Fantechi

- big idea: given $\left\{\begin{array}{l}\text{right}\\ \text{left}\end{array}\right.$ ex. ftor $F: \mathcal{A} \to \mathcal{B}$
we can extend to $\left\{\begin{array}{l}\mathbb{L}F : D^-(\mathcal{A}) \to D^-(\mathcal{B})\\ \mathbb{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})\end{array}\right.$

if the ab. cat $\mathcal{A}$ has enough $\left\{\begin{array}{l}F\text{-proj}\\ F\text{-inj}\end{array}\right.$

i.e. every object in $\mathcal{A}$ is $\left\{\begin{array}{l}\text{quot of an } F\text{-proj}\\ \text{subobj} \text{ -- " -- } F\text{-inj}\end{array}\right.$

Lemma let X proj sch$/\mathbb{k} = \bar{\mathbb{k}}$. let $\mathcal{F} \in Coh_X$, choose $\mathcal{O}_X(1)$ very ample. Then $\exists n_0$ s.t. $\forall n \geq n_0$ $\mathcal{F}(n)$ is gen by global secns

Pf. (hint) $X \hookrightarrow \mathbb{P}^N$. $M = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$ is grad. mod. over $S = \mathbb{k}[x_0, \dots, x_N]$ and $\mathcal{F}$ coh, $\Rightarrow M$ fin. gen. Pick $s \in \Gamma(X, \mathcal{F}(n_0))$, then $x_0 s, \dots, x_N s \in \Gamma(X, \mathcal{F}(n_0+1))$. □

Cor. With same assumptions, $\forall n \geq n_0$ $\exists r > 0$ and $\mathcal{O}_X(-n)^{\oplus r} \twoheadrightarrow \mathcal{F}$

Pf $\mathcal{F}(n)$ gen by glob secns $\Leftrightarrow$
$$\Gamma(X, \mathcal{F}(n)) \otimes_{\mathbb{k}} \mathcal{O}_X \twoheadrightarrow \mathcal{F}(n) \quad / - \otimes_{\mathcal{O}_X} \mathcal{O}_X(-n)$$
$$\Rightarrow \underbrace{\Gamma(X, \mathcal{F}(n))}_{\simeq \mathbb{k}^{\oplus r} \text{ for some } r} \otimes_{\mathbb{k}} \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{F}$$

Rmk Fix $n > 0$. Then $H^i(\mathbb{P}^n, \mathcal{O}(m)) = 0$,
except $\left\{\begin{array}{lll} i = 0 & m \geq 0 & H^0(\mathbb{P}^n, \mathcal{O}(m)) \simeq \mathbb{k}[x_0, \dots, x_n]_m \\ i = n & m \leq -n-1 & H^n(\mathbb{P}^n, \mathcal{O}(-n-1-m)) \simeq H^0(\mathbb{P}^n, \mathcal{O}(m))^\vee \end{array}\right.$

Rmk On a sep. sch., cohom can be computed using Čech coh. on any open affine $\Rightarrow \forall \mathcal{F} \in Qcoh_{\mathbb{P}^n}$ $\forall i > n$, $H^i(\mathbb{P}^n, \mathcal{F}) = 0$

Cor If X proj. sch $\mathcal{F} \in Qcoh_X \Rightarrow H^i(X, \mathcal{F}) = 0$ $\forall i > \dim X$

Cor If $X = \mathbb{P}^n$ then $H^n : Qcoh_X \to Vect$ is right-exact

Prop Let $\mathcal{A} = \mathrm{Coh}_{\mathbb{P}^n}$. Then $\mathcal{A}$ has enough H^n-proj.

