STOCHASTIC FEM

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1st Assignment

A) The Karhunen-Loeve Expansion

Stochastic field : $E(x) = 10(1 + f(x))$

↳ not a stochastic process, i.e. $X(t)$

We will compute the KL expansion of $f(x)$, since it is a zero mean, stationary Gaussian with unit variance.

From the autocorrelation function of f, $R_f = e^{-|\tau|/2}$, we know that the correlation length is $b = 2$.

The field is defined over $x \in [0,5]$, so after a shift of $X = 2.5$:

$$\mathcal{D}' = \mathcal{D} - X = [-2.5, 2.5]$$

Hence $a = 2.5$.

→ 1. Then using the formulas for the eigenvalues and eigenfunctions for a Gaussian process we create realizations for $f(x)$ and using $E(x) = 10(1+f(x))$ we create realizations of $E(x)$. We keep 40 terms in the expansion, i.e. $M = 40$.
The calculations are shown in the Mathematica notebook.
There were 5000 (R=5000) realizations generated and at the end 20 arbitrary realizations are shown.

→ 2. We only keep eigenvalues larger than some threshold. Here, $\lambda < 0.0001576851 \lambda_{max}$

→ 3. We also calculate the ensemble average in the Mathematica notebook.
For example, at $x = 2$, we have an average of 10.0014 and a variance of 74.4098. These values would converge to 10 and 100, respectively, since $f(x)$ mean and unit variance. (as $M \to \infty$, $R \to \infty$.)

1st Assignment: Stochastic FEM

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A: The KL expansion

Parameters from the problem

```
b = 2;
a = 2.5;
M = 40; (* Number of terms in the KL expansion. *)
R = 5000; (* Number of realizations. *)
```

Eigenvalues and eigenfunctions for f(x)

```
For[n = 0, n < M, n = n + 1;
 If[OddQ[n],
  sol = NSolve[{1/b - x Tan[x a] == 0, (n - 1) Pi/a ≤ x ≤ (n - 1/2) Pi/a}, x];
  w[n] = Part[x /. sol, 1];
  λ[n] = 2 b/(1 + w[n]^2 b^2);
  c[n] = 1/Sqrt[a + Sin[2 w[n] a]/(2 w[n])];
  φ[n][x_] := c[n] Cos[w [n] x],
  sol = NSolve[{1/b Tan[x a] + x = 0, (n - 1/2) Pi/a ≤ x ≤ (n) Pi/a}, x];
  w[n] = Part[x /. sol, 1];
  λ[n] = 2 b/(1 + w[n]^2 b^2);
  c[n] = 1/Sqrt[a - Sin[2 w[n] a]/(2 w[n])];
  φ[n][x_] := c[n] Sin[w[n] x]
 ]
]
```

Random variables $\xi(\theta)$

```
For[i = 0, i < R, i = i + 1;
 ξ[i] = RandomVariate[NormalDistribution[], M]
]
```

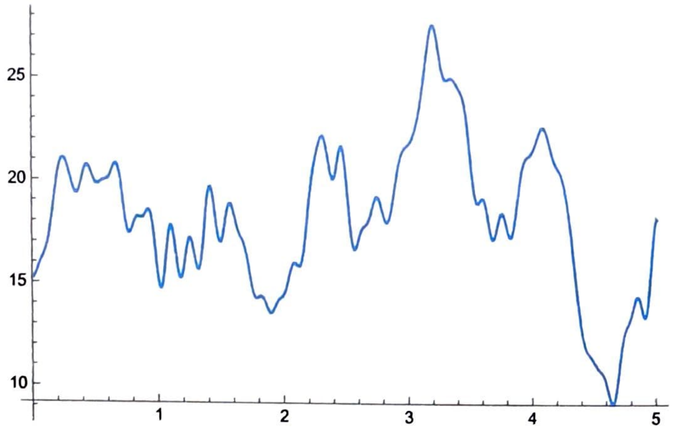
Realization of f(x) and E(x)

```
RealizationF[i_, x_] := Sum[Sqrt[λ[n]] φ[n][x - 2.5] ξ[i][[n]], {n, 1, M}];
Realization[i_, x_] := 10 (1 + RealizationF[i, x]);
```

1.2

Example plot of a realization of E(x)

```
Plot[Realization[567, x], {x, 0, 5}]
```



Ensemble averages and variances

```
EnsembleAverage[x_] := Mean[Table[Realization[i, x], {i, 1, R}]]
EnsembleVariance[x_] := Variance[Table[Realization[i, x], {i, 1, R}]]
```

Variance error

```
VarianceError[x_] := 10^2 - 10^2 Sum[λ[n] φ[n][x - 2.5]^2, {n, 1, M}]
```

