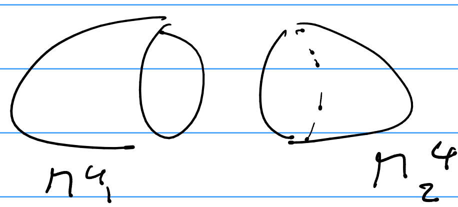
Gauge @ IGAP

6d invariants – Putrov, cont'd

Rmk VOA modules form a braided monoidal cat wrt "fusion"
– under some simplicity assumptions, a modular tensor cat

– so $M_\lambda^{u(1)}[M^4]$-modules form a $MTC_{u(1)}[M^3]$ depending only on $\partial M^4 = M^3$

– consider



$$\partial M_1^4 = M^3$$
$$\partial M_2^4 = \overline{M}^3$$

$$\Rightarrow VOA_{u(1)}\Big[M^4 = M_1^4 \underset{M_3}{\sqcup} M_2^4\Big] = \bigoplus_{\mu \in H_1(M^3,\mathbb{Z})} M_\mu^{u(1)}[M_1^4] \otimes M_{-\mu}^{u(1)}[M_2^4]$$

Seiberg–Witten invariants

– M^4 closed, w/ $Spin^{\mathbb{C}}$ – lift of $SO(4)$-bdl
– L cpx lbdl assoc. to $U(1)$-pbd
– $W_\pm$ cpx rk 2 bdles assoc. to $U(2)_\pm$

$$Spin^{\mathbb{C}}(4) = \frac{Spin(4)\times U(1)}{\mathbb{Z}_2} \longrightarrow \frac{SU(2)\times SU(2)\times U(1)}{\mathbb{Z}_2} \begin{array}{l} \nearrow U(2)_\pm \\ \searrow U(1) \end{array}$$

– $H_1(M^4,\mathbb{Z}) = 0$
– then $Spin^{\mathbb{C}}(M^4) \cong \{\lambda = c_1(L) \in H^2(M,\mathbb{Z}) \mid \lambda \equiv w_2(TM^4) \bmod 2\}$
– $W_+ \otimes \overline{W_+} \cong (\Lambda^{2,+}T^*M^4 \otimes \mathbb{C}) \oplus \underbrace{\mathbb{C}}_{\text{trivial bdl.}}$

- take $A \in \mathrm{Conn}(L)$, which also defines conn's on $W_\pm$, and $\psi \in \Gamma(W_+)$
- SW eqn's:

$$\begin{cases} F_A^+ = (\psi \otimes \bar{\psi})_+ & \in \Gamma(\Lambda^{2,+}T^*M) \\ \not{D}_A \psi = 0 & \in \Gamma(W_-), \\ & \text{since } \not{D}_A : W_+ \to W_- \end{cases}$$

- solutions $\in B \subset (\mathrm{Conn}(L) \times \Gamma(W_+))|_{\psi \neq 0}$

$$\mathcal{G} = \{M^4 \to U(1)\}$$
free action

- so $\mathcal{M}(\lambda) := B/\mathcal{G}$

Thm For generic metric, $b_2 > 1$, $\mathcal{M}(\lambda)$ is orientable mfd of expected

$$\mathrm{vd}\,\mathcal{M}(\lambda) = \frac{1}{4}(\lambda^2 - (3\sigma + 2\chi))$$

- consider univ. lb'd

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ M^4 \times \mathcal{M}(\lambda) \xrightarrow{pr} \mathcal{M}(\lambda) \end{array}$$

s.t. $\mathcal{L}|_{M^4} = L$, and $u = pr_*([pt] \cup c_1(\mathcal{L})) \in H^2(\mathcal{M}(\lambda), \mathbb{Z})$

- define $SW : \mathrm{Spin}^c(M^4) \to \mathbb{Z}$

$$\lambda \mapsto SW(\lambda) = \int_{\mathcal{M}(\lambda)} u^{\frac{\dim \mathcal{M}}{2}}$$

- simple type conj.:
$\mathcal{M}(\lambda) = \emptyset$, unless $\mathrm{vd}\,\mathcal{M}(\lambda) = 0$, if $b_2^+ > 1$

- now consider multimonopole eqn's

$$\begin{cases} F_A = \sum_{i=1}^{N} (\psi^i \otimes \overline{\psi}^i)_+ \\ \not{D}_A \psi^i = 0 , \quad \forall i = 1, \dots, N \end{cases}$$

where $\psi^i \in F(W_+)$

- solutions $B_N \subset \left(\mathrm{Conn}(L) \times \Gamma(W_+)^N \right) \Big|_{\substack{\text{not all} \\ \psi^i \equiv 0}}$

$\mathcal{M}_N := B_N / \mathcal{G}$

$\Rightarrow vd(M_N(\lambda)) = \dfrac{N(\lambda^2 - \sigma) - 2(\chi + \sigma)}{4}$

- problem: $\mathcal{M}_N$ not cpt. even if $b_2^+ > 1$

- let M^4 be spin, $\sigma < 0$, $L = \underline{\mathbb{C}}$
  $\Rightarrow \exists$ harmonic spinor $\xi \in \Gamma(W_+)$,
  $\not{D} \xi = 0$
  $\Rightarrow \exists$ solution $\psi^1 = \alpha \cdot \xi$, $\psi^2 = \alpha \underset{\text{(exists)}}{J} \xi$, $\psi^{i > 2} = 0$
  s.t.
  $$0 = \sum_{i=1}^{N} (\psi^i \otimes \overline{\psi}^i)_+$$
  $$\not{D}_\psi = 0$$

- Dedushenko-Gukov-P.:

$\exists$ $\mathbb{T} \subset SU(N)$ action on m.sp

$$\Rightarrow ESW_N(\lambda, z) = \mathbb{T}\text{-equiv} \int_{\mathcal{M}(\lambda)} 1$$

$$= \sum_{\substack{C \subset \mathcal{M}_N(\lambda) \\ \mathbb{T}\text{-fixed pt.}}} \int_C \frac{1}{Eu_{\mathbb{T}}(N_C)}$$

$$= SW(\lambda) \sum_{i=1}^{N} \frac{1}{\prod_{j \neq i} (z_j - z_i)^{\frac{1}{8}(\lambda^2 - 3)}}$$

assuming simple-type conj.

- consistent w Physics prediction

$$ESW(\lambda, z) = \langle \lambda | \sum_{i=1}^{N} S(z_i) \prod_{j \neq i} S_+(z_j) | 0 \rangle$$

where $S, S_+ \in VOA_{u(1)}[\Lambda^4]$,

$$S(z) = \sum_{\lambda \in Spin^c(\Lambda^4)} u_\lambda(z)\, SW(\lambda)$$

$$S_+(z)\, u_\lambda(w) = \frac{1}{(z-w)^{\frac{1}{8}(\lambda^2 - 3)}} + :S_+(z) u_\lambda(w):$$