STOCHASTIC FEM

F. I. Giasemis

1st Assignment

A) The Karhunen-Loeve Expansion

Stochastic field : $E(x) = 10(1 + f(x))$

We will compute the KL expansion of $f(x)$, since it is a zero mean, stationary Gaussian with unit variance.

From the autocorrelation function of f, $R_f = e^{-|\tau|/2}$ we know that the correlation length is $b=2$.

The field is defined over $x \in [0,5]$, so after a shift $X = 2.5$:

$$D' = D - X = [-2.5, 2.5]$$

Hence, $a = 2.5$.

1. Then using the formulas for the eigenvalues and eigenfunctions for a Gaussian process we create realizations for $f(x)$ and using $E(x) = 10(1+f(x))$ we create realizations of $E(x)$. We keep $M=40$ terms in the expansion. The calculations are shown in the Mathematica notebook, with $R=5000$ realizations generated.

2. We only keep eigenvalues larger than some threshold, i.e. $\lambda > \lambda_{threshold}$. Here, this threshold is $0.0016\,\lambda_{even}[1]$ and $0.00068\,\lambda_{odd}[1]$.

3. We also calculate the ensemble average and variance. For example, for $x=2$, we have an average of 9.89485 and a variance of 94.9847. Since, $f(x)$ is of unit variance, $E(x) = 10(1+f(x))$ is of variance 10^2 ($var(aX) = a^2 var(X)$). So we expect the above two values to converge to 10 and 100, respectively.

1.1

1st Assignment: Stochastic FEM

F. I. Giasemis

A: KL expansion

Parameters from the problem

```
b = 2;
a = 2.5;
M = 40; (* Number of terms in the KL expansion. *)
R = 5000; (* Number of realizations. *)
```

Eigenvalues and eigenfunctions for f(x)

```
For[n = 0, n < M/2, n = n + 1;
 sol = NSolve[{1/b - x Tan[x a] == 0, (n - 1) Pi/a ≤ x ≤ (n - 1/2) Pi/a}, x];
 wodd[n] = Part[x /. sol, 1];
 λodd[n] = 2 b/(1 + wodd[n]^2 b^2);
 codd[n] = 1/Sqrt[a + Sin[2 wodd[n] a]/(2 wodd[n])];
 φodd[n][x_] := codd[n] Cos[wodd[n] x];

 sol = NSolve[{1/b Tan[x a] + x == 0, (n - 1/2) Pi/a ≤ x ≤ (n) Pi/a}, x];
 weven[n] = Part[x /. sol, 1];
 λeven[n] = 2 b/(1 + weven[n]^2 b^2);
 ceven[n] = 1/Sqrt[a - Sin[2 weven[n] a]/(2 weven[n])];
 φeven[n][x_] := ceven[n] Sin[weven[n] x]
]
```

Random variables $\xi(\theta)$

```
For[i = 0, i < R, i = i + 1;
 ξ[i] = RandomVariate[NormalDistribution[], M]
]
```

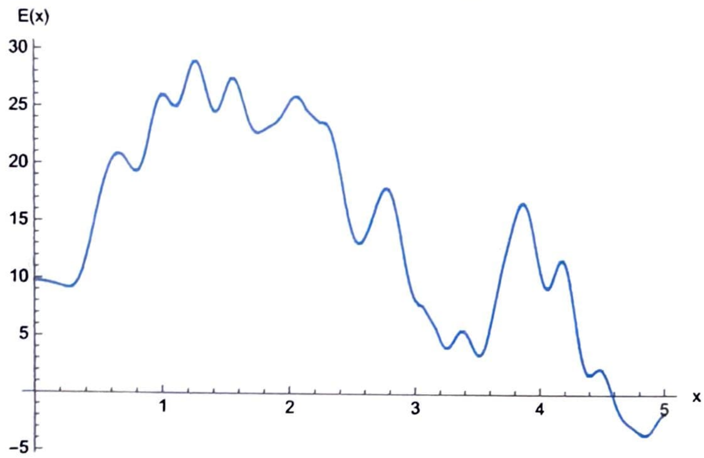
Realization of f(x) and E(x)

```
RealizationF[i_, x_] := Sum[Sqrt[λodd[n]] φodd[n][x - 2.5] ξ[i][[n]], {n, 1, M/2}] +
   Sum[Sqrt[λeven[n]] φeven[n][x - 2.5] ξ[i][[M/2 + n]], {n, 1, M/2}];
Realization[i_, x_] := 10 (1 + RealizationF[i, x]);
```

