Tanzini

Topological field theories

- "operative" def of field theories:
  - theory of maps from Σ domain Riemann mfd (world-volume) to tgt mfd M (target space)
  - set $\dim_{\mathbb{R}} \Sigma := d$ (in this course) $0, 1, 2$
    - 0: finite-dim or 0-dim QFT
    - 1: S^1 or $[a,b]$ quantum mechanics
    - 2: strings

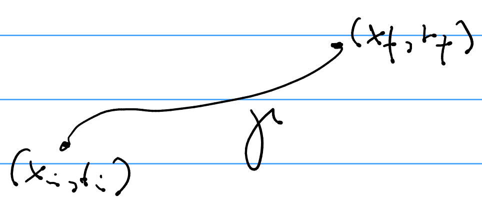
- and "topological"?
  1) one can only compute metric-independent quantities (independent of both Σ, M metrics)
  11) the semi-classical limit is exact

  $\Rightarrow$ TQFT $\rightsquigarrow$ enumerative invariants

- how to study them?

- AXIOMATIC APPROACH (cobordisms) [Atiyah-Segal]
  - $\dim_{\mathbb{R}} M = 2, 3, \dots$?
- PATH INTEGRAL APPROACH (heuristic!)
  - very flexible
  - gives hints on calculating top. inv.
    - $\rightarrow$ and on dualities (e.g. mirror symmetry)

- classical mechanics:



$$\gamma : I \to M, \quad I = [t_i, t_f]$$

$$\delta S[\gamma] = 0$$

- but in q.m. $\Delta x \Delta p \geq \frac{\hbar}{2}$
- so look at probability $\langle x_i t_i | x_f t_f \rangle = \int_{C_{\ldots}} [D\gamma]\, e^{\frac{i}{\hbar} S[\gamma]}$

$\hookrightarrow \left\{ \begin{array}{l} \text{space of maps } \gamma \in \mathrm{Map}(I, \mathbb{R}) \\ \gamma(t_i) = x_i \\ \gamma(t_f) = x_f \end{array} \right\}$

  - if semi-cl. approx <u>exact</u>, then ∞-dim integral collapses to integration over moduli space of solns. to partial diff. eqns

- main reference: Mirror symmetry, AMS

- stationary phase

$$I(s) := \int_{\mathbb{R}} dx\, e^{isf(x)} g(x), \quad f, g \in C^\infty(\mathbb{R})$$

- asymptotics? (as $s \to +\infty$)
- suppose $f(x)$ has isolated extremum at x_0 and $f''(x_0) \neq 0$

$$\Rightarrow f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots$$

- leading term:

$$I_0(s) = g(x_0)\, e^{isf(x_0)} \int_{\mathbb{R}} dx\, e^{\frac{i}{2} s f''(x_0)(x-x_0)^2}$$
$$= g(x_0) \exp\left[i \left(s f(x_0) + \mathrm{sign}(f''(x_0)) \frac{\pi}{4} \right) \right] \left(\frac{2\pi}{s |f''(x_0)|} \right)^{1/2}$$

<u>exercise</u>: prove it (hint: contour integral)

- n dimensions:

$$I(s) = \int d^n x \; g(x)\, e^{isf(x)}$$

- again, $\nabla f|_{x_0} = 0$, $\mathrm{Hess} f|_{x_0}$ nondeg.
- put $f(x) = f(x_0) + \sum_{ij=1}^{n} f_{ij}(x_0)(x-x_0)_i (x-x_0)_j$

$$I_0(s) = g(x_0)\, e^{isf(x_0)} \left(\frac{2\pi}{s}\right)^{n/2} \frac{e^{i\sigma\frac{\pi}{4}}}{|\det \mathrm{Hess} f|_{x_0}|^{1/2}}$$

where σ = signature of Hess := # of pos. eigenvalues − # of neg. —"—

- example: two-sphere S^2
  - $g(x,y,z) = 1$
    $f(x,y,z) = z \longrightarrow$ extrema N & S poles

  - N: $z = 1 - \frac{1}{2}(x^2+y^2) + \ldots$
  - S: $z = -1 + \frac{1}{2}(x^2+y^2) + \ldots$

$$I_0(s) = \frac{2\pi}{s}\Big(\underbrace{e^{i(-2)\frac{\pi}{4}} e^{is}}_{N} + \underbrace{e^{i(2)\frac{\pi}{4}} e^{-is}}_{S}\Big)$$

$$= 4\pi \frac{\sin(s)}{s}$$

- note that $\int_{S^2} dA\, e^{is\cdot z}\Big|_{s=0} = 4\pi$ ✓
- full integration?

$$I(s) = \int (-d(\cos\vartheta)\, d\varphi)\; e^{is\cos\vartheta}$$

$$= -2\pi \int_{+1}^{-1} d(\cos\vartheta)\, e^{is\cos\vartheta}$$

$$= -2\pi \frac{1}{is}\left(e^{is} - e^{is}\right) = 4\pi \frac{\sin(s)}{s} \;✓$$

Equivariant cohomology [Berline – Vergne
B. Getzler: Heatkernel & Disc op §VII
Atiyah – Bott: Moment maps & equiv. coh.]

- $G \circlearrowright M$:
  i) if action free, M/G is smooth mfd,
  we set $H^\bullet_G(M) = H^\bullet(M/G)$
  ii) action not free $\Rightarrow$ stacky points $(pt, stab_{pt})$

$\Rightarrow$ introduce univ. bdl EG:
  i) contractible space (so we don't change coh. of M)
  ii) G-action free

$$\Rightarrow H^\bullet_G(M) := H^\bullet(M \times_G EG)$$
$$= H^\bullet\left(\frac{M \times EG}{G}\right)$$

- example: $M = pt$, $H^n(pt, \mathbb{R}) = \begin{cases} \mathbb{R} & , n = 0 \\ 0 & , n > 0 \end{cases}$

- but equiv. coh:

$$H^\bullet_G(pt) = H^\bullet(EG/G) = H^\bullet(BG)$$

- example: $G = U(1) \simeq S^1$
$E S^1 \simeq S^{2n+1}$
$B S^1 \simeq \mathbb{C}P^n$ as $n \to \infty$

$$H_{S^1}(pt) = \mathbb{C}[t], \quad t \in H^2(\mathbb{C}P^\infty, \mathbb{C})$$

Cartan model for equiv. coh.

- $G \circlearrowright M$ acts on $f \in C^\infty(M)$ as $h \cdot f(x) = f(h^{-1}x)$
- for $L \in \mathfrak{g}$ denote by v the generating v.f.

$$(v \cdot f)(x) := \frac{d}{d\varepsilon} f(e^{-\varepsilon L} x)\Big|_{\varepsilon = 0}, \quad v = v^a T_a^i \partial_i$$

- $\mathbb{C}[\mathfrak{g}]$ alg. of cpx val. fns on $\mathfrak{g}$
- consider $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$
  $\to G$-action $\quad (h \cdot \alpha)(x) = h\,\alpha(h^{-1}x)$

Def. Equivariant dif. forms satisfy $\alpha(hx) = h\alpha(x)$.

$\to$ invariant under G-action!

Def. Equiv. exterior differential

$$d_v \alpha := d\alpha + i \xi\, \iota_v \alpha$$

$\nwarrow$ formal param

- need to check!
- first define grading:

$$\deg(P \otimes \beta) = 2 \deg P + \deg_{dR} \beta$$

$$d_v : \Omega^\bullet(M, \mathfrak{g}) \to \Omega^{\bullet+1}(M, \mathfrak{g})$$

$$d_v^2 = i\xi(d\,\iota_v + \iota_v d) = i\xi\,\mathcal{L}_v$$

$\to$ vanishes on equivariant forms

- example S^2: sympl. form $\omega = d\cos\vartheta\, d\varphi$.
  $\to S^1 \circlearrowright S^2$ generated by $v = \xi \frac{\partial}{\partial\varphi}$
  $\to \omega(\xi) = \omega + \xi R(\vartheta)$

$$0 = d_v \omega(\xi) = \left(d + i\xi\, \iota_{\frac{\partial}{\partial\varphi}}\right)\left(\omega + \xi R(\vartheta)\right) = i\xi\left(\iota_{\frac{\partial}{\partial\varphi}}\omega - i\,dR(\vartheta)\right)$$

$\Rightarrow R(\vartheta) = i \cos \vartheta \rightarrow$ the height function!

Tanzini

Localisation formulae

- poly forms $\alpha(\xi) \doteq \alpha_n + \alpha_{n-2} + \dots + \alpha_0$ (← even case)
- also $0 = d_v \alpha(\xi) = (d + i\xi \partial_v)(\alpha_n + \dots)$

$$= \underbrace{d\alpha_n}_{=0, \text{ say}} + \underbrace{i\xi\, i_v \alpha_n + d\alpha_{n-2}}_{\text{relates them}} + \dots$$

→ so the α_i are not independent

$$\int_M \alpha(M) = \left(\frac{-2\pi}{i\xi}\right)^{\ell} \sum_{\substack{\{p\} \\ \text{fixed pts}}} \frac{\alpha_0(\xi)(p)}{(\det L_p)^{1/2}},$$

$\begin{pmatrix} \partial M = \phi, \\ M \text{ cpt.} \end{pmatrix}$

$L_p \subset \mathfrak{G}(T_pM)$. If we split $T_pM = \bigoplus_{k=1}^{\ell} V_k$,

$$(\det L_p)^{1/2} = e(N_p) = (-)^{\frac{\lambda(p)}{2}} \prod_k |v_k|$$

$\lambda(p) \leftarrow$ # neg v_k

$e(N_p)$: equiv. Euler class of normal bdl

- Example. $G = S^1$, $v = \frac{\partial}{\partial \varphi}$

$$x^i(\varphi) = x^i_0 + R^i{}_j(\varphi)(x - x_0)^j$$

$$v = \sum_{i=1} v_i (x_i \partial_{y_i} - y_i \partial_{x_i}), \quad g = \sum_{i=1} (dx_i^2 + dy_i^2)$$

Pf. of localisation formula

- pick S^1-inv metric as above
- let $\psi := \frac{1}{2} g(v, -)$, and define

$$\beta(\xi) = d_v \psi = d\psi + i\frac{\xi}{2}\|v\|^2$$

- locally, $\psi = \frac{1}{2} \sum_{k=1}^{l} v_k (x_k dy_k - y_k dx_k)$

so $\beta(\xi) = \sum_{k=1}^{l} v_k\, dx_k \wedge dy_k + i\frac{\xi}{2} \sum_{k=1}^{l} v_k^2 (x_k^2 + y_k^2)$

- now note that $\int_M \alpha(\xi) = \int_M \alpha(\xi) e^{is\beta(\xi)}$, since we are adding d_v-exact terms.
- so take $s \to \infty$ limit:

$$\int \alpha(\xi) \underset{s\to\infty}{\simeq} \int_M \alpha(M) e^{is\beta(\xi)}$$

$$= \alpha_0(\xi)(x_p) \prod_{k=1}^{\ell} is\,v_k \underbrace{\int dx_k \wedge dy_k\, e^{-s\xi \|v\|^2}}_{\frac{2\pi}{s\xi (v_k^{(p)})^2}}$$

$$= \frac{\alpha_0(\xi)(x_p)}{v_1^{(p)} \cdots v_\ell^{(p)}} \left(-\frac{2\pi}{i\xi}\right)^{\ell}. \quad \square$$

- another take:

Lemma if $\alpha(\xi)$ d_v-closed, then outside the 0-set of v its top dR component is closed.

$\to$ so $\int_M \alpha(\xi) = \lim_{\varepsilon\to 0} \int_{M \setminus \bigsqcup_{\text{fixed pts}} B(pt, \varepsilon)} \cdots = \cdots$

Duistermaat-Heckmann thm

- symplectic mfd (M, ω).
- let v Hamiltonian, i.e. $\exists H$, $i_v\omega + dH = 0$
- $\mathcal{L} := \frac{\omega^\ell}{\ell!} = [e^\omega]_{top}$ Liouville form
- then $I(\xi) = \int_M \mathcal{L} e^{i\xi H} = \left(\frac{-2\pi}{i\xi}\right)^{\ell} \sum_p \frac{e^{i\xi H(x_p)}}{\lambda_1^{(p)} \cdots \lambda_\ell^{(p)}}$

- this follows if we define $\omega(\xi) := \omega + i\xi H$,
→ then $d_v \omega(\xi) = i\xi(i_v \omega + dH) = 0$
- so $I(\xi) = \int_M \alpha(\xi) = \int_M [e^{\omega(\xi)}]_{top} = \int_M \frac{\omega^k}{k!} e^{i\xi H}$

- example: S^2, $\omega = d\cos\theta\, d\varphi$, $H = \cos\theta$, $v = \partial_\varphi$

$$I(\xi) = \int \omega e^{i\xi z} = \left(\frac{-2\pi}{i\xi}\right)\left(\frac{e^{i\xi z_N}}{\upsilon_N} + \frac{e^{i\xi z_S}}{\upsilon_S}\right)$$
$$= 4\pi \frac{\sin\xi}{\xi}$$

- noncpt. example: $\mathbb{R}^2$
$\omega = dx\,dy$, $v = x\partial_y - y\partial_x$, $\omega(\xi) = dx\,dy + i\xi \frac{x^2+y^2}{2}$

$$\Rightarrow \int_{\mathbb{R}^2} e^{i\omega(\xi)} = i \int_{\mathbb{R}^2} dx\,dy\, e^{-\frac{x^2+y^2}{2}\xi} = \frac{2\pi i}{\xi}$$

- Exercise. $S^2 \simeq \mathbb{CP}^1$

$\omega_{FS} = \frac{i}{2\pi} \frac{dz\, d\bar{z}}{(1+|z|^2)^2}$. v. field $z \mapsto e^{i\varepsilon} z$, $\bar{z} \mapsto e^{-i\varepsilon}\bar{z}$

I) construct equiv. extension, compute equiv. volume
II) $\mathbb{CP}^n$, $\omega_{FS} = i\partial\bar{\partial}\log|z|^2$, $z_j \mapsto e^{i\varepsilon_j} z_j$ (look at charts)
→ note that each affine chart has a fixed pt!

