Multiphase flows
Lecture 2

Equation of motion of a single particle, forces on individual particles

In this lecture we will:

- Analyze the equation of motion of an individual particle
- Identify and discuss forces acting on individual particles
- Discuss specific features related to bubbles

But first:

Let us (re)visit the concepts of Eulerian and Lagrangian frames of reference – fundamental to understand continuum mechanics (including fluid mechanics)

Lagrangian framework: evolution of properties associated with the $i-th$ particle (X, V, R *(radius)* – Lagrangian coordinates)

$$\frac{\mathrm{d}\boldsymbol{X}_{(i)}}{dt} = \boldsymbol{V}_{(i)} \qquad \frac{\mathrm{d}\boldsymbol{V}_{(i)}}{dt} = \boldsymbol{A}_{(i)} \qquad \frac{\mathrm{d}R_{(i)}}{dt} = \Theta_{(i)}$$

$\boldsymbol{A}_{\boldsymbol{i}}$ – acceleration experienced by the $i-th$ particle

$\Theta_{(i)}$ - rate of change of radius due to interphase mass transfer (e.g. change of phase)

Lagrangian vs. Eulerian reference frames

Eulerian reference frame

$$\vec{u}(\vec{x}, t)$$

Eulerian velocity field at location $\vec{x}$ and time t

Lagrangian reference frame

$$\vec{X}\left(\overrightarrow{X_0}, t\right)$$

Position of element $\overrightarrow{X_0}$ at time t

There is something fundamental we have to learn and adopt here

Communication between Lagrangian and Eulerian frameworks

Essence of everything we do in continuum mechanics

The property associated with a fluid particle at a point is the same as the (field) property at a point.

The *Lagrangian (or material)* property associated with a fluid particle at a point is the same as the *Eulerian (or spatial)* property evaluated at the point.

Essential: Lagrangian displacement field

$$\boxed{\vec{x} = \vec{X}(\vec{X}_o, t)}$$

X_0: fluid element

$\boldsymbol{x}$: "action" that displaces the fluid element X_0

How it works: take the density

Lagrangian

Eulerian

$$R(\vec{X}_o, t) = \rho(\underbrace{\vec{X}[\vec{X}_o, t]}_{\vec{x}}, t)$$

Density of the fluid element (fluid particle) X_0

…or the velocity

Eulerian velocity field

$$u(\underbrace{\vec{X}(\overrightarrow{X_0},t)}_{\vec{x}},t) = \frac{\partial \vec{X}}{\partial t}(\overrightarrow{X_0},t)$$

Now we can study:

Equation of motion of a single particle

- Assumptions
- Some hints on the derivation procedure
- Analysis of the forces
- Differences between different types of particles

Equation of motion of a single particle (Maxey and Riley, 1983)

Index p – particle
Index f - fluid

Action of the fluid on the particle surface

$$m_P \frac{d\boldsymbol{u}_P}{dt} = \sum \boldsymbol{F}_i \Longleftrightarrow m_P \frac{d\boldsymbol{u}_P}{dt} = -\int_{S_P} \sigma_{fij} n_{pj} \mathrm{d}S + m_P \boldsymbol{g}$$

$$m_P \frac{d\boldsymbol{u}_P}{dt} = \sum \boldsymbol{F}_i \Longleftrightarrow m_P \frac{du_{P,i}}{dt} = \oint_S \underbrace{\left[-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]}_{\text{Stress tensor of the fluid phase}} n_j dS + m_P g_i$$

Stress tensor of the fluid phase
(to be evaluated on the surface of the sphere)

Fundamental assumptions in the process of derivation:

- Incompressible flow
- Translational motion
- Unbounded domain - no influence from walls
- Small sphere – smaller than the smallest length scale of the flow
- $Re_P \ll 1$ – creeping flow (defined with the particle Reynolds number)

$$Re_P = \frac{\rho_f \left(u_f - u_P\right) d_{\mathrm{p}}}{\mu_f}$$

Some indication on a possible way to do the derivation - Decomposition of the stress tensor

$$-\int_{S_p} \sigma_{fij} n_{1j} dS = \underbrace{\int_{S_p} \overset{o}{\sigma}_{fij}\, n_{2j} dS}_{\text{Term 1}} + \underbrace{\int_{S_p} \delta\sigma_{fij} n_{2j} dS}_{\text{Term 2}}$$

Term 1- Stress tensor on a surface of a particle (if the particle were not present)

Term 2 – Perturbation (due to the presence of a particle)

Result – identification of different forces

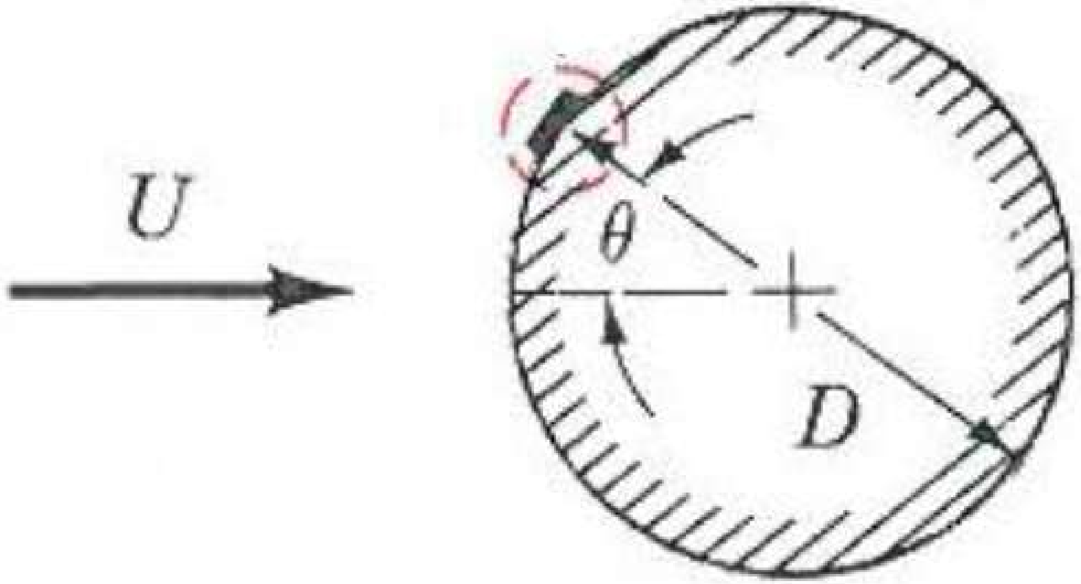
- Drag force
- Lift force
- History force
- Added mass force
- Pressure gradient force
- Force due to Brownian motion
- …

Important to have in mind:

- Analytical solution available only for the Stokes regime (creeping flow)
- Consideration of heat and mass transfer requires solution of two additional PDE (e.g. droplet diameter and droplet temperature)
- What happens for higher particle Re numbers?

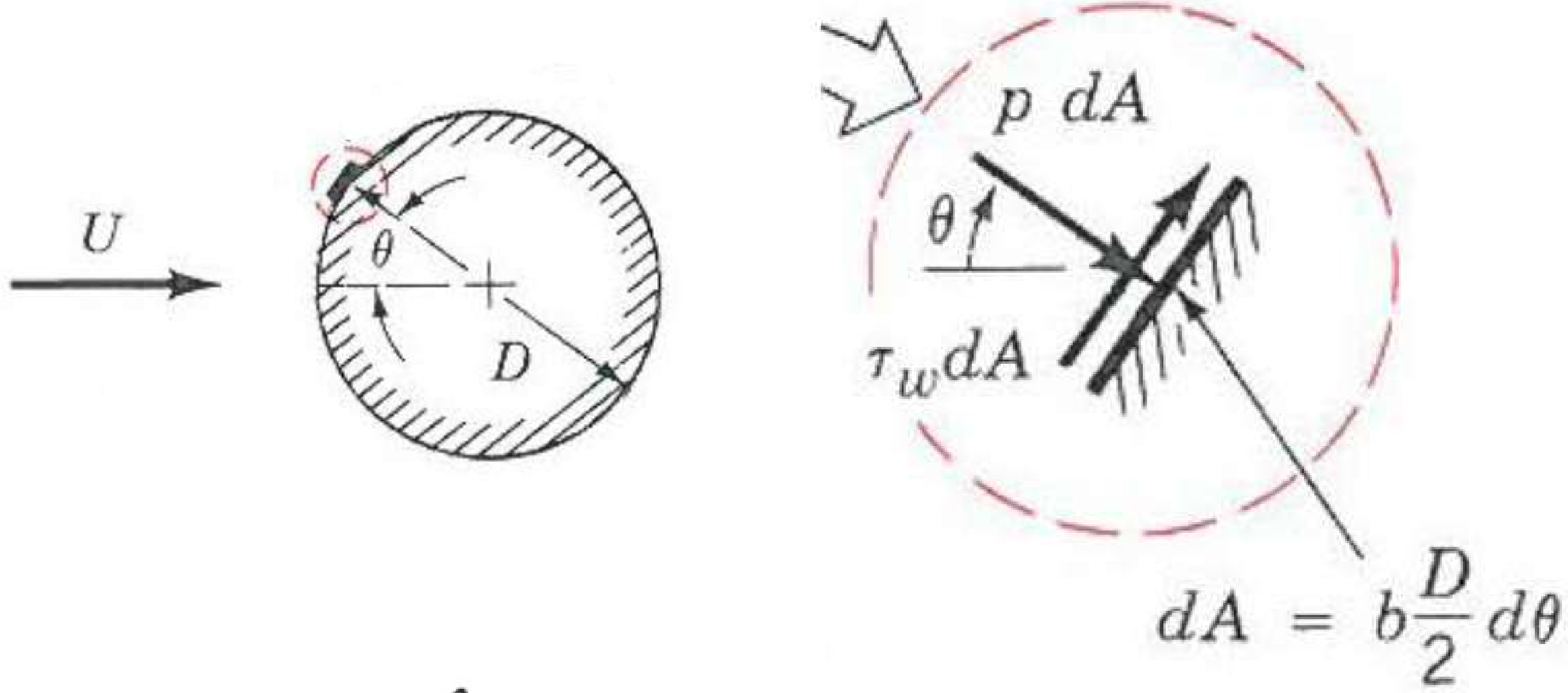
Fluid – particle interaction force (normal and tangential components)

