

# Category and $\infty$ -Category Theory in Context

## From Analysis to Higher Structures

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# Preface



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# Notation and Conventions



# **Catalogue of Common Categories and Constructions**



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# References and Further Reading