Category and ∞-Category Theory in Context

From Analysis to Higher Structures

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Chapter 5

Categories and Basic Examples

5.1 Definition of a Category

In this section we isolate the abstract pattern shared by many familiar mathematical worlds: sets and functions, groups and homomorphisms, topological spaces and continuous maps, and so on. A *category* is a structure whose basic data consist of *objects* and *morphisms* (or *arrows*) between them, together with a way to compose morphisms that behaves like the composition of functions.

Definition 5.1 (Category)**.** A *category* $\mathcal{C}$ consists of the following data:

1. A collection (often a class) of *objects*. We write $\mathrm{Ob}(\mathcal{C})$ for the collection of all objects of $\mathcal{C}$.

2. For each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set

$$\mathrm{Hom}_{\mathcal{C}}(A, B)$$

   called the *hom-set* from A to B. Its elements are called *morphisms* (or *arrows*) from A to B. If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ we write $f : A \to B$.

3. For each triple of objects A, B, C, a specified *composition map*

$$\circ \ : \ \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C), \qquad (g, f) \longmapsto g \circ f,$$

   which assigns to any composable pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ a composite arrow $A \xrightarrow{g \circ f} C$.

4. For each object A an arrow

$$\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$$

   called the *identity morphism* on A.

These data are required to satisfy the following axioms.
(Associativity). Whenever we have arrows

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D,$$

the two possible composites agree:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

as arrows $A \to D$.

(Identity laws). For any arrow $f : A \to B$ we have

$$f \circ \mathrm{id}_A = f \qquad \text{and} \qquad \mathrm{id}_B \circ f = f.$$

Remark 5.2 (Small and locally small categories). In practice the collection of objects of a category is often too large to form a set; in this case we say that $\mathcal{C}$ is a *large* category. If $\mathrm{Ob}(\mathcal{C})$ is a set, we say that $\mathcal{C}$ is *small*. A category is called *locally small* if for every pair of objects A, B the hom-class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set. Most categories we meet in ordinary mathematics are at least locally small.

Remark 5.3 (Notation for hom-sets). It is common to abbreviate $\mathrm{Hom}_{\mathcal{C}}(A, B)$ simply as $\mathcal{C}(A, B)$, or even $\mathrm{Hom}(A, B)$ when the ambient category is clear from context.

Definition 5.4 (Isomorphism). Let $\mathcal{C}$ be a category and let A, B be objects of $\mathcal{C}$. A morphism $f : A \to B$ is called an *isomorphism* if there exists a morphism $g : B \to A$ such that

$$g \circ f = \mathrm{id}_A \qquad \text{and} \qquad f \circ g = \mathrm{id}_B.$$

In this case g is called an *inverse* of f, and we say that A and B are *isomorphic*, writing $A \cong B$.

Lemma 5.5 (Uniqueness of identity). *For each object A in a category $\mathcal{C}$, the identity morphism id_A is unique.*

Lemma 5.6 (Uniqueness of inverses). *If a morphism $f : A \to B$ in a category $\mathcal{C}$ has an inverse, then this inverse is unique.*

Example 5.7 (A discrete category). Let S be any set. We can form a category $\mathcal{D}_S$, called the *discrete category on S*, whose objects are the elements of S, and whose only morphisms are identity morphisms:

$$\mathrm{Hom}_{\mathcal{D}_S}(x, y) = \begin{cases} \{\mathrm{id}_x\}, & \text{if } x = y, \\ \varnothing, & \text{if } x \neq y. \end{cases}$$

The axioms are trivially satisfied, so $\mathcal{D}_S$ is a category.

Example 5.8 (An indiscrete category). Let S be any set. We can also form a category $\mathcal{I}_S$, called the *indiscrete category on S*, whose objects are the elements of S, and for every pair $x, y \in S$ there is exactly one morphism $x \to y$. Composition is forced to be the unique possible choice, and the identity morphisms are the unique endomorphisms $x \to x$.

Exercises for Section 5.1

Exercise 5.9 (Prove Lemma 5.5). Prove Lemma 5.5: for each object A in a category $\mathcal{C}$, the identity morphism id_A is unique.

Exercise 5.10 (Prove Lemma 5.6). Prove Lemma 5.6: if a morphism $f : A \to B$ in a category $\mathcal{C}$ has an inverse, then this inverse is unique.

Exercise 5.11 (Set as a category). Let **Set** denote the category whose objects are all sets and whose morphisms are functions between sets. Show that the data of **Set** satisfy the axioms of Definition 5.1. In particular, check explicitly:

1. the associativity of composition of functions;
2. the identity laws for the identity function on each set.

Exercise 5.12 (A “not quite” category)**.** Suppose we keep the same objects and morphisms as in the category **Set**, but we try to define the composition of $g : B \to C$ and $f : A \to B$ by the pointwise formula

$$(g \star f)(a) = g(a) + f(a)$$

whenever this makes sense. Explain in detail why this new operation $\star$ does *not* make **Set** into a category. Which of the axioms of Definition 5.1 fails?

Exercise 5.13 (Isomorphisms in **Set**)**.** Show that a morphism $f : A \to B$ in the category **Set** is an isomorphism if and only if it is a bijective function. (You may use standard results from set theory about left and right inverses.)

Exercise 5.14 (Isomorphism as an equivalence relation)**.** Let $\mathcal{C}$ be any category. Show that “A is isomorphic to B” defines an equivalence relation on the class of objects of $\mathcal{C}$: it is reflexive, symmetric, and transitive.

5.2 Standard Examples: Set, Top, Grp, Ring, Vect, Ban

In this section we collect a first family of concrete categories. Each of them consists of some familiar mathematical structures as objects and the usual “structure–preserving maps” as morphisms. In all cases, composition of morphisms is just the usual composition of functions, and identities are the usual identity maps on each object. The fact that these data satisfy the axioms from Section 5.1 will be left to the exercises.

Example 5.15 (The category **Set**)**.** The category **Set** has as objects all (small) sets and as morphisms all functions between sets. For two sets X and Y we write

$$\mathbf{Set}(X, Y)$$

for the set of all functions $X \to Y$. Composition and identities are exactly the ones from elementary set theory.

Intuitively, **Set** is the “background category” in which most of classical mathematics takes place: many other categories will be obtained by taking sets equipped with extra structure and restricting to maps that respect this structure.

Example 5.16 (The category **Top**)**.** The category **Top** has as objects all topological spaces and as morphisms all continuous maps between them. For topological spaces X and Y we denote by $\mathbf{Top}(X, Y)$ the set of continuous maps $X \to Y$.

Intuitively, **Top** remembers not only the underlying sets of points but also how these points cluster and vary continuously. Morphisms in **Top** are the maps that do not tear or glue points in a discontinuous way.

Example 5.17 (The category **Grp**)**.** The category **Grp** has as objects all (not necessarily commutative) groups and as morphisms all group homomorphisms. Thus a morphism $f : G \to H$ in **Grp** is a function of the underlying sets that satisfies

$$f(gh) = f(g)f(h) \quad \text{and} \quad f(e_G) = e_H$$

for all $g, h \in G$, where e_G and e_H denote the identity elements.

From the categorical point of view, *groups* are the algebraic structures and *homomorphisms* are exactly the maps that preserve all the operations and equations that define those structures.

Example 5.18 (The category **Ring**)**.** The category **Ring** has as objects all rings and as morphisms all ring homomorphisms. For definiteness we adopt the following convention: objects are rings with a multiplicative unit 1, not necessarily commutative, and a ring homomorphism $f : R \to S$ is required to preserve 0, 1, addition and multiplication:

$$f(0_R) = 0_S, \quad f(1_R) = 1_S, \quad f(a+b) = f(a) + f(b), \quad f(ab) = f(a)f(b).$$

In this way **Ring** is another example of a category of "sets with algebraic structure" and structure–preserving maps. Forgetting the ring structure gives a natural functor **Ring** $\to$ **Set** sending each ring to its underlying set.

Example 5.19 (The category $\mathbf{Vect}_k$)**.** Fix a field k. The category $\mathbf{Vect}_k$ has as objects all vector spaces over k and as morphisms all linear maps between them. A morphism $T : V \to W$ in $\mathbf{Vect}_k$ is a function satisfying

$$T(v + v') = T(v) + T(v'), \qquad T(\lambda v) = \lambda T(v)$$

for all $v, v' \in V$ and all scalars $\lambda \in k$.

Conceptually, $\mathbf{Vect}_k$ is the basic habitat of linear algebra. Many constructions in analysis and geometry can be seen as living inside this category or inside suitable subcategories, such as the category of finite-dimensional vector spaces.

Example 5.20 (The category **Ban**)**.** The category **Ban** has as objects all Banach spaces (complete normed vector spaces) over $\mathbb{R}$ or $\mathbb{C}$, and as morphisms all bounded linear operators between them. Thus a morphism $T : X \to Y$ in **Ban** is a linear map that is continuous with respect to the norms; equivalently, there exists $C \geq 0$ such that

$$\|T(x)\|_Y \leq C\|x\|_X \quad \text{for all } x \in X.$$

From the categorical viewpoint, **Ban** refines $\mathbf{Vect}_k$ by adding analytic information: not only do we keep the linear structure, but we also keep the norm and the completeness, and we restrict to morphisms that respect this analytic structure. Many constructions in functional analysis can be phrased as categorical statements inside **Ban**.

These six categories share a common pattern: their objects are sets equipped with some kind of additional structure (topology, group operation, ring operations, linear structure, norm), and their morphisms are functions that preserve this structure in the obvious sense. Verifying that the identity maps and compositions of such structure–preserving maps are again structure–preserving is straightforward and illustrates how the abstract definition of a category captures "mathematics done with functions" in a unified language.

5.3 Preorders, posets, and small categories

In this section we explain how familiar ordered structures can be seen as special kinds of small categories. From this point of view a preorder is a "thin" category: between any two objects there is at most one morphism. Posets appear when we additionally identify objects that are mutually comparable in both directions.

5.3.1 Preorders and partial orders

Definition 5.21 (Preorder)**.** A *preorder* is a pair $(P, \leq)$ consisting of a set P and a binary relation $\leq$ on P such that

1. **reflexivity:** for every $x \in P$ we have $x \leq x$;
2. **transitivity:** for all $x, y, z \in P$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 5.22 (Partial order and poset)**.** A *partial order* is a preorder $(P, \leq)$ that also satisfies

(3) **antisymmetry:** for all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.