1.2

Example plot of a realization of E(x)

```
Plot[Realization[567, x], {x, 0, 5}, AxesLabel → {"x", "E(x)"}]
```



Ensemble averages and variances

```
EnsembleAverage[x_] := Mean[Table[Realization[i, x], {i, 1, R}]]
EnsembleVariance[x_] := Variance[Table[Realization[i, x], {i, 1, R}]]
```

Example calculation of ensemble average and variance

```
EnsembleAverage[2]
EnsembleVariance[2]
```

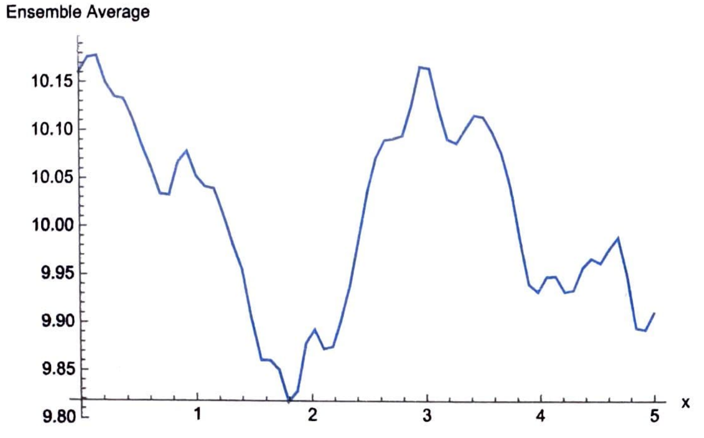
```
9.89485

94.9847
```

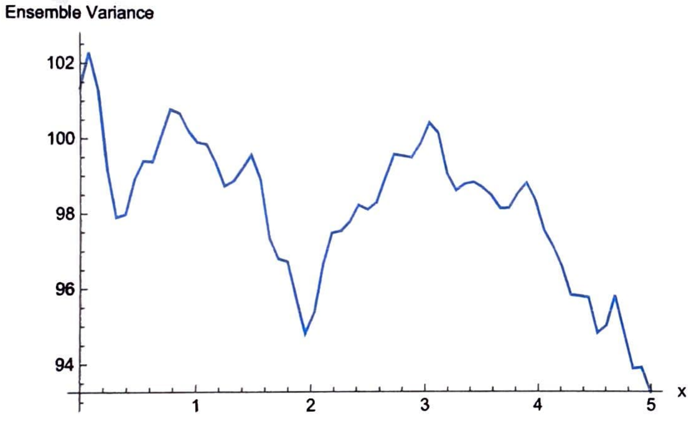
1.3

Plot of ensemble average and variance

```
Plot[EnsembleAverage[x], {x, 0, 5}, PlotPoints → 2, AxesLabel → {"x", "Ensemble Average"}]
```



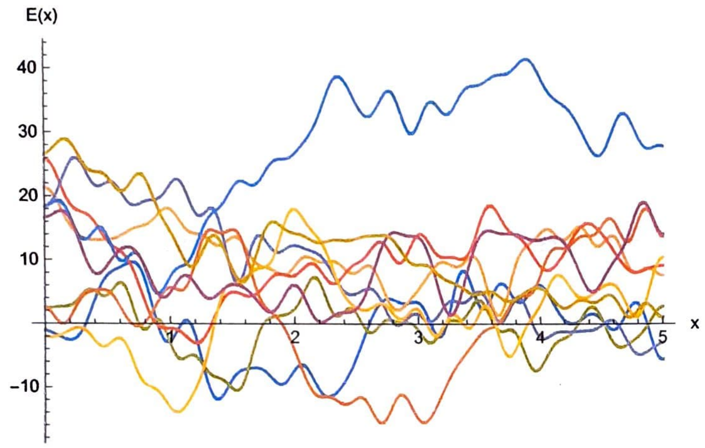

```
Plot[EnsembleVariance[x], {x, 0, 5}, PlotPoints → 2, AxesLabel → {"x", "Ensemble Variance"}]
```



