Fantechi

- schedule: 3x weekly till Easter, then break
  until May 30

- new timetable:

Wed 9–11	Thu 14–16	Fri 9–11
136	136	136

- plan: review ab. cats., complexes,
  exactness/cohom., (half) exact functors
  - primary exs: sheaves of modules
    over (not necc.) ringed spaces
- content of course: derived categories,
  derived functors in more general way
  than Hartshorne ("nobody has ever seen an injective object!", also usually lack of projectives)
  Ext, Tor, Rf_*, Lf^*, Spect. sequ. (Leray, local-to-global Ext)
  coh. and base changes, Grothendieck–Serre duality

Examples of NC rings/sheaves of rings

- ex 1 - G finite gp, k field.
  $\mathrm{Rep}_k(G)$ cat. of G-representations,
  objects V k-vsp $G \to GL(V)$,
  morphisms $V \to W$ k-lin. G-equiv. maps
  - modules over $k[G]$ NC group alg
  - product induced by G-product & linearity:
    $a, b \in k[G]$. $a = \sum_G a_g[g]$, $b = \sum b_g[g]$
    $\Rightarrow a \cdot b = \sum_{g,h \in G} a_g b_h [g \cdot h]$
- exercise: check that $k[G]$ is k-alg
  check $\mathrm{Mod}_{k[G]} \simeq \mathrm{Rep}(G)$ canonically

- ex 2. - let X be C^∞-mfd, C^∞_X sheaf of $C^\infty_{\mathbb{R}}$-funcs on X.
  - $OP(C^\infty_X)$ sheaf of lin. maps $C^\infty_X \circlearrowleft$
  - (very big) NC sheaf of NC $\mathbb{R}$-alg where product is composition
  - possesses an interesting subalg: differential operators, subalg gen by $f \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_r}}$ (Kashiwara-Schapira)

- recall: A nc. ring, $[a,b] := ab - ba$.
- let
  $D^{\leq k}_X$ = $\mathbb{R}$-lin subsheaf of $OP(C^\infty_X)$ gen by $f \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_r}}$ with $r \leq k$
- $D_X = \bigcup_k D^{\leq k}_X$, $D^{\leq k}_X \subset D^{\leq k+1}_X$
- exercise:
  1) $D^0_X = C^\infty_X$
  ii) $D^{\leq k}_X = \{ \varphi \in OP(C^\infty_X) \mid \forall f \in C^\infty_X, [f, \varphi] \in D^{\leq k-1}_X \}$

Def. Let $\mathcal{C}$ cat. We always assume $\forall x, y \in Ob\,\mathcal{C} \Rightarrow Mor_{\mathcal{C}}(x,y)$ is a set.
To every $x \in Ob\,\mathcal{C}$ associate $h_x : \mathcal{C}^{op} \to Set$ functor $h_x(y) := Mor_{\mathcal{C}}(y,x)$,
$f: y_1 \to y_2 \rightsquigarrow h_x(f) = Mor_{\mathcal{C}}(y_2,x) \to Mor_{\mathcal{C}}(y_1,x)$,
$g \mapsto g \circ f$

- Recall: $\mathcal{C}, \mathcal{C}'$ cats. $Fun(\mathcal{C}, \mathcal{C}')$ cat with objects functors $\mathcal{C} \to \mathcal{C}'$, mor := nat transf $\alpha : F \Rightarrow G$

i.e. datum $\forall x \in Ob\,\mathcal{C}$ of a mor. $\alpha(x)$ in $\mathcal{C}'$

s.t. $\forall f: x_1 \to x_2$,
$$\begin{tikzcd} F(x_1) \arrow[r, "\alpha(x_1)"] \arrow[d, "F(f)"'] & G(x_1) \arrow[d, "G(f)"] \\ F(x_2) \arrow[r, "\alpha(x_2)"] & G(x_2) \end{tikzcd}$$
commutes.

- <u>exercise</u> Show $\alpha: F \Rightarrow G$ iso in $Fun(\mathcal{C}, \mathcal{C}')$
$\iff \forall x \in Ob(\mathcal{C})$, $\alpha(x)$ iso in $\mathcal{C}'$

- <u>example</u>. $\mathcal{V} \in Vect_k$. Construct nat trans
$id_{\mathcal{V}} \Rightarrow DD$, $DD: \mathcal{V} \to \mathcal{V}$
$$V \mapsto V^{\vee\vee} = Hom(Hom(V,k),k)$$

<u>Def.</u> $F: \mathcal{C}^{op} \to Set$ is *representable* if
$\exists\, x \in Ob\,\mathcal{C}$ $\wedge$ $\exists$ nat transf $h_x \to F$.
We say x *represents* F.

<u>Lemma.</u> (Yoneda) Let $F: \mathcal{C}^{op} \to Set$, $x \in Ob\,\mathcal{C}$.
Then the natural map
$$Mor(h_x, F) \to F(x)$$
$$\alpha: h_x \to F \mapsto \alpha(x)(id_x)$$
where $\alpha(x): h_x(x) \to F(x)$,
is a bijection.

<u>Cor.</u> Assume we are given nat eq
$$\begin{tikzcd} h_{x_1} \arrow[r, "\alpha_1"] \arrow[d, dashed, "\beta"'] & F \\ h_{x_2} \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

Then $\exists!\, \varphi: x_1 \xrightarrow{\sim} x_2$ in $\mathcal{C}$
such that $\overset{\beta}{\dashrightarrow}$ defined by $\varphi \in h_{x_2}(x_1) = Mor(x_1, x_2)$
makes diagram commute.

<u>Pf.</u> α_1, α_2 isos in $Fun(\mathcal{C}^{op}, Set) \Rightarrow \beta := \alpha_2^{-1} \circ \alpha_1$ well-def.
iso in $Fun(\mathcal{C}^{op}, Set) \Rightarrow$ by Yoneda it is defined by $\varphi: x_1 \to x_2$

φ is iso since (by Yoneda) corr. to β^{-1}. $\square$

- examples. $\mathcal{C} = (Sch/_k \vee C^\infty \text{ mfds})$, V f.d. vsp, $r \in \mathbb{N}$. $G(r,V)$ represents $G : (Sch/_k)^{op} \to (Set)$, $G(X) = \{rk = r$ subbdls of $V \times X\}$.
  - for $S_2 \subseteq V \times X_2 \Rightarrow \varphi^* S_2 = S_2 \times_{X_2} X_1$, $X_1 \xrightarrow{\varphi} X_2$

- exercise. Let $\mathcal{C}$ be $C^\infty_{\mathbb{R}}$-mfds or $Sch/_k$.

  Define $F : \mathcal{C}^{op} \to Set$ by $F(X) = C^\infty(X)$ or $\mathcal{O}_X(X)$

  i) define F on morphisms, demonstrate functoriality
  ii) show it is representable & find representative ob.

- how to use repr. in alg. geom?
  i) to define schemes: Fix $P \in \mathbb{Q}[t]$, $X \subseteq \mathbb{P}^N$ closed.
     define $Hilb^P(X)$ to be scheme repr. the functor
     $Q(S) = \{ Z \subseteq X \times S \text{ closed} \mid Z \text{ flat over } S,$
     i.e. $\forall s \in S$, $Z_s \subseteq X$ has Hilb. polyn. $= P\}$

  Thm (Grothendieck) Q represented by a proj. $sch/_k$.

  ii) describe props of schemes or morph. of sch in terms of Yoneda functors
  - e.g. val criterion: $f : X \to Y$ proper
    $\Longleftrightarrow$ $\forall$ comm. diag.

$$\begin{tikzcd} \eta = \operatorname{Spec} K(A) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ h_X(c) \quad c = \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with A valuation domain, $K(A)$ quot field,

$\exists! \xrightarrow{\omega}$ making everything commute

- so $h_X(c) \to h_X(\eta) \times_{h_Y(\eta)} h_Y(c)$ bijects

What is an abel. category?

- Grothendieck, Tohoku is great!
  - has a few superfluous assumptions for us

Def. An *additive category* $\mathcal{C}$ is a cat s.t. $\forall x, y \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(x,y)$ has a structure of an abelian gp. & $\mathrm{Mor}(x,y) \times \mathrm{Mor}(y,z) \to \mathrm{Mor}(x,y)$ is bilinear.

Def. If A add.cat., $\varphi: X \to Y \in \mathrm{Mor}\,A$, a *kernel* of φ is a mor $i: K \to X$ st. $\forall Z \in \mathrm{Ob}\,\mathcal{C}$ the following sqn. of ab. gps is exact:

$$0 \to \mathrm{Hom}(Z,K) \xrightarrow{i} \mathrm{Hom}(Z,X) \xrightarrow{\varphi} \mathrm{Hom}(Z,Y),$$

i.e. $\mathrm{Hom}(Z,K) = \{\psi \in \mathrm{Hom}(Z,K) \mid \varphi \circ \psi = 0 \in \mathrm{Hom}(Z,Y)\}$

i.e. $h_K \subseteq h_X$ is a subfunctor.

- by Yoneda, if kernel exists, then unique up to iso which commutes w $(-) \hookrightarrow X$.

- exercise: define coker of $X \to Y$
  - hint coYoneda

Def. An *abelian category* is an additive cat. such that

1) finite direct sums exist
ii) ker & coker exist
iii) $\forall f: X \to Y$ mor, $\exists$ morphisms $K \xrightarrow{i} X \xrightarrow{\pi} P \xrightarrow{j} Y \xrightarrow{p} C$ s.t.

$i = \ker f$, $\pi = \operatorname{coker} i$, $p = \operatorname{coker} f$, $j = \ker p$, $j \circ \pi = f$.

