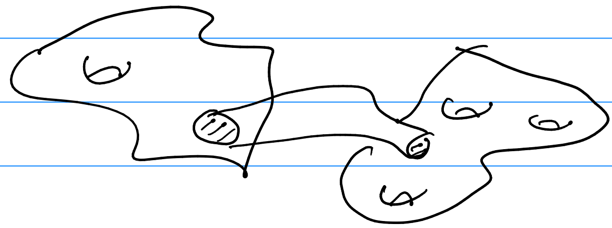
Stecconi

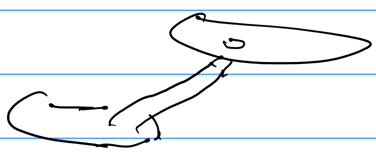
What is ... a surface made of?

- surface: 2D smooth mfd
- closed: cpt $\partial = \emptyset$

Def: M, N surfaces, conn. sum
$M \supset D_1$, $N \supset D_2$
$$M \# N = (M \setminus D_1 \sqcup N \setminus D_2) \sqcup_\partial \mathcal{C}yl$$



- $M \# N$ is in general well-defined
at least one of M and N is nonorientable,
or reversible
→ the problem is that we can do



or

→ reversible means $\exists$ an orientation reversing diffeo (e.g. 1-torus)
→ commutative, associative w unit ($\mathbb{S}$)

Thm Any closed connected surface is diffeo to $\mathbb{T}^{\# p} \# \mathbb{P}^{\# q}$ or $\mathbb{S} := \mathbb{T}^{\# 0} \# \mathbb{P}^{\# 0}$
unique up to $\mathbb{P} \# \mathbb{P} \# \mathbb{T} \cong \mathbb{T} \# \mathbb{P}$

Def. $f: M \to \mathbb{R}$ smooth is **Morse** if
$\partial_x f|_p = \partial_y f|_p = 0 \Rightarrow \det \partial_i \partial_j f \neq 0$

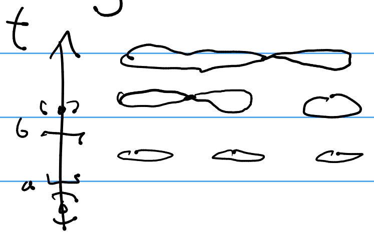
Lemma. (Morse lemma) $\forall p$ crit. pts $\exists\, U_p, x, y$
s.t. $f|_{U_p} = $
- $f(p) + x^2 + y^2$ → index 0
- $f(p) + x^2 - y^2$ → index 1
- $f(p) - x^2 - y^2$ → index 2

- we say f is **generic Morse** if $\forall p, q$ c.p. $f(p) \neq f(q)$

Thm $\exists$ a gen. Morse func.

Pf. - embed $M \hookrightarrow \mathbb{R}^N \ni a$
- let $f_a : M \to \mathbb{R}$
  $p \to a^T p$
- f_a is Morse if a is a reg value of $NM \longrightarrow \mathbb{R}^N$
  $(p, v) \longmapsto v$

$f: \Sigma \to \mathbb{R}$ gen. Morse
t reg value $\Rightarrow f^{-1}(t) \cong \sqcup S^1$



Lemma 1 if $[a,b] \not\ni$ crit. pt.,
$f^{-1}([a,b]) \cong f^{-1}(a) \times [a,b]$
$f^{-1}(a) \cong \sqcup S^1$

Pf. $v(p) = \frac{\nabla f}{\|\nabla f\|^2}$ flow φ wrt a metric
$f^{-1}(a) \times [a,b] \longrightarrow \Sigma$
$(p,t) \longmapsto \varphi^{t-a}(p)$

Lemma 2 p crit. pt. of index 0 or 2, $t = f^{-1}(p)$.

Then $f^{-1}(B_\varepsilon(t)) \cong \mathbb{D} \sqcup (\sqcup \mathcal{C}yl)$

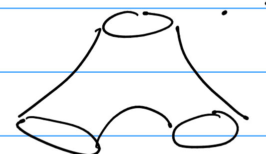
Pf. Let $\mathcal{O}_p$ x, y. $f|_{\mathcal{O}_p} = t \pm (x^2+y^2)$

$\Rightarrow \{f = c\} \cong S^1$, $\{t-\varepsilon \le f \le t+\varepsilon\} = \mathbb{D}$

Lemma 3. index of $p = 1$, $f(p) = t$.