1.4

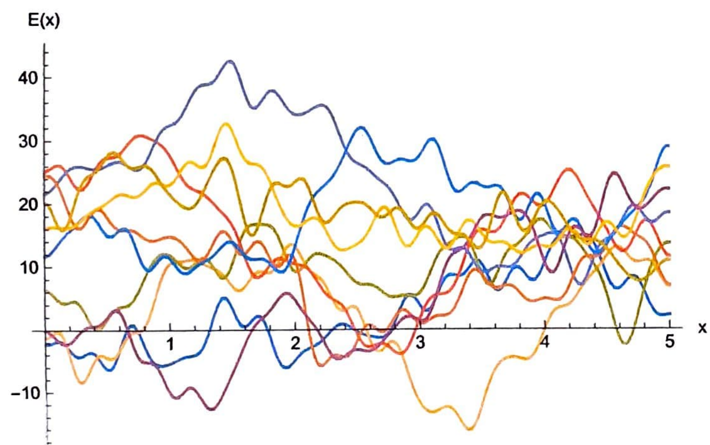
Plot of 10 realizations

```
list = {};
For[i = 0, i < 10, i = i + 1;
 AppendTo[list, Realization[i, x]]
]
Plot[list, {x, 0, 5}, AxesLabel → {"x", "E(x)"}]
```



Plot of 10 realizations

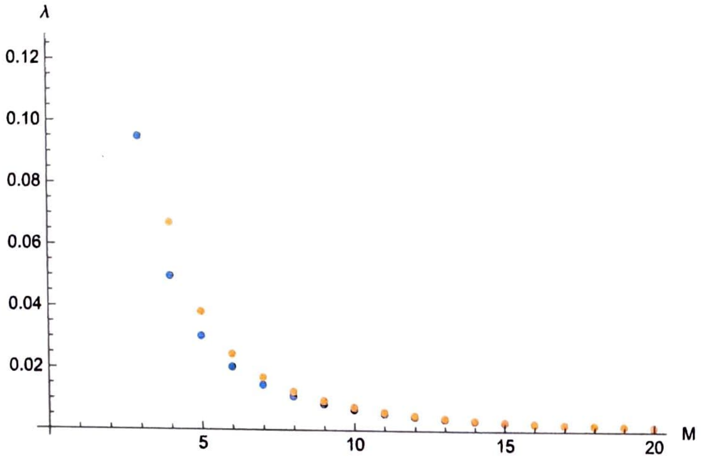
```
list = {};
For[i = 4300, i < 4310, i = i + 1;
 AppendTo[list, Realization[i, x]]
]
Plot[list, {x, 0, 5}, AxesLabel → {"x", "E(x)"}]
```



1.5

Number of terms in the KL expansion

```
ListPlot[{Table[λeven[n], {n, 1, M}], Table[λodd[n], {n, 1, M}]}, AxesLabel → {"M", "λ"}]
(* Justifying the number of terms in the KL expansion. *)
```




```
λeven[20] / λeven[1] (* Keep terms only with λ > λ_threshold etc. Here,
λ_even_threshold = 0.0016. *)
λodd[20] / λodd[1]
```

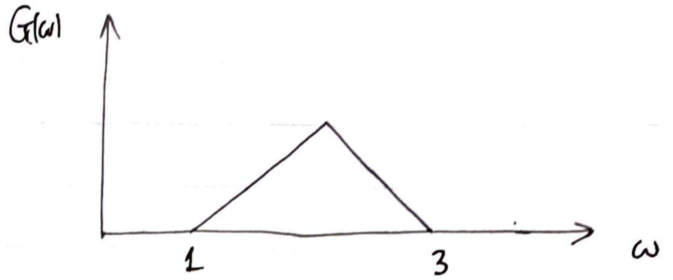
0.00159695

0.000680913

B Spectral Representation Method

Zero-mean Gaussian process: $X(t)$, $t \in [0, 10]$

Power spectrum: $G(\omega) = \begin{cases} \omega - 1 ; & 1 \leq \omega \leq 2 \\ 3 - \omega ; & 2 < \omega \leq 3 \\ 0 ; & \text{otherwise} \end{cases}$



From the above, the cutoff frequency is $\omega_u = 3$.

