2nd Assignment: Stochastic FEM

F. I. Giasemis

Deterministic FEM

The stiffness matrix for one element

```
a = 0.05;
b = 0.05;
ν = 0.3;
EY = 10^5;
t = 0.2;
r = a / b;
ρ = (1 - ν) / 2;
μ = 3 (1 + ν) / 2;
λ = 3 (1 - 3 ν) / 2;
P = 10;

kinput =
  EY t / (12 (1 - ν^2)) {{4 / r + 4 ρ r, 0, 0, 0, 0, 0, 0, 0}, {μ, 4 r + 4 ρ / r, 0, 0, 0, 0, 0, 0},
    {-4 / r + 2 ρ r, λ, 4 / r + 4 ρ r, 0, 0, 0, 0, 0}, {-λ, 2 r - 4 ρ / r, -μ, 4 r + 4 ρ / r, 0, 0, 0, 0},
    {-2 / r - 2 ρ r, -μ, 2 / r - 4 ρ r, λ, 4 / r + 4 ρ r, 0, 0, 0},
    {-μ, -2 r - 2 ρ / r, -λ, -4 r + 2 ρ / r, μ, 4 r + 4 ρ / r, 0, 0},
    {2 / r - 4 ρ r, -λ, -2 / r - 2 ρ r, μ, -4 / r + 2 ρ r, λ, 4 / r + 4 ρ r, 0},
    {λ, -4 r + 2 ρ / r, μ, -2 r - 2 ρ / r, -λ, 2 r - 4 ρ / r, -μ, 4 r + 4 ρ / r}};
kinput // TraditionalForm
```

$$\begin{pmatrix}
9890.11 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
3571.43 & 9890.11 & 0. & 0. & 0. & 0. & 0. & 0. \\
-6043.96 & 274.725 & 9890.11 & 0. & 0. & 0. & 0. & 0. \\
-274.725 & 1098.9 & -3571.43 & 9890.11 & 0. & 0. & 0. & 0. \\
-4945.05 & -3571.43 & 1098.9 & 274.725 & 9890.11 & 0. & 0. & 0. \\
-3571.43 & -4945.05 & -274.725 & -6043.96 & 3571.43 & 9890.11 & 0. & 0. \\
1098.9 & -274.725 & -4945.05 & 3571.43 & -6043.96 & 274.725 & 9890.11 & 0. \\
274.725 & -6043.96 & 3571.43 & -4945.05 & -274.725 & 1098.9 & -3571.43 & 9890.11
\end{pmatrix}$$

```
k = kinput + Transpose[kinput] -
   DiagonalMatrix[{kinput[[1, 1]], kinput[[1, 1]], kinput[[1, 1]], kinput[[1, 1]],
     kinput[[1, 1]], kinput[[1, 1]], kinput[[1, 1]], kinput[[1, 1]]}];
k // TraditionalForm
```

$$\begin{pmatrix} 9890.11 & 3571.43 & -6043.96 & -274.725 & -4945.05 & -3571.43 & 1098.9 & 274.725 \\ 3571.43 & 9890.11 & 274.725 & 1098.9 & -3571.43 & -4945.05 & -274.725 & -6043.96 \\ -6043.96 & 274.725 & 9890.11 & -3571.43 & 1098.9 & -274.725 & -4945.05 & 3571.43 \\ -274.725 & 1098.9 & -3571.43 & 9890.11 & 274.725 & -6043.96 & 3571.43 & -4945.05 \\ -4945.05 & -3571.43 & 1098.9 & 274.725 & 9890.11 & 3571.43 & -6043.96 & -274.725 \\ -3571.43 & -4945.05 & -274.725 & -6043.96 & 3571.43 & 9890.11 & 274.725 & 1098.9 \\ 1098.9 & -274.725 & -4945.05 & 3571.43 & -6043.96 & 274.725 & 9890.11 & -3571.43 \\ 274.725 & -6043.96 & 3571.43 & -4945.05 & -274.725 & 1098.9 & -3571.43 & 9890.11 \end{pmatrix}$$