Omissions in course

i) all set-theoretic questions
ii) strict use of def. of ab. cat.
- ∄ embedding thm guaranteeing that every ab. cat is an ab. subcat of cat. of modules ?̥ (Freyd)

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Def. An additive cat is abel. if

i) every mor has ker. & coker

ii) finite direct sums exist

iii) every mor $X \xrightarrow{f} Y$ induces

$$K \xrightarrow{i} X \xrightarrow{\pi} P \xrightarrow{j} Y \xrightarrow{p} C$$

st. $i = \ker f$, $j = \operatorname{coker} f$, $\pi = \operatorname{coker} i$, $p = \ker p$, $j \circ \pi = f$

- exercise. Show iii) functorial, i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \text{ comm.} \Rightarrow \begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C \arrow[d] \\ K' \arrow[r] & X' \arrow[r] & P' \arrow[r] & Y' \arrow[r] & C' \end{tikzcd} \text{ comm., unique}$$

Def. Let $\mathcal{C}$ any cat, $X, Y \in \mathrm{Ob}\,\mathcal{C}$. A product for X and Y is an object $P = X \times Y$ which represents $h_X \times h_Y : \mathcal{C}^{op} \to Set$, $h_X \times h_Y(S) = h_X(S) \times h_Y(S)$

- in other words, to give mor $S \to X \times Y$ is the same as giving $S \to X$ and $S \to Y$
- exercise: see it in Top, Grp, Mfd, ...
- exercise: $\exists$ canonical projs. $X \leftarrow X \times Y \to Y$ (use Yon.).

Def. $\mathcal{C}$ cat, $X \in \mathrm{Ob}\,\mathcal{C}$. Put $h^X : \mathcal{C} \to Set$, $h^X(Y) = \mathrm{Mor}(X, Y)$.

- exercise: define corepresentable functors, state co-Yoneda.

Def. Let $\mathcal{C}$ cat, $X, Y \in \mathrm{Ob}\,\mathcal{C}$. A coproduct Z of X and Y corepresents $h^X \times h^Y$.

- exercise: as before. What about comm. rings?

Prop Let $\mathcal{A}$ add. cat satisfying 1) and [1.1]. Then $\forall X, Y \in \mathrm{Ob}\,\mathcal{A}$, a coproduct exists if and only if a product exists. Further, they are equal.

Pf. Assume $\exists$ product P. Then

$$\begin{tikzcd} & X \arrow[d, "{(\mathrm{id}_X, 0)}"'] \arrow[dr, equal] & \\ Y \arrow[r, "{(0, \mathrm{id}_Y)}"'] \arrow[dr, equal] & P \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \\ & Y & \end{tikzcd}$$

- exercise: $\pi_X = \mathrm{coker}\, i_Y$, $i_Y = \mathrm{coker}\, \pi_X$ and $(X \Leftrightarrow Y)$

Given $Z \in \mathrm{Ob}\,\mathcal{C}$,

$$\mathrm{Hom}(P, Z) \longrightarrow \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)$$
$$f \longmapsto (f \circ i_X, f \circ i_Y)$$

is bijection, so P direct sum.

Check converse.

Rmk. $\mathcal{A}$ abel. $\Longleftrightarrow$ $\mathcal{A}^{op}$ abel.

Def. Let $\mathcal{A}$ abel. cat. A (possibly inf.) sequence of objects and morphisms indexed by morphisms indexed by $\mathbb{Z}$

$$\cdots \longrightarrow A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} A_{i+2} \longrightarrow \cdots$$

is called a *complex* if $\forall i$, $d_{i+1} \circ d_i = 0$, if makes sense. Its i-th cohomology (group) object is $\ker d_{i+1} / \mathrm{Im}\, d_i$

$$\begin{tikzcd} A_1 \arrow[r, "d_1"] & A_2 \arrow[r, "d_2"] \arrow[dl, hook'] \arrow[dr, two heads] & A_3 \\ \operatorname{Im} d_1 = P_1 & & K_2 = \ker d_2 \end{tikzcd}$$

, $\ker d_1 = K_1 \xrightarrow{i_1} A_1 \to P_1 = \operatorname{coker} i_1$

$d_2 \circ d_1 = 0$

$$\begin{tikzcd} A_1 \arrow[r, "d_1"] \arrow[dr] & A_2 \\ & K_2 \arrow[u] \end{tikzcd} , \quad \begin{tikzcd} K_1 \arrow[r] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & A_1 \arrow[r] \arrow[d] & A_2 \\ & K_2 \arrow[ur] & \end{tikzcd}$$

$\Rightarrow P_1 \to K_2$. Def. $H_1(A_\bullet) = \operatorname{coker}(P_1 \to K_2)$

Notation. we use cohom. indexing A^i.

Def. A cpx $A^\bullet = (\cdots \to A^i \to A^{i+1} \to \cdots)$ is called exact at i if $h^i(A^\bullet) = 0$. It is called exact if $\forall i$ $h^i(A^\bullet) = 0$.

Def. Let $A^\bullet, B^\bullet$ complexes with same index interval I. A morphism $\varphi: A^\bullet \to B^\bullet$ is the datum: $\forall i \in I$ $\varphi_i: A^i \to B^i$ s.t. if $i, i+1 \in I$, then

$$\begin{tikzcd} A^i \arrow[r, "d_A^i"] \arrow[d, "\varphi_i"'] & A^{i+1} \arrow[d, "\varphi_{i+1}"] \\ B^i \arrow[r, "d_B^i"] & B^{i+1} \end{tikzcd}$$

commutes.

We call φ a (co) chain map.

- exercise: show I-complexes constitute a cat.

Lemma. $\varphi: A^\bullet \to B^\bullet$ mor $\Rightarrow h^i(\varphi): h^i(A) \to h^i(B)$ whenever $i-1, i, i+1 \in I$.

Def. The cat $C(\mathcal{A})$ has as objects $\mathbb{Z}$-cpxs in $\mathcal{A}$ and chain maps as morphisms.

- exercise: $\forall i \in \mathbb{Z}$, $h^i : C(\mathcal{A}) \to \mathcal{A}$, $A^\bullet \mapsto h^i(A^\bullet)$, $\varphi \mapsto h^i(\varphi)$ is an additive functor (respects abel. structure on mor)

Def. Let $\varphi, \psi : A^\bullet \to B^\bullet \in \operatorname{Mor} \mathcal{C}(\mathcal{A})$.
We say φ is homotopic to ψ if $\exists$ $\forall i \in \mathbb{Z}$ $\alpha_i : A^i \to B^{i-1}$ mor in $\mathcal{A}$ s.t. $\forall i \in \mathbb{Z}$ $\psi^i - \varphi^i = \alpha^{i+1} \circ d_A^i + d_B^{i-1} \circ \alpha^i$.
In this case we call $\alpha = (\alpha_i)_{i \in \mathbb{Z}}$ a **homotopy** from φ to ψ.

$$\begin{tikzcd}
\arrow[r] & A^{i-1} \arrow[r, "d_A^i"] \arrow[dl] \arrow[d, shift right] \arrow[d, shift left] & A^i \arrow[r, "d_A^i"] \arrow[dl, "\alpha^i"'] \arrow[d, shift right] \arrow[d, shift left] & A^{i+1} \arrow[r] \arrow[dl, "\alpha^{i+1}"'] \arrow[d, shift right] \arrow[d, shift left] & {} \arrow[dl] \\
\arrow[r] & B^{i-1} \arrow[r, "d_B^i"'] & B^i \arrow[r] & B^{i+1} \arrow[r] & {}
\end{tikzcd}$$

- philosophical q.: what is the natural structure on $C(\mathcal{A})$?
- higher category theory, due to homotopies.

Lemma. Assume $\varphi, \psi : A^\bullet \to B^\bullet$ homotopic mors in $C(\mathcal{A})$. Then $\forall i \in \mathbb{Z}$, $h^i(\varphi) = h^i(\psi)$.

Pf. (We use objects since it is simpler. Diagram chasing CAN BE DONE, tho.)
Let $[x] \in h^i(A)$, meaning $x \in Z^i(A) = \ker d_A^i$.
$h^i(\psi)[x] = [\psi_i(x)]$. Now
$\psi_i(x) - \varphi_i(x) = \alpha^{i+1}(d_i x) + d^i(\alpha^{i-1} x)$
so $[\psi_i(x)] = [\varphi_i(x)]$.

Def: A morphism φ in $C(\mathcal{A})$ is called a quasi-isomorphism if $\forall i \in \mathbb{Z}$, $h^i(\varphi)$ is isom.

Rmk. Analoguous to weak-equivalence in alg. top.

Rmk. Like top. spaces, $C(\mathcal{A})$ also has a natural structure: model cat.

- why qisos? example:

- let A (Noeth) comm. ring, B fin.gen A-alg.
  Define module of kähler differentials $\Omega_{B/A}$ by saying it represents a functor
  $(\mathrm{Mod}_B)^{op} \to (\mathrm{Sets})$, $M \mapsto \mathrm{Der}_A(B, M)$,
  or by saying $B = A[x_1, \dots x_n] / (f_1, \dots f_r)$
  and
  $$\bigoplus_{j=1}^{r} Be_j \xrightarrow{\frac{\partial f_j}{\partial x_i}} \bigoplus_{i=1}^{n} B \cdot dx_i \to \Omega_{B/A} \to 0 \text{ exact.}$$

- the first approach is great, but the 2^{nd} possibly depends on basis?