1. We create $R = 5000$ realizations of $X(t)$, as shown in the Mathematica notebook. We keep $M = 200$ terms.

2. We also calculate the ensemble average and variance. For example at $t = 5$, the average is 0.0107552 and the variance is 1.03731. These values would converge to 0 and 1, respectively, since X has zero mean, and from $G(\omega)$ we find that $\int G(\omega)\, d\omega = 1$, which means that $\sigma^2 = 1$. In other words, the variance of X is 1.

3. The temporal average and variance are also calculated. For example for the 2000 th realization, these are −0.000967937 and 0.968735, respectively. We can expect that these values will approach 0 and 1, even more for larger M and R.

2.1

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B: Spectral representation method

Parameters

```
ωu = 3; (* Cutoff frequency. *)
M = 200; (* Number of terms in the expansion. *)
R = 5000; (* Number of realizations. *)
```

Terms in the expansion

```
A[0] = 0;
ω[0] = 0;
Δω = ωu / M;
For[n = 0, n < M - 1, n = n + 1;
 ω[n] = n Δω;
 If[ω[n] < 1,
   G[ω_] := 0;
   A[n] = Sqrt[G[ω[n]] Δω];
  ]
  If[1 ≤ ω[n] ≤ 2,
   G[ω_] := ω - 1;
   A[n] = Sqrt[G[ω[n]] Δω];
  ]
  If[2 < ω[n] ≤ 3,
   G[ω_] := 3 - ω;
   A[n] = Sqrt[G[ω[n]] Δω];
  ]
]
```

Random variables Φ

```
For[i = 0, i < R, i = i + 1;
 Φ[i] = RandomVariate[UniformDistribution[{0, 2 Pi}], M - 1]
]
```

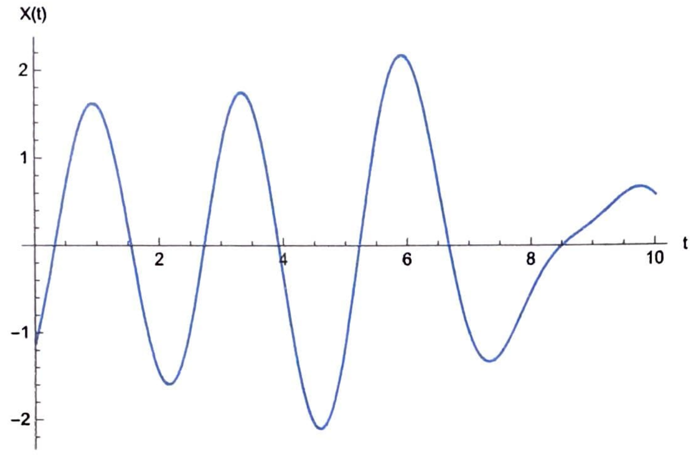
Realization

```
Realization[i_, t_] := Sqrt[2] Sum[A[n] Cos[ω[n] t + Φ[i][[n]]], {n, 1, M - 1}];
```

2.2

Example plot of a realization of X(t)

```
Plot[Realization[4578, t], {t, 0, 10}, AxesLabel → {"t", "X(t)"}]
```



Ensemble averages and variances

```
EnsembleAverage[t_] := Mean[Table[Realization[i, t], {i, 1, R}]]
EnsembleVariance[t_] := Variance[Table[Realization[i, t], {i, 1, R}]]
```

Example calculation of ensemble average and variance

```
EnsembleAverage[5]
EnsembleVariance[5]
```

```
0.00342159

1.00844
```

Temporal average and variance from a single realization

```
TempAverage[i_] := NIntegrate[Realization[i, t], {t, 0, 10}] / 10
TempVariance[i_] := NIntegrate[Realization[i, t]^2, {t, 0, 10}] / 10 -
   (NIntegrate[Realization[i, t], {t, 0, 10}] / 10)^2
```

Example calculation of temporal average and variance

```
TempAverage[2000]
TempVariance[2000]
```