Assembling the global stiffness matrix

```
dim = (40 + 1) (10 + 1) 2;
global[i_, j_, m_, n_] := (
  kglobal = ConstantArray[0, {dim, dim}];

  kglobal[[2 i - 1, 2 i - 1]] += k[[1, 1]];
kglobal[[2 i - 1, 2 i]] += k[[1, 2]];
kglobal[[2 i - 1, 2 j - 1]] += k[[1, 3]];
kglobal[[2 i - 1, 2 j]] += k[[1, 4]];
kglobal[[2 i - 1, 2 m - 1]] += k[[1, 5]];
kglobal[[2 i - 1, 2 m]] += k[[1, 6]];
kglobal[[2 i - 1, 2 n - 1]] += k[[1, 7]];
kglobal[[2 i - 1, 2 n]] += k[[1, 8]];
kglobal[[2 i, 2 i - 1]] += k[[2, 1]];
kglobal[[2 i, 2 i]] += k[[2, 2]];
kglobal[[2 i, 2 j - 1]] += k[[2, 3]];
kglobal[[2 i, 2 j]] += k[[2, 4]];
kglobal[[2 i, 2 m - 1]] += k[[2, 5]];
kglobal[[2 i, 2 m]] += k[[2, 6]];
kglobal[[2 i, 2 n - 1]] += k[[2, 7]];
kglobal[[2 i, 2 n]] += k[[2, 8]];
kglobal[[2 j - 1, 2 i - 1]] += k[[3, 1]];
kglobal[[2 j - 1, 2 i]] += k[[3, 2]];
kglobal[[2 j - 1, 2 j - 1]] += k[[3, 3]];
kglobal[[2 j - 1, 2 j]] += k[[3, 4]];
kglobal[[2 j - 1, 2 m - 1]] += k[[3, 5]];
kglobal[[2 j - 1, 2 m]] += k[[3, 6]];
kglobal[[2 j - 1, 2 n - 1]] += k[[3, 7]];
kglobal[[2 j - 1, 2 n]] += k[[3, 8]];

  kglobal[[2 j, 2 i - 1]] += k[[4, 1]];
kglobal[[2 j, 2 i]] += k[[4, 2]];
kglobal[[2 j, 2 j - 1]] += k[[4, 3]];
kglobal[[2 j, 2 j]] += k[[4, 4]];
kglobal[[2 j, 2 m - 1]] += k[[4, 5]];
kglobal[[2 j, 2 m]] += k[[4, 6]];
```

```
kglobal[[2 j, 2 n - 1]] += k[[4, 7]];
kglobal[[2 j, 2 n]] += k[[4, 8]];

  kglobal[[2 m - 1, 2 i - 1]] += k[[5, 1]];
kglobal[[2 m - 1, 2 i]] += k[[5, 2]];
kglobal[[2 m - 1, 2 j - 1]] += k[[5, 3]];
kglobal[[2 m - 1, 2 j]] += k[[5, 4]];
kglobal[[2 m - 1, 2 m - 1]] += k[[5, 5]];
kglobal[[2 m - 1, 2 m]] += k[[5, 6]];
kglobal[[2 m - 1, 2 n - 1]] += k[[5, 7]];
kglobal[[2 m - 1, 2 n]] += k[[5, 8]];

  kglobal[[2 m, 2 i - 1]] += k[[6, 1]];
kglobal[[2 m, 2 i]] += k[[6, 2]];
kglobal[[2 m, 2 j - 1]] += k[[6, 3]];
kglobal[[2 m, 2 j]] += k[[6, 4]];
kglobal[[2 m, 2 m - 1]] += k[[6, 5]];
kglobal[[2 m, 2 m]] += k[[6, 6]];
kglobal[[2 m, 2 n - 1]] += k[[6, 7]];
kglobal[[2 m, 2 n]] += k[[6, 8]];

  kglobal[[2 n - 1, 2 i - 1]] += k[[7, 1]];
kglobal[[2 n - 1, 2 i]] += k[[7, 2]];
kglobal[[2 n - 1, 2 j - 1]] += k[[7, 3]];
kglobal[[2 n - 1, 2 j]] += k[[7, 4]];
kglobal[[2 n - 1, 2 m - 1]] += k[[7, 5]];
kglobal[[2 n - 1, 2 m]] += k[[7, 6]];
kglobal[[2 n - 1, 2 n - 1]] += k[[7, 7]];
kglobal[[2 n - 1, 2 n]] += k[[7, 8]];

  kglobal[[2 n, 2 i - 1]] += k[[8, 1]];
kglobal[[2 n, 2 i]] += k[[8, 2]];
kglobal[[2 n, 2 j - 1]] += k[[8, 3]];
kglobal[[2 n, 2 j]] += k[[8, 4]];
kglobal[[2 n, 2 m - 1]] += k[[8, 5]];
kglobal[[2 n, 2 m]] += k[[8, 6]];
kglobal[[2 n, 2 n - 1]] += k[[8, 7]];
kglobal[[2 n, 2 n]] += k[[8, 8]];
  Return[kglobal])
```