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exercise $(Sch/\Bbbk)$, functor $S \mapsto \Gamma(S, \mathcal{O}_S)$ from $(Sch)^{op} \to (Set)$, on mor $S \xrightarrow[\varphi]{} S'$,
$\Gamma(S', \mathcal{O}_{S'}) \xrightarrow{\varphi^*} \Gamma(S, \mathcal{O}_S)$.

- $\Gamma(\mathcal{O}_-)$ is repr by $\mathbb{A}^1_\Bbbk$
  - take $x \in \Gamma(\mathbb{A}^1_\Bbbk, \mathcal{O}_{\mathbb{A}^1_\Bbbk})$, then
    $$\begin{aligned} Mor(S, \mathbb{A}^1_\Bbbk) &\longrightarrow \Gamma(S, \mathcal{O}_S) \\ \varphi &\longmapsto \varphi^*(x) \end{aligned}$$
    bijects
  - $f \in \Gamma(S, \mathcal{O}_S)$, $\Gamma(\mathbb{A}^1_\Bbbk, \mathcal{O}_{\mathbb{A}^1_\Bbbk}) \xrightarrow[\exists! \varphi_f^*]{} \Gamma(S, \mathcal{O}_S)$

    s.t. $f = \varphi_f^*(x)$
- for $S \in ob\, Sch/\Bbbk$, A $\Bbbk$-alg, $\varphi \mapsto \varphi^*$ gives
  $$Mor_{Sch/\Bbbk}(S, Spec A) \simeq Hom_{\Bbbk\text{-alg}}(A, \Gamma(S, \mathcal{O}_S))$$

exercise For $\mathcal{A}$ add. cat. define the 0 functor
$0 : \mathcal{A}^{op} \to Set$, $A \mapsto pt$.
A zero object is an object representing the 0 functor, called $0_\mathcal{A}$.
$\forall A \in ob\, \mathcal{A}$, $Hom(A, 0_\mathcal{A}) = 0$ (final obj)

- claim $Z_A := \ker(A \xrightarrow{id_A} A)$ is zero object
  Pf. $Hom(B, Z_A) = \{\varphi : B \to A \mid id_A \circ \varphi = 0\} = \{0\}$
- noting that $id_0 \in Hom(0,0) = \{0\} \Rightarrow id_0 = 0$,
  $\psi : 0 \to B \Rightarrow \psi = \psi \circ id_0 = 0$, so $0_\mathcal{A}$ is also an initial object

- In abel cat $\mathcal{A}$, $A \oplus B$ is the coproduct of A, B, i.e. $\forall A, B \in ob\,\mathcal{A}$, $\exists\ A \xrightarrow{i} A \oplus B \xleftarrow{j} B$ s.t. $\forall C \in ob\,\mathcal{A}$, $\mathrm{Hom}(A \oplus B, C) \xrightarrow{\sim} \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C)$

$$\varphi \longmapsto (\varphi \circ i, \varphi \circ j)$$

- claim: $A \oplus B$ is product.
  - take

$$\begin{tikzcd} A \arrow[r, "i"] & A \oplus B \arrow[r, "\pi"] & \mathrm{coker}(i) \\ & B \arrow[u, "j"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

  - claim β is iso.
  - we'll finish tomorrow

exercise Let $\Bbbk$ base comm. ring, $A \to B$ hom. of fin. gen. comm. $\Bbbk$-algs s.t. it factors as $A \to P \to B$, P = free A-alg i.e. $P = A[x_1, \dots, x_n]$, $P \to B$ surjects with ker I

- we associate to this an element of $C(B)$, $C(B) = C(\mathrm{Mod}_B)$

$$\dots \to 0 \to I/I^2 \to \Omega_{P/A} \otimes_A B \to 0 \dots \to$$

Lemma. $d : P \to \Omega_{P/A}$:
 i) maps I^2 to zero in $\Omega_{P/A} \otimes_P B$
 ii) induced hom $\underset{I \otimes_P B}{\overset{I/I^2}{\shortparallel}} \to \Omega_{P/A} \otimes_P B$ is B-linear

Pf. i) $I^2 = \langle f, g \rangle$, $f, g \in I$, but $d(fg) = f\,dg + df \cdot g = 0$ in $\Omega_{P/A} \otimes_P B$ since $f, g \mapsto 0$ in B

ii) $f \in P$, $[f] \in B$, $[g] \in I/I^2$, $g \in I$,
$d([f][g]) = [f]\,d[g]$ in $\Omega_{P/A} \otimes_P B$
$d(fg) = df\,g + \underbrace{f\,dg}_{= 0 \text{ in } \Omega_{P/A} \otimes_P B}$ in $\Omega_{P/A}$

- if we had two factors P_1, P_2,
then

$$\begin{tikzcd} & P_1 \arrow[d] \arrow[dr] & \\ A \arrow[ur] \arrow[r] \arrow[dr] & P_1 \otimes_A P_2 \arrow[r] & B \\ & P_2 \arrow[u] \arrow[ur] & \end{tikzcd}$$

- focus on upper part (symmetry)
- put $P = A[x_1 \ldots x_n, y_1, \ldots, y_m]$

$P_1 = A[x_1 \ldots x_n]$,

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & P \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I_1 \arrow[r] \arrow[u, hook] & P_1 \arrow[r] \arrow[u, hook] & B \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

gives morphism in $C(B)$ since

$$\begin{tikzcd} I \arrow[r] & \Omega_{P/A} \\ I_1 \arrow[r] \arrow[u] & \Omega_{P_1/A} \arrow[u] \end{tikzcd} \Longrightarrow \begin{tikzcd} I/I^2 \arrow[r] & \Omega_{P/A} \otimes_P B \\ I_1/I_1^2 \arrow[r] \arrow[u] & \Omega_{P_1/A} \otimes_{P_1} B \arrow[u] \end{tikzcd}$$

- claim: iso on ker, coker

- so we get

$$\begin{tikzcd} & I/I^2 \to \Omega_{P/A} \otimes_P B & \\ I_1/I_1^2 \to \Omega_{P_1/A} \otimes_{P_1} B \arrow[ur, "\text{q iso}"] & & I_2/I_2^2 \to \Omega_{P_2/A} \otimes_{P_2} B \arrow[ul, "\text{q-iso}"'] \end{tikzcd}$$

- are these two objects "isomorphic"?
  - this motivates derived categories
- let's get back to claim
  - put $Q_i = P[y_1, \ldots y_i]$ s.t. $Q_0 = P$, $Q_m = P_1$,
  let $J_i = \ker Q_i \to P$.

- look at

$$\begin{array}{ccc} J_0/J_0^2 & \longrightarrow & \Omega_{Q_0/A} \otimes_{Q_0} B \\ \downarrow & & \downarrow \\ J_1/J_1^2 & \longrightarrow & \Omega_{Q_1/A} \otimes_{Q_1} B \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ J_n/J_n^2 & \longrightarrow & \Omega_{Q_n/A} \otimes_{Q_n} B \end{array}$$

- we focus on showing each inner square is quiso

$$\begin{array}{ccc} Q & \overset{\pi}{\twoheadrightarrow} & B \\ \downarrow & \nearrow_{\tilde{\pi}} & \\ Q[y] & & \end{array}$$

Q free f.g. A-alg

$\tilde{\pi}$ is det. by π and $\tilde{\pi}(y) = b_0 \in B$,
choose $f_0 \in Q$ s.t. $\pi(f_0) = b_0$

- we can show that

$$\begin{array}{ccc} & I/I^2 & \quad (y - f_0) \in I_1 \\ & \downarrow & \\ B & \longrightarrow & I_1/I_1^2 \\ b & \longmapsto & b(y - f_0) \end{array}$$

so $I/I^2 \otimes B \to I_1/I_1^2$ iso

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Yesterday

- $\mathcal{A}$ ab.cat, $A \oplus B$ also $A \times B$.
- $A \oplus B$ means $\forall C \in \text{ob}\,\mathcal{A}$,
  $A \xrightarrow{i} A \oplus B \xleftarrow{j} B$ s.t. $\text{Hom}(A \oplus B, C) \to \text{Hom}(A, C) \times \text{Hom}(B, C)$
  sending $\lambda \mapsto (\lambda \circ i, \lambda \circ j)$ bijects
- we want dual statement with $A \xleftarrow{p} A \times B \xrightarrow{q} B$
- step 1

$$A \xrightarrow{i} A \oplus B \xrightarrow{\pi} \text{coker}(i) =: B',$$

$$\beta : B \to B', \quad \beta = \pi \circ j$$

- pick $C \in \text{ob}\,\mathcal{A}$, then

$$\text{Hom}(B', C) = \{\lambda \in \text{Hom}(A \oplus B, C) \mid \lambda \circ i = 0\}$$
$$\downarrow -\circ\beta$$
$$\text{Hom}(B, C)$$

- since $\lambda \mapsto (0, \lambda \circ j)$ bijects, $-\circ\beta$ bijects, so by co-Yoneda, equivalent functors $\Rightarrow \beta$ bijects
- step 2
- fix $C \in \text{ob}\,\mathcal{A}$, want $\gamma \mapsto (p \circ \gamma, q \circ \gamma)$ bijects.
- explicit inverse:
  - for $C \overset{\varphi \to A}{\underset{\psi \to B}{}} \xrightarrow{\gamma} A \oplus B$, $\gamma = i \circ \varphi + j \circ \psi$.
- exercise: $\gamma^{-1} = (\varphi, \psi)$.

