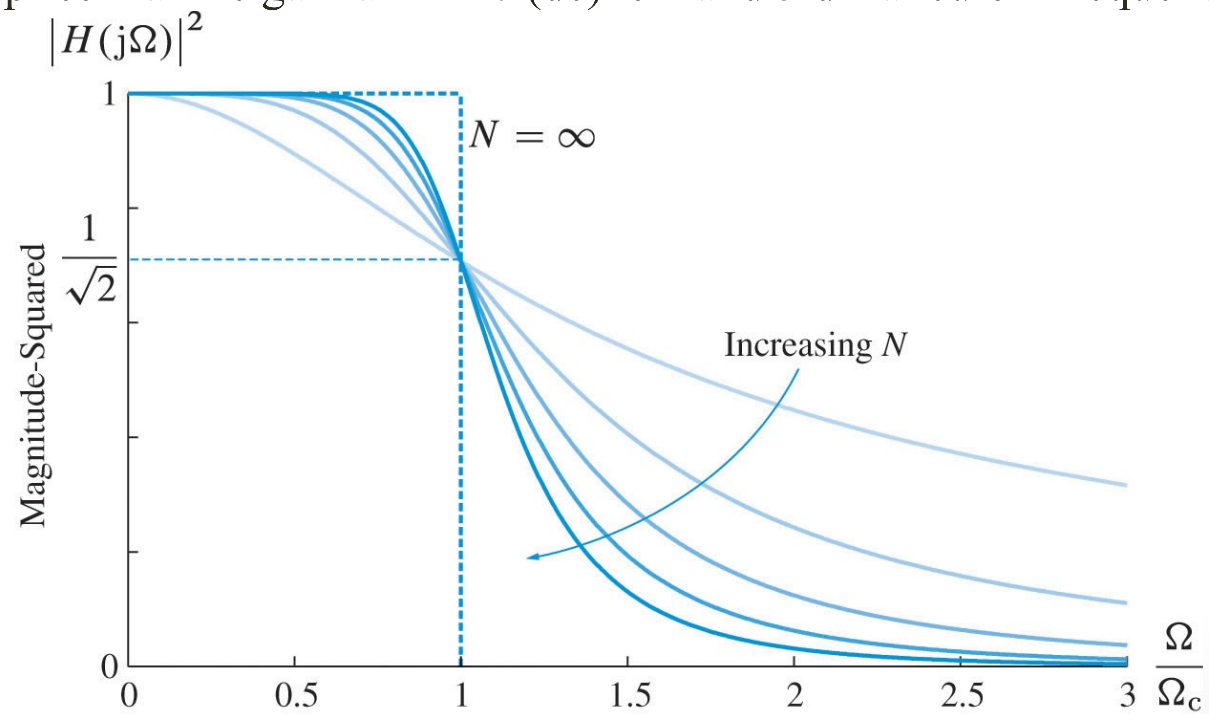
Butterworth suggested that $V(\Omega) = (\Omega/\Omega_c)^{2N}$ be used as an approximation:

$$|H_B(j\Omega)|^2 \triangleq \frac{1}{1+(\Omega/\Omega_c)^{2N}}, \quad N = 1, 2, \ldots$$

For every value of N we have

$$|H_B(j0)|^2 = 1, \quad |H_B(j\Omega_c)|^2 = 1/2, \quad \text{and} \quad |H_B(j\infty)|^2 = 0.$$

This implies that the gain at $\Omega = 0$ (dc) is 1 and 3 dB at cutoff frequency Ω_c.



The magnitude of Butterworth functions of various orders.

IIR filter design

The Butterworth approximation

The Taylor series expansion of Butterworth magnitude-squared response about $\Omega = 0$ can be found from the series $1/(1+x)=1 - x + x^2 - x^3 + \cdots$, where $|x| < 1$, by letting $x = (\Omega/\Omega_c)^2$. Therefore, the error in the passband is given by

$$E_c^2(j\Omega) = 1 - |H_B(j\Omega)|^2 = (\Omega/\Omega_c)^{2N} - (\Omega/\Omega_c)^{4N} + \cdots$$

The first $(2N-1)$ derivatives with respect to Ω are 0 at $\Omega = 0$. Thus, Butterworth filters are also called maximally flat magnitude filters.

For frequencies $|\Omega| \gg \Omega_c$ we have the asymptotic approximation

$$|H_B(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}} \simeq \frac{1}{(\Omega/\Omega_c)^{2N}}$$

IIR filter design

The Butterworth approximation

Pole locations:

The poles of $H_B(s)H_B(-s)$ are found by solving the equation

$$1 + (s/\mathrm{j}\Omega_c)^{2N} = 0 \quad \text{or} \quad (s/\mathrm{j}\Omega_c)^{2N} = -1 = e^{\mathrm{j}(2k-1)\pi}$$

$s_k = \sigma_k + \mathrm{j}\Omega_k$ for any even or odd value of N

$$\sigma_k = \Omega_c \cos\theta_k,$$

$$\Omega_k = \Omega_c \sin\theta_k,$$

where

$$\theta_k \triangleq \frac{\pi}{2} + \frac{2k-1}{2N}\pi. \qquad k = 1, 2, \ldots, 2N$$

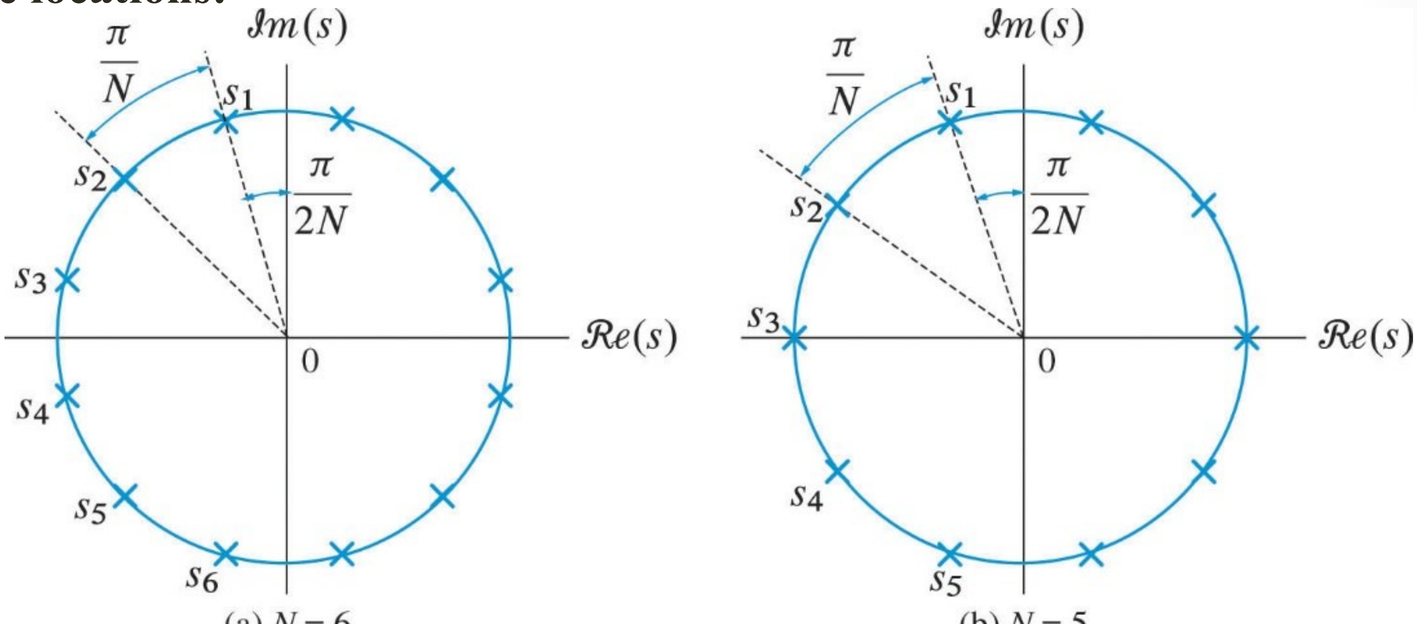
$$\rightarrow \qquad |s_k|^2 = \sigma_k^2 + \Omega_k^2 = \Omega_c^2 \cos^2\theta_k + \Omega_c^2 \sin^2\theta_k = \Omega_c^2.$$

- The poles of a Butterworth filter lie on a circle with radius Ω_c and are equiangularly spaced with angular separation π/N.
- Poles never fall on the imaginary axis.