Let us get the forces in *any chosen two* directions



Munson et al., 2010
Van der Akker, 2009

Fluid – particle interaction force (look in more detail

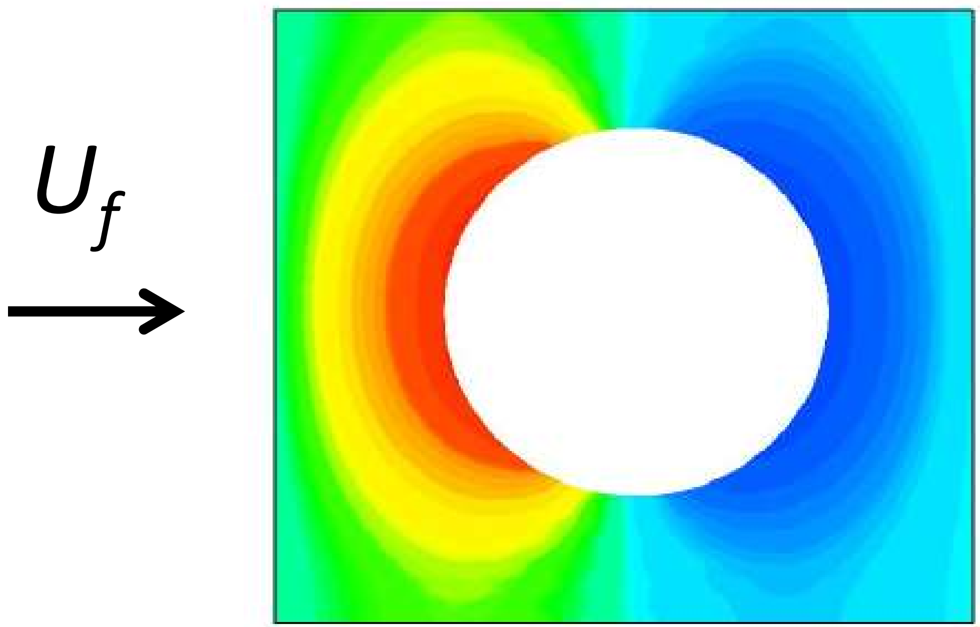


$$\int dF_x = \int p \cos\theta \, dA + \int \tau_w \sin\theta \, dA$$ → **Drag**

$$\int dF_y = -\int p \sin\theta \, dA + \int \tau_w \cos\theta \, dA$$ → **Lift**

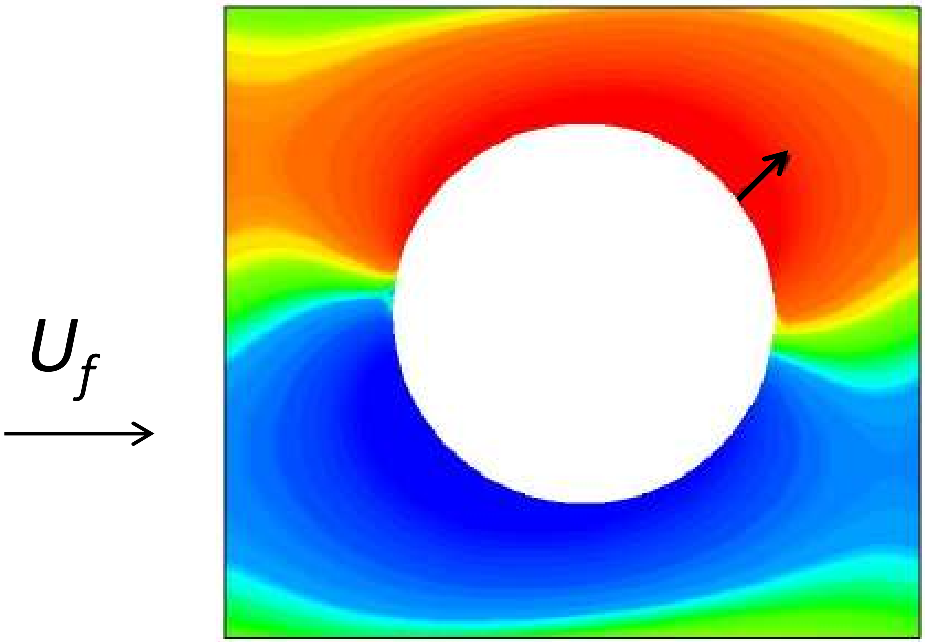
Fluid – particle interaction force: interesting situations

Distribution of pressure – flow over ***a stationary particle***

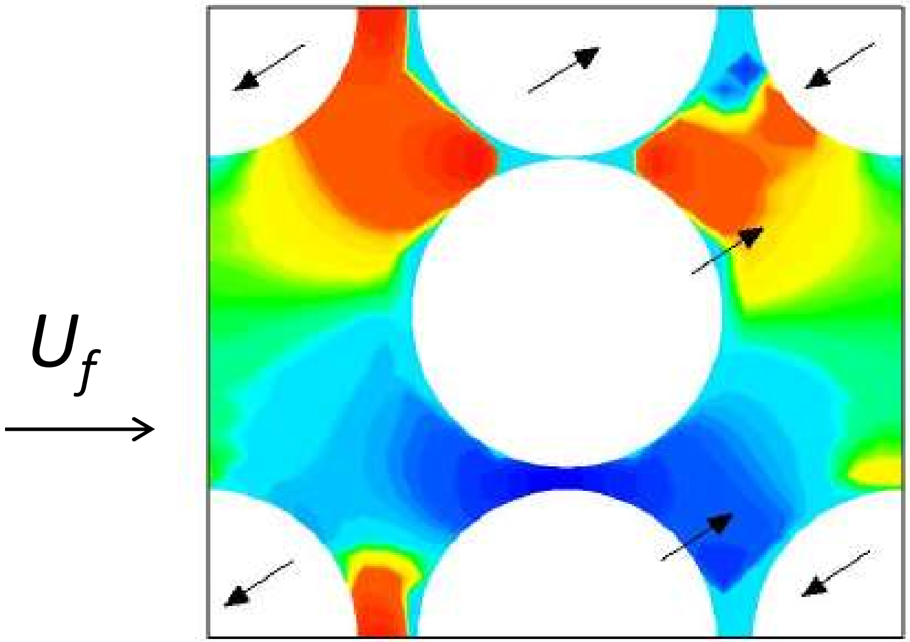


What happens in other cases (contours of pressure are shown)?

A particle **moving in a different direction** than the mean fluid flow



Collection of particles **moving in a different direction** than the mean fluid flow



Drag force

For highly viscous flows: viscous drag (Stokes law) – only this one an analytical solution

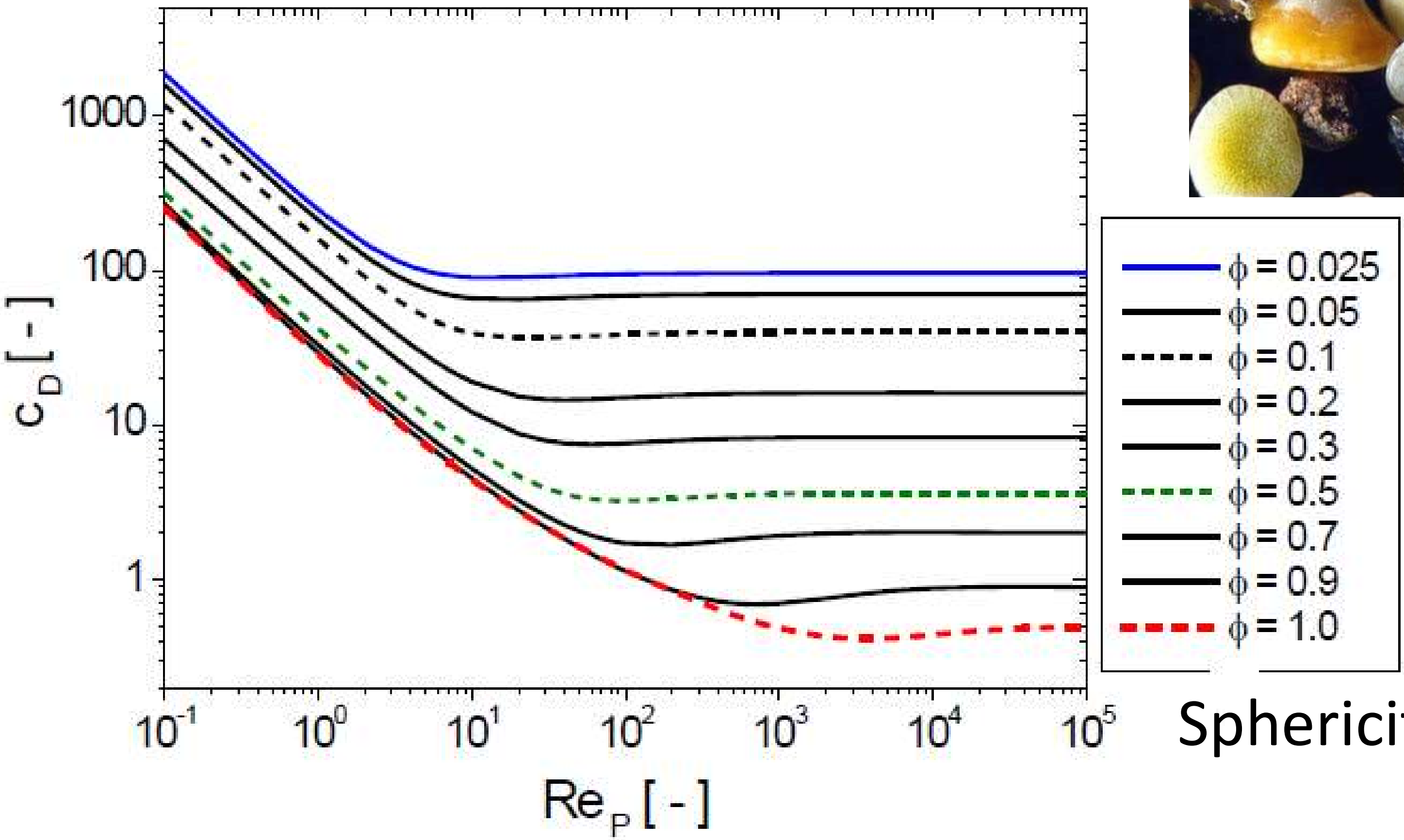
$$\boldsymbol{F_D} = 3\pi\,\mu_f\,d_P\left(\boldsymbol{u}_f - \boldsymbol{u}_\mathrm{P}\right)$$

- Form drag: $1/3\ FD$
- Skin (friction) drag: $2/3\ FD$

What else can affect the drag force? See next slides.