Tanzini

Supermanifold

$\longrightarrow$ $(M, A) =: M^{m|n}$, where M mfd, $\dim_{\mathbb{R}} M = m$; A graded algebra, $\dim A = n$

- consider $E \to M$ vector bdl, $A := \Lambda^\bullet E$
- $E = TM$. ΠTM tautological smfd (π parity reversal)
- $A = A^+ \oplus A^-$, $\mathbb{Z}_2$-graded comm. alg.

$$[\alpha, \beta] = \alpha\beta - (-)^{\deg\alpha \deg\beta} \beta\alpha$$

- local coords: $(x_1, \dots, x_m | \underbrace{\psi_1 \dots \psi_n}_{\text{Graßmann}})$

- superfields:

$$f(x, \psi) = f_0(x) + f^a(x)\psi_a + f^{ab}\psi_a\psi_b + \dots + f^{1\dots n}\psi_1 \cdots \psi_n$$

- Berezin integration:

$$\int \psi \, d\psi = 1, \quad \int d\psi = 1$$

$$\int \psi_1 \cdots \psi_n \, d\psi_1 \cdots d\psi_n = 1, \quad \int \psi_1 \cdots \hat{\psi}_i \cdots \psi_n \, d\psi_1 \cdots d\psi_n = 0$$

Gaussian integration

A $n\times n$ sym. real. (positive?)

$$\int_{\mathbb{R}^n} d^n x \, e^{-\frac{1}{2}(x, A x)} = (2\pi)^{N/2} (\det A)^{-1/2}$$

- if 0 modes present, use $\det' A = \prod$ (nonzero eigenval.)

H $n\times n$ hermitian:

$$\int_{\mathbb{C}^n} \prod_i \frac{dz_i d\bar{z}_i}{2\pi i} e^{-z^\dagger H z} = (\det H)^{-1}$$

skew-sym.

- for fermions, integrate $\omega_{ij}\psi_i\psi_j$

$$\int \exp\left[\tfrac{1}{2}\psi^t\omega\psi\right] d\psi_1\cdots d\psi_{2m} = \mathrm{Pf}(\omega)$$

where $\frac{1}{m!}\left(\frac{1}{2}\psi^t\omega\psi\right)^m = \psi_1\cdots\psi_{2m}\cdot\mathrm{Pf}(\omega)$

- more explicitly:

$$\mathrm{Pf}(\omega) = \frac{1}{2^m m!}\sum_{\sigma\in S_{2m}} \mathrm{sign}(\sigma)\prod_{i=1}^m \omega_{\sigma(2i-1)\sigma(2i)}$$

$$\to \mathrm{Pf}(\omega)^2 = \det\omega$$

$$\int e^{\bar\psi^t\omega\psi}\prod_a d\psi_a d\bar\psi_a = \det\omega$$

Localisation on smfds.

- Supergroup: Lie grp w/ $\mathbb{Z}_2$-graded generators

- Q odd: $v\cdot y = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{-\varepsilon Q}y$, $\varepsilon^2 = ?$

Prop. Given a Q-inv sfunc f, only fixed pts contribute to the integral over E.

Pf. Suppose action free. Then E/F smooth. So $\int_E f = \int_F d\varepsilon \int_{E/F} f = 0$, because $\int d\varepsilon = 0$.

- 0-dim QFT w. susy:

$$Z = \int_{\mathbb{R}^{1|2}} dx\, d\psi_1\, d\psi_2 \exp\left[-S(x,\psi_1,\psi_2)\right]$$

$$S(x,\psi_1,\psi_2) = S_0(x) + \psi_1\psi_2 S_1(x)$$

- let $S_0(x) = \frac{1}{2}(h')^2$, $S_1(x) = -h''$ for $h \in C^\infty(\mathbb{R})$

$$\delta_\varepsilon x = \varepsilon^1\psi_1 + \varepsilon^2\psi_2$$
$$\delta_\varepsilon \psi_1 = \varepsilon^2 h'$$
$$\delta_\varepsilon \psi_2 = -\varepsilon^1 h'$$

$\Rightarrow \delta_\varepsilon S = 0$!

- suppose δ_ε has <u>no</u> fixed pts.
$\Leftrightarrow h' \neq 0$
- pick substitution

$$\left.\begin{aligned} \hat{x} &= x - \frac{\psi_1\psi_2}{h'} \\ \hat{\psi}_1 &= \psi_1 - h'\frac{\psi_1}{h'} = 0 \\ \hat{\psi}_2 &= \psi_1 + \psi_2 \end{aligned}\right\} \quad \varepsilon^1 = \varepsilon^2 = -\psi_1/h'$$

- Exercise: show the Jacobian of is 1.

- but now $Z = \int d\hat{x}\, d\hat{\psi}_1\, d\hat{\psi}_2 \exp\left[-S(\hat{x},0,\hat{\psi}_1)\right] = 0.$

- Z is counting crit. pts :

$$h(x) = h(x_c) + \frac{1}{2} h''(x_c)(x-x_c)^2 + \dots$$
$$h'(x) = h''(x_c)(x-x_c) + \dots$$

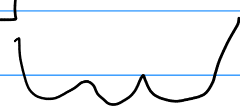
$$Z = \sum_{\{x_c\}} \frac{1}{\sqrt{2\pi}} \int dx\, d\psi_1\, d\psi_2 \exp\left[-\frac{1}{2} h''(x_c)^2 (x-x_c)^2 + h''(x_c)\psi_1\psi_2\right]$$

redefinition

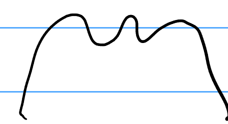
$$= \sum_{\{x_c\}} \frac{h''(x_c)}{|h''(x_c)|} = \sum_{\{x_c\}} \mathrm{Sign}(\det \mathrm{Hess}\, h)$$

- supp. $h(x) \in \mathbb{R}[x]$

$\deg h(x)$ → even

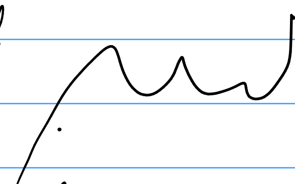


$Z = +1$



$Z = -1$

$\deg h(x)$ → odd



$Z = 0$

- Invariant under deformations, as long as we don't change sign of highest power

- explicitly :

$$Z = \frac{1}{\sqrt{2\pi}} \int dx\, d\psi_1\, d\psi_2 \exp\left[-\frac{1}{2}(h')^2 + h''\psi_1\psi_2\right]$$
$$= \frac{1}{\sqrt{2\pi}} \int \frac{dx\, h''}{d(h')} \exp^{-\frac{1}{2}(h')^2} = \{y = h'\}$$
$$= \frac{D}{\sqrt{2\pi}} \int_{\mathbb{R}} dy\, e^{-y^2} = D$$, where D counts preimages of $y = h'(x)$

Deformation invariance

- $f = \delta_\varepsilon g \Rightarrow \int f e^{-S} = \int \delta_\varepsilon g \cdot e^{-S} = \int \delta_\varepsilon (g e^{-S}) = 0$

- $h \mapsto h + s$, s small

$$S(h+s) = S(h) + \delta_s S, \quad \delta_s S = s' h' - s'' \psi_1 \psi_2$$
$$= \delta_\varepsilon (s'(x) \psi_1), \quad \varepsilon^1 = \varepsilon^2 = 2$$

Rmk. classical soln $\Longleftrightarrow$ susy fixed pt !?

Duistermaat-Heckmann.

- $(M, \omega) \rightsquigarrow$ taut. smfd. $\pi(TM)$ $M^{n|n}$

Tanzini

SUSY or Top. q. mech.

- Hilb. sp. ⇝ complete unitary vect. sp.
  - → $|\alpha\rangle \in \mathcal{H}$, $\langle\beta| \in \mathcal{H}^*$
  - → bra-ket $\langle\_|\_\rangle : \mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C}$ is
    - i) sesquilinear
    - ii) $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$
    - iii) $\langle\alpha|\alpha\rangle \geq 0$ with = iff $|\alpha\rangle = 0$

- supersym. Hilb. sp. is $\mathbb{Z}_2$-graded $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$
  - i) Q degree 1 operator (supercharge), $Q^2 = 0$
    - → also $Q^\dagger$
  - ii) $H := \frac{1}{2}\{Q, Q^\dagger\}$
  - iii) $(-)^F$ is $\begin{matrix} +1 \\ -1 \end{matrix}$ on $\begin{matrix} \mathcal{H}_B \\ \mathcal{H}_F \end{matrix}$, $[(-)^F, Q] = -Q$.

- consequences:
  - i) $[(-)^F, H] = [Q, H] = [Q^\dagger, H] = 0$
    - → follows from graded Jacobi and $\{Q, Q\} = 0$
  - ii) $H \geq 0$. Also, $H|\alpha\rangle = 0$ iff $Q|\alpha\rangle = Q^\dagger|\alpha\rangle = 0$.
    - → let $\beta = Q|\alpha\rangle$, $\gamma = Q^\dagger|\alpha\rangle$. Then
      $\langle\alpha|H|\alpha\rangle = \frac{1}{2}(\|\beta\|^2 + \|\gamma\|^2) \geq 0$.

- note that $Q_1 \equiv Q + Q^\dagger$ is iso $\mathcal{H}_B^{E>0} \xrightarrow[\sim]{Q_1} \mathcal{H}_F^{E>0}$
  so states come in boson-fermion pairs
  - → not true for ground states.

- Witten index: $\Omega := \dim \mathcal{H}_B^{E=0} - \dim \mathcal{H}_F^{E=0}$
  - invariant as long as deformations don't change boundary conds!
- claim: $\Omega = \operatorname{tr}_{\mathcal{H}} (-)^F e^{-\beta H}$, easy to see.

Example.

- 1D: $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})\,\bar{\psi} = \mathcal{H}_B \oplus \mathcal{H}_F$

- $[p, x] = -i$, $\{\psi, \bar{\psi}\} = 1$
- representation $p = -i\partial_x$, $\psi = \partial_{\bar{\psi}}$
  $Q := \bar{\psi}(ip + h'(x))$
  $\to Q^{+} := \psi(-ip + h'(x))$, $h \in C^\infty(\mathbb{R})$ superpotential

$$2H = p^2 + (h')^2 + i\bar{\psi}\psi[p, h'] - i\psi\bar{\psi}[p, h']$$
$$= p^2 + (h')^2 + h''(\bar{\psi}\psi - \psi\bar{\psi})$$

- susy g.s.? $Q|\alpha\rangle = Q^+|\alpha\rangle = 0$

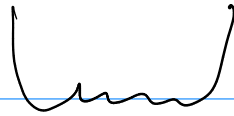
  $\to$ write it as $\varphi = \varphi_B(x) + \varphi_F(x)\,\psi$

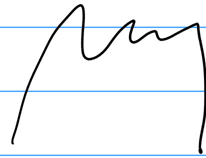
  $\to$ write that as $\varphi = \begin{pmatrix} \varphi_B \\ \varphi_F \end{pmatrix}$

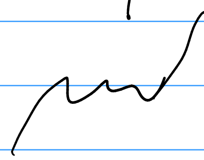
  $\to$ now $Q = \begin{pmatrix} 0 & 0 \\ \partial_x + h' & 0 \end{pmatrix}$, $Q^+ = \begin{pmatrix} 0 & -\partial_x + h' \\ 0 & 0 \end{pmatrix}$

- so $\begin{matrix} (\partial_x + h')\varphi_B = 0 \\ (-\partial_x + h')\varphi_F = 0 \end{matrix} \Rightarrow \varphi(x) = \begin{pmatrix} c_B e^{-h(x)} \\ c_F e^{+h(x)} \end{pmatrix}$

  $\to$ but $\varphi \in L^2(\mathbb{R}) \oplus \bar{\psi} L^2(\mathbb{R})$

- if $h(x) \xrightarrow[x\to\pm\infty]{} +\infty$  1 bosonic g.s., $\Omega = +1$

- if $h(x) \xrightarrow[x\to\pm\infty]{} -\infty$  1 fermionic g.s., $\Omega = -1$

- if $h(x) \xrightarrow[x\to\pm\infty]{} \mathrm{sgn}(x)\,\infty$  no susy vacua, $\Omega = 0$

- now put $h \mapsto \lambda h$ and $\lambda >> 0$.

$$h(x) = \frac{1}{2}\omega x^2 \Rightarrow V(x) = \frac{1}{2}\omega^2 x^2 = h'(x)^2$$

$\rightarrow$ so $\varphi_{\omega>0} = e^{-\frac{1}{2}\omega x^2}$ (standard h.o.)

$\varphi_{\omega<0} = e^{-\frac{1}{2}|\omega| x^2}\bar{\psi}$

- $E_{bos} = (n + 1/2)|\omega|$

- $H_f = \frac{1}{2}\omega[\bar{\psi}, \psi] = \frac{\omega}{2}\begin{pmatrix} -1 & \\ & +1 \end{pmatrix}$

so, $E_{ferm} = \pm\frac{|\omega|}{2}$

$\omega > 0$: B: 0, $|\omega|$, $2|\omega|$, ...
F: $\times$, $|\omega|$, $2|\omega|$, ...

$\omega < 0$: B: $\times$, $|\omega|$, $2|\omega|$, ...
F: 0, $|\omega|$, $2|\omega|$, ...

$$\mathrm{Tr}_B\, e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)|\omega|}$$

$$\mathrm{Tr}_F\, e^{-\beta H} = e^{-\beta\omega/2} + e^{\beta\omega/2}$$

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} = \frac{e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}}}{e^{\frac{\beta|\omega|}{2}} - e^{-\frac{\beta|\omega|}{2}}} = \mathrm{cth}\left(\frac{\beta|\omega|}{2}\right)$$

- but $\Omega = \mathrm{Tr}_{\mathcal{H}} (-)^F e^{\beta H} = \dfrac{e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}}}{e^{\frac{\beta|\omega|}{2}} - e^{-\frac{\beta|\omega|}{2}}} = \dfrac{\omega}{|\omega|}$

- $\mathbb{R}^{N|N}$, $\varphi(\bar\psi, x) = \sum\limits_{\substack{b_1 \dots b_n \\ \in \{0,1\}^n}} \phi_{b_1 \dots b_n}(x) (\bar\psi^1)^{b_1} \dots (\bar\psi^n)^{b_n}$

$Q = \sum\limits_I \bar\psi^I (i p_I + \partial_I h)$

$H = \sum\limits_I \frac{1}{2} p_I^2 + \frac{1}{2}(\partial_I h)^2 + \frac{1}{2} \sum\limits_{I,J} [\bar\psi_I, \psi_J] \partial_I \partial_J h$

- cannot solve in general due to couplings

$h \mapsto \lambda h$, $\lambda \to \infty$ gives

$\varphi_0 = \exp\left[-\frac{1}{2} \sum\limits_I |\omega_I| (x^I)^2\right] \prod\limits_{I : \omega_I < 0} \bar\psi^I$

- $\# I : \omega_I < 0$ is called Morse index $\lambda(p)$
  $\to$ h Morse function

$\Rightarrow$ $\Omega = (-)^{\lambda(p)}$

$\Omega = \sum\limits_{p, h'(p) = 0} (-)^{\lambda(p)} = \chi(Q - cpx)$

Tanzini.

