Тихомиров

- $\nabla_A : L^2_3(\Omega^0(\mathrm{ad}\,P)) \longrightarrow L^2_2(\Omega^1(\mathrm{ad}\,P))$
  $\nabla_A^* : L^2_2(\Omega^1(\mathrm{ad}\,P)) \longrightarrow L^2_1(\Omega^0(\mathrm{ad}\,P))$
  $\Delta_A := \nabla^*_A \circ \nabla_A : L^2_3(\Omega^0(\mathrm{ad}\,P)) \longrightarrow L^2_1(\Omega^0(\mathrm{ad}\,P))$
  is elliptic.
- $A \in \mathcal{A}_2^*$ $(\mathrm{Stab}\,A = \mathbb{Z}_2)$, $f : \mathcal{G}_3 \to \mathcal{A}_2, g \mapsto g^*A$
  $\to (f_*)_{id} = \nabla_A$
- $\ker \nabla_A = \{0\} \Rightarrow \ker \Delta_A = \{0\}$
- further, $\begin{cases} \mathrm{coker}\,\nabla_A^* = \ker \nabla_A \\ \mathrm{coker}\,\nabla_A = \ker \nabla_A^* \end{cases}$ means

  Δ_A bijects
  - not that surprising — L^2_3 and L^2_1 are separable Hilbert spaces, and there is only one such space up to iso (ℓ^2)

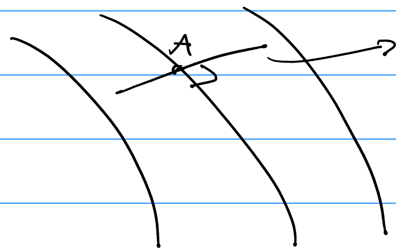
- let $H = L^2_3(\Omega^0(\mathrm{ad}\,P)) = \mathrm{Lie}(\mathcal{G})$, then
  $\exp : H \to \mathcal{G}$ gives $(f_*)_{id}(s) = \frac{d}{dt}(f(\exp ts))|_{t=0}$
  $= ds + [\omega_A, s] = \nabla_A s$

Prop (Implicit func thm for Banach mfds)
Let $\hat{E}_1, \hat{E}_2, \hat{F}$ Banach mfds, $f : \hat{E}_1 \times \hat{E}_2 \to \hat{F}$ smooth with differential at point $x^0 = (x_1^0, x_2^0) \in \hat{E}_1 \times \hat{E}_2$ $(f_*)_{x^0} = (f_{1*}, f_{2*}) : E_1 \oplus E_2 \to F$ where $E_i = T_{x_0}\hat{E}_i$, $i = 1, 2$, $F = T_{f(x_0)}\hat{F}$.
Assume $f_{2*} : E_2 \longrightarrow F$ is invertible.
Then $\exists$ $\mathcal{U}_i \subset \hat{E}_i$ open nbhds of x_i^0, $i = 1, 2$, and $\exists$ smooth map $h : \mathcal{U}_1 \to \mathcal{U}_2$ s.t.
$f(x_1, h(x_2)) = f(x_1^0, x_2^0)$ for $x_i \in \mathcal{U}_i$.

$$\mathcal{U} \times \tilde{\mathcal{G}}_3$$

$$\|$$

- $\tilde{\mathcal{G}}_3 = \mathcal{G}_3 / \mathbb{Z}_2$, $\mathcal{U} \times_{Stab} \mathcal{G}_3 \xrightarrow{\sim} \mathcal{O} \xhookrightarrow{open} \mathcal{A}_2^*$



$X_A = \{ A + a \mid \nabla_A^* a = 0 \}$
$\overset{?}{\subset} \mathcal{U}$?

$f : \mathcal{A}_2^* \times \tilde{\mathcal{G}}_3 \longrightarrow L^2_1(\Omega^0)$, $(A + a, g) \mapsto \nabla_A^* (g^{-1})^* a$

- $(x_1^0, x_2^0) = (A, id)$
- $(f_*)_{(A, id)} : L^2_2(\Omega^1) \oplus L^2_3(\Omega^0) \longrightarrow L^2_1(\Omega^0)$

$(a, u) = \nabla_A^* (-\nabla_A u + a)$

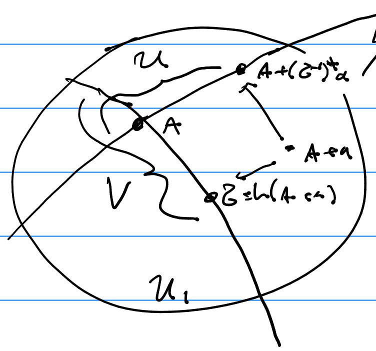
$f_{*2} = -\nabla_A^* \circ \nabla_A = -\Delta_A$ invertible