A *partially ordered set*, or *poset* for short, is a set equipped with a partial order. We often write simply P for a poset $(P, \leq)$ when the relation is understood.

Example 5.23 (Basic examples of preorders and posets)**.**

1. The usual order $(\mathbb{N}, \leq)$ or $(\mathbb{R}, \leq)$ is a poset.
2. Given any set S, its power set $\mathcal{P}(S)$ becomes a poset when equipped with the inclusion relation $\subseteq$.
3. On the set $\mathbb{N}$ of natural numbers, the divisibility relation $m \mid n$ defines a partial order: we declare $m \leq n$ if m divides n.
4. If P is any set, the relation $x \leq y$ defined by $x = y$ for all $x, y \in P$ is a partial order called the *discrete* order.
5. More generally, given an equivalence relation $\sim$ on a set P, we obtain a preorder by setting $x \leq y$ whenever $x \sim y$. This preorder fails to be antisymmetric unless the equivalence classes are singletons.

Remark 5.24 (Intuition)*.* A preorder encodes a notion of "being at most as large as" or "being at most as informative as" without forcing distinct elements that are mutually related to coincide. Passing from a preorder to a poset amounts to collapsing elements that are indistinguishable from the point of view of the relation.

5.3.2 Preorders as thin categories

Definition 5.25 (Thin category)**.** A category $\mathcal{C}$ is called *thin* if for every pair of objects $X, Y \in \mathcal{C}$ the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has at most one element.

Example 5.26 (A preorder as a thin category)**.** Let $(P, \leq)$ be a preorder. We may regard P as a category, denoted by the same symbol, by declaring:

- the objects of P are the elements of the set P;
- for $x, y \in P$ there is a unique morphism $x \to y$ if $x \leq y$, and no morphism otherwise;
- identities and composition are forced by the reflexivity and transitivity of $\leq$.

This construction always produces a small thin category.

Remark 5.27 (Categorical reading of the order)*.* In the category associated to a preorder $(P, \leq)$, the statement $x \leq y$ is literally equivalent to the existence of a morphism $x \to y$. Thus the entire preorder structure is encoded in the pattern of morphisms between objects.

5.3.3 Thin categories as preorders

Conversely, every small thin category determines a preorder on its set of objects by declaring that $X \leq Y$ whenever there exists a morphism $X \to Y$.

Proposition 5.28 (Preorders and thin categories)**.** *Giving a preorder* $(P, \leq)$ *is equivalent to giving a small thin category* $\mathcal{C}$*. More precisely:*

1. *From any preorder* $(P, \leq)$ *we obtain a small thin category as in Example 5.26.*
2. *From any small thin category* $\mathcal{C}$ *we obtain a preorder on* $\mathrm{Ob}(\mathcal{C})$ *by declaring* $X \leq Y$ *if and only if* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *is nonempty.*

These two constructions are inverse to one another.

Remark 5.29 (Intuition: "categorifying" a preorder). A thin category can be thought of as a "categorified" preorder: the yes/no information "$x \leq y$ or not" has been upgraded to the existence of a morphism $x \to y$, but there is no additional choice of which morphism because each hom-set contains at most one element. No higher-dimensional structure is present, so the category still behaves like a one-dimensional order.

5.3.4 Posets as thin skeletal categories

To single out posets among preorders, we must identify those thin categories in which objects that are "equivalent in both directions" are literally equal.

Definition 5.30 (Skeletal category)**.** A category $\mathcal{C}$ is called *skeletal* if whenever two objects $X, Y \in \mathcal{C}$ are isomorphic, then $X = Y$.

Proposition 5.31 (Posets as thin skeletal categories)**.** *A poset is the same thing as a small category that is both thin and skeletal. More explicitly:*

1. *If* $(P, \leq)$ *is a poset, then the associated category described in Example 5.26 is thin and skeletal.*
2. *If* $\mathcal{C}$ *is a small thin skeletal category, then the preorder on* $\mathrm{Ob}(\mathcal{C})$ *defined by* $X \leq Y$ *if and only if* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *is nonempty is in fact a partial order.*

Remark 5.32 (Antisymmetry as "no non-trivial isomorphisms"). In a thin category, the existence of morphisms $X \to Y$ and $Y \to X$ forces them to be inverse to each other, so X and Y are isomorphic. Requiring the category to be skeletal then identifies such objects, which exactly matches the antisymmetry condition in the definition of a poset.

5.3.5 Monotone maps as functors

The equivalence between preorders and thin categories extends from objects to morphisms if we interpret functions that preserve order as functors.

Definition 5.33 (Monotone map)**.** Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be preorders. A function $f : P \to Q$ is *monotone*, or *order-preserving*, if for all $x, y \in P$,

$$x \leq_P y \quad \Rightarrow \quad f(x) \leq_Q f(y).$$

Proposition 5.34 (Monotone maps as functors)**.** *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be preorders, regarded as thin categories. Then:*

1. *Any monotone map $f : P \to Q$ induces a functor of categories $F : P \to Q$ that is the identity on objects and sends the unique morphism $x \to y$ (when $x \leq_P y$) to the unique morphism $f(x) \to f(y)$ (when $f(x) \leq_Q f(y)$).*
2. *Conversely, any functor $F : P \to Q$ between the associated thin categories is determined by its action on objects and is automatically monotone with respect to the induced preorders.*

In particular, the category whose objects are preorders and whose morphisms are monotone maps is equivalent to the full subcategory of small categories whose objects are thin categories.

Remark 5.35 (Order-theoretic phenomena as categorical phenomena)*.* Once preorders and posets are viewed as thin (skeletal) categories, many familiar notions from order theory—such as minima, maxima, infima, suprema, and Galois connections—can be recognized as special cases of limits, colimits, and adjunctions in category theory. This perspective allows general categorical theorems to be applied directly to ordered structures.

5.4 Opposite Categories and the Duality Philosophy

5.4.1 Main Text: Definitions, Examples, Intuition (no proofs)

Definition 5.36 (Opposite category)**.** Given a category C, its *opposite category* C^{op} is defined by:

- $\mathrm{Ob}(C^{op}) = \mathrm{Ob}(C)$;
- for $X, Y \in C$, $\mathrm{Hom}_{C^{op}}(X, Y) := \mathrm{Hom}_C(Y, X)$;
- identities are unchanged: $(\mathrm{id}_X)^{op} = \mathrm{id}_X$;
- composition is reversed: if $f : X \to Y$ and $g : Y \to Z$ in C, then in C^{op}

$$f^{op} : Y \to X, \quad g^{op} : Z \to Y, \quad \text{and} \quad (g \circ f)^{op} = f^{op} \circ g^{op}.$$

Remark 5.37 (Duality as "reverse all arrows")*.* Any definition, construction, or statement that can be expressed purely in terms of objects, morphisms, identities, and composition has a *formal dual*: replace C by C^{op} and reverse every arrow. If a statement is true in C, its dual statement is true in C^{op}.

Example 5.38 (Terminal vs. initial objects)**.** An object 1 is *terminal* in C if for every X there exists a unique morphism $X \to 1$. Dually, an object 0 is *initial* if for every X there exists a unique morphism $0 \to X$. Thus, terminal objects in C become initial objects in C^{op} (and vice versa).

Definition 5.39 (Binary product)**.** Given objects $A, B \in C$, a *product* of A and B consists of an object $A \times B$ equipped with *projections*

$$\pi_A : A \times B \to A, \qquad \pi_B : A \times B \to B,$$

such that for every object X and every pair of morphisms $f : X \to A$, $g : X \to B$, there exists a unique morphism $\langle f, g\rangle : X \to A \times B$ satisfying

$$\pi_A \circ \langle f, g\rangle = f, \qquad \pi_B \circ \langle f, g\rangle = g.$$

Definition 5.40 (Binary coproduct)**.** Given objects $A, B \in C$, a *coproduct* of A and B consists of an object $A \amalg B$ equipped with *injections*

$$\iota_A : A \to A \amalg B, \qquad \iota_B : B \to A \amalg B,$$

such that for every object X and every pair of morphisms $f : A \to X$, $g : B \to X$, there exists a unique morphism $[f, g] : A \amalg B \to X$ satisfying

$$[f, g] \circ \iota_A = f, \qquad [f, g] \circ \iota_B = g.$$

Example 5.41 (Set: product and disjoint union)**.** In **Set**, the product is the cartesian product $A \times B$ with projections. The coproduct is the *disjoint union* $A \amalg B$, which can be realized as

$$A \amalg B \;\cong\; (A \times \{0\}) \cup (B \times \{1\}),$$

so elements carry a tag recording whether they came from A or from B.

Remark 5.42 (Products/coproducts as terminal/initial objects in cone categories)*.* The slogan "a product is a terminal object" is literally true, but *not* in C itself. It is true in a *cone category* whose objects are cones into (A, B). Dually, coproducts are initial objects in a *cocone category* whose objects are cocones out of (A, B). This explains why products come with maps to A and B (they are part of the object-data in that cone category).

Definition 5.43 (Cone and cocone categories for a pair (A, B))**.** Fix $A, B \in C$.

- The *cone category* $\mathrm{Cone}(A, B)$ has objects $(X, f : X \to A, g : X \to B)$. A morphism $(X, f, g) \to (Y, f', g')$ is a morphism $m : X \to Y$ in C such that

$$f = f' \circ m, \qquad g = g' \circ m.$$

- The *cocone category* $\mathrm{Cocone}(A, B)$ has objects $(X, f : A \to X, g : B \to X)$. A morphism $(X, f, g) \to (Y, f', g')$ is a morphism $m : X \to Y$ in C such that

$$f' = m \circ f, \qquad g' = m \circ g.$$

Then $A \times B$ is a terminal object in $\mathrm{Cone}(A, B)$ (if it exists), and $A \amalg B$ is an initial object in $\mathrm{Cocone}(A, B)$ (if it exists).

Remark 5.44 (Higher-arity products and coproducts)*.* For objects $A_1, \ldots, A_n$, an n-ary product $\prod_{i=1}^{n} A_i$ (resp. coproduct $\coprod_{i=1}^{n} A_i$) is defined by the same universal property with n projections (resp. n injections).