Rmk. By induction $A_1 \oplus \cdots \oplus A_n \simeq A_1 \times \cdots \times A_n$.
But untrue for infinite I s.t. $\forall i \in I$, $A_i \in \text{ob}\,\mathcal{A}$,
even if $\bigoplus_{i \in I} A_i$ and $\prod_{i \in I} A_i$ both exist.
E.g. in $\mathcal{A} = \text{Mod}_R$,

$$\prod_{i \in I} A_i = \{(a_i)_{i \in I} \mid \begin{smallmatrix} a_i \in A_i \\ \forall i \in I \end{smallmatrix}\} \supsetneq \bigoplus_{i \in I} A_i = \{\begin{smallmatrix}\text{same, but} \\ \text{fin. many } a_i\text{'s} \neq 0\end{smallmatrix}\}$$

- R noeth rings $\Rightarrow$ $\mathcal{A}$ subcat of Mod_R of f.g. R-mod.
  $\rightarrow$ shows arb $\oplus$'s or Π's don't exist

- A comm. ring, $P = A[x_1 \dots x_n]$, $B = P/I$,
  $Q = P[y]$, $B = Q/J$,
$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, hook] & B \\ Q \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

- claim
$$\begin{tikzcd} I/I^2 \arrow[r] \arrow[d] & \Omega_{P/A} \otimes_P B \\ J/J^2 \arrow[r] & \Omega_{Q/A} \otimes_Q B \end{tikzcd} \quad \text{is qiso}$$

- Step 1
  - $\exists f_0 \in P$ s.t. $\tilde{\pi}(y) = \pi(f_0)$, $f_0 = 0 \Leftrightarrow \tilde{\pi}(y) = 0$.
  - what can be said of J/J^2 wrt I/I^2 in that case?
  - $\Omega_{P/A} = \bigoplus_{i=1}^{n} P\,dx^i$, $\Omega_{Q/A} = \bigoplus_{i=1}^{n} Q\,dx^i \oplus Q\,dy$

$$Q = P \oplus Py \oplus Py^2 \oplus \dots$$
$$\cup\!\mathsf{I}$$
$$J = I \oplus Py \oplus Py^2 \oplus \dots$$
$$\cup\!\mathsf{I}$$
$$J^2 = I^2 \oplus Iy \oplus Py^2 \oplus \dots$$

$$\Rightarrow J/J^2 = I/I^2 \oplus By$$

- Step 2
  - gen. case, let $\tilde{Q} = P[z]$, $\alpha(z) = y - f_0$, $\alpha : \tilde{Q} \xrightarrow{\sim} Q$
  - then
$$\begin{matrix} I/I^2 \to \Omega_{P/A} \otimes_P B \\ \downarrow \\ \tilde{J}/\tilde{J}^2 \to \Omega_{\tilde{Q}/A} \otimes_{\tilde{Q}} B \\ \downarrow \\ J/J^2 \to \Omega_{Q/A} \otimes_Q B \end{matrix} \quad \begin{matrix} \}\ \text{qiso} \\ \\ \}\ \text{iso} \Rightarrow \text{qiso} \end{matrix}$$

$\square$

Recall $X \to Y \in \operatorname{mor} Sch/_{\operatorname{Spec} A}$, A noeth,
X, Y f.g. $/\operatorname{Spec} A \rightsquigarrow [I/I^2 \to \Omega_{P/\mathbb{Z}} \otimes_P B]$
"unique up to qiso" in $C(Mod_B) = C(QCoh_X)$

Def. Let $\mathcal{C}$ be cat, $Q \subseteq Mor\,\mathcal{C}$. We say that the **localization** of $\mathcal{C}$ at Q is a functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ s.t.

i) $\forall \varphi \in Q$, $F(\varphi)$ iso

ii) $\forall G : \mathcal{C}' \to \mathcal{C}''$ s.t. $\forall \varphi \in Q$, $G(\varphi)$ is iso $\wedge$ $\exists! H : \mathcal{C}' \to \mathcal{C}''$ s.t. $G = H \circ F$

iii) $F|_{ob} : ob\,\mathcal{C} \to ob\,\mathcal{C}'$ is bijective

- exercise: if localisation exists, it is unique up to canonical iso,

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "\hat{F}"] & \\ \mathcal{C}' & & \hat{\mathcal{C}}' \arrow[ll, dashed, "\exists!"] \end{tikzcd}$$

Problems

q i) does $F : \mathcal{C} \to \mathcal{C}'$ exist?

q ii) if yes, how explicit is $\mathcal{C}'$?

a i) not in general due to size issues.

a ii) usualy not at all.

Thm Let $\mathcal{A}$ abel. cat. Then the localisation of $K(\mathcal{A})$ at $Q = \{q\text{-ison}\}$ exists, and we call it the **derived cat.** $D(\mathcal{A})$ of $\mathcal{A}$.
Given $A^\bullet, B^\bullet \in ob\, C(\mathcal{A}) = ob\, K(\mathcal{A}) = ob\, D(\mathcal{A})$, any mor in $D(\mathcal{A})$ can be written as

"roof" $\begin{tikzcd} & \tilde{A} \arrow[dl, "\text{qiso in } C(A)"'] \arrow[dr, "\text{mor in } C(A)"] & \\ A & & B \end{tikzcd}$ or "sink" $\begin{tikzcd} A \arrow[dr, "\text{mor in } C(A)"'] & & B \arrow[dl, "\text{qiso in } C(A)"] \\ & \tilde{B} & \end{tikzcd}$

- sketch of "roof" composition

$$\begin{tikzcd} & \tilde{A} \arrow[dl, "\varphi"', "q.i.s"] \arrow[dr, "\alpha"] & & \tilde{B} \arrow[dl, "\psi"', "q.i.s"] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

- idea: define $\hat{A} := \tilde{A} \underset{B}{\times} \tilde{B}$ by

$$\hat{A}^n = \ker\left(\tilde{A}^n \oplus \tilde{B}^n \xrightarrow{(\varphi, -\psi)} B^n \right),$$
$$\hat{d} = (\tilde{d}_A, \tilde{d}_B) : \tilde{A}^n \oplus \tilde{B}^n \longrightarrow \tilde{A}^{n+1} \oplus \tilde{B}^{n+1}$$

Rmk. Let $\mathcal{F}: \mathcal{A} \to \mathcal{A}'$ additive functor.
$\mathcal{F}$ induces $C(\mathcal{F}): C(\mathcal{A}) \to C(\mathcal{A}')$, $K(\mathcal{F}): \dots$
by $C(\mathcal{F})(A^\bullet) = (\dots \to F(A^n) \xrightarrow{F(d^n)} F(A^{n+1}) \to \dots)$
But in general, does not induce $D(\mathcal{F})$.

- e.g. $\mathcal{A} = \mathrm{Mod}_{\mathbb{C}[t]}$, $\mathcal{A}' = \mathrm{Mod}_{\mathbb{C}}$, $F(M) := M \underset{\mathbb{C}[t]}{\otimes} \mathbb{C}$.

- $\mathbb{C}$ is $\mathbb{C}[t]$-alg via $1.t = 0$,
- $\{0\} \xhookrightarrow{i} \mathbb{A}^1_{\mathbb{C}}$, $F = i^*: \mathrm{QCoh}(\mathbb{A}^1) \to \mathrm{QCoh}(pt)$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}[t] \arrow[r, "\cdot t"] \arrow[d] & \mathbb{C}[t] \arrow[r] \arrow[d, "t \mapsto 0"] & 0 \\ & 0 \arrow[r] & \mathbb{C} & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[r, "\cdot 0"] \arrow[d] & \mathbb{C} \arrow[d, equal] \\ 0 \arrow[r] & \mathbb{C} \end{tikzcd} \quad \text{not q.i.s.o}$$

Rmk. If $\mathcal{F}: \mathcal{A} \to \mathcal{A}'$ exact, then "obvious" $D(\mathcal{F})$ really works

- let $\mathcal{A}$ ab. cat, define full subcat of $C(\mathcal{A}), K(\mathcal{A}), D(\mathcal{A})$ s.t. $\forall\, a \leq b$, $a, b \in \mathbb{Z} \cup \{\pm\infty\}$,
$\mathrm{Ob}\, C^{[a,b]}(\mathcal{A}) = \{ A \in \mathrm{ob}\, C(\mathcal{A}) \mid h^i = 0 \text{ if } i \neq [a,b] \}$,
similarly with $K^{[a,b]}(\mathcal{A})$, $D^{[a,b]}(\mathcal{A})$

- also, $\mathrm{Ob}\, C^b(\mathcal{A}) = \bigcup_{-\infty < a \leq b < +\infty} \mathrm{Ob}\, C^{[a,b]}(\mathcal{A})$.
bounded,
$\mathrm{Ob}\, C^+(\mathcal{A}) = \bigcup_{-\infty < a} \mathrm{Ob}\, C^{[a,\infty]}(\mathcal{A})$, $\mathrm{Ob}\, C^-(\mathcal{A}) = \cdots$

- define functor $\tau_{\geq n} : C(\mathcal{A}) \longrightarrow C^{[n,\infty]}(\mathcal{A})$
by $A' := \tau_{\geq n} A$, $A'_i = \begin{cases} A_i & \text{if } i > n \\ A_n / d A_{n-1} & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$

so that
$$\begin{tikzcd} A'_n = A_n / d A_{n-1} \arrow[rr, "\bar{d}"] & & A_{n+1} = A_{n+1} \\ & A_n \arrow[ul, "d_n"] \arrow[ur, "d_n"'] & \end{tikzcd}$$

- exercises: i) $\exists$ nat map $A \xrightarrow{\alpha} \tau_{\geq n} A$ in $C(\mathcal{A})$
ii) $h^i(\alpha)$ is iso $\forall i \geq n$
iii) if $\varphi : A \to B$ qiso, $\tau_{\geq}(\varphi)$ qiso
iv) $\tau_{\geq}$ induces functors on $K(\mathcal{A})$, $D(\mathcal{A})$

- let $\overline{C}^{[n,\infty]}(\mathcal{A})$ = full subcat of cpx s.t. $A^i = 0$ $\forall i < n$. Then $\tau_{\geq n}$ is left adj of inclusion $\overline{C}^{[n,\infty]}(\mathcal{A}) \to C(\mathcal{A})$.

