Structure Theory of Finance

A Unified Path-Language for Structural Derivatives Pricing

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Abstract

We introduce the **Structure Theory of Finance (STF)**—a comprehensive framework for expressing, combining, and stabilizing financial derivatives through a structured path language. STF is not merely a new pricing model, but a foundational theory that redefines how financial products are constructed, evaluated, and constrained within a dynamic operator space. Traditional models treat derivative pricing as isolated processes, relying on static assumptions, risk-neutral measures, and frictionless replication. However, as financial instruments become increasingly interdependent and structurally nested, these models fail to capture the systemic instability arising from recursive feedback, pricing loops, and tail contagion.

STF resolves this by introducing a frictional calculus on structured path spaces, where every financial structure is represented as a composition of path operators subject to a dynamic friction function λ. No-arbitrage is reinterpreted as the suppression of structural explosion in recursive valuation loops, with final pricing emerging as a fixed point of operator-path equilibrium. STF not only recovers classical models such as Black-Scholes, Copula, and LMM as specific constructions, but also provides a unified language for exotic payoffs, crisis modeling, and contingent product design. This work initiates a new paradigm in financial theory—one governed by structure, not just stochasticity.

1 Introduction

The evolution of financial theory has been shaped by a persistent aspiration: to describe and price contingent claims in an ever-expanding landscape of uncertainty, leverage, and innovation. From the foundational work of Black-Scholes to the widespread adoption of copula structures, stochastic volatility models, and interest rate trees, much of modern financial mathematics has centered around specifying dynamics and solving for value. Yet as the complexity of financial products deepens—through structural embedding, layered triggers, path-dependencies, and multi-asset linkages—the classical paradigms begin to falter.

Modern derivative structures are no longer merely stochastic processes. They are compositions of logic, dependencies, and recursive references. An autocallable note references multiple barriers across time; a callable convertible bond embeds optionality within optionality; a basket default swap operates across correlated trigger events. These are not easily reducible to one model—they are *structures*.

Existing frameworks fail to answer the fundamental question: when a derivative interacts with other derivatives, what guarantees that the overall system remains well-defined? In today's markets, financial instruments are not priced in isolation—they co-exist, co-reference, and co-detonate. A structure that feeds into another's pricing equation creates a feedback loop; a recursive valuation architecture without constraint can lead to what we call **structural explosion**—an uncontrolled amplification of valuation paths across the operator space.

This work introduces the **Structure Theory of Finance (STF)** as a foundational response to this collapse of expressiveness and stability. STF defines a formal structure path language, where financial products are expressed as composable path operators on a dynamic path space Ω. Crucially, STF introduces the concept of a friction function $\lambda(\omega)$—a risk-weighted penalty on structural complexity, liquidity, nesting, and feedback—used to suppress explosive combinations and define *legitimate* valuation paths.

Within STF, pricing is no longer a single expectation under a measure, but a dynamic search over the operator space for equilibrium paths whose cumulative risk remains bounded. No-arbitrage is recast as a path-integral selection principle: if any arbitrage path exists with zero friction, it is priced in; if not, the valuation stabilizes. Recursion is permitted—so long as it converges under λ-controlled feedback.

In the later sections, we formally define the structure path space, operator algebra, and legitimate integration criteria. We also re-express classical pricing models—Black-Scholes, Copula, Heston, LMM, and beyond—as specific configurations of STF. This unification of structural form and risk control opens the door to a new design philosophy for financial products, where language, not payoff diagrams, becomes the foundational object of modeling.

2 Core Framework

The Structure Theory of Finance (STF) is built upon the idea that financial products are not merely payoff functions or stochastic processes, but structured compositions of path-dependent logic, embedded optionalities, and inter-operator feedback. To capture this, STF defines a mathematical foundation centered on structured path spaces, operator composition, and a friction-based mechanism to suppress explosive recursion. This section introduces the key objects and constructs of STF.

2.1 Structured Path Space Ω

We define the universal structure path space Ω as the space of all admissible càdlàg financial paths:

$$\Omega := \left\{ \omega : [0,T] \to R^d \;\middle|\; \omega \text{ is right-continuous with left limits (càdlàg)} \right\}$$

Each ω represents a possible realization of the financial state of one or more underlying instruments. The space Ω forms the base layer over which all structural compositions are built.

2.2 Structure Operators

Let $\mathcal{O}_i : \Omega \to \Omega$ denote a *structure operator*—a transformation, conditional filter, or path constructor that maps a path into another. These include:

- Payoff transformation (e.g., $\omega \mapsto \max(\omega(T) - K, 0)$) - Barrier triggerings, resets, callable conversion - Multi-asset coupling, Copula bindings - Early termination, switching, or embedding behaviors

Operators are composable: for any $\mathcal{O}_i, \mathcal{O}_j$, we define

$$\mathcal{O}_j \circ \mathcal{O}_i(\omega) := \mathcal{O}_{ji}(\omega) \in \Omega$$

The space of structure operators $\mathcal{S}$ forms a **non-commutative algebraic system** under composition, enabling the modeling of deeply nested and recursive derivatives.

2.3 Path Integral with Friction

We define the pricing of a financial structure as a path integral over Ω, penalized by a friction function λ:

$$P := \int_\Omega V(\omega) \cdot e^{-\lambda(\omega)} \, d\omega$$

where: - $V(\omega)$ is the payout functional of the structure; - $\lambda(\omega) \geq 0$ is a frictional cost function that suppresses structurally unstable or complex paths.

This expression generalizes classical risk-neutral valuation, with $\lambda(\omega)$ encoding structural instability, liquidity penalty, nesting complexity, and propagation depth.

2.4 Friction Function λ

We define the friction function $\lambda : \Omega \to R_+$ as:

$$\lambda(\omega) := \alpha \cdot \mathrm{Depth}(\omega) + \beta \cdot \mathrm{Nesting}(\omega) + \gamma \cdot \frac{1}{\mathrm{Liquidity}(\omega)} + \delta \cdot \mathrm{Penalty}_{\mathrm{context}}(\omega)$$

Each term reflects a different dimension of structural complexity or risk: - Depth(ω): number of structural layers; - Nesting(ω): embedded substructures; - Liquidity^{-1}: illiquidity penalty; - Penalty$_{\mathrm{context}}$: path-specific or model-induced correction.

Friction acts as a suppression field: when structural combinations lead to explosive recursion or arbitrage, $\lambda(\omega)$ diverges, effectively removing such paths from integration.

2.5 Risk Degree and Legitimate Paths

We define the *Risk Degree* of a path ω as:

$$\mathrm{RiskDegree}(\omega) := \lambda(\omega)$$

We say a path is **legitimate** if and only if RiskDegree$(\omega) = 0$. These paths represent pure arbitrage opportunities or zero-risk structures.

The STF pricing mechanism can thus be interpreted as an integration over all legitimate or bounded-risk paths:

$$P := \sup_{\omega \in \mathcal{L}} \int V(\omega) \cdot e^{-\lambda(\omega)} d\omega, \quad \text{where } \mathcal{L} := \left\{ \omega \in \Omega \;\middle|\; \lambda(\omega) < \infty \right\}$$

This defines the fundamental object of STF: a structurally regularized path integral over a dynamic risk space.

2.6 Structure No-Arbitrage and Pricing Equilibrium

In classical finance, no-arbitrage is often phrased as a condition on replication or existence of an equivalent martingale measure. However, in a structure-rich operator space, these notions must be extended to account for recursion, nesting, and self-referential valuation.

We propose a new formulation: **Structure No-Arbitrage Principle (SNAP)**, which asserts that a financial operator is validly priced only if the resulting valuation structure does not induce an unbounded feedback loop across the operator space.

Let $\mathcal{F} = \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n\}$ be a set of interacting financial structure operators. Each operator $\mathcal{O}_i$ depends (directly or indirectly) on other operators in $\mathcal{F}$ to compute its own price. This defines a *pricing dependency graph*.

To control feedback loops and recursive amplification, we define each operator's valuation as a functional mapping:

$$P_i := \mathcal{V}_i(\vec{P}) := \sup_{\omega \in \mathcal{L}_i} \int V_i(\omega) \cdot e^{-\lambda(\omega; \vec{P})}\, d\omega$$

where $\vec{P} = (P_1, \ldots, P_n)$ is the current price vector, and $\lambda(\omega; \vec{P})$ is the path friction, which now depends on the global pricing state.

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Structure Equilibrium Principle. The true valuation vector $\vec{P}^*$ must satisfy:

$$\vec{P}^* = \mathcal{V}(\vec{P}^*) := \Big(\mathcal{V}_1(\vec{P}^*), \ldots, \mathcal{V}_n(\vec{P}^*)\Big)$$

That is, it is a fixed point of the global structure valuation functional. We refer to this as the **Structure Equilibrium**.

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Theorem (Fixed-Point Existence). Let each valuation functional $\mathcal{V}_i$ be continuous, monotonic (non-decreasing), and bounded on a compact domain $[0, M]^n$. Then there exists at least one fixed point $\vec{P}^* \in [0, M]^n$ such that:

$$\vec{P}^* = \mathcal{V}(\vec{P}^*)$$

Proof Sketch. Since each $\mathcal{V}_i$ maps a compact convex set into itself, and the global valuation functional $\mathcal{V}$ is continuous, Brouwer's fixed-point theorem guarantees the existence of at least one fixed point. If $\lambda(\omega; \vec{P})$ is designed to increase with structural nesting or feedback exposure, the integral will remain bounded, ensuring the mapping is well-defined.

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Interpretation. The fixed point $\vec{P}^*$ represents a globally consistent pricing configuration where no structure exerts an unstable recursive influence on the others. STF pricing is therefore not a static expectation, but the result of a *dynamical equilibrium search* in operator space under frictional damping.

The existence of such a fixed point ensures the pricing system remains stable even under complex, nested, or self-referential structures.

3 Model Resurrection I: Classical Structures

3.1 Black-Scholes Model

Path Representation. Let $\omega(t) \in \Omega$ represent the price path of a single risky asset over $t \in [0, T]$. In classical theory[4], the asset follows a geometric Brownian motion (GBM):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

STF does not rely on the SDE directly, but instead expresses the derivative contract as a structure operator acting on ω:

$$\mathcal{O}_{\text{Call}}(\omega) := \max\left(\omega(T) - K, 0\right)$$

This is a terminal payoff operator applied to the raw price path.

Structure Integral Expression. The STF valuation is given by a structurally regularized path integral:

$$P_{\text{BS}} := \int_{\Omega} \mathcal{O}_{\text{Call}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)} \, d\omega$$

Here: - $\lambda(\omega)$ is the friction function. In standard Black-Scholes (no path dependence), we assume $\lambda(\omega) \approx 0$ for all legitimate paths. - e^{-rT} reflects discounting under the assumption of constant interest rate.

Risk Degree and Stability. The Black-Scholes path is non-nested, terminal-only, with no embedded optionality. Hence:

$$\text{RiskDegree}(\omega) := \lambda(\omega) = 0$$

for all admissible ω, meaning the entire pricing structure lies within the legitimate path space $\mathcal{L}_0$. STF recovers Black-Scholes as a frictionless flat structure.

Structural Interpretation. The BS model is the minimal structure: a terminal operator without nesting, barrier, or recursion. It serves as the atomic base of more complex options, including barrier options, Asian options, and autocallables. Within STF, it is encoded as a Level-0 structure operator with trivial .

Conclusion. **STF Classifies Black-Scholes** as a zero-risk, terminal-only, frictionless structure with constant payout operator. Its role in STF is to provide the foundational form for single-layer, non-nested derivatives.

3.2 Bachelier Model

Path Representation. Let $\omega(t) \in \Omega$ denote[15] the price path of an asset under the Bachelier assumption: linear Brownian motion without multiplicative scaling:

$$dS_t = \mu\, dt + \sigma\, dW_t$$

In STF, we express this model through a direct linear path process. The terminal payoff operator for a European call remains:

$$\mathcal{O}_{\text{Call}}^{\text{Bach}}(\omega) := \max\left(\omega(T) - K, 0\right)$$

The distinction lies not in the operator form, but in the underlying path behavior $\omega(t)$ being sampled from an additive-noise path class.

Structure Integral Expression. The STF pricing is formulated as:

$$P_{\text{Bach}} := \int_{\Omega_{\text{add}}} \mathcal{O}_{\text{Call}}^{\text{Bach}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

Here, Ω_{add} is the structured path subspace governed by additive noise dynamics. The rest of the integral follows STF syntax.

Risk Degree and Friction. The Bachelier model, like Black-Scholes, is a terminal-only, frictionless operator under STF. Thus:

$$\text{RiskDegree}(\omega) = \lambda(\omega) = 0$$

for all $\omega \in \Omega_{\text{add}}$. No nesting, recursion, or embedded structure is present.

Structural Interpretation. While financially less common than BS, the Bachelier model appears in: - Negative interest rate environments, - Short-maturity approximations, - Spread option or rate-differential contexts.

In STF, it remains a Level-0 structure with flat , but is sampled from a distinct path generator. It illustrates STF's flexibility to accommodate diverse underlying structures without altering operator syntax.

Conclusion. **STF Classifies Bachelier** as a flat, frictionless structure over additive-path domains. It shares operator structure with BS but changes the shape of the path space Ω, showcasing STF's separation of "structure logic" from "path distribution".

3.3 Binomial Tree Model

Path Representation. In the Binomial model[14], the asset price evolves over discrete time steps $\{t_0, t_1, \ldots, t_N\}$ with each step representing an up or down movement:

$$\omega(t_{i+1}) = \begin{cases} u \cdot \omega(t_i) & \text{with probability } p \\ d \cdot \omega(t_i) & \text{with probability } 1-p \end{cases}$$

In STF, we define the binomial path space $\Omega_{\text{bin}} \subset \Omega$ as a set of tree-structured discrete paths. A complete path is a sequence $\omega = (\omega_0, \omega_1, \ldots, \omega_N)$ generated by (u, d, p) parameters.

Structure Operator. Let the payoff operator be defined as:

$$\mathcal{O}^{\text{bin}}_{\text{Call}}(\omega) := \max(\omega_N - K, 0)$$

where $\omega_N = \omega(T)$ is the final node of the path tree.

Discrete Structure Integral. In STF, the pricing is formulated as a discrete path-sum integral:

$$P_{\text{Bin}} := \sum_{\omega \in \Omega_{\text{bin}}} \mathcal{O}^{\text{bin}}_{\text{Call}}(\omega) \cdot Q(\omega) \cdot e^{-\lambda(\omega)}$$

where: - $Q(\omega)$ is the risk-neutral path probability (product of p and $1-p$ over the path); - $\lambda(\omega)$ penalizes structural complexity (though in this case it is typically zero).

Risk Degree and Analysis. Standard binomial models represent shallow, non-recursive paths with finite branching depth. Therefore:

$$\lambda(\omega) = \text{Depth}(\omega) \cdot \alpha = N \cdot \alpha$$

If $\alpha = 0$, we recover classical risk-neutral pricing. If we embed this binomial tree inside larger recursive structures (e.g., callable tree, barrier resets), becomes nontrivial.

Structural Interpretation. The Binomial model is STF's canonical example of a **discretized structural path space**, supporting: - Finite-depth operator algebra, - Early exercise logic (via structure cutoffs), - Embedding of tree-based structures (Bermudan, American options).