Drag force - Effect of particle shape?

C_D: the drag force coefficient (to be explained later)

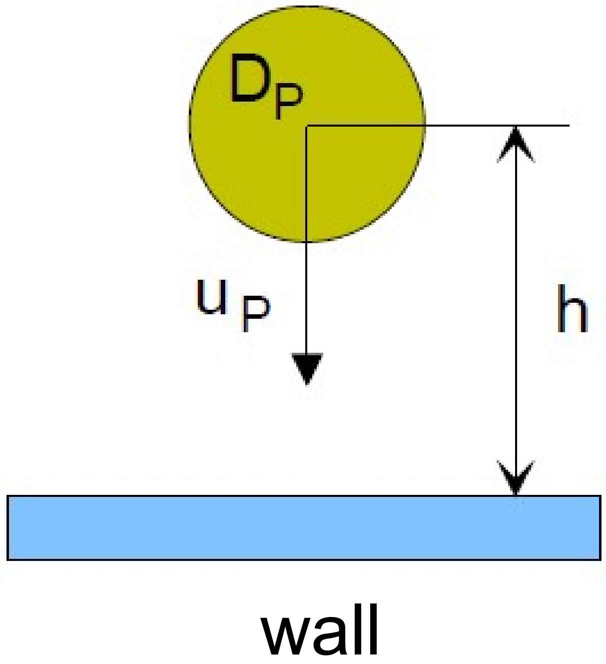


Sphericity

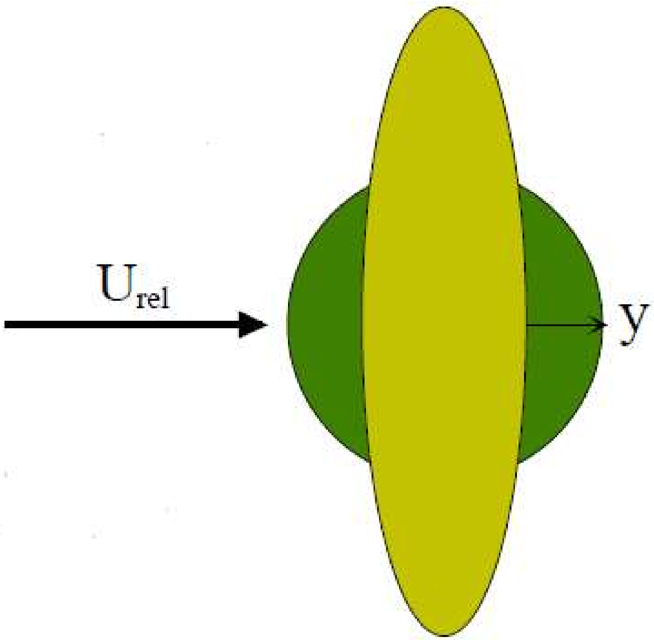
Drag force - Effect of wall?

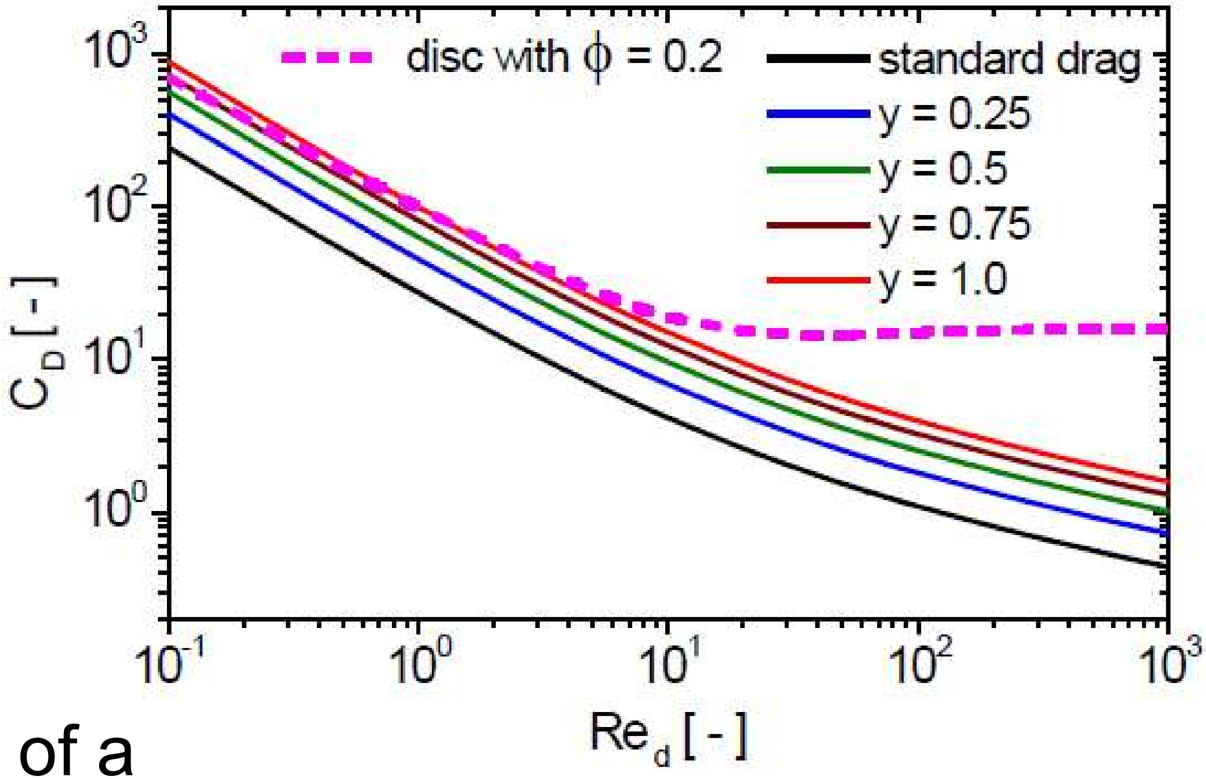
Drag force increases – introduction of correction coefficients in modelling

$$\frac{C_D}{C_{D,Stokes}} = 1 + const\,\frac{d_p}{h}$$



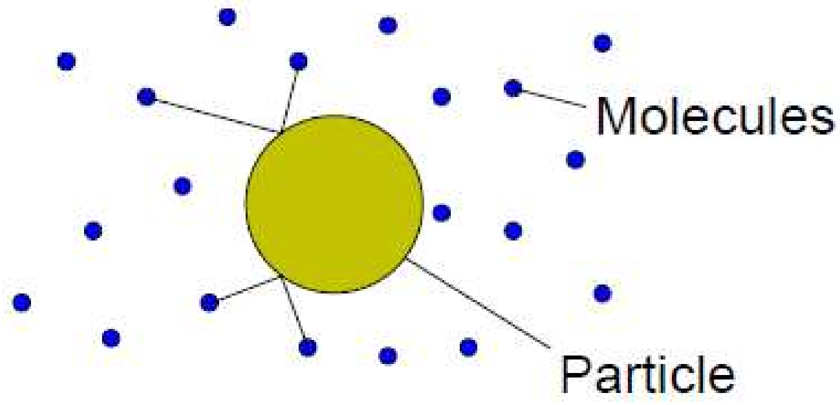
Drag force – Oscillating and/or distorted particles





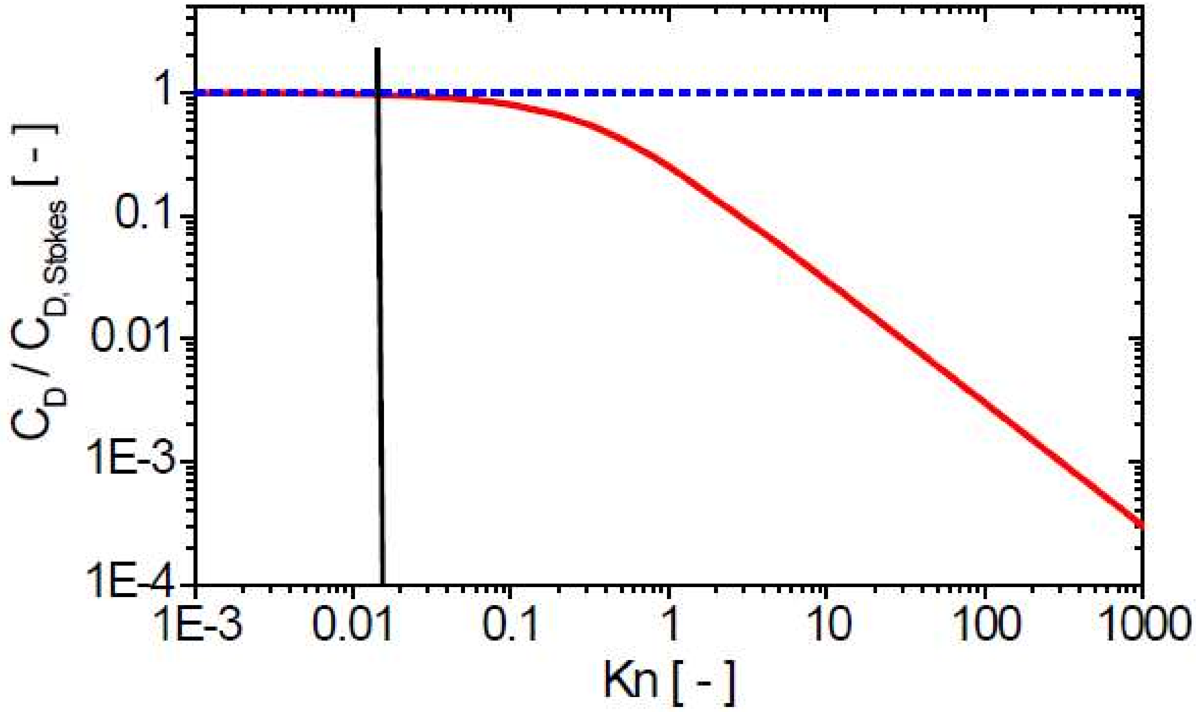
C_D: function of the form of a droplet

Drag force – what if the surrounding flow is not a continuum?