IIR filter design

The Butterworth approximation

Pole locations:



Pole locations of Butterworth magnitude-squared function in the *s*-plane for: (a) even *N*, and (b) odd *N*. Poles on the left-half plane correspond to a stable system.

We form the stable system function $H_B(s)$ by choosing poles for $k = 1, 2, \ldots, N$, which clearly lie in the left-half plane:

$$H_B(s) = \frac{\Omega_c^N}{(s - s_1)(s - s_2)\ldots(s - s_N)}.$$

IIR filter design

The Butterworth approximation

Design procedure

Suppose we wish to design a Butterworth lowpass filter specified by the parameters Ω_p, A_p, Ω_s, and A_s. The design process consists of determining the parameters N and Ω_c so that

$$\frac{1}{1+(\Omega_p/\Omega_c)^{2N}} \geq \frac{1}{1+\epsilon^2} \quad \text{or} \quad (\Omega_p/\Omega_c)^{2N} \leq \epsilon^2,$$

$$\frac{1}{1+(\Omega_s/\Omega_c)^{2N}} \leq \frac{1}{A^2} \quad \text{or} \quad (\Omega_s/\Omega_c)^{2N} \geq A^2 - 1.$$

$$\rightarrow \qquad \Omega_s^N \geq \Omega_c^N \sqrt{A^2-1} \geq \Omega_p^N \frac{\sqrt{A^2-1}}{\epsilon}.$$

$$\rightarrow \qquad N \geq \frac{\ln \beta}{\ln \alpha},$$

where $$\alpha \triangleq \frac{\Omega_s}{\Omega_p}, \quad \beta \triangleq \frac{1}{\epsilon}\sqrt{A^2-1} = \frac{\sqrt{10^{A_s/10}-1}}{\sqrt{10^{A_p/10}-1}}.$$

IIR filter design

The Butterworth approximation

Design procedure

- The value of N is chosen as the largest integer satisfying $N \geq \frac{\ln \beta}{\ln \alpha}$,
- The frequency Ω_c can be chosen anywhere in the interval

$$\Omega_p(10^{A_p/10} - 1)^{-1/(2N)} \leq \Omega_c \leq \Omega_s(10^{A_s/10} - 1)^{-1/(2N)}$$

To ensure a smaller ripple in the passband, we choose Ω_c using the right limit.

IIR filter design

The Butterworth approximation

Example: Design procedure – Butterworth approximation

$$-6 \text{ dB} \le 20\log_{10}|H(j\Omega)| \le 0, \quad 0 \le |\Omega| \le 2\frac{\text{rad}}{\text{sec}},$$

$$20\log_{10}|H(j\Omega)| \le -20 \text{ dB}, \quad 3\frac{\text{rad}}{\text{sec}} \le |\Omega| < \infty.$$

Thus $A_p = 6$ and $A_s = 20$.

$$\longrightarrow \qquad \epsilon = \sqrt{10^{0.1(6)}} = 1.7266 \quad \text{and} \quad A = 10^{0.05(20)} = 10.$$

Step-1 Compute the parameters α and β

$$\alpha = \frac{3}{2} = 1.5; \quad \beta = \frac{1}{1.7266}\sqrt{10^2 - 1} = 5.7628.$$

Step-2 Compute order N

$$N = \left\lceil \frac{\ln(5.7628)}{\ln(1.5)} \right\rceil = \lceil 4.3195 \rceil = 5.$$

Step-3 Determine 3 dB cutoff frequency Ω_c

the lower values of Ω_c is $2(10^{6/10} - 1)^{-1/(10)} = 1.7931$
and the upper value of Ω_c is $3(10^{20/10} - 1)^{-1/(10)} = 1.8948$,
We choose the upper value $\Omega_c = 1.8948$ rad/s, which satisfies the specifications at Ω_s and provides a smaller ripple at Ω_p.

IIR filter design

The Butterworth approximation

Example: Design procedure – Butterworth approximation

Step-4 Compute pole locations

The poles of $H_B(j\Omega)$ are located on a circle of radius c = 1.8948 at angles

$$\theta_k = \frac{\pi}{2} + \frac{2k-1}{10}\pi = 0.4\pi + 0.2k\pi,$$

$$s_k = 1.8948\cos(0.4\pi + 0.2k\pi) + j1.8948\sin(0.4\pi + 0.2k\pi),\ k = 1, \ldots, 5.$$

Step-5 Compute the system function $H_B(j\Omega)$

$$H_B(j\Omega) = \frac{1.8948^5}{\Pi_{k=1}^{5}(s - s_k)}$$

$$= \frac{24.42}{s^5 + 6.13s^4 + 18.80s^3 + 35.61s^2 + 41.71s + 24.42}$$

IIR_Butterworth.py

IIR filter design

The Chebyshev approximation

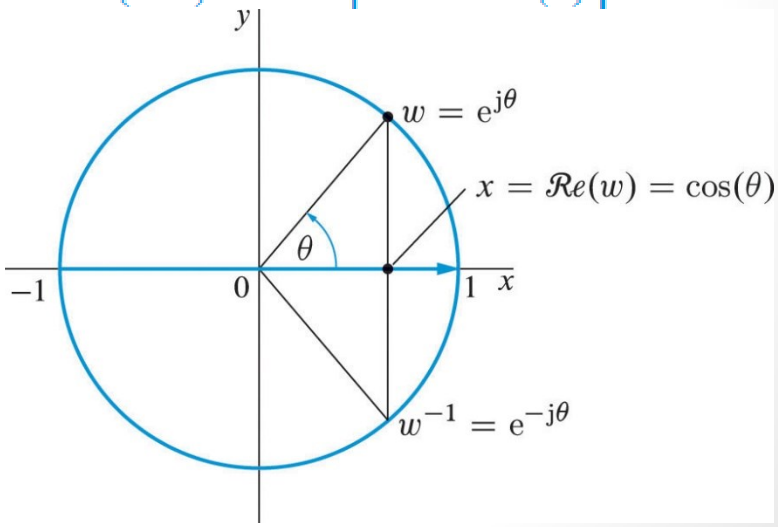
The Chebyshev approximation is optimum according to the minimax criterion which results in equiripple behavior.

We note that for each $x \in [-1, 1]$, there is a complex number w on the unit circle, say $w = e^{j\theta}$, such that

$$x = \mathcal{R}e(w) = \frac{1}{2}\left(w + w^{-1}\right) = \cos\theta \in [-1, 1]$$

The mth-order *Chebyshev polynomial*, denoted by $T_m(x)$, is defined by

$$T_m(x) \triangleq \mathcal{R}e\left(w^m\right) = \frac{1}{2}\left(w^m + w^{-m}\right) = \cos(\theta m) = \cos\left[m \cos^{-1}(x)\right]$$



Recall the trigonometric identity

$$\cos[(m+1)\theta] = 2\cos(\theta)\cos(m\theta) - \cos[(m-1)\theta], \quad m \geq 1$$

$$\rightarrow T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x). \quad m \geq 1$$

With $T_0(x) = 1$ and $T_1(x) = x$

IIR filter design

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The Chebyshev approximation is optimum according to the minimax criterion which results in equiripple behavior.

$$T_m(x) \triangleq \mathcal{R}e\left(w^m\right) = \frac{1}{2}\left(w^m + w^{-m}\right) = \cos(\theta m) = \cos\left[m\cos^{-1}(x)\right]$$

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x). \quad m \geq 1$$

Low order Chebyshev polynomials.

Order	Polynomial $T_m(x) = \cos[m\cos^{-1}(x)]$
0	$T_0(x) = 1$
1	$T_1(x) = x$
2	$T_2(x) = 2x^2 - 1$
3	$T_3(x) = 4x^3 - 3x$
4	$T_4(x) = 8x^4 - 8x^2 + 1$
5	$T_5(x) = 16x^5 - 20x^3 + 5x$
6	$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

Note: $|T_m(x)| \leq 1$ for $x \in [-1, 1]$, even though its leading coefficient is as large as 2^{m-1}.