Remark 5.45 (Universal properties as "equation systems")*.* A universal property can be viewed as a family of commutative-diagram equations parameterized by *all test objects X*, with a unique solution morphism for each test datum. This "for all X" naturality is the beginning of the Yoneda-style philosophy: objects are determined by their behavior under all possible probes X.

Remark 5.46 (Existence is not automatic). Universal properties guarantee uniqueness *if* something exists, but do not guarantee existence. Some categories lack certain products/coproducts (e.g. products may fail in a category where morphisms must be injective).

Remark 5.47 (Transporting universal properties along functors). A functor $F : C \to D$ need not preserve products or terminal objects. If F *does* preserve products, then $F(A \times B) \cong F(A) \times F(B)$ in D. More generally, right adjoints preserve limits (including products), while left adjoints preserve colimits. If structure is lost under F, one may either re-take the universal construction in D, or enlarge/completion-build a category where the desired universal objects exist.

5.4.2 Exercises (treat universal statements as problems)

Exercise 5.48 (Opposite reverses composition)**.** Given composable $X \xrightarrow{f} Y \xrightarrow{g} Z$ in C, prove $(g \circ f)^{op} = f^{op} \circ g^{op}$.

Exercise 5.49 (Terminal $\leftrightarrow$ initial)**.** Prove: 1 is terminal in C iff it is initial in C^{op}.

Exercise 5.50 (Products as terminal cones)**.** Define $\mathrm{Cone}(A, B)$ precisely and prove: $(A \times B, \pi_A, \pi_B)$ is terminal in $\mathrm{Cone}(A, B)$ iff $A \times B$ satisfies the usual product universal property.

Exercise 5.51 (Coproducts as initial cocones)**.** Define $\mathrm{Cocone}(A, B)$ precisely and prove: $(A \amalg B, \iota_A, \iota_B)$ is initial in $\mathrm{Cocone}(A, B)$ iff $A \amalg B$ satisfies the usual coproduct universal property.

Exercise 5.52 (n-ary generalization)**.** Write the universal property of a ternary product $A \times B \times C$ and dualize it to obtain the ternary coproduct property.

Exercise 5.53 (Uniqueness up to unique isomorphism)**.** Assume P and Q are both products of A and B. Construct canonical morphisms $u : P \to Q$ and $v : Q \to P$ and prove u, v are inverse isomorphisms.

Exercise 5.54 (Representing property for products)**.** Show that a product $A \times B$ is equivalently characterized by a natural isomorphism

$$\mathrm{Hom}(X, A \times B) \cong \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

natural in X.

Exercise 5.55 (Non-existence example)**.** Consider the category **FinInj** of finite sets and injective maps. Explain why the product of A and B typically fails to exist when $|B| > 1$.

Exercise 5.56 (Preservation under functors)**.** Let $F : C \to D$ be a right adjoint. Prove that if $A \times B$ exists in C, then $F(A \times B)$ is (canonically) a product of $F(A)$ and $F(B)$ in D.

5.4.3 Solutions and Author's Notes (standard proofs + learning log)

Solution to Exercise 5.4.1. In C^{op}, $f^{op} : Y \to X$ and $g^{op} : Z \to Y$ are composable, hence $f^{op} \circ g^{op} : Z \to X$. By definition of composition in C^{op}, this composite corresponds to $(g \circ f)^{op}$.

Solution to Exercise 5.4.2. 1 terminal in C means: for all X, $\exists!\ X \to 1$ in C. But $\mathrm{Hom}_{C^{op}}(1, X) = \mathrm{Hom}_C(X, 1)$, so this is equivalent to: for all X, $\exists!\ 1 \to X$ in C^{op}, i.e. 1 is initial in C^{op}.

Solution to Exercise 5.4.3. Assume (P, π_A, π_B) is terminal in $\mathrm{Cone}(A, B)$. Given $f : X \to A$ and $g : X \to B$, the triple (X, f, g) is an object of $\mathrm{Cone}(A, B)$. Terminality gives a unique morphism $m : (X, f, g) \to (P, \pi_A, \pi_B)$, i.e. a unique $m : X \to P$ in C with $f = \pi_A \circ m$ and $g = \pi_B \circ m$. This is exactly the product universal property. Conversely, the product universal property implies terminality by the same unpacking.

Solution to Exercise 5.4.4. Identical, dualized: initiality in $\mathrm{Cocone}(A, B)$ is equivalent to the coproduct universal property.

Solution to Exercise 5.4.5. Ternary product: an object $A \times B \times C$ with projections π_A, π_B, π_C such that for all X and $f : X \to A$, $g : X \to B$, $h : X \to C$ there exists a unique $m : X \to A \times B \times C$ with $\pi_A \circ m = f$, $\pi_B \circ m = g$, $\pi_C \circ m = h$. Dualize by reversing arrows to obtain: coproduct $A \amalg B \amalg C$ with injections and a unique mediator $[f, g, h]$.

Solution to Exercise 5.4.6. Let P, Q be products with projections π_A^P, π_B^P and π_A^Q, π_B^Q. By the product property of Q, applied to (π_A^P, π_B^P), there is a unique $u : P \to Q$ with $\pi_A^Q \circ u = \pi_A^P$ and $\pi_B^Q \circ u = \pi_B^P$. Similarly, obtain $v : Q \to P$. To show $v \circ u = \mathrm{id}_P$, compare morphisms $v \circ u$ and id_P from P to P: both have the same composites with π_A^P and π_B^P, hence by uniqueness in the product property of P they are equal. Similarly $u \circ v = \mathrm{id}_Q$.

Solution to Exercise 5.4.7. Define the map $\Phi_X : \mathrm{Hom}(X, A \times B) \to \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$ by $\Phi_X(u) = (\pi_A \circ u, \pi_B \circ u)$. The product universal property provides the inverse map $(f, g) \mapsto \langle f, g \rangle$. Naturality follows from functoriality of precomposition.

Solution to Exercise 5.4.8. In **FinInj**, morphisms must be injective. If a product P of A and B existed, the projection $\pi_A : P \to A$ would have to be injective. But for any set-theoretic candidate behaving like $A \times B$, the projection to A collapses different B-coordinates, hence fails to be injective when $|B| > 1$. This obstructs existence of products.

Solution to Exercise 5.4.9. If F is right adjoint, it preserves limits. In particular it preserves products: given $A \times B$ with projections, the cone $(F(A \times B) \to F(A), F(B))$ is terminal among cones over $(F(A), F(B))$.

Author's Notes / Learning Log (from today's discussion)

- I first internalized C^{op} as "same objects, all arrows reversed", and noticed composition must reverse order: $(g \circ f)^{op} = f^{op} \circ g^{op}$. This felt like an automatic bookkeeping rule once the arrow directions are fixed.
- The duality philosophy clicked when I saw *terminal* $\leftrightarrow$ *initial* by literally reversing arrows in the definition.

- Confusion point: "a product is terminal, so why does it still have maps to A and B?" Resolution: it is terminal *in a different category* (the cone category) where *objects are triples* $(X, X \to A, X \to B)$. So the two projections are not extra; they are part of the object-data.

- I drew a diagram and realized I had actually built the *cocone category*: objects $(X, A \to X, B \to X)$ and morphisms $m : X \to Y$ satisfying $f' = m \circ f$, $g' = m \circ g$. In that world, the universal object is the coproduct $A \amalg B$ as an *initial* object. This helped me see that "cone vs cocone" is literally just the direction of the legs.

- I liked thinking of universal properties as "systems of equations" whose *unknown* is the mediator morphism. A subtlety: it is not one fixed equation system; it is a *family* of equation systems indexed by all test objects X (and all possible input data $f, g, \ldots$). The invariant is the *natural* way solutions are produced across all probes.

- This naturally led to the representability viewpoint: $A \times B$ represents the functor $X \mapsto \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$. I felt this as "the object is determined by all its observational behaviors", which already smells like Yoneda.

- New conceptual question: what happens under a functor $F : C \to D$? If F does not preserve limits, then the universal property may be lost ("information loss"). A clean sufficient condition is being a right adjoint (hence limit-preserving). If not preserved, one can re-take the limit in D, or enlarge the setting via a free completion that forces existence of limits.

5.5 Subcategories and Full Subcategories

Motivation and viewpoint (informal)

Rather than focusing on a particular subcategory, we often study the *space of all subcategories* of a category $\mathcal{C}$, ordered by inclusion. This "global" viewpoint packages many constructions as lattice-theoretic operations (meets/joins), and aligns with the intuition: *more requirements* $\Rightarrow$ *fewer models*, suggesting an order-reversing relationship between "constraints" and "solutions."

5.5.1 Definitions

Definition 5.57 (Subcategory)**.** A *subcategory* $\mathcal{D} \subseteq \mathcal{C}$ consists of

- a subclass of objects $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$,

- for each $A, B \in \mathrm{Ob}(\mathcal{D})$, a subset $\mathrm{Hom}_{\mathcal{D}}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$,

such that $\mathcal{D}$ is a category in its own right and the inclusion on objects and morphisms is compatible with identities and composition:

$$\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{D}}(A, A), \qquad g \circ f \in \mathrm{Hom}_{\mathcal{D}}(A, C) \text{ whenever } f \in \mathrm{Hom}_{\mathcal{D}}(A, B),\ g \in \mathrm{Hom}_{\mathcal{D}}(B, C).$$

Definition 5.58 (Wide subcategory)**.** A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is *wide* if $\mathrm{Ob}(\mathcal{D}) = \mathrm{Ob}(\mathcal{C})$.

Definition 5.59 (Full subcategory)**.** A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is *full* if for all objects $A, B \in \mathrm{Ob}(\mathcal{D})$,

$$\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B).$$

Equivalently, a full subcategory is determined uniquely by its class of objects.