The global stiffness matrix

```
K = ConstantArray[0, {dim, dim}];
For[index = 0, index ≤ 408, index += 1,
 i = 1 + index;
 j = 2 + index;
 m = 43 + index;
 n = 42 + index;
 If[Divisible[i, 41] == False, K = K + global[i, j, m, n]]
]
```

Boundary conditions

```
fix[x_] := (
  For[i = 1, i ≤ dim, i += 1,
   K[[x, i]] = 0;
   K[[i, x]] = 0;];
  K[[x, x]] = 10^10;)
```

```
For[x = 1, x ≤ 821, x += 82,
 fix[x]]
For[x = 2, x ≤ 822, x += 82,
 fix[x]]
```

External forces

```
F = ConstantArray[0, dim];
F[[dim]] = -P;
```

Solve the system

```
U = Inverse[K].F;
```

Checking the results

```
U[[902]]
```

-0.135265

```
U[[901]]
```

0.025131

```
U[[82]]
```

-0.132377

```
U[[81]]
```

-0.0236291

```
U[[492]]
```

-0.132579

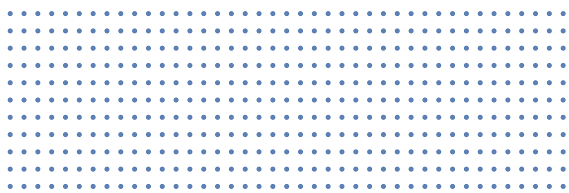
```
U[[491]]
```

-0.0000864896

Visualisation

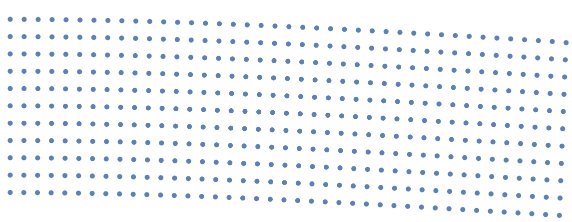
Before applying the force

```
points = {};
For[j = 1, j ≤ 11, j++,
  For[i = 1, i ≤ 41, i++,
   AppendTo[points, 0.1 {i, j} - {0.1, 0.1}];
  ]];
ListPlot[points, PlotRange → {{-.5, 4.5}, {-0.5, 1.5}}, Axes → False, AspectRatio → 1 / 2]
```



After applying the force

```
newpoints = ConstantArray[{0, 0}, 451];
For[k = 1, k ≤ 451, k++,
 newpoints[[k]] = points[[k]] + {U[[2 k - 1]], U[[2 k]]};
]
ListPlot[newpoints, PlotRange → {{-.5, 4.5}, {-0.5, 1.5}}, Axes → False, AspectRatio → 1 / 2]
```



KL expansion

Parameters from the problem

```
b = 3;
a = 2;
M = 8; (* Number of terms in the KL expansion. *)
R = 5000; (* Number of realizations. *)
```

Eigenvalues and eigenfunctions for f(x)

```
For[n = 0, n < M / 2, n = n + 1;
 sol = NSolve[{1 / b - x Tan[x a] == 0, (n - 1) Pi / a ≤ x ≤ (n - 1 / 2) Pi / a}, x];
 wodd[n] = Part[x /. sol, 1];
 λodd[n] = 2 b / (1 + wodd[n]^2 b^2);
 codd[n] = 1 / Sqrt[a + Sin[2 wodd[n] a] / (2 wodd[n])];
 φodd[n][x_] := codd[n] Cos[wodd [n] x];

 sol = NSolve[{1 / b Tan[x a] + x == 0, (n - 1 / 2) Pi / a ≤ x ≤ (n) Pi / a}, x];
 weven[n] = Part[x /. sol, 1];
 λeven[n] = 2 b / (1 + weven[n]^2 b^2);
 ceven[n] = 1 / Sqrt[a - Sin[2 weven[n] a] / (2 weven[n])];
 φeven[n][x_] := ceven[n] Sin[weven[n] x]
]
```