Recall $\mathcal{A}, \mathcal{B}$ abel cat, $\mathcal{A}$ enough injectives, $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ left exact. Define $R^i F: \mathcal{A} \to \mathcal{B}$ $\forall i \geq 0$ s.t. $\forall A \in \mathrm{Ob}\,\mathcal{A}$ $\forall$ $0 \to A \to I_0 \to I_1 \to \dots$ inj resn of A, $R^i F(A) = h^i(F(I_0 \to I_1 \to \dots))$

- with assumptions unchanged, we want: $\exists R\mathcal{F}: D^+(\mathcal{A}) \to D^+(\mathcal{B})$, uniquely determined by demanding if $(I^\bullet \in D^+(\mathcal{A})$ is cpx$) \wedge (I^n$ is inj $\forall n \in \mathbb{Z}) \wedge (\exists n_0$ s.t. $I^n = 0$ $\forall n < n_0)$ then $RF(I^\bullet) = (\dots \to F(I^n) \to F(I^{n+1}) \to \dots)$

Rmk. $0 \to A \xrightarrow{i} I^0 \xrightarrow{d} I^1 \to \dots$ inj resn

$$\Longleftrightarrow \quad \begin{array}{ccccccccc} 0 & \to & A & \to & 0 & \to & 0 & \to & \\ \downarrow 0 & & \downarrow i & & \downarrow 0 & & \downarrow 0 & & \\ 0 & \to & I^0 & \to & I^1 & \to & I^2 & \to & \end{array} \quad \text{qiso}$$

Intermediate step. Let $\mathcal{J}$ all injectives in $\mathcal{A}$
$C^+_{\mathcal{J}}(\mathcal{A}) \subseteq C^+(\mathcal{A})$ full subcat
- claim: induces $D^+_{\mathcal{J}}(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$.
This is equiv. of cats.

- replace "enough injs" by "enough acyclics"
 - note that injectives are acyclic for left ex. functors since $\forall$ $0 \to J' \to J \to J'' \to 0$ inj res in $\mathcal{A}$, $\forall F: \mathcal{A} \to \mathcal{B}$ lexact $\Rightarrow$ $0 \to F(J') \to F(J) \to F(J'') \to 0$

- look at Gel'fand-Manin, or ch. on derived cats from Kashiwara-Shapira Sheaves on mfds

Fantechi

- we will be following Kashiwara-Shapira

- let $\mathcal{A}, \mathcal{B}$ ab. cats, $F: \mathcal{A} \to \mathcal{B}$ right exact.
  - e.g.
    - A ring, $\mathcal{A} = \mathcal{B} = \mathrm{Mod}_A$, M A-mod $F(N) = N \otimes_A M$
    - X top. sp., $\mathcal{A} = \mathcal{B} = \mathcal{O}_X$-mod, $F(\mathcal{G}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$
    - X scheme, $\mathcal{A} = \mathcal{B} = (\mathrm{QCoh}_X) \vee (X \text{ noeth} \Rightarrow \mathrm{Coh}\, X)$
      $f: X \to Y \Rightarrow f^*: (\mathcal{O}_Y\text{-mod}, \mathrm{Qcoh}_Y, \mathrm{Coh}_Y) \to (Y \leftrightarrow X)$
      - for QCoh, $f^*(\mathcal{F}) = f^{-1}(\mathcal{F}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$
    - $\mathcal{A}$ ab. cat, $\mathcal{B} = Ab$, $M \in \mathrm{ob}\, \mathcal{A}$
      $\mathrm{Hom}_{\mathcal{A}}(-, M): \mathcal{A}^{op} \to Ab$
- recall $X \xrightarrow{f} Y$ proper (proj) mor of sch. f.g./k
  $n = \max\{\dim f^{-1}(y) \mid y \in Y\}$, $R^n f_*: \mathrm{Coh}\, X \to \mathrm{Coh}\, Y$
  is right exact.
  - for $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, $R^i f_*$ long ex. sq. ends
    with $\to R^n f_* \mathcal{F} \to R^n f_* \mathcal{F}'' \to 0 = R^{n+1} f_* \mathcal{F}'$

Def. A full add. subcat P of $\mathcal{A}$ is called a cat of F-projectives or F-cyclics if
1) ever ob of $\mathcal{A}$ is the quotient of an ob in P
2) given $0 \to M' \to M \to M'' \to 0$ exact in $\mathcal{A}$, (*)
   if $M, M'' \in \mathrm{ob}\, P$, so is M'
3) if in (*) M, M', M'' in P, then
   $0 \to F(M') \to F(M) \to F(M'') \to 0$ exact.

Thm Assume an F-proj P exists. Then
1) $D^-(\mathcal{A})^P \to D^-(\mathcal{A})$ is eq of cats, where
   $D^-(\mathcal{A})^P$ full subcat of cpx's $A^\bullet$ s.t. $(A^i = 0\ \forall i >> 0) \wedge (\forall i \in \mathbb{Z}\ A^i \in P)$

ii) the naive functor $F : \mathcal{C}^-(\mathcal{A})^P \to \mathcal{C}^-(\mathcal{B})$ induces a functor $LF : D^-(\mathcal{A})^P \to D^-(\mathcal{B})$ and hence $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$

iii) $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ doesn't depend on choice of P

Def When thm applies, we call $F : \mathcal{A} \to \mathcal{B}$ def'd by $\mathcal{A} \to D^-(\mathcal{A}) \xrightarrow{LF} D^-(\mathcal{B}) \xrightarrow{h^i} \mathcal{B}$ the i-th left der fctor of F.

$A \mapsto \cdots \to 0 \to A \to 0 \to 0 \to \cdots$

- exercise: A noeth comm. ring, M fixed f.g A-mod, $\mathcal{A} = \mathrm{FGMod}_A$, $F(N) = N \otimes_A M$

  i) show proj A-mod in $\mathcal{A}$ are a cat of F-proj (N proj $\iff \widetilde{N}$ loc. free on $\mathrm{Spec}\, A$)

  ii) compute L^iF where $A = \mathbb{C}[x,y]/\langle xy \rangle$, $M = \mathbb{C}$; $L^iF(N)$, $N = \Omega_{A/\mathbb{C}}$, M

Rmk. We can analoguously define F-injs for F left exact

- back to foundational material
- functor $[n] : C(\mathcal{A}) \circlearrowleft$ for $n \in \mathbb{Z}$ s.t. for $A^\bullet \in \mathcal{C}(\mathcal{A})$, $(A[n])^i = A^{n+i}$, d maybe gets a sign

Prop i) $[n]$ is autoequivalence w inverse $[-n]$

ii) $[n]$ sends $C^+(\mathcal{A}), C^b(\mathcal{A}), C^-(\mathcal{A})$ to their respective selves

iii) $[n]$ induces fctor $K(\mathcal{A}) \circlearrowleft$

iv) $h^i(A[n]) = h^{i+n}(A)$

v) if $A \xrightarrow{\varphi} B$ in $C(\mathcal{A})$, φ qiso iff $\varphi[n]$ qiso

Cor. $\forall n \in \mathbb{Z}$, $[n]$ induces $D(\mathcal{A}) \circlearrowleft$

Pf:
$$\begin{tikzcd} K(A) \arrow[r, "{[n]}"] \arrow[d] & K(A) \arrow[d] \\ D(A) \arrow[r, dashed] & D(A) \end{tikzcd}$$
if φ qiso, $\varphi[n]$ qiso, in $K(A)$ by univ property $\exists!$ $D(A) \circlearrowleft$

- let $\mathcal{A}, \mathcal{B}$ ab. cat $F: \mathcal{A} \to \mathcal{B}$ exact
- exercise: F exact $\Longleftrightarrow$ F commutes w ker, coker and with (co)limits

Lemma Let $F: \mathcal{A} \to \mathcal{B}$ exact, $A^\bullet \in \mathcal{C}(\mathcal{A})$
$\forall n \in \mathbb{Z}$, $Z^n(A) = \ker(A^n \to A^{n+1})$
$B^n(A) = \operatorname{Im}(A^{n-1} \to A^n)$
$= \operatorname{coker}(\ker d_{n-1} \to A^n)$
$h^n(A) := \operatorname{coker}(B^n(A) \to Z^n(A))$
Then $\exists$ natural iso $F(Z^n(A)) \simeq Z^n(F(A))$,
$F(B^n(A)) \simeq B^n(F(A))$, $F(h^n(A)) \simeq h^n(F(A))$
where $F(A) \in D(\mathcal{B})$ $(F(A))^i = F(A^i)$, $d_i(F(A)) = F(d_i)$

Cor If F exact, $\varphi: A_1^\bullet \to A_2^\bullet$ mor in $\mathcal{C}(\mathcal{A})$, then φ qis $\Rightarrow$ $F(\varphi)$ qis

Cor If $F: \mathcal{A} \to \mathcal{B}$ exact, F induces $K(\mathcal{A}) \to K(\mathcal{B})$, $D(\mathcal{A}) \to D(\mathcal{B})$.