Definition 5.60 (Full subcategory generated by a set of objects)**.** Given a class of objects $S \subseteq \mathrm{Ob}(\mathcal{C})$, the *full subcategory on* S, denoted $\mathcal{C}|_S$, is defined by

$$\mathrm{Ob}(\mathcal{C}|_S) = S, \qquad \mathrm{Hom}_{\mathcal{C}|_S}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B) \text{ for all } A, B \in S.$$

Definition 5.61 (Subcategory generated by a set of morphisms)**.** Let M be a collection of morphisms in $\mathcal{C}$. The *subcategory generated by* M, denoted $\langle M \rangle$, is the smallest subcategory of $\mathcal{C}$ that contains every morphism in M (and hence contains all identities on sources/targets of morphisms in M and is closed under composition).

5.5.2 Examples and intuition

Example 5.62 (Wide but not full: the core groupoid of **Set**)**.** Let $\mathcal{C} = \mathbf{Set}$. Define $\mathcal{D}$ to have all sets as objects, but only bijections as morphisms. Then $\mathcal{D}$ is wide, and it is a groupoid (all morphisms invertible), hence not full in **Set**. This is sometimes described as "groupoidifying" **Set** by discarding non-invertible maps.

Example 5.63 (Full but not wide)**.** Fix any object $A \in \mathcal{C}$. The full subcategory $\mathcal{C}|_{\{A\}}$ has one object and morphisms $\mathrm{End}_{\mathcal{C}}(A)$. Unless $\mathcal{C}$ has only one object, this is full but not wide.

Remark 5.64 ("Skinny vs. fat" and an interval of choices). Fix a set of morphisms M and let S be the set of all sources and targets of morphisms in M. Then $\langle M \rangle$ is the *skinniest* subcategory containing M (minimal morphisms forced by identities and composition), while the full subcategory $\mathcal{C}|_S$ is the *fattest* subcategory with object set S (all morphisms among S). Every subcategory $\mathcal{D}$ with $\mathrm{Ob}(\mathcal{D}) = S$ and $M \subseteq \mathrm{Mor}(\mathcal{D})$ lies in the "sandwich" interval

$$\langle M \rangle \subseteq \mathcal{D} \subseteq \mathcal{C}|_S.$$

5.5.3 Subcategories as a lattice (statements only)

Definition 5.65 (The poset of subcategories)**.** Let $\mathrm{Sub}(\mathcal{C})$ denote the collection of subcategories of $\mathcal{C}$, ordered by inclusion.

Proposition 5.66 (Meets are intersections)**.** *For subcategories $\mathcal{D}, \mathcal{E} \subseteq \mathcal{C}$, the objectwise-and-morphismwise intersection $\mathcal{D} \cap \mathcal{E}$ is a subcategory. Moreover, it is the greatest lower bound (meet) of $\mathcal{D}$ and $\mathcal{E}$ in* $\mathrm{Sub}(\mathcal{C})$*.*

Proposition 5.67 (Unions need not be subcategories)**.** *In general, the objectwise-and-morphismwise union $\mathcal{D} \cup \mathcal{E}$ of two subcategories need not be a subcategory, since it may fail to be closed under composition.*

Proposition 5.68 (Joins as intersection of all upper bounds)**.** *Given a family of subcategories $\{\mathcal{D}_i\}_{i \in I}$, define*

$$\bigvee_{i \in I} \mathcal{D}_i := \bigcap \Big\{ \mathcal{E} \subseteq \mathcal{C} \ \Big|\ \forall i \in I,\ \mathcal{D}_i \subseteq \mathcal{E} \Big\}.$$

Then $\bigvee_{i\in I}\mathcal{D}_i$ *is the least upper bound (join) of the family in* $\mathrm{Sub}(\mathcal{C})$*. Equivalently,* $\bigvee_{i\in I}\mathcal{D}_i$ *is the subcategory generated by the union of all objects and morphisms appearing in the* $\mathcal{D}_i$*.*

Remark 5.69 (A deferred analogy: sieves as "information refinement")*.* We briefly compared subcategory-closure under composition with sieve-closure under precomposition. These are different closure strengths; the sieve viewpoint will be revisited later when developing Grothendieck topologies.

5.5.4 Exercises

Exercise 5.70. Prove Proposition 5.66.

Exercise 5.71. Construct an explicit example showing Proposition 5.67. (Hint: use a category containing $A \xrightarrow{f} B \xrightarrow{g} C$ and consider subcategories supported on $\{A, B\}$ and $\{B, C\}$.)

Exercise 5.72. Show that a full subcategory is uniquely determined by its class of objects.

Exercise 5.73. In **Set**, show that the subcategory with all objects and only bijections as morphisms is wide but not full.

Exercise 5.74. Prove Proposition 5.68.

Exercise 5.75. Let M be a set of morphisms and S its set of endpoints. Verify the inclusions $\langle M\rangle \subseteq \mathcal{C}|_S$ and characterize the class of subcategories $\mathcal{D}$ satisfying $\langle M\rangle \subseteq \mathcal{D} \subseteq \mathcal{C}|_S$.

5.5.5 Solutions and Author's Notes

[Solution to Exercise 5.70] Let $\mathcal{D}, \mathcal{E} \subseteq \mathcal{C}$ be subcategories. Define $\mathcal{D} \cap \mathcal{E}$ by intersecting objects and hom-sets. Identities: if A is an object in the intersection, then id_A lies in both $\mathcal{D}$ and $\mathcal{E}$, hence in the intersection. Composition: if f and g lie in the intersection and are composable, then $g \circ f$ lies in $\mathcal{D}$ and in $\mathcal{E}$, hence in the intersection. Therefore $\mathcal{D} \cap \mathcal{E}$ is a subcategory. It is clearly a lower bound; if $\mathcal{X}$ is any other lower bound, then $\mathcal{X} \subseteq \mathcal{D}$ and $\mathcal{X} \subseteq \mathcal{E}$, so $\mathcal{X} \subseteq \mathcal{D} \cap \mathcal{E}$.

I recognized the proof as the categorical analogue of "the intersection of subgroups is a subgroup": closure properties are inherited because membership in the intersection means membership in each factor simultaneously.

[Solution to Exercise 5.71] Let $\mathcal{C}$ have objects A, B, C and non-identity morphisms $f : A \to B$ and $g : B \to C$ with composite $g \circ f : A \to C$. Let $\mathcal{D}$ be the subcategory on objects $\{A, B\}$ containing f (and identities), and $\mathcal{E}$ the subcategory on objects $\{B, C\}$ containing g. Then $f \in \mathcal{D} \cup \mathcal{E}$ and $g \in \mathcal{D} \cup \mathcal{E}$, but $g \circ f \notin \mathcal{D} \cup \mathcal{E}$ since $A \to C$ does not appear in either $\mathcal{D}$ or $\mathcal{E}$. Hence $\mathcal{D} \cup \mathcal{E}$ is not closed under composition.

I initially conflated "union" and "intersection" by analogy with subgroups; the correction is: intersections behave well, unions generally do not unless one contains the other. For categories, the failure mechanism is almost always "missing composites."

[Solution to Exercise 5.72] Let $\mathcal{D} \subseteq \mathcal{C}$ be full and put $S = \mathrm{Ob}(\mathcal{D})$. By fullness, for any $A, B \in S$, the hom-set $\mathrm{Hom}_{\mathcal{D}}(A, B)$ must equal $\mathrm{Hom}_{\mathcal{C}}(A, B)$. Thus the morphisms of $\mathcal{D}$ are uniquely forced by S, so $\mathcal{D} = \mathcal{C}|_S$.

My guiding slogan: "choose objects freely; morphisms are automatic." Full subcategories are therefore best indexed by object-classes.

[Solution to Exercise 5.73] Let $\mathcal{D}$ have all sets as objects and only bijections as morphisms. Identities are bijections, and composites of bijections are bijections, so $\mathcal{D}$ is a wide subcategory of **Set**. It is not full because for many pairs (A, B) there exist non-bijective functions $A \to B$ in **Set** which are not present in $\mathcal{D}$.

This example matches my "groupoidification" intuition: keeping all objects but discarding non-invertible morphisms turns a category into its maximal groupoid core.

[Solution to Exercise 5.74] Let $\mathcal{U}$ be the intersection of all upper bounds of the family $\{\mathcal{D}_i\}$. Intersections of subcategories are subcategories, hence $\mathcal{U}$ is a subcategory. Each $\mathcal{D}_i$ is contained in every upper bound, hence in the intersection, so $\mathcal{U}$ is an upper bound. If $\mathcal{X}$ is any upper bound, then $\mathcal{U} \subseteq \mathcal{X}$ by definition of intersection, so $\mathcal{U}$ is the least upper bound.

This "steal a trick" definition is exactly the lattice move: *join = intersection of all upper bounds*. It fits my preference for studying the whole poset $\mathrm{Sub}(\mathcal{C})$ rather than a single subcategory.

[Solution to Exercise 5.75] Let S be the set of sources and targets of morphisms in M. By construction, $\langle M \rangle$ has object set contained in S and consists of morphisms built from M using identities and composition; hence every morphism in $\langle M \rangle$ has endpoints in S and thus lies in $\mathcal{C}|_S$, so $\langle M \rangle \subseteq \mathcal{C}|_S$. A subcategory $\mathcal{D}$ lies in the interval $\langle M \rangle \subseteq \mathcal{D} \subseteq \mathcal{C}|_S$ precisely when: (i) $\mathrm{Ob}(\mathcal{D}) = S$ (or at least contains S, depending on conventions), (ii) $M \subseteq \mathrm{Mor}(\mathcal{D})$, and (iii) $\mathcal{D}$ is closed under identities and composition.

My "fat vs skinny sandwich" picture: $\langle M \rangle$ is the minimal morphism-closure forced by M (skinny), while $\mathcal{C}|_S$ is the maximal option once objects S are fixed (fat), and the genuinely interesting world lives in between, as all possible "implementations" of constraints. I also tried to relate this to sieve-style "information refinement"; I postponed the formal comparison until sieves are developed systematically.

5.6 Exercises and Notes

Chapter 6

Functors and Natural Transformations

6.1 Covariant and Contravariant Functors

Intuition

A category packages "things" (objects) together with "processes" (morphisms) that can be composed. A *functor* is a structure-preserving translation between categories: it sends objects to objects and morphisms to morphisms, in a way compatible with identities and composition. In the same way that a group homomorphism preserves the group operation, a functor preserves the categorical "operation" of composition.