Random variables $\xi(\theta)$

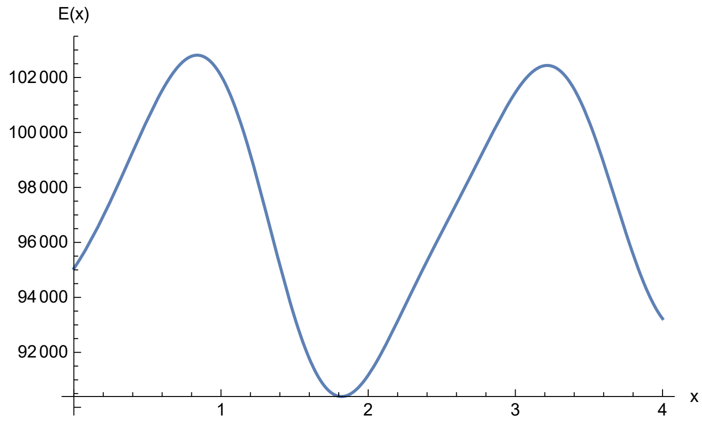
```
For[i = 0, i < R, i = i + 1;
 ξ[i] = RandomVariate[NormalDistribution[], M]
]
```

Realization of f(x) and E(x)

```
RealizationF[i_, x_] := Sum[Sqrt[λodd[n]] φodd[n][x - 2] ξ[i][[n]], {n, 1, M / 2}] +
   Sum[Sqrt[λeven[n]] φeven[n][x - 2.5] ξ[i][[M / 2 + n]], {n, 1, M / 2}];
Realization[i_, x_] := 10^5 (1 + 0.1 RealizationF[i, x]);
```

Example plot of a realization of E(x)

```
Plot[Realization[567, x], {x, 0, 4}, AxesLabel → {"x", "E(x)"}]
```



Ensemble averages and variances

```
EnsembleAverage[x_] := Mean[Table[Realization[i, x], {i, 1, R}]]
EnsembleVariance[x_] := Variance[Table[Realization[i, x], {i, 1, R}]]
```

Example calculation of ensemble average and variance

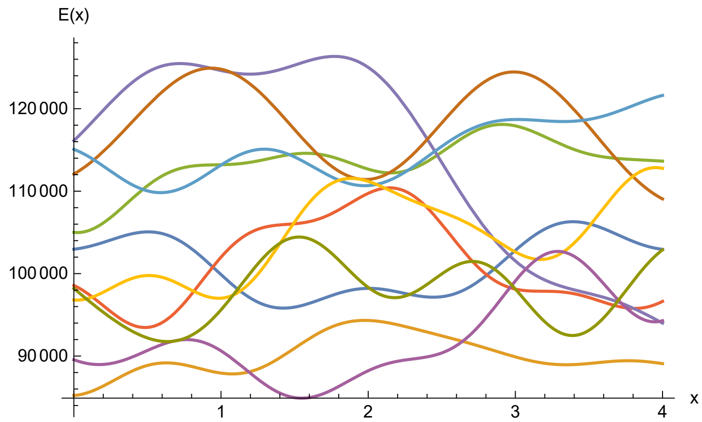
```
EnsembleAverage[2]
EnsembleVariance[2]
```

100 093.