Example calculation of ensemble average, variance and variance error

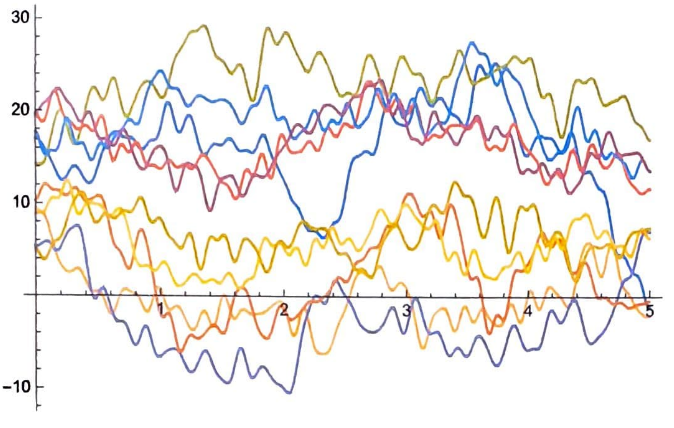
```
EnsembleAverage[2]
EnsembleVariance[2]
VarianceError[2]
```

```
9.97326

75.0572

23.155
```

1.3

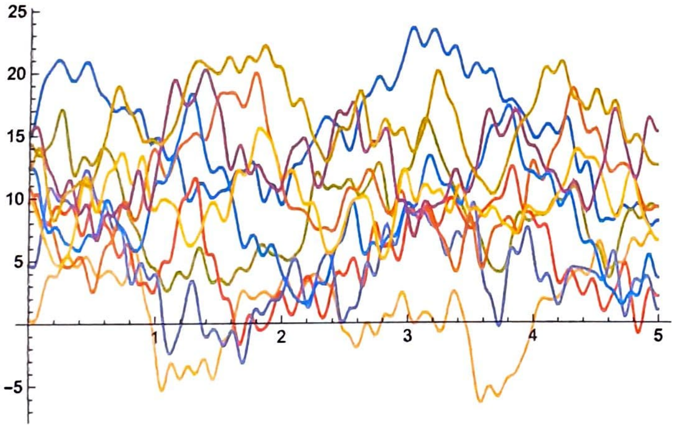
Plot of 10 realizations

```
list = {};
For[i = 0, i < 10, i = i + 1;
 AppendTo[list, Realization[i, x]]
]
Plot[list, {x, 0, 5}]
```



Plot of 10 realizations

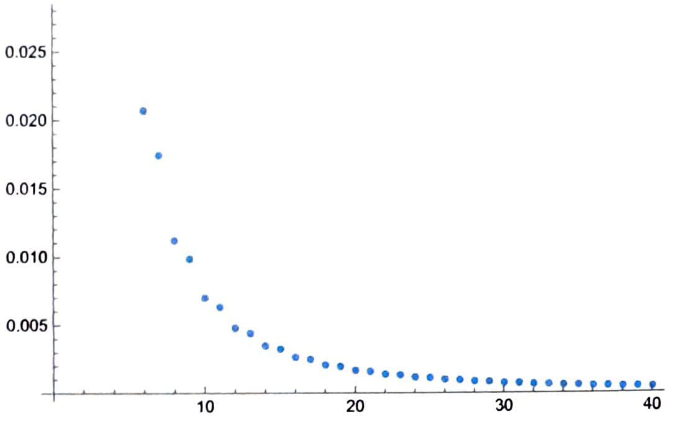
```
list = {};
For[i = 4300, i < 4310, i = i + 1;
 AppendTo[list, Realization[i, x]]
]
Plot[list, {x, 0, 5}]
```



(1.4)

Number of terms in the KL expansion

```
ListPlot[Table[λ[n], {n, 1, M}]]
```




```
λ[M] / λ[1]
0.000157685
```

B) Spectral Representation Method

Zero-mean Gaussian process : $X(t)$, $t \in [0,10]$ (sec).

Power spectrum: $G(\omega) = \begin{cases} \omega-1 , & 1 \leq \omega \leq 2 \\ 3-\omega , & 2 < \omega \leq 3 \\ 0 , & \text{otherwise} \end{cases}$

From the above the cutoff frequency is $\omega_u = 3$.

→ 1. We create $R=5000$ realizations of $X(t)$, as shown in the Mathematica notebook. We keep $M=200$ terms.

→ 2. We also calculate the ensemble average and variance. For example at $t=5$, the ensemble average is 0.00650853 and the variance 0.986566. These values would converge to 0 and 1 respectively, since X has zero mean, and from $G(\omega)$ we find that $\int G(\omega)\,d\omega = 1$, which means that $\sigma^2 = 1$. In other words, the variance of $X(t)$ is 1.

→ 3. The temporal average and variance are also calculated. For example for the 4000 th realization, these are -0.00159382 and 1.14668, respectively. We can expect that these values would approach 0 and 1 even more for larger and larger M, R.