IIR filter design

The Chebyshev approximation

The Chebyshev approximation is optimum according to the minimax criterion which results in equiripple behavior.

Chebyshev lowpass filter approximation

$$|H_C(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)},$$

where $T_N(x)$, $x = \Omega/\Omega_c$, is the Nth order Chebyshev polynomial:

$$T_m(x) \triangleq \mathcal{R}e\left(w^m\right) = \frac{1}{2}\left(w^m + w^{-m}\right) = \cos(\theta m) = \cos\left[m\cos^{-1}(x)\right]$$

where $w = e^{j\theta}$ is a complex number on the unit circle.

Recall the trigonometric identity

$$\cos[(m+1)\theta] = 2\cos(\theta)\cos(m\theta) - \cos[(m-1)\theta], \quad m \geq 1$$

$$\rightarrow \quad T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x). \quad m \geq 1$$

IIR filter design

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$$T_m(x) \triangleq \mathcal{R}e(w^m) = \frac{1}{2}\left(w^m + w^{-m}\right) = \cos(\theta m) = \cos\left[m\cos^{-1}(x)\right]$$

where $w = e^{j\theta}$ is a complex number on the unit circle.

Since $|T_N(x)| \le 1$ for $|x| \le 1$ we have $|T_N(\Omega/\Omega_c)| \le 1$ for $|\Omega| \le \Omega_c$.
If we choose $\epsilon^2 \ll 1$, the approximation error in the passband is given by:

$$E_C^2(\Omega/\Omega_c) = 1 - \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)} \simeq \epsilon^2 T_N^2(\Omega/\Omega_c). \quad |\Omega| \le \Omega_c$$

IIR filter design

The Chebyshev approximation

The weighted error $(1/\epsilon)E_C(\Omega/\Omega_c)$ can be expressed as a single Chebyshev polynomial $T_N(\Omega/\Omega_c)$, therefore the optimum equiripple lowpass filter approximation within the entire passband is provided by

$$|H_C(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)},$$

The leading term of $T_N(x)$ is $2^{N-1}x^N$, therefore the values of $T^2_N(x)$ grow very fast for $|x| > 1$. Thus, in the stopband we have $T^2_N(\Omega/\Omega_c) \gg 1$ or equivalently $|H_C(j\,\Omega)|^2 \ll 1$, for $|\Omega| > \Omega_c$.

Low order Chebyshev polynomials.

Order	Polynomial $T_m(x) = \cos[m\cos^{-1}(x)]$
0	$T_0(x) = 1$
1	$T_1(x) = x$
2	$T_2(x) = 2x^2 - 1$
3	$T_3(x) = 4x^3 - 3x$
4	$T_4(x) = 8x^4 - 8x^2 + 1$
5	$T_5(x) = 16x^5 - 20x^3 + 5x$
6	$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

IIR filter design

The Chebyshev approximation

The formula used for $T_N(x)$ in the passband

$$T_N(x) = \cos(N \cos^{-1} x), \quad |x| \le 1$$

is not valid for $|x| > 1$.

By replacing the trigonometric functions by their hyperbolic counterparts:

$$\cosh(x) \triangleq \frac{1}{2}(e^x + e^{-x}), \quad \cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\sinh(x) \triangleq \frac{1}{2}(e^x - e^{-x}), \quad \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Then we have

$$x = \cos(j\phi) = (e^{j(j\phi)} + e^{-j(j\phi)})/2 = \cosh\phi \text{ or } \phi = \cosh^{-1} x.$$

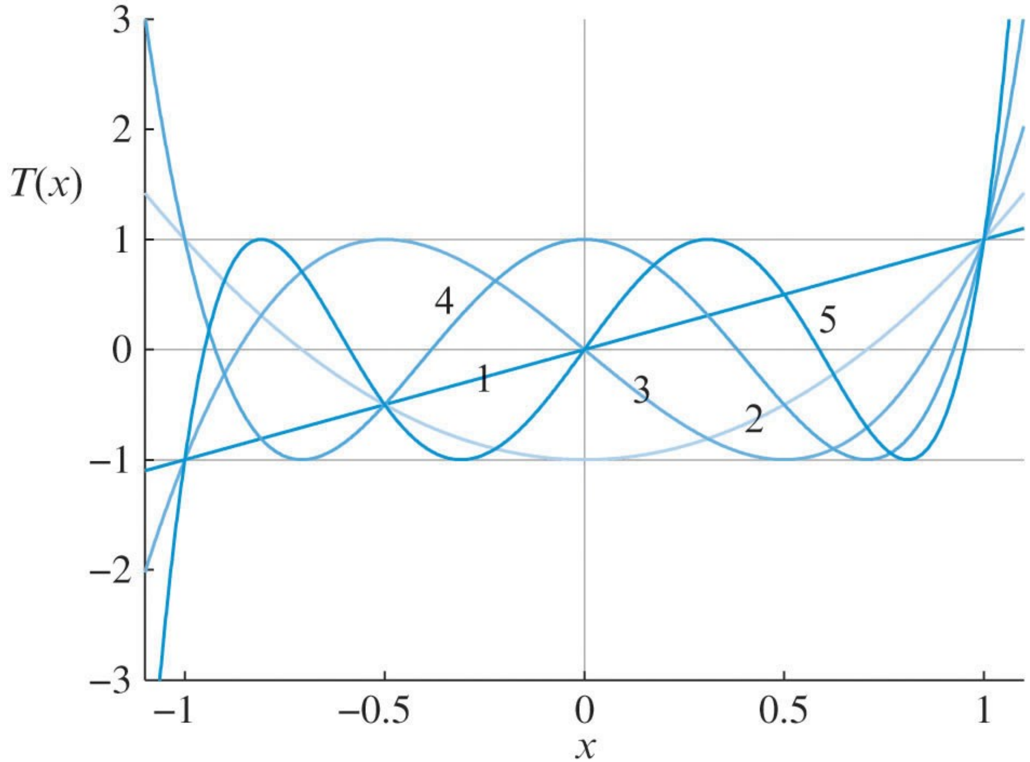
$$\cos(N\phi) = \cos(Nj\phi) = \cosh(N\phi)$$

This leads to the formula

$$T_N(x) = \cosh(N \cosh^{-1} x). \quad |x| > 1$$

IIR filter design

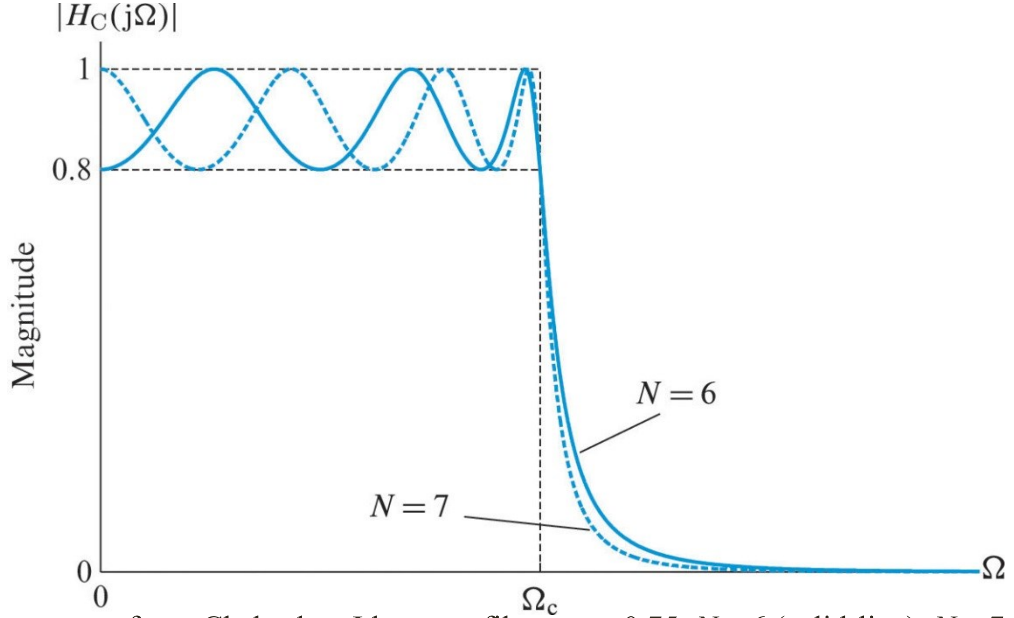
The Chebyshev approximation



Graphs of Chebyshev polynomials $T_N(x)$ for $N = 1, 2, \ldots, 5$.