A particle "sees" individual molecules

Kn – particle Knudsen number (particle diameter/mean free path)



Transverse forces can come from:

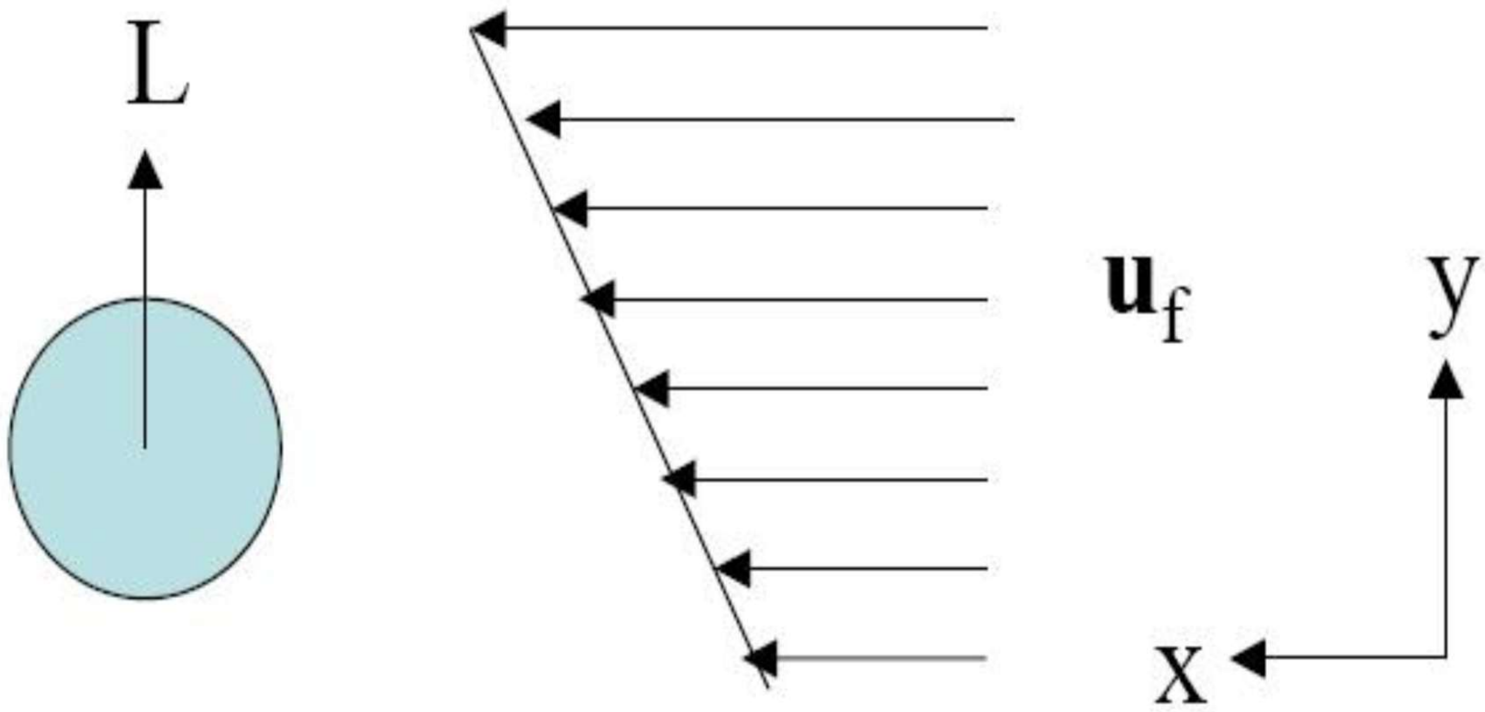
if the flow is non uniform – presence of a velocity gradient

if the particle is rotating (velocity gradient, collisions)

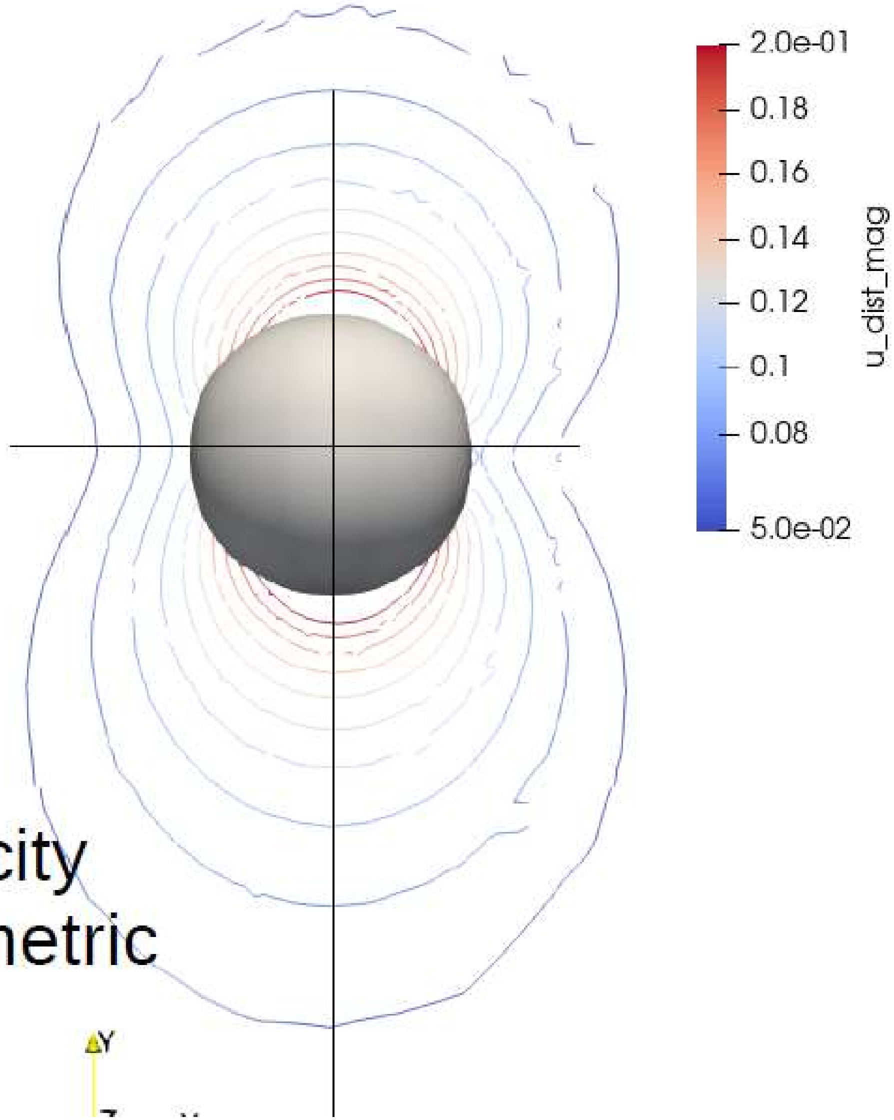
if the particle moves in the vicinity of a wall

Transverse forces

Saffmann force – due to presence of a velocity gradient (shear)



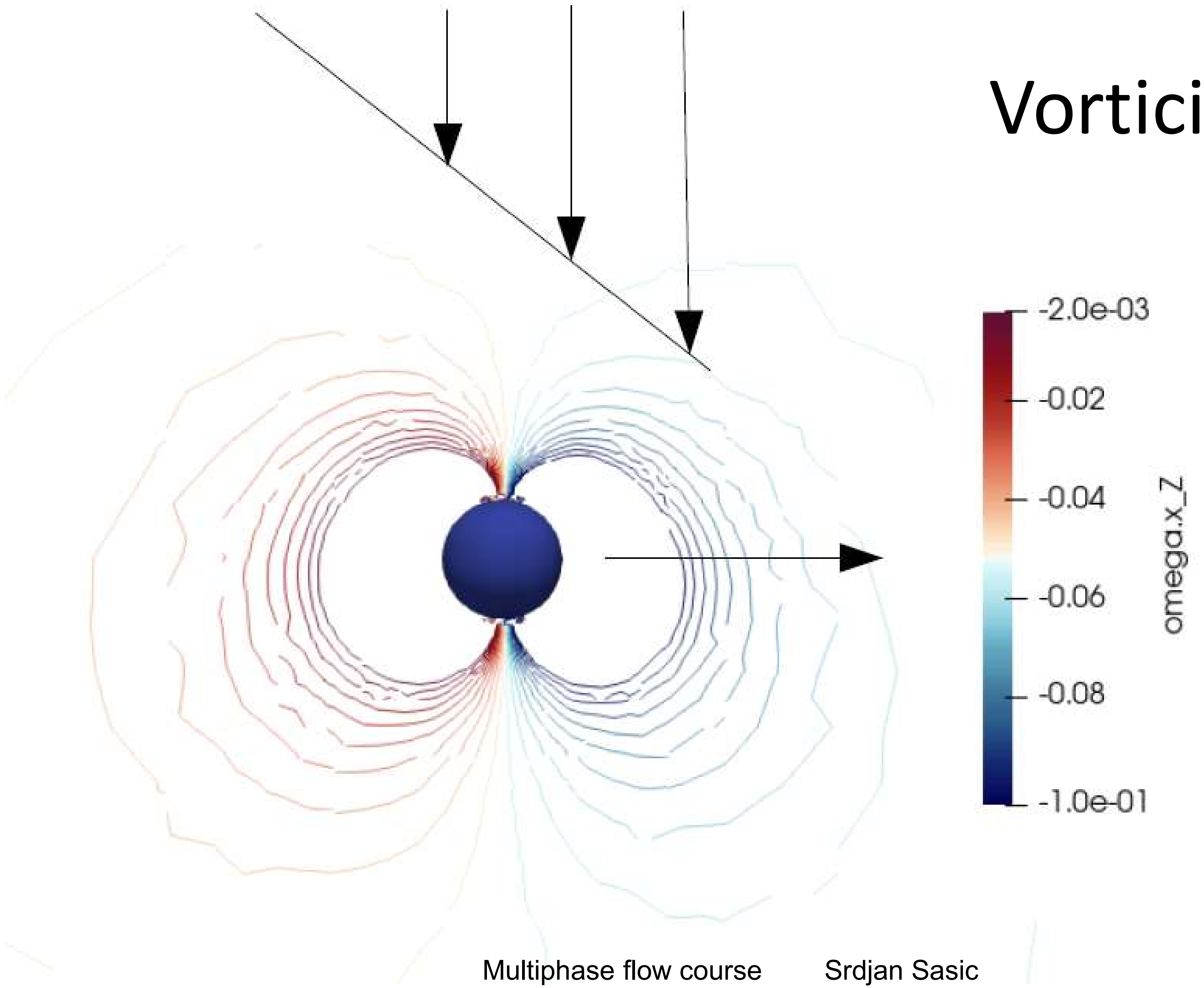
Why do we have a force if we have such a disturbance velocity field?