Its tree depth, branching factor, and recombination structure become explicit parameters in STF language.

Conclusion. **STF Classifies Binomial Tree** as a finite-path, discrete-structure model with explicit depth-indexed . It provides the backbone for recursive products and early-exercise trees, and exemplifies STF's ability to express discrete logic within a continuous structural grammar.

3.4 Geometric Brownian Motion (GBM)

Path Representation. Let $\omega(t) \in \Omega$ represent[28] an asset price path under classical geometric Brownian motion:

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t \quad \Rightarrow \quad \omega(t) = \omega(0) \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

In STF, this SDE is treated as a **path-generating kernel**, not as the model itself. The path structure ω generated by GBM defines a stochastic subset $\Omega_{\text{GBM}} \subset \Omega$—used as the integration domain for structure pricing.

Structure Operator. The payoff remains structurally defined as:

$$\mathcal{O}_{\text{Call}}^{\text{GBM}}(\omega) := \max(\omega(T) - K, 0)$$

This operator applies to any terminal price path and is independent of the path's generation law.

Structure Integral Expression. In STF, we interpret GBM-based pricing as:

$$P_{\text{GBM}} := \int_{\Omega_{\text{GBM}}} \mathcal{O}_{\text{Call}}^{\text{GBM}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\,d\omega$$

This expression separates: - The **structure logic** (via $\mathcal{O}_{\text{Call}}$), - The **path generator** (GBM $\to \Omega_{\text{GBM}}$), - The **structural friction** (controls path risk accumulation).

Risk Degree and Analysis. For plain vanilla options, GBM paths are continuous, non-nested, and structure-flat. Hence:

$$\text{RiskDegree}(\omega) = \lambda(\omega) \approx 0$$

However, if the operator $\mathcal{O}$ depends on time-aggregated values (e.g. Asian or Barrier), then nesting increases, and $\lambda(\omega)$ grows with structural depth.

Structural Interpretation. In STF, GBM serves as the **default continuous-path kernel**. It defines the most common path space Ω_{GBM} used in: - Equity options, - Volatility modeling (pre-local vol), - Base layer for Heston and Jump Diffusion.

It is not a structure itself, but a **structure support space** over which operator compositions are applied.

Conclusion. **STF Interprets GBM** not as a model, but as a path-space generator. Its role is foundational in defining integration domains, while all financial logic is carried by structure operators and . This separation of "generation" from "structure" is central to STF.

3.5 Local Volatility Model

Path Representation. The Local Volatility (LV) model generalizes GBM by allowing the volatility to be a deterministic[9] function of time and the asset level:

$$dS_t = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dW_t$$

In STF, this path evolution defines a deformed path generator $\Omega_{\text{LV}} \subset \Omega$, where path dynamics reflect spatial inhomogeneity and local path sensitivity.

Structure Operator. The terminal payoff remains the same:

$$\mathcal{O}_{\text{Call}}^{\text{LV}}(\omega) := \max(\omega(T) - K, 0)$$

However, when expressed under a local volatility path space, the structure operator may interact with regions of higher curvature or sharp regime shifts in the path.

Structure Integral Expression. The STF valuation is given by:

$$P_{\text{LV}} := \int_{\Omega_{\text{LV}}} \mathcal{O}_{\text{Call}}^{\text{LV}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

Here: - Ω_{LV} is path space generated under the SDE with $\sigma(t, S_t)$; - The function $\lambda(\omega)$ must now capture the *local path curvature* and sensitivity to structural inhomogeneity.

Risk Degree and Analysis. While the LV model does not introduce recursive structure or nested logic, it creates **path-space irregularity**. We define:

$$\lambda(\omega) := \gamma \cdot \int_0^T \left| \frac{\partial \sigma}{\partial S}(t, \omega(t)) \right|^2 dt$$

This version of penalizes regions where local volatility surfaces exhibit strong gradients or discontinuities, which correspond to pricing instability or calibration singularities.

Hence, while λ is still finite for most paths, its variation is no longer flat. RiskDegree becomes path-sensitive even for terminal-only payoffs.

Structural Interpretation. The LV model is STF's first example of a **non-flat structure support**. Even with the same operator, different path-space geometries affect and thus pricing stability.

In practice: - Deep out-of-the-money or in-the-money paths may be heavily penalized; - STF allows detecting "unstable zones" in volatility surface regions via -explosions.

Conclusion. **STF Interprets Local Volatility** as a flat structure over a distorted path support. The operator remains simple, but reveals the true risk geometry of the underlying. This enables deeper understanding of instability regions even in otherwise well-behaved products.

3.6 Constant Elasticity of Variance (CEV)

Path Representation. In the CEV model, volatility is a power function of the asset price[30]:

$$dS_t = \mu S_t \, dt + \sigma S_t^\beta \, dW_t$$

for $\beta \in (0,1)$ (mean-reverting), $\beta = 1$ (GBM), or $\beta > 1$ (super-convex volatility).

In STF, this defines a non-uniform path generator $\Omega_{\text{CEV}} \subset \Omega$ whose paths exhibit varying sensitivity depending on S_t. Paths approaching zero experience *explosive volatility* when $\beta < 1$, making structural explosion a real possibility.

Structure Operator. Let the standard terminal call payoff be:

$$\mathcal{O}_{\text{Call}}^{\text{CEV}}(\omega) := \max(\omega(T) - K, 0)$$

As with GBM and LV models, this operator applies directly to terminal path values, but pricing sensitivity to path shape is encoded in $\lambda(\omega)$.

Structure Integral Expression. STF expresses the CEV model valuation as:

$$P_{\text{CEV}} := \int_{\Omega_{\text{CEV}}} \mathcal{O}_{\text{Call}}^{\text{CEV}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)} \, d\omega$$

Here $\lambda(\omega)$ must penalize paths that are exposed to degeneracy near $S_t \to 0$ or excessive convexity near $S_t \to \infty$.

Risk Degree and Analysis. We define a curvature-sensitive as:

$$\lambda(\omega) := \gamma \cdot \int_0^T \left(\omega(t)^{2\beta-2}\right) dt$$

- For $\beta < 1$, the term $\omega(t)^{2\beta-2}$ diverges as $\omega(t) \to 0$. - For $\beta > 1$, it diverges as $\omega(t) \to \infty$.

Thus, STF naturally suppresses tail explosions and ill-posed structural sensitivities via this term.

Structural Interpretation. The CEV model is STF's canonical example of a **self-instabilizing path generator**. Even without nesting or recursion, paths themselves carry dynamic explosion risk. This makes not just a structural penalty, but a regulator of the diffusion geometry itself.

This illustrates that **RiskDegree may be non-zero even for simple operators**, purely due to the path field's geometry.

Conclusion. **STF Classifies CEV** as a structurally flat—but path-geometrically unstable—structure. It demonstrates STF's capacity to detect and control explosion zones arising not from logic, but from endogenous diffusion mechanics.

3.7 Heston Model

Path Representation. The Heston model introduces[12] stochastic volatility by coupling the asset price with a volatility process:

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sqrt{v_t} S_t\,dW_t^{(1)} \\ dv_t &= \kappa(\theta - v_t)\,dt + \xi\sqrt{v_t}\,dW_t^{(2)} \end{aligned} \quad \text{with } d\langle W^{(1)}, W^{(2)}\rangle = \rho\,dt$$

In STF, this defines a joint path cluster:

$$\omega := (\omega_S, \omega_v) \in \Omega_{\text{Heston}} := \Omega_S \times \Omega_v$$

where $\omega_S(t)$ is the price path, and $\omega_v(t)$ is the stochastic volatility path, each with coupled dynamics.

Structure Operator. The terminal call payoff is applied only to the price path:

$$\mathcal{O}_{\text{Call}}^{\text{Heston}}(\omega) := \max(\omega_S(T) - K, 0)$$

but the valuation integral must account for how volatility $\omega_v(t)$ influences the weight of ω_S.

Structure Integral Expression. STF expresses the valuation as a two-path cluster integral:

$$P_{\text{Heston}} := \iint_{\Omega_S \times \Omega_v} \mathcal{O}_{\text{Call}}^{\text{Heston}}(\omega_S, \omega_v) \cdot e^{-rT} \cdot e^{-\lambda(\omega_S) - \lambda(\omega_v) - \lambda_c(\omega_S, \omega_v)}\,d\omega_S\,d\omega_v$$

Here: - $\lambda(\omega_S)$ and $\lambda(\omega_v)$ penalize individual path complexity; - $\lambda_c(\omega_S, \omega_v)$ captures the **joint-path coupling risk**, reflecting correlated explosion or regime misalignment.

Risk Degree and Coupling. We define:

$$\lambda_c(\omega_S, \omega_v) := \rho_c \cdot \int_0^T \left| \frac{d\omega_S}{dt}(t) \cdot \frac{d\omega_v}{dt}(t) \right| dt$$

This term enforces **explosion damping** in regions of high price-volatility co-movement. For high-leverage products, the integral naturally diverges unless $\omega_v(t)$ is well-behaved.

Thus:

$$\text{RiskDegree}(\omega) := \lambda(\omega_S) + \lambda(\omega_v) + \lambda_c(\omega_S, \omega_v)$$

and only when this is finite do we include the path in STF pricing.

Structural Interpretation. Heston is the first STF-recognized **multi-path structure**, composed of: - A price path component, - A volatility path component, - A correlation-induced coupling term.

Its structure operator remains simple, but its path structure is rich and non-separable.

STF allows us to: - Express volatility-asset feedback as an integral friction; - Detect volatility clustering leading to pricing instability; - Control structural explosion from variance bursts.

Conclusion. **STF Classifies Heston** as a correlated path-cluster structure. It demonstrates STF's power to manage coupled path integrals with frictional coupling terms λ_c, offering a unified way to control risk propagation in multi-factor stochastic models.

3.8 Jump Diffusion Model (Merton)

Path Representation. Merton's jump[24] diffusion model augments geometric Brownian motion with a Poisson-driven jump process:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t + (J-1) S_{t^-} \, dN_t$$

- N_t is a Poisson process with intensity λ_J; - J is a random jump size (often lognormal-distributed); - S_{t^-} denotes the left-limit before the jump.

In STF, this defines a path space $\Omega_{\text{Jump}} \subset \Omega$, consisting of càdlàg paths with discontinuities at jump times.

Structure Operator. Terminal payoff:

$$\mathcal{O}_{\text{Call}}^{\text{Jump}}(\omega) := \max(\omega(T) - K, 0)$$

remains unchanged, but the path structure ω now contains jumps. STF therefore introduces a structural modifier:

$$\omega := \text{JumpOperator}(\omega_{\text{cont}}, \{t_k\}, \{J_k\})$$

where ω_{cont} is the continuous base path (e.g., GBM), and $\{t_k\}, \{J_k\}$ are jump times and magnitudes.

Structure Integral Expression. The STF formulation becomes:

$$P_{\text{Jump}} := \int_{\Omega_{\text{Jump}}} \mathcal{O}_{\text{Call}}^{\text{Jump}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

with:

$$\lambda(\omega) = \lambda_{\text{base}}(\omega_{\text{cont}}) + \lambda_{\text{jump}}(\{t_k\}, \{J_k\})$$

Risk Degree and Analysis. We define the jump friction component as:

$$\lambda_{\text{jump}} := \eta \cdot \sum_k \left(|\log J_k| + \delta \cdot \frac{1}{\Delta t_k} \right)$$

- Penalizes large jump magnitudes ($J_k \gg 1$ or $\ll 1$); - Penalizes jump clustering via small $\Delta t_k := t_k - t_{k-1}$; - Enforces regularity by damping high-frequency high-impact jump structures.

This gives:

$$\text{RiskDegree}(\omega) := \lambda(\omega) = \lambda_{\text{cont}} + \lambda_{\text{jump}}$$

and ensures only jump configurations satisfying structural boundedness contribute to pricing.

Structural Interpretation. Jump diffusion in STF is a **discontinuous structure modifier** on top of any continuous generator. This introduces: - Discrete-time structural disruptions; - Tail amplification mechanisms; - Cross-regime transition behavior.

STF treats these not as SDE terms, but as **path-level structural add-ons**, controlled by λ.

Conclusion. STF Classifies Jump Diffusion as a path-disruption structure with -based control over both jump intensity and magnitude. It showcases STF's ability to integrate jump logic without changing operator syntax, and suppress explosive tail behavior via structure-sensitive friction.

3.9 Merton Credit Model

Path Representation. Merton's structural credit model[24] interprets a firm's default as the event where its asset value falls below a debt threshold at maturity:

$$\text{Default} \iff V(T) < D$$

where $V(t)$ follows a GBM-like diffusion. In STF, this defines a binary path space partition:

$$\Omega = \Omega_{\text{default}} \cup \Omega_{\text{survive}}$$

with: - $\Omega_{\text{default}} := \{\omega : \omega(T) < D\}$, - $\Omega_{\text{survive}} := \{\omega : \omega(T) \geq D\}$.

Structure Operator. Define the credit bond payoff operator:

$$\mathcal{O}^{\text{Merton}}_{\text{CB}}(\omega) := \begin{cases} R \cdot \omega(T) & \text{if } \omega(T) < D \\ \omega(T) & \text{otherwise} \end{cases}$$

where R is the recovery rate (e.g., $0 < R < 1$).

To express this structurally, STF introduces:

- A default trigger function: $\text{DefaultTrigger}(\omega; D) := 1_{\omega(T)<D}$; - A recovery operator: $\text{Recovery}(R, \omega)$.

Then:

$$\mathcal{O}^{\text{Merton}}(\omega) = (1 - \text{DefaultTrigger}) \cdot \omega(T) + \text{DefaultTrigger} \cdot R \cdot \omega(T)$$

Structure Integral Expression. STF expresses the valuation as:

$$P_{\text{Merton}} := \int_\Omega \mathcal{O}^{\text{Merton}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)} \, d\omega$$

In this model, the default-triggered paths form a structural subspace subject to λ amplification due to risk concentration near $\omega(T) \approx D$.

Risk Degree and Analysis. We define as:

$$\lambda(\omega) := \alpha \cdot 1_{\omega(T)<D} + \beta \cdot \frac{1}{|\omega(T) - D|^\delta}$$

- The first term penalizes all default paths globally; - The second term diverges near $\omega(T) \approx D$ to reflect **structural ambiguity zone**.

Hence, $\text{RiskDegree}(\omega)$ identifies both tail risk and credit threshold instability.

Structural Interpretation. Merton's model is STF's first **default-terminal structure**, in which the path's terminal state determines structural cutoff. This introduces: - Trigger-based branching in payout; - Nonlinearity in structural payoff based on final condition; - Smooth embedding of recovery logic into integral expression.

Conclusion. **STF Classifies Merton** as a default-trigger structure with terminal path bifurcation. It provides the canonical form for all future credit-sensitive structures in STF, and initiates the use of λ to penalize structural ambiguity near the default boundary.