IIR filter design

The Chebyshev approximation



Magnitude responses of two Chebyshev I lowpass filters. $\varepsilon = 0.75$, $N = 6$ (solid-line), $N = 7$ (dashed-line).

An Nth-order prototype lowpass Chebyshev I filter has the following basic properties:

1. For $|\Omega| \leq \Omega_c$, $|H_C(j\Omega)|^2$ has equiripple behavior between $\frac{1}{1+\varepsilon^2}$ and 1.
2. For $|\Omega| \geq \Omega_c$, $|H_C(j\Omega)|^2$ decreases monotonically toward zero.
3. From the definition of Chebyshev polynomials we have

Normalization condition: $$|H_C(j0)|^2 = \begin{cases} 1, & N \text{ odd} \\ 1/(1+\epsilon^2), & N \text{ even} \end{cases}$$ $$|H_C(j\Omega_c)|^2 = \frac{1}{1+\epsilon^2}$$

For $|\Omega| \gg \Omega_c$ $$|H_C(j\Omega)|^2 \simeq \left[\epsilon^2 2^{2(N-1)} (\Omega/\Omega_c)^{2N}\right]^{-1}$$

IIR filter design

The Chebyshev approximation

Pole locations

The poles of the product $H_C(s)H_C(-s)$ are obtained by solving the Equation:

$$T_N(s/\mathrm{j}\Omega_c) = \pm\mathrm{j}/\epsilon.$$

$$\longrightarrow \quad T_N(s/\mathrm{j}\Omega_c) = \cos[N\cos^{-1}(s/\mathrm{j}\Omega_c)] = \pm\mathrm{j}/\epsilon.$$

We define

$$w \triangleq u + \mathrm{j}v = \cos^{-1}(s/\mathrm{j}\Omega_c).$$

Considering $\sinh(x) = -\mathrm{j}\sin(\mathrm{j}x)$.

$$\longrightarrow \quad \cos[N(u+\mathrm{j}v)] = \cos(Nu)\cosh(Nv) - \mathrm{j}\sin(Nu)\sinh(Nv) = \pm\mathrm{j}/\varepsilon$$

Equating the real parts:

$$\longrightarrow \quad \cos(Nu)\cosh(Nv) = 0.$$

$\cosh(Nv) \geq 1$ for all values of Nv, $\quad \rightarrow \quad \cos(Nu) = 0$

$$u_k = \frac{2k-1}{N}\frac{\pi}{2}, \quad k = 1, 2, \ldots, 2N.$$

IIR filter design

The Chebyshev approximation

Equating the imaginary parts and recognizing that for all values of *u*, $\sin(Nu) = \pm 1$

$$v = -\frac{1}{N}\sinh^{-1}\frac{1}{\epsilon} \triangleq -\phi.$$

$$\longrightarrow \quad s_k = j\Omega_c\cos(u_k + jv) = \Omega_c\sin(u_k)\sinh(v) + j\Omega_c\cos(u_k)\cosh(v).$$

Thus, the poles $s_k = \sigma_k + j\Omega_k$

$$\sigma_k = -[\Omega_c\sinh(\phi)]\sin u_k, \quad \Omega_k = [\Omega_c\cosh(\phi)]\cos u_k.$$

$$\sigma_k = [\Omega_c\sinh(\phi)]\cos(\theta_k),$$

$$\Omega_k = [\Omega_c\cosh(\phi)]\sin(\theta_k),$$

$$\longrightarrow \quad \theta_k = \frac{\pi}{2} + \frac{2k-1}{2N}\pi, \quad k = 1, 2, \ldots, 2N,$$

we assign to $H_C(s)$ the poles located on the left-half plane ($\sigma_k < 0$),

$$H_C(s) = \frac{G}{\prod_{k=1}^{N}(s - s_k)}, \quad G = \prod_{k=1}^{N}(-s_k) \times \begin{cases} 1/\sqrt{1+\epsilon^2}, & N \text{ even} \\ 1, & N \text{ odd} \end{cases}$$

Where *G* is selected to satisfy the normalization condition.

IIR filter design

The Chebyshev approximation

After some algebraic manipulations we obtain

$$a \triangleq \sinh(\phi) = \frac{1}{2}(\gamma - \gamma^{-1}),$$

$$b \triangleq \cosh(\phi) = \frac{1}{2}(\gamma + \gamma^{-1}),$$

where

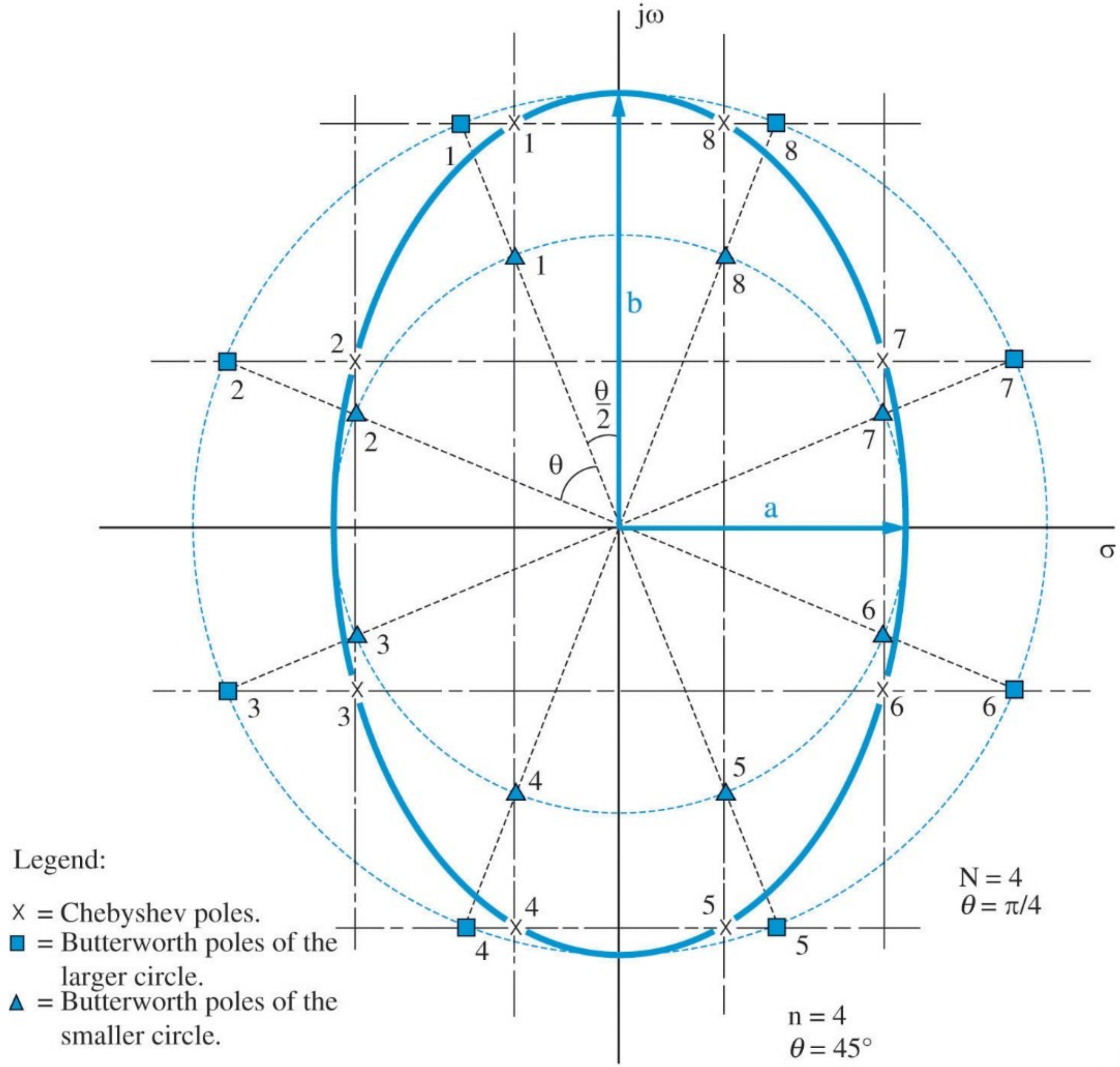
$$\gamma \triangleq \left(1/\epsilon + \sqrt{1 + 1/\epsilon^2}\right)^{1/N}$$

$$\sin^2\theta_k + \cos^2\theta_k = 1 \quad \longrightarrow \quad \left(\frac{\sigma_k}{\Omega_c a}\right)^2 + \left(\frac{\Omega_k}{\Omega_c b}\right)^2 = 1$$

If we drop the index k and let σ and Ω have any values, we note that this equation is the equation for an ellipse with major semi-axis $\Omega_c b$ and minor semi-axis $\Omega_c a$. Since $b > a$, the major axis of the ellipse lies along the j axis.