SQM.

- on a manifold M, odd elements of H.lb. superspace

$\wr S$

differential forms

$\to$ so in particular we use that iso

to write

$$\langle \varphi^{(1)} | \varphi^{(2)} \rangle = \int \overline{\varphi^{(1)}} \wedge * \varphi^{(2)}$$

$$= \sum_{F=0}^{m} \int d^m x \sqrt{g}\, g^{I_1 J_1} \cdots g^{I_F J_F} \overline{\varphi}^{(1)}_{I_1 \cdots I_F} \varphi^{(2)}_{J_1 \cdots}$$

- free ptcl $(h=0)$ $\Rightarrow$ $Q = \overline{\psi}^I \nabla_I$

$\wr S$

d_{dR}

thus $H = \frac{1}{2}\{Q, Q^+\} \xrightarrow{\sim} \frac{1}{2}\Delta = \frac{1}{2}\{d, d^+\}$

- susy g.s. $\to$ $Q|\psi\rangle = Q^+|\psi\rangle = 0$

$\wr S$

harmonic forms on M.

harmonic forms

- given $b_F(M) = \dim H^F(M)$

- therefore

$$\Omega = \sum_{F=0}^{m} (-)^F b_F = \chi(M)$$

Digression. Let $P \to M$ $SO(2n)$-principal bdl.

- define Euler class $e(P) = \frac{1}{(2\pi)^n} Pf(F)$

curvature

- fact: $\chi(P) = \int_M e(P)$. In particular for $P = TM$,

$(2\pi)^n \chi(M) = \int_M Pf(R)$ gen. Gauss-Bonnet

- SUSY path int:

$$Z = \int [d\varphi \, d\psi \, d\bar{\psi}] \, e^{-S(\varphi,\psi,\bar{\psi})} = \mathrm{Tr}\,(-)^F e^{-\beta H}$$

$\{\varphi : S^1_\beta \to M,$
$\varphi(0) = \varphi(\beta)$
$\psi(0) = \psi(\beta)$
$\bar{\psi}(0) = \bar{\psi}(\beta)\}$

$\leftarrow$ we could pick antiperiodic conditions for fermions since we always deal w/ bilinears, but then we do not reproduce $\mathrm{Tr}\,(-)^F e^{-\beta H}$.

$$-\mathcal{L} = \frac{1}{2} g_{IJ} \dot{\varphi}^I \dot{\varphi}^J + g_{IJ} \bar{\psi}^I \underbrace{\overset{\varphi^*(\nabla_\mu)}{\overset{\shortparallel}{D_\tau}} \psi^J}_{\partial_\tau \psi^I + \Gamma^I{}_{JK} \partial_\tau \varphi^J \psi^K} + \frac{1}{2} R_{IJKL} \psi^I \bar{\psi}^K \psi^J \bar{\psi}^L$$

$$\delta \mathcal{L} = d_\tau (\cdots)$$

$$\delta \varphi^I = \epsilon \bar{\psi}^I - \bar{\epsilon} \psi^I$$
$$\delta \psi^I = \epsilon (-\dot{\varphi}^I - \Gamma^I{}_{JK} \psi^J \bar{\psi}^K)$$
$$\delta \bar{\psi}^I = \epsilon (\dot{\varphi}^I - \Gamma^I{}_{JK} \bar{\psi}^J \psi^K)$$

$$\delta(-) = 0 \Rightarrow \dot{\varphi}^I = 0 \to \text{constant maps } (\varphi_0, \psi_0, \underbrace{\bar{\psi}_0)}_{\substack{\Gamma^I_{JK}=0 \\ \text{first order}}}$$

$$Z = \int D\varphi_0 D\psi_0 D\bar{\psi}_0 \prod_{n \neq 0} D\varphi_n D\psi_n D\bar{\psi}_n \exp\Big[-\oint d\tau \Big(\frac{1}{2} g_{IJ} \dot{\varphi}^I \dot{\varphi}^J - g_{IJ} \bar{\psi}^I \partial_\tau \psi^J + \frac{1}{2} R_{IJKL} \psi_0^I \bar{\psi}_0^K \psi_0^J \bar{\psi}_0^L \Big)\Big]$$

where $\varphi(\tau) = \sum_{n \in \mathbb{Z}} \varphi_n e^{in\tau}$ etc.

Gaussian approx: lowest order

$$\left[\det{}' \left(-\frac{d^2}{d\tau^2}\right)\right]^{-1/2} \left(\det{}' \frac{d}{d\tau}\right) = 1$$

$$\Big[\prod_{n \neq 0} n^2\Big]^{-1/2} = \Big[\prod_{n>0} n^2\Big]^{-1}$$

$$\prod_{n \neq 0} in = \prod_{n>0} (in)(-in) = \prod_{n>0} n^2$$

Digression 2. (ζ-func. regularization)

- G hermitian op. w. discrete spectrum $\{\lambda_n\}$

$$\zeta_G(s) := \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{dt}{t} t^s \operatorname{Tr}\left(P_\perp e^{-tG}\right)$$

$$\overset{\text{(formally)}}{=} \sum_n \frac{1}{\lambda_n^s}$$

- then $\det' G := e^{-\zeta'_G(0)}$

Exercise: do this for spectrum of $-\frac{d^2}{dz^2}$ on $\frac{d}{dz}$.

$$Z = \int_{\Pi(TM)} \prod_{I=1}^{n} D\varphi_0^I D\psi_0^I D\bar{\psi}_0^I \, e^{-\frac{1}{2} R_{IJKL} \bar{\psi}^I \bar{\psi}^J \psi^K \psi^L}$$

$$= \int_M \operatorname{Pf}(R) = \chi(M)$$

Adding a superpotential.

- $\varphi: S^1 \to M$, $h: M \to \mathbb{R}$.

- operator formalism: $Q_h \doteq e^{-h} Q e^h = \bar{\psi}(\partial + h')$

$$\Big\downarrow \text{S}$$

$$d_h \doteq e^{-h} d e^h$$

- on the Hilb space, $|\alpha\rangle \mapsto e^{-h}|\alpha\rangle$

- we will be adding $\mathcal{L}_h \doteq \frac{1}{2} g^{IJ} \partial_I h \partial_J h + D_I \partial_J h \bar{\psi}^I \psi^J$ to our previous Lagrangian

- new fixed pts: constant maps to the crit pts of h

$$\Delta_B = -g_{IJ}\frac{d^2}{d\tau^2} + (D_I\partial_K h\, D_J \partial^K h)$$

$$\Delta_F = g_{IJ}\frac{d}{d\tau} + D_I\partial_J h$$

<u>$n\neq 0$</u>

$\to$ (B): $$\prod_{n\neq 0}\left(g_{IJ} n^2 + (D_I\partial_K h\, D_J\partial^K h)|_{\varphi_0}\right)^{-1/2}$$

$$= \prod_{n>0}\left(\text{—''—}\right)^{-1}$$

(F): $$\prod_{n\neq 0}\left(i n g_{IJ} + D_I\partial_J h|_{\varphi}\right) = \prod_{n>0}\left(i n g_{IJ} + \ldots\right)\left(-i n g_{IJ} + \ldots\right)$$

$$= \prod_{n>0}\left(n^2 g_{IJ} + D_I\partial_K h\, D_J\partial^K h\right)$$

<u>$n=0$</u>

$$\frac{\det D_I\partial_J h}{\left(\det(D_I\partial_K h\, D_J\partial^K h)\right)^{1/2}} = \text{sign}\det \text{Hess}\, h|_{\varphi_0}$$

$$\Rightarrow Z_{h\neq 0} = \sum_{\{\varphi_c:\ \partial h|_{\varphi_c}=0\}} \text{sign}\det\text{Hess}\, h = \chi(M) \text{ by Poincaré–Hopf}$$

Tanzini.

Morse theory

Def. $f: M \to \mathbb{R}$ ($\dim_{\mathbb{R}} M = m$) is Morse if $\mathrm{Hess}(f)$ is nondeg. at all crit. pts.

Lemma (Morse) In the nbhd of a crit. pt. x_c $\exists$ coordinates $\{x\}$ s.t.

$$f(x) - f(x_c) = -x_1^2 \cdots - x_{\lambda(p)}^2 + x_{\lambda(p)+1}^2 \cdots + x_n^2$$

where $\lambda(p) = \#$ neg. eigenvalues of $\mathrm{Hess}(f)|_{x_c}$

Cor. i) crit. pts are isolated
ii) M cpt $\Rightarrow$ $\#$ crit. pts finite

Exercise: prove corollary.

- topology: Poincaré polynomial $P_t(M) = \sum_p b_p t^p$,
$P_{-1}(M) = \chi(M)$

- let $M_p := \#$ cr. pts w/ p neg. eigenvalues (of Hessian)

$\to$ then: i) $\chi(M) = \sum_p (-)^p M_p$

ii) $M_p \geq b_p$ (weak Morse ineq.)

iii) $\sum_p M_p t^p - P_t(M) = (1+t) \sum_p Q_p t^p$, $Q_p \geq 0$ (strong Morse ineq.)

- back to S.Q.M.

- $\varphi : S^1 \longrightarrow M$, susy fixed pts $\dot{\varphi} = 0$
  $\varphi_c / \nabla h(\varphi_c) = 0$

$$\delta \psi^I = \varepsilon (\dot{\varphi}^I + \lambda g^{IJ} \partial_J h) = 0$$

$$0 \leq \int_{S^1} dt \, \frac{1}{2} \left| \frac{d\varphi^I}{dt} \pm \lambda g^{IJ} \partial_J h \right|^2$$

$$= \int_{S^1} dt \, \frac{1}{2} g_{IJ} \dot{\varphi}^I \dot{\varphi}^J + \frac{\lambda^2}{2} g^{IJ} \partial_I h \partial_J h$$

$$\underbrace{\pm \lambda \int_{S^1} \underbrace{\partial_I h(\varphi) \dot{\varphi}^I \, dt}_{\frac{dh}{dt}}}_{\text{vanishes for } S^1 \text{ but for } \Sigma \text{ w } \partial\Sigma \neq \phi \text{ doesn't}}$$

- for $\varphi : \mathbb{R} \to M$,

$$\lambda \int_{\mathbb{R}} dh = S(+\infty) - S(-\infty)$$
$$= \lambda \left| h(\varphi_c(+\infty)) - h(\varphi_c(-\infty)) \right|$$

- so $S_{SQM} \geq \lambda \left| h(\varphi_c(+\infty)) - h(\varphi_c(-\infty)) \right|$

$$\dot{\varphi}^I \pm \lambda g^{IJ} \partial_J h(\varphi) = 0$$

(anti) instantons or gradient flow lines

$\to e^{-\lambda}$, nonperturbative wrt $\mathcal{O}(\frac{1}{\lambda})$

- $Q = e^{-\lambda h} d\, e^{\lambda h}$, $Q^+ = e^{\lambda h} d^+ e^{-\lambda h}$

$$H = \frac{1}{2}\{Q, Q^+\} = \frac{1}{2}\Delta + \frac{1}{2}\lambda \nabla_I \partial_J h [\bar{\psi}^I, \psi^J]$$
$$+ \frac{\lambda^2}{2} g^{IJ} \partial_I h \partial_J h$$

- $\lambda \to +\infty$:

$$H(x_c) = \underbrace{\sum_{I=1}^{m} \frac{1}{2} p_I^2 + \frac{1}{2}\lambda^2 c_I^2 (x^I)^2 + \frac{1}{2}\lambda c_I [\bar{\psi}^I, \psi_I]}_{H_0} + O\left(\frac{1}{\lambda}\right)$$

- at i-th critical pt:

$$|a_i\rangle = e^{-\lambda \sum_I |c_I| x_I^2} \prod_{J: c_J < 0} \bar{\psi}^J |0\rangle$$ ← N.B. this means that at x_c we get $\mu(x_c)$-forms!

- $\# J =: \mu(p)$ Morse index

$$|a_i\rangle \in \Omega^{\mu(p_i)} \otimes \mathbb{C} =: X_{\mu_i}$$

$\underline{Q|a_i\rangle \neq 0}$, $0 \neq \langle a_j\, Q\, a_i\rangle$
no longer a ground state!

$$\langle a_j\, Q\, a_i\rangle = \int_M \underset{\mu_j}{\bar{a}_j} \wedge * \underbrace{(d + dh\wedge) a_i}_{\mu_i + 1}$$

$\Rightarrow \mu_j - \mu_i = 1$

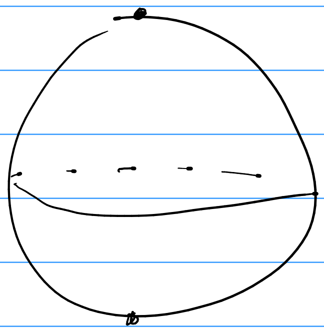
$\to \langle a_i Q a_j\rangle \neq 0$ gives $\mu_i - \mu_j = 1$

- introduce differential $\delta : X^\mu \to X^{\mu+1}$

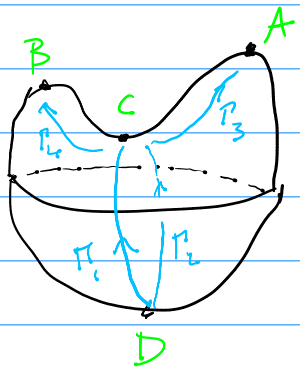
$$n(a_i, a_j) = \sum_\Gamma n_\Gamma, \quad n_\Gamma = \pm 1$$ ← Γ = path connecting crit pts differing by 1 dimension, n_Γ is direction of path

$$Q|a_i\rangle = \sum_{a_j} n(a_i, a_j) |a_j\rangle$$

$$\langle a_i \, Q \, a_j \rangle = \sum_{\Gamma : i \to j} n_\Gamma \, e^{-\lambda |h(i) - h(j)|}$$



$h^0(S^2) = 1$
$h^2(S^2) = 1$
$h^1(S^2) = 0$



$$Q|D\rangle = n^{CD}_{\Gamma_1} + n^{CD}_{\Gamma_2} = 0$$

$$Q|C\rangle = e^{-\lambda} n^{AC}_{\Gamma_3} |A\rangle + e^{-\lambda} n^{BC}_{\Gamma_4} |B\rangle$$
$$= e^{-\lambda} (|A\rangle - |B\rangle)$$

$$Q|A\rangle = Q|B\rangle = 0$$

$Q^\dagger |A\rangle \sim |C\rangle$, $Q^\dagger |B\rangle \sim -|C\rangle$

$Q^\dagger(|A\rangle + |B\rangle) = 0$

→ so in fact we get, again, 2 g.s.