Disturbance velocity field almost symmetric

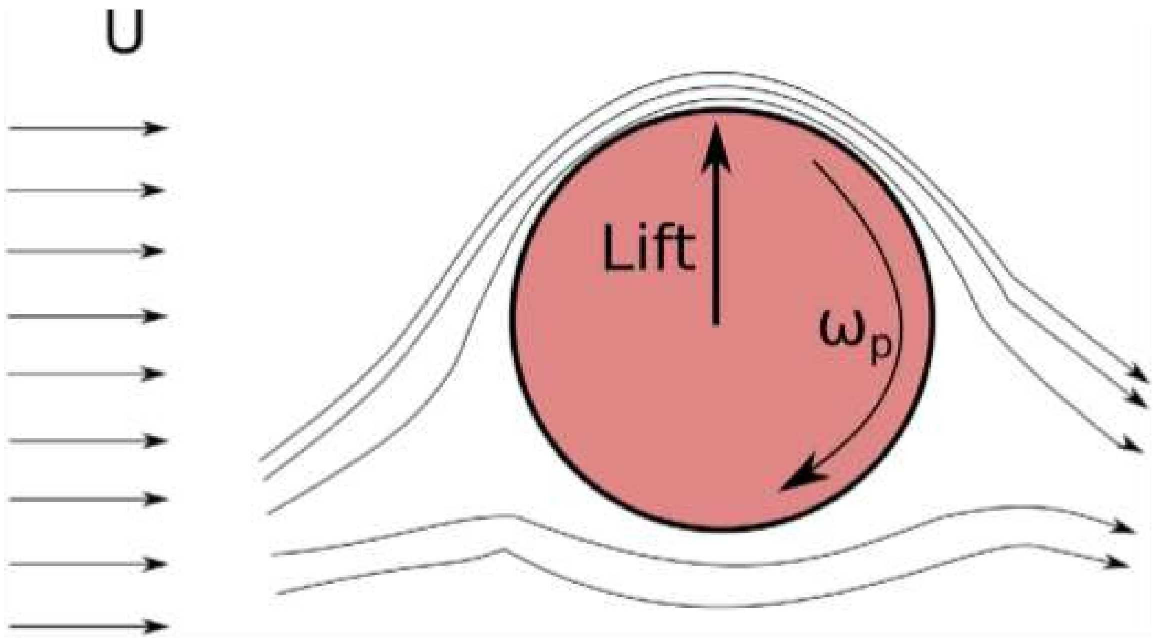
Something else is not symmetrical

Vorticity!



Transverse forces

Magnus force – due to perturbation of the mean flow caused by angular motion of the particle



Saffmann lift force – presence of a velocity gradient in the direction normal to the main flow

$$\mathbf{F}_{saff} = 1.61\, d_P^2 \sqrt{\rho_f \mu_f}\, \frac{\left(\mathbf{u}_f - \mathbf{u}_\mathrm{P}\right) \times \boldsymbol{\omega}_f}{\sqrt{\left|\omega_f\right|}}; \quad \boldsymbol{\omega}_f = rot\, \mathbf{u}_f$$

Magnus lift force – particle rotating in a fluid flow (rotation introduced due to, e.g., collisions)

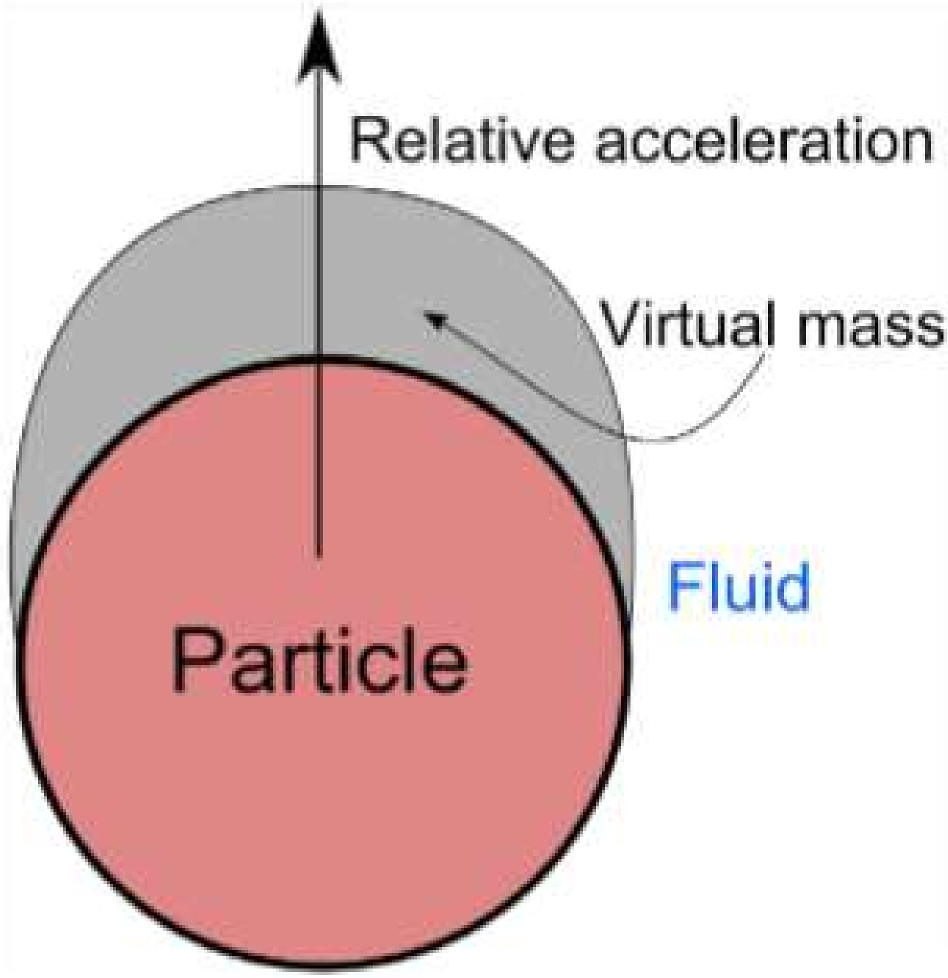
$$\mathbf{F}_{mag} = \frac{\pi}{8} d_P^3\, \rho_f \left(\frac{1}{2} \nabla \times \mathbf{u}_\mathrm{f} - \boldsymbol{\omega}_P \right) \times \left(\mathbf{u}_f - \mathbf{u}_\mathrm{P}\right)$$

Two important forces will be introduced now:

- Added mass force (virtual mass force)
- History (Basset force)

Sometimes they are referred to as: **unsteady drag** (**acceleration of the relative velocity**)

Added mass force – acceleration of a certain fraction of the surrounding fluid



Inertia added to the system

Crowe, 2012

Added mass force (Virtual mass force)

– Expression and analysis

$$\boldsymbol{F_A} = \frac{1}{2} m_P \frac{\rho_f}{\rho_P} \frac{d}{dt}\left(\boldsymbol{u_f} - \boldsymbol{u_p}\right)$$

Conclude when (i.e. in what types of flows) can this force be neglected – based on the density ratio

It can be shown: $$\boxed{\rho_{virt} = \rho_p + \frac{1}{2}\rho_F}$$

History (Basset) force – delay in the boundary layer development with changing relative velocity – hydrodynamic memory

Basset kernel

$$\boldsymbol{F}_H = \sqrt{\pi\rho_f\mu_f}\,\frac{m_P}{\rho_P d_P}\int_0^t K(t-\tau)\,\frac{d}{dt}\left(\mathbf{u}_f - \boldsymbol{u}_p\right)$$

$$\boldsymbol{F}_H = \sqrt{\pi\rho_f\mu_f}\,\frac{m_P}{\rho_P d_P}\int_0^t \underbrace{\frac{1}{\sqrt{1-\tau}}\,\frac{d}{dt}}\left(\mathbf{u}_f - \boldsymbol{u}_p\right)$$

Integration along the entire trajectory for each time step of calculation

History (Basset) force – some info:

Inclusion of the history force transforms Newton's second law for the particle from an ODE to an integro-differential equation (not explicit in u_p or du_p/dt)

Effects most pronounced for high frequency unsteady flows when the fluid-to-particle density ratio is high, and for acceleration from rest in a quiescent fluid

Pressure gradient force (presence of a local pressure gradient) and the force due to shear

$$F_P = \frac{m_P}{\rho_P}\left(-\nabla p + \mu_f \nabla \tau_{shear}\right)$$

from the Navier-Stokes equations

$$-\nabla p + \nabla \tau_{shear} = \rho_f \left(\frac{D u_f}{D t} - g\right)$$

The total pressure force is:

$$F_P = m_P \frac{\rho_f}{\rho_P}\left(\frac{D\, u_\mathbf{f}}{D\, t} - g\right)$$