IIR filter design

The Chebyshev approximation



Geometrical construction of Chebyshev filter poles.

IIR filter design

The Chebyshev approximation

Design procedure

we wish to design a Chebyshev lowpass filter specified by the parameters Ω_p, A_p, Ω_s, and A_s

For equiripple response in the passband we choose $\Omega_c = \Omega_p$. Thus, the constraint on the stopband is given by

$$\frac{1}{1+\epsilon^2 T_N^2(\Omega_s/\Omega_p)} \leq \frac{1}{A^2} \quad \text{or} \quad T_N(\Omega_s/\Omega_p) \geq \frac{1}{\epsilon}\sqrt{A^2-1}.$$

$\Omega_s/\Omega_p > 1$

$$\left.\begin{array}{l} \\ T_N(x) = \cosh(N\cosh^{-1}x). \quad |x|>1 \end{array}\right\} \cosh\left[N\cosh^{-1}(\Omega_s/\Omega_p)\right] \geq \frac{1}{\epsilon}\sqrt{A^2-1}.$$

$$\rightarrow \quad N \geq \frac{\cosh^{-1}(\beta)}{\cosh^{-1}(\alpha)} = \frac{\ln(\beta+\sqrt{\beta^2-1})}{\ln(\alpha+\sqrt{\alpha^2-1})},$$

$$\alpha = \frac{\Omega_s}{\Omega_p}, \quad \beta = \frac{1}{\epsilon}\sqrt{A^2-1} = \frac{\sqrt{10^{A_s/10}-1}}{\sqrt{10^{A_p/10}-1}}.$$

IIR filter design

The Chebyshev approximation

Scipy functions

scipy.signal.cheb1ord(*wp***,** *ws***,** *gpass***,** *gstop***,** *analog=False***)**

scipy.signal.cheby1(*N***,** *rp***,** *Wn***,** *btype='low'***,** *analog=False***,** *output='ba'***)**

Example

Design procedure – Chebyshev I approximation

The specifications of the analog lowpass filter is given by

$$-6\text{ dB} \le 20\log_{10}|H(j\Omega)| \le 0, \quad 0 \le |\Omega| \le 2\tfrac{\text{rad}}{\text{sec}},$$

$$20\log_{10}|H(j\Omega)| \le -20\text{ dB}, \quad 3\tfrac{\text{rad}}{\text{sec}} \le |\Omega| < \infty.$$

$$20\log_{10}\left(\sqrt{1+\epsilon^2}\right) = A_\text{p} \quad \text{and} \quad 20\log_{10}(A) = A_\text{s}$$

$\longrightarrow$ $\varepsilon = 1.7266$ and $A = 10$

IIR filter design

The Chebyshev approximation

Step-1 Compute the parameters α and β using

$$\alpha = \frac{3}{2} = 1.5, \quad \beta = \frac{1}{1.7266}\sqrt{10^2 - 1} = 5.7628.$$

Step-2 Compute order *N* and round upwards to the nearest integer:

$$N = \left\lceil \frac{\ln\left(5.7628 + \sqrt{5.7628^2 - 1}\right)}{\ln\left(1.5 + \sqrt{1.5^2 - 1}\right)} \right\rceil = \lceil 2.5321 \rceil = 3.$$

Step-3 Set $\Omega_c = \Omega_p$ and compute *a* and *b*

$$\Omega_c = \Omega_p = 2; \quad \gamma = \left(1/1.7266 + \sqrt{1 + 1/1.7266^2}\right)^{1/3} = 1.2016,$$

$$a = \frac{1}{2}(1.2016 - 1/1.2016) = 0.1847,$$

$$b = \frac{1}{2}(1.2016 + 1/1.2016) = 1.0169.$$

IIR filter design

The Chebyshev approximation

Step-4 Compute the pole locations

$$s_1 = (0.1847)(2)\cos(\tfrac{\pi}{2} + \tfrac{\pi}{6}) + \mathrm{j}(1.0169)(2)\sin(\tfrac{\pi}{2} + \tfrac{\pi}{6})$$
$$= -0.1847 - \mathrm{j}1.7613,$$
$$s_2 = (0.1847)(2)\cos(\tfrac{\pi}{2} + \tfrac{3\pi}{6}) + \mathrm{j}(1.0169)(2)\sin(\tfrac{\pi}{2} + \tfrac{3\pi}{6})$$
$$= -0.3693,$$
$$s_3 = (0.1847)(2)\cos(\tfrac{\pi}{2} + \tfrac{5\pi}{6}) + \mathrm{j}(1.0169)(2)\sin(\tfrac{\pi}{2} + \tfrac{5\pi}{6})$$
$$= -0.1847 + \mathrm{j}1.7613.$$

Step-5 Compute the filter gain G and the system function $H_C(\mathrm{j}\Omega)$

$$G = -(-0.1847 - \mathrm{j}1.7613)(-0.3693)(-0.1847 + \mathrm{j}1.7613)(1)$$
$$= 1.1584,$$
$$H_C(s) = \frac{1.1584}{(s + 0.1847 + \mathrm{j}1.7613)(s + 0.3693)(s + 0.1847 - \mathrm{j}1.7613)}$$
$$= \frac{1.1584}{s^3 + 0.7387s^2 + 3.2728s + 1.1584}.$$

