Fantechi

exercise $(Sch/\mathbb{k})$, functor $S \mapsto \Gamma(S, \mathcal{O}_S)$
from $(Sch)^{op} \to (Set)$, on mor $S \xrightarrow[\varphi]{} S'$,
$\Gamma(S', \mathcal{O}_{S'}) \xrightarrow{\varphi^*} \Gamma(S, \mathcal{O}_S)$.
- $\Gamma(\mathcal{O}_-)$ is repr by $\mathbb{A}^1_{\mathbb{k}}$
  - take $x \in \Gamma(\mathbb{A}^1_{\mathbb{k}}, \mathcal{O}_{\mathbb{A}^1_{\mathbb{k}}})$, then
$$\begin{aligned} Mor(S, \mathbb{A}^1_{\mathbb{k}}) &\longrightarrow \Gamma(S, \mathcal{O}_S) \\ \varphi &\longmapsto \varphi^*(x) \end{aligned}$$
  bijects
  - $f \in \Gamma(S, \mathcal{O}_S)$, $\Gamma(\mathbb{A}^1_{\mathbb{k}}, \mathcal{O}_{\mathbb{A}^1_{\mathbb{k}}}) \xrightarrow[\exists! \varphi_f^*]{} \Gamma(S, \mathcal{O}_S)$

  s.t. $f = \varphi_f^*(x)$
- for $S \in ob\, Sch/\mathbb{k}$, A $\mathbb{k}$-alg, $\varphi \mapsto \varphi^*$ gives
$$Mor_{Sch/\mathbb{k}}(S, Spec A) \simeq Hom_{\mathbb{k}\text{-alg}}(A, \Gamma(S, \mathcal{O}_S))$$

exercise For $\mathcal{A}$ add. cat. define the 0 functor
$0 : \mathcal{A}^{op} \to Set$, $A \mapsto pt.$
A zero object is an object representing the 0 functor, called $0_{\mathcal{A}}$.
$\forall A \in ob\, \mathcal{A}$, $Hom(A, 0_{\mathcal{A}}) = 0$ (final obj)
- claim $Z_A := \ker(A \xrightarrow{id_A} A)$ is zero object
  Pf. $Hom(B, Z_A) = \{\varphi : B \to A \mid id_A \circ \varphi = 0\} = \{0\}$
- noting that $id_0 \in Hom(0, 0) = \{0\} \Rightarrow id_0 = 0$,
  $\psi : 0 \to B \Rightarrow \psi = \psi \circ id_0 = 0$, so $0_{\mathcal{A}}$ is also an initial object

- In abel cat $\mathcal{A}$, $A \oplus B$ is the coproduct of A, B,
i.e. $\forall A, B \in ob\,\mathcal{A}$, $\exists\ A \xrightarrow{i} A \oplus B \xleftarrow{j} B$ s.t.
$\forall C \in ob\,\mathcal{A}$, $Hom(A \oplus B, C) \xrightarrow{\sim} Hom(A, C) \times Hom(B, C)$

$$\varphi \longmapsto (\varphi \circ i, \varphi \circ j)$$

- claim: $A \oplus B$ is product.
  - take

$$\begin{tikzcd} A \arrow[r, "i"] & A \oplus B \arrow[r, "\pi"] & \operatorname{coker}(i) \\ & B \arrow[u, "j"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

  - claim β is iso.
- we'll finish tomorrow

exercise Let $\Bbbk$ base comm. ring, $A \to B$ hom. of fin. gen. comm. $\Bbbk$-algs s.t. it factors as $A \to P \to B$, P = free A-alg i.e. $P = A[x_1, \dots, x_n]$
$P \to B$ surjects with ker I
- we associate to this an element of $\overset{C(\mathrm{Mod}_B)}{\overset{\|}{C(B)}}$,

$$\cdots \to 0 \to I/I^2 \to \Omega_{P/A} \otimes_A B \to 0 \cdots \to$$

Lemma. $d : P \to \Omega_{P/A}$:
1) maps I^2 to zero in $\Omega_{P/A} \otimes_P B$
11) induced hom $\underset{\underset{I \otimes_P B}{\|}}{I/I^2} \to \Omega_{P/A} \otimes_P B$ is B-linear