Definition (Covariant functor)

Given categories $\mathcal{C}$ and $\mathcal{D}$, a (covariant) functor

$$F : \mathcal{C} \to \mathcal{D}$$

consists of:

- an assignment on objects $X \mapsto F(X)$,
- an assignment on morphisms $f : X \to Y \mapsto F(f) : F(X) \to F(Y)$,

such that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \qquad F(g \circ f) = F(g) \circ F(f)$$

whenever $X \xrightarrow{f} Y \xrightarrow{g} Z$ are composable.

Definition (Contravariant functor)

A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor

$$F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}.$$

Equivalently, it assigns to each $f : X \to Y$ in $\mathcal{C}$ a morphism

$$F(f) : F(Y) \to F(X)$$

in $\mathcal{D}$, satisfying $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and

$$F(g \circ f) = F(f) \circ F(g).$$

Examples

Example 6.1 (Forgetful functors)**.** The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ sends a group to its underlying set and a group homomorphism to its underlying function. Similarly, there are forgetful functors $\mathbf{Top} \to \mathbf{Set}$, $\mathbf{Vect}_k \to \mathbf{Set}$, etc.

Example 6.2 (Inverse image on power sets is contravariant)**.** For a function $f : X \to Y$, the inverse image map

$$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X), \qquad B \mapsto f^{-1}(B)$$

reverses direction. This defines a contravariant functor

$$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}.$$

Example 6.3 (Hom-functors)**.** Fix an object $A \in \mathcal{C}$.

- The covariant hom-functor $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}$ sends

$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(A, X), \qquad (f : X \to Y) \mapsto \Big(h \mapsto f \circ h\Big).$$

- The contravariant hom-functor $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ sends

$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, A), \qquad (f : X \to Y) \mapsto \Big(k \mapsto k \circ f\Big).$$

A first look ahead: natural transformations (no proofs)

Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\mu : F \Rightarrow G$ can be viewed as a family of morphisms in $\mathcal{D}$

$$\mu(X) : F(X) \to G(X) \qquad (X \in \mathcal{C}),$$

subject to a compatibility condition for every morphism $f : X \to Y$ in $\mathcal{C}$. This compatibility is often presented as the commutativity of a square.

Author's notes: folding vs unfolding

In practice, I find it helpful to alternate between two viewpoints:

- **Folded (compressed) viewpoint.** Treat $\mu : F \Rightarrow G$ as a single "meta-morphism" between functors (a morphism in the functor category $[\mathcal{C}, \mathcal{D}]$). In this view, diagrams and equalities are conceptually simple: μ is one arrow at the meta-level.

- **Unfolded (expanded) viewpoint.** To *compute* or *check* anything, I must expand μ into its components $\mu(X)$ and verify the compatibility equation for every morphism $f : X \to Y$. This is the moment when an abstract definition becomes concrete constraints (a system of equations).

Author's notes: a "3D" commutative picture

My mental model of naturality is not merely a 2D commutative square, but a 3D picture: the functor F produces one layer of structure, the functor G produces another layer, and the natural transformation provides vertical edges linking the two layers. Naturality says that the corresponding side face commutes.

$$\begin{tikzcd}
 & G(X) \arrow[rr, "G(f)"] \arrow[dd, "\mu(X)"] & & G(Y) \arrow[dd, "\mu(Y)"] \\
X \arrow[ur, "G"] \arrow[rr, "f"] \arrow[dr, "F"'] & & Y \arrow[ur, "G"'] \arrow[dr, "F"] & \\
 & F(X) \arrow[rr, "F(f)"'] & & F(Y)
\end{tikzcd}$$

In the unfolded algebraic form (for covariant functors), the commutativity of the relevant face is the equation

$$\mu(Y) \circ F(f) = G(f) \circ \mu(X).$$

"Folded" or "unfolded" are two complementary interfaces to the same idea.

A methodological remark: *folding* (layering) complicated structures into objects

In practice, categorical data quickly becomes too dense to draw directly in the "native" object–arrow picture. A recurring strategy is to *fold* a complicated configuration (together with its constraints) into a single *object* of a newly constructed category, so that we can continue reasoning with ordinary commutative diagrams. When actual computations are needed, we *unfold* the object back into its components and verify the relevant equalities by type-checking and diagram chasing.

The general pattern. Whenever a piece of structure in $\mathcal{C}$ is cumbersome to handle directly, we build a category $\mathcal{E}$ whose objects *are* the structures we want to study, and whose morphisms encode the appropriate notion of compatibility. Then we work in $\mathcal{E}$ using the same categorical grammar (objects, arrows, commutative diagrams), postponing low-level bookkeeping until we explicitly unfold.

Examples of folding.

- *Folding arrows:* the arrow category $\mathrm{Arr}(\mathcal{C}) \cong \mathcal{C}^{[1]}$ treats a morphism $f : X \to Y$ in $\mathcal{C}$ as an object; a morphism in $\mathrm{Arr}(\mathcal{C})$ is a commutative square.

- *Folding functors:* the functor category $[\mathcal{C}, \mathcal{D}] = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ treats functors as objects and natural transformations as morphisms, thereby providing a clean stage for comparing functorial constructions.

- *Folding diagrams:* for any indexing category J, the functor category $\mathcal{C}^J$ treats J-shaped diagrams in $\mathcal{C}$ as objects and natural transformations as morphisms. This allows us to regard an entire diagram (together with its built-in relations) as a single point in a higher-level picture.

Folded vs. unfolded viewpoints.

- *Folded viewpoint:* we manipulate a complex configuration as a single object in a suitable category of structures, drawing simple diagrams at the new level.
- *Unfolded viewpoint:* we expand that object into components (objects, arrows, and specified commutativities) and verify conditions by checking that the only well-typed composites agree.

This "fold/unfold" workflow is a systematic way to *layer* categorical reasoning: whenever a discussion becomes high-dimensional, we move up one level by packaging the relevant data and constraints into objects of a new category.

6.2 Composition of Functors and Identity Functor

6.2.1 Composition of functors

Definition 6.4 (Composite functor)**.** Given categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and functors

$$F : \mathcal{C} \to \mathcal{D}, \qquad G : \mathcal{D} \to \mathcal{E},$$

their *composite* is the functor

$$G \circ F : \mathcal{C} \to \mathcal{E}$$

defined by

- on objects: $(G \circ F)(c) := G(F(c))$;
- on morphisms: for $f : c \to c'$ in $\mathcal{C}$,

$$(G \circ F)(f) := G(F(f)) : G(F(c)) \to G(F(c')).$$

A functor acts like a "structure-preserving function" between categories. The composite $G \circ F$ means "first apply F, then apply G", on both objects and morphisms.

Proposition 6.5 (Functoriality of composition)**.** *If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, then $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor.*

Example 6.6. Let $\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$ be the forgetful functor sending a group to its underlying set, and let $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{Set}$ send a set to its power set (as a set). Then $\mathcal{P} \circ U : \mathbf{Grp} \to \mathbf{Set}$ sends a group to the power set of its underlying set.

6.2.2 Identity functor

Definition 6.7 (Identity functor)**.** For any category $\mathcal{C}$, the *identity functor*

$$\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$

is defined by

- on objects: $\mathrm{Id}_{\mathcal{C}}(c) := c$;
- on morphisms: $\mathrm{Id}_{\mathcal{C}}(f) := f$.

$\mathrm{Id}_{\mathcal{C}}$ does nothing: it leaves every object and morphism unchanged. It plays the role of a "unit" for functor composition.

Proposition 6.8 (Identity laws)**.** *For any functor $F : \mathcal{C} \to \mathcal{D}$,*

$$F \circ \mathrm{Id}_{\mathcal{C}} = F, \qquad \mathrm{Id}_{\mathcal{D}} \circ F = F.$$

6.2.3 A categorical viewpoint

Remark 6.9. The composition of functors is associative, and identity functors act as units. Thus, one can organize categories as objects and functors as morphisms in a "category of categories".

Example 6.10 (A commuting diagram for functoriality of composition)**.** For composable morphisms $c \xrightarrow{f} c' \xrightarrow{g} c''$ in $\mathcal{C}$, the functoriality of F and G can be summarized by the following commuting diagram:

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d] & c' \arrow[r, "g"] \arrow[d, "F"] & c'' \arrow[d] \\
Fc \arrow[r, "Ff"] \arrow[d] & Fc' \arrow[r, "Fg"] \arrow[d, "G"] & Fc'' \arrow[d] \\
GFc \arrow[r, "GFf"] & GFc' \arrow[r, "GFg"] & GFc''
\end{tikzcd}$$

Exercise 6.11. Prove Proposition *Functoriality of composition*.

Exercise 6.12. Prove Proposition *Identity laws*.

In our discussion, the commuting diagram is a visual shorthand for an equality of composites. For $f : c \to c'$ and $g : c' \to c''$, the key chain of equalities is

$$(GF)(g \circ f) = G(F(g \circ f)) = G(Fg \circ Ff) = G(Fg) \circ G(Ff) = (GF)g \circ (GF)f.$$

When one of the functors is an identity functor, the same pattern collapses to the unit laws.

6.3 Natural Transformations and Naturality Squares

6.3.1 Motivation: morphisms between functors

Given functors $F, G : \mathcal{C} \to \mathcal{D}$, it is often useful to compare them in a way that is compatible with all morphisms of $\mathcal{C}$. A *natural transformation* is precisely a coherent family of comparison maps between F and G.

6.3.2 Natural transformations

Definition 6.13 (Natural transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation*

$$\alpha : F \Rightarrow G$$

consists of a family of morphisms in $\mathcal{D}$, one for each object $c \in \mathcal{C}$,

$$\alpha_c : F(c) \to G(c),$$

such that for every morphism $f : c \to c'$ in $\mathcal{C}$, the *naturality square* commutes:

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\alpha_c"'] & F(c') \arrow[d, "\alpha_{c'}"] \\ G(c) \arrow[r, "G(f)"'] & G(c'). \end{tikzcd}$$

Equivalently, for all $f : c \to c'$,

$$G(f) \circ \alpha_c \;=\; \alpha_{c'} \circ F(f).$$

Remark 6.14 (Coherence as "loop constraints"). Naturality can be read as a constraint on a loop: there are two routes from $F(c)$ to $G(c')$ around the square, and naturality demands that these two composites agree. In this sense, *commutative diagrams encode constraint equations* ("every specified loop must commute").

