Antonini

- $\hat{G}$ Pontryagin dual of G LCA (loc. cpt. a.G.) grp
- G discrete $\Rightarrow \hat{G}$ cpt; G cpt. $\Rightarrow \hat{G}$ discrete
- e.g. $G = \mathbb{Z}$, $\pi : \mathbb{Z} \to U(1)$ characters
  - take $\pi(1) =: \zeta \in U(1)$, so $\pi(n) = \zeta^n$
    and $\hat{\mathbb{Z}} = U(1)$
- keeping $\pi = \pi_\zeta$, take $f \in L^1(\mathbb{Z})$, $(f_n)_{n\in\mathbb{Z}}$
  and put $\hat{\pi}_\zeta(f) = \int_{\mathbb{Z}} f_n \pi_\zeta(n)\, \delta_{\mathbb{Z}}$

$$= \sum_{n\in\mathbb{Z}} f_n \zeta^n$$

so $\hat{\pi}_\zeta(f) = \hat{f}(\zeta)$ where $\hat{f} : U(1) \to \mathbb{C}$
given as $\hat{f}(z) = \sum_{n\in\mathbb{Z}} f_n z^n$

- $\|f\|_* = \sup_{\zeta\in U(1)} \|\pi_\zeta(f)\| = \|\hat{f}\|_\infty$

Order & positivity

- $C \subset V$, V vsp.
- C cone if $x \in C, a > 0 \Rightarrow ax \in C$.
- C convex cone if $x, y \in C, a, b > 0 \Rightarrow ax + by \in C$
- C flat if $\exists x \in C$ nonzero s.t. $-x \in C$
- C salient if not flat

$\to$ a convex cone is salient iff $C \cap (-C) \subseteq \{0\}$.

Thm A C^*-alg, $h \in A$ s.a. TFAE

a)
1) $Sp_A h \subseteq \mathbb{R}_+$
2) $\exists k \in A, k = k^*$ s.t. $h = k^2$
3) $\exists x \in A$ s.t. $x^* x = h$

$\Big\}$ definition of positive element

b) the positive elements form a salient convex cone (containing zero)

Pf. 1a) $\iff (\exists t \in \mathbb{R}_+) \ \|t - h\| \leq t$ (assuming A unital)

- now we have an induced order on A:
  $a \geq b$ if $a - b \in A_+$ ← pos. elements of A
- this is compatible w. vsp-struct. and of course $a \geq 0$ iff $a \in A_+$

- $(\forall a \in A_+)(\exists! \, b \in A_+) \ b^2 = a$
- $-t \leq a \leq t$ for $t \geq 0$ iff $\|a\| \leq t$

- let $a, b \in A$:
  i) $0 \leq a \leq b \implies \|a\| \leq \|b\|$
  ii) $a \leq b \implies x^* a x \leq x^* b x$

- if A unital:
  i) $x \in A$, $y \in A^{-1}$. $x^* x \leq y^* y \iff \|x y^{-1}\| \leq 1$,
  ii) $x, y \in A^{-1} \cap A_+$. $x \leq y \iff y^{-1} \leq x^{-1}$

Approximate units

- B Banach alg., a (bilat.) approx. id. is a family (net) $(u_i)_{i \in \Lambda} \subset B$, Λ filtered (directed) set s.t. $\lim_{i \in \Lambda} u_i x = x = \lim_{i \in \Lambda} x u_i$ $(*)$

- directed set $\Lambda \neq \emptyset$ with relation $(\leq)$, reflexive & transitive
  s.t. $a, b \in \Lambda \Rightarrow \exists c \quad a \leq c \wedge b \leq c$

- $(*)$: $\lim_{i \in \Lambda} u_i = u$ if $\forall \varepsilon > 0 \ \exists c \in \Lambda$ s.t.
  $j \geq c \Rightarrow \|u - u_j\| \leq \varepsilon$

Def. C^*alg A is called σ-unital when $\exists$ an approx unit which is a sequence.

- unital $\Rightarrow$ σ-unital $\Leftarrow$ separable

- both H ∞-dim $\Rightarrow$ $\mathbb{B}(H)$ nonsep. $\wedge$ unital
  $\mathbb{K}(H)$ sep. $\wedge$ nonunital
  $\mathbb{B}(H)/\mathbb{K}(H)$, Calkin alg., unital $\wedge$ nonsep
- $C_0(X)$ is σ-unital iff X σ-cpt.