- ex. 1 $X \xrightarrow{f} Y$ proj mor, assume factors as
$X \xrightarrow{i} X \times \mathbb{P}^n \xrightarrow{\pi} Y$, i cl. emb, π proj, $f_* = \pi_* \circ i_*$
$Rf_*: D(\mathrm{QCoh}\,X) \to D(\mathrm{QCoh}\,Y)$

- ex 2. X sch, $\mathcal{A} = \mathcal{O}_X$-mod or $QCoh_X$ or Coh_X, $\mathcal{L} \in Pic(X)$. Then $\otimes \mathcal{L} : \mathcal{A} \to \mathcal{A}$ is exact and induces autoequiv. of $D(\mathcal{A})$ w inv $\otimes \mathcal{L}^\vee$
- research q.: given X sch, $\mathcal{A}$ as above. Classify autoequivalences of $D(\mathcal{A})$.

as triang. cats.

- for R^iF, $0 \to M' \to M \to M'' \to 0$ gives long ex. sq. for F left exact if enough inj.
- what about L^iF? similar, but induced by structure of triang. cats

Fantechi

- Recall: $\mathcal{A}$ ab. cat, $M \in ob\,\mathcal{A}$ proj if $\forall\; N \overset{\varphi}{\twoheadrightarrow} N'$ surj, given $M \to N'$ $\exists$ lift making

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[dr] & N \arrow[d] \\ & N' \end{tikzcd}$$

commute

Rmk In Mod_A every free is proj

Cor In Mod_A, every obj is quot of a proj

Cor Given any cpx $M^\bullet \in \mathcal{C}^-(Mod_A)$ $\exists$ $F^\bullet \to M^\bullet$ quiso s.t. each F^i is free, therefore proj.

Examples $A = \mathbb{C}[x,y]$, $M^\bullet = [\dots 0 \to \mathbb{C} \to 0 \to \dots]$
$= [\mathbb{C}]$

$A/(x,y)$ mod-structure

- $\mathbb{A}^2$, $Mod_A = Qcoh(\mathbb{A}^2)$
- $\mathcal{O}_p$, $p \in \mathbb{A}^2$ origin

$$0 \leftarrow \mathbb{C} \leftarrow A \leftarrow (x,y) \leftarrow 0$$

$\overset{\text{is}}{I_p}$, $m_p = I_p$

$$0 \leftarrow \mathcal{O}_p \leftarrow \mathcal{O}_{\mathbb{A}^2} \leftarrow \mathcal{I}_p \leftarrow 0$$

$$\dim_{\mathbb{C}}(I_p/I_p^2) = 2$$

$\shortparallel$

$I_p \otimes \mathcal{O}_p/m_p$

$$0 \leftarrow \mathbb{C} \xleftarrow{\alpha} A^{\oplus 2} \leftarrow A \leftarrow 0$$

$xf + yg \quad (f,g) \quad (y,-x)$

$xf + yg = 0 \iff xyf_1 = -xyg_1 \overset{\text{domain}}{\iff} f_1 = -g_1$
$f,g \in \mathbb{C}[x,y]$ UFD $\Rightarrow$ $g = xg_1$, $f = yf_1$

$\Rightarrow (f,g) \in \ker \alpha$
$\shortparallel$
$(yf_1, -xf_1)$

Rmk i) special case of Koszul
ii) resol of len 2

Thm (Hilbert) A comm. ring, noeth, any A-mod.
M has resln of length $\equiv \dim \operatorname{Spec} A$
(nonsing.)

- $0 \leftarrow I_p \leftarrow A^{\oplus 2} \leftarrow A \leftarrow 0$ also a resn.
- I_p has no torsion on nonsing. affine sfc, a coh sh. has len 1 free resl $\Leftrightarrow$ torsion free

- $\operatorname{Tor}^i_A(M,M) = ?$ $\operatorname{Tor}^A_i(-,M) = L^i(- \otimes_A M)$
$M = A/I_p$
$h^{-i}([F^{-2} \to F^{-1} \to F^0] \otimes_A M) = \operatorname{Tor}^A_i(\ ,M)$

$A^{\oplus 5} \otimes_A N = N^{\oplus 5}$, $[F] \otimes_A M = [M \xrightarrow[(-y,x)]{} M^{\oplus 2} \xrightarrow[(x,y)]{} M]$

$= [\mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C}]$, on M, mult by x,y is zero.
$\Rightarrow \operatorname{Tor}^i_A(M,M) = \begin{cases} \mathbb{C}, & i = 0,1,2 \\ 0, & \text{otherwise} \end{cases}$

- what about $\mathbb{P}^2$? can I construct loc. free resoln similarly of $\mathcal{O}_p \in \operatorname{Coh}_{\mathbb{P}^2}$
$0 \leftarrow \mathcal{O}_p \leftarrow \mathcal{O}_{\mathbb{P}^2} \leftarrow \mathcal{J}_p \leftarrow 0$
- $\operatorname{Hom}(\mathcal{O}_p, -)$:
$0 \to \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{J}_p) \to \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}) \to \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_p)$

$$\begin{array}{ccccccc} 0 & \to & \Gamma(\mathcal{J}_p) & \longrightarrow & \Gamma(\mathcal{O}_{\mathbb{P}^2}) & \to & \Gamma(\mathcal{O}_p) \\ & & \| & & \| & & \\ & & 0 & & \mathbb{C} & \xrightarrow{\sim} & \mathbb{C} \end{array}$$

- want map $\underset{\text{loc.free}}{\mathcal{F}} \to \mathcal{O}_{\mathbb{P}^2}$ whose img is (x,y)

$x, y \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2})$

$\Rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow[(x,y)]{} \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_p \to 0$

$\qquad\qquad \text{extend} \Downarrow \text{to}$

$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_p \to 0$

Example $C \hookrightarrow \mathbb{P}^3$ sat. normal / twisted cubic

image of 3^{rd} deg Veronese $\mathbb{P}^1 \to \mathbb{P}^3$

$(s,t) \mapsto (s^3, s^2t, st^2, t^3)$

- compute loc. free resn of $\mathcal{Q}_C \in \mathrm{Coh}_{\mathbb{P}^3}$

$I_C = \{ \mathrm{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 1 \} = (Q_1, Q_2, Q_3)$

where $Q_1 = \det\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}$, $Q_2 = \det\begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \end{pmatrix}$, $Q_3 = \dots$

$Q_i \in \mathbb{C}[x_0, \dots, x_3]$ hom. poly. of deg 2

$0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_{\mathbb{P}^3} \xleftarrow[(Q_1,Q_2,Q_3)]{\alpha} \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 3}$

- what is $\ker \alpha$?

$\neg f$ s.t. $(Q_2 f, -Q_1 f, 0) \in \ker \alpha$ (koszul)

$\det\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix} = 0 \quad \forall x_0, \dots, x_3$

$\Rightarrow x_1 Q_3 + x_2 Q_2 + x_3 Q_1 = 0$

$\qquad x_0 Q_3 + x_1 Q_2 + x_2 Q_1 = 0$ relations

$$0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_{\mathbb{P}^3} \xleftarrow{(Q_1,Q_2,Q_3)} \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \xleftarrow{\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}} \mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 2} \leftarrow 0$$

$$f, g \mapsto \begin{pmatrix} x_0 f + x_1 g \\ x_1 f + x_2 g \\ x_2 f + x_3 g \end{pmatrix}$$

- check!
- useful "criterion":
  - pick pt $\notin C$ in $\mathbb{P}^3$
    - fiber in $\mathcal{O}_C$ is zero,

      so we get $0 \leftarrow \mathcal{O}_{\mathbb{P}^3} \leftarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \leftarrow \mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 2} \leftarrow 0$

      finite sqn of vector spaces

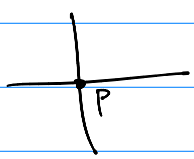
      $\Rightarrow \sum (-1)^i \operatorname{rk} V_i = 0$, which is true here

if $f = x_1 h$, $g = -x_0 h$

$\Rightarrow (x_1^2 - x_0 x_2) h = 0$

- $\exists$ finite free resn in nonsing affine case by Hilbert syzygy thm

$C = \operatorname{Spec} \mathbb{C}[x,y]/xy$, $A = \dfrac{\mathbb{C}[x,y]}{xy}$



$M = A/(x,y)$, $\widetilde{M} = \mathcal{O}_p$

- rmk: as v.sp., $A = \mathbb{C} \oplus x\mathbb{C}[x] \oplus y\mathbb{C}[y]$

$$0 \leftarrow M \leftarrow A \xleftarrow[(x,y)]{\alpha} A^{\oplus 2}$$

- $\ker \alpha$? f, g s.t. $xf + yg = 0$ in A?