6.3.3 Examples in Set

Example 6.15 (Diagonal map)**.** Let $F = \mathrm{Id}_{\mathbf{Set}}$ and let $G : \mathbf{Set} \to \mathbf{Set}$ be the functor $G(X) = X \times X$, $G(f) = f \times f$. Define $\Delta_X : X \to X \times X$ by $\Delta_X(x) = (x, x)$. Then $\Delta = \{\Delta_X\}$ is a natural transformation

$$\Delta : \mathrm{Id}_{\mathbf{Set}} \Rightarrow (-) \times (-).$$

Naturality is the equation $(f \times f) \circ \Delta_X = \Delta_Y \circ f$ for all $f : X \to Y$.

Example 6.16 (Projection map)**.** Fix a set A. Define $F_A : \mathbf{Set} \to \mathbf{Set}$ by $F_A(X) = X \times A$ and $F_A(f) = f \times \mathrm{id}_A$. The projections $\pi_X : X \times A \to X$ assemble into a natural transformation

$$\pi : F_A \Rightarrow \mathrm{Id}_{\mathbf{Set}}.$$

6.3.4 Composition and identities

Definition 6.17 (Vertical composition)**.** Given natural transformations

$$F \overset{\alpha}{\Rightarrow} G \overset{\beta}{\Rightarrow} H$$

their *vertical composite* $\beta \circ \alpha : F \Rightarrow H$ is defined componentwise by

$$(\beta \circ \alpha)_c := \beta_c \circ \alpha_c.$$

Definition 6.18 (Identity natural transformation)**.** For a functor $F : \mathcal{C} \to \mathcal{D}$, the *identity natural transformation*

$$1_F : F \Rightarrow F$$

has components $(1_F)_c := \mathrm{id}_{F(c)}$.

6.3.5 Natural isomorphisms

Definition 6.19 (Natural isomorphism)**.** A natural transformation $\alpha : F \Rightarrow G$ is a *natural isomorphism* if each component $\alpha_c : F(c) \to G(c)$ is an isomorphism in $\mathcal{D}$.

Proposition 6.20 (Inverse of a natural isomorphism)**.** *If $\alpha : F \Rightarrow G$ is a natural isomorphism, then the componentwise inverses*

$$(\alpha^{-1})_c := (\alpha_c)^{-1} : G(c) \to F(c)$$

assemble into a natural transformation $\alpha^{-1} : G \Rightarrow F$.

Remark 6.21 (Opposite vs. inverse)*.* In the opposite category $[\mathcal{C}, \mathcal{D}]^{op}$, every arrow reverses direction formally. This is different from an *inverse* natural transformation, which exists only when the components are invertible (i.e. for natural isomorphisms).

6.3.6 The functor category

Proposition 6.22 (Functor category)**.** *For categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $[\mathcal{C}, \mathcal{D}]$ whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations. Composition and identities are given by vertical composition and identity natural transformations.*

6.3.7 2-dimensional picture in Cat

Remark 6.23 (Natural transformations as 2-cells)*.* Viewing **Cat** as a 2-category:

- 0-cells are categories,
- 1-cells are functors,
- 2-cells are natural transformations.

A natural transformation $\alpha : F \Rightarrow G$ can be pictured as a *2-dimensional filler* (a "face") whose boundary is a naturality square. The naturality equation expresses that the boundary composites agree.

6.3.8 Author's method: "pointification" (folding constraints into points)

Remark 6.24 (Viewpoint elevation by folding)*.* A recurring technique is to *raise the viewpoint* by treating structured data as a single "point":

- In $\mathcal{C}$, the basic units are objects and morphisms.
- In $[\mathcal{C}, \mathcal{D}]$, a *functor* is regarded as a new "point" (object), already packaging the functoriality constraints (preservation of identities and composition).
- A *natural transformation* is regarded as a new "arrow" between such points, already packaging the naturality constraints (commuting squares for all f).

When working at the higher viewpoint, one may manipulate these packaged points/arrows without constantly expanding their internal equations. When needed, one "unpacks" a point (or arrow) to recover the underlying constraint equations, possibly generating new commutative loops that must be checked.

Exercises

Exercise 6.25. Prove that the diagonal maps $\Delta_X : X \to X \times X$ define a natural transformation $\Delta : \mathrm{Id}_{\mathbf{Set}} \Rightarrow (-) \times (-)$.

Exercise 6.26. Prove that if $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are natural transformations, then the componentwise composite $(\beta \circ \alpha)_c = \beta_c \circ \alpha_c$ is natural.

Exercise 6.27. Prove Proposition "Inverse of a natural isomorphism".

Exercise 6.28. Prove Proposition "Functor category": show that functors and natural transformations form a category $[\mathcal{C}, \mathcal{D}]$ under vertical composition.

Solutions and Author's Thoughts

Solution to Exercise 1. Let $f : X \to Y$. We must show $(f \times f) \circ \Delta_X = \Delta_Y \circ f$. For each $x \in X$,

$$((f \times f) \circ \Delta_X)(x) = (f \times f)(x, x) = (f(x), f(x)) = \Delta_Y(f(x)) = ((\Delta_Y \circ f)(x)).$$

Hence the functions agree, so the square commutes for all f. □

Solution to Exercise 2. Fix $f : c \to c'$ in $\mathcal{C}$. Using naturality of α and β,

$$H(f) \circ (\beta_c \circ \alpha_c) = (H(f) \circ \beta_c) \circ \alpha_c = (\beta_{c'} \circ G(f)) \circ \alpha_c = \beta_{c'} \circ (G(f) \circ \alpha_c) = \beta_{c'} \circ (\alpha_{c'} \circ F(f)) = (\beta_{c'} \circ \alpha_{c'}) \circ F(f).$$

Thus $\beta \circ \alpha$ is natural. □

Solution to Exercise 3. Assume $\alpha : F \Rightarrow G$ is a natural isomorphism. Define $(\alpha^{-1})_c = (\alpha_c)^{-1}$. From naturality $G(f) \circ \alpha_c = \alpha_{c'} \circ F(f)$, postcompose by $(\alpha_c)^{-1}$ and precompose by $(\alpha_{c'})^{-1}$ to obtain

$$(\alpha_{c'})^{-1} \circ G(f) = F(f) \circ (\alpha_c)^{-1}.$$

Rewriting, $F(f) \circ (\alpha^{-1})_c = (\alpha^{-1})_{c'} \circ G(f)$, which is precisely the naturality condition for $\alpha^{-1} : G \Rightarrow F$. □

Solution to Exercise 4. Define $[\mathcal{C}, \mathcal{D}]$ to have objects the functors $\mathcal{C} \to \mathcal{D}$ and morphisms the natural transformations. Define composition by $(\beta \circ \alpha)_c = \beta_c \circ \alpha_c$ and identities by $(1_F)_c = \mathrm{id}_{F(c)}$. Associativity and identity laws follow from associativity and identities in $\mathcal{D}$, checked componentwise. Naturality of composites was established in Exercise 2, so composition is well-defined. Hence $[\mathcal{C}, \mathcal{D}]$ is a category. □

Remark 6.29 (Author's reflection: "constraints live on boundaries"). While learning, it was helpful to treat commutative diagrams as *constraint equations*: whenever a new arrow is introduced, new "loops" (or, in the 2-categorical picture, new boundaries of faces) appear, and coherence asks that specified boundaries commute. The "pointification" viewpoint then allows one to work at a higher level (functors as points, natural transformations as arrows/2-cells), only unpacking internal constraints when necessary.

6.4 6.4 Natural Isomorphisms and Equivalences of Categories

6.4.1 Natural isomorphisms

Definition 6.30 (Natural isomorphism)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\alpha : F \Rightarrow G$ is a *natural isomorphism* if each component $\alpha_c : Fc \to Gc$ is an isomorphism in $\mathcal{D}$. We write $F \cong G$ when such an α exists.

Remark 6.31 (Isomorphisms in a functor category)*.* Natural transformations are morphisms in the functor category $[\mathcal{C}, \mathcal{D}]$. A natural isomorphism is precisely an isomorphism in $[\mathcal{C}, \mathcal{D}]$.

Proposition 6.32 (Pointwise inverse gives inverse natural transformation)**.** *If $\alpha : F \Rightarrow G$ is a natural isomorphism, then the componentwise inverses $\alpha_c^{-1} : (Gc \to Fc)$ assemble into a natural transformation $\alpha^{-1} : G \Rightarrow F$, and $\alpha^{-1} \circ \alpha = 1_F$ and $\alpha \circ \alpha^{-1} = 1_G$ in $[\mathcal{C}, \mathcal{D}]$.*

Example 6.33 (Double opposite)**.** For any category $\mathcal{C}$, there is a canonical isomorphism of categories $(\mathcal{C}^{op})^{op} \cong \mathcal{C}$. The underlying data is objectwise identity, and morphisms are reversed twice.

Example 6.34 (Finite-dimensional double dual)**.** Let $\mathbf{Vect}_{fd}$ be the category of finite-dimensional vector spaces over a field. The canonical map $V \to V^{**}$ is natural in V and is an isomorphism. Hence $(-)^{**} \cong \mathrm{Id}_{\mathbf{Vect}_{fd}}$. (Contrast: for infinite-dimensional spaces it fails to be an isomorphism in general.)

6.4.2 Equivalences of categories

Definition 6.35 (Equivalence of categories)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF, \qquad \epsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}.$$

In this situation G is called a *quasi-inverse* of F.

Remark 6.36 (What is being "weakened")*.* An isomorphism of categories requires strict equalities $GF = \mathrm{Id}_{\mathcal{C}}$ and $FG = \mathrm{Id}_{\mathcal{D}}$. An equivalence relaxes these equalities to *natural isomorphisms.*

Definition 6.37 (Fully faithful and essentially surjective)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is

- *faithful* if for all $c, c' \in \mathcal{C}$, the function $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is injective;
- *full* if the same function is surjective;
- *fully faithful* if it is bijective;
- *essentially surjective on objects* if for every $d \in \mathcal{D}$ there exists $c \in \mathcal{C}$ and an isomorphism $Fc \cong d$ in $\mathcal{D}$.

