

Category and ∞ -Category Theory in Context

From Analysis to Higher Structures

Shen Han

November 28, 2025

Contents

Preface	vii
I Preliminaries: Logic, Algebra and Topology	1
1 Logic and Set Theory Foundations	3
1.1 Propositional and First-Order Logic	3
1.2 Sets, Relations and Equivalence Relations	3
1.3 Maps, Injectivity, Surjectivity and Bijections	3
1.4 Orders, Partial Orders and Lattices	3
1.5 Choice Axiom, Zorn's Lemma and Maximal Elements	3
1.6 Exercises and Notes	3
2 Basic Algebraic Structures	5
2.1 Groups and Group Homomorphisms	5
2.2 Rings, Ideals and Ring Homomorphisms	5
2.3 Vector Spaces and Linear Maps	5
2.4 Modules and Free Objects	5
2.5 Abelian Groups and \mathbb{Z} -Modules	5
2.6 Exercises and Notes	5
3 Point-Set Topology and Metric Spaces	7
3.1 Metric Spaces, Open and Closed Sets	7
3.2 Topological Spaces: Axiomatic Definition	7
3.3 Continuous Maps and Homeomorphisms	7
3.4 Product Topology and Subspace Topology	7
3.5 Compactness and Connectedness	7
3.6 Relation Between Metric and Topological Spaces	7
3.7 Exercises and Notes	7
4 Linear Analysis and Functional Examples	9
4.1 Normed and Banach Spaces	9
4.2 Bounded Linear Operators and Operator Norm	9
4.3 Inner Product and Hilbert Spaces	9
4.4 Function Spaces: $C(X)$, L^p , ℓ^p	9
4.5 Structured Maps: Self-Adjoint, Compact Operators, etc.	9
4.6 Exercises and Notes	9

II	Basic Category Theory	11
5	Categories and Basic Examples	13
5.1	Definition of a Category	13
5.2	Standard Examples: Set, Top, Grp, Ring, Vect, Ban	13
5.3	Preorders, Posets and Small Categories	13
5.4	Opposite Categories and Duality Philosophy	13
5.5	Subcategories and Full Subcategories	13
5.6	Exercises and Notes	13
6	Functors and Natural Transformations	15
6.1	Covariant and Contravariant Functors	15
6.2	Composition of Functors and Identity Functor	15
6.3	Natural Transformations and Naturality Squares	15
6.4	Natural Isomorphisms and Equivalences of Categories	15
6.5	Catas a 2-Category: First Look	15
6.6	Exercises and Notes	15
7	Representable Functors and Yoneda's Lemma	17
7.1	Hom-Functors and Representable Functors	17
7.2	Yoneda Lemma and Its Proof	17
7.3	The Yoneda Embedding	17
7.4	Characterizing Structures via Representability	17
7.5	Examples in Set, Top, Ab, Ban	17
7.6	Exercises and Notes	17
8	Limits, Colimits and Universal Properties	19
8.1	Diagrams, Cones and Cocones	19
8.2	Terminal Objects, Products, Pullbacks, Equalizers	19
8.3	Initial Objects, Coproducts, Pushouts, Coequalizers	19
8.4	Complete and Cocomplete Categories	19
8.5	Category of Elements and Pointwise Descriptions	19
8.6	Computing Limits and Colimits in Concrete Categories	19
8.7	Exercises and Notes	19
9	Adjoint Functors	21
9.1	Definition of Adjunction: Unit and Counit	21
9.2	Hom-Set Characterization of Adjunctions	21
9.3	Classical Examples: Free/Forgetful, Tensor/Hom, etc.	21
9.4	Adjoints and (Co)Limits	21
9.5	Reflective Subcategories and Constructions via Adjoints	21
9.6	Exercises and Notes	21
10	Monads and Their Algebras	23
10.1	Monads as Monoid Objects in Endofunctors	23
10.2	Monads Arising from Adjunctions	23
10.3	Algebras for a Monad and Eilenberg–Moore Categories	23
10.4	Monadicity and Reconstructing Categories	23
10.5	Classical Examples: List, Probability and Algebraic Monads	23