→ $|D\rangle$ same as before, but the two new maxima are actually only admissable in superposition.

$$\varphi^I(t) = \gamma^I(t) + \xi^I$$
$$\hookrightarrow \dot{\gamma}^I - \lambda g^{IJ}\partial_J h(\gamma) = 0$$

$$S = \lambda |h(x_i) - h(x_j)| + \int_{\mathbb{R}} \left(\tfrac{1}{2}|D_-\xi|^2 - D_-\bar{\psi}^I \psi_I \right) dt$$

$$D_-\xi^I := D_t \xi^I - \lambda^{IJ} D_J \partial_K h \, \xi^K$$

- so in the path integral we will have a situation $\int (\ker D_-)(\ker D_-)^+$
- we also note $\mathrm{Ind}\, D_- = \dim \ker D_- - \dim \operatorname{coker} D$
$$= \mu_i - \mu_j = +1$$
- also assume $\dim \ker D_+ = 0$ (genericity assumption)

$$\int dt_0 \prod_I d\bar{\psi}_0^I \prod_{n \neq 0} d\xi_n^I \, d\psi_n^I \, d\bar{\psi}_n^I \, \exp^{-S}$$

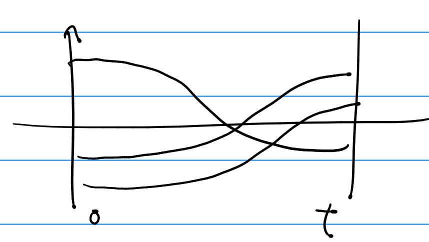
$$Q(\partial_I h) = D_J \partial_I h \, \bar{\psi}^I \psi^J = 0$$

$$\frac{\langle \partial_I h \bar{\psi}^I \rangle}{|h(\varphi_i) - h(\varphi_j)|} = \frac{1}{|\cdots|} \int dt_0 \prod_I d\psi_0^I \, \partial_I h \, \bar{\psi}_0^I \frac{\det' D_-}{\sqrt{\det D_-^+ D_-}} e^{-\lambda|h(\varphi_i) - h(\varphi_j)|}$$
$$= \frac{h(\varphi_i) - h(\varphi_j)}{|\text{—}\,''\,\text{—}|} = \pm 1$$

$$H(h) : T_x M \circlearrowleft, \quad v^I \to g^{IJ} D_J \partial_K h \, v^K$$

$$D_- = D_t - H(h)$$
- for $f_{I_{\mp}}(t) := e_I(t) \exp\left[\pm \int_0^t \lambda_I(t')\, dt'\right]$

we get $D_- f_I = 0$



Def. Morse func. is *perfect* if cr. pts have Morse index differing by at least 2
$\iff$ cbdry op $\delta \equiv 0$.

- example of p.M.f: moment map of circle action
$$i_v \omega = df$$

Rmk. Instead of looking at $Q \rightsquigarrow$ deRham, getting Witten index $= \chi(M)$, we can look at Atiyah-Singer index thm by putting $\bar{\psi}^I = \psi^I$.

$$S(\varphi, \psi) = \frac{1}{2} g_{IJ} \dot{\varphi}^I \dot{\varphi}^J + \psi^I D_t \psi^J g_{IJ}$$

Exercise: Prove that path int. of this action gives $\mathrm{Ind}(\not{D}) = \int_M \hat{A}(M)$
where $\hat{A}(M) = \prod_i \frac{x_i}{\operatorname{sh} x_i}$

- N.B. $\langle a_j Q a_i \rangle \underset{\lambda \to \infty}{\sim} \frac{1}{h(\varphi_i) - h(\varphi_j) + O(1/\lambda)} \langle a_j \underbrace{[Q, h]}_{\partial_I h \bar{\psi}^I} a_i \rangle$

and this localises

$\to$ this is why we computed $\langle \partial_I h \bar{\psi}^I \rangle$, it is precisely the overlap! (besides being susy inv)

Tanzini.

- TQFT ⇝ theory of maps $\underset{\substack{\downarrow \\ \text{worldvolume}}}{\Sigma_d} \xrightarrow{\varphi} \underset{\text{tgt. space}}{M}$
  - $\underline{d=0}$ | path int ⟷ ordinary int over M.
  - $\underline{d=1}$ | p.i ⟷ int. over loop space LM

    t

  - $\underline{d=2}$ | Σ called world sheet, strings

- to get localization in $d=2$, we need 2d susy

Spinors

- let $V \cong \mathbb{C}^d$ equipped w/ sym. bilinear form g
- define $\mathrm{Spin}(V)$ as the $\mathbb{Z}_2$-extension of $SO(V)$, i.e. $0 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \to SO(V) \to 0$

Def. let S be cpx Dirac spinor module of $\mathrm{Spin}(V)$, $\dim_{\mathbb{C}} S = 2^{[\frac{d}{2}]}$

- if d odd, S irred
- $S = S^+ \oplus S^-$, S^+, S^- Weyl modules (in $d=2$)

Def. Clifford algebra $\doteq T(V)/\langle v \cdot v - g(v,v) \rangle$

- pick $\{\gamma_\mu\}_{\mu=1\ldots d}$ $2^{[d/2]} \times 2^{[d/2]}$ matrices, provided we pick a basis for V, s.t. $\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathbb{1}$ → module on $Cl(V)$
  - called Dirac matrices

$\mathbb{R}^{2|4}$

- ϑ_α, $\alpha = +,-$ i.e. $\vartheta_\alpha = (\vartheta_+, \vartheta_-)$.
- $\bar{\vartheta}_\alpha := (\vartheta_\alpha)^+$, $\bar{\vartheta}^\alpha := \varepsilon^{\alpha\beta} \bar{\vartheta}_\beta$, $\varepsilon^{+-} = 1$

- $(\gamma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $(\gamma^2)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\{\gamma^m, \gamma^n\} = 2\delta^{mn}$

- Lorentz: $\vartheta \mapsto \exp\left[\frac{i}{2}\omega^{mn} S_{mn}\right]\vartheta$, $S_{mn} = \frac{i}{4}[\gamma_m, \gamma_n]$

  - explicitly, $\vartheta^\pm \mapsto e^{\pm i \frac{\omega}{2}} \vartheta^\pm$, $\bar\vartheta^\pm \mapsto e^{\mp i \frac{\omega}{2}} \bar\vartheta^\pm$ for $\omega_{mn} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$

- superfields: $\varphi(x^m, \vartheta^\pm, \bar\vartheta^\pm) = f(x^m) + \vartheta^+ f_+(x^m) + \bar\vartheta^+ g_+(x^m) + \dots + \vartheta^+\vartheta^-\bar\vartheta^+\bar\vartheta^- F(x^m)$

  - 2^4 components

- supercharges $\{Q_\alpha, \bar Q_\beta\} = 2i\gamma^m{}_{\alpha\beta}\partial_m$

  - realisation: $\begin{cases} Q_\alpha = \frac{\partial}{\partial\vartheta^\alpha} - i\gamma^m{}_{\alpha\beta}\bar\vartheta^\beta\partial_m \\ \bar Q_\alpha = -\frac{\partial}{\partial\bar\vartheta^\alpha} + i\vartheta^\beta\gamma^m{}_{\beta\alpha}\partial_m \end{cases}$

- Weyl modules $\begin{cases} Q_\pm = \frac{\partial}{\partial\vartheta^\pm} + \bar\vartheta^\pm\partial_\pm \\ \bar Q_\pm = -\frac{\partial}{\partial\bar\vartheta^\pm} - \vartheta^\pm\partial_\pm \end{cases}$, $\partial_\pm := \frac{1}{2}(\partial_2 \pm i\partial_1)$ Dolbeault on $\mathbb{R}^2 \cong \mathbb{C}$

  $\{Q_\pm, \bar Q_\pm\} = -2\partial_\pm = H \pm P$

  - in $d=1$ we only had $\{Q, \bar Q\} \sim H$, since there was no Lorentz group, now $P^m = (H, P)$

- SUSY transformation:

$$\delta\varphi(x^m, \vartheta, \bar\vartheta) = \underbrace{\left[\xi_\alpha Q^\alpha + \bar\xi^\alpha \bar Q_\alpha\right]}_{\xi_+ Q_- - \xi_- Q_+ - \bar\xi_+ \bar Q_- + \bar\xi_- \bar Q_+} \varphi$$

$\mathcal{N}=(2,2)$ superalgebra

$$\{Q_\pm, \bar{Q}_\pm\} = 2\partial_\pm$$

$$\{\bar{Q}_+, \bar{Q}_-\} = Z \quad , \quad \{Q_+, Q_-\} = Z^*$$
$$\{Q_-, \bar{Q}_+\} = \tilde{Z} \quad \{Q_+, \bar{Q}_-\} = \tilde{Z}^*$$
$$Q_\pm^2 = \bar{Q}_\pm^2 = 0$$
$$[iM, \overset{\leftrightarrow}{Q}_\pm] = \mp i \overset{\leftrightarrow}{Q}_\pm$$

$$[iF_V, Q_\pm] = -iQ_\pm \quad , \quad [iF_V, \bar{Q}_\pm] = i\bar{Q}_\pm$$
$$[iF_A, Q_\pm] = \mp i Q_\pm \quad , \quad [iF_A, \bar{Q}_\pm] = \pm i \bar{Q}_\pm$$

- vector sym. $\varphi(x^m, \vartheta, \bar{\vartheta}) \mapsto e^{i\alpha q_V}\varphi(x^m, e^{-i\alpha}\vartheta, e^{i\alpha}\bar{\vartheta})$
- axial sym. $\varphi(x^m, \vartheta, \bar{\vartheta}) \mapsto e^{i\alpha q_A}\varphi(x^m, e^{\mp i\beta}\vartheta^\pm, e^{\pm i\beta}\bar{\vartheta}^\pm)$

- superfields are reducible reps.

IRREPS of $\mathcal{N}=(2,2)$

- superspace derivatives $\begin{cases} D_\alpha = \frac{\partial}{\partial\vartheta^\alpha} + i\gamma^m_{\alpha\beta}\bar{\vartheta}^\beta\partial_m \\ \bar{D}_\alpha = -\frac{\partial}{\partial\bar{\vartheta}^\alpha} - i\vartheta^\beta\gamma^m_{\beta\alpha}\partial_m \end{cases}$

$$\Rightarrow \begin{cases} D_\pm = \frac{\partial}{\partial\vartheta^\pm} - \bar{\vartheta}^\pm\partial_\pm \\ \bar{D}_\pm = -\frac{\partial}{\partial\bar{\vartheta}^\pm} + \vartheta^\pm\partial_\pm \end{cases}$$

- chiral superfields $\bar{D}_\pm\varphi = 0$

- can be written as $\varphi(y^\pm, \vartheta^\pm)$ where
$y^\pm := x^\pm - \vartheta^\pm\bar{\vartheta}^\pm$
- $\bar{D}_\pm y^\pm = \frac{\partial}{\partial\bar{\vartheta}^\pm}(\vartheta^\pm\bar{\vartheta}^\pm) + \vartheta^\pm\partial_\pm x^\pm = 0$, $\bar{D}_\pm\vartheta^\pm = 0$

$$\varphi(y^\pm, \vartheta^\pm) = \varphi(y^\pm) + \vartheta^+\psi_+ + \vartheta^-\psi_- + \vartheta^+\vartheta^- F(y^\pm)$$

- twisted chiral superfields:

$$\overline{D}_+ U = D_- U = 0$$

- can be written as $U(\tilde{y}^\pm, \vartheta^+, \overline{\vartheta}^-)$
  where $\tilde{y}^\pm := x^\pm \mp \vartheta^\pm \overline{\vartheta}^\pm$

- chiral and twisted chiral superfields get exchanged under mirror sym.!
- and axial $\longleftrightarrow$ vector sym.

Tanzini.

$\mathcal{N}=(2,2)$ nonlinear σ model.

- maps $\Sigma_d \xrightarrow{\varphi} M$. general comments on spin in this case
- spinors: Dirac modules of $Spin(V)$
  - Spin acting on $SL(2,\mathbb{F})$ are
    $\mathbb{F}=\mathbb{R}$ (for $d=3$), $\mathbb{C}$ (for $d=4$),
    $\mathbb{H}$ (for $d=6$), $\mathbb{O}$ (for $d=10$)

- fields of (minimal) susy nonlin. σ model:

$$\varphi \in \mathcal{C}^\infty(\Sigma_d, M)$$
$$\psi \in \mathcal{C}^\infty(\Sigma_d, \Pi S \otimes_{\mathbb{F}} \varphi^*(TM))$$

↪ in $d=1$, $\Sigma_1 = S^1$
$\varphi \in LM$
$\psi \in \mathcal{C}^\infty(S^1, \varphi^*(TM))$

- so, susy twists the geometry! $D_t \psi^I = \partial_t \psi^I + \Gamma^I{}_{JK} \partial^J \varphi\, \psi^K$ s.t. $\varphi^*(TM)$

- TM endowed w left multiplication by $\mathbb{F}$
  - trivial only for $\mathbb{F}=\mathbb{R}$
  - for $\mathbb{F}=\mathbb{C}$ we need almost cpx. structure
    $J: TM \circlearrowleft$, $J^2 = -\mathrm{id}_{TM}$
  - for others, some other tensor controls it
  - → we want its covariant derivative to vanish, which is ⟺ to ∇ being Levi-Civita
    - → so for $\mathbb{F}=\mathbb{C}$, M is Kähler,
      for $\mathbb{F}=\mathbb{H}$, M is Hyperkähler

$\dim_{\mathbb{R}} M = 2n$, holonomy group $SO(2n) \xrightarrow[\text{to}]{\text{reduces}} U(n)$
$\dim_{\mathbb{R}} M = 4k$, — " — $Sp(k)$

Thm minimal susy nonlin. σ models $\exists$ iff $\left\{\begin{array}{l} d=3 \ldots M \text{ Riemannian} \\ d=4 \ldots M \text{ Kähler} \\ d=6 \ldots M \text{ Hyperkähler} \end{array}\right.$ look at Freed, Deligne?

- for $\mathcal{N}=(2,2)$ susy we need $\dim_{\mathbb{R}} \Sigma = 4$,
since $\dim S = 2^{[\frac{d}{2}]}$

- assume Σ closed orientable Riem. sfc.