Theorem 6.38 (Characterization of equivalences)**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective on objects.*

Remark 6.39 (Uniqueness up to isomorphism)*.* Quasi-inverses are not unique on the nose, but any two quasi-inverses of the same equivalence are naturally isomorphic.

Example 6.40 (Skeletal subcategory)**.** A *skeleton* of a category $\mathcal{D}$ is a full subcategory $\mathcal{D}_{sk} \hookrightarrow \mathcal{D}$ containing exactly one object from each isomorphism class. The inclusion $\mathcal{D}_{sk} \to \mathcal{D}$ is an equivalence.

6.4.3 Intuition: "graph is all you need" via changing tests

[Equations vs existence] A commutative diagram efficiently encodes *equations* between composites. Statements of the form "there exists g such that ... " are *quantified* and are not literally forced by a picture. A standard move is to replace existence-claims by *extra structure*: instead of saying "f is invertible", we *specify* an inverse g and then record the two triangle equations $gf = 1$ and $fg = 1$ diagrammatically.

[Different viewpoints = different tests] Many properties of a morphism become "visible" only after choosing the right testing language. For instance, a continuous map $f : X \to Y$ in **Top** can be tested by the induced inverse-image map on opens $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$, which preserves unions and finite intersections. The slogan is: *choose the appropriate subobjects/tests, then encode the property as preservation of structure in diagrams.*

[Relative notions of equivalence] Notions of "equivalence" depend on what we decide to test and what we decide to ignore. Fixing a family of tests $\{T_i\}$ (functorial invariants) determines a class W of morphisms that look invertible under all T_i. Conversely, specifying a class W and formally inverting it (localization) produces a setting where exactly the morphisms in W become isomorphisms. In this sense, "tests" and "inverted morphisms" are two faces of the same idea.

6.4.4 Exercises

Exercise 6.41 (Inverse of a natural isomorphism)**.** Prove Proposition 6.32.

Exercise 6.42 (Equivalence implies fully faithful + essentially surjective)**.** Assume $F : \mathcal{C} \to \mathcal{D}$ has a quasi-inverse G and natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$. Prove that F is fully faithful and essentially surjective.

Exercise 6.43 (Fully faithful + essentially surjective implies equivalence)**.** Assume F is fully faithful and essentially surjective. Construct a quasi-inverse G (using a choice of representatives) and prove Theorem 6.38.

Exercise 6.44 (Skeleta)**.** Show that the inclusion of a skeleton $\mathcal{D}_{sk} \hookrightarrow \mathcal{D}$ is an equivalence.

Exercise 6.45 (Testing and relative equivalences)**.** Let $\{T_i : \mathcal{C} \to \mathcal{D}_i\}_{i \in I}$ be a family of functors and let $W = \{f \mid T_i(f) \text{ is an isomorphism for all } i\}$. Check that W contains identities and is closed under composition. (Think: what extra hypotheses would make W satisfy a "2-out-of-3" property?)

6.4.5 Solution sketches and author's notes

Solution sketch to Exercise 6.41. To show α^{-1} is natural, start from the naturality square for α: for each $f : c \to c'$ in $\mathcal{C}$,

$$Gf \circ \alpha_c = \alpha_{c'} \circ Ff.$$

Use that α_c and $\alpha_{c'}$ are invertible and compose with inverses on the left/right to rewrite the same equation into the naturality condition for α^{-1}.

Solution sketch to Exercise 6.42. For full faithfulness: use the quasi-inverse G to transfer a morphism $u : Fc \to Fc'$ back to $\mathcal{C}$ via Gu, then correct it with η. For essential surjectivity: given $d \in \mathcal{D}$, use $Gd \in \mathcal{C}$ and the component $\epsilon_d : FGd \to d$.

Solution sketch to Exercise 6.43. Pick for each $d \in \mathcal{D}$ an object $Gd \in \mathcal{C}$ and an isomorphism $\epsilon_d : FGd \cong d$ (choice uses essential surjectivity). Define G on morphisms using full faithfulness of F: transport a morphism $d \to d'$ to a morphism $FGd \to FGd'$ and then uniquely lift it back to $\mathcal{C}(Gd, Gd')$. Then build $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ from the bijection on hom-sets.

Author's notes (to be expanded).

- **Diagrams vs quantifiers.** Diagrams excel at encoding equations between composites. Existence statements can be "diagrammatized" by promoting existence to *chosen data* (e.g. choose an inverse) and then recording only equations.
- **Changing tests / changing subobjects.** Looking at a map in **Set** tests only its underlying function (e.g. bijectivity), whereas looking through opens (or other structured tests) reveals continuity as a structure-preservation condition.
- **Relative equivalences.** Different equivalence relations arise by fixing some tests and relaxing others: a "notion of sameness" is a design choice about which structure is observed.

6.5 Cat as a 2-Category: whiskering, horizontal composition, and pasting

6.5.1 Guiding idea: "loops become equations" (in the strict world)

A recurring theme in our study is that once we *package* structure into a higher-level object, then any *admissible loop* in the resulting diagrammatic language becomes a *constraint*. In a *strict* setting, "constraint" means *equality* (commutativity of a diagram). This viewpoint scales well: rather than tracking every object-level corner, we track *typed edges* (morphisms) and use *pasting* to derive large commutative diagrams from smaller ones.

6.5.2 2-categories (minimal definition)

A *(strict) 2-category* $\mathcal{K}$ consists of:

- **0-cells** (objects) $A, B, \ldots$
- **1-cells** $f, g : A \to B$
- **2-cells** $\eta : f \Rightarrow g$ between parallel 1-cells

equipped with:

- **vertical composition** of 2-cells (inside a hom-category),
- **horizontal composition** (composition along 1-cells),
- strict associativity/unitality for both compositions, plus the **interchange law**.

We will not dwell on the full axiom list here; the important point is that 2-dimensional pasting diagrams are *meaningful* and yield well-typed composites.

6.5.3 The fundamental example: Cat

There is a canonical strict 2-category **Cat**:

- 0-cells: (small) categories $\mathcal{C}, \mathcal{D}, \ldots$
- 1-cells: functors $F : \mathcal{C} \to \mathcal{D}$
- 2-cells: natural transformations $\alpha : F \Rightarrow G$

Vertical composition

Given $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, the vertical composite $\beta \circ \alpha : F \Rightarrow H$ is defined componentwise:

$$(\beta \circ \alpha)_c := \beta_c \circ \alpha_c.$$

Whiskering (pre/post composition by a functor)

The term *whiskering* is purely pictorial: composing a 2-cell with a 1-cell attaches a "thin tail" (a *whisker*) to a 2-dimensional cell.

Post-whiskering. Given $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{E}$, define

$$H\alpha : HF \Rightarrow HG \qquad \text{by} \qquad (H\alpha)_c := H(\alpha_c).$$

Pre-whiskering. Given $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ and $K : \mathcal{B} \to \mathcal{C}$, define

$$\alpha K : FK \Rightarrow GK \qquad \text{by} \qquad (\alpha K)_b := \alpha_{K(b)}.$$

Horizontal composition of natural transformations

Let $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ and $\beta : H \Rightarrow J : \mathcal{D} \to \mathcal{E}$. Their horizontal composite

$$\beta * \alpha : HF \Rightarrow JG$$

may be defined at $c \in \mathcal{C}$ by either of the following (a priori different-looking) formulas:

$$(\beta * \alpha)_c := J(\alpha_c) \circ \beta_{F(c)} : HF(c) \to JG(c),$$
$$(\beta * \alpha)_c := \beta_{G(c)} \circ H(\alpha_c) : HF(c) \to JG(c).$$

These two expressions are equal by the naturality of β applied to the morphism $\alpha_c : F(c) \to G(c)$ in $\mathcal{D}$.

6.5.4 Pasting and "derived" commutative diagrams

A key practical principle is that one rarely needs to draw a huge outer loop explicitly. Instead, we build it by pasting smaller commutative squares whose commutativity is already known (e.g. naturality squares of α and β). The only thing that must be checked for the pasting to be *legal* is *typing*: the shared boundary 1-cell must match on both sides.

Typing-first viewpoint ("use f as the point"). To verify naturality of a composite transformation, we focus on a single morphism $f : c \to c'$. In the horizontal-composition proof, the crucial shared edge is

$$JF(f) = J(Ff),$$

which is the gluing boundary between the β-naturality square (for Ff) and the J-image of the α-naturality square (for f).

6.5.5 Strict vs. weak: why higher coherence appears

In **Cat**, composition is strictly associative, so changing parentheses costs nothing: there is no need for extra "associator" data. In a *bicategory* (a weak 2-category), associativity of 1-cell composition holds only up to a specified *invertible 2-cell*

$$a_{f,g,h} : (f \circ g) \circ h \Rightarrow f \circ (g \circ h).$$

In higher weak settings (e.g. tricategories), associativity at the level of 2-cells may itself be weakened, requiring invertible 3-cells, and so on. This is not an "external meta-proof" phenomenon: the extra coherence data and coherence laws are part of the *internal structure definition*.

6.5.6 Diagram gallery: vertical/horizontal composition, whiskering, and pasting

A. A 2-cell in Cat

A natural transformation $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ can be depicted as a 2-cell:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "F"] \arrow[r, shift right=2, "G"'] \arrow[r, phantom, "\alpha"] & \mathcal{D} \end{tikzcd}$$

B. Vertical composition (stacking 2-cells)

Given $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, their vertical composite $\beta \circ \alpha : F \Rightarrow H$ is shown by stacking:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=4, "F"] \arrow[r, shift left=2, phantom, "\alpha"] \arrow[r] \arrow[r, shift right=2, phantom, "\beta"] \arrow[r, shift right=4, "H"'] & \mathcal{D} \end{tikzcd} \qquad \text{(think: } \alpha \text{ on top of } \beta\text{)}$$

C. Whiskering (adding "whiskers")

Post-whiskering by $H : \mathcal{D} \to \mathcal{E}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "HF"] \arrow[r, bend right, "HG"'] \arrow[r, phantom, "\Downarrow H\alpha"] & \mathcal{E} \end{tikzcd}$$

Pre-whiskering by $K : \mathcal{B} \to \mathcal{C}$ is analogous.