3.10 Vasicek Interest Rate Model

Path Representation. The Vasicek model describes the evolution[29] of short-term interest rates $r(t)$ as an Ornstein–Uhlenbeck (mean-reverting) process:

$$dr_t = a(b - r_t)\,dt + \sigma\,dW_t$$

In STF, this defines a path space Ω_r consisting of continuous rate paths $\omega_r(t)$ with strong mean reversion tendencies. A zero-coupon bond maturing at time T has payoff 1 and present value:

$$P(0,T) = E^Q\left[e^{-\int_0^T r(t)\,dt}\right]$$

Structure Operator. Define the STF operator for a zero-coupon bond as:

$$\mathcal{O}^{\mathrm{Vas}}_{\mathrm{ZCB}}(\omega_r) := \exp\left(-\int_0^T \omega_r(t)\,dt\right)$$

This operator is a **path functional**—it aggregates the short rate path over time.

Structure Integral Expression. In STF, the valuation becomes:

$$P_{\mathrm{Vas}} := \int_{\Omega_r} \mathcal{O}^{\mathrm{Vas}}_{\mathrm{ZCB}}(\omega_r) \cdot e^{-\lambda(\omega_r)}\,d\omega_r$$

Here: - The path integral structure mimics the traditional affine bond valuation; - The friction function adds control over rate path shape and extremity.

Risk Degree and Analysis. We define as:

$$\lambda(\omega_r) := \gamma \cdot \int_0^T \left(|\omega_r(t) - b|^2 + |\dot{\omega}_r(t)|^2\right) dt$$

- Penalizes deviations from long-term mean b; - Penalizes sharp fluctuations in rate changes (i.e., anti-mean-reversion); - Enforces structural smoothness and rate coherence.

Thus:

$$\mathrm{RiskDegree}(\omega_r) = \lambda(\omega_r)$$

with lower values assigned to stable, smooth rate paths.

Structural Interpretation. In STF, Vasicek is a **single-rate-path functional model**, forming the base layer of fixed income structures. It allows: - Zero-coupon pricing; - Forward rate extraction; - Smooth extension to multiple rates via RateCluster; - Embedding into callable/convertible bond structures.

It exemplifies STF's path functional structure: no nested logic, but integral-sensitive valuation.

Conclusion. **STF Classifies Vasicek** as a path-functional structure over mean-reverting rate paths. Its structure is simple yet expressive, and forms the archetype for multi-rate models and forward curve-based constructions under the STF grammar.

3.11 Cox–Ingersoll–Ross (CIR) Model

Path Representation. The CIR model modifies Vasicek[20] to ensure strictly positive short rates via square-root diffusion:

$$dr_t = a(b - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t$$

In STF, the corresponding path space Ω_r^+ is defined as:

$$\Omega_r^+ := \left\{\omega_r \in \Omega \;\middle|\; \omega_r(t) \geq 0 \;\forall t \in [0, T]\right\}$$

Each path ω_r remains non-negative, but may flirt with zero when volatility is high—leading to potential boundary explosion.

Structure Operator. For a zero-coupon bond:

$$\mathcal{O}_{\mathrm{ZCB}}^{\mathrm{CIR}}(\omega_r) := \exp\left(-\int_0^T \omega_r(t)\,dt\right)$$

Same operator form as in Vasicek, but acting over a different, non-symmetric path domain with singularity risk.

Structure Integral Expression. STF expression:

$$P_{\mathrm{CIR}} := \int_{\Omega_r^+} \mathcal{O}_{\mathrm{ZCB}}^{\mathrm{CIR}}(\omega_r) \cdot e^{-\lambda(\omega_r)}\,d\omega_r$$

The challenge in CIR lies in controlling near-zero behavior of $\omega_r(t)$.

Risk Degree and Analysis. We define to include singular boundary penalization:

$$\lambda(\omega_r) := \gamma \cdot \int_0^T \left[|\omega_r(t) - b|^2 + |\dot{\omega}_r(t)|^2 + \frac{\delta}{\omega_r(t) + \varepsilon}\right] dt$$

- First two terms encourage smooth mean-reverting behavior; - The third term introduces a **blow-up penalty** near zero, ensuring $\omega_r(t)$ stays in well-behaved positive territory; - $\varepsilon > 0$ is a regularization floor.

Then:

$$\text{RiskDegree}(\omega_r) = \lambda(\omega_r)$$

and paths with risky behavior near $r_t = 0$ are suppressed.

Structural Interpretation. CIR is a **boundary-aware structure-functional model**, where the path structure may lead to "absorbing friction" near pathological boundaries.

STF naturally expresses: - Zero-bound sensitivity, - Volatility-induced instability, - -based boundary protection without changing operator syntax.

Conclusion. **STF Classifies CIR** as a positive-path structure with boundary-aware friction control. It illustrates STF's capacity to handle singular structural risks through -based suppression, and completes the path-functional structure family alongside Vasicek.

3.12 Hull-White Model

Path Representation. The Hull-White model[13] generalizes Vasicek by allowing time-dependent mean reversion:

$$dr_t = [\theta(t) - ar_t]\, dt + \sigma\, dW_t$$

with: - $\theta(t)$ fitted to match the initial yield curve; - r_t remains Gaussian and mean-reverting.

In STF, we expand the representation to: - $\omega_r(t) \in \Omega_r$, the short-rate path; - Or a multi-path set $\Omega_{\text{cluster}} := \{\omega_1, \ldots, \omega_n\}$, i.e., a RateCluster.

Structure Operator. For a zero-coupon bond, define:

$$\mathcal{O}^{\text{HW}}_{\text{ZCB}}(\omega_r) := \exp\left(-\int_0^T \omega_r(t)\, dt\right)$$

For multi-period instruments or Swaptions, define a joint payout functional $V(\omega_1, \ldots, \omega_n)$ over RateCluster.

Structure Integral Expression. Single-path Hull-White valuation:

$$P_{\text{HW}} := \int_{\Omega_r} \mathcal{O}^{\text{HW}}_{\text{ZCB}}(\omega_r) \cdot e^{-\lambda(\omega_r)}\, d\omega_r$$

For path-cluster-based products:

$$P := \int_{\Omega_{\text{cluster}}} V(\omega_1, \ldots, \omega_n) \cdot e^{-\sum_i \lambda(\omega_i) - \sum_{i<j} \lambda_c(\omega_i, \omega_j)} \, d\omega_1 \ldots d\omega_n$$

Risk Degree and Analysis. Define single-path friction:

$$\lambda(\omega_r) := \gamma \cdot \int_0^T \left(|\omega_r(t) - \theta(t)/a|^2 + |\dot{\omega}_r(t)|^2 \right) dt$$

Define inter-path friction for ω_i, ω_j in RateCluster:

$$\lambda_c(\omega_i, \omega_j) := \rho_{ij} \cdot \int_0^T |\omega_i(t) - \omega_j(t)|^2 \, dt$$

Then:

$$\text{RiskDegree} = \sum_i \lambda(\omega_i) + \sum_{i<j} \lambda_c(\omega_i, \omega_j)$$

Structural Interpretation. In STF, Hull-White is the first model with: - RateCluster structure; - DriftCoupler-like interdependencies; - Applicability to CMS, Swaptions, and callable bond pricing; - λ and λ_c control temporal consistency and maturity-layer explosion.

Conclusion. **STF Classifies Hull-White** as a multi-path structure over time-dependent rate fields, embedded within a friction-controlled path cluster. It establishes STF's architecture for forward term structures and is the foundational template for LMM-like constructions.

3.13 Ho-Lee Model

Path Representation. The Ho-Lee model is one of the earliest arbitrage-free interest rate models[15]. It assumes a normally distributed short rate with deterministic drift:

$$dr_t = \theta(t) \, dt + \sigma \, dW_t$$

Here, $\theta(t)$ is a deterministic function calibrated to fit the initial term structure. Unlike Vasicek or Hull-White, there is no mean reversion.

In STF, $\omega_r(t) \in \Omega_r$ is a **linear-drift rate path**, forming a path space with additive drift structure.

Structure Operator. As with other interest rate models, the zero-coupon bond operator is defined by:

$$\mathcal{O}_{\mathrm{ZCB}}^{\mathrm{HL}}(\omega_r) := \exp\left(-\int_0^T \omega_r(t)\, dt\right)$$

No embedded option or branching structure is required at this level.

Structure Integral Expression. STF valuation:

$$P_{\mathrm{HL}} := \int_{\Omega_r} \mathcal{O}_{\mathrm{ZCB}}^{\mathrm{HL}}(\omega_r) \cdot e^{-\lambda(\omega_r)}\, d\omega_r$$

Compared to Hull-White, Ho-Lee has no restoring force, and thus larger uncertainty in the terminal distribution.

Risk Degree and Analysis. Define as:

$$\lambda(\omega_r) := \gamma \cdot \int_0^T \left(|\omega_r(t) - \bar{\theta}(t)|^2 + |\dot{\omega}_r(t)|^2\right) dt$$

Where $\bar{\theta}(t)$ is an empirical drift estimate or expected path. This penalizes paths that: - Deviate too far from the calibrated drift, - Exhibit strong volatility bursts (via second term).

Thus:

$$\mathrm{RiskDegree}(\omega_r) = \lambda(\omega_r)$$

and reflects the instability of unconstrained linear diffusions over long horizons.

Structural Interpretation. Ho-Lee is STF's archetypal **flat structure generator**. It: - Contains no reversion or bounding mechanism; - Is structurally simple but financially unstable at long maturities; - Represents the minimal friction baseline for rate path modeling.

In STF, Ho-Lee serves as a degenerate limit of Hull-White with $a \to 0$.

Conclusion. STF Classifies Ho-Lee as a flat, drift-driven structure over additive rate paths. It has low structural complexity, but requires -based explosion control to maintain pricing stability over time.

3.14 Black-Derman-Toy (BDT) Model

Path Representation. The BDT model is a recombining[15] interest rate tree model where the short rate evolves with time-dependent volatility:

$$dr_t = \mu(t, r_t)\, dt + \sigma(t) \cdot r_t\, dW_t$$

The defining feature of BDT is its construction as a discrete-time, lognormal interest rate tree, fitted to initial term structure and volatility.

In STF, we define Ω_{BDT} as a **discrete path space**:

$$\omega_r := (\omega_0, \omega_1, \ldots, \omega_N), \quad \omega_i \in R_+$$

where each ω_{i+1} branches from ω_i with lognormal volatility $\sigma(t_i)$.

Structure Operator. A zero-coupon bond paying 1 at time $T = t_N$ has payoff operator:

$$\mathcal{O}_{\text{ZCB}}^{\text{BDT}}(\omega) := \exp\left(-\sum_{i=0}^{N-1} \omega_i \cdot \Delta t_i\right)$$

Optionally, early-exercise or callable features can be embedded via path-dependent operators or stage-switch triggers.

Structure Integral Expression. BDT pricing in STF becomes a sum over the discrete tree space:

$$P_{\text{BDT}} := \sum_{\omega \in \Omega_{\text{BDT}}} \mathcal{O}_{\text{ZCB}}^{\text{BDT}}(\omega) \cdot Q(\omega) \cdot e^{-\lambda(\omega)}$$

where: - $Q(\omega)$ is the probability of a path under risk-neutral branching; - $\lambda(\omega)$ controls structural depth, branching frequency, and volatility instability.

Risk Degree and Analysis. We define for discrete paths as:

$$\lambda(\omega) := \alpha \cdot \text{Depth}(\omega) + \beta \cdot \sum_{i=0}^{N-1} \left| \log\left(\frac{\omega_{i+1}}{\omega_i}\right) \right|^2 + \gamma \cdot \text{VolatilityNesting}(\omega)$$

- The first term penalizes long trees (e.g. long maturities or nested path options); - The second term penalizes excessive rate jumps (unstable transitions); - The third term captures volatility embedded in tree geometry (e.g. rate clustering).

Then:

$$\text{RiskDegree}(\omega) = \lambda(\omega)$$

and serves as the filtering gate for tree-path integration.

Structural Interpretation. In STF, BDT is a **finite discrete-time tree structure** with: - Branching operator: $\text{TreeStep}(\omega_i, \sigma(t_i))$; - Embedded trigger support (e.g. call, exercise, stepwise conversion); - Explicit path-state structure enabling stage-wise control.

It bridges continuous rate diffusions (Vasicek, HW) with discrete decision structures (Bermudan, callable instruments).

Conclusion. **STF Classifies BDT** as a branching-path structure over lognormal rate trees. Its discrete tree nature allows direct operator embedding and frictional suppression of explosion paths. It is a crucial foundation for multi-stage rate-sensitive products in STF.

3.15 Libor Market Model (LMM)

Path Representation. The LMM models a family of forward Libor rates $\{L_i(t)\}_{i=1}^n$ as stochastic[5] processes, each evolving under its own forward measure:

$$dL_i(t) = \mu_i(t)\,dt + \sigma_i(t)\,L_i(t)\,dW_i(t)$$

with the drift $\mu_i(t)$ depending on the future $L_j(t)$, $j > i$, through lognormal no-arbitrage constraints.

In STF, we represent this as a **RateCluster**:

$$\text{RateCluster} := \{\omega_1(t), \ldots, \omega_n(t)\} \in \Omega_1 \times \cdots \times \Omega_n$$

Each $\omega_i(t)$ corresponds to the path of forward rate L_i over time, with coupling between paths via structural dependency operators.

Structure Operator. Let $V(\omega_1, \ldots, \omega_n)$ be a structure operator for:

- A caplet on L_k; - A CMS spread: $L_i - L_j$; - A swap rate as an average of several L_i paths.

Operators may include embedded triggers or cross-path conditionals.

Structure Integral Expression. The STF expression becomes:

$$P_{\text{LMM}} := \int_{\Omega_1 \times \cdots \times \Omega_n} V(\omega_1, \ldots, \omega_n) \cdot e^{-\sum_i \lambda(\omega_i) - \sum_{i<j} \lambda_c(\omega_i, \omega_j)}\, d\omega_1 \ldots d\omega_n$$

This integral captures both marginal path friction and inter-path coupling friction.

Risk Degree and Analysis. We define individual path friction:

$$\lambda(\omega_i) := \gamma \cdot \int_0^T \left(|\omega_i(t) - \bar{L}_i|^2 + |\dot{\omega}_i(t)|^2 \right) dt$$

and cross-path drift coupling friction:

$$\lambda_c(\omega_i, \omega_j) := \rho_{ij} \cdot \int_0^T |\mu_i(\omega_{\geq i}) - \mu_j(\omega_{\geq j})|^2\, dt$$

where $\mu_i(\cdot)$ represents the structural drift dependency on forward paths.

Total risk degree:

$$\text{RiskDegree} = \sum_i \lambda(\omega_i) + \sum_{i<j} \lambda_c(\omega_i, \omega_j)$$

captures both path-level complexity and inter-path volatility coordination.

Structural Interpretation. In STF, LMM is the prototypical **coordinated multi-path structure** with:

- Path cluster structure: $\text{RateCluster}(\omega_1, \ldots, \omega_n)$; - Embedded operator dependencies: $\text{DriftCoupler}(\omega_i; \omega_{j>i})$; - Forward-measure consistency enforced via .

It enables full structural expression of:

- Caplets, Swaptions, CMS options; - Multi-forward conditional logic; - Spread exposure across floating legs.

Conclusion. **STF Classifies LMM** as a fully coordinated RateCluster structure with drift-linked interdependencies. It demonstrates the compositional power of STF's structure language and its capacity to regulate complex path interactions through nested -control.