10.6 Exercises and Notes	23
11 Kan Extensions and “All Concepts are Kan Extensions”	25
11.1 Left and Right Kan Extensions: Definitions	25
11.2 Kan Extensions as (Co)Limits	25
11.3 Adjunctions and Kan Extensions	25
11.4 Derived Functors as Kan Extensions	25
11.5 Conceptual Examples of Kan Extensions	25
11.6 Exercises and Notes	25
12 Categorical Perspectives on Classical Theorems	27
12.1 Revisiting Classical Constructions via Yoneda	27
12.2 Rewriting Theorems Using (Co)Limits and Adjoints	27
12.3 From Concrete Categories Back to Abstract Categories	27
12.4 Towards Homotopy and Higher Categories	27
12.5 Exercises and Notes	27
III Bridges to Homotopy Theory: Enriched, 2- and Model Categories	29
13 Enriched Category Theory	31
13.1 Definition of Enriched Categories	31
13.2 Examples: Ab -, Vect - and Ban -Enriched Categories	31
13.3 Enriched Functors and Natural Transformations	31
13.4 Enriched (Co)Limits and Adjunctions	31
13.5 Enriched Structures in Analysis and Geometry	31
13.6 Exercises and Notes	31
14 2-Categories and Pseudonatural Transformations	33
14.1 2-Categories: Objects, 1-Morphisms, 2-Morphisms	33
14.2 Strict vs Weak 2-Categories	33
14.3 2-Functors, 2-Natural Transformations and Modifications	33
14.4 Cat as a 2-Category Revisited	33
14.5 From 2-Categories to Bicategories: Intuition	33
14.6 Exercises and Notes	33
15 Abstract Homotopy Theory and Model Categories	35
15.1 Homotopy Categories and Weak Equivalences	35
15.2 Model Categories: Fibrations, Cofibrations, Weak Equivalences	35
15.3 Localization and the Homotopy Category $Ho(\mathcal{C})$	35
15.4 Standard Examples: Topological Spaces, Chain Complexes, etc.	35
15.5 Model Categories and ∞ -Categories: A Bridge	35
15.6 Exercises and Notes	35

IV	Basic ∞-Category Theory	37
16	∞-Cosmoi and the Homotopy 2-Category	39
16.1	The Idea of an ∞ -Cosmos	39
16.2	Construction of the Homotopy 2-Category	39
16.3	Equivalences and Weak Equivalences	39
16.4	Examples: Quasicategories, Complete Segal Spaces (Overview)	39
16.5	Exercises and Notes	39
17	Adjunctions and (Co)Limits in ∞-Categories I	41
17.1	Mapping Spaces and Hom-Objects in ∞ -Categories	41
17.2	Definition of ∞ -Adjunctions	41
17.3	∞ -Limits and ∞ -Colimits	41
17.4	Comparison with 1-Categorical Notions	41
17.5	Exercises and Notes	41
18	Comma ∞-Categories and Fibrations	43
18.1	Comma ∞ -Categories	43
18.2	(Co)Limits in Comma ∞ -Categories	43
18.3	(Co)Cartesian Fibrations in the ∞ -Setting	43
18.4	Grothendieck Construction in the ∞ -World	43
18.5	Exercises and Notes	43
19	Adjunctions and (Co)Limits in ∞-Categories II	45
19.1	Adjunctions via Comma Objects	45
19.2	Kan Extensions in ∞ -Categories	45
19.3	Homotopy (Co)Limits and ∞ -(Co)Limits	45
19.4	Completeness and Cocompleteness in ∞ -Categories	45
19.5	Exercises and Notes	45
20	Fibrations and the ∞-Yoneda Lemma	47
20.1	Cartesian and Cocartesian Fibrations	47
20.2	Fibrations and Varying Coefficients	47
20.3	∞ -Yoneda Lemma and the ∞ -Yoneda Embedding	47
20.4	Representability and Homotopy Representability	47
20.5	Exercises and Notes	47
V	Calculus of Modules in ∞-Category Theory	49
21	∞-Modules and Double Fibrations	51
21.1	Modules in the ∞ -Categorical Setting	51
21.2	Double Fibrations and Their Structure	51
21.3	Morphisms of Modules and Composition	51
21.4	Exercises and Notes	51

22 Kan Extensions via Module Calculus	53
22.1 Describing Kan Extensions Using Modules	53
22.2 Beck–Chevalley Conditions in the ∞ -Context	53
22.3 Change-of-Base and Variable Coefficients	53
22.4 Exercises and Notes	53
23 Adjunctions, (Co)Limits and the Calculus of Modules	55
23.1 Adjunctions via Modules	55
23.2 (Co)Limits Expressed in Module Language	55
23.3 Applications to Geometry and Homotopy Theory (Sketches)	55
23.4 Exercises and Notes	55
VI Model Independence and Further Structures	57
24 Model-Independent ∞-Category Theory	59
24.1 Models of ∞ -Categories	59
24.2 Quillen Equivalences Between Models	59
24.3 Model-Independent Constructions	59
24.4 Exercises and Notes	59
25 ∞-Categories and Interfaces to Geometry and Analysis	61
25.1 ∞ -Topoi and Higher Geometry (Overview)	61
25.2 Banach-World and Analytic Shadows in ∞ -Categories	61
25.3 Interfaces to Homotopy Theory, QFT and Neuroscience Modelling	61
25.4 Open Problems and Research Directions	61
25.5 Exercises and Notes	61
Notation and Conventions	63
Catalogue of Common Categories and Constructions	65
Selected Solutions and Additional Comments	67
References and Further Reading	69