- $Q_+, \bar{Q}_+$ sections of $K_\Sigma^{1/2}$ $\quad (K_\Sigma = T^{\vee(1,0)}\Sigma)$
  $Q_-, \bar{Q}_-$ — " — $\bar{K}^{1/2}_\Sigma$

- $\Sigma = \mathbb{C}$, $\mathbb{R}^{2|4}$

$$S = \int d^2z \, d^2\vartheta \, d^2\bar{\vartheta} \, K(\varphi^i, \bar{\varphi}^{\bar{\imath}}) + \int d^2z \, d\vartheta^+ d\vartheta^- \, W(\varphi^i)\big|_{\bar{\vartheta}^{\pm}=0} + c.c.$$

- K real, called D-term
- W holom., called superpotential or F-term

- since M Kähler, write
$$ds^2 = g_{i\bar{j}} \, d\varphi^i \odot d\varphi^{\bar{j}}$$
$$\omega = \tfrac{1}{2} i \, g_{i\bar{j}} \, d\varphi^i \wedge d\varphi^{\bar{j}}, \quad d\omega = 0$$

$$d\omega = 0 \implies \begin{matrix} \partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}} \\ \partial_{\bar{k}} g_{i\bar{j}} = \partial_{\bar{j}} g_{i\bar{k}} \end{matrix} \implies g_{i\bar{j}} = \frac{\partial^2 K}{\partial\varphi^i \partial\varphi^{\bar{j}}}$$

Rmk $\quad K(\varphi, \bar{\varphi}) \mapsto K(\varphi, \bar{\varphi}) + f(\varphi) + \bar{f}(\bar{\varphi})$
does not change the metric
$\to$ in the Lagrangian it gives
total derivatives

Exercise. show that
$$\mathcal{L} = \int d^4\vartheta \, K(\varphi^i, \bar{\varphi}^{\bar{\imath}}) = g_{i\bar{j}} \partial_z \varphi^i \partial_{\bar{z}} \varphi^{\bar{j}} + i g_{i\bar{j}} \bar{\psi}^{\bar{j}}_- D_z \psi_-^i$$
$$+ i g_{i\bar{j}} \bar{\psi}_+^{\bar{j}} D_{\bar{z}} \psi_+^i + R_{i\bar{j}k\bar{l}} \psi_+^i \psi_+^{\bar{j}} \psi_-^k \bar{\psi}_-^{\bar{l}}$$
$$+ g_{i\bar{j}} (F^i - \Gamma^i_{jk} \psi_+^j \psi_-^k)(F^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \bar{\psi}_-^{\bar{k}} \psi_+^{\bar{l}})$$

$$D_{\bar{z}} \psi^i := \partial_{\bar{z}} \psi^i + \Gamma^i_{jk} \partial_{\bar{z}} \varphi^j \psi^k$$

- however, it turns out only $\Sigma \simeq \mathbb{T}^2$ admits global sections !
  → we solve this by top. twist

- R-symmetries
  - vector: $Q_\pm \mapsto e^{-i\alpha} Q_\pm$
  - axial: $Q_\pm \mapsto e^{\mp i\beta} Q_\pm$

- topological twist:

$$U(1)'_E = \begin{cases} \mathrm{diag}(U(1)_E \times U(1)_V) & \text{A-twist} \\ \mathrm{diag}(U(1)_E \times U(1)_A) & \text{B-twist} \end{cases}$$

- redefinition of the spin connection

$$\partial_z + \underset{\text{spin}}{\underset{\uparrow}{\omega_z}} + \underset{U(1)_R}{\underset{\uparrow}{A_z}}$$

	Σ model $U(1)_V$	$U(1)_A$	$U(1)_E$	A model	B model
φ	0	0	0	φ	φ
ψ^i_-	-1	1	1	$\chi^i \in \Gamma(\varphi^* T^{1,0}M)$	$\rho^i_z \in \Gamma(\varphi^*(T^{1,0}M) \otimes K_\Sigma)$
$\bar\psi^{\bar i}_+$	1	1	-1	$\bar\chi^{\bar i} \in \Gamma(\varphi^* T^{0,1}M)$	$-\frac{1}{2}(\bar\upsilon + \bar\eta)^{\bar i} \in \Gamma(\varphi^* T^{0,1}M)$
$\bar\psi^{\bar i}_-$	1	-1	-1	$\rho^{\bar i}_z \in \Gamma(\varphi^*(T^{0,1}M) \otimes K_\Sigma)$	$\frac{1}{2}(\bar\upsilon - \bar\eta)^{\bar i} \in \Gamma(\varphi^* T^{0,1}M)$
ψ^i_+	-1	-1	-1	$\rho^i_{\bar z} \in \Gamma(\varphi^*(T^{1,0}M) \otimes \bar K_\Sigma)$	$\rho^i_{\bar z} \in \Gamma(\varphi^*(T^{1,0}M) \otimes \bar K_\Sigma)$

- A twist: Q_-, $\bar Q_+$ are scalars
  B twist: $\bar Q_+$, $\bar Q_-$ are scalars

- define

$$Q_A := \bar Q_+ + Q_- \quad \rightsquigarrow \varphi : \Sigma \to M \quad \text{(symplectic)}$$
$$Q_B := \bar Q_+ + \bar Q_- \qquad \text{((Calabi-Yau))}$$

Rmk. $\mathbb{Z}_2$ automorphism $Q_- \leftrightarrow \bar Q$, $F_V \leftrightarrow F_A$ ⇒ exchanges A & B models

Tanzini

- $\mathcal{N}=(2,2)$ nonlin σ model
- $S = \int d^2 z\, d^4\vartheta\, K(\vartheta^I, \bar{\vartheta}^J) + \int d^2 z\, d\vartheta^+ d\vartheta^- W(\varphi^I)$

real func, Kähler potential of tgt mfd M

holom field, superpotential

- top. twist: $U(1)_E' = \begin{cases} \mathrm{diag}(U(1)_E \otimes U(1)_V) & \text{A} \\ \mathrm{diag}(U(1)_E \otimes U(1)_A) & \text{B} \end{cases}$ models

- basically, if $\psi_- \xmapsto[U(1)_E]{} e^{i\alpha}\psi_-$, $\psi_- \xmapsto[U(1)_V]{} e^{-i\beta}\psi_-$, say, then

$$\psi_- \mapsto \mathrm{diag}(e^{i\alpha} \otimes_{\mathbb{C}} e^{-i\beta})\psi_- = e^{i(\alpha-\alpha)}\psi_-.$$

- noting that $\vartheta_\pm \xmapsto[U(1)_V]{} e^{-i\alpha}\vartheta_\pm$, $\vartheta_\pm \xmapsto[U(1)_A]{} e^{\pm i\beta}\vartheta_\pm$,

to preserve $U(1)_V$, either $W(\varphi^I) \equiv 0$, or is quasihomogeneous of charge 2, because $d\vartheta_+ d\vartheta_- \mapsto e^{-2i\alpha} d\vartheta_+ d\vartheta_-$

Axial anomaly

- toy model $S = \int_{\mathbb{T}^2} d^2 z\, i(\bar{\psi}_+ D_z \psi_+ + \bar{\psi}_- D_{\bar{z}} \psi_-)$, $\psi_\pm = \Gamma(\mathbb{T}^2, E \otimes S_\pm)$

$D_z = \partial_z + A_z$, $D_{\bar{z}} = \partial_{\bar{z}} + A_{\bar{z}}$ where $A = A_z dz + A_{\bar{z}} d\bar{z}$ Hermitean conn.

- suppose $K := \int_{\mathbb{T}^2} c_1(E) > 0$, then $\mathrm{Ind}\, D_{\bar{z}} = K > 0$, so the measure cannot be invariant under axial symmetry? There is a zero-modes mismatch?

$\#\,\psi_-$ zero modes $> \#\,\bar{\psi}_-$ zero modes

- genericity assumption $\mathrm{Ker}\, D_{\bar z} = \emptyset$ gives

$$\text{any correlator} = \int \prod_{a=1}^{k} \underset{\substack{\uparrow \\ \text{not ok}}}{d\psi_-^{(0)}} \int \prod_{n\neq 0} d\psi_-^{(n)} d\bar\psi_-^{(n)} \cdots$$

e.g. $\langle \psi_-(z_1)\cdots\psi_-(z_k)\bar\psi_+(w_1)\cdots\bar\psi_+(w_k)\rangle$
is $U(1)_V$ inv, but gains
$e^{2ik\beta}$ factor under $U(1)_A$

- anomaly breaks $U(1)_A \longrightarrow \mathbb{Z}_k$

- expanding $\psi_\pm, \bar\psi_\mp$ in eigenfuncs of $D_{\bar z}^\dagger D_{\bar z}, D_z^\dagger D_z$

$$\bar\psi_- = \sum_{n\geq 1} b_n \bar\varphi_-^n \qquad \psi_- = \sum_{a=1}^{k} c_{0a}\varphi_-^{0a} + \sum_{n\geq 1} c_n \varphi_-^n$$

$$\psi_+ = \sum_{n\geq 1} \tilde b_n \varphi_+^n \qquad \bar\psi_+ = \sum_{a=1}^{k} \tilde c_{0a}\bar\varphi_+^{0a} + \sum_{n\geq 1} \tilde c_n \bar\varphi_+^n$$

gives

$$D\psi D\bar\psi\, e^{-S} = \prod_{a=1}^{k} dc_{0a}\, d\tilde c_{0a} \prod_{n\geq 1} db_n\, dc_n\, d\tilde b_n\, d\tilde c_n\, e^{-\sum_{n\geq 1}\lambda_n(b_n c_n + \tilde b_n \tilde c_n)}$$

- for $\mathcal{N}=(2,2)$ nonlinear σ model,
$\mathcal{L} = -2i\, g_{I\bar J}\, \bar\psi_-^{\bar J} D_{\bar z} \psi_-^I + \text{conj.}$

but $k = \int_\Sigma c_1(\varphi^*(T^{(1,0)}M)) = \int_\Sigma \varphi^* c_1(M) \neq 0$ in general

$\Rightarrow U(1)_A$ preserved if $c_1(M)=0$
$\Rightarrow (M \text{ Kähler}) \wedge (c_1(M)=0) = (M \text{ is Calabi-Yau})$

- our cases:

	$U(1)_V$	$U(1)_A$	
CY	✓	✓	
$C_1(M)\neq 0$	✓	×	
$W\neq 0$	×	✓	(Landau-Ginzburg)
$W\neq 0$ but quasihom. deg 2	✓	✓	

- to understand A, B models, understand susy fixed pts
- recall

$$\delta\phi = \left[\xi_+ Q_- - \xi_- Q_+ - \bar\xi_+ \bar Q_- + \bar\xi_- \bar Q_+\right]\phi$$

for superfield $\phi^I = \varphi^I + \theta^+\psi_+^I + \bar\theta^-\bar\psi_-^I + \ldots$ gives in components

$$\delta\varphi^I = \xi_+\psi_-^I - \xi_-\psi_+^I$$
$$\delta\varphi^{\bar I} = -\bar\xi_+\bar\psi_-^{\bar I} + \bar\xi_-\bar\psi_+^{\bar I}$$
$$\delta\psi_+^I = 2\bar\xi_-\partial_{\bar z}\varphi^I + \xi_+\left(\Gamma^I{}_{JK}\psi_+^J\psi_-^K - \tfrac{1}{2}g^{I\bar L}\partial_{\bar L}\bar W\right)$$
$$\delta\psi_-^I = -2\bar\xi_+\partial_z\varphi^I + \xi_-\left(\quad -\!\!\;\;\text{''}\;\;- \quad\right)$$
$$\delta\bar\psi_+^{\bar I} = -2\xi_-\partial_{\bar z}\varphi^{\bar I} + \bar\xi_+\left(\Gamma^{\bar I}{}_{\bar J\bar K}\bar\psi_-^{\bar J}\bar\psi_+^{\bar K} - \tfrac{1}{2}g^{\bar JL}\partial_L W\right)$$
$$\delta\bar\psi_-^{\bar I} = 2\xi_+\partial_z\bar\varphi^{\bar I} + \bar\xi_-\left(\quad -\!\!\;\;\text{''}\;\;- \quad\right)$$

- for A-model, $Q_A = \bar Q_+ + Q_-$, so set
$\xi_+ = \bar\xi_- = 1$, $\xi_- = \bar\xi_+ = 0$

$$Q_A\varphi^I = \chi^I,\quad Q_A\varphi^{\bar I} = \chi^{\bar I} \qquad (\psi^I \mapsto \chi^I \text{ in A model})$$
$$Q_A\chi^I = 0,\quad Q_A\chi^{\bar I} = 0$$
$$Q_A\, \rho_{\bar z}^I = 2\partial_{\bar z}\varphi^I + \Gamma^I{}_{JK}\,\rho_{\bar z}^J\chi^K$$
$$Q_A\, \rho_z^{\bar I} = 2\partial_z\varphi^{\bar I} + \Gamma^{\bar I}{}_{\bar J\bar K}\,\rho_z^{\bar J}\chi^{\bar K}$$

- fixed pts? $\chi^I = \chi^{\bar{I}} = 0$, $\partial_{\bar{z}} \varphi^I = \partial_z \varphi^{\bar{I}} = 0$

→ holomorphic maps $\varphi : \Sigma \to M$

- B model ($W = 0$), $Q_B = \bar{Q}_+ + \bar{Q}_-$, $\bar{\zeta}_- = 1$, $\bar{\zeta}_+ = -1$, $\zeta_+ = \zeta_- = 0$

$$Q_B \varphi^I = 0, \quad Q_B \varphi^{\bar{I}} = \bar{\eta}^{\bar{I}}$$

$$Q_B \eta^I = 0, \quad Q_B \bar{\eta}^{\bar{I}} = 0$$

$$\left.\begin{aligned} Q_B \, \varsigma^I_z &= -2\, \partial_z \varphi^I \\ Q_B \, \varsigma^{\bar{I}}_{\bar{z}} &= -2\, \partial_{\bar{z}} \varphi^I \end{aligned}\right\} \quad Q_B \, \varsigma^I = -2\, d\varphi^I$$

- fixed pts? $d\varphi^I = 0$ constant maps

Tanzini

- recap: looked at $\mathcal{N}=(2,2)$ susy in 2d, nonlin σ models $\Sigma \xrightarrow{\varphi} M$, we required M to be (generalised) Kähler
- top. twist $\longrightarrow$ A model, vector R sym
  $\searrow$ B model, axial R sym.
  - for B model we require $c_1(M)=0$ as to not break axial R-sym
- $Q_A = \bar{Q}_+ + Q_-$, $Q_B = \bar{Q}_+ + \bar{Q}_-$ are now nilpotent scalars

Observables.