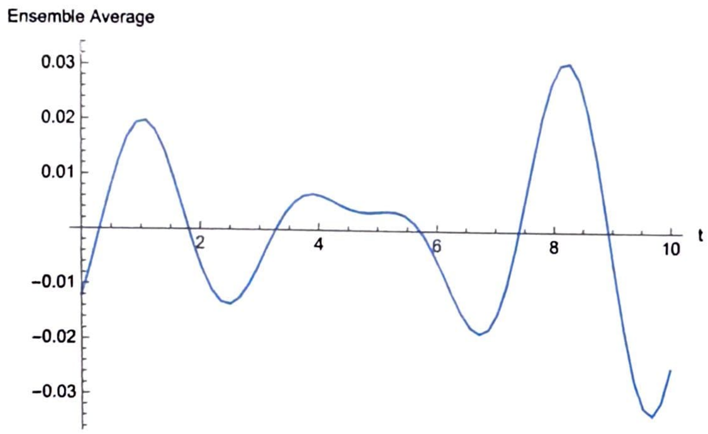
```
-0.000967937

0.968735
```

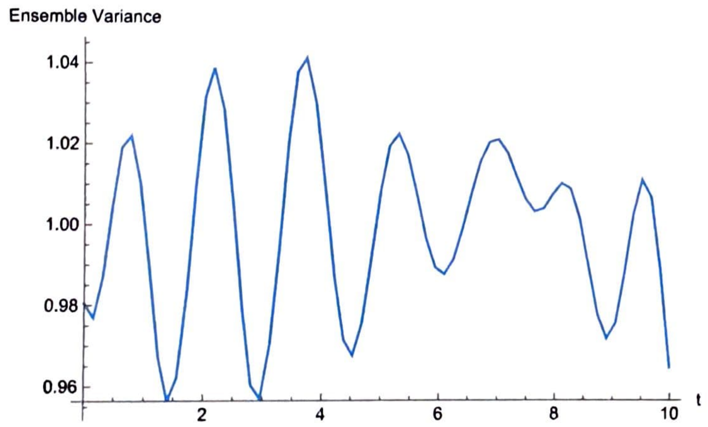
2.3

Plot of ensemble average and variance

```
Plot[EnsembleAverage[t], {t, 0, 10}, PlotPoints → 2, AxesLabel → {"t", "Ensemble Average"}]
```



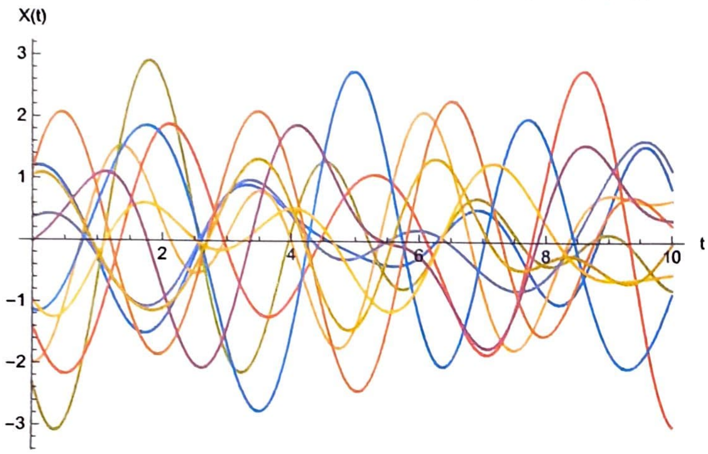

```
Plot[EnsembleVariance[t], {t, 0, 10},
 PlotPoints → 2, AxesLabel → {"t", "Ensemble Variance"}]
```



2.4

Plot of 10 realizations

```
list = {};
For[i = 0, i < 10, i = i + 1;
 AppendTo[list, Realization[i, t]]
]
Plot[list, {t, 0, 10}, AxesLabel → {"t", "X(t)"}]
```



Plot of 10 realizations

```
list = {};
For[i = 4100, i < 4110, i = i + 1;
 AppendTo[list, Realization[i, t]]
]
Plot[list, {t, 0, 10}, AxesLabel → {"t", "X(t)"}]
```