Preface

Part I

Preliminaries: Logic, Algebra and Topology

Chapter 1

Logic and Set Theory Foundations

1.1 Propositional and First-Order Logic

1.2 Sets, Relations and Equivalence Relations

1.3 Maps, Injectivity, Surjectivity and Bijections

1.4 Orders, Partial Orders and Lattices

1.5 Choice Axiom, Zorn's Lemma and Maximal Elements

1.6 Exercises and Notes

Chapter 2

Basic Algebraic Structures

2.1 Groups and Group Homomorphisms

2.2 Rings, Ideals and Ring Homomorphisms

2.3 Vector Spaces and Linear Maps

2.4 Modules and Free Objects

2.5 Abelian Groups and \mathbb{Z} -Modules

2.6 Exercises and Notes

Chapter 3

Point-Set Topology and Metric Spaces

- 3.1 Metric Spaces, Open and Closed Sets
- 3.2 Topological Spaces: Axiomatic Definition
- 3.3 Continuous Maps and Homeomorphisms
- 3.4 Product Topology and Subspace Topology
- 3.5 Compactness and Connectedness
- 3.6 Relation Between Metric and Topological Spaces
- 3.7 Exercises and Notes

Chapter 4

Linear Analysis and Functional Examples

4.1 Normed and Banach Spaces

4.2 Bounded Linear Operators and Operator Norm

4.3 Inner Product and Hilbert Spaces

4.4 Function Spaces: $C(X)$, L^p , ℓ^p

4.5 Structured Maps: Self-Adjoint, Compact Operators, etc.

4.6 Exercises and Notes

Part II

Basic Category Theory

Chapter 5

Categories and Basic Examples

5.1 Definition of a Category

5.2 Standard Examples: Set, Top, Grp, Ring, Vect, Ban

5.3 Preorders, Posets and Small Categories

5.4 Opposite Categories and Duality Philosophy

5.5 Subcategories and Full Subcategories

5.6 Exercises and Notes

Chapter 6

Functors and Natural Transformations

- 6.1 Covariant and Contravariant Functors
- 6.2 Composition of Functors and Identity Functor
- 6.3 Natural Transformations and Naturality Squares
- 6.4 Natural Isomorphisms and Equivalences of Categories
- 6.5 Categories as a 2-Category: First Look
- 6.6 Exercises and Notes

Chapter 7

Representable Functors and Yoneda's Lemma

7.1 Hom-Functors and Representable Functors

7.2 Yoneda Lemma and Its Proof

7.3 The Yoneda Embedding

7.4 Characterizing Structures via Representability

7.5 Examples in Set, Top, Ab, Ban

7.6 Exercises and Notes

Chapter 8

Limits, Colimits and Universal Properties

- 8.1 Diagrams, Cones and Cocones
- 8.2 Terminal Objects, Products, Pullbacks, Equalizers
- 8.3 Initial Objects, Coproducts, Pushouts, Coequalizers
- 8.4 Complete and Cocomplete Categories
- 8.5 Category of Elements and Pointwise Descriptions
- 8.6 Computing Limits and Colimits in Concrete Categories
- 8.7 Exercises and Notes

Chapter 9

Adjoint Functors

- 9.1 Definition of Adjunction: Unit and Counit
- 9.2 Hom-Set Characterization of Adjunctions
- 9.3 Classical Examples: Free/Forgetful, Tensor/Hom, etc.
- 9.4 Adjoints and (Co)Limits
- 9.5 Reflective Subcategories and Constructions via Adjoints
- 9.6 Exercises and Notes

Chapter 10

Monads and Their Algebras

- 10.1 Monads as Monoid Objects in Endofunctors
- 10.2 Monads Arising from Adjunctions
- 10.3 Algebras for a Monad and Eilenberg–Moore Categories
- 10.4 Monadicity and Reconstructing Categories
- 10.5 Classical Examples: List, Probability and Algebraic Monads
- 10.6 Exercises and Notes

Chapter 11

Kan Extensions and “All Concepts are Kan Extensions”

11.1 Left and Right Kan Extensions: Definitions

11.2 Kan Extensions as (Co)Limits

11.3 Adjunctions and Kan Extensions

11.4 Derived Functors as Kan Extensions

11.5 Conceptual Examples of Kan Extensions

11.6 Exercises and Notes

Chapter 12

Categorical Perspectives on Classical Theorems

- 12.1 Revisiting Classical Constructions via Yoneda
- 12.2 Rewriting Theorems Using (Co)Limits and Adjoints
- 12.3 From Concrete Categories Back to Abstract Categories
- 12.4 Towards Homotopy and Higher Categories
- 12.5 Exercises and Notes