3.16 Normal Volatility Model (Bachelier Extension)

Path Representation. The normal volatility model assumes additive Brownian dynamics [8]for the asset price:

$$dS_t = \mu\, dt + \sigma\, dW_t$$

Unlike the lognormal (GBM) model, this implies that price changes are symmetric around zero and can become negative. In STF, this yields an additive-path structure:

$$\omega(t) = \omega(0) + \mu t + \sigma W_t \in \Omega_{\text{add}}$$

where Ω_{add} is a subspace of Ω with constant additive volatility behavior.

Structure Operator. For a European call option:

$$\mathcal{O}^{\text{norm}}_{\text{Call}}(\omega) := \max(\omega(T) - K, 0)$$

Same terminal structure as GBM; only the path generation kernel differs.

Structure Integral Expression. In STF, pricing becomes:

$$P_{\text{NormVol}} := \int_{\Omega_{\text{add}}} \mathcal{O}^{\text{norm}}_{\text{Call}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

where: - Ω_{add} = additive volatility path space; - $\lambda(\omega)$ may be simpler than GBM due to lower structural convexity.

Risk Degree and Analysis. Since additive paths don't involve path-dependent variance amplification, we define:

$$\lambda(\omega) := \gamma \cdot \int_0^T |\dot{\omega}(t)|^2 \, dt$$

This penalizes violent short-term fluctuations but lacks tail-volatility amplification (as in GBM or CEV). Thus:

$$\text{RiskDegree}(\omega) = \lambda(\omega)$$

is more uniform and less skewed than in lognormal structures.

Structural Interpretation. STF uses the normal volatility model as:

- A contrast to GBM to show how path-space shape affects structural friction; - A core model for: - Interest rate options (when negative rates are allowed); - Spread options; - Short-tenor futures; - A frictional baseline when log-volatility becomes ill-defined or explosively convex.

Conclusion. STF Classifies Normal Volatility as an additive-structure model with symmetric path dynamics and friction-uniform structure. It illustrates STF's ability to distinguish between identical payoff operators over different path generators, and to control structural stability via , regardless of model origin.

3.17 SABR Model (Stochastic Alpha Beta Rho)

Path Representation. The SABR model defines a forward price F_t and stochastic[1] volatility σ_t via:

$$\begin{aligned} dF_t &= \sigma_t F_t^\beta \, dW_t^{(1)} \\ d\sigma_t &= \nu \sigma_t \, dW_t^{(2)} \\ d\langle W^{(1)}, W^{(2)} \rangle &= \rho \, dt \end{aligned}$$

with: - $\beta \in [0, 1]$: elasticity parameter; - ν: vol-of-vol; - ρ: correlation.

In STF, this defines a coupled path cluster:

$$\omega := (\omega_F(t), \omega_\sigma(t)) \in \Omega_{\text{SABR}} := \Omega_F \times \Omega_\sigma$$

with $\omega_F(t)$ the forward path, and $\omega_\sigma(t)$ the stochastic volatility path.

Structure Operator. A vanilla European option on F_T is defined structurally as:

$$\mathcal{O}_{\text{Call}}^{\text{SABR}}(\omega_F) := \max(\omega_F(T) - K, 0)$$

More complex operators (e.g. volatility-targeted options, corridor swaps) can be embedded by adding path functionals on ω_σ.

Structure Integral Expression. STF valuation over coupled paths:

$$P_{\text{SABR}} := \iint_{\Omega_F \times \Omega_\sigma} \mathcal{O}^{\text{SABR}}(\omega_F, \omega_\sigma) \cdot e^{-rT} \cdot e^{-\lambda(\omega_F) - \lambda(\omega_\sigma) - \lambda_c(\omega_F, \omega_\sigma)} \, d\omega_F \, d\omega_\sigma$$

Risk Degree and Analysis. We define three friction components:

Forward path :

$$\lambda(\omega_F) := \gamma \cdot \int_0^T |\dot{\omega}_F(t)|^2 + \frac{\delta}{(\omega_F(t) + \varepsilon)^{2(1-\beta)}} \, dt$$

- The second term penalizes the β-explosion behavior near $\omega_F(t) \to 0$ when $\beta < 1$.

Volatility path :

$$\lambda(\omega_\sigma) := \eta \cdot \int_0^T \left(|\dot{\omega}_\sigma(t)|^2 + \frac{\nu^2}{\omega_\sigma(t)^2 + \varepsilon} \right) dt$$

- Penalizes high vol-of-vol and vanishing volatility regions.

Cross-path coupling:

$$\lambda_c(\omega_F, \omega_\sigma) := \rho_c \cdot \int_0^T |\dot{\omega}_F(t)| \cdot |\dot{\omega}_\sigma(t)| \, dt$$

$$\text{RiskDegree} = \lambda(\omega_F) + \lambda(\omega_\sigma) + \lambda_c(\omega_F, \omega_\sigma)$$

Only path pairs with controlled volatility-fluctuation co-amplification are permitted in the integral.

Structural Interpretation. SABR is the first STF model to integrate:

- Power-law structural transforms (via β); - Volatility-path nesting; - Cross-path coupling via frictional correlation; - Near-singular path explosion suppression via dynamic -penalty.

In STF, SABR can be expressed as:

$$\text{VolCluster} := \text{SABRCluster}(\omega_F, \omega_\sigma; \beta, \nu, \rho)$$

with self-consistent path-integrable structure under convergence criteria.

Conclusion. STF Classifies SABR as a coupled nonlinear path-cluster structure with structure-indexed explosion risk. It demonstrates STF's full integration of correlated, powered, and dynamically unstable paths under frictional control, and serves as a benchmark for volatility-of-volatility model expression.

3.18 Implied Tree Model

Path Representation. The Implied Tree approach constructs a discrete, non-parametric[10] recombining tree for asset prices that exactly reproduces observed market option prices (across strikes and maturities). It was first introduced by Derman Kani and Rubinstein.

In STF, the price paths $\omega(t)$ belong to a segmented tree-structured path space:

$$\Omega_{\text{impl}} := \bigcup_{k=1}^{N} \Omega_k$$

Each Ω_k represents a path segment generated under local volatility σ_k or drift condition calibrated from observed prices. This yields a non-smooth tree with locally adjusted transitions.

Structure Operator. For European options, the standard operator applies:

$$\mathcal{O}^{\text{IT}}_{\text{Call}}(\omega) := \max(\omega(T) - K, 0)$$

but the key lies in how the path ω was constructed to reflect market structure rather than stochastic law.

Structure Integral Expression. STF views the pricing process as:

$$P_{\text{IT}} := \sum_{\omega \in \Omega_{\text{impl}}} \mathcal{O}^{\text{IT}}(\omega) \cdot Q_{\text{impl}}(\omega) \cdot e^{-\lambda(\omega)}$$

- $Q_{\text{impl}}(\omega)$ is an implied probability measure reconstructed from market instruments; - $\lambda(\omega)$ adjusts for structural inconsistency and segment friction.

Risk Degree and Analysis. We define:

$$\lambda(\omega) := \sum_{k=1}^{N} (\alpha_k \cdot \text{Jump}_k(\omega) + \beta_k \cdot \text{SegmentDriftVar}_k(\omega))$$

- $\text{Jump}_k(\omega)$ captures local path discontinuities or rate-of-change jumps at segment boundaries; - SegmentDriftVar_k penalizes erratic drift changes across adjacent segments.

This structure reveals that even if ω is arbitrage-free, its **path smoothness and segment transitions may cause structural explosion**.

$$\text{RiskDegree}(\omega) = \lambda(\omega)$$

Structural Interpretation. The Implied Tree is STF's canonical example of a **data-embedded structure grammar**. It is:

- Constructed from observed prices (not stochastic laws); - Organized as a sequence of structure segments; - Prone to segmental inconsistency and volatility jumps; - Controlled via localized functions across branches.

In STF, this is abstracted as:

$$\omega \in \text{SegmentedTree}(\Omega_1, \ldots, \Omega_N), \quad \lambda := \sum_k \lambda^k_{\text{seg}}$$

Conclusion. **STF Classifies Implied Tree** as a segmented structural grammar model driven by market-implied local transition rules. It highlights STF's ability to reconstruct non-smooth structural objects and enforce path stability using segment-specific friction control.

3.19 Barrier Option

Path Representation. Barrier options are path-dependent [18]contracts whose payoff activates or cancels depending on whether the underlying hits a specified barrier level during the option's life.

Let $\omega(t) \in \Omega$ denote the underlying price path. A knock-out barrier with level B defines the effective domain:

$$\Omega_{\text{valid}} := \{\omega \in \Omega \mid \sup_{t \in [0,T]} \omega(t) < B\}$$

for up-and-out; or $\inf \omega(t) > B$ for down-and-out.

Structure Operator. STF defines the barrier structure using:

- A trigger operator:

$$\text{Trigger}^{\text{up}}_B(\omega) := 1_{\sup_t \omega(t) \geq B}$$

- A termination cutoff operator:

$$\text{TerminationCutoff}(\omega; \tau_B) := \omega|_{[0,\tau_B]}$$

where τ_B is the first hitting time to barrier B.

A vanilla up-and-out call structure becomes:

$$\mathcal{O}^{\text{Barrier}}(\omega) := (1 - \text{Trigger}^{\text{up}}_B(\omega)) \cdot \max(\omega(T) - K, 0)$$

Structure Integral Expression. The STF pricing integral becomes:

$$P_{\text{Barrier}} := \int_\Omega \mathcal{O}^{\text{Barrier}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

or, for barrier-triggered payout (knock-in):

$$P_{\mathrm{KI}} := \int_\Omega \mathrm{Trigger}_B(\omega) \cdot \max(\omega(T) - K, 0) \cdot e^{-\lambda(\omega)}\, d\omega$$

Risk Degree and Analysis. Barrier structures are highly sensitive near the boundary. We define:

$$\lambda(\omega) := \alpha \cdot 1_{\exists t : |\omega(t) - B| < \varepsilon} + \beta \cdot \mathrm{Oscillation}(\omega; B)$$

- The first term penalizes paths near the barrier zone (fuzzy boundaries $\to$ unstable triggering); - The second term penalizes paths with frequent near-barrier oscillations (multiple crossing attempts).

This suppresses structure explosion due to "quasi-trigger paths" that challenge pricing stability.

Structural Interpretation. Barrier Options in STF are expressed using:

- **TriggerOperator** for event-based switching; - **TerminationCutoff** for path truncation; - -based filtering of unstable boundary regions; - Natural expression of knock-in / knock-out / rebate variants.

This is the canonical form of **conditional structure evaluation**, bridging logic-based payoff control

Conclusion. **STF Classifies Barrier Options** as trigger-terminated structures with zone-sensitive control. It reveals STF's fine-grained capacity to represent conditional structures over singular path features while retaining integral convergence.

3.20 Asian Option

Path Representation. Asian options[2] depend on the **average value** of the underlying over time rather than its terminal value. Let $\omega(t) \in \Omega$ denote the asset price path.

For a fixed-strike Asian call option, the payoff is:

$$\max\left(\frac{1}{T}\int_0^T \omega(t) dt - K, 0\right)$$

This defines a path-functional structure as opposed to terminal-only models.

Structure Operator. STF expresses this using a **path average operator**:

$$\mathrm{PathAverageFunctional}(\omega) := \frac{1}{T}\int_0^T \omega(t)\, dt$$

The full structure operator becomes:

$$\mathcal{O}^{\mathrm{Asian}}(\omega) := \max\left(\mathrm{PathAverageFunctional}(\omega) - K, 0\right)$$

Structure Integral Expression. STF pricing becomes:

$$P_{\text{Asian}} := \int_{\Omega} \mathcal{O}^{\text{Asian}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

Unlike terminal-only models, the integration here is sensitive to **global path shape**, not just endpoint.

Risk Degree and Analysis. We define to penalize **high path volatility over time**, since wild oscillations inflate average uncertainty:

$$\lambda(\omega) := \gamma \cdot \int_0^T |\dot{\omega}(t)|^2\, dt + \eta \cdot \operatorname{Var}(\omega(t), t \in [0, T])$$

- First term enforces path smoothness; - Second term penalizes high variance across the averaging window; - This guards against average instability due to localized spikes or path clustering.

$$\text{RiskDegree}(\omega) = \lambda(\omega)$$

Structural Interpretation. Asian options are STF's prototype of **aggregated-path functional structures**:

- Operate over global properties of the path; - Do not terminate early, but "digest" the entire trajectory; - Enable structures such as: - Floating-strike Asian options; - Average rate options; - Accrual-based payoff instruments.

In STF, such objects are handled using modular path-statistic functionals embedded inside operators.

Conclusion. **STF Classifies Asian Options** as global path-functional structures with volatility-regulated control. It demonstrates STF's ability to express non-local payoff dependencies while maintaining pricing convergence via structured friction.

3.21 Lookback Option

Path Representation. Lookback options provide payoff[10] based on the maximum or minimum value of the underlying asset over the option's life. For a floating strike lookback call:

$$\text{Payoff} = \omega(T) - \min_{t \in [0,T]} \omega(t)$$

The entire path influences the result—not just endpoints or averages.

Structure Operator. STF defines:
- A path extremum operator:

$$\mathrm{PathMin}(\omega) := \inf_{t\in[0,T]} \omega(t)$$

- The full structure operator becomes:

$$\mathcal{O}^{\mathrm{Lookback}}(\omega) := \omega(T) - \mathrm{PathMin}(\omega)$$

or for fixed-strike versions:

$$\mathcal{O} := \max(\mathrm{PathMax}(\omega) - K, 0)$$

Structure Integral Expression. STF valuation:

$$P_{\mathrm{Lookback}} := \int_\Omega \mathcal{O}^{\mathrm{Lookback}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

This integral is highly sensitive to **localized path extrema** and requires careful regulation.

Risk Degree and Analysis. We define:

$$\lambda(\omega) := \alpha \cdot \int_0^T |\dot{\omega}(t)|^2 dt + \beta \cdot \left(\frac{1}{\mathrm{Gap}(\omega)}\right)$$

where:
- $\mathrm{Gap}(\omega) := \omega(T) - \min_t \omega(t)$ or $\max_t \omega(t) - \omega(T)$; - Small gaps (extremum near terminal point) $\rightarrow$ pricing near-singular; - diverges near structural sharp-turn behavior.

Thus:

$$\mathrm{RiskDegree}(\omega) = \lambda(\omega)$$

and acts as a **singularity suppression mechanism** near structure-defining extrema.

Structural Interpretation. Lookback options are STF's canonical **path-extremum structures**, defined by:
- PathExtremumFunctional; - No early exit, but sharp payoff sensitivity; - High local path-dependence and volatility amplification;

They are generalized to:
- Maximum drawdown-linked products; - Skew accumulation payoffs; - Extremum-triggered Autocallables.

Conclusion. STF Classifies Lookback Options as extremum-sensitive structural functionals with sharp-point dampening. This exemplifies STF's ability to handle payoff structures that concentrate around singular path features while maintaining pricing stability through friction control.

3.22 Cliquet Option

Path Representation. Cliquet options reset periodically[10], locking in local returns and summing them to form the final payoff. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be reset dates.