Pf. consider subcat of sheaves which "only have H^n", by which we mean $\mathcal{F}$ s.t. $H^i(\mathcal{F}) = 0$ $\forall i \neq n$.
If $\mathcal{F}_1, \mathcal{F}_2$ are such, so is $\mathcal{F}_1 \oplus \mathcal{F}_2$.
If $\mathcal{F}, \mathcal{F}''$ — " — and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$,
so is $\mathcal{F}'$ by long ex. seqn. $\square$

- given $\mathcal{F} \in \mathrm{Coh}_{\mathbb{P}^n}$, we can find a resn s.t. $0 < N_0 \leq N_i \leq \dots$,
$$\dots \longrightarrow \mathcal{O}(-N_i)^{\oplus r_i} \longrightarrow \mathcal{O}(-N_0)^{\oplus r_0} \longrightarrow \mathcal{F} \to 0 \quad (*)$$
- if $0 \to \xi \to \mathcal{O}(-N_{n-1})^{\oplus r_{n-1}} \to \dots \to \mathcal{O}(-N_0)^{\oplus r_0} \to \mathcal{F} \to 0$
exact, then $H^i(\xi) = 0$ $\forall i \neq n$?
By Hilbert syzygy thm applied to open $x_i \neq 0$ in $\mathbb{P}^n$, $\Gamma(x_i \neq 0, \xi)$ is proj $\Leftrightarrow$ $\xi|_{x_i \neq 0}$ loc. free $\Rightarrow$ ξ loc. free

Aim $R\Gamma : D^b(\mathrm{Coh}_{\mathbb{P}^n}) \to D^b(\mathrm{Vect}_{\mathbb{K}})$ can be computed in terms of $LH^n : D^b(\mathrm{Coh}_{\mathbb{P}^n}) \to D^b(\mathrm{Vect}_{\mathbb{K}})$

Prop Let $0 \to \xi_{-n} \to \xi_{-(n-1)} \to \dots \to \xi_0 \to \mathcal{F} \to 0$
ex. seqn in $\mathrm{Coh}_{\mathbb{P}^n}$ s.t. $H^i(\xi_j) = 0$ $\forall i \neq 0$.
Then $\forall i = 0, \dots, n$, $H^i(\mathbb{P}^n, \mathcal{F}) \underset{can}{\simeq} h^{i-n}(H^n(\xi_{-n}) \to \dots \to H^n(\xi_0))$

Pf. i) if $n = 0$, $\xi_0 \xrightarrow{\sim} \mathcal{F}$
ii) if $i = n$, $h^0(\dots) = \mathrm{coker}(H^n(\xi_{-1}) \to H^n(\xi_0)) \underset{can}{\simeq} H^n(\mathcal{F})$
iii) $i < n$, use induction on m. Let $\mathcal{G} = \ker(\xi_0 \to \mathcal{F})$
i.e. $0 \to \mathcal{G} \to \xi_0 \to \mathcal{F} \to 0$, ...

Cor $H^i(\mathbb{P}^n, \mathcal{F})$ is fin. dim. $\forall \mathcal{F} \in \mathrm{Coh}_{\mathbb{P}^n}$.

- we also have nice results like Serre vanishing, $\exists n_0$ s.t. $H^i(\mathcal{F}(n)) = 0$ for $\mathcal{F} \in Coh_X$, X proj, for $\forall n \geq n_0$.