Part III

Bridges to Homotopy Theory: Enriched, 2- and Model Categories

Chapter 13

Enriched Category Theory

13.1 Definition of Enriched Categories

13.2 Examples: Ab-, Vect- and Ban-Enriched Categories

13.3 Enriched Functors and Natural Transformations

13.4 Enriched (Co)Limits and Adjunctions

13.5 Enriched Structures in Analysis and Geometry

13.6 Exercises and Notes

Chapter 14

2-Categories and Pseudonatural Transformations

- 14.1 2-Categories: Objects, 1-Morphisms, 2-Morphisms
- 14.2 Strict vs Weak 2-Categories
- 14.3 2-Functors, 2-Natural Transformations and Modifications
- 14.4 Catas a 2-Category Revisited
- 14.5 From 2-Categories to Bicategories: Intuition
- 14.6 Exercises and Notes

Chapter 15

Abstract Homotopy Theory and Model Categories

15.1 Homotopy Categories and Weak Equivalences

15.2 Model Categories: Fibrations, Cofibrations, Weak Equivalences

15.3 Localization and the Homotopy Category $Ho(\mathcal{C})$

15.4 Standard Examples: Topological Spaces, Chain Complexes, etc.

15.5 Model Categories and ∞ -Categories: A Bridge

15.6 Exercises and Notes

Part IV

Basic ∞ -Category Theory

Chapter 16

∞ -Cosmoi and the Homotopy 2-Category

16.1 The Idea of an ∞ -Cosmos

16.2 Construction of the Homotopy 2-Category

16.3 Equivalences and Weak Equivalences

16.4 Examples: Quasicategories, Complete Segal Spaces (Overview)

16.5 Exercises and Notes

Chapter 17

Adjunctions and (Co)Limits in ∞ -Categories I

- 17.1 Mapping Spaces and Hom-Objects in ∞ -Categories
- 17.2 Definition of ∞ -Adjunctions
- 17.3 ∞ -Limits and ∞ -Colimits
- 17.4 Comparison with 1-Categorical Notions
- 17.5 Exercises and Notes

Chapter 18

Comma ∞ -Categories and Fibrations

18.1 Comma ∞ -Categories

18.2 (Co)Limits in Comma ∞ -Categories

18.3 (Co)Cartesian Fibrations in the ∞ -Setting

18.4 Grothendieck Construction in the ∞ -World

18.5 Exercises and Notes

Chapter 19

Adjunctions and (Co)Limits in ∞ -Categories II

19.1 Adjunctions via Comma Objects

19.2 Kan Extensions in ∞ -Categories

19.3 Homotopy (Co)Limits and ∞ -(Co)Limits

19.4 Completeness and Cocompleteness in ∞ -Categories

19.5 Exercises and Notes

Chapter 20

Fibrations and the ∞ -Yoneda Lemma

20.1 Cartesian and Cocartesian Fibrations

20.2 Fibrations and Varying Coefficients

20.3 ∞ -Yoneda Lemma and the ∞ -Yoneda Embedding

20.4 Representability and Homotopy Representability

20.5 Exercises and Notes

Part V

Calculus of Modules in ∞ -Category Theory

Chapter 21

∞ -Modules and Double Fibrations

21.1 Modules in the ∞ -Categorical Setting

21.2 Double Fibrations and Their Structure

21.3 Morphisms of Modules and Composition

21.4 Exercises and Notes

Chapter 22

Kan Extensions via Module Calculus

22.1 Describing Kan Extensions Using Modules

22.2 Beck–Chevalley Conditions in the ∞ -Context

22.3 Change-of-Base and Variable Coefficients

22.4 Exercises and Notes

Chapter 23

Adjunctions, (Co)Limits and the Calculus of Modules

23.1 Adjunctions via Modules

23.2 (Co)Limits Expressed in Module Language

23.3 Applications to Geometry and Homotopy Theory (Sketches)

23.4 Exercises and Notes

Part VI

Model Independence and Further Structures

Chapter 24

Model-Independent ∞ -Category Theory

24.1 Models of ∞ -Categories

24.2 Quillen Equivalences Between Models

24.3 Model-Independent Constructions

24.4 Exercises and Notes

Chapter 25

∞ -Categories and Interfaces to Geometry and Analysis

- 25.1 ∞ -Topoi and Higher Geometry (Overview)
- 25.2 Banach-World and Analytic Shadows in ∞ -Categories
- 25.3 Interfaces to Homotopy Theory, QFT and Neuroscience Modelling
- 25.4 Open Problems and Research Directions
- 25.5 Exercises and Notes

Notation and Conventions

Catalogue of Common Categories and Constructions

Selected Solutions and Additional Comments

References and Further Reading