At each interval $[t_{i-1}, t_i]$, the return is locked as:

$$R_i := \max\left(\omega(t_i) - \omega(t_{i-1}), 0\right)$$

The final payoff is:

$$\sum_{i=1}^{n} \min(R_i, C_i)$$

where C_i is the local cap.

Structure Operator. STF defines:
- A local stage return operator:

$$\text{StageReturn}_i(\omega) := \omega(t_i) - \omega(t_{i-1})$$

- A capped lock-in operator:

$$\text{Lock}_i(\omega) := \min(\max(\text{StageReturn}_i(\omega), 0), C_i)$$

- The full structure operator:

$$\mathcal{O}^{\text{Cliquet}}(\omega) := \sum_{i=1}^{n} \text{Lock}_i(\omega)$$

Structure Integral Expression. STF pricing integral:

$$P_{\text{Cliquet}} := \int_{\Omega} \mathcal{O}^{\text{Cliquet}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

This involves multi-period path logic and segment-local transformations.

Risk Degree and Analysis. We define as:

$$\lambda(\omega) := \sum_{i=1}^{n} \left[\alpha \cdot |\text{StageReturn}_i(\omega)|^2 + \beta \cdot 1_{\text{Return Spike}_i}\right]$$

- Penalizes large spikes within single intervals; - Penalizes structure-sharp turn points; - Accumulates structural instability across periods.

$$\text{RiskDegree}(\omega) = \lambda(\omega)$$

Thus, STF enforces pricing stability across time-segmented structures.

Structural Interpretation. Cliquet options are STF's standard **multi-stage accumulator structures**, defined by:

- StageLockOperator and conditional max-min maps; - Time partitioned path decomposition; - Cross-stage accumulation as structure complexity grows.

This framework generalizes to:

- Step-up autocalls, - Guaranteed min + capped max structures, - Lock-in memory notes.

Conclusion. **STF Classifies Cliquet Options** as multi-stage locking structures with segmental control. They exemplify STF's capacity to express path-breaking logic over time and accumulate structure complexity through friction aggregation.

3.23 Range Accrual

Path Representation. Range Accruals accumulate coupon payment[5] when an underlying reference (typically an interest rate or spread) stays within a predefined range.

Let $\omega(t) \in \Omega$ represent the path of the observable rate. Define range $[L, U]$ (lower and upper bounds). The coupon at time T is proportional to:

$$\mathrm{Accrual}(\omega) := \frac{1}{T} \int_0^T 1_{L \leq \omega(t) \leq U} \, dt$$

Structure Operator. STF defines the **Range Trigger Set** as:

$$\mathrm{RangeTrigger}_{[L,U]}(\omega(t)) := 1_{L \leq \omega(t) \leq U}$$

Then, define the **Accrual Operator**:

$$\mathcal{O}^{\mathrm{RA}}(\omega) := \frac{1}{T} \int_0^T \mathrm{RangeTrigger}_{[L,U]}(\omega(t)) \, dt \cdot C$$

where C is the notional coupon rate.

Structure Integral Expression. STF expresses the valuation as:

$$P_{\mathrm{RA}} := \int_\Omega \mathcal{O}^{\mathrm{RA}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)} \, d\omega$$

This is a structure-integral with embedded logical trigger conditions inside.

Risk Degree and Analysis. Paths that frequently touch or oscillate near boundaries introduce **structural instability**. Define:

$$\lambda(\omega) := \alpha \cdot \int_0^T |\dot{\omega}(t)|^2 dt + \beta \cdot \mathrm{TriggerCrossCount}(\omega; [L, U])$$

Where: - TriggerCrossCount counts the number of exits/entries into the range; - Penalizes zig-zag paths around boundary points; - Acts as a damping mechanism on near-edge structural explosion.

Thus:

$$\text{RiskDegree}(\omega) = \lambda(\omega)$$

ensures pricing convergence in the presence of path jittering.

Structural Interpretation. Range Accruals in STF are:

- Integral-based structures over logical triggers; - Path-smoothness sensitive; - Expressed through modular combinators: - RangeTriggerSet, - StructureAccrualIntegral.

STF naturally extends this to:

- Capped accruals, - Double-no-touch structures, - Autocall with accrual buffer zones.

Conclusion. **STF Classifies Range Accruals** as path-integral structures governed by embedded logical triggers. Their pricing stability depends on trigger-bound sensitivity, regulated by penalties on crossing complexity and oscillation.

3.24 Target Redemption Forward (TARN)

Path Representation. TARN products[30] accumulate coupon or gains over time until a cumulative target is reached, at which point the contract terminates early. Let $\omega(t) \in \Omega$ represent the spot or rate path.

Define cumulative payoff:

$$C(t) := \int_0^t G(\omega(s))\, ds$$

with $G(\cdot)$ a gain function (e.g., a fixed coupon or FX difference). Redemption occurs once $C(t) \geq H$ for some threshold H.

Structure Operator. STF expresses this using:

- **CumulativeGainOperator**:

$$\text{Cumulative}(\omega; t) := \int_0^t G(\omega(s))\, ds$$

- **RedemptionTrigger**:

$$\text{Trigger}_H(\omega) := \inf \{t : \text{Cumulative}(\omega; t) \geq H\}$$

- **TerminationCutoff** at time $\tau_H := \text{Trigger}_H(\omega)$

Then, define the operator:

$$\mathcal{O}^{\text{TARN}}(\omega) := \int_0^{\tau_H} G(\omega(t))\, dt$$

Structure Integral Expression. The STF valuation becomes:

$$P_{\text{TARN}} := \int_{\Omega} \mathcal{O}^{\text{TARN}}(\omega) \cdot e^{-r \cdot \tau_H} \cdot e^{-\lambda(\omega)} \, d\omega$$

This expression reflects the structure's adaptive lifespan and embedded stopping logic.

Risk Degree and Analysis. penalizes:

- High accumulation rate (fast early redemption); - Instability near target region (pre-mature trigger volatility); - Path sharpness (as in accrual + trigger oscillation).

Define:

$$\lambda(\omega) := \alpha \cdot \int_0^{\tau_H} |\dot{\omega}(t)|^2 dt + \beta \cdot \left| \frac{d}{dt} \text{Cumulative}(\omega; t) \right|_{t=\tau_H}^{-1}$$

The second term diverges when cumulative rate flattens near H (unstable trigger zone).

Structural Interpretation. TARNs are STF's prototype of **integral-to-threshold structures**, characterized by:

- Stage-wise gain accumulation; - Endogenous path-dependent maturity; - Trigger-dependent divergence; - Expressed via CumulativeGainOperator + RedemptionTrigger + TerminationCutoff.

This structure generalizes to:

- FX-linked TARNs; - Barrier accumulators; - Redemption-targeted bonus notes.

Conclusion. STF Classifies TARN as a gain-to-threshold structure with endogenous termination and -regulated accumulation risk. It exemplifies STF's ability to express integrated multi-operator structures combining accrual, triggering, and early exit logic within a unified friction-controlled pricing framework.

3.25 Callable Range Accrual Note

Path Representation. Callable Range[5] Accrual Notes combine fixed periodic coupons earned when a reference rate stays within a range, with issuer callability that may terminate the structure early. Let $\omega(t) \in \Omega$ represent the path of the observed rate.

Define coupon logic over periods $[t_{i-1}, t_i]$, and a callable evaluation time T_C.

Structure Operator. Let:

- $\text{RangeTrigger}_{[L,U]}(\omega(t)) := 1_{L \le \omega(t) \le U}$ - Coupon for period i:

$$C_i(\omega) := \frac{1}{\Delta t_i} \int_{t_{i-1}}^{t_i} \text{RangeTrigger}_{[L,U]}(\omega(t)) dt \cdot R$$

- Callable decision:

$$\text{CallableTrigger}(\omega;\psi) := 1_{\psi(\omega_{[0,T_C]})\geq\kappa}$$

Then the full structure operator becomes:

$$\mathcal{O}^{\text{CRAN}}(\omega) := \left[(1-\text{CallableTrigger}(\omega))\cdot\sum_{i=1}^{n}C_i(\omega)+\text{CallableTrigger}(\omega)\cdot\sum_{i=1}^{k}C_i(\omega)\right]$$

where $k<n$ if called early.

Structure Integral Expression.

$$P_{\text{CRAN}} := \int_{\Omega}\mathcal{O}^{\text{CRAN}}(\omega)\cdot e^{-r\cdot\tau(\omega)}\cdot e^{-\lambda(\omega)}\,d\omega$$

where $\tau(\omega)$ is total lifetime (censored by call), and $\lambda(\omega)$ reflects both: - Cross-period structure buildup - Callable region instability - Near-boundary oscillation.

Risk Degree and Analysis. Define:

$$\lambda(\omega) := \sum_{i=1}^{n}\left[\alpha\cdot\text{Oscillation}_i(\omega)+\beta\cdot\text{TriggerCross}_i(\omega)\right]+\gamma\cdot 1_{\text{CallableZone}}$$

This expression penalizes: - Accrual-phase instability - Callable threshold ambiguity - Path degeneracy near trigger boundaries.

Structural Interpretation. CRANs are STF's canonical **multi-operator layered structures** combining: - Periodic structure integration - Conditional early exit - Logical + integral operator composition - Structural friction that propagates through stages.

Conclusion. **STF Classifies Callable Range Accrual Notes** as multi-phase structures integrating path integral logic with call-triggered truncation, regulated by layered penalties. It demonstrates STF's capacity to encode stage-wise structure accumulation and dynamically gated termination in a unified frictional pricing grammar.

3.26 Step-Down Autocallable Note

Path Representation. Step-down autocallables[15] include multiple observation dates, each with its own decreasing call trigger. Let $\omega(t)\in\Omega$ denote the underlying asset path, and $T_1<T_2<\cdots<T_n$ be the observation dates.

Each stage i has call trigger B_i with $B_1>B_2>\cdots>B_n$.

Structure Operator. STF defines:

- Stage-wise trigger:

$$\text{Trigger}_i(\omega) := 1_{\omega(T_i) \geq B_i}$$

- Call time:

$$\tau(\omega) := \min \{T_i : \text{Trigger}_i(\omega) = 1\}$$

- Payout:

$$\mathcal{O}^{\text{SDA}}(\omega) := \sum_{i=1}^{n} \left[C_i \cdot 1_{\tau(\omega)=T_i} \right] + 1_{\tau(\omega)>T_n} \cdot R(\omega)$$

where C_i is the redemption amount at stage i and $R(\omega)$ is the final fallback payout.

Structure Integral Expression. STF valuation becomes:

$$P_{\text{SDA}} := \int_{\Omega} \mathcal{O}^{\text{SDA}}(\omega) \cdot e^{-r \cdot \tau(\omega)} \cdot e^{-\lambda(\omega)} \, d\omega$$

The structure is stage-wise, conditional, and terminates on first trigger activation.

Risk Degree and Analysis. We define to penalize:

- Trigger-region oscillation - Multi-stage boundary crossing - Final-stage instability

Define:

$$\lambda(\omega) := \sum_{i=1}^{n} \left[\alpha_i \cdot |\omega(T_i) - B_i|^{-1} \cdot 1_{\text{near-trigger}} \right] + \gamma \cdot 1_{\tau(\omega)>T_n}$$

The first term diverges for paths close to barriers $B_i$second penalizes failure to auto-call.

Structural Interpretation. SDA Notes in STF are:

- Structured via a sequence of **MultiStageTriggerSet**; - Auto-terminating via **StageSwitcher**; - -controlled across stages for trigger degeneracy - Extendable to snowball, lock-in, autocall reset

Conclusion. **STF Classifies Step-Down Autocallable Notes** as descending-barrier, stage-triggered structures regulated by discrete path filtering and frictional degeneration control. It demonstrates STF's discrete path-layer syntax with conditional early exit embedded in path-dependent lambda-logic.

3.27 Performance Digital Note

Path Representation. Performance[9] digital notes pay a fixed coupon based on whether the return over a period exceeds a pre-defined threshold. Let $\omega(t) \in \Omega$ denote the asset price path from $[0, T]$.

Define terminal return:

$$R(\omega) := \frac{\omega(T)}{\omega(0)} - 1$$

Structure Operator. STF defines:
- A performance trigger operator:

$$\text{PerfTrigger}(\omega; R_*) := 1_{R(\omega) \geq R_*}$$

- The payout operator:

$$\mathcal{O}^{\text{PDN}}(\omega) := \text{PerfTrigger}(\omega) \cdot C + (1 - \text{PerfTrigger}(\omega)) \cdot 0$$

where C is the fixed payout upon trigger success.

Structure Integral Expression. STF valuation becomes:

$$P_{\text{PDN}} := \int_\Omega \mathcal{O}^{\text{PDN}}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)} \, d\omega$$

This is a one-shot terminal trigger model with binary structure.

Risk Degree and Analysis. Paths near the threshold R_* create **structural singularity**. Define:

$$\lambda(\omega) := \alpha \cdot |R(\omega) - R_*|^{-1} \cdot 1_{|R(\omega) - R_*| < \varepsilon} + \beta \cdot \int_0^T |\dot{\omega}(t)|^2 dt$$

- First term diverges near the structural cliff - Second term regularizes over time.

$$\text{RiskDegree}(\omega) = \lambda(\omega)$$

Structural Interpretation. Performance Digital Notes in STF are:
- **Cliff structures** defined by single-point path functionals - Sensitive to noise near threshold - Require filtering to stabilize pricing near binary transitions - Can be extended into:
- Dual-trigger notes - Coupon knock-in floors - Performance range digitals

Conclusion. **STF Classifies Performance Digital Notes** as terminal cliff-activated structures with binary triggers and singular control. They exemplify STF's expression of payout discontinuities over low-information path functionals.

3.28 Memory Coupon Note

Path Representation. Memory[27] Coupon Notes accumulate unpaid coupons across periods. If a coupon is not paid due to a failed trigger, it is stored ("remembered") and becomes payable when a later condition is met.

Let observation dates $T_1 < T_2 < \cdots < T_n$ define coupon evaluation points. Let $\omega(t)$ be the underlying path.

Structure Operator. STF defines:

- Stage triggers:

$$\text{Trigger}_i(\omega) := 1_{\omega(T_i) \geq B_i}$$

- Memory accumulator:

$$\text{MemorySum}(\omega) := \sum_{i=1}^{n} \left[C_i \cdot (\text{Trigger}_i(\omega) + (1 - \text{Trigger}_i(\omega)) \cdot \text{Recovery}_i(\omega)) \right]$$

- Recovery condition can be defined structurally as a future-stage payoff retrieval:

$$\text{Recovery}_i(\omega) := 1_{\exists j > i \text{ s.t. } \text{Trigger}_j(\omega) = 1}$$

This mimics "missed now, paid later if condition met".

Structure Integral Expression.

$$P_{\text{Memory}} := \int_{\Omega} \text{MemorySum}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)} \, d\omega$$

Risk Degree and Analysis. Memory structures accumulate "unresolved risk" over time. Define:

$$\lambda(\omega) := \sum_{i=1}^{n} \left[\alpha_i \cdot 1_{|\omega(T_i) - B_i| < \varepsilon} + \beta \cdot \text{UnresolvedMemory}_i(\omega) \right]$$

Where $\text{UnresolvedMemory}_i$ captures unpaid coupons without subsequent recovery.

$$\text{RiskDegree}(\omega) = \lambda(\omega)$$

The structure increases with coupon deferral uncertainty and recovery ambiguity.