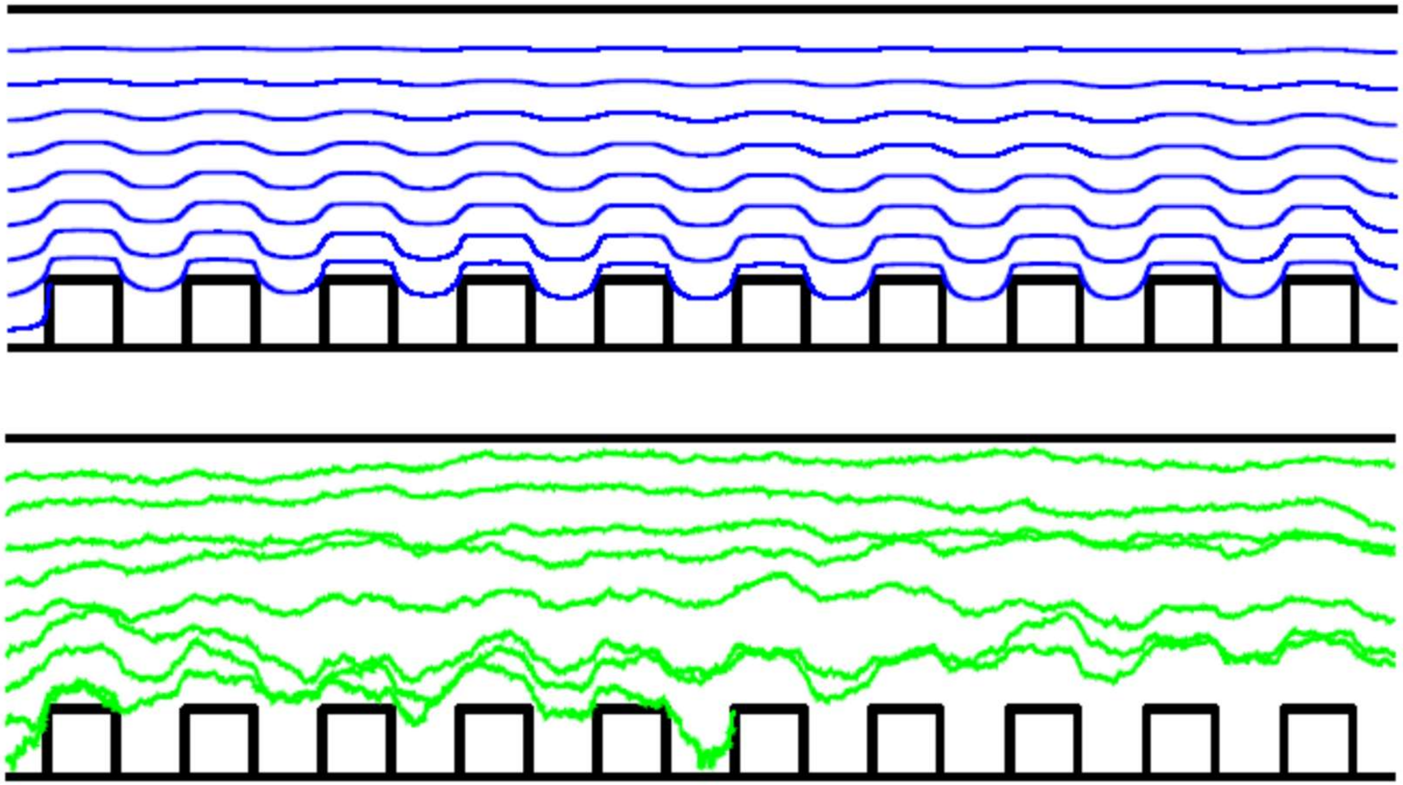
When is this force important?

Pressure gradient force

"For a small sphere, small compared to the scale of the spatial variations of the undisturbed flow, the effect of the undisturbed fluid stresses both from pressure and viscosity is to produce the same net force as would act on a fluid sphere of the same size. This force must equal the product of the fluid mass and local fluid acceleration."

$$F_P = m_P \frac{\rho_f}{\rho_P}\left(\frac{D\, u_{\mathbf{f}}}{D\, t} - g\right)$$

Force due to Brownian motion
(sub-micron particles*)



Drag only

Drag + Brownian

*: Mind whether particles are in gas or liquid

Force due to Brownian motion

$$F_{Brownian} = \xi m_p \sqrt{\frac{216\mu k_B T}{\pi d_p{}^5 \rho_p{}^2 C_c \Delta t}}$$

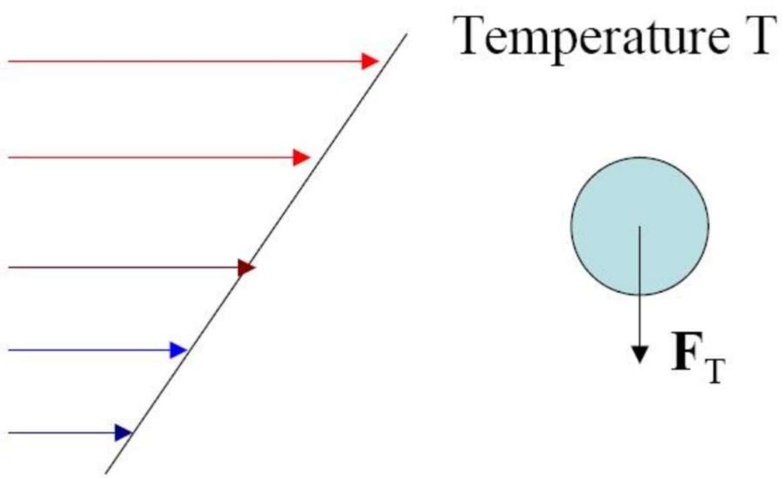
ξ - random number
k_B – Boltzmann constant
T – temperature
C_c – Cunningham correction

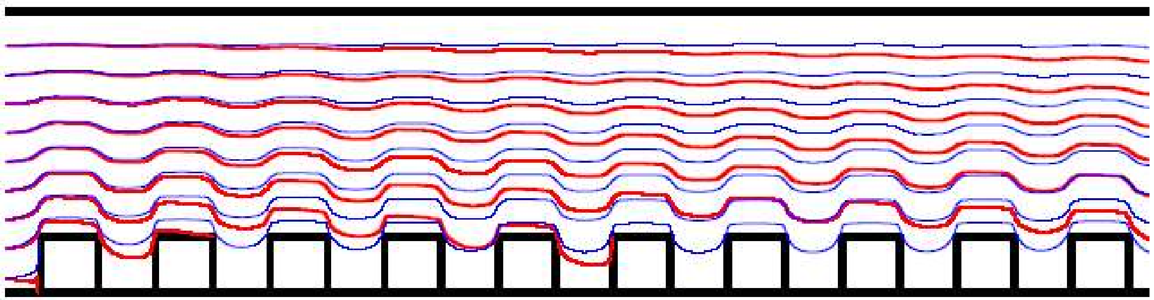
- Can be modelled as a white-noise processes using random numbers
- Transforms Newton's second law for the particle from an ODE to a stochastic differential equation
- Requires very short time steps and many particles

Thermophoretic force

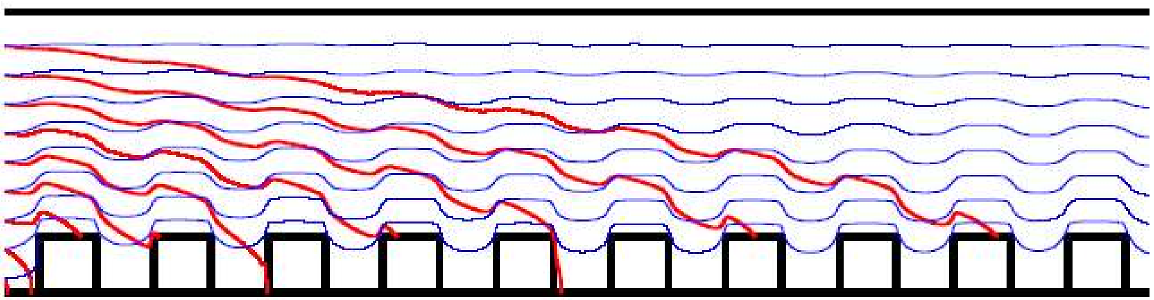
$$F_T = -\frac{6\pi d_p \mu^2 C_s (K + C_t \mathrm{Kn})}{\rho(1+3C_m \mathrm{Kn})(1+2K+2C_t \mathrm{Kn})}\frac{1}{T}\frac{\partial T}{\partial x}$$

Talbot et al., 1980





$$\frac{dT}{dh} = 1 \;\; K/mm$$



$$\frac{dT}{dh} = 10 \;\; K/mm$$

Some relevant questions: how good/general is our equation of motion?

Multiple particles: What is the effect of the neighbouring particles?

What is the effect of the carrier phase flow field (e.g. turbulence)?

What about particle collisions and their effect on the interaction force(s)?

Fundamental question: what happens for $Re_p > 1$ and $Re_p >> 1$

1. The same forces as recognized before, but with coefficients introduced

2. Drag coefficient (C_D), Lift coefficient (C_L), Added Mass coefficients (C_{AM}), History force coefficient (C_H)

3. For the drag force – successful

4. Other forces with varying success

Generalization of the forces (drag)

$$F_D = \frac{1}{2}\rho_f \frac{d_p^2 \pi}{4} C_D \left| u_f - u_p \right| \left(u_f - u_p \right)$$

$\Downarrow$ Comparison with the Stokes drag

$$C_D = \frac{24}{Re_P}$$

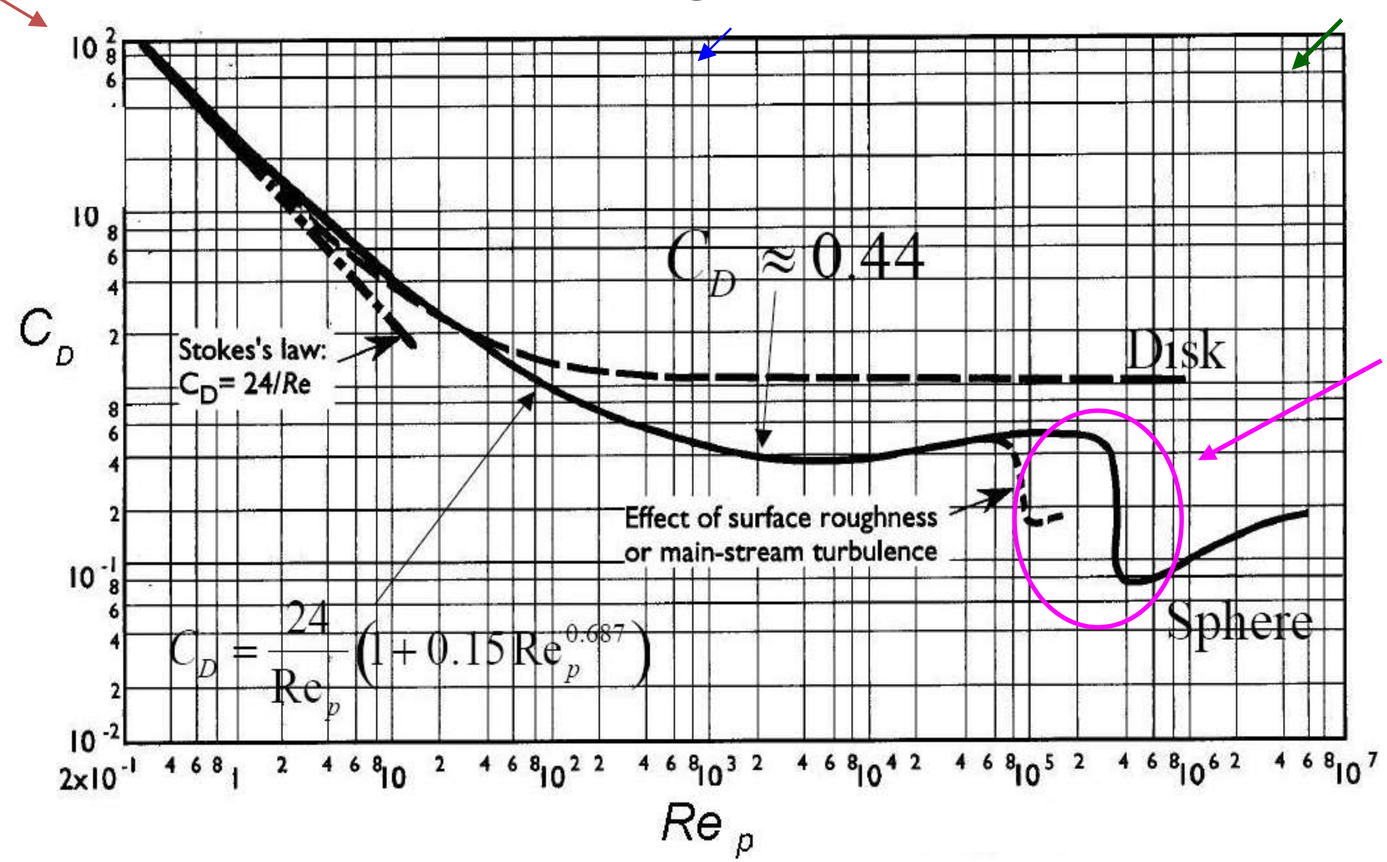
$$Re_P = \frac{\rho_f \, (u_\mathrm{f} - u_P) \, d_\mathrm{p}}{\mu_f}$$