1.10931×10^{8}

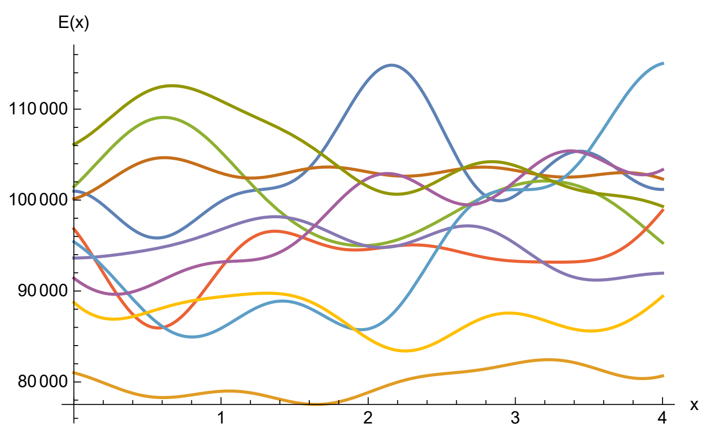
Plot of 10 realizations

```
list = {};
For[i = 0, i < 10, i = i + 1;
 AppendTo[list, Realization[i, x]]
]
Plot[list, {x, 0, 4}, AxesLabel → {"x", "E(x)"}]
```



Plot of 10 realizations

```
list = {};
For[i = 4300, i < 4310, i = i + 1;
 AppendTo[list, Realization[i, x]]
]
Plot[list, {x, 0, 4}, AxesLabel → {"x", "E(x)"}]
```



Stochastic FEM

The stiffness matrix for one element

```
aelem = 0.05;
belem = 0.05;
ν = 0.3;
t = 0.2;
r = aelem / belem;
ρ = (1 - ν) / 2;
μ = 3 (1 + ν) / 2;
λ = 3 (1 - 3 ν) / 2;

kinput = t / (12 (1 - ν^2)) {{4 / r + 4 ρ r, 0, 0, 0, 0, 0, 0, 0}, {μ, 4 r + 4 ρ / r, 0, 0, 0, 0, 0, 0},
    {-4 / r + 2 ρ r, λ, 4 / r + 4 ρ r, 0, 0, 0, 0, 0}, {-λ, 2 r - 4 ρ / r, -μ, 4 r + 4 ρ / r, 0, 0, 0, 0},
    {-2 / r - 2 ρ r, -μ, 2 / r - 4 ρ r, λ, 4 / r + 4 ρ r, 0, 0, 0},
    {-μ, -2 r - 2 ρ / r, -λ, -4 r + 2 ρ / r, μ, 4 r + 4 ρ / r, 0, 0},
    {2 / r - 4 ρ r, -λ, -2 / r - 2 ρ r, μ, -4 / r + 2 ρ r, λ, 4 / r + 4 ρ r, 0},
    {λ, -4 r + 2 ρ / r, μ, -2 r - 2 ρ / r, -λ, 2 r - 4 ρ / r, -μ, 4 r + 4 ρ / r}};
k = kinput + Transpose[kinput] - DiagonalMatrix[
    {kinput[[1, 1]], kinput[[1, 1]], kinput[[1, 1]], kinput[[1, 1]],
     kinput[[1, 1]], kinput[[1, 1]], kinput[[1, 1]], kinput[[1, 1]]}];

klocal[x_] := (
  EY = f[x];
  Return[EY k])
```