Pf. 1) $I^2 = \langle f g \rangle$, $f, g \in I$, but $d(fg) = f\,dg + df \cdot g = 0$ in $\Omega_{P/A} \otimes_P B$ since $f, g \mapsto 0$ in B
11) $f \in P$, $[f] \in B$, $[g] \in I/I^2$, $g \in I$,
$d([f][g]) = [f]\,d[g]$ in $\Omega_{P/A} \otimes_P B$
$d(fg) = df\,g + \underbrace{f\,dg}_{= 0 \text{ in } \Omega_{P/A} \otimes_P B}$ in $\Omega_{P/A}$

- if we had two factors P_1, P_2, then

$$\begin{tikzcd} & P_1 \arrow[d] \arrow[dr, two heads] & \\ A \arrow[ur] \arrow[r] \arrow[dr] & P_1 \otimes_A P_2 \arrow[r, two heads, dashed] & B \\ & P_2 \arrow[u] \arrow[ur, two heads] & \end{tikzcd}$$

- focus on upper part (symmetry)
- put $P = A[x_1 \ldots x_n, y_1, \ldots, y_m]$
  $P_1 = A[x_1 \ldots x_n]$,

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & P \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I_1 \arrow[r] \arrow[u, hook] & P_1 \arrow[r] \arrow[u, hook] & B \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

gives morphism in $C(B)$ since

$$\begin{tikzcd} I \arrow[r] & \Omega_{P/A} \\ I_1 \arrow[r] \arrow[u] & \Omega_{P_1/A} \arrow[u] \end{tikzcd} \Longrightarrow \begin{tikzcd} I/I^2 \arrow[r] & \Omega_{P/A} \otimes_P B \\ I_1/I_1^2 \arrow[r] \arrow[u] & \Omega_{P_1/A} \otimes_{P_1} B) \arrow[u] \end{tikzcd}$$

- claim: iso on ker, coker

- so we get

$$\begin{tikzcd} & I/I^2 \to \Omega_{P/A} \otimes_P B & \\ I_1/I_1^2 \to \Omega_{P_1/A} \otimes_{P_1} B \arrow[ur, "\text{qiso}"] & & I_2/I_2^2 \to \Omega_{P_2/A} \otimes_{P_2} B \arrow[ul, "\text{q-iso}"'] \end{tikzcd}$$

- are these two objects "isomorphic"?
  - this motivates derived categories
- let's get back to claim
  - put $Q_i = P[y_1, \ldots y_i]$ s.t. $Q_0 = P$, $Q_m = P_1$,
    let $J_i = \ker Q_i \to P$.

- look at

$$\begin{array}{ccc} J_0/J_0^2 & \longrightarrow & \Omega_{Q_0/A} \otimes_{Q_0} B \\ \downarrow & & \downarrow \\ J_1/J_1^2 & \longrightarrow & \Omega_{Q_1/A} \otimes_{Q_1} B \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ J_m/J_m^2 & \longrightarrow & \Omega_{Q_m/A} \otimes_{Q_m} B \end{array}$$

- we focus on showing each inner square is quiso

$$\begin{array}{ccc} Q & \overset{\pi}{\twoheadrightarrow} & B \\ \downarrow & \nearrow_{\tilde{\pi}} & \\ Q[y] & & \end{array}$$

Q free f.g. A-alg

$\tilde{\pi}$ is det. by π and $\tilde{\pi}(y) = b_0 \in B$,
choose $f_0 \in Q$ s.t. $\pi(f_0) = b_0$

- we can show that

$$\begin{array}{ccc} & & I/I^2 \\ & & \downarrow \\ B & \longrightarrow & I_1/I_1^2 \\ b & \longmapsto & b(y-f_0) \end{array}$$

$(y - f_0) \in I_1$

so $I/I^2 \otimes B \to I_1/I_1^2$ iso