IIR filter design

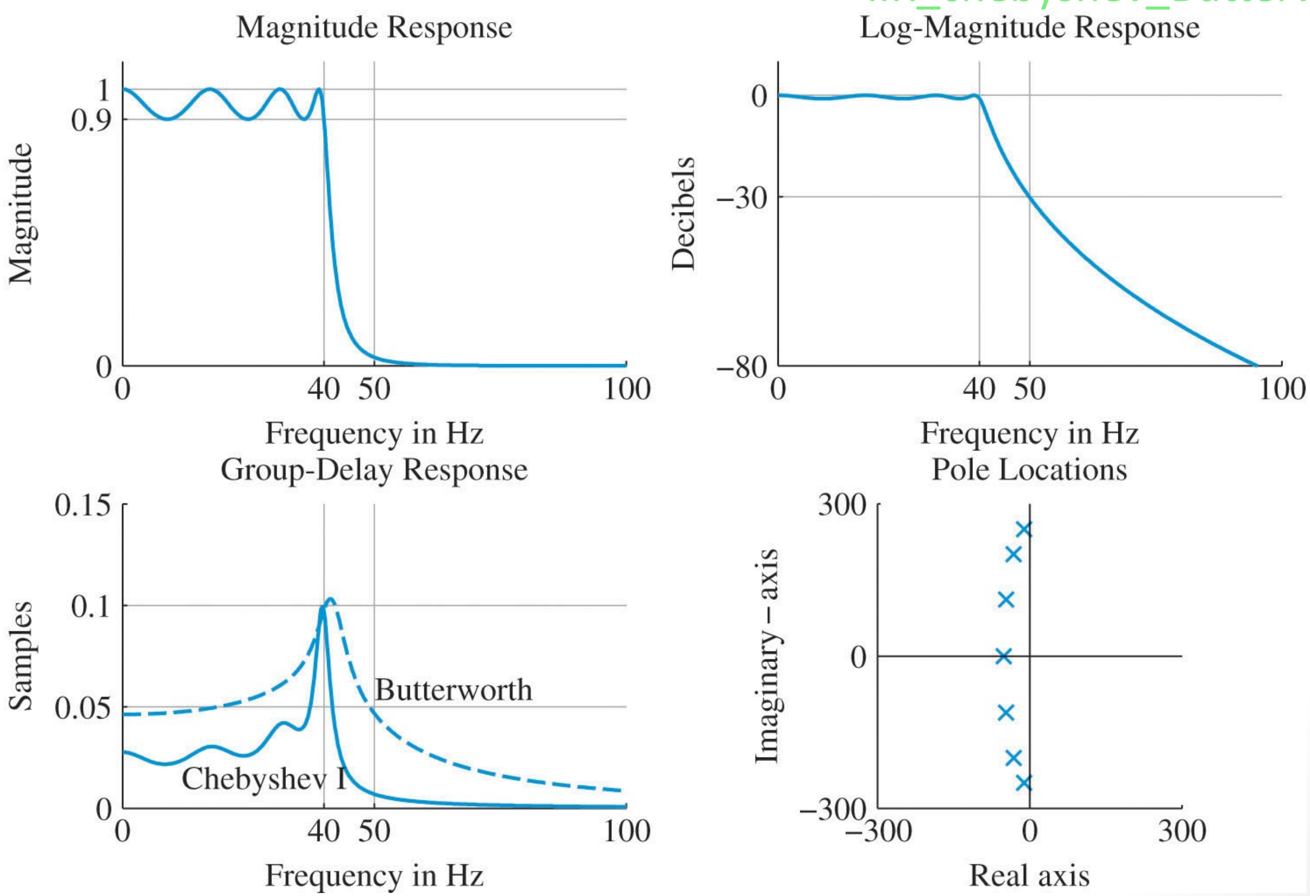
The Chebyshev approximation

Example: Chebyshev I filter design

Consider the analog filter specifications:

Passband edge: F_p = 40 Hz, Passband ripple: A_p = 1dB,
Stopband edge: F_s = 50 Hz, Stopband attenuation: A_s = 30 dB.

IIR_chebyshev_Butterworth.py



Design plots for the seventh-order lowpass Chebyshev I filter

IIR filter design

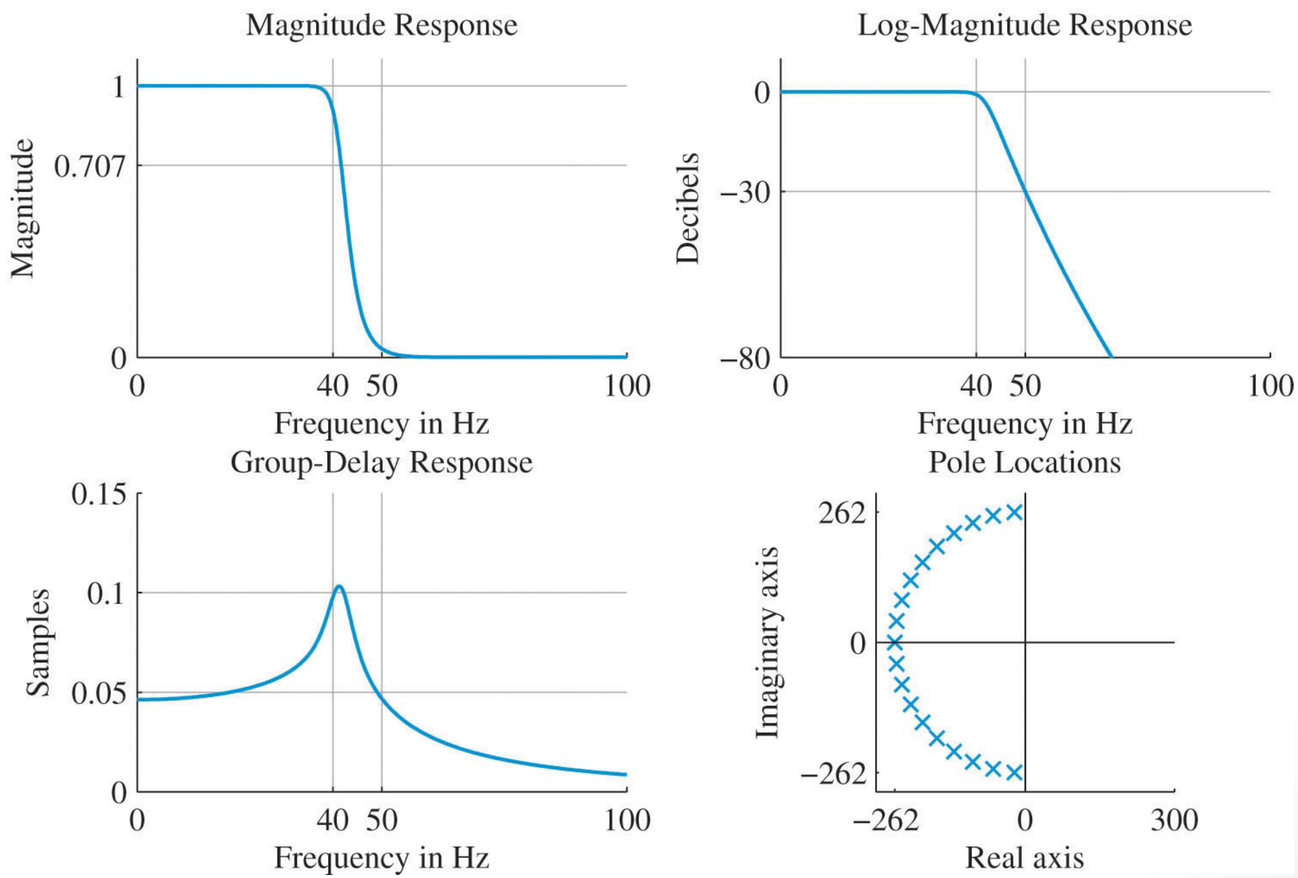
The Chebyshev approximation

Example: Butherworth filter design

Consider the analog filter specifications:

Passband edge: F_p = 40 Hz, Passband ripple: A_p = 1dB,
Stopband edge: F_s = 50 Hz, Stopband attenuation: A_s = 30 dB.

IIR_chebyshev_Butterworth.py



Design plots for the 19th-order lowpass Butterworth filter.

IIR filter design

The Chebyshev approximation

Example: Chebyshev I filter design

Consider the analog filter specifications:

Passband edge: F_p = 40 Hz, Passband ripple: A_p = 1dB,
Stopband edge: F_s = 50 Hz, Stopband attenuation: A_s = 30 dB.

IIR_chebyshev_Butterworth.py

Butterworth:

- In the magnitude response plot the magnitude at 41.7 Hz is down to 3 dB $\equiv 1/\sqrt{2}$
- In the log magnitude plot the response at F_s = 50 Hz is exactly 30 dB.
- The group-delay response shows a nonlinear but smooth function.

Chebyshev:

- The group-delay response is more nonlinear than that of the Butterworth design as shown in the group-delay plot.
- The Chebyshev I design meets the given specification using a much smaller order of 7 compared to 19 for the Butterworth design.

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Each transformation is equivalent to a mapping function $s = f(z)$ from the s-plane to the z-plane. Any useful mapping should satisfy three desirable conditions:

- A rational $H_c(s)$ should be mapped to a rational $H(z)$ (realizability):

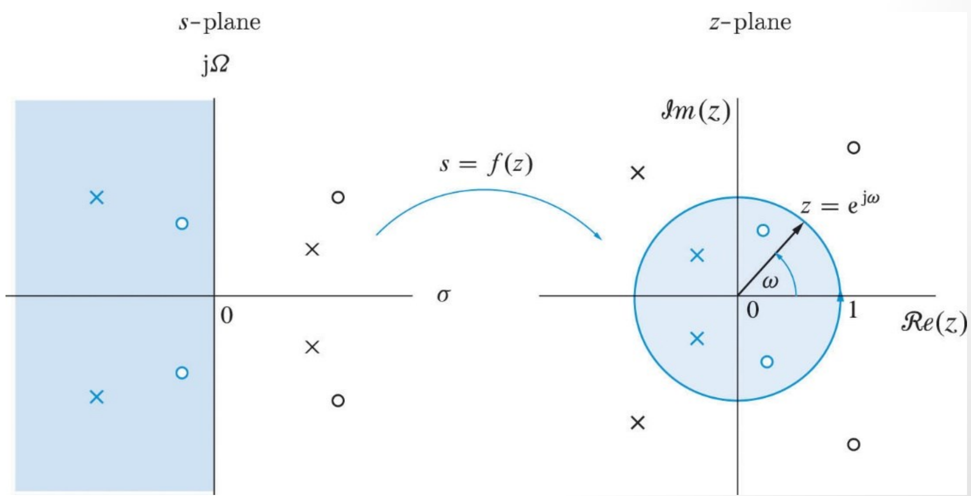
  Rational $H_c(s) \rightarrow$ Rational $H(z)$.