Assembling the global stiffness matrix

```
dim = (40 + 1) (10 + 1) 2;
global[i_, j_, m_, n_] := (
  x = Mod[i, 41] 2 aelem - aelem;
  kglobal = ConstantArray[0, {dim, dim}];
kglobal[[2 i - 1, 2 i - 1]] += klocal[x][[1, 1]];
kglobal[[2 i - 1, 2 i]] += klocal[x][[1, 2]];
kglobal[[2 i - 1, 2 j - 1]] += klocal[x][[1, 3]];
kglobal[[2 i - 1, 2 j]] += klocal[x][[1, 4]];
kglobal[[2 i - 1, 2 m - 1]] += klocal[x][[1, 5]];
kglobal[[2 i - 1, 2 m]] += klocal[x][[1, 6]];
kglobal[[2 i - 1, 2 n - 1]] += klocal[x][[1, 7]];
kglobal[[2 i - 1, 2 n]] += klocal[x][[1, 8]];
kglobal[[2 i, 2 i - 1]] += klocal[x][[2, 1]];
kglobal[[2 i, 2 i]] += klocal[x][[2, 2]];
kglobal[[2 i, 2 j - 1]] += klocal[x][[2, 3]];
kglobal[[2 i, 2 j]] += klocal[x][[2, 4]];
kglobal[[2 i, 2 m - 1]] += klocal[x][[2, 5]];
kglobal[[2 i, 2 m]] += klocal[x][[2, 6]];
```

```
kglobal[[2 i, 2 n - 1]] += klocal[x][[2, 7]];
kglobal[[2 i, 2 n]] += klocal[x][[2, 8]];
kglobal[[2 j - 1, 2 i - 1]] += klocal[x][[3, 1]];
kglobal[[2 j - 1, 2 i]] += klocal[x][[3, 2]];
kglobal[[2 j - 1, 2 j - 1]] += klocal[x][[3, 3]];
kglobal[[2 j - 1, 2 j]] += klocal[x][[3, 4]];
kglobal[[2 j - 1, 2 m - 1]] += klocal[x][[3, 5]];
kglobal[[2 j - 1, 2 m]] += klocal[x][[3, 6]];
kglobal[[2 j - 1, 2 n - 1]] += klocal[x][[3, 7]];
kglobal[[2 j - 1, 2 n]] += klocal[x][[3, 8]];
kglobal[[2 j, 2 i - 1]] += klocal[x][[4, 1]];
kglobal[[2 j, 2 i]] += klocal[x][[4, 2]];
kglobal[[2 j, 2 j - 1]] += klocal[x][[4, 3]];
kglobal[[2 j, 2 j]] += klocal[x][[4, 4]];
kglobal[[2 j, 2 m - 1]] += klocal[x][[4, 5]];
kglobal[[2 j, 2 m]] += klocal[x][[4, 6]];
kglobal[[2 j, 2 n - 1]] += klocal[x][[4, 7]];
kglobal[[2 j, 2 n]] += klocal[x][[4, 8]];
kglobal[[2 m - 1, 2 i - 1]] += klocal[x][[5, 1]];
kglobal[[2 m - 1, 2 i]] += klocal[x][[5, 2]];
kglobal[[2 m - 1, 2 j - 1]] += klocal[x][[5, 3]];
kglobal[[2 m - 1, 2 j]] += klocal[x][[5, 4]];
kglobal[[2 m - 1, 2 m - 1]] += klocal[x][[5, 5]];
kglobal[[2 m - 1, 2 m]] += klocal[x][[5, 6]];
kglobal[[2 m - 1, 2 n - 1]] += klocal[x][[5, 7]];
kglobal[[2 m - 1, 2 n]] += klocal[x][[5, 8]];
kglobal[[2 m, 2 i - 1]] += klocal[x][[6, 1]];
kglobal[[2 m, 2 i]] += klocal[x][[6, 2]];
kglobal[[2 m, 2 j - 1]] += klocal[x][[6, 3]];
kglobal[[2 m, 2 j]] += klocal[x][[6, 4]];
kglobal[[2 m, 2 m - 1]] += klocal[x][[6, 5]];
kglobal[[2 m, 2 m]] += klocal[x][[6, 6]];
kglobal[[2 m, 2 n - 1]] += klocal[x][[6, 7]];
kglobal[[2 m, 2 n]] += klocal[x][[6, 8]];
kglobal[[2 n - 1, 2 i - 1]] += klocal[x][[7, 1]];
kglobal[[2 n - 1, 2 i]] += klocal[x][[7, 2]];
kglobal[[2 n - 1, 2 j - 1]] += klocal[x][[7, 3]];
kglobal[[2 n - 1, 2 j]] += klocal[x][[7, 4]];
kglobal[[2 n - 1, 2 m - 1]] += klocal[x][[7, 5]];
kglobal[[2 n - 1, 2 m]] += klocal[x][[7, 6]];
kglobal[[2 n - 1, 2 n - 1]] += klocal[x][[7, 7]];
kglobal[[2 n - 1, 2 n]] += klocal[x][[7, 8]];
kglobal[[2 n, 2 i - 1]] += klocal[x][[8, 1]];
kglobal[[2 n, 2 i]] += klocal[x][[8, 2]];
kglobal[[2 n, 2 j - 1]] += klocal[x][[8, 3]];
kglobal[[2 n, 2 j]] += klocal[x][[8, 4]];
kglobal[[2 n, 2 m - 1]] += klocal[x][[8, 5]];
kglobal[[2 n, 2 m]] += klocal[x][[8, 6]];
kglobal[[2 n, 2 n - 1]] += klocal[x][[8, 7]];
kglobal[[2 n, 2 n]] += klocal[x][[8, 8]];
Return[kglobal])
```