$\overset{IFT}{\Longrightarrow} \exists\, \mathcal{U}_1 \xhookrightarrow{open} \mathcal{A}_2^*$, $\mathcal{U}_2 \xhookrightarrow{open} \tilde{\mathcal{G}}_3$

and $h : \mathcal{U}_1 \longrightarrow \mathcal{U}_3$, $A + a \mapsto g$

- $b = (g^{-1})^* a$, $\nabla_A^* (g^{-1})^* a = 0$, $b \in X_A \cap \mathcal{U}_1 =: \mathcal{U}$

$V := \mathcal{U}_2$



$X_A = \{ A + a \mid \nabla_A^* a = 0 \}$

$\varphi : \mathcal{U} \times V \xrightarrow{\sim} \operatorname{im} \varphi =: \mathcal{O} \xhookrightarrow{open} \mathcal{U}_1 \subset \mathcal{A}_2^*$

$(A + b, g) \mapsto A + g^* b$

$\varphi^{-1} : \mathcal{O} \longrightarrow \mathcal{U} \times V$

$A + a \longmapsto \left(A + (g^{-1})^* b,\ g = h(A + a) \right)$

- $\mathcal{A}_2 \supset \tilde{\mathcal{M}} = \tilde{\mathcal{M}}(P) = \{ A \in \mathcal{A}_2 \mid * \overset{\Omega^2(\mathrm{ad}P)}{\overset{\cup}{F_A}} = -F_A \}$

$*: \Omega^i \longrightarrow \Omega^{4-i}$, $\alpha \wedge *\beta = (\alpha, \beta)\, d\mathrm{Vol}$,

$*^2 = (-)^{i(4-i)}\,\mathrm{id}$

- so for 2-forms on 4-mfd we have splitting

$$\Omega_x^2 B = \Lambda^2 T_x^* B = \Omega^2_{+,x} \oplus \Omega^2_{-,x}$$

→ bases $\Omega^2_{\pm,x} = \left\{ \begin{array}{l} e_1 \wedge e_2 \pm e_3 \wedge e_4 \\ e_1 \wedge e_3 \pm e_4 \wedge e_2 \\ e_1 \wedge e_4 \pm e_2 \wedge e_3 \end{array} \right.$

$$\begin{aligned} \Omega^2(\mathrm{ad}B) &= \Gamma(\Lambda^2 T^* B \otimes \mathrm{ad}P) \\ &= \Omega^2_+(\mathrm{ad}P) \oplus \Omega^2_-(\mathrm{ad}P) \\ \overset{\cup}{F_A} \quad &= \quad (F_A^+ \quad , \quad F_A^-) \end{aligned}$$

with $*F_A^\pm = \pm F_A^\pm$.

Def A is called anti-self-dual (ASD) (respectively SD) if $F_A = F_A^-$ ($F_A = F_A^+$).

- $\tilde{\mathcal{M}}$ is $\mathcal{G}$-invariant, elements of $\mathcal{M} = \tilde{\mathcal{M}}/\mathcal{G}$ are called instantons

- $F_{A+a} = F_A + \nabla_A a$

$$\begin{tikzcd} & \Omega^2(\mathrm{ad}P) \arrow[dr, "P_+"] & \\ \mathcal{A} \arrow[ur, "A \mapsto F_A"] \arrow[rr, "\psi"'] & & \Omega^2_+(\mathrm{ad}P) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ A \arrow[rr, mapsto] & & P_+(F_A) \end{tikzcd}$$

- $\psi_* = P_+ \nabla_A : \Omega^1(\mathrm{ad}P) \longrightarrow \Omega^2_+(\mathrm{ad}P)$

- if A is ASD, $D_+ F_A = 0$, so $\mathcal{U}_*$ is not Fredholm (kernel not fin. dim)
- however, it is, when restricted to χ_A?
- $\Omega^1(\mathrm{ad}\,P) = \mathrm{im}\,\nabla_A \oplus \ker \nabla_A^*$

- now, take $\begin{array}{c} T^*B \\ \downarrow p \\ B \end{array}$, look at

$$0 \to \Gamma(E_1) \xrightarrow{L_1} \Gamma(E_2) \xrightarrow{L_2} \Gamma(E_3) \to 0$$

and lift to

$$0 \to p^*\Gamma(E_1) \xrightarrow{\mathrm{Symb}_{L_1}} p^*\Gamma(E_2) \xrightarrow{\mathrm{Symb}_{L_2}} \Gamma(E_3) \to 0 \qquad (*)$$

where $\mathrm{Symb}_{L_1}(s) := L_1\left(\frac{1}{k!}(g - g(x))^k \tilde{s}(x)\right)\Big|_x$

if L_1 diff op. of order k, where $\tilde{s} \in \Gamma(E_1)$ any s.t. $\tilde{s}(x) = s$, $g \in C^\infty(B)$ any s.t. $dg|_x = v$.

- we call this an elliptic cpx if $(*)$ is exact.