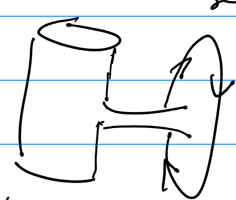
Then

1°) $f^{-1}(B_\varepsilon(t)) \cong (\sqcup \mathcal{C}yl) \sqcup \underbrace{\mathcal{C}yl \# \mathbb{D}}$



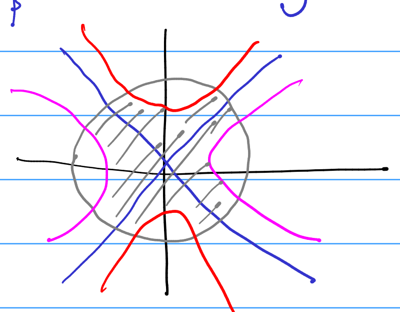
pair of pants

II°) $f^{-1}(B_\varepsilon(t)) \cong (\sqcup \mathcal{C}yl) \sqcup \underbrace{\mathcal{C}yl \# \mathbb{P}}$



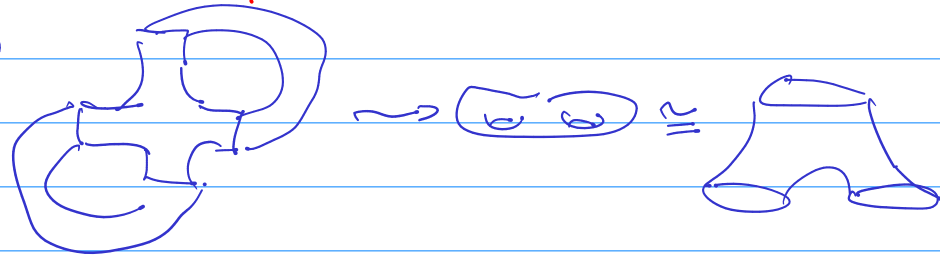
alien's ear

Pf. $f|_{\mathcal{O}_p} = t + x^2 - y^2$

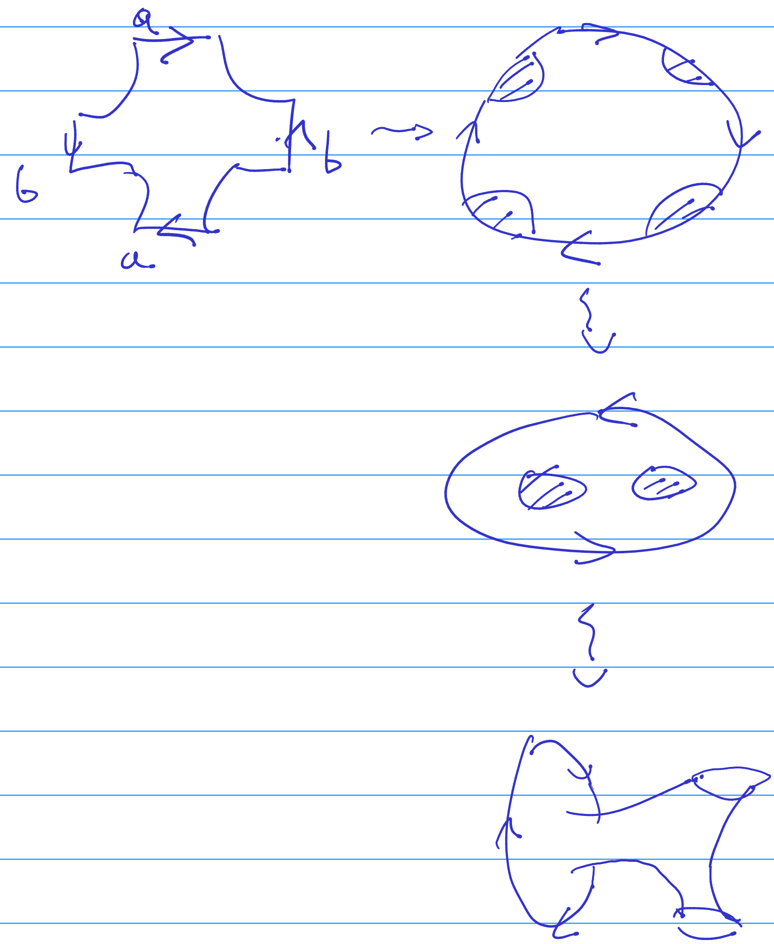


$= f^{-1}(B_\varepsilon(t)) \cap \mathcal{O}_p$

1°)

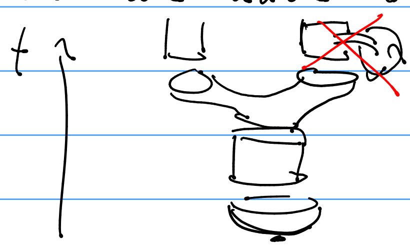


11°)



$\mathbb{P} \setminus \mathbb{D} \setminus \mathbb{D}$

- now we have all "building blocks"



→ doesn't work → but substitute $Cyl \# \mathbb{P}$ with Cyl but pay attntn to attachment

- Σ is nonorientable iff $\exists$ Möb $\hookrightarrow \Sigma$

$\Sigma' = \Sigma \setminus \text{Möb} \sqcup_{\partial \mathbb{D}} \mathbb{D}$, $\Sigma = \Sigma' \# \mathbb{P}$

- now look at

$\mathbb{T}^{\# g}$