The global stiffness matrix

```
Kglobal := (
  Kglobal = ConstantArray[0, {dim, dim}];
  For[index = 0, index ≤ 408, index += 1,
   i = 1 + index;
   j = 2 + index;
   m = 43 + index;
   n = 42 + index;
   If[Divisible[i, 41] == False, Kglobal += global[i, j, m, n]]
  ];
  Return[Kglobal])
```

Boundary conditions

```
fix[x_] := (
  For[i = 1, i ≤ dim, i += 1,
   KGLOBAL[[x, i]] = 0;
   KGLOBAL[[i, x]] = 0;];
  KGLOBAL[[x, x]] = 10^10;)
```

```
fixall := (
  For[x = 1, x ≤ 821, x += 82,
   fix[x]];
  For[x = 2, x ≤ 822, x += 82,
   fix[x]];)
```

External forces

```
F = ConstantArray[0, dim];
P = RandomVariate[NormalDistribution[10, 2], 1];
F[[dim]] = -P[[1]];
```

Solve the system

```
U[R_] := (
  F = ConstantArray[0, dim];
  P = RandomVariate[NormalDistribution[10, 2], 1];
  F[[dim]] = -P[[1]];

  f[x_] := Realization[R, x];

  KGLOBAL = Kglobal;

  fixall;

  sol = Inverse[KGLOBAL].F;

  Return[sol];)
```

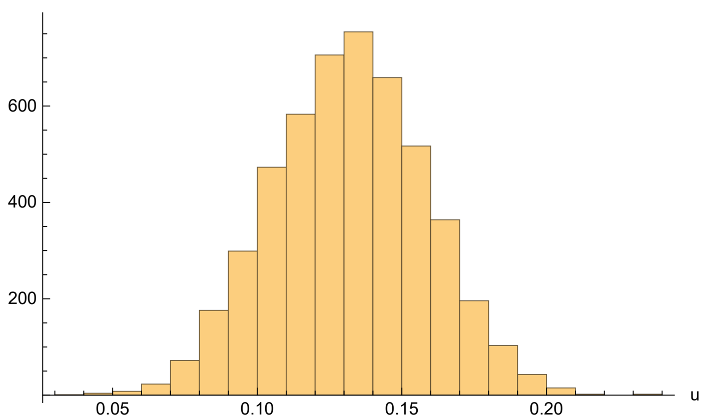
Histogram of response u

```
Quiet[u = Table[-U[real][[82]], {real, 1, 5000}];]
```

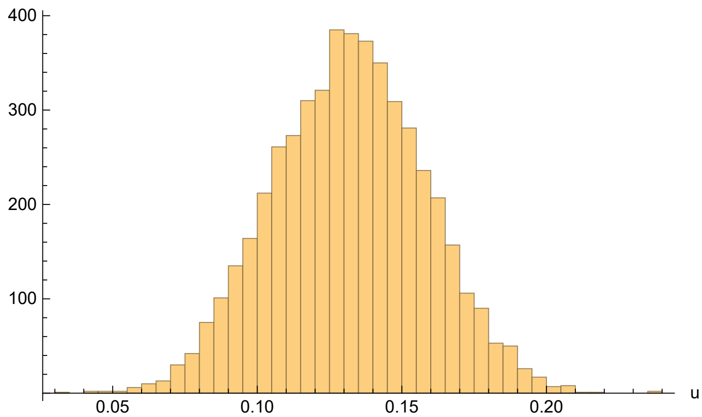
```
Export["response_data.txt", u, "CSV"]
```

```
response_data.txt
```

```
Histogram[u, {0.01}, AxesLabel → {"u", ""}]
```



```
Histogram[u, {0.005}, AxesLabel → {"u", ""}]
```



Empirical pdf

```
Histogram[u, {0.005}, "PDF", AxesLabel → {"u", ""}]
```