  $\rightarrow f, g \in \mathbb{C}[x,y]$, so $(xf + yg)$ multiple of $xy \Rightarrow (y|f) \wedge (x|g)$

$\rightarrow f = y f_1$, $g = x g_1$, $f_1 \in \mathbb{C}[y]$, $g_1 \in \mathbb{C}[x]$ unique

so $xf + yg = xyf_1 + xyg_1 = 0$ in A

- $\ker \alpha$ is gen by $\begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}$

$$0 \leftarrow M \leftarrow A \xleftarrow[(x,y)]{} A^{\oplus 2} \xleftarrow[\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}]{\beta} A^{\oplus 2} \xleftarrow[\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}]{} A^{\oplus 2} \leftarrow \cdots \text{ to } \infty$$

and beyond

- taking $-\otimes_A M$ gives

$$0 \leftarrow \mathbb{C} \xleftarrow{0} \mathbb{C}^2 \xleftarrow{0} \mathbb{C}^2 \leftarrow \cdots$$

so taking homology gives

Cor $\mathrm{Tor}_*^i(M,M) = \begin{cases} \mathbb{C}, & i=0 \\ \mathbb{C}^{\oplus 2}, & i>0 \end{cases}$

- so M does not have finite free resn as A-module, we can't get rid of Tors
- we are doomed without Hilbert's protection

Fact X aff sing sch $\Rightarrow \exists$ objects in Coh_X without finite free resns
In fact, you can always find one of the form $\mathcal{O}_p$ if $\Gamma(X, \mathcal{O}_X)$ f.g. $/\mathbb{k} = \bar{\mathbb{k}}$, $p \in X(\mathbb{k})$

- same X, A. look at $\Omega_X = \Omega_{A/\mathbb{C}}$
- $X \hookrightarrow \mathbb{A}$ cl. emb

$$\Rightarrow I_X / I_X^2 \xrightarrow[\alpha]{} \Omega_{\mathbb{A}^2}/X \to \Omega_X \to 0$$

$$I_X \leq \underset{\underset{R}{\parallel}}{\mathbb{C}[x,y]},\ I_X = R \cdot xy,\ I_X/I_X^2 = A \cdot (xy)$$

- α induced by $d: R \to \Omega_R$, $d(xy) = dx\, y + x\, dy$
$\Omega_{\mathbb{A}^2} = R\,dx \oplus R\,dy$, $\Omega_{\mathbb{A}^2}/X = A\,dx \oplus A\,dy$

$$A \xrightarrow[(x\cdot,\, y\cdot)]{\alpha} A\,dx \oplus A\,dy \longrightarrow \Omega_X \to 0$$

$\mathbb{C}[x]$ $\mathbb{C}[y]$

- ker α? $f \in A$, $f = c + xf_1 + yf_2$ ($xf_1 \in \mathbb{C}[x]$, $yf_2 \in \mathbb{C}[y]$)
  $\alpha(f) = (yf, xf) = (y(c + yf_2), x(c + xf_1))$
  $= 0 \iff c + yf_2 = 0,\ c + xf_1 = 0 \iff c = f_1 = f_2 = f = 0$
- so ker α empty, $0 \to A \xrightarrow{\alpha} A\,dx \oplus A\,dy \to \Omega_X \to 0$

- recall, $X \hookrightarrow Y$ cl. emb. of schemes, $\mathcal{J} = \mathcal{J}_{X/Y}$
  $(*) \Rightarrow \mathcal{J}/\mathcal{J}^2 \to \Omega_{Y/X} \to \Omega_X \to 0$ in Coh_X

[Ha] If X, Y nonsing then α injects and $(*)$ sh. ex. seq of loc free sh

Def $X \hookrightarrow Y$ cl. emb. is regular of cod r if locally near each pt of X, $\mathcal{I}_X \subseteq \mathcal{O}_Y$ is gen by len r reg seq.

Fact If $X \hookrightarrow Y$ reg, $\mathcal{J}/\mathcal{J}^2$ is loc. fr. of rk r. Moreover, if Y nonsing $\wedge$ X reduced then $(*)$ is exact on left ($\iff$ ker $\alpha = 0$ $\Rightarrow$ Ω_X has a loc free resn of len 1)

Cor Let $X = \operatorname{Spec} \mathbb{C}[x,y]/x \cdot y$. Then $\operatorname{Ext}^1(\Omega_X, \mathcal{O}_X) = \mathbb{C}$, $\operatorname{Ext}^{i>1}(\Omega_X, \mathcal{O}_X) = 0$.

Pf $\operatorname{Hom}(-, A): \mathrm{Mod}_A^{op} \to \mathrm{Mod}_A$ right exact, Mod_A has enough proj.
Apply to

$$\underset{-1}{A} \xrightarrow{\binom{y}{x}} \underset{0}{A^{\oplus 2}} \rightsquigarrow \underset{0}{A^{\oplus 2}} \xrightarrow[(y,\,x)]{\alpha} \underset{1}{A}$$

- $\ker \alpha = \{(f,g) \in A^{\oplus 2} \mid yf + xg = 0\}$
$= x\,\mathbb{C}[x] \oplus y\,\mathbb{C}[y]$
- so $\operatorname{coker} \alpha = \mathbb{C} = \operatorname{Ext}^1(\Omega_X, \mathcal{O}_X)$
- note $\operatorname{Hom}(\Omega_X, \mathcal{O}_X) = \Gamma(X, \underline{\underline{T_X}})$

where $T_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X)$ $(= \vartheta_X)$

Exercise For any scheme X (loc. fin. / $\mathbb{k} = \bar{\mathbb{k}}$), $\Gamma(T_X)$ are tgt sp. to $\operatorname{Aut}(X)$.

- $\operatorname{Ext}^1(\Omega_X, \mathcal{O}_X)$ are 1^{st} order deformations of X
- informally, $X = \{xy = 0\} \subseteq \mathbb{A}^2$
so $\{xy + \varepsilon f = 0\}_{\varepsilon^2 \to 0} \subseteq \mathbb{A}^2 \times \mathbb{C}[\varepsilon]/\varepsilon^2$
is deformation
- but if $f = gx + hy$,
$xy + \varepsilon f = (x + \varepsilon f)(y + h\varepsilon) + O(\varepsilon^2)$
so nothing changes
$\to$ so $f \in \mathbb{C}$

Fautech 1

- today: Kashiwara Schapira §1.

Lemma $\mathcal{A}$ ab. cat $\Rightarrow C(\mathcal{A})$ ab. cat.

Pf sketch. Let $A^\bullet \xrightarrow{\varphi} B^\bullet$ in $C(\mathcal{A})$.
Define kernel as $\ker\varphi = (K^\bullet, d_A|_{K^\bullet})$
where $K^n := \ker(A^n \xrightarrow{\varphi^n} B^n)$.
Define direct sum $(A \oplus B)^\bullet \rightsquigarrow (A^n \oplus B^n, d^n = \begin{pmatrix} d_A^n & \\ & d_B^n \end{pmatrix})$

Prop To every s.e.s $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$
in $C(\mathcal{A})$ we can associate natural maps $H^n(C) \to H^{n+1}(A)$ s.t.

i) $\cdots \to H^n(A) \to H^n(B) \to H^n(C) \to H^{n+1}(A) \to \cdots$
is exact

ii) the definition is functorial, i.e. given comm. diag in $C(\mathcal{A})$ w/ exact rows

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \to & \tilde{A} & \to & \tilde{B} & \to & \tilde{C} & \to & 0 \end{array}$$

then $\forall n \in \mathbb{Z}$

$$\begin{array}{ccc} H^n(C) & \longrightarrow & H^{n+1}(A) \\ \downarrow h^n(\gamma) & & \downarrow h^{n+1}(\alpha) \\ H^n(\tilde{C}) & \longrightarrow & H^{n+1}(\tilde{A}) \end{array}$$

commutes

Def Let $A^\bullet \xrightarrow{\varphi} B^\bullet$ in $C(\mathcal{A})$. We associate to it $M(\varphi) \in ob\, C(\mathcal{A})$ and morphisms

$$A^\bullet \xrightarrow[\varphi]{} B^\bullet \xrightarrow[\alpha(\varphi)]{} M(\varphi) \xrightarrow[\beta(\alpha)]{} A[1]$$

We call $M(\varphi)$ the mapping cone of φ.

Warning Convention used: in $A[k]$, $d^n[k] := (-)^k d^{n+k}$

We let $M(\varphi)^n := A[1]^n \oplus B^n$,

$$d^n_{M(\varphi)} = \begin{pmatrix} d^n_{A[1]} & \\ \varphi^{n+1} & d^n_B \end{pmatrix}, \text{ and}$$

$$\alpha(\varphi) = \begin{pmatrix} 0 \\ id_{B^n} \end{pmatrix}, \quad \beta(\varphi) = \begin{pmatrix} id_{A[1]^n} & 0 \end{pmatrix}$$

Lemma (informal) The triangles

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \arrow[dl, "\alpha(\varphi)"] \\ M(\varphi) \arrow[u, "+1"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M(\alpha(\varphi)) \arrow[r, "+1"] & B \arrow[dl, "\alpha(\varphi)"] \\ M(\varphi) \arrow[u] & \end{tikzcd}$$

"are the same" in $K(\mathcal{A})$ (but not in $C(\mathcal{A})$).

Lemma Given $A^\bullet \xrightarrow{\varphi} B^\bullet$ in $C(\mathcal{A})$ $\exists$ $A[1] \xrightarrow{\psi} M(\varphi)$ s.t.