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B: Spectral representation method

Parameters

```
ωu = 3; (* Cutoff frequency. *)
M = 200; (* Number of terms in the expansion. *)
R = 5000; (* Number of realizations. *)
```

Terms in the expansion

```
A[0] = 0;
ω[0] = 0;
Δω = ωu / M;
For[n = 0, n < M - 1, n = n + 1;
  ω[n] = n Δω;
  If[ω[n] < 1,
    G[ω_] := 0;
    A[n] = Sqrt[G[ω[n]] Δω];
   ]
   If[1 ≤ ω[n] ≤ 2,
    G[ω_] := ω - 1;
    A[n] = Sqrt[G[ω[n]] Δω];
   ]
   If[2 < ω[n] ≤ 3,
    G[ω_] := 3 - ω;
    A[n] = Sqrt[G[ω[n]] Δω];
   ]
]
```

Random variables Φ

```
For[i = 0, i < R, i = i + 1;
 Φ[i] = RandomVariate[UniformDistribution[{0, 2 Pi}], M - 1]
]
```

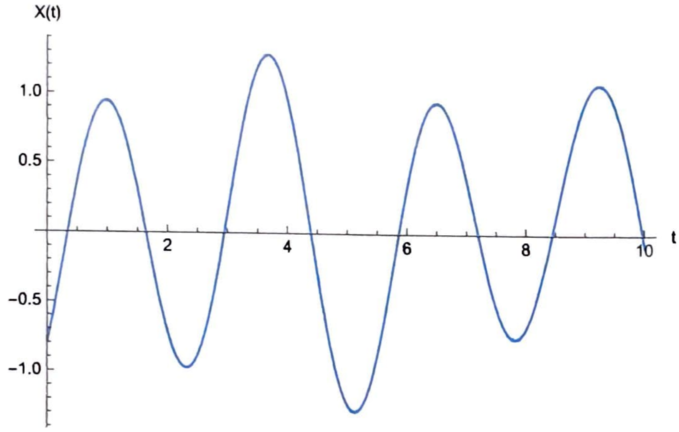
Realization

```
Realization[i_, t_] := Sqrt[2] Sum[A[n] Cos[ω[n] t + Φ[i][[n]]], {n, 1, M - 1}];
```

2.2

Example plot of a realization of X(t)

```
Plot[Realization[4578, t], {t, 0, 10}, AxesLabel → {"t", "X(t)"}]
```



Ensemble averages and variances

```
EnsembleAverage[t_] := Mean[Table[Realization[i, t], {i, 1, R}]]
EnsembleVariance[t_] := Variance[Table[Realization[i, t], {i, 1, R}]]
```

Example calculation of ensemble average and variance

```
EnsembleAverage[5]
EnsembleVariance[5]
```

```
0.00650853

0.986566
```

Temporal average and variance from a single realization

```
TempAverage[i_] := NIntegrate[Realization[i, t], {t, 0, 10}] / 10
TempVariance[i_] := NIntegrate[Realization[i, t]^2, {t, 0, 10}] / 10 -
  (NIntegrate[Realization[i, t], {t, 0, 10}] / 10)^2
```

Example calculation of temporal average and variance

```
TempAverage[4000]
TempVariance[4000]
```

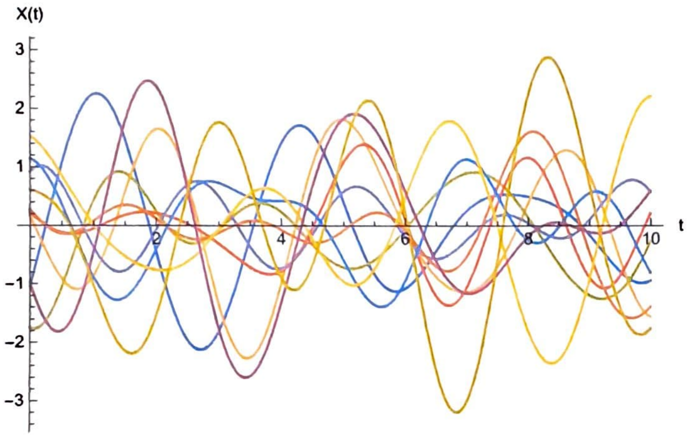
```
-0.00159382

1.14668
```

(2.3)

Plot of 10 realizations

```
list = {};
For[i = 0, i < 10, i = i + 1;
 AppendTo[list, Realization[i, t]]
]
Plot[list, {t, 0, 10}, AxesLabel → {"t", "X(t)"}]
```



Plot of 10 realizations

```
list = {};
For[i = 4100, i < 4110, i = i + 1;
 AppendTo[list, Realization[i, t]]
]
Plot[list, {t, 0, 10}, AxesLabel → {"t", "X(t)"}]
```