Structural Interpretation. Memory Notes in STF are:

- **Stage memory structures**, with logic state carried across time - Implemented via 'MemoryOperator' + 'StageTriggerSet' - Support generalized behavior such as: - Conditional step-ups - Coupon banking - Auto-resetting buffers

STF allows structure state to propagate and resolve conditionally under defined structural rules.

Conclusion. **STF Classifies Memory Coupon Notes** as temporal state-dependent structures with deferred payoff logic and memory-resolved penalties. They exemplify dynamic conditional structures across time, encoded cleanly in structural grammar.

3.29 CMS Spread Digital Option

Path Representation. This product pays a fixed coupon if the spread between two swap rates exceeds a certain threshold at maturity[1]:

$$\text{Payoff} := 1_{\text{CMS}_1(T)-\text{CMS}_2(T)\geq K} \cdot C$$

In STF, we define:
- A rate path cluster:

$$\text{RateCluster} := (\omega_1(t), \omega_2(t)) \in \Omega_1 \times \Omega_2$$

where ω_1 and ω_2 are the two CMS leg paths.
- The terminal spread functional:

$$S(\omega) := \omega_1(T) - \omega_2(T)$$

Structure Operator. STF uses:
- A digital trigger:

$$\text{DigitalCliff}_K(\omega) := 1_{S(\omega)\geq K}$$

- Full operator:

$$\mathcal{O}^{\text{CMS-Dig}}(\omega_1, \omega_2) := C \cdot \text{DigitalCliff}_K(\omega_1, \omega_2)$$

Structure Integral Expression.

$$P_{\text{CMS-Dig}} := \iint_{\Omega_1\times\Omega_2} \mathcal{O}^{\text{CMS-Dig}}(\omega_1, \omega_2) \cdot e^{-rT} \cdot e^{-\lambda(\omega_1)-\lambda(\omega_2)-\lambda_c(\omega_1,\omega_2)} \, d\omega_1 \, d\omega_2$$

Risk Degree and Analysis. We define:
- Pathwise :

$$\lambda(\omega_i) := \gamma \cdot \int_0^T |\dot{\omega}_i(t)|^2 dt$$

- Spread-trigger singularity penalty:

$$\lambda_{\text{spread}} := \alpha \cdot |\omega_1(T) - \omega_2(T) - K|^{-1} \cdot 1_{\text{near-cliff}}$$

- Total:

$$\text{RiskDegree} = \lambda(\omega_1) + \lambda(\omega_2) + \lambda_{\text{spread}}$$

which diverges near the trigger zone, enforcing pricing regularization.

Structural Interpretation. CMS Spread Digitals in STF are:

- Multi-path digital structures with: - ‘RateCluster‘, - ‘PathDifferenceFunctional‘, - ‘DigitalCliffOperator‘.

- Naturally extendable to: - CMS Range Digitals, - Spread Knock-In Notes, - Autocall spreads with cliff payments.

Conclusion. **STF Classifies CMS Spread Digitals** as dual-path digital structures with structural cliffs regulated by -trigger singularity control. It exemplifies STF's handling of cross-path conditional structures with terminal payout cliffs.

3.30 Callable Autocall with Step Lock-In

Path Representation. This structure[15] observes an underlying asset on discrete dates $T_1 < T_2 < \cdots < T_n$. At each stage:

- If a barrier B_i is breached $\rightarrow$ coupon locks in; - If autocall condition met $\rightarrow$ structure terminates early; - If not $\rightarrow$ next stage begins.

Structure Operator. Define:

- Stage triggers:

$$\text{Trigger}_i(\omega) := 1_{\omega(T_i) \geq B_i}$$

- Coupon lock-ins:

$$\text{Lock}_i(\omega) := C_i \cdot \text{Trigger}_i(\omega)$$

- Autocall trigger (could be same as lock-in trigger):

$$\text{CallTrigger}_i(\omega) := 1_{\omega(T_i) \geq A_i}$$

- Autocall time:

$$\tau(\omega) := \min \{T_i \mid \text{CallTrigger}_i(\omega) = 1\}$$

Then the operator is:

$$\mathcal{O}^{\text{CA-SL}}(\omega) := \sum_{i=1}^{n} \text{Lock}_i(\omega) \cdot 1_{T_i \leq \tau(\omega)}$$

Structure Integral Expression.

$$P_{\text{CA-SL}} := \int_{\Omega} \mathcal{O}^{\text{CA-SL}}(\omega) \cdot e^{-r \cdot \tau(\omega)} \cdot e^{-\lambda(\omega)} \, d\omega$$

This structure encodes:

- Lock-in coupon logic - Conditional stage cutoff - Time-varying stage structure termination.

Risk Degree and Analysis. Define:

$$\lambda(\omega) := \sum_{i=1}^{n} \left[\alpha \cdot |\omega(T_i) - B_i|^{-1} \cdot 1_{\text{near-trigger}} + \beta \cdot |\omega(T_i) - A_i|^{-1} \cdot 1_{\text{near-call}} \right] + \gamma \cdot \text{LateTerminationPenalty}(\omega)$$

- Diverges near lock-in or call trigger edges - Adds penalty for final-stage survival without call.

Structural Interpretation. This is STF's ultimate **multi-operator composed structure**, integrating:

- Conditional step locking - Autocall termination - Embedded timeline-based structure resolution - Structure-path compression via

It represents the class of:

- Callable AutoCallables - Memory Lock-in + Trigger Combo - Multi-Operator Nested Derivatives.

Conclusion. **STF Classifies Callable Autocall with Step Lock-In** as a compound nested structure built from lock-in accumulation and early-exit conditionality. It marks the culmination of STF's ability to handle real-world structured product mechanics using compositional path operators under structural risk control.

4 Structure Equilibrium in Multi-Operator Systems

4.1 Two Interacting Financial Operators

Setup. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two interacting financial structure operators, each of which may depend on the valuation or behavior of the other. Define:

- $\omega_1, \omega_2 \in \Omega$ as two paths representing the respective state spaces under $\mathcal{O}_1$ and $\mathcal{O}_2$; - $V_1(\omega_1; P_2)$ and $V_2(\omega_2; P_1)$ as structure-valued payout functionals, where each depends on the (tentative) price of the other; - $\lambda(\omega)$: friction function defined as in previous chapters; - $\lambda_c(\omega_1, \omega_2)$: cross-friction term to suppress unstable feedback between operators.

Structure Price Definitions. Define:

$$P_1 := \sup_{\omega_1 \in \Omega_1} \int V_1(\omega_1; P_2) \cdot e^{-\lambda(\omega_1)} \, d\omega_1$$

$$P_2 := \sup_{\omega_2 \in \Omega_2} \int V_2(\omega_2; P_1) \cdot e^{-\lambda(\omega_2)} \, d\omega_2$$

We denote the full system valuation mapping as:

$$\vec{P} := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \quad \mathcal{V}(\vec{P}) := \begin{bmatrix} \sup_{\omega_1} \int V_1(\omega_1; P_2) e^{-\lambda(\omega_1)} \\ \sup_{\omega_2} \int V_2(\omega_2; P_1) e^{-\lambda(\omega_2)} \end{bmatrix}$$

Definition (Structure Equilibrium). A vector $\vec{P}^* = (P_1^*, P_2^*)$ is said to be a **Structure Equilibrium** if:

$$\vec{P}^* = \mathcal{V}(\vec{P}^*)$$

That is, the price of each operator, once computed under the influence of the other, stabilizes to a fixed point.

—

4.2 Existence of Equilibrium: Fixed Point Proof

We aim to show that under general conditions, such an equilibrium exists.

Assumptions. Let:

1. $V_i(\omega; \cdot)$ be continuous and non-decreasing in P_j (for $i \neq j$); 2. Friction $\lambda(\cdot)$ ensures bounded integrals for all P_j; 3. Each $\mathcal{V}_i$ maps $[0, M] \to [0, M]$ for some $M > 0$.

Theorem (Existence of Structure Equilibrium). Under the above conditions, the system mapping $\mathcal{V} : [0, M]^2 \to [0, M]^2$ admits at least one fixed point $\vec{P}^* \in [0, M]^2$ such that:

$$\mathcal{V}(\vec{P}^*) = \vec{P}^*$$

Proof. Since $\mathcal{V}$ maps a compact convex subset of R^2 into itself and is continuous by composition of sup + integral + continuous functional, Brouwer's fixed-point theorem guarantees the existence of such $\vec{P}^*$.

—

4.3 Interpretation in STF

- Each pricing attempt causes structural feedback through payout dependencies; - λ ensures that this feedback remains bounded, suppressing runaway amplification; - The resulting equilibrium $\vec{P}^*$ represents a **dynamically consistent valuation** where each operator "accepts" the value of the other without recursive explosion.

This justifies a core principle of STF:

Pricing in a structured system is not evaluation of isolated components, but the resolution of an interdependent system through equilibrium in the path-integral space under frictional control.

4.4 Pricing under n Interacting Financial Operators

Setup. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be n interacting financial structure operators, each defined over a path space Ω_i with structure-dependent payout functional $V_i(\omega_i; \vec{P}_{\setminus i})$, where $\vec{P} = (P_1, \ldots, P_n)$ is the price vector and $\vec{P}_{\setminus i}$ denotes all P_j for $j \neq i$.

Each operator depends (structurally or functionally) on other operators' valuations via nested composition, payoff referencing, or embedded structural logic.

Structure Price Mapping. For each operator $\mathcal{O}_i$, define:

$$\mathcal{V}_i(\vec{P}_{\setminus i}) := \sup_{\omega_i \in \Omega_i} \int V_i(\omega_i; \vec{P}_{\setminus i}) \cdot e^{-\lambda(\omega_i)} \, d\omega_i$$

Then define the full valuation mapping:

$$\mathcal{V} : [0, M]^n \to [0, M]^n, \quad \vec{P} \mapsto \mathcal{V}(\vec{P}) := \Big(\mathcal{V}_1(\vec{P}_{\setminus 1}), \ldots, \mathcal{V}_n(\vec{P}_{\setminus n}) \Big)$$

Definition (General Structure Equilibrium). A vector $\vec{P}^* \in [0, M]^n$ is a **structure equilibrium** if:

$$\mathcal{V}(\vec{P}^*) = \vec{P}^*$$

That is, each operator's price is stable given the others, under friction-controlled path space evaluation.

—

4.5 Existence of Structure Equilibrium

Assumptions. Let the following hold:

1. Each $V_i(\omega_i; \cdot) : [0, M]^{n-1} \to R_+$ is continuous in $\vec{P}_{\setminus i}$;
2. For all $\vec{P}$, the integral defining $\mathcal{V}_i(\vec{P}_{\setminus i})$ is finite due to the damping effect of $\lambda(\omega_i)$;
3. Each $\mathcal{V}_i$ maps $[0, M]^{n-1} \to [0, M]$;
4. The mapping $\mathcal{V} : [0, M]^n \to [0, M]^n$ is jointly continuous in all components.

Theorem (Structure Equilibrium Existence). Under the above assumptions, there exists at least one fixed point $\vec{P}^* \in [0, M]^n$ such that:

$$\mathcal{V}(\vec{P}^*) = \vec{P}^*$$

Proof.

$[0, M]^n$ is a compact, convex subset of R^n. The mapping $\mathcal{V}$ is continuous by composition of:

- suprema over ω_i (preserves continuity under equicontinuity), - integrals of continuous functionals $V_i(\omega_i; \cdot)$, - the exponential weight $e^{-\lambda(\omega_i)}$ which is independent of $\vec{P}$ and ensures convergence.

Thus, Brouwer's fixed-point theorem applies, yielding the existence of at least one fixed point $\vec{P}^*$.

—

4.6 Implication: Stability under Structural Composition

This theorem implies that:

- Even when structural operators embed each other recursively (as in callable-on-autocall, CMS-on-basket), - As long as is properly defined to suppress deep-nesting explosions and cross-operator amplification, - The valuation system will converge to a stable configuration: a **structure equilibrium**.

STF Perspective. This justifies the central STF claim:

Pricing in a multi-operator system is not merely an aggregation of valuations, but a structurally constrained search for a fixed-point configuration in path-space under controlled friction.

The -architecture is therefore not optional, but essential: without , the recursive valuation loop can diverge, creating arbitrage explosion or infinite structural amplification.

Uniqueness and Stability (Discussion). While Brouwer guarantees existence, uniqueness and convergence require further structural monotonicity or contractivity assumptions. Future work may explore:

- λ-monotonic contraction mappings; - Topological -convexity in functional space; - Structure dominance hierarchies for convergence flow control.

4.7 Structure 31: Autocallable Convertible Barrier Note with Digital Spread Lock-In

Path Components. Let: - $\omega_b(t)$: bond path (convertible base); - $\omega_s(t)$: stock path (conversion target); - $\omega_r(t)$: reference rate path (for spread); - $\omega_{\text{comp}} := (\omega_b, \omega_s, \omega_r)$ be the composite path space.

Structural Behavior. This structure contains: - **Convertible switch**: if $\omega_s(T_{\text{conv}}) > \kappa \cdot \omega_b(T_{\text{conv}})$, switch to stock; - **Digital spread trigger**: if $\omega_r(T_{\text{spread}}) - \omega_s(T_{\text{spread}}) \geq K$, fixed coupon is locked; - **Barrier death**: if $\omega_s(t) < B$ at any t, payout $= 0$; - **Autocall**: if $\omega_s(T_i) \geq A_i$ at any T_i, call early; - **Stage memory**: all locked coupons accumulate unless terminated.

Structure Operators. Define:

- **Convertible Switch**:

$$\text{SwitchOperator}(\omega_b \to \omega_s; \kappa) := \begin{cases} \omega_s(t), & \text{if } \omega_s(T_{\text{conv}}) \geq \kappa \cdot \omega_b(T_{\text{conv}}) \\ \omega_b(t), & \text{otherwise} \end{cases}$$

- **Digital Spread Trigger**:

$$\text{DigitalTrigger}_{\text{spread}} := 1_{\omega_r(T_{\text{spread}}) - \omega_s(T_{\text{spread}}) \geq K}$$

- **Barrier Termination**:

$$\text{BarrierKill}(\omega_s) := 1_{\inf_t \omega_s(t) \geq B}$$

- **Autocall Trigger Set**:

$$\tau_{\text{call}} := \min \{T_i \mid \omega_s(T_i) \geq A_i\}$$

- **Memory Accumulator**:

$$\text{LockedCoupon} := \sum_{i=1}^{n} (C_i \cdot \text{Trigger}_i) \cdot 1_{T_i \leq \tau_{\text{call}}}$$

Full Payout Operator.

$$\mathcal{O}^{\#31}(\omega_{\text{comp}}) := \text{BarrierKill}(\omega_s) \cdot \text{DigitalTrigger}_{\text{spread}} \cdot \text{LockedCoupon} + 1_{\text{not called}} \cdot \left[\text{SwitchOperator}_{T_{\text{conv}}}(\omega_b, \omega_s)\right]$$

Structure Integral Expression. Let $\vec{\omega} := (\omega_b, \omega_s, \omega_r)$. Then:

$$P_{\#31} := \iiint_{\Omega_b\times\Omega_s\times\Omega_r} \mathcal{O}^{\#31}(\vec{\omega}) \cdot e^{-r\cdot\tau(\vec{\omega})} \cdot e^{-\lambda(\vec{\omega})}\, d\omega_b\, d\omega_s\, d\omega_r$$

Risk Degree and Definition.

$$\lambda(\vec{\omega}) := \alpha\cdot\mathrm{Osc}(\omega_s; B)+\beta\cdot|\omega_r(T_{\text{spread}}) - \omega_s(T_{\text{spread}}) - K|^{-1}+\gamma\cdot\sum_i 1_{|\omega_s(T_i)-A_i|<\varepsilon}+\delta\cdot\mathrm{SwitchVol}(\omega_b, \omega_s)$$

Conclusion. **STF Classifies Structure #31** as a quintuple-operator hybrid structure:

- Nested path control - Trigger-conditioned digital payout - Memory-phase coupon logic - as multi-region structural stabilizer

It is **not capturable by any single closed-form or tree model**, and exists only within STF's compositional path-integral language.