Drag force: friction + form drag

Stokes regime

Transition region

Newton regime



Transition to a turbulent boundary layer (drag crisis)

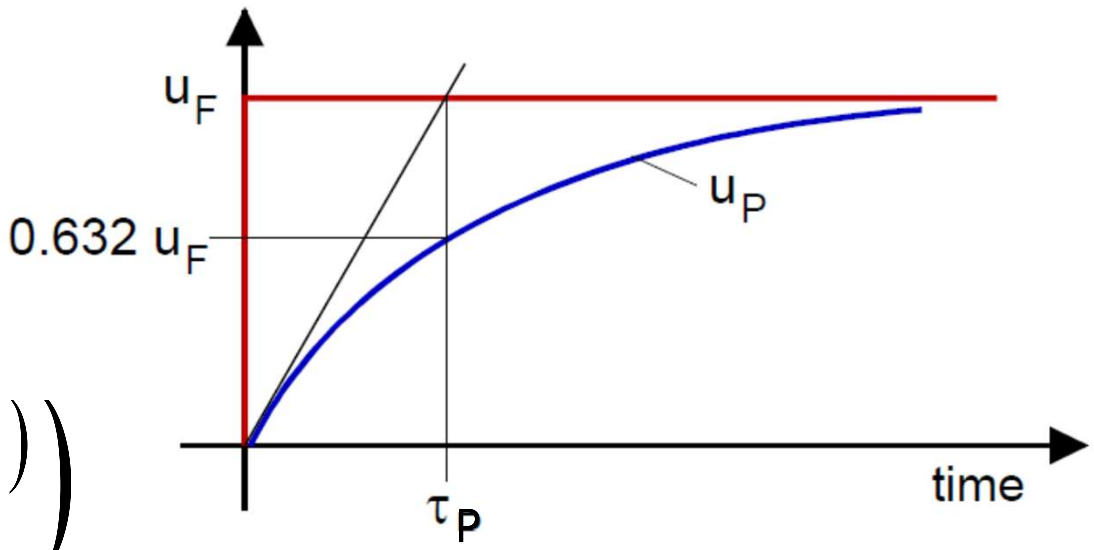
Particle response time - derivation

We start with $$m_P \frac{du_P}{dt} = \frac{1}{2}\rho_f \frac{d_p^2 \pi}{4} C_D \left|u_f - u_p\right| \left(u_f - u_p\right)$$

$$\frac{du_P}{dt} = \frac{18\mu_f}{\rho_p d_p^2} \frac{C_D \operatorname{Re}_p}{24} \left(u_f - u_p\right) \Longrightarrow \frac{du_P}{dt} = \frac{\left(u_f - u_p\right)}{\tau_p}$$

Stokes response time and correction

$$u_P = u_f \left(1 - e^{-\left(t/\tau_p\right)}\right)$$



Generalization of the forces (virtual mass and lift, similar for other forces)

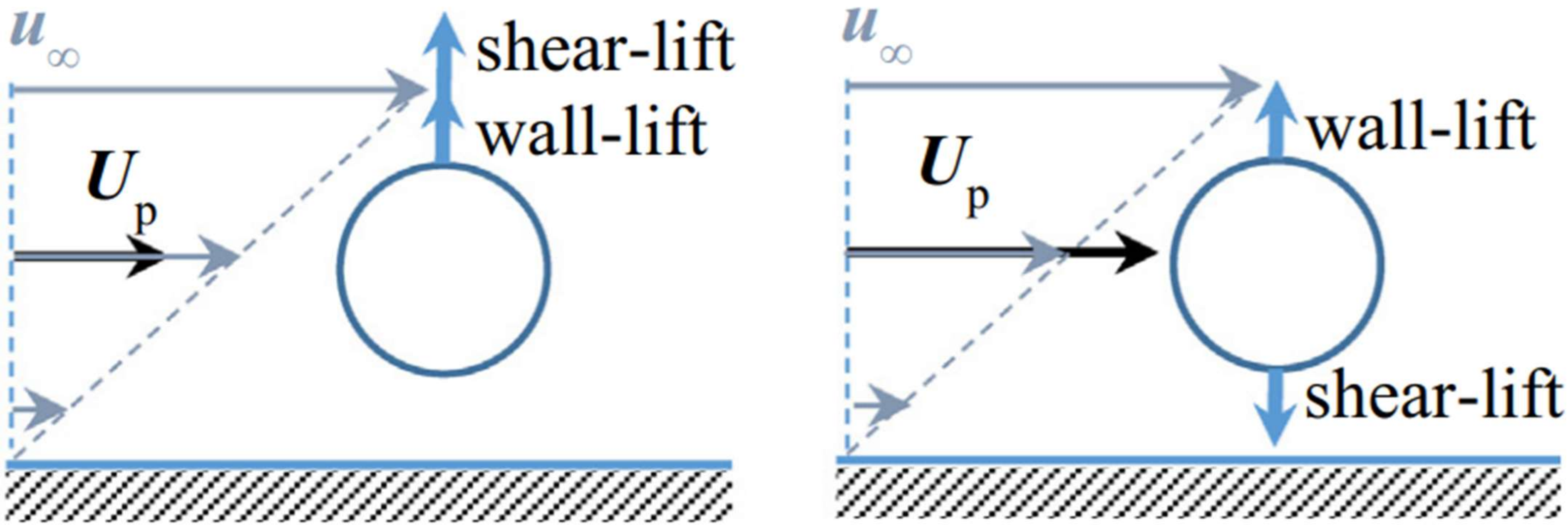
$$\boldsymbol{F}_{\mathbf{vm}} = C_{vm}\rho_l\alpha_g\alpha_l\left(\frac{D\mathbf{u_l}}{Dt} - \frac{D\mathbf{u_g}}{Dt}\right)$$

$$\boldsymbol{F}_{\mathbf{lift}} = C_L\rho_l\alpha_g\alpha_l\left(\mathbf{u_g} - \mathbf{u_l}\right) \times \underbrace{\left(\nabla \times \mathbf{u_l}\right)}_{\omega_l}$$

l, *g*: liquid, gas

Vorticity of the liquid phase

Other things are important as well – influence of the wall



- Corrections of the lift force due to the presence of the wall
- Change of direction (what is different on the sketch above?)

Specific features related to bubbles

Important to have in mind

Absence of a rigid interface between a bubble and fluid

⬇

Consequence: Reduction of the drag force (compared to the corresponding solid particle)

Contamination of the surface of a bubble

⬇

Consequence: Rigid interface again created?

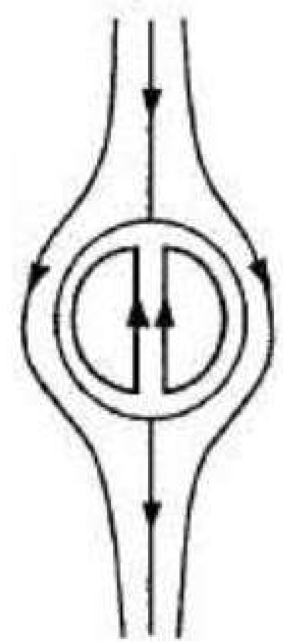
Internal circulation as a fundamental phenomenon (function of viscosity ratio)

Tendency to internal circulation given by:

$$\frac{3\kappa + 2}{3\kappa + 3}$$

1 = particle behaves as a rigid body
2/3 = “full” internal circulation

$$\kappa = \frac{viscosity\ of\ particle\ fluid}{viscosity\ of\ carrier\ fluid}$$



Drag force – what happens for bubbles and drops?

Hadamard-Rybczynski drag law – stresses on the surface induce internal motion - drag coefficient decreased

$$C_{D,HR} = \frac{24}{Re_p}\left(\frac{2/3 + \mu_d/\mu_c}{1 + \mu_d/\mu_c}\right)$$

d- dispersed phase
c – continuous (carrier) phase

Droplet in air : the Stokes law is recovered $(24/Rep)$

Bubble in liquid: drag is reduced by $1/3$

Bubble deformation and oscillation

Weber number: **(inertia/surface tension)**

$$We_b = \frac{\rho_f \, u_{rel}^2 \, d_{eq}}{\sigma}$$

d_{eq}: Equivalent hydraulic diameter

u_{rel}: relative velocity between the phases

σ - surface tension

Can bubbles maintain spherical shape?

$$E_O = \frac{We_b}{Fr_b} = g \frac{\left|\rho_f - \rho_b\right| \, d_{eq}{}^2}{\sigma}$$

Eötvös number **(buoyancy/surface tension)**

Bubble-shape diagram

Capillary number

(viscous/surface tension)