- for any C^* $\Lambda := \{a \in A_+ \mid \|a\| < 1\}$ is directed and is a bounded approx. unit

Cor. $I \subset A$ closed bilateral ideal. Then $a \in I \Rightarrow a^* \in I$.

- I closed ideal $\Rightarrow$ A/I is C^*-alg with quot. norm $\|a + I\|_{A/I} := \inf_{z \in a+I} \{\|z\|\}$

Representations

- let A $*$-alg.

- a representation is a $*$-morphism
  $A \xrightarrow{\pi} \mathbb{B}(H_\pi)$
  - $\pi(a)\pi(b) = \pi(ab)$, $\pi(a^*) = \pi(a)^*$

- Invariant subspaces
  $F \subset H_\pi$ linear subspace is invariant
  if $\pi(a)x \in F$ $\forall x \in F$ $\forall a \in A$

- $F \subset H_\pi$ invariant
  $\Rightarrow$ $\overline{F}$ is invariant ($\pi(a)$ is cont., for fixed $a \in A$)
  $\Rightarrow$ $F^\perp := \{ y \in H_\pi \mid \langle y, x \rangle = 0 \ \forall x \in F \}$
  is invariant

- if F is closed inv. subspace, by restriction
  we get $\pi_F : A \to \mathbb{B}(F)$

- direct sums: $\bigoplus_{i \in I} H_i$ = collection of $(x_i)_{i \in I}$
  s.t. $\sum_{i \in I} \langle x_i, x_i \rangle < \infty$.

- label $\sum_{i \in I} \langle x_i, y_i \rangle =: \langle (x_i), (y_i) \rangle$

- now $\forall i$ define $A \xrightarrow{\pi_i} \mathbb{B}(H_{\pi_i})$ assuming
  $\forall a \in A$, $\sup\{ \| \pi_i(a) \| \} < \infty$

- $\exists!$ rep. $\pi := \bigoplus_{i \in I} \pi_i : A \to \mathbb{B}(\bigoplus H_{\pi_i})$
  s.t. $\pi(a)((x_i)_{i \in I}) = (\pi_i(x_i))_{i \in I}$, and $\|\pi(a)\| := \sup_{i \in I} \|\pi_i(a)\|$

- commutant: $S \subset \mathbb{B}(H)$, then
$$S' := \{ T \in \mathbb{B}(H) \mid Tx = xT \ \forall x \in S \}$$

- intertwining operators.
$\pi_i : A \to \mathbb{B}(H_{\pi_i})$, $i \in \{1, 2\}$
$$\mathrm{Hom}(\pi_1, \pi_2) := \{ T \in \mathcal{L}(H_{\pi_1}, H_{\pi_2}) \mid T\pi_1(a) = \pi_2(a) T \ \forall a \in A \}$$
$\mathrm{Hom}(\pi, \pi) =: \mathrm{End}(\pi) = (\pi(A))'$
$T \in \mathrm{Hom}(\pi_1, \pi_2)$, $T^* \in \mathrm{Hom}(\pi_2, \pi_1)$,
$t \in \mathrm{Hom}(\pi_1, \pi_2)$, $s \in \mathrm{Hom}(\pi_2, \pi_3)$
$\Rightarrow s \circ t \in \mathrm{Hom}(\pi_1, \pi_3)$
$\mathrm{Hom}(\pi_1, \pi_2) \subset \mathcal{L}(H_{\pi_1}, H_{\pi_2})$ weakly closed
- we can see this by fixing $x \in H_{\pi_1}$, $y \in H_{\pi_2}$ and writing $\langle \pi_2(a^*)x, Ty \rangle = \langle x, T\pi_1(a) y \rangle$

Prop $A \xrightarrow{\pi} \mathbb{B}(H)$, $E \subset H$ closed subsp. with its projection $P : H \circlearrowleft$.
Then E invariant $\iff P \in (\pi(A))' = \mathrm{End}(\pi)$

Pf. $P \in \mathrm{End}(\pi) \Rightarrow \forall T \in \pi(A)$, $TP = PT$
$\Rightarrow TP = PTP \iff E$ inv. for T
Conversely, if E inv then also $E^\perp$ inv
so $\begin{cases} (1-P)TP = 0 \\ (1-(1-P))T(1-P) = 0 = -PT(1-P) \end{cases}$
$\Rightarrow TP = PTP = PT \ \forall T$ $\square$

Def $\pi_1 \sim \pi_2$ if $\exists$ a unitary $\in \mathrm{Hom}(\pi_1, \pi_2)$

- if E closed and inv for π
  then $\pi \sim \pi_E \oplus \pi_{E^\perp}$