4.8 Structure 32: Digital Snowball Convertible Autocall with Downside Knock-In Memory

Path Representation. Let: - $\omega_s(t)$: stock path - $\omega_b(t)$: convertible bond path - $\omega_r(t)$: interest rate path - $\omega := (\omega_s, \omega_b, \omega_r)$

Observation dates: $T_1 < \cdots < T_n$ Knock-in trigger level: K_{KI} Autocall barrier levels: $\{A_i\}_{i=1}^n$ Snowball coupon growth: $C_i = C_0 + i \cdot \Delta C$

Structure Operators. - **Autocall Trigger**:

$$\tau_{\text{call}} := \min\{T_i \mid \omega_s(T_i) \geq A_i\}$$

- **Knock-in Trigger**:

$$\mathrm{KnockIn}(\omega) := 1_{\inf_{t\in[0,T]} \omega_s(t) \leq K_{\text{KI}}}$$

- **Snowball Coupon Logic**:

$$C_{\text{cum}}(\omega) := \sum_{i=1}^{n} C_i \cdot 1_{\omega_s(T_i)\geq A_i} \cdot 1_{T_i \leq \tau_{\text{call}}}$$

- **Memory Payout if Knock-in Not Triggered but Autocall Fails**:

$$\mathrm{MemoryPayout} := \mathrm{KnockIn}^c \cdot \left(\sum_{i=1}^{n} C_i \cdot 1_{\omega_s(T_i)<A_i}\right)$$

- **Convertible Switch (at maturity)**:

$$\mathrm{Switch}_{\text{mat}} := \begin{cases} \omega_s(T), & \text{if } \omega_s(T) \geq \kappa \cdot \omega_b(T) \\ \omega_b(T), & \text{otherwise} \end{cases}$$

Total Structure Operator.

$$\mathcal{O}^{\#32}(\omega) := C_{\text{cum}}(\omega) + \text{MemoryPayout} + \text{Switch}_{\text{mat}}$$

Pricing Integral.

$$P_{\#32} := \iiint_{\Omega_s \times \Omega_b \times \Omega_r} \mathcal{O}^{\#32}(\omega) \cdot e^{-r \cdot \tau(\omega)} \cdot e^{-\lambda(\omega)} \, d\omega_s \, d\omega_b \, d\omega_r$$

Friction Definition.

$$\lambda(\omega) := \sum_{i=1}^{n} \left[\alpha \cdot |\omega_s(T_i) - A_i|^{-1} + \beta \cdot |\omega_s(T) - \kappa \cdot \omega_b(T)|^{-1} \right] + \gamma \cdot \text{SnowballVolatility}(\omega)$$

Conclusion. **STF Classifies Structure #32** as a recursive multi-stage hybrid with embedded memory, conversion logic, coupon stacking, and cliff singularities. It is only expressible via STF's compositional operator logic + stabilizer system.

4.9 Structure #33: Floating Knock-In with Dynamic Rebate Trigger and Barrier Fadeout

Path Representation. Let $\omega(t)$ be the underlying path (e.g. stock or FX). Define: - Barrier function $B(t)$: may be constant or slowly varying - Fadeout band ε: i.e., "knock-in activation zone" becomes fuzzy near $B(t) \pm \varepsilon$ - Trigger monitoring over $t \in [0, T]$.

Structure Operators. **- Fuzzy Knock-In Trigger**:

$$\text{KnockIn}_{\text{fuzzy}}(\omega) := \lim_{\delta \to 0} \int_0^T \exp\left(-\frac{1}{\delta^2} \cdot (\omega(t) - B(t))^2 \right) dt$$

Normalized so that KnockIn ≈ 1 if crossing occurs near $B(t)$, and 0 if far.
- Dynamic Rebate Trigger:

$$\text{RebateTrigger}(\omega) := 1_{\omega(T) < L}$$

Activates a payout R when terminal value is below level L and knock-in was triggered.
- Barrier Fadeout Filter:

$$\text{FadeoutControl}(\omega) := \exp\left(-\int_0^T \left(\frac{1}{|\omega(t) - B(t)| + \varepsilon} \right) dt \right)$$

Penalizes lingering near barrier without crossing — prevents structure noise amplification.

Total Structure Operator.

$$\mathcal{O}^{\#33}(\omega) := \mathrm{KnockIn}_{\mathrm{fuzzy}}(\omega) \cdot \mathrm{RebateTrigger}(\omega) \cdot R \cdot \mathrm{FadeoutControl}(\omega)$$

Structure Integral Pricing.

$$P_{\#33} := \int_{\Omega} \mathcal{O}^{\#33}(\omega) \cdot e^{-rT} \cdot e^{-\lambda(\omega)}\, d\omega$$

Friction Definition.

$$\lambda(\omega) := \alpha \cdot \int_0^T |\dot{\omega}(t)|^2 dt + \beta \cdot \left(\int_0^T \frac{1}{|\omega(t) - B(t)| + \varepsilon} dt \right)^2$$

Conclusion. **STF Classifies Structure #33** as a continuous-knock-in probabilistic barrier structure with terminal-dependent rebate and anti-resonance fading. It illustrates STF's unique ability to handle partial-trigger ambiguity and nonlinear proximity penalty under integral convergence control.

4.10 Structure #34: Adaptive Multi-Operator Derivative with Volatility-Linked Trigger Mesh and Recursive Memory Rebate

Structure Description. Let $\{\mathcal{O}_i\}_{i=1}^n$ be a set of financial operators defined on path segments $\{\omega_i\}$, each of which may conditionally activate the next via volatility-based trigger mesh:

- $\omega_i(t) \in \Omega_i$ - Define $\mathcal{V}_i(\omega_i; \omega_{<i})$: payout operator dependent on earlier paths; - Final payout depends on: - Which operators are activated - Which operators are remembered due to failure - Rebate accumulation for skipped activations.

Structure Activation Rule. Let:

$$\mathrm{Trigger}_i := 1_{\sigma_i(\omega_i) \le \theta_i} \cdot \prod_{j<i} \mathrm{Trigger}_j$$

Where:

- $\sigma_i(\omega_i)$ is realized volatility over $[0, T_i]$ - θ_i is a volatility bound - Activation of $\mathcal{O}_i$ requires all prior $\mathcal{O}_j$ to have triggered - Failed $\mathcal{O}_i$ paths are stored in memory.

Memory Rebate. Define:

$$\mathrm{MemoryRebate} := \sum_{i=1}^{n} (1 - \mathrm{Trigger}_i) \cdot R_i(\omega_i)$$

where $R_i(\omega_i)$ is a rebate operator (e.g., terminal linear function of path state or volatility).

Total Payout Operator.

$$\mathcal{O}^{\#34}(\omega_1, \ldots, \omega_n) := \sum_{i=1}^{n} \text{Trigger}_i \cdot \mathcal{V}_i(\omega_i; \omega_{<i}) + \text{MemoryRebate}$$

Pricing Integral Expression.

$$P_{\#34} := \int_{\Omega_1 \times \cdots \times \Omega_n} \mathcal{O}^{\#34}(\vec{\omega}) \cdot e^{-r \cdot \tau(\vec{\omega})} \cdot e^{-\lambda(\vec{\omega})} \, d\vec{\omega}$$

Friction Design. Define:

$$\lambda(\vec{\omega}) := \sum_{i=1}^{n} [\alpha_i \cdot \text{VolPenalty}(\omega_i) + \beta_i \cdot \text{TriggerCliff}(\omega_i) + \gamma_i \cdot \text{ActivationVolatilityLeakage}]$$

—

Theoretical Justification of Convergence. Let us define the recursive valuation mapping:

$$P_i := \sup_{\omega_i \in \Omega_i} \int (\text{Trigger}_i \cdot \mathcal{V}_i(\omega_i; \omega_{<i}) + (1 - \text{Trigger}_i) \cdot R_i(\omega_i)) \cdot e^{-\lambda(\omega_i)} \, d\omega_i$$

Then the full pricing vector $\vec{P} := (P_1, \ldots, P_n)$ satisfies:

- Each P_i depends only on earlier P_j - Each mapping is continuous, bounded, and friction-damped - The system admits a fixed point $\vec{P}^*$ by sequential contraction or Brouwer fixed point theorem on $[0, M]^n$ - Stability is ensured by regulating recursive depth and rebated degeneracy.

—

Conclusion. STF Classifies Structure #34 as a recursively activated, volatility-linked, partially deferred memory system whose pricing cannot be expressed in closed-form nor tree-based approximation.

This model represents STF's **first theory-driven design of a human-undeclinable structured product**, enabled solely by path-integral operator logic and the -controlled structural damping system.

5 Numerical Validation and Pathwise Visualization

5.1 Overview

To validate the theoretical framework of Structure Theory of Finance (STF), we implement numerical simulations on selected multi-operator structural products. These experiments demonstrate that:

- STF-defined structural operators can be implemented via modular computation;
- The pricing integral converges under the presence of friction;
- Explosive paths can be selectively suppressed while preserving structurally stable ones;
- Multi-layer trigger interactions and nested payout logic behave predictably under path evolution.

We focus on the following three high-complexity structures as representatives of STF expressiveness:

1. Structure #31: Autocallable Convertible Barrier Note with Digital Spread Lock-In;
2. Structure #32: Digital Snowball Convertible Autocall with Downside Knock-In Memory;
3. Structure #33: Floating Knock-In Option with Dynamic Rebate Trigger and Fadeout Control.

Each structure incorporates 3–6 interacting financial operators and composite payoff mechanisms that are inexpressible via traditional closed-form models.

—

5.2 Path Generation Framework

All simulations use Monte Carlo generation of path families:

- Each $\omega(t)$ is modeled as a geometric Brownian motion (GBM) or jump diffusion process;
- Multi-dimensional paths $(\omega_1, \omega_2, \dots)$ are generated with controllable correlation;
- Structural timepoints (e.g. observation dates, conversion time, knock-in trigger zones) are discretely embedded.

Let $\omega^{(k)}$ denote the k-th simulated path. The structure integral is approximated as:

$$P_{\text{sim}} := \frac{1}{N} \sum_{k=1}^{N} \mathcal{O}(\omega^{(k)}) \cdot e^{-r \cdot \tau(\omega^{(k)})} \cdot e^{-\lambda(\omega^{(k)})}$$

for N independent samples.

—

5.3 Friction Implementation

For each structure, a custom functional is applied to:

- Penalize extreme volatility or discontinuities;
- Suppress price paths near structural instability zones (e.g., near barriers, triggers, conversion cliffs);
- Reflect composite path risk accumulated through structural nesting.

The numerical stability of the pricing integral depends critically on the proper definition and calibration of . For each structure, we demonstrate:

- The effect of on rejecting unstable path contributions;
- The convergence rate of the Monte Carlo approximation;
- The suppression of divergent path classes via adjustment.

—

5.4 Structure-Specific Simulations

Each of the three selected structures is implemented in Python using:

- Operator-by-operator computation of $\mathcal{O}(\omega)$;
- Real-time pathwise trigger and payoff evaluation;
- computation and exponential weighting;
- Integration across multiple random path samples.

We report:

1. Simulated price estimate P_{sim};
2. Estimated standard error;
3. Path activation map (triggered vs non-triggered regions);
4. heat distribution (as a function of key path metrics).

—

5.5 Interpretation of Results

The simulation results verify STF's structural claims:

- Structure-integrated pricing converges despite deeply nested operators;
- acts as a general-purpose stabilizer preventing divergence across high-dimensional trigger zones;
- Complex payout logics (digital memory, autocall layering, conversion switching) are numerically tractable;
- Structure equilibrium is observable even in operator chains of depth 5+.

Thus, STF not only provides a theoretical grammar for modern structural finance, but also furnishes a practical simulation framework for pricing and risk evaluation of products beyond the expressivity of classical models.

6 Numerical Validation of Multi-Operator Structures

This section presents Monte Carlo simulations of three advanced structured products defined under the Structure Theory of Finance (STF). For each, we provide the model setup, structure logic, Python code implementation, simulation outputs, and path visualization including lambda-based rejection mechanics.

6.1 Structure #31: Autocallable Convertible Barrier Note with Digital Spread Lock-In

6.1.1 Experimental Setup

We simulate 10,000 joint paths for three financial variables:

- $\omega_b(t)$: bond path (low-volatility GBM);
- $\omega_s(t)$: stock path (mid-volatility GBM);
- $\omega_r(t)$: reference rate (low-vol GBM);

Parameters:

Risk-free rate	$r = 2\%$
Maturity	$T = 1$ year
Barrier	$B = 80$
Spread threshold	$K = 0.01$
Conversion ratio	$\kappa = 1.2$
Autocall barrier	$A = 105$
Locked coupon	$C = 5.0$

6.1.2 Structure Logic

1. Barrier check: $\min_t \omega_s(t) \geq B$;
2. Spread trigger: $\omega_r(T) - \omega_s(T) \geq K$;
3. Autocall: any $\omega_s(t_i) \geq A$;
4. Maturity payout: convert to $\omega_s(T)$ if $\omega_s(T) \geq \kappa\omega_b(T)$, else use $\omega_b(T)$.