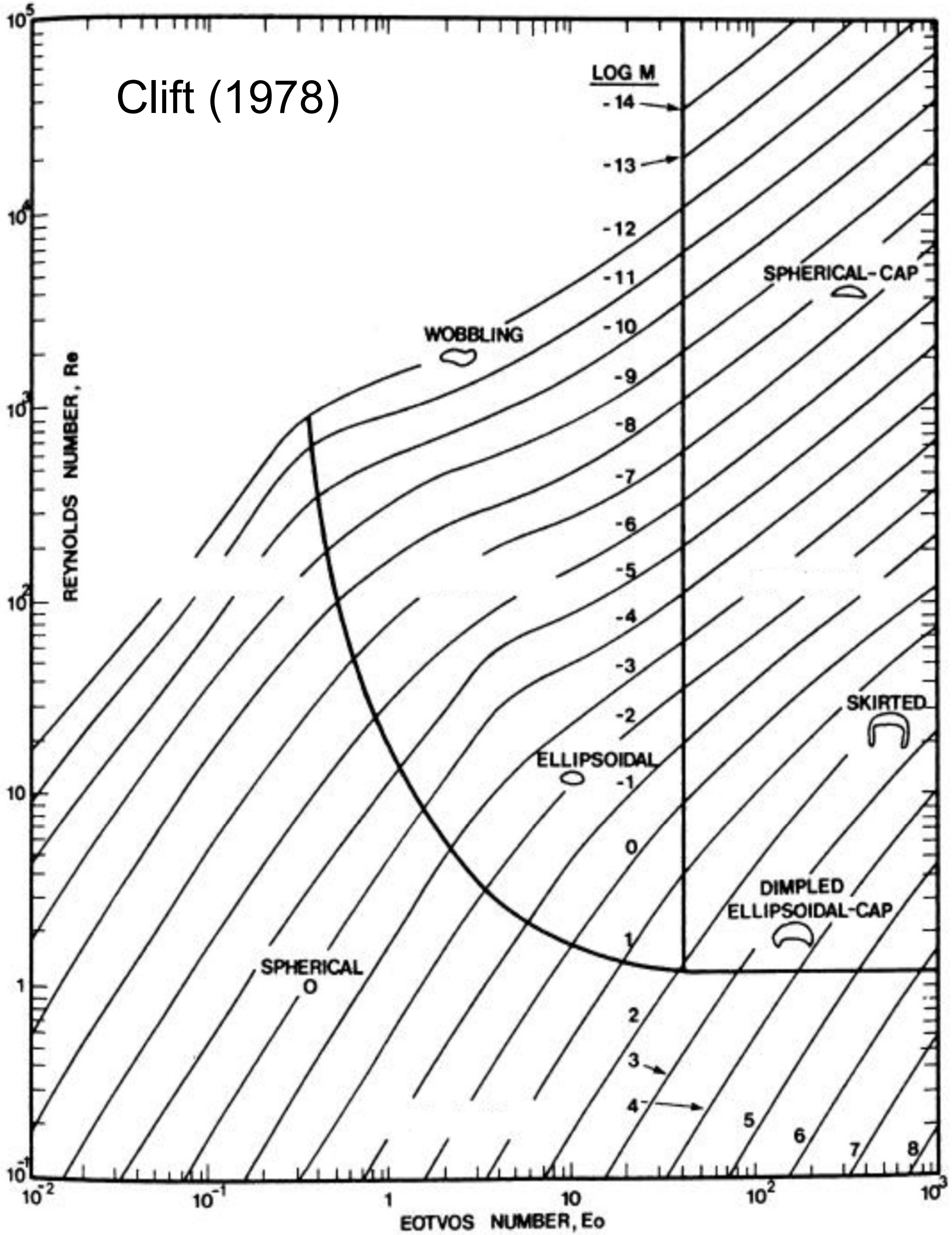
$$Ca = \frac{\mu_f U}{\sigma} = \frac{We}{\mathrm{Re}}$$

Morton number

$$Mo = \frac{g {\mu_f}^4 |\rho_f - \rho_b|}{\rho_f^2 \sigma^3}$$

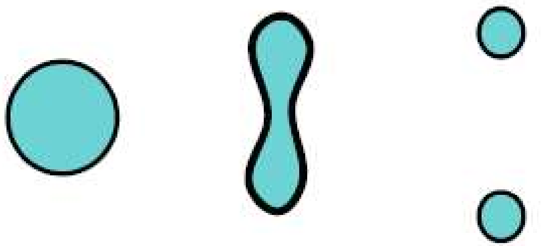
$$Mo = (EoWe\text{^}2)/Re\text{^}4$$

Clift (1978)

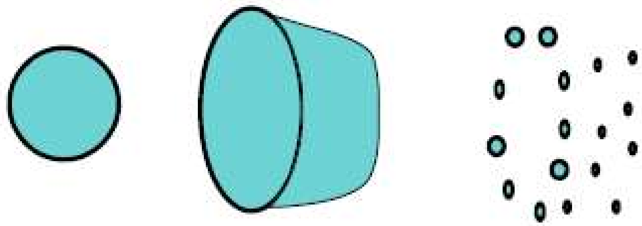


Example: (secondary) breakup of droplets as a function of *We*

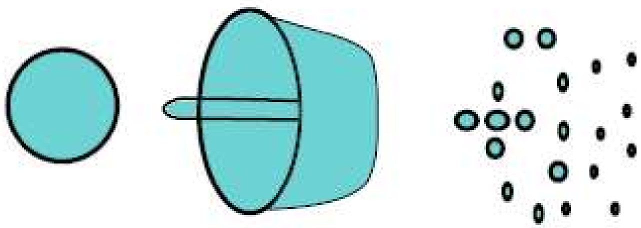
Wierzba, 1990



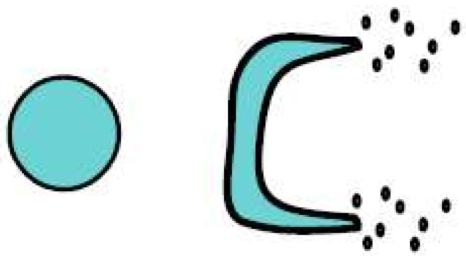
(a) Vibrational breakup, $We_d \approx 12$



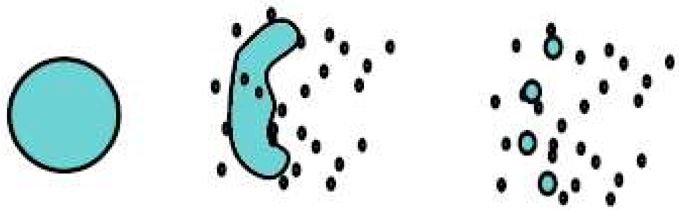
(b) Bag breakup, $We_d < 20$



(c) Bag / streamer breakup, $We_d < 50$

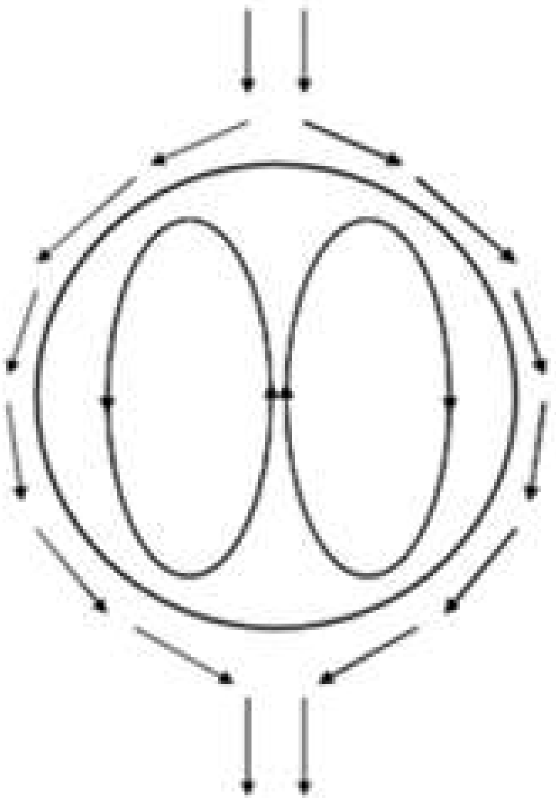


(d) Stripping breakup, $We_d < 100$

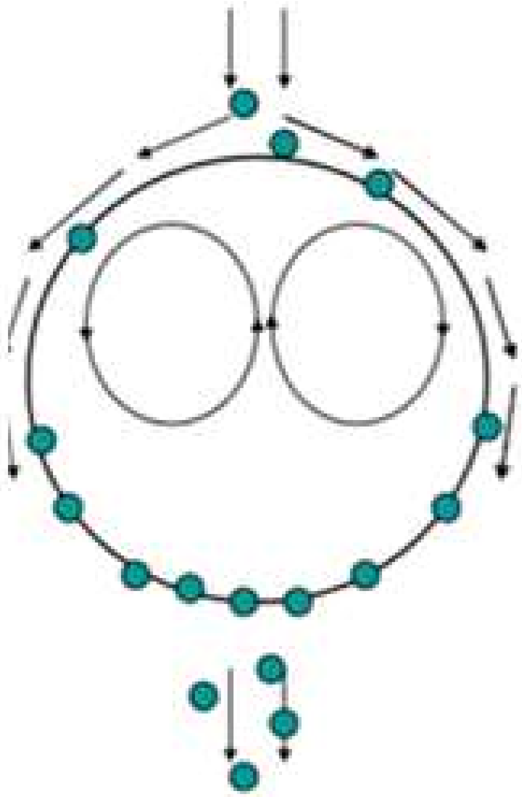


(e) Catastrophic breakup, $We_d > 100$

Drag force as a function of level of contamination



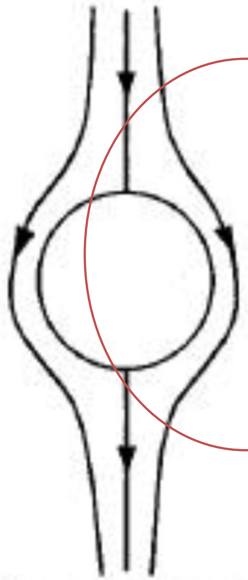
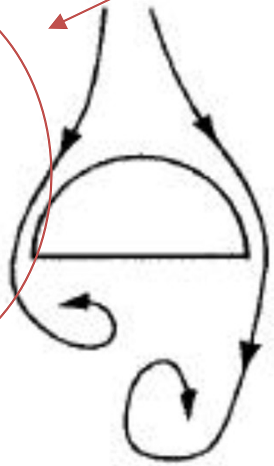
Pure liquid with free-slip boundary condition at bubble interface



Slightly contaminated liquid with limited circulation in the bubble

Fully contaminated liquid: no internal field (no-slip boundary condition at bubble interface)

Bubble behaviours - summary

Shape	spherical		non-spherical
Motion	rectilinear		fluctuating
Purity	pure	contaminated	both
Flow pattern			
Governing effects	viscosity	viscosity	surface tension and gravity
Relevant dimension-less number	Re	Re	Eo

Rigid interface re-created?

Tomyama (1998)