- recall $\{Q_\pm, \bar{Q}_\pm\} = -2\partial_\pm$, $\{\bar{Q}_+, \bar{Q}_-\} = Z$, $\{Q_-, \bar{Q}_+\} = \tilde{Z}$ with $Z, \tilde{Z}$ central

Rmk. $\tilde{Z}$ breaks $U(1)_A$, Z breaks $U(1)_V$
- so we put $Z = \tilde{Z} = 0$

- since $\langle Q_{A,B} - \rangle = 0$ automatically, $Q_{A,B}$-cohomology gives nontrivial observables

B model.
- $Q_B = \bar{Q}_+ + \bar{Q}_-$
- recall $\bar{Q}_\pm = -\frac{\partial}{\partial \bar{\vartheta}^\pm} - \vartheta^\pm \partial_\pm$, $\bar{D}_\pm = -\frac{\partial}{\partial \bar{\vartheta}^\pm} + \vartheta^\pm \partial_\pm$

- note that first component of chiral superfield is Q_B-closed:
$$\bar{Q}_\pm \varphi = \bar{Q}_\pm \Phi \Big|_{\vartheta^\pm, \bar{\vartheta}^\pm = 0} = (\bar{D}_\pm - 2\vartheta^\pm \partial_\pm)\Phi \Big|_0 = 0$$

- so B-model observables are functs of φ

A model

- observables are lowest comps of twisted chiral sfields

Chiral ring

- note we can define chiral ring, since $\Phi_1 \cdot \Phi_2$ chiral if Φ_1, Φ_2 are
  - basis $\{\varphi_i\}_{i=1}^{d}$, $\varphi_i \varphi_j = C^k{}_{ij}\varphi_k + (Q\text{-boundary})$
  - since unit $\varphi_0 = 1$ chiral,
    $\varphi_0 \varphi_j = C^k_{0j}\varphi_k \Rightarrow C^k_{0j} = \delta^k{}_j$
  - associativity gives $C^m_{il} C^l_{jk} = C^l_{ij} C^m_{lk}$
  - Frobenius algebra

Properties

1) Independence from insertion pt on Σ
- if $Q_B \mathcal{O}_B = 0$,

$$-2\partial_{\bar z}\mathcal{O}_B = [H+P, \mathcal{O}_B] = [\{Q_+, \bar Q_+\}, \mathcal{O}_B]$$

$$\overset{\text{sJac.}}{=} \{[Q_+, \mathcal{O}_B], \bar Q_+\} - \{Q_+, [\bar Q_-, \mathcal{O}_B]\}$$

$$\overset{\substack{\text{sJac}\\ \text{on 2nd}}}{=} \{\bar Q_+, [Q_+, \mathcal{O}_B]\} - [\underbrace{\{Q_+, \bar Q_-\}}_{=0}, \mathcal{O}_B]$$

$$+ \{\bar Q_-, [Q_+, \mathcal{O}_B]\}$$

$$= \{Q_B, [Q_+, \mathcal{O}_B]\}$$

- therefore, $\partial_{\bar z_j} \langle \prod \mathcal{O}_i(z_j, \bar z_j)\rangle = \langle \mathcal{O}_B \cdots \rangle = 0$

- also note that $[Q_+, \mathcal{O}_B]$ can be thought of as "$\int \mathcal{O}^{(1)}$", with $\mathcal{O}^{(1)}$ a 1-form observable

Descent equations

- for $\mathcal{O}^{(0)}$ Q_B-closed, we get
  $d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\}$ where $\mathcal{O}^{(1)} = \{Q_+ + Q_-, \mathcal{O}^{(0)}\}$
- continue: $d\mathcal{O}^{(1)} = \{Q, \mathcal{O}^{(2)}\}$, $\mathcal{O}^{(2)} = \{Q_+, [Q_-, \mathcal{O}^{(0)}]\}$

- so susy charges $Q_\pm$ act as ladder ops

II) Independence from tgt sp. metric

- we vary F-term since it contains tgt metric as kähler potential

$$\int d^4\vartheta \, \Delta K \overset{\text{Berezin rules}}{=} \frac{\partial}{\partial\vartheta^+}\frac{\partial}{\partial\bar\vartheta^-}\frac{\partial}{\partial\vartheta^+}\frac{\partial}{\partial\vartheta^-}\Delta K \Big|_{\vartheta^\pm = \bar\vartheta^\pm = 0}$$

$$\sim \{\underbrace{\bar Q_+,}_{\text{add } \bar Q_- \text{ for free}} [\bar Q_-, \int d\vartheta^+ d\vartheta^- \Delta K]\}\Big|_{\vartheta^+ = \vartheta^- = 0} = \{Q_B, \dots\}$$

III) indep. from twisted chiral deformations

$$\int d^2 z \sqrt{h}\, d\vartheta^+ d\bar\vartheta^- \Delta\tilde{w} \sim \int d^2 z \sqrt{h}\, \{Q_+, [\underbrace{\bar Q_-,}_{\text{add } \bar Q_+} \Delta w]\}$$

$$\sim \{Q_B, \dots\}$$

IV) ... antichiral deformations

$$\int d^2 z \sqrt{h}\, d^2\bar\vartheta \, \Delta\bar w(\bar\varphi) \sim \int \dots \{\bar Q_+, [\bar Q_-, \Delta\bar w)\}$$

$$\sim \{Q_B, \dots\}$$

v) dependance on chiral sector

$$\int \sqrt{h}\, d^2 z \int d^2\theta\, \Delta W \sim \int \cdots \underbrace{\{ Q_+ , [Q_- , \Delta W]\}}_{= \Delta W^{(2)}}$$

- for A model, depends on twi. chiral, but not on chiral

- what are struct constants?
- on 2-sphere study $S^2 \to M$

$C_{ijk} := \langle \varphi_i \varphi_j \varphi_k \rangle$, let $C_{ij0} = \langle \varphi_i \varphi_j \rangle \equiv \eta_{ij}$

$$= \langle \varphi_i C^{\ell}{}_{jk} \varphi_\ell \rangle = C^{\ell}{}_{jk} \eta_{i\ell}$$

- $\partial_\ell \langle \varphi_i \varphi_j \varphi_k \rangle = \langle \varphi_i \varphi_j \varphi_k \int d^2 z \sqrt{h}\, \mathcal{O}^{(2)}_\ell \rangle$

where $\mathcal{O}^{(2)}_\ell = \{ Q_+ , [Q_- , \varphi_\ell]\}$

- use $PSL(2,\mathbb{C})$ to fix 3 pts

$\Rightarrow \partial_\ell C_{ijk} = \partial_i C_{\ell jk}$ WDVV eqns

$\Rightarrow C_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}$, $\mathcal{F}$ prepotential

Tanzini

- we saw how the chiral ring defines a unital $\mathbb{C}$-alg. w Frobenius property
  - A $\rightsquigarrow$ twisted chiral, B $\rightsquigarrow$ chiral

A model (with $W \equiv 0$)

- $\Sigma \xrightarrow{\varphi} M$, $Q_A(\varphi^I, \varphi^{\bar I}) = (\chi^I, \chi^{\bar I})$, $Q_A(\chi^I, \chi^{\bar I}) = 0$

$$Q_A \, s_{\bar z}^{I} = \partial_{\bar z}\varphi^I + \Gamma^I{}_{JK}\, s_{\bar z}^J \chi^K$$
$$Q_A \, s_z^{\bar I} = \partial_z \varphi^{\bar I} + \Gamma^{\bar I}{}_{\bar J \bar K}\, s_z^{\bar J} \chi^{\bar K}$$

- identifying $\chi^I \rightsquigarrow dz^I$, $Q_A \rightsquigarrow d = \partial + \bar\partial$ we get

$$\underbrace{Q_A\, \omega(\varphi)_{z_1 \dots z_p \bar z_1 \dots \bar z_q} \chi^{i_1} \dots \chi^{i_p} \chi^{\bar i_1} \dots \chi^{\bar i_q} = 0}_{\omega^{(p,q)} \text{ closed}}$$

- fixed locus, $Q_A s_{\bar z}^I = \partial_{\bar z}\varphi^I = 0, \dots = \partial_z \varphi^{\bar I} = 0$ Cauchy-Riemann, so $\Sigma \to \varphi_*(\Sigma) \in H_2(M, \mathbb{Z})$

- $\omega \in H^{1,1}(M)$ kähler form,

$$\int_\Sigma \varphi^*(\omega) = \int_\Sigma (-i) \overbrace{\omega_{i\bar j}}^{i g_{i\bar j}} \left(\partial_z \varphi^i \partial_{\bar z}\varphi^{\bar j} - \partial_{\bar z}\varphi^i \partial_z \varphi^{\bar j}\right) d^2 z$$

- but $S = \int_\Sigma g_{i\bar j}\left(\partial_z\varphi^i \partial_{\bar z}\varphi^{\bar j} + \partial_{\bar z}\varphi^i \partial_z \varphi^{\bar j}\right) d^2 z$

$$= \underbrace{\int_\Sigma g_{i\bar j}\, \partial_{\bar z}\varphi^i \partial_z \varphi^{\bar j}}_{=0 \text{ for holomorphic}} + \int_\Sigma \varphi^*(\omega) \geq \underset{\deg(\varphi)}{\overset{\shortparallel}{K}}$$

- for strings, introduce B-field
  $\Rightarrow \omega_{\mathbb{C}} = \omega + iB$

- on the tgt mfd we look at the Kähler cone, given by cycles C, D, M (in say $d=3$) satisfying $\int_C \omega > 0$, $\int_D \omega^2 > 0$, $\int_M \omega^3 > 0$.

B model

- $Q_B(\varphi^i, \varphi^{\bar{i}}, \theta^i, \eta^{\bar{i}}, \rho^i_z, \rho^{\bar{i}}_{\bar{z}})$

$$\shortparallel$$

$$(0, \eta^{\bar{i}}, 0, 0, \partial_z \varphi^i, \partial_{\bar{z}} \varphi^{\bar{i}})$$

$\rightarrow \eta^{\bar{i}} \rightsquigarrow dz^{\bar{i}}$, $\theta_i \rightsquigarrow \frac{\partial}{\partial z^i}$

- so $\omega_{\bar{i}_1 \ldots \bar{i}_p}{}^{j_1 \ldots j_q} \eta^{\bar{i}_1} \ldots \eta^{\bar{i}_p} \theta_{j_1 \ldots j_q}$

$$\wr$$

$$\omega_{\bar{i}_1 \ldots \bar{i}_p}{}^{j_1 \ldots j_q} dz^{\bar{i}_1} \ldots dz^{\bar{i}_p} \partial_{j_1} \ldots \partial_{j_q}$$

has to lie in $H^{0,p}(M, \Lambda^q T^{1,0} M)$

LG model (B model w/ $W \neq 0$)

- change wrt B model: $Q_{LG} \theta^{\bar{i}} = g^{\bar{i}j} \partial_j W$
- here $M = \mathbb{C}^n$
- Q_{LG} cohomology $\simeq \mathbb{C}[\varphi^1 \ldots \varphi^n] / (\partial_j W)$

- for B model, fixed pts = constant maps
- for LG, const. maps to crit. pts of W!

Calabi-Yau moduli space

- CY_n is Kähler with, equivalently:
  - $c_1(CY_n) = 0$
  - holonomy $\subset SU(n)$
  - trivial canonical bdl

- Hodge diamond:
  - $h^{n,0} = 1$
  - $h^{1,0} = h^{0,1} = 0$ by simple connectedness
  - $H^1(\mathcal{J}) \simeq H^2(\mathcal{J} \times K_M)^\vee \Rightarrow h^{2,0} = h^{0,2} = 0$
  - for 3-folds:

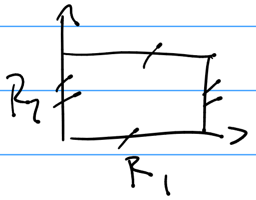
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & & 1 & & & \end{array}$$

  $\rightarrow$ by symmetry, only depends on h_{11}, h_{21}

- for cpx structure $\Omega^{3,0}$,
  $$\Omega + \delta\Omega = \Omega + \mu^a{}_{\bar b}\, \Omega_{abc}\, dz^{\bar b} dz^b dz^c,$$
  with $\mu \in H^1(M, TM)$
- so cpx deformation moduli dimension is $h_{21} \underbrace{+1}_{\text{rescaling of } \Omega}$

$\Rightarrow$ (Tian-Todorov) no obstructions

- simple example of cpt CY is $\mathbb{T}^2 \simeq \mathbb{C}/\Lambda$



A: kähler modulus $\sim R_1 R_2$

B: cpx modulus $\sim i\, R_2/R_1$

- for $\mathcal{N}=(2,2)$
$$Q_- \longleftrightarrow \overline{Q}$$
$$F_V \longleftrightarrow F_A$$
$$(z \longleftrightarrow \tilde{z})$$

- note, you can also twist $\mathcal{N}=(0,2)$ susy, although geometrically still unclear

Tanzini

Cohomological field theory [Witten 90's, Trieste]

- top. theory of coh. type
  = intersection theory on mod. sp. of solns of PDEs
- moduli space = zero locus of ∞-dim vect bdl
  $$\mathcal{M} = \{ \varphi \in \mathcal{E} \mid s(\varphi) = 0 \} / G$$
- path int = Mathai - Quillen repr of Thom class
  of the ∞-dim vect bdl
  - morally, $Pf(R)\, e^{-\varepsilon \| s \|^2}$
- equiv. coh wrt G

- physically we have:

  - Space of fields
  - action $\leftrightarrow$ (BPS) e.o.m. / choice of section
    is Q-exact
    $\Rightarrow T_{\mu\nu}$ is Q-exact
  - symmetries $\leftrightarrow$ G-equiv coh

- A model fits

we called this $S_{\bar z}$?

$$S_A = \overbrace{\int_\Sigma \varphi^*(\omega + iB)}^{S_{top}} + \overbrace{i \int d^2 z\, Q_A \big[g_{i\bar j} (\bar\psi^{i}_{\bar z}\, \partial_z \varphi^{\bar j} + \overbrace{\bar\psi^{\bar j}_{z}\, \partial_{\bar z} \varphi^{i}}^{=0})}^{S_{g.f.}}$$

$$= i \int d^2 z \Big[\tfrac{1}{2} g_{\mu\nu}\, \partial_z \varphi^\mu \partial_{\bar z} \varphi^\nu + g_{i\bar j} (\bar\psi^{i}_{\bar z}\, D_z \psi^{\bar j} + \bar\psi^{\bar j}_{z}\, D_{\bar z} \psi^{i})$$

$$- R_{i\bar i j \bar j}\, \bar\psi^{i}_{\bar z}\, \bar\psi^{\bar i}_{z}\, \psi^{j}_{\bar z}\, \psi^{\bar j}_{z}$$

Q_A - fixed pts
$$\partial_z \varphi^{\bar i} = \partial_{\bar z} \varphi^{i} = 0 \quad \rightsquigarrow \mathcal{H}ol(M)$$
$$D_z \psi^{\bar i} = D_{\bar z} \psi^{i} = 0 \quad \rightsquigarrow TM$$
$$D_{\bar z} \bar\psi^{\bar i}_{z} = D_z \bar\psi^{i}_{\bar z} = 0 \quad \rightsquigarrow \text{obstructions}$$

$$\int D\varphi D\psi D\bar\psi\, e^{-S_{top}-S_{g.f.}} = \int_{\mathcal{M},\ker D_{\bar z},\ker D_{\bar z}^\dagger} dm\, d\psi^{(0)} d\bar\psi^{(0)} e^{-S_{top}} \cancel{\frac{\det'\Delta_f}{\det'\Delta_\xi}}^{=1}$$

bosonic part – mod. sp. of hol maps $\Sigma_g \xrightarrow{\varphi} M$

$$\mathcal{M}_g(M, C) = \bigsqcup_d \mathcal{M}_g(M,\beta),$$

where $d \in \mathbb{N}^{b_2(M)}$, $\beta = [\varphi_*(\Sigma_g)] \in H_2(M, \mathbb{Z})$

and $\beta = \sum_{i=1}^{b_2(M)} d_i [S_i]$, $[S_i]$ basis of $H_2(M,\mathbb{Z})$

– $e^{-S_{top}} = e^{-\int_{\Sigma_g}\varphi^*(\omega+iB)} = q^\beta = \prod_{i=1}^{b_2(M)} q_i^{d_i}$,

where $q_i = e^{-t_i}$, t_i cpx kähler param.