- let $F: Coh_X^{op} \to Vect_{\mathbb{k}}$, $F = Hom_{\mathcal{O}_X}(-, \mathcal{F})$ is left exact on Coh_X^{op}
- want to show that $\{ \mathcal{G} \in Coh_X \mid \mathcal{E}xt^i(\mathcal{G}, \mathcal{F}) = 0 \ \forall i > 0 \}$
  1) is closed under $\oplus$ in Coh_X^{op} (and therefore in Coh_X)
  ii) given $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ ex. in Coh_X^{op}
    $\Leftrightarrow$ $0 \to \mathcal{G}'' \to \mathcal{G} \to \mathcal{G}' \to 0$ in Coh_X, if $\mathcal{G}', \mathcal{G}$ ok
    then also $\mathcal{G}''$, using $\mathcal{E}xt^i(\mathcal{G}', \mathcal{F}) \to \mathcal{E}xt^i(\mathcal{G}, \mathcal{F}) \to \mathcal{E}xt^i(\mathcal{G}'', \mathcal{F}) \to \dots$.
  iii) in Coh_X^{op} $\forall \mathcal{G} \ \exists \tilde{\mathcal{G}}$ F-proj and $0 \to \mathcal{G} \to \tilde{\mathcal{G}}$ in Coh_X^{op}
    $\Leftrightarrow$ $\forall \mathcal{G} \ \exists \tilde{\mathcal{G}} \to \mathcal{G}$ in Coh_X s.t. $\mathcal{E}xt^i(\tilde{\mathcal{G}}, \mathcal{F}) = 0 \ \forall i > 0$
  - choose $n_0 \geq 0$ from Serre vanishing.
    $\forall n \geq n_0 \ \exists r \geq 0$, $\mathcal{O}_X(-n)^{\oplus r} \to \mathcal{G} \to 0$.
    Then
$$\begin{aligned}
&\mathcal{E}xt^i(\mathcal{O}_X(-n)^{\oplus r}, \mathcal{F}) \\
&\quad \parallel \\
&\mathcal{E}xt^i(\mathcal{O}_X(-n), \mathcal{F})^{\oplus r} \\
&\quad \wr\mathrm{S} \\
&H^i(X, \mathcal{F} \otimes (\mathcal{O}_X(-n)^\vee))^{\oplus r} \\
&\quad \wr\mathrm{S} \\
&H^i(X, \mathcal{F}(n))^{\oplus r} = 0.
\end{aligned}$$

Cohomology & base change

- X, Y loc. of finite type $/ \mathbb{k} = \overline{\mathbb{k}}$
- $f: X \to Y$ proj mor (usually we take proper, but this case can be reduced to proj), $\mathcal{F} \in Coh_X$ flat over Y, e.g. f flat, $\mathcal{F}$ loc. free
  f smooth (in alg. geom sense, in diff. geom étale)
  $\Rightarrow$ flat $A[x_1, \dots x_{n+r}]/(f_1, \dots f_r)$, $rk \frac{\partial f_i}{\partial x_j} < r$

- let $y \in Y(\mathbb{k})$

Thm 1) $\exists$ nat. map

$$R^i f_* \mathcal{F} \otimes \overset{\text{res. fld at } y}{\mathbb{k}(y)} \xrightarrow[\pi_{i,y}]{} H^i(X_y, \mathcal{F}|_{X_y})$$

$$\begin{tikzcd} R^i f_* \mathcal{F} \otimes \mathbb{k}(y) \arrow[d, "\wr"] \arrow[r, "\pi_{i,y}"] & H^i(X_y, \mathcal{F}|_{X_y}) \arrow[d, "\wr"] \\ \alpha^* R^i f_* \mathcal{F} & R^i g_*(\beta^* \mathcal{F}) \end{tikzcd}$$

in

$$\begin{tikzcd} X_y \arrow[r, "\beta"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathbb{k}(y) \arrow[r, "\alpha"] & Y \end{tikzcd}$$

ii) if map surjects for some $y_0 \in Y$, then $\exists U$ open nbhd of y_0 in Y s.t. it surjects $\forall y \in U$, and bijects.

iii) assuming $\pi_{i,y}$ does surject, TFAE $\pi_{i-1,y}$ surj $\iff R^i f_* \mathcal{F}$ is loc. free near y

Rmks

- assuming $\dim X_y \le n$ $\forall y \in Y$, $\underset{= 0}{R^{n+1} f_* \mathcal{F} \otimes \mathbb{k}(y)} \to \underset{= 0}{H^{n+1}(X_y, \mathcal{F}|_{X_y})}$
- $\pi_{n,y}$ surj $\iff R^{n+1} f_* \mathcal{F}$ loc. free
  $\to$ of course, since it is zero!
- $i = 0$ says $f_* \mathcal{F} \otimes \mathbb{k}(y) \xrightarrow[\pi_{0,y}]{} H^0(X_y, \mathcal{F}|_{X_y})$
- if $\pi_{0,y}$ surj., TFAE $\pi_{-1,y}$ surj $\iff f$ ...