D. Horizontal composition (side-by-side pasting)

Let $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ and $\beta : H \Rightarrow J : \mathcal{D} \to \mathcal{E}$. Their horizontal composite $\beta * \alpha : HF \Rightarrow JG$ can be pictured as:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "HF"] \arrow[r, bend right, "JG"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \mathcal{E} \end{tikzcd}$$

E. Component-level "point tracking" using a single morphism $f : c \to c'$

To verify naturality, we focus on one $f : c \to c'$ and track only typed edges. The key pasted rectangle for horizontal composition is:

$$\begin{tikzcd} HF(c) \arrow[r, "\beta_{F(c)}"] \arrow[d, "HF(f)"'] & JF(c) \arrow[r, "J(\alpha_c)"] \arrow[d, "JF(f)"] & JG(c) \arrow[d, "JG(f)"] \\ HF(c') \arrow[r, "\beta_{F(c')}"'] & JF(c') \arrow[r, "J(\alpha_{c'})"'] & JG(c') \end{tikzcd}$$

The outer rectangle expresses the naturality of $\beta * \alpha$, and it is *derived* by pasting the two commuting squares along the shared edge $JF(f)$.

F. The interchange law (a 2-dimensional "compatibility square")

In a strict 2-category, vertical and horizontal compositions satisfy the interchange law:

$$(\beta' \circ \beta) * (\alpha' \circ \alpha) \;=\; (\beta' * \alpha') \circ (\beta * \alpha),$$

whenever the types match (i.e. all composites exist). This expresses that "pasting a grid" does not depend on whether we compose row-wise first or column-wise first.

6.5.7 Weak 2-categories (bicategories): diagrams with associators and unitors

A. Associator and unitors are invertible 2-cells (not equalities)

In a bicategory, rebracketing is implemented by a specified invertible 2-cell:

$$a_{f,g,h} : \; (f \circ g) \circ h \Rightarrow f \circ (g \circ h).$$

It can be depicted as a 2-cell between *different* 1-cells:

$$
\begin{tikzcd}
A \arrow[r, bend left, "(f\circ g)\circ h"] \arrow[r, bend right, "f\circ(g\circ h)"'] \arrow[r, phantom, "\Downarrow a_{f,g,h}"] & D
\end{tikzcd}
$$

Similarly, unit laws are witnessed by invertible 2-cells (left/right unitors)

$$\ell_f :\ 1 \circ f \Rightarrow f, \qquad r_f :\ f \circ 1 \Rightarrow f.$$

B. Coherence diagrams (as picture-constraints)

The associators and unitors are not arbitrary: they satisfy coherence laws. Two standard ones are the pentagon and triangle (stated as commutative diagrams of 2-cells).

Pentagon (associators).

$$
\begin{tikzcd}
((f\circ g)\circ h)\circ k \arrow[r, "a_{f,g,h}\circ 1_k"] \arrow[d, "a_{f\circ g,h,k}"'] & (f\circ(g\circ h))\circ k \arrow[r, "a_{f,g\circ h,k}"] & f\circ((g\circ h)\circ k) \\
(f\circ g)\circ(h\circ k) \arrow[rr, "a_{f,g,h\circ k}"'] & & f\circ(g\circ(h\circ k)) \arrow[u, "1_f\circ a_{g,h,k}"']
\end{tikzcd}
$$

Triangle (associator + unitor).

$$
\begin{tikzcd}
(f\circ 1)\circ g \arrow[r, "a_{f,1,g}"] \arrow[dr, "r_f\circ 1_g"'] & f\circ(1\circ g) \arrow[d, "1_f\circ \ell_g"] \\
& f\circ g
\end{tikzcd}
$$

Intuition (for later higher-categorical upgrades). In the strict world, "loops become equalities." In weak worlds, the same loops are often filled by higher invertible cells, and coherence laws ensure that different pastings yield compatible fillings.

6.5.8 Exercises

Exercise 6.46 (Naturality of post-whiskering)**.** Let $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{E}$. Prove that $H\alpha : HF \Rightarrow HG$ is a natural transformation.

Exercise 6.47 (Naturality of horizontal composition)**.** Let $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ and $\beta : H \Rightarrow J : \mathcal{D} \to \mathcal{E}$. Prove that $\beta * \alpha : HF \Rightarrow JG$ defined by $(\beta * \alpha)_c := J(\alpha_c) \circ \beta_{F(c)}$ is natural.

Exercise 6.48 (Two formulas agree)**.** With α, β as above, prove that

$$J(\alpha_c) \circ \beta_{F(c)} = \beta_{G(c)} \circ H(\alpha_c)$$

for all $c \in \mathcal{C}$.

Exercise 6.49 (Pasting lemma in **Cat**)**.** Formulate precisely a "two squares paste to a rectangle" statement for commutative diagrams in a category and prove it. Emphasize the typing condition on the shared edge.

Exercise 6.50 (Strict vs. weak: what breaks?)**.** Explain (without proof) why in a bicategory one cannot replace the associator $a_{f,g,h}$ by an equality, and why coherence constraints (pentagon/triangle) become necessary.

6.5.9 Solutions and author's notes

Solution to Exercise 6.5.1 (standard proof)

Take $f : c \to c'$ in $\mathcal{C}$. We must show

$$(H\alpha)_{c'} \circ HF(f) = HG(f) \circ (H\alpha)_c.$$

By definition $(H\alpha)_c = H(\alpha_c)$ and functoriality of H gives

$$(H\alpha)_{c'} \circ HF(f) = H(\alpha_{c'}) \circ H(Ff) = H(\alpha_{c'} \circ Ff),$$

$$HG(f) \circ (H\alpha)_c = H(Gf) \circ H(\alpha_c) = H(Gf \circ \alpha_c).$$

Since α is natural, $\alpha_{c'} \circ Ff = Gf \circ \alpha_c$. Applying H yields the desired equality.

Author's notes. The large loop in $\mathcal{E}$ is unnecessary. The minimal workflow is: *(i)* write the naturality square for α once, *(ii)* apply H to both sides, *(iii)* use functoriality to re-expand.

Solution to Exercise 6.5.2 (standard proof)

Fix $f : c \to c'$. We need

$$(\beta * \alpha)_{c'} \circ HF(f) = JG(f) \circ (\beta * \alpha)_c.$$

Expand the definition:

$$(\beta * \alpha)_{c'} \circ HF(f) = \Big(J(\alpha_{c'}) \circ \beta_{F(c')}\Big) \circ HF(f) = J(\alpha_{c'}) \circ \Big(\beta_{F(c')} \circ HF(f)\Big).$$

By naturality of β applied to $Ff : F(c) \to F(c')$:

$$\beta_{F(c')} \circ H(Ff) = J(Ff) \circ \beta_{F(c)}.$$

Thus

$$(\beta * \alpha)_{c'} \circ HF(f) = J(\alpha_{c'}) \circ J(Ff) \circ \beta_{F(c)} = J(\alpha_{c'} \circ Ff) \circ \beta_{F(c)}.$$

Using naturality of α:

$$\alpha_{c'} \circ Ff = Gf \circ \alpha_c,$$

hence

$$J(\alpha_{c'} \circ Ff) \circ \beta_{F(c)} = J(Gf \circ \alpha_c) \circ \beta_{F(c)} = J(Gf) \circ J(\alpha_c) \circ \beta_{F(c)} = JG(f) \circ (\beta * \alpha)_c.$$

Author's notes. This is the prototypical "typed pasting" argument: the shared edge is $JF(f) = J(Ff)$, and the outer rectangle commutes because it is the pasting of two commuting squares. When diagrams feel too large, zoom in on the shared edge.

Solution to Exercise 6.5.3 (standard proof)

Apply naturality of $\beta : H \Rightarrow J$ to the morphism $\alpha_c : F(c) \to G(c)$ in $\mathcal{D}$:

$$J(\alpha_c) \circ \beta_{F(c)} = \beta_{G(c)} \circ H(\alpha_c).$$

Author's notes. This identity explains why the two "obvious" factorizations of $(\beta * \alpha)_c$ coincide: either change the *outer functor* first (via β at $F(c)$) and then change the object (via $J(\alpha_c)$), or change the object first (via $H(\alpha_c)$) and then change the outer functor (via β at $G(c)$).

Solution to Exercise 6.5.4 (standard proof)

A typical statement: if the left and right squares in a pasted diagram commute and the shared edge matches as a well-typed morphism, then the outer rectangle commutes. Proof: write both outer composites and rewrite one into the other using the commutativity equations of the two squares, plus associativity of composition.

Author's notes. No "existence/uniqueness" is needed: pasting does not *create* new morphisms, it compares two already-defined composites. The only nonnegotiable requirement is typing compatibility of the glued boundary.

Solution to Exercise 6.5.5 (discussion)

In a weak setting, associativity is implemented by specified invertible higher cells rather than equality. Coherence laws ensure that different ways of rebracketing and pasting yield the same resulting higher cell (up to the next level), preventing ambiguity in diagram evaluation.

Author's notes. A useful slogan: *strict* means "loops give equations"; *weak* means "loops give higher fillers" (invertible cells), and coherence organizes all fillers into a consistent system.

6.6 Exercises and Notes

Chapter 7

Representable Functors and Yoneda's Lemma

7.1 Hom-Functors and Representable Functors

7.2 Yoneda Lemma and Its Proof

7.3 The Yoneda Embedding

7.4 Characterizing Structures via Representability

7.5 Examples in Set, Top, Ab, Ban

7.6 Exercises and Notes

Chapter 8

Limits, Colimits and Universal Properties

8.1 Diagrams, Cones and Cocones

8.2 Terminal Objects, Products, Pullbacks, Equalizers

8.3 Initial Objects, Coproducts, Pushouts, Coequalizers

8.4 Complete and Cocomplete Categories

8.5 Category of Elements and Pointwise Descriptions

8.6 Computing Limits and Colimits in Concrete Categories

8.7 Exercises and Notes

Chapter 9

Adjoint Functors

9.1 Definition of Adjunction: Unit and Counit

9.2 Hom-Set Characterization of Adjunctions

9.3 Classical Examples: Free/Forgetful, Tensor/Hom, etc.

9.4 Adjoints and (Co)Limits

9.5 Reflective Subcategories and Constructions via Adjoints

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