– axial R-sym anomaly:

$\# \psi^{(0)} - \# \bar\psi^{(0)} = \dim\ker D_{\bar z} - \dim\ker D_{\bar z}^\dagger = \operatorname{ind} D_{\bar z}$

$= \dim H^0(\varphi^*(TM)) - \dim H^1(\varphi^*(TM))$

$= \int_{\Sigma_g} ch(\varphi^*(TM))\, td(T\Sigma)$

$= \dim_{\mathbb{R}} M\cdot(1-g) + \int_{\Sigma_g}\varphi^*(c_1(TM))$

– so we need to insert some observables to offset this, $\langle \prod_k O_k \rangle_A$

– $\sum \text{Axial}(O_k) = 2 \operatorname{Ind} D_{\bar z}$

$\to$ <0 if $g>1$ & (and CY)

$\to$ solution: integrate over all cpx structures. Something is wrong with rigidity of cpx structs

– topological strings, $\dim \bar M_g = 3g-3$, $T M_g = H^1(T\Sigma_g)$

parametrised by Beltrami diffs $\mu^i{}_{\bar z}{}^{z}$, $\bar\mu^i{}_{z}{}^{\bar z}$, $i=1,\dots,3g-3$

$$F_g = \int_{\bar M_g} \langle \prod_{i=1}^{3g-3} G_+(\mu_i) \prod_{l=1}^{3g-3} G_-(\bar\mu^l) \rangle_A, \text{ with } \begin{array}{l} T_{++} = \{Q, G_+\} \\ T_{--} = \{Q, G_-\}\end{array}$$

- with this in mind, index becomes

$$2(\dim_{\mathbb{R}} M - 3)\cdot(1-g) + 2\int_{\Sigma_g} \varphi^*(c_1(TM))$$

$\to = 0$ for CY3?

$$F_g = \sum_{\beta} N_{\beta}^{g} q^{\beta}$$

$\downarrow$

"number" of hol maps of deg β from Σ to M

$\Rightarrow \in \mathbb{Q}$ due to bdry of $\overline{M}_{g,n}$

counts maps up to automorphism

- Gromov-Witten inv

Tanzini

A model

- $S_A = \int_{\Sigma_g} \varphi^*(\omega + i B) + \{Q_A, V\}$
- fixed locus of $Q_A \rightsquigarrow \varphi^i, \varphi^{\bar{i}} \in \mathcal{H}ol(M)$
  $\psi^i, \psi^{\bar{i}} \in TM$, $\psi_{\bar{z}}{}^{i}, \psi_z{}^{\bar{i}} \in$ obs
- $\mathrm{vd}\, \mathcal{M} = (\dim_{\mathbb{C}} M - 3)(1-g) + \int_{\Sigma_g} \varphi^*(c_1(TM)) =: K > 0$
- generic case:
  $\#\ \bar{\psi}$ zero modes $= 0$ i.e. $\ker D^+_{\bar{z}} = \{0\}$

- $\dim_{\mathbb{C}} \mathcal{M}_{\Sigma}(M, \beta) = K$

- $\mathcal{O}_i(x_i)$ are pullbacks of $\omega_i \in H^\bullet(M)$
  via evaluation maps on distinguished
  pts $\{p_i\}$, $ev_i : \mathcal{M}_{\Sigma_g}(M, \beta) \longrightarrow M$
  $\varphi \longmapsto \varphi(p_i)$

  and the ω_i are Poincaré duals to
  cycles $D_i \hookrightarrow M$

- alles zusammen gibt es

$$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_n(p_n) \rangle_A = \sum_{\beta} q^{\beta} \underbrace{\int_{M_{\Sigma_g}(M,\beta)} ev_1^* \omega_1 \wedge \cdots \wedge ev_n^* \omega_n}$$

so $N^g_\beta(D_1, \ldots, D_n) := \{ \#$ of $\varphi \in \mathcal{H}ol(M) \mid \varphi(p_i) \in D_i\ \forall i,\ [\varphi^*(\Sigma_g)] = \beta \}$ $\equiv N^g_\beta(D_1 \ldots D_n)$

$q^\beta = \prod_i e^{-t_i n_i}$

$\omega_i \cdot \beta > 0$, the Kähler form res. to hol. map is pos. semidef

- in large vol limit only $\beta = 0$ contribution survives:
$\beta = 0 \Rightarrow \varphi$ constant map $\Rightarrow \mathcal{M} = M$
$ev_i \equiv id_M \ \forall i$
$k = \dim_{\mathbb{C}} M$, so only genus 0 allowed

$$\langle O_1 \cdots O_n \rangle^{const}_{g=0} = \int_M \omega_1 \wedge \cdots \wedge \omega_n, \text{ classical int. theory}$$

A model chiral ring $\xrightarrow[\varphi = const]{g=0}$ cohomology ring of M

$C_{abc} = \#(D_a \cap D_b \cap D_c)$
$\eta_{ab} = \langle 1\, O_a\, O_b \rangle = \int_M \omega_a \wedge \omega_b$

- nongeneric case, $\ker D_{\bar z}^{+} \neq \{0\}$
$D_z \bar\psi_{\bar z \bar i} = D_{\bar z} \bar\psi_{z i} = 0$, $\bar\psi_{z i} \in H^0(\Sigma, K \otimes \varphi^*(T^*M))$
- let $h^0(\text{previous}) =: l$
$\rightarrow \dim \mathcal{M}_{\Sigma_g}(M, \beta) = k + l$
$\# \chi$ z.m.

- from $\Gamma^i{}_{jk} \partial_{\bar z} \varphi^j$ terms in $\bar\psi D_{\bar z} \psi$ we get also for 0-modes

$$S = \int_\Sigma d^2 z\, R_{i \bar i j \bar j}\, \bar\psi^{(0)\bar i}_{z}\, \bar\psi^{(0)\bar j}_{\bar z}\, \psi^{(0)i}\, \psi^{(0)j}$$

$$- \bar\psi_{\bar z \bar j} \chi^k \partial_z \varphi^l R^{\bar j i}{}_{\bar k l}\, G^{z \bar z}{}_i{}^j\, \chi^k \partial_{\bar z} \varphi^l R^n{}_{j k \bar l}\, \psi_{z n}$$

is $\langle \bar\psi, F_\upsilon \bar\psi \rangle \Rightarrow \mathrm{Pf}\, F_\upsilon \Rightarrow e(\upsilon)$

- put $e(\upsilon)$ inside $\int_{\mathcal{M}_{\Sigma_g}(M,\beta)} \underbrace{ev_1^* \omega_1 \wedge \cdots \wedge ev_n^* \omega_n}_{(k,k)\text{-form}} \wedge \underbrace{e(\upsilon)}_{(l,l)\text{-form}}$

A twisted nonlin σ model for $\mathbb{P}^1$

- $H^0(\mathbb{P}^1) = H^2(\mathbb{P}^2) = \mathbb{C}$, $\int_{\mathbb{P}^1} H = 1$,
  $1, P \in H^0$, $H, Q \in H^2$
  $\eta_{\alpha\beta} = \int_{\mathbb{P}^1} \omega_\alpha \wedge \omega_\beta = \begin{cases} 1 & \text{for } (\alpha,\beta) \in \{(P,Q),(Q,P)\} \\ 0 & \text{otherwise} \end{cases}$

- 3pt function

$$\langle Q\,Q\,Q \rangle = \underbrace{\sum_\beta q^\beta}_{\sum_{n\in\mathbb{N}} q^n \text{ since } \exists\, 1 \text{ class}} \langle Q\,Q\,Q\rangle_\beta$$

$$\dim_{\mathbb{C}} \mathbb{P}^1 (1-g) + \int_{\Sigma_g} \varphi^*(c_1(\mathbb{P}^1)) = 1-g+2\overset{\deg \varphi^*(\Sigma_g)}{n}$$

$g=0$, $\mathbb{P}^1_{w.s.} \to \mathbb{P}^1_{tgt}$, axial Q charge $=2$
so dim of $Q^3 = 6$, must match vd(M)

so $6 = 2(1+2n) \Rightarrow n = 1$

$$\Rightarrow \langle Q\,Q\,Q\rangle = e^{-t}$$

quantum coh. ring of $\mathbb{P}^1$

$C_{PPP} = 0$, $C_{PPQ} = \eta_{PQ} = 1$, $C_{QQQ} = e^{-t}$, $C_{PQQ} = 0$

Exercise. Show $PP = P$, $PQ = Q = QP$, $QQ = e^{-t}P$

- rk 2 vbdl on $\mathbb{P}^1$ s.t. tot. sp. is CY
  - splits $\mathcal{L}_1 \oplus \mathcal{L}_2 \longrightarrow \mathbb{P}^1$,
    tot sp being $\mathrm{tot}\left(\mathcal{O}(-n_1) \oplus \mathcal{O}(-n_2)\right)_{\mathbb{P}^1}$
  - locally $w' = z^n w$ for $z' = 1/z$
    so if we demand can. bdl to be
    trivial, $dw_1' dw_2' dz' = z^{n_1+n_2-2} dw_1 dw_2 dz$
    $\Rightarrow n_1 = n_2 = 1$
  - so $\mathrm{tot}\left(\mathcal{O}(-1)^{\oplus 2}\right)$
    $\rightarrow$ conifold $xy - wz = 0$

- let d = degree of $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$
  $$z = [x:y] \mapsto m = [s:t]$$

  with explicit: $d = 1$, $m = \dfrac{az+b}{cz+d}$

  $d > 1$, $\dfrac{s}{t} = \dfrac{\sum_{i=0}^{d} a_i x^i y^{d-i}}{\cdots b_i \cdots}$

  w/ mod. sp. $(a_i, b_i) \subset \mathbb{P}^{2d+1}$

- Aspinwall-Morrison [CMP '93],
  $$C_{abc}(t) = [D_a] \cap [D_b] \cap [D_c] + \sum_{k=1}^{\infty} \sum_{d|k} \frac{k^3}{d^3} q^k$$

Tanzini

- corr. functions: chiral ring for LG B model
- chiral ring $\mathbb{C}[\varphi^1, \ldots, \varphi^n]/\langle \partial_i W \rangle$
- fixed pts: $d\varphi = 0$, $\partial_i W = 0 \rightsquigarrow$ const. maps to crit pts of W
  - supposing finite number, label by $\{y_i\}_{i=1,\ldots N}$

$(f_1, \ldots f_s) \mapsto \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_s}$

$$\langle \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_s} \rangle = \sum_{j=1}^{N} \langle \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_s} \rangle \big|_{y_j}$$

$$S = \int d^2 z \Big(g_{i\bar{j}} h^{\alpha\beta} \partial_\alpha \varphi^i \partial_\beta \varphi^{\bar{j}} \sqrt{h} - i g_{i\bar{j}} \psi^{\bar{j}} D_{\bar{z}} \rho_z^i + i g_{i\bar{j}} \psi^{\bar{j}} D_z \rho_{\bar{z}}^{i}$$

$$- \tfrac{1}{2} R_{i\bar{j}k\bar{l}}\, \rho_{\bar{z}}^i \rho_z^k \psi^{\bar{j}} \psi^{\bar{l}}$$

$$+ \tfrac{1}{8} g^{\bar{j}l} \partial_l W \partial_{\bar{j}} \overline{W} + \tfrac{1}{4} D_i \partial_j W \rho_{\bar{z}}^i \rho_z^j + D_{\bar{i}} \partial_{\bar{j}} \overline{W} \psi^{\bar{i}} \psi^{\bar{j}} \Big)$$

- nonconst. modes produce det' factors that cancel
- constant modes:
  1) 1 const. mode for $\varphi^i, \varphi^{\bar{i}}, \psi^i, \psi^{\bar{i}}$
  2) g const. modes for $\rho_z^i, \rho_{\bar{z}}^i$

$$\int d^n\varphi\, d^n\bar{\varphi}\, e^{-\frac{1}{4}\varphi^0 \Delta \varphi^0} = |\det \partial_i \partial_j W|^{-2}(y_i)$$

$$\int d^n\psi\, d^n\bar{\psi} \cdots \sim \det \partial_{\bar{i}} \partial_{\bar{j}} \overline{W}\, (y_i)$$

$$\int d^{n\cdot g}\rho_z\, d^{n\cdot g}\rho_{\bar{z}} \cdots \sim (\det \partial_i \partial_j W)^g (y_i)$$

$$\Rightarrow \langle \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_s} \rangle = \sum_{\{y_j\}} f_1(y_j) \cdots f_s(y_j) (\det \partial_i \partial_j W)^{g-1} (y_j)$$

$$C_{ijk} = \sum_{\{dW=0\}} \frac{f_i f_j f_k}{\det \partial_i \partial_j W}, \quad \eta_{ij} = \sum_{\{dW=0\}} \frac{f_i f_j}{\det \partial_i \partial_j W}$$

- sinh-Gordon model with $M =$ cylinder, $z \in \mathbb{C}^*$
has $W = z + z^{-1} e^{-t}$
$\to W' = 1 - e^{-t}/z^{*2} = 0 \Rightarrow z^* = \pm e^{-t/2}$

- Hessian: $z\partial_z(z \partial_z W) = z + z^{-1} e^{-t} \big|_{z^* = \pm 2e^{-t/2}}$

- chiral ring gen. by $1, z, z^2 = e^{-t}$

$\Rightarrow \langle 1 1 1 \rangle_0 = \frac{1}{2e^{-t/2}} + \frac{1}{-2e^{-t/2}} = 0$

$\langle 1 1 z \rangle_0 = \frac{e^{-t/2}}{2e^{-t/2}} + \frac{-e^{-t/2}}{-2e^{-t/2}} = 1$

$\langle 1 z z \rangle_0 = 0$

$\langle z z z \rangle_0 = \frac{e^{-3t/2}}{2e^{-t/2}} + \frac{-e^{-3t/2}}{-2e^{-t/2}} = e^{-t}$

- same as $\mathbb{P}^1$ maps in A model!

