chaos

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CO24

1 Introduction

This practical investigates the solution of an interesting set of differential equations using the Runge-Kutta method. The solutions, which show apparently chaotic behaviour, are of a type with applications in several areas of physics.

You cannot do this practical if you have already done CO31 — *Structure of white dwarf stars*.

2 Physical description

In another problem (CO23 — *solitons*) we use a particular non-linear equation to illustrate the maintenance of coherent structures and even the emergence of coherent structures from a disordered state. Here, by contrast, we illustrate generation of chaos, which is being actively researched in theoretical physics, applied mathematics and fluid mechanics. Many seemingly complicated systems with many apparent degrees of freedom can be reduced to much simpler systems with only a few real degrees of freedom. Lorenz generated a now famous set of equations illustrating chaotic behaviour and a 'strange attractor', which we will investigate further below. For more detail on the historical background and physical motivation of these equations, see [1] and [3], whereas [4] is a good introductory text on chaos.

The Lorenz equations for the time dependent variables y_1, y_2 and y_3 are

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = a(y_2 - y_1) \tag{1}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = ry_1 - y_2 - y_1y_3 \tag{2}$$

$$\frac{\mathrm{d}y_3}{\mathrm{d}t} = y_1y_2 - by_3 \tag{3}$$

The set of equations has stationary solutions, where the LHS are zero, at the points

$$y_1 = y_2 = \pm\sqrt{b(r-1)} \qquad y_3 = (r-1) \qquad y_1 = y_2 = y_3 = 0$$

but the approach to these solutions from an arbitrary starting point depends in a remarkable way on the value of r.

(a) If $r < 1$ the only real stationary point is the origin.

(b) If $r > 1$ all three stationary points exist, but the origin is unstable, so you will never reach it unless you start exactly on it. The manner of approach to the other two solutions depends on the value of r. The system becomes less and less certain of which one it has chosen as r is increased, until for r greater than about 24 it will never converge on either, but will wander for ever between their neighbourhoods. This is the behaviour referred to as a *strange attractor*; it is bounded but non-periodic.

The system of equations 1–3 can be used to illustrate why weather forecasting is a difficult problem. The equations of motion of the atmosphere are far more complex, but they show a qualitatively similar behaviour, and in that case we are interested in the detailed track (e.g. the variations of temperature and pressure with time).

3 Numerical approach

There is an extensive literature on the numerical solution of sets of first order differential equations; see for example [5]. In that book the authors remark that Runge-Kutta methods nearly always work, but are not usually fastest; they will be the first choice for a new problem or where one has no reason to expend much time on efficiency.

One well-known Runge-Kutta algorithm, derived in full in [5], can be explained as follows. We can express equations 1–3 as

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = f_i(y_1, y_2, y_3) \tag{4}$$

Starting from the values of y_i at some time t we compute the values of y_i at a slightly later time $t + \delta t$. By repeated application of this process we obtain solutions at times $t + m\delta t$ (m an integer). Each step is identical. We let $y_{i,0}$ be the values of y_i at the beginning of the step, say at time t, and $y_{i,4}$ be the values at the end of the step, i.e. at time $t + \delta t$.

We require several intermediate values of the f_i's between t and $t + \delta t$. All components of y_i and f_i must be calculated at each intermediate point; otherwise the form is as in the first year problem D2 part B. We will speak of $y_{i,n}$, $f_{i,n}$, where at the nth intermediate point in our calculation of one step we evaluate the set of $f_{i,n}$'s from the set of $y_{i,n}$'s.

Calculate the following:

(a) $f_{i,0}$ using the initial values $y_{i,0}$

(b) $f_{i,1}$ using $y_{i,1} = y_{i,0} + \frac{f_{i,0}\delta t}{2}$

(c) $f_{i,2}$ using $y_{i,2} = y_{i,0} + \frac{f_{i,1}\delta t}{2}$

(d) $f_{i,3}$ using $y_{i,3} = y_{i,0} + f_{i,2}\delta t$ N.B. no 2!

The step is then completed using:

$$y_{i,4} = y_{i,0} + \frac{(f_{i,0} + 2f_{i,1} + 2f_{i,2} + f_{i,3})\delta t}{6}$$

The values $y_{i,4}$ then become $y_{i,0}$ in the next step.

4 Computation

Write a program to implement the algorithm of section 3 using equations 1–3 as the problem to be solved. Suitable values are $a = 10$, $b = 8/3$ and $\delta t = 0.05$. Your program should ask for a value of n, the number of steps; of y_1, y_2 and y_3, the starting point, and of r. Your program should be able to plot the variables against time, or against each other. Plots of y_2 against y_3 and of y_1 against t are of particular interest. Try a range of values of r covering all the regimes.

▶ **Discuss your results with a demonstrator.**

To illustrate the 'weather forecasting' phenomenon, try the case $r = 28$ and plot y_1 against t for some initial point, e.g. $(4, 5, 6)$. Then repeat with a small 'measurement error' e.g. $(4.01, 5.01, 6.01)$.

▶ **Comment on your results.**

▶ **You now have a program that can easily be adapted to integrate other sets of equations. Look in the references (e.g. [4], [6]) for suitable examples, and discuss this and other possibilities with a demonstrator.**

This practical is assessed via a report (refer to AD34 — *the art of scientific report writing*). When you have written your report, take it to a demonstrator for marking when the lab is open (Monday and Tuesday 1000–1700 for all of Michaelmas Term and weeks 3–4 in Trinity Term — marking is not possible in Hilary Term). Consider including the following in your write-up:

The relative advantages and disadvantages of the RK4 method, its mathematical origin, its associated global and local errors and its suitability in solving a system of a large number of coupled equations. How might such a system be solved? You may wish to consult our short option on numerical methods[7].

Be clear on what chaos is and think about what the quantities $y1, y2$ and $y3$ represent. If you were given a set of coupled differential equations, how would you test to see whether they were chaotic?

Don't forget that you will also need to keep good records of your progress during the practical, usually by a combination of commenting your code and notes in your logbook.

References

[1] J. Gleick, *Chaos: Making a New Science*, Vintage, 1997.

[2] I. M. Stewart, *Does God Play Dice?: The New Mathematics of Chaos*, 2nd edition, Penguin, 1997.

[3] N. Hall (editor), *The New Scientist Guide to Chaos*, Penguin, 1992.

[4] G. L. Baker, J. P. Gollub, *Chaotic Dynamics: An Introduction*, 2nd edition, CUP, 1996. doi: 10.1017/CBO9781139170864

[5] W. H. Press et al. *Numerical Recipes: the Art of Scientific Computing*, Cambridge University Press (2007).

[6] P. G. Drazin, *Nonlinear Systems*, CUP, 1992. doi: 10.1017/CBO9781139172455

[7] A. O'Hare, *Numerical Methods for Physicists*, Oxford Physics, 2005[1].

[1] `http://www-teaching.physics.ox.ac.uk/computing/NumericalMethods/nummethods.html`