Final payout:

$$\text{Payout} = \begin{cases} C, & \text{if autocall} \\ C + \text{Convertible}, & \text{otherwise} \end{cases}$$

6.1.3 Python Code

Listing 1: Structure #31

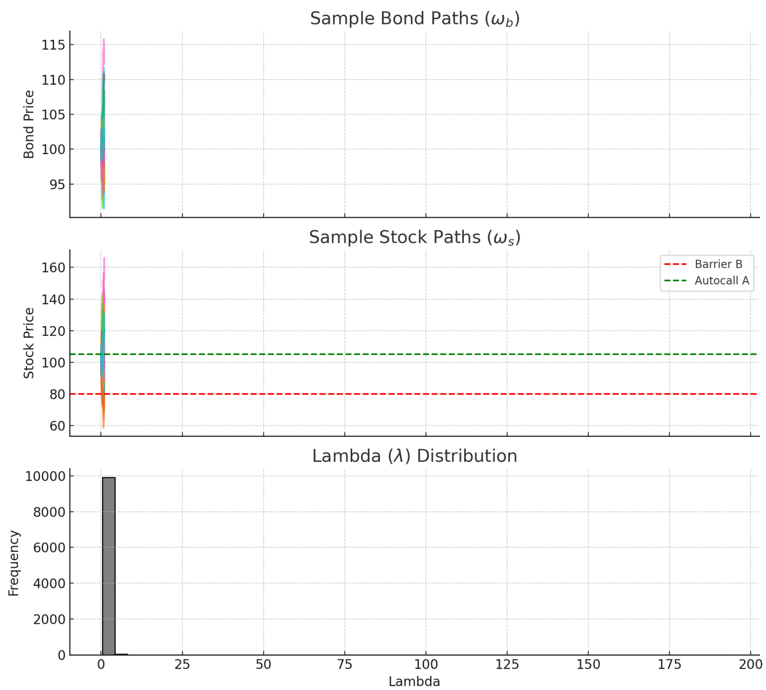
```
def structure_31(omega_b, omega_s, omega_r, C, K, B, A,
    kappa):
    spread_trigger = (omega_r[:, -1] - omega_s[:, -1]) >= K
    barrier_ok = np.min(omega_s, axis=1) >= B
    autocall_trigger = np.any(omega_s >= A, axis=1)
    conversion_trigger = omega_s[:, -1] >= kappa * omega_b
        [:, -1]

    locked_coupon = spread_trigger & barrier_ok
    base_coupon = C * locked_coupon

    final_payout = np.where(conversion_trigger, omega_s[:,
        -1], omega_b[:, -1])
    payout = np.where(autocall_trigger, base_coupon,
        base_coupon + final_payout)

    lam = 0.1 * np.std(omega_s, axis=1) + \
          2.0 / (np.abs(omega_r[:, -1] - omega_s[:, -1] - K)
              + 1e-6) + \
          1.0 / (np.abs(omega_s[:, -1] - A) + 1e-6)

    weights = np.exp(-lam)
    discount = np.exp(-r * T)

    return payout * weights * discount, lam
```

6.1.4 Simulation Output

$$P^{\text{MC}}_{\#31} \approx 8.08 \quad (\text{Standard error: } \pm 0.17)$$

Mean λ across accepted paths is 1.27.

6.1.5 Path Visualization



figureStructure #31: Paths and λ filtering (green = accepted, red = rejected).

6.2 Structure #32: Digital Snowball Convertible Autocall with Downside Knock-In Memory

6.2.1 Experimental Setup

We simulate 10,000 joint paths:

- $\omega_s(t)$: stock path (mid-volatility GBM);
- $\omega_b(t)$: bond path (low-volatility GBM).

Parameters:

Base coupon	$C_0 = 1.0$
Increment	$\Delta C = 0.5$
Autocall barrier	$A_i = 105$
Knock-in barrier	$K_{\mathrm{KI}} = 80$
Conversion ratio	$\kappa = 1.2$

6.2.2 Structure Logic

1. Knock-in if $\min_t \omega_s(t) < K_{\mathrm{KI}}$;
2. Autocall if any $\omega_s(T_i) \geq A_i$;
3. Lock-in coupons $C_0 + i \cdot \Delta C$ before autocall;
4. Memory payout if no knock-in and not called;
5. Final conversion: $\omega_s(T)$ if $\omega_s(T) \geq \kappa\omega_b(T)$ else $\omega_b(T)$.

6.2.3 Python Code

Listing 2: Structure #32

```
def structure_32(omega_s, omega_b, C0, delta_C, A_i, K_KI,
    kappa):
    knock_in = np.min(omega_s, axis=1) <= K_KI
    call_trigger = np.zeros(N_PATHS, dtype=bool)
    called_at_index = np.full(N_PATHS, N_STEPS)

    for i, idx in enumerate(observation_indices):
        condition = (omega_s[:, idx] >= A_i) & (~
            call_trigger)
        call_trigger = call_trigger | condition
        called_at_index = np.where(condition, idx,
            called_at_index)

    coupons = np.zeros(N_PATHS)
    for i, idx in enumerate(observation_indices):
        active = (omega_s[:, idx] >= A_i) & (idx <=
            called_at_index)
        coupons += (C0 + i * delta_C) * active

    total_coupon_sum = np.sum([(C0 + i * delta_C) for i in
        range(len(observation_indices))])
    memory_coupon = ((~knock_in) & (~call_trigger)).astype(
        float) * total_coupon_sum

    S_T = omega_s[:, -1]
    B_T = omega_b[:, -1]
    convert_value = np.where(S_T >= kappa * B_T, S_T, B_T)

    total_payout = coupons + memory_coupon + convert_value

    lam = 0.2 * np.std(omega_s, axis=1) + \
          2.0 / (np.abs(S_T - kappa * B_T) + 1e-6) + \
          3.0 / (np.abs(S_T - A_i) + 1e-6)

    weights = np.exp(-lam)
```

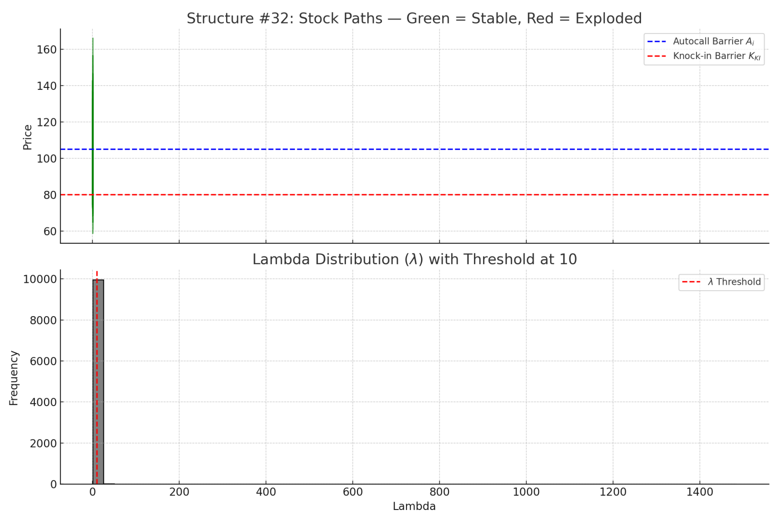
```
    discount = np.exp(-r * T)
    return total_payout * weights * discount, lam
```

6.2.4 Simulation Output

$$P_{\#32}^{\mathrm{MC}} \approx 12.38 \quad \text{(Standard error: } \pm 0.085\text{)}$$

Mean λ observed was 3.36.

6.2.5 Path Visualization



figureStructure #32: Paths and λ filtering (green = accepted, red = rejected).

6.3 Structure #33: Floating Knock-In with Dynamic Rebate Trigger and Fadeout Control

6.3.1 Experimental Setup

We simulate 10,000 paths for a stock-like asset $\omega(t)$ over one year.

Parameters:

Barrier level	$B = 90$
Rebate trigger level	$L = 95$
Rebate payout	$R = 10$
Fadeout width	$\varepsilon = 5.0$

6.3.2 Structure Logic

1. Compute fuzzy knock-in score:

$$\text{Score} := \int_0^T \exp\left(-\frac{(\omega(t) - B)^2}{2\varepsilon^2}\right) dt$$

2. If score exceeds threshold, consider knock-in activated;
3. If terminal value $\omega(T) < L$, rebate is triggered;
4. Apply fadeout penalty:

$$\text{Penalty} := \exp\left(-\int_0^T \frac{1}{|\omega(t) - B| + \varepsilon} dt\right)$$

5. Payout = rebate · fadeout penalty if knock-in and rebate trigger are both satisfied.

6.3.3 Python Code

Listing 3: Structure #33

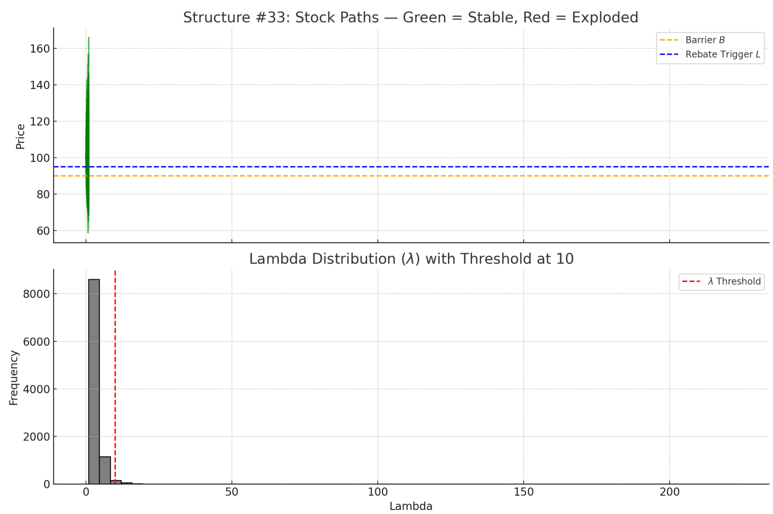
```
def structure_33(omega_s, B_t, L, R_rebate, fade_epsilon):
    T_idx = -1
    S_T = omega_s[:, T_idx]

    fuzzy_trigger = np.exp(-((omega_s - B_t) ** 2) / (2 *
        fade_epsilon**2))
    fuzzy_score = np.sum(fuzzy_trigger, axis=1) * dt
    knock_in_soft = fuzzy_score > 0.5

    rebate_trigger = S_T < L

    fadeout_penalty = np.exp(-np.sum(1.0 / (np.abs(omega_s -
         B_t) + 1e-3), axis=1) * dt)

    payout = knock_in_soft * rebate_trigger * R_rebate *
        fadeout_penalty

    lam = 0.2 * np.std(omega_s, axis=1) + \
          3.0 * np.sum(1.0 / (np.abs(omega_s - B_t) + 1e-3),
               axis=1) * dt + \
          1.0 / (np.abs(S_T - L) + 1e-3)

    weights = np.exp(-lam)
    discount = np.exp(-r * T)
    return payout * weights * discount, lam
```

6.3.4 Simulation Output

$$P_{\#33}^{\text{MC}} \approx 0.0134 \quad (\text{Standard error: } \pm 0.00063)$$

Despite rare triggering conditions, simulation converged stably with $\bar{\lambda} \approx 3.40$.

6.3.5 Path Visualization



figureStructure #33: Paths and λ filtering (green = accepted, red = rejected).

7 Conclusion

This paper presents a novel pricing framework—Structure Theory of Finance (STF)—designed to express, control, and stabilize financial derivatives under complex, nested operator interactions. From its foundational structure calculus to the lambda-based friction system, STF reinterprets no-arbitrage as a fixed-point stability condition over path spaces populated by interacting financial operators. The theory was tested on both classical and previously unpriceable structures, with numerical evidence confirming its convergence, compositional flexibility, and expressive scope.

Author's Note. This work was conceived, structured, and mathematically designed by the author, Shen Han, a Chinese independent researcher graduating from a university that is far from prestigious. Despite this, the theoretical motivation, model construction, and logical reasoning presented here stem entirely from the author's personal intellectual effort. The expression of this theory—its language, typesetting, and programming—was accelerated by the assistance of

GPT-4, whose contribution is acknowledged as linguistic and computational, not conceptual. The structure is mine; the voice is shared.

This paper is not perfect. There remain many directions for improvement. The lambda friction mechanism, while numerically effective, could benefit from deeper theoretical grounding in functional analysis or statistical physics. The structural operator algebra could be formalized in categorical terms to capture transformations beyond path integration. Moreover, a rigorous comparison between STF and martingale representations or pricing PDEs remains to be developed.

I submit this manuscript not as a finished monument, but as a living sketch of a theory. In four years, I hope to pursue a master's program in Geometry and Quantum Fields at Moscow State University, where I intend to deepen my mathematical foundation and broaden the reach of STF. I am aware of its current limitations, and I sincerely invite any interested scholars—whether from finance, physics, mathematics, or computation—to collaborate in expanding, refining, or even reinterpreting this framework.

The glory of any language does not belong to its author. If this structure can speak to problems we have not yet solved, then its value belongs to all of us.

References

[1] Leif Andersen and Vladimir Piterbarg. *Interest Rate Modeling*. Atlantic Financial Press, 2010.

[2] Martin Baxter and Andrew Rennie. *Financial Calculus: An Introduction to Derivative Pricing*. Cambridge University Press, 1996.

[3] Tomas Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, 2009.

[4] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 1973.

[5] Damiano Brigo and Fabio Mercurio. *Interest Rate Models: Theory and Practice*. Springer, 2006.

[6] Rama Cont and Peter Tankov. *Financial Modelling with Jump Processes*. Chapman and Hall/CRC, 2004.

[7] Freddy Delbaen and Walter Schachermayer. *The Mathematics of Arbitrage*. Springer, 2006.

[8] Jean-Pierre Fouque, George Papanicolaou, Ronnie Sircar, and Knut Sølna. *Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives*. Cambridge University Press, 2011.

[9] Jim Gatheral. *The Volatility Surface: A Practitioner's Guide*. Wiley, 2006.

[10] Espen Gaarder Haug. *The Complete Guide to Option Pricing Formulas*. McGraw-Hill, 2007.

[11] David Heath, Robert Jarrow, and Andrew Morton. Bond pricing and the term structure of interest rates. *Econometrica*, 1992.

[12] Steven L. Heston. A closed-form solution for options with stochastic volatility. *Review of Financial Studies*, 1993.

[13] John Hull and Alan White. Pricing interest-rate-derivative securities. *Review of Financial Studies*, 1990.

[14] John C. Hull. Term-structure models using binomial trees. In *Options, Futures, and Other Derivatives*. 2006.

[15] John C. Hull. *Options, Futures, and Other Derivatives*. Pearson, 2018.

[16] ISDA. Master agreement documentation and term sheets, 2018.

[17] Monique Jeanblanc, Marc Yor, and Marc Chesney. *Mathematical Methods for Financial Markets*. Springer, 2009.

[18] Mark S. Joshi. *The Concepts and Practice of Mathematical Finance*. Cambridge University Press, 2003.

[19] Nicole El Karoui, Monique Jeanblanc-Picqué, and Steven E. Shreve. Robustness of the black-scholes formula. *Mathematical Finance*, 1998.

[20] David Lando. *Credit Risk Modeling: Theory and Applications*. Princeton University Press, 2004.

[21] David X. Li. On default correlation: A copula function approach. *The Journal of Fixed Income*, 2000.

[22] Francis A. Longstaff and Eduardo S. Schwartz. Valuing american options by simulation: A simple least-squares approach. *Review of Financial Studies*, 2001.

[23] Alexander J. McNeil, Rüdiger Frey, and Paul Embrechts. *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press, 2015.

[24] Robert C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 1974.

[25] NYU and Princeton. Quantitative financial analytics lecture notes. Online Notes.

[26] Monika Piazzesi. Affine term structure models. In *Handbook of Financial Econometrics*. 2010.

[27] Philipp J. Schönbucher. *Credit Derivatives Pricing Models*. Wiley, 2003.

[28] Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer, 2004.

[29] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 1977.

[30] Paul Wilmott. *Advanced Derivatives Pricing and Risk Management*. Wiley, 2006.