- how to describe using field theory?
- since $\mathbb{CP}^{N-1} \simeq S^{2N-1}/U(1)$, we want to have a 2d susy theory with a quotient tgt mfd
- basically, $U(1)$ bdl:

$L = -\sum_{i=1}^{N} |D_\mu \varphi_i|^2 - U(\varphi)$ where $D_\mu = \partial_\mu + v_\mu$,
$U(\varphi) = \frac{e^2}{2}\left(\sum_{i=1} |\varphi_i|^2 - r\right)^2$

- v_μ auxiliary, its e.o.m. gives

$$\sum_i (D_\mu \bar{\varphi}^i)\varphi^i - \bar{\varphi}^i(D_\mu \varphi^i) = 0$$

$$\Rightarrow v_\mu = \frac{i}{2} \frac{\sum_{i=1}^N \bar{\varphi}_i \partial_\mu \varphi_i - \partial_\mu \bar{\varphi}_i \varphi_i}{\sum |\varphi_i|^2}$$

- when this is inserted back in the action, we obtain a nonlinear σ-model with a certain induced metric in the IR.
- we call L prior to that a linear σ-model, simply because the metric is flat here
- but eliminating v_μ gives

$$g^{FS} = \frac{\sum_{i=1}^{N-1} |dz_i|^2}{1+\sum |z_i|^2} - \frac{\sum |\bar{z}_i \, dz_i|}{1+\sum |z_i|^2}, \quad z_i := \frac{\varphi_i}{\varphi_N}$$

$\longrightarrow$ NσM on $\mathbb{C}P^{N-1}$ $\quad g_{ij}(\varphi)\partial_\mu \varphi^i \partial^\mu \varphi^j$

- susy?
- here we had local $U(1)$, $\varphi_i(x) \mapsto e^{i\gamma(x)}\varphi_i(x)$

$$L = \int d^2\vartheta \, d^2\bar{\vartheta} \; \bar{\Phi}\Phi$$

$\Phi \mapsto e^{iA}\Phi =: \Phi'$ gives $\bar{\Phi}'\Phi' = \bar{\Phi}\, \underline{e^{-i(\bar{A}-A)}}\, \Phi$

- to absorb it introduce vector sfield V, $V \mapsto V + i(A-\bar{A})$, s.t. $\underline{\bar{\Phi}' e^{V'} \Phi' = \bar{\Phi} e^{V} \Phi}$.

- V contains $U(1)$ conn., but also scalars, fermions

- supercurvature $\Sigma := \bar{D}_+ D_- V$
  - this is a twisted chiral field since $\bar{D}_+ \Sigma = D_- \Sigma = 0$
  - put $\tilde{y} = x + \bar{\vartheta}^+ \vartheta^-$

$$\Sigma(\tilde{y}) = \sigma(\tilde{y}) + i\vartheta^+ \lambda_+(\tilde{y}) - i\bar{\vartheta}^- \lambda_-(\tilde{y}) + \vartheta^+ \bar{\vartheta}^- [D(\tilde{y}) - F_{12}(\tilde{y})]$$

where $F_{\mu\nu} = \partial_{[\mu} v_{\nu]}$

$$L_{kin} = \int d^4\vartheta \, \bar{\Phi} e^V \Phi$$

$$L_{gauge} = -\frac{1}{2e^2} \int d^4\vartheta \, \Sigma \bar{\Sigma}$$

$\tilde{W}_{FI,\vartheta} = -t\Sigma$, $t := r - i\vartheta$ gives

$$L_{FI,\vartheta} = \frac{1}{2}\left(-t \int d^2\tilde{\vartheta} \, \Sigma + c.c.\right) = -rD + \vartheta F_{12}$$

- in general for $U(1)^k$ we get

$$L = \int d^4\vartheta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^{\sum_{a=1}^{k} Q_{ia} V_a} \Phi_i - \sum_{a,b=1}^{k} \frac{1}{2e^2_{a,b}} \bar{\Sigma}_a \Sigma_b \right) + \frac{1}{2} \int d^2\tilde{\vartheta} \sum_{a=1}^{k} (-t_a \Sigma_a) + c.c.$$

- so Q_{ia} crucially determines tgt space
- this L is invariant under $U(1)_V \times U(1)_A$ unless $\sum_i Q_{ia} \neq 0$

- terms containing D field: $\frac{1}{2e^2}D^2 + D(|\varphi|^2 - r_0)$

  - we see that completing the square for Gaussian integration gives us potential for φ

  - however, $\langle|\varphi|^2\rangle = \int_{\mu}^{\Lambda_{UV}} \frac{d^2k}{k^2} \sim \log\frac{\Lambda_{UV}}{\mu}$

    so effectively $\frac{1}{2e^2}D^2 + D\left(\log\frac{\Lambda_{UV}}{\mu} - r_0\right)$

    so $r_0 = r + \log\frac{\Lambda_{UV}}{\mu}$, $r(\mu) \sim \log\frac{\mu}{\Lambda_{UV}}$

- recall that usually the term
  $2i\bar{\psi}_- D_{\bar{z}}\psi_- + 2i\bar{\psi}_+ D_z \psi_+$ breaks R-sym,

$$D\psi D\bar{\psi} \mapsto e^{-2ik\alpha} D\psi D\bar{\psi}$$

  where $k = \operatorname{ind} D_{\bar{z}} = \frac{i}{2\pi}\int c_1(\mathcal{F})$

  - but we can negate this by $\vartheta \mapsto \vartheta - 2\alpha$
    but this breaks $U(1)_A$ to $\mathbb{Z}_2$

- generally $r_a(\mu) = \sum_i Q_{ia} \log\frac{\mu}{\Lambda}$

$$\vartheta_a \mapsto \vartheta_a - 2\left(\sum Q_{ia}\right)\alpha$$

Tanzini

- we saw: (susy) gauged linear σ-model (GLSM) $\longrightarrow$ nonlin σ-model (NLSM)

$\underline{\mathbb{C}P^{N-1}}$, $U(1)$ gauge theory w N chiral fields $\phi_1 \dots \phi_N$

$$L = \int d^4\theta \left(\sum_{i=1}^{N} \bar{\phi}_i e^V \phi_i - \frac{1}{2e^2} \bar{\Sigma} \Sigma \right) + \frac{1}{2}\left(-t \int d^2\tilde{\theta}\, \Sigma + c.c. \right)$$

$$= \sum_{j=1}^{N} \Big(-D^\mu \bar{\varphi}_j D_\mu^{(v)} \varphi_j + i \bar{\psi}_{j-} D_{\bar{z}}^{(v)} \psi_{j-} + i \bar{\psi}_{j+} D_{\bar{z}}^{(v)} \psi_{j-}$$

$$- |\sigma|^2 |\varphi_j|^2 - \bar{\psi}_{j-} \sigma \psi_{j+} - \bar{\psi}_{j+} \bar{\sigma} \psi_{j-} - i \bar{\varphi}_j \lambda_- \psi_{j+}$$

$$+ i \bar{\varphi}_j \lambda_- \psi_{j+} + i \bar{\psi}_{j+} \lambda_+ \varphi_j - i \bar{\psi}_{j-} \bar{\lambda}_+ \varphi_j$$

$$+ \frac{1}{2e^2}\left(-\partial^\mu \bar{\sigma} \partial_\mu \sigma + i \bar{\lambda}_- \partial_{\bar{z}} \lambda_- + i \bar{\lambda}_+ \partial_z \lambda_+ + \bar{F}_{12} F^{12} \right)$$

$$+ \vartheta F_{12} - \frac{e^2}{2} \left(\sum_{i=1}^{N} |\varphi_i|^2 - r \right)^2$$

- vacua. $U = \sum_i |\sigma|^2 |\varphi_i|^2 + \frac{e^2}{2} \left(\sum |\varphi_i|^2 - r \right)^2$

- cases
  - $r > 0$: $\sum |\varphi_i|^2 = r$ $\quad S^{2N-1}$, $\sigma = 0$
  - $r = 0$: $\varphi_i \equiv 0 \ \forall i$, σ unconstrained
  - $r < 0$: no solutions.

- we focus on $r > 0$, $\mathbb{C}P^{N-1} \simeq S^{2N-1}/U(1)$

- for σ we have $\underbrace{\sum |\varphi_i|^2 |\sigma|^2}_{= r |\sigma|^2} - \frac{1}{2e^2} |\partial \sigma|^2 + \dots$

which means σ has mass $e\sqrt{2r}$
$\quad \to$ and by susy same for $\lambda_\pm$, v

- $e \to +\infty$ limit:
  - φ_i span $\mathbb{C}P^{N-1}$, $\sum_{i=1}^{N} \overline{\varphi}_{i\pm}\varphi_i = \sum_i \overline{\varphi}_i \varphi_{i\pm} = 0$

$$v_\mu = \frac{i}{2} \frac{\sum_i \overline{\varphi}_i \partial_\mu \varphi_i - (\partial_\mu \overline{\varphi}_i)\varphi_i}{\sum |\varphi_i|^2}$$

$$\sigma = -\frac{\sum_i \overline{\varphi}_{i+}\varphi_{i-}}{\sum |\varphi_i|^2}$$

- $ds^2 = \frac{r}{2\pi} g^{FS}$

- on the tgt mfd $\mathbb{C}P^{N-1}$ $\exists$ gauge conn. A of a line bdl L whose $c_1(L)$ generates $H^2(\mathbb{C}P^{N-1}, \mathbb{Z})$
  $c_1(L) = -\frac{1}{2\pi} dA = -\frac{1}{2\pi}\omega_{FS}$

  $\frac{\theta}{2\pi}\int \underset{\varphi^*(\omega_{FS})}{\varphi^*(dA)}$ B field coupling

- quantum level: $r(\mu') = r(\mu) - N \log \mu/\mu'$

$$\left.\begin{aligned} R^{FS}_{i\bar{j}} &= N g^{FS}_{i\bar{j}} \\ g_{i\bar{j}} &= \frac{r}{2\pi} g^{FS}_{i\bar{j}} \\ R_{i\bar{j}} &= \frac{r}{2\pi} R^{FS}_{i\bar{j}} \end{aligned}\right\} \quad g_{i\bar{j}}(\mu') = \frac{1}{2\pi}\left(r - N\log \frac{\mu}{\mu'}\right) g^{FS}_{i\bar{j}}$$

$\hookrightarrow$ form of metric survives (one loop) corrections

- If we started with $r=0$, we wouldn't have gotten $\mathbb{C}P^{N-1}$, but quantum corrections fix this.

- $\omega - iB = \frac{r - i\vartheta}{2\pi}\omega^{FS}$, the cpx kähler modulus is a twisted chiral field

Toric mfds

- $U(1)^k = \prod_a U(1)_a$, N chiral sfields, $Q_{1a} - Q_{Na}$, $1/e^2_{a,b} = \delta_{a,b} 1/e_a^2$

- $U = \sum_{i=1}^{N} |Q_{ia}\sigma_a|^2 |\varphi_i|^2 + \sum_{a=1}^{k} \frac{e_a^2}{2}\left(\sum_{i=1}^{N} |\varphi_i|^2 Q_{ia} - r_a\right)^2$

$r_a > 0$ ... $\sum_i Q_{ia}|\varphi_i|^2 = r$, $a = 1 \ldots k$ (*)

- $X_r = \{(\varphi_1 \ldots \varphi_n) \mid (*) \text{ holds}\} / U(1)^k$
  symplectic quotient

- as cpx mfd $X_r \cong X_p = (\mathbb{C}^N \setminus P)/(\mathbb{C}^*)^k$
  Git quotient

  where $P \subset \mathbb{C}^N$ is locus of $\mathbb{C}^N$ whose $(\mathbb{C}^*)^k$ orbit does not contain solns to (*)

- $X_r \cong X_p$ is a toric manifold

  - $U(1)^N \circlearrowright \mathbb{C}^N \rightsquigarrow U(1)^{N-k}$ free & trans. action on open dense sets of X_r

  - equiv., $(\mathbb{C}^*)^{N-k}$ ..... of X_p

- example: conifold, tot. sp. of $(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$ bdl over $\mathbb{P}^1$

- GLSM: $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ $U(1)$ with
  charges $(1, 1, -1, -1)$

- condition is $|\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 - |\varphi_4|^2 = r$

  $r > 0$ ... φ_1, φ_2 hom. coords on $\mathbb{P}^1$
  $r < 0$ ... φ_3, φ_4 — " —
  $r = 0$ ... $x = \varphi_1 \varphi_3$, $y = \varphi_1 \varphi_4$
  $z = \varphi_2 \varphi_3$, $w = \varphi_2 \varphi_4$
  obey $xy = zw$, conifold

- if $c_1(X_r)$ $\begin{cases} > 0 & \text{Fano} \\ \geq 0 & \text{nef} \\ = 0 & \text{conformal} \end{cases}$

- B field v.e.v. is nonzero so it resolves the singularity

- turning on W i.e. F-terms induces hypersfcs on tgt mfd
- in $\mathbb{C}\mathbb{P}^{N-1}$, $G(\varphi_1 \cdots \varphi_N) = \sum_{i_1 \cdots i_d \in I} \alpha_{i_1 \cdots i_d} \varphi_{i_1} \cdots \varphi_{i_d}$

  genericity $\overset{\text{fails if}}{\Rightarrow}$ $G = 0$, further $\varphi_1 = \cdots = \varphi_N = 0$

  smooth hyper sfc M, $G(\varphi_1 \cdots \varphi_N) = 0$, $c_1(M) = (N-d)[H]|_M$
- for $N = d$ $r > 0$ CY hypersfc
  $r < 0$ LG model w $\mathbb{Z}_N$ sym.