- The imaginary axis of the s-plane is mapped on the unit circle of the z-plane:

  $\{s = j\Omega | -\infty < \Omega < \infty\} \rightarrow \{z = e^{j\omega} \mid -\pi < \omega \leq \pi\}$.

- The left-half s-plane is mapped into the interior of the unit circle of the z-plane:

  $\{s | Re(s) < 0\} \rightarrow \{z | \, |z| < 1\}$.



IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Impulse-invariance transformation

Sampling the impulse response of a continuous-time filter

$$h[n] \triangleq T_{\mathrm{d}}\, h_{\mathrm{c}}(nT_{\mathrm{d}}),$$

$$\rightarrow \qquad H(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{k=-\infty}^{\infty} H_{\mathrm{c}}\left(\mathrm{j}\frac{\omega}{T_{\mathrm{d}}} + \mathrm{j}\frac{2\pi}{T_{\mathrm{d}}}k\right)$$

The fundamental difference: $H(\mathrm{e}^{\mathrm{j}\omega})$ is periodic whereas $H_{\mathrm{c}}(\mathrm{j}\Omega)$ is nonperiodic.

If the continuous-time filter is bandlimited, that is,

$$H_{\mathrm{c}}(\mathrm{j}\Omega) = 0, \ |\Omega| \geq \pi/T_{\mathrm{d}}$$

then, we have

$$H(\mathrm{e}^{\mathrm{j}\omega}) = H_{\mathrm{c}}\left(\mathrm{j}\frac{\omega}{T_{\mathrm{d}}}\right), \quad |\omega| \leq \pi$$

In general, the impulse-invariance mapping causes aliasing.

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Mapping for the impulse-invariance transformation

we start with the partial fraction expansion of $H_c(s)$, which for $M < N$ is given by

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}.$$

For simplicity we assume that the poles are distinct. Taking the inverse Laplace transform yields the impulse response of the continuous-time filter

$$h_c(t) = \sum_{k=1}^{N} A_k \, e^{s_k t} \, u(t).$$

Hence the impulse response of the discrete-time filter is given by

$$h[n] = T_d h_c(nT_d) = \sum_{k=1}^{N} T_d A_k \left(e^{s_k T_d}\right)^n u[n],$$

and the system function of the discrete-time system is therefore given by

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} \sum_{k=1}^{N} T_d A_k \left(e^{s_k T_d}\right)^n z^{-n}$$

Transformation of continuous-time filters to discrete-time IIR filters

Mapping for the impulse-invariance transformation

Assuming that $\left| e^{s_k T_d} \right| < 1$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}.$$

we conclude that, for single poles, *H(z)* is obtained from $H_c(s)$ by using the following mapping

$$\frac{1}{s - s_k} \longrightarrow \frac{T_d}{1 - e^{s_k T_d} z^{-1}} = \frac{T_d}{1 - p_k z^{-1}},$$

where

$$p_k \triangleq e^{s_k T_d}$$

maps the poles of the continuous-time filter to the poles of the discrete-time filter.

Note:
This mapping relates the locations of the poles of $H_c(s)$ and *H(z)* but not the locations of the zeros.

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Mapping for the impulse-invariance transformation

It is obvious that the mapping $s = f(z)$ corresponding to impulse invariance is $s = \ln(z)/T_d$ or

$$z = e^{sT_d}$$

Since $s = \sigma + j\Omega$ and $z = re^{j\omega}$

$$r = e^{\sigma T_d},$$

$$\omega = \Omega T_d.$$

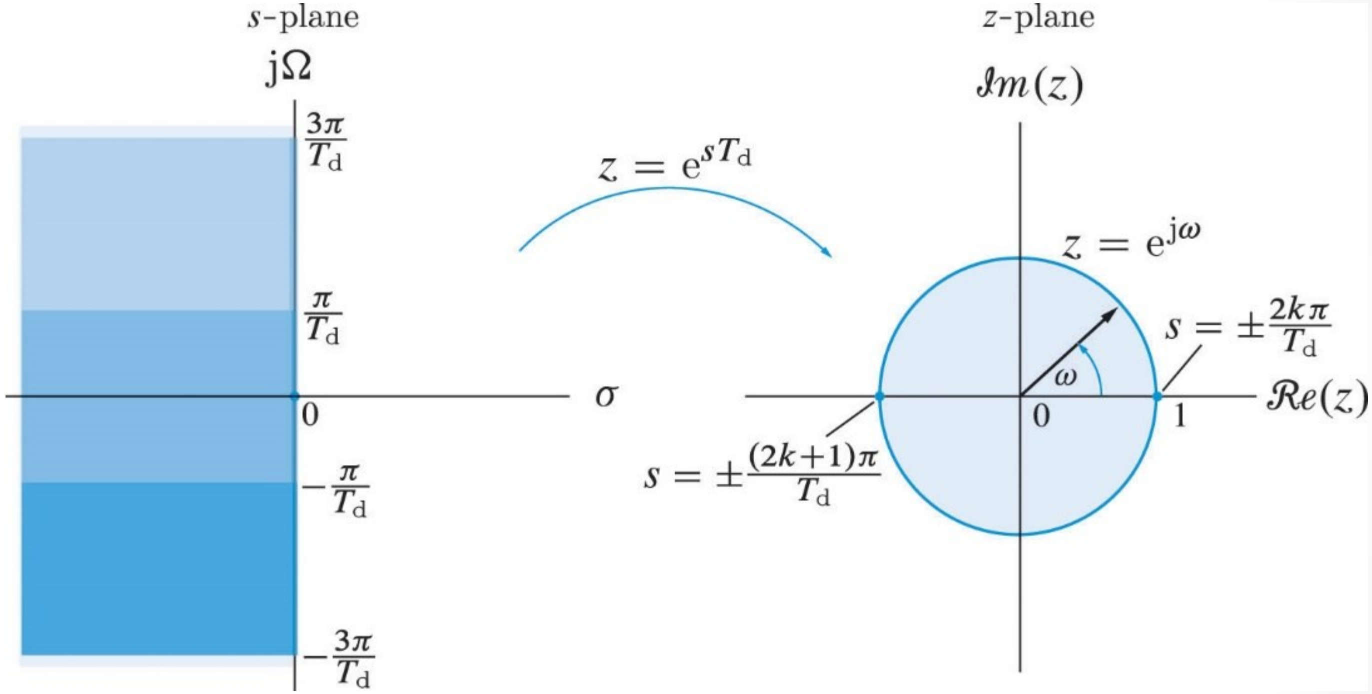
$\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$ → the left-half s-plane is mapped inside the unit circle of the z-plane.

$\sigma = 0$ yields $r = 1$ → the frequency axis $s = j\Omega$ is mapped on the unit circle (this is not one-to-one)

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Mapping for the impulse-invariance transformation



The mapping from the *s*-plane to the *z*-plane corresponding to the impulse-invariance transformation.

The source of the aliasing effect is that the mapping of $z = e^{sT_d}$ is not one-to-one

$$H(z)|_{z=e^{sT_d}} = \sum_{k=-\infty}^{\infty} H_c\left(s + j\frac{2\pi k}{T_d}\right)$$

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Example

$$h_c(t) = e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} H_c(s) = \frac{1}{s+2}, \quad \mathcal{R}e(s) > -2$$

$T_d = 0.1$

$$h[n] = 0.1h_c(0.1n) = 0.1e^{-0.2n}u[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z) = \frac{0.1}{1-e^{-0.2}z^{-1}}, \quad |z| > e^{-0.2}$$

$$H(z) = \frac{0.1}{1-0.8187z^{-1}}, \quad |z| > 0.8187$$

the mapping preserves stability and the lowpass characteristic of the magnitude response.

Consider next the first-order highpass filter obtained by

$$H_c'(s) = 1 - H_c(s) = 1 - \frac{1}{s+2} = \frac{s+1}{s+2}. \quad \rightarrow \quad h_c'(t) = \delta(t) - e^{-2t}u(t).$$

- we cannot sample $h'_c(t)$ at $t = 0$.

Impulse-invariance can be applied to lowpass and bandpass filters that have strictly proper system functions, and can not be used for systems with improper system functions, like highpass and bandstop filters

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Design procedure

Suppose we wish to design a digital lowpass filter *H(z)* specified by the parameters ω_p, A_p, ω_s, and A_s.

- Start by choosing the design sampling interval T_d which is arbitrary, and then map ω_p into $\Omega_p = \omega_p/T_d$ and ω_s into $\Omega_s = \omega_s/T_d$.
- Next, design the equivalent analog filter $H_c(s)$ using the Butterworth or Chebyshev I approximations that satisfies the specifications.
- Perform a partial fraction expansion on the rational function H_c and map its poles $\{s_k\}$ into digital poles $\{p_k\}$ using $z = e^{sT_d}$
- Finally, assemble the desired digital filter system function *H(z)* using

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}.$$

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Example: Impulse-invariance transformation – Butterworth

Design a lowpass digital Butterworth filter to satisfy specifications:

Passband edge: $\omega_p = 0.25\pi$ rad, Passband ripple: $A_p = 1$ dB,

Stopband edge: $\omega_s = 0.4\pi$ rad, Stopband attenuation: $A_s = 30$ dB.

Step-1 Choose design sampling interval T_d. Let $T_d = 0.1$ s.

Step-2 Compute the equivalent analog filter band edge frequencies.

$$\Omega_p = \frac{0.25\pi}{0.1} = 7.8540 \quad \text{and} \quad \Omega_s = \frac{0.4\pi}{0.1} = 12.5664.$$

Step-3 Design the analog lowpass filter $H_c(s)$.

Step-4 Transform $H(s)$ into $H(z)$.

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

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Step-3 Design the analog lowpass filter $H_c(s)$.

Step-4 Transform $H(s)$ into $H(z)$.

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Bilinear transformation

The *bilinear transformation* is an invertible one-to-one nonlinear mapping between the *s*-plane and the *z*-plane defined by

$$s = f(z) \triangleq \frac{2}{T_\mathrm{d}} \frac{1 - z^{-1}}{1 + z^{-1}}.$$

T_d does *not* have any useful interpretation as a sampling interval because the bilinear transformation does *not* involve any sampling operation.

Each occurrence of s in $H_c(s)$ is replaced by the above transformation function, or more formally:

$$H(z) = H_\mathrm{c}(s)\Big|_{s=\frac{2}{T_\mathrm{d}} \frac{1-z^{-1}}{1+z^{-1}}}.$$

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

The bilinear transformation can be implemented by individually mapping the zeros and poles of $H_c(s)$, which results in:

$$H(z) = G \frac{(1+z^{-1})^{N-M} \prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})},$$

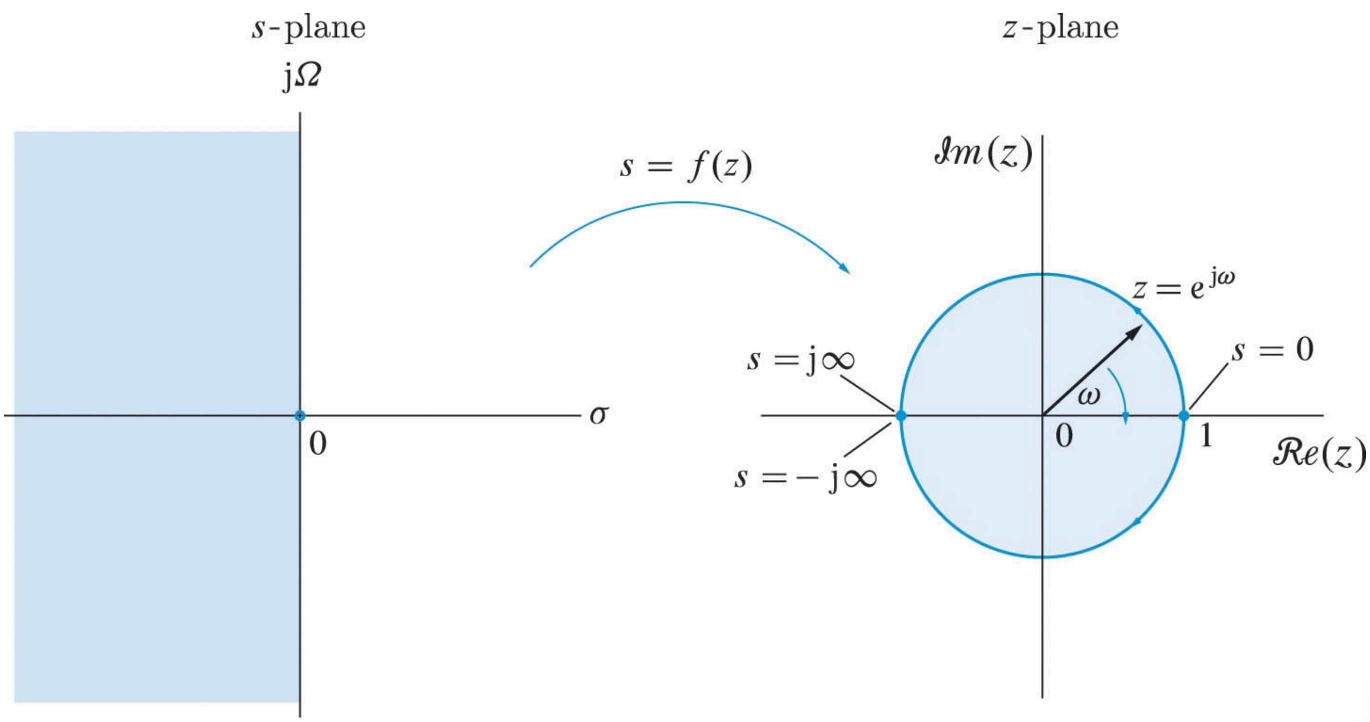
where

$$z_k = \frac{1 + T_d \zeta_k/2}{1 - T_d \zeta_k/2}, \; p_k = \frac{1 + T_d s_k/2}{1 - T_d s_k/2}, \; G = \frac{\beta_0 \left(\frac{T_d}{2}\right)^{N-M} \prod_{k=1}^{M} \left(1 - \zeta_k \frac{T_d}{2}\right)}{\prod_{k=1}^{N} \left(1 - s_k \frac{T_d}{2}\right)}.$$

- The mapping is a rational function, so a rational $H_c(s)$ always gives a rational $H(z)$.
- The bilinear mapping preserves the order of the system (number of poles N), but increases the number of zeros from M to N (when $N > M$) by placing $(N - M)$ zeros at $z = -1$.

IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters



IIR filter design

Transformation of continuous-time filters to discrete-time IIR filters

Example:

Consider the following analog filter system function:

$$H_c(s) = \frac{5(s+2)}{(s+3)(s+4)} = \frac{5s+10}{s^2+7s+12}.$$

Let $T_d = 2$ or $T_d/2 = 1$.

The zeros of $H(z)$ are given by:

$$z_1 = \frac{1+(-2)}{1-(-2)} = -\frac{1}{3} \quad \text{and} \quad z_2 = -1 \quad (\because 1+z^{-1} \text{ factor}),$$

$$p_1 = \frac{1+(-3)}{1-(-3)} = -\frac{1}{2} \quad \text{and} \quad p_2 = \frac{1+(-4)}{1-(-4)} = -\frac{3}{5},$$

$$G = \frac{5(1)(1-(-2))}{(1-(-3))(1-(-4))} = \frac{3}{4}.$$

$$\rightarrow \qquad H(z) = \frac{3}{4}\frac{(1+z^{-1})(1+\frac{1}{3}z^{-1})}{(1+\frac{1}{2}z^{-1})(1+\frac{3}{5}z^{-1})} = \frac{0.75+z^{-1}+0.25z^{-2}}{1+1.1z^{-1}+0.3z^{-2}}.$$