1) ψ is iso in $K(\mathcal{A})$

ii) the following commutes in $K(\mathcal{A})$

$$\begin{tikzcd} B \arrow[r, "\alpha(\varphi)"] \arrow[d, equal] & M(\varphi) \arrow[r, "\beta(\varphi)"] \arrow[d, equal] & A[1] \arrow[r, "-\varphi[1]"] \arrow[d, "\psi"] & B[1] \arrow[d, equal] \\ B \arrow[r, "\alpha(\varphi)"'] & M(\varphi) \arrow[r, "\alpha(\alpha(\varphi))"'] & M(\alpha(\varphi)) \arrow[r, "\beta(\alpha(\varphi))"'] & B[1] \end{tikzcd}$$

Pf sketch. $M(\alpha(\varphi))^n = B[1]^n \oplus M(\varphi)^n$
$= B[1]^n \oplus A[1]^n \oplus B^n$

So define $\psi^n : A[1]^n \to M(\alpha(\varphi))^n$
by $\psi^n = (-\varphi[1]^n, \mathrm{id}_{A[1]^n}, 0)$.
Also define $\tilde{\psi}^n : M(\alpha(\varphi))^n \to A[1]^n$
by $\tilde{\psi}^n = (0, \mathrm{id}_{A[1]}, 0)$ and note

$$\tilde{\psi} \circ \psi = \mathrm{id}_{A[1]}, \quad \psi \circ \tilde{\psi} = \begin{pmatrix} 0 & -\varphi[1] & 0 \\ 0 & \mathrm{id}_{A[1]} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and show iso in $K(\mathcal{A})$ □

Def. A *triangle* in $K(\mathcal{A})$ is seq. of mor

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

A triangle is called *distinguished*: if it is isomorphic to

$$\tilde{A}^\bullet \xrightarrow{\varphi} \tilde{B}^\bullet \to M(\varphi) \to \tilde{A}^\bullet[1]$$

where a morphism is
$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & A[1] \\ \downarrow \alpha & \circlearrowright & \downarrow \beta & \circlearrowright & \downarrow \gamma & \circlearrowright & \downarrow \alpha \\ \tilde{A} & \to & \tilde{B} & \to & \tilde{C} & \to & \tilde{A}[1] \end{array}$$

Cor (of lemma) If $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\lambda} A[1]$ distinguished then so is
$$B \xrightarrow{\psi} C \xrightarrow{\lambda} A[1] \xrightarrow{-\varphi[1]} B[1]$$

- we now formalise dist. triangles

Prop Distinguished triangles obey the following

(TR0) If a triangle isom. to a dist. tri.,
then it is dist.

(TR1) $\forall A \in \mathrm{ob}\, K(\mathcal{A})$, $A \xrightarrow{id} A \to 0 \to A[1]$
is dist.

(TR2) any mor $\varphi : A \to B$ can be included
in a dist tr $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\chi} A[1]$

(TR3) $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\chi} A[1]$ distinguished
$\Leftrightarrow B \xrightarrow{\psi} C \xrightarrow{\chi} A[1] \xrightarrow{-\varphi[1]} B[1]$ distinguished

(TR4) Given comm diag $\exists$ mor of dist tr

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\ \widetilde{A} \arrow[r] & \widetilde{B} \arrow[r] & \widetilde{C} \arrow[r] & \widetilde{A}[1] \end{tikzcd}$$

(TR5) Octahedron.

Def A *triangulated category* $\mathcal{T}$ is
an add. cat with autoeq. $[1]$ (and $[k] = [1] \circ \cdots \circ [1]$, $[-k] \ldots$)
and a collection of triangles called
distinguished which satisfy (TR0)-(TR5).

- key structure on derived cats.: triangulated cat
- today we show this structure on $K(\mathcal{A})$

Philosophy $F : \mathcal{A} \to \mathcal{B}$ half-ex. among ab. cats
$\Rightarrow$ derived fctor $RF : D^+(A) \to D^+(B)$ unique, if exists.
$LF : D^-(A) \to D^-(B)$

Lemma If $0 \to A^\bullet \xrightarrow{\varphi} B^\bullet \xrightarrow{\psi} C^\bullet \to 0$ ex. seq. in $C(\mathcal{A})$ then $\exists$ nat map $M(\varphi) \to C$, $A[1] \oplus B \xrightarrow{(0,\psi)} C$ on objects, which is iso in $K(\mathcal{A})$.

Def Let $\mathcal{T}$ triang. cat, $\mathcal{A}$ ab. cat. A **cohomological functor** $F: \mathcal{T} \to \mathcal{A}$ is additive functor s.t. $\forall$ dist. tr
$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{\eta} X[1]$,
$F(X) \xrightarrow{F(\varphi)} F(Y) \xrightarrow{F(\psi)} F(Z)$ is exact in $\mathcal{A}$.

Exercise Using (TR3) show (F coh. funct)
$\Rightarrow \cdots \to F(X) \to F(Y) \to F(Z) \to F(X[1]) \xrightarrow{-F(\varphi[1])} F(Y[1]) \to \cdots$
is exact

Prop If $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{\eta} X[1]$ is dist. tr, then $\psi \circ \varphi = 0$

Pf
$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1] \end{tikzcd}$$
by (TR1&4)

thm let $\mathcal{T}$ tr. cat, $W \in \mathrm{ob}\,\mathcal{T}$. Then $\mathrm{Hom}_{\mathcal{T}}(W, -): \mathcal{T} \to (Ab)$ is cohomological.

Rmk. Same for representables

Pf. $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{\eta} X[1]$ dist. tr
$\Rightarrow \forall W$ $h^W(X) \to h^W(Y) \to h^W(Z)$ exact in Ab.
$\alpha \longmapsto \varphi \circ \alpha$
$\beta \longmapsto \psi \circ \beta$

$\beta : W \to Y$ s.t. $\psi \circ \beta = 0$,

$$\begin{tikzcd} W \arrow[r, "id"] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[r, "0"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & W[1] \arrow[d, dashed] \\ X \arrow[r, "\varphi"'] & Y \arrow[r, "\psi"'] & Z \arrow[r, "\lambda"] & X[1] \end{tikzcd} \quad \square$$

Prop $\mathcal{A}$ ab. cat $\Rightarrow$ $H^0 : K(\mathcal{A}) \to \mathcal{A}$ cohom. f.

Pf. Enough to show for dist. tr

$$X \xrightarrow[\varphi]{} Y \xrightarrow[\psi]{} M(\varphi) \xrightarrow[\lambda]{} X[1],$$

$$H^0(Y) \to H^0(M(\varphi)) \to H^0(X[1])$$

exact. Follows from:

Lemma If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ ex in $C(\mathcal{A})$ then $\exists$ nat map $M(\varphi) \to C$ iso in $K(\mathcal{A})$

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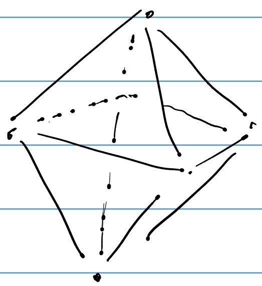
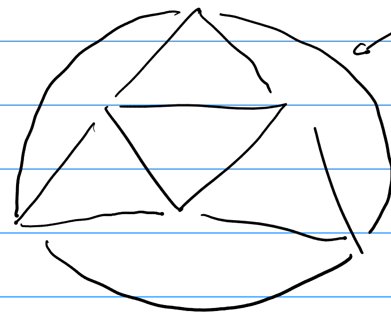
- (TR5) octahedron
- recall what an octahedron is:

 $\rightsquigarrow$ 

"outer" triangle

- start with $X \xrightarrow{f} Y \xrightarrow{g} Z$, $g \circ f = :h$

and draw

$$\begin{tikzcd}
M_f \arrow[dr, "+1"'] \arrow[ddrr, bend right=60, blue] & & Y \arrow[ll] \arrow[dr, "g"] & & M_g \arrow[ll, "+1"'] \arrow[llll, bend right=30, blue, "+1"'] \\
& X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \arrow[ur] \arrow[dl] & \\
& & M_h \arrow[ul, "+1"] \arrow[uurr, bend right=60, blue] & &
\end{tikzcd}$$

we demand that all triangles commute, and that the outer blue triangle is distinguished

$$M_f \to M_h \to M_g \to M_f[1]$$

Rmk We proved if $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \to X[1]$ d.t then $\psi \circ \varphi = 0$. But in $C(A)$, $X \xrightarrow{\varphi} Y \to M(\varphi)$ is nonzero

Lemma In a triag. cat., given d.t. morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\varphi"] & Y \arrow[r] \arrow[d, "\psi"] & Z \arrow[r] \arrow[d, "\vartheta"] & X[1] \arrow[d, "\varphi"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd},$$

if φ and ψ iso, then ϑ iso.

Rmk By axioms, $\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed, "\exists\vartheta"] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$

Rmk However, this ϑ is not unique or canonical.

Pf. By coYoneda embedding,
$\mathrm{Hom}(W, Z) \to \mathrm{Hom}(W, Z')$, $\varphi \mapsto \vartheta \circ \varphi$
is bijective. Also remembering $[1]$ is auto eq., φ, ψ iso $\Rightarrow$ $\varphi[1], \psi[1]$ iso.
$\mathrm{Hom}(W, -)$ is cohomological, so gives exact seq.

$$\begin{tikzcd} h^W(X) \arrow[r] \arrow[d, "\wr", "\varphi\circ -"'] & h^W(Y) \arrow[r] \arrow[d, "\wr", "\psi\circ -"'] & h^W(Z) \arrow[r] \arrow[d, "\vartheta"] & h^W(X[1]) \arrow[r] \arrow[d, "\wr", "\varphi[1]\circ -"'] & h^W(Y[1]) \arrow[d, "\wr", "\psi\circ -"'] \\ h^W(X') \arrow[r] & h^W(Y') \arrow[r] & h^W(Z') \arrow[r] & h^W(X'[1]) \arrow[r] & h^W(Y'[1]) \end{tikzcd}$$

so ϑ iso by diagram chasing (5-lemma).

Recall For $\mathcal{A}$ ab.cat. we "defined" $D(\mathcal{A})$ as localisation of $K(\mathcal{A})$ at quisos.
Categorical nonsense says it exists, but it's not simple to describe.

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{C} \\ D(\mathcal{A}) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

- there are some assumptions we are given, tho

Def Let $\mathcal{C}$ cat. A multiplicative system S is a collection of mor in $\mathcal{C}$ s.t.

i) all identities are in S

ii) $f \in S \wedge g \in S \wedge \exists g \circ f \Rightarrow g \circ f \in S$

iii) given
$$\begin{tikzcd} X \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$
in S, we can add
$$\begin{tikzcd} & Y' \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$
and this commutes

iv) assuming $f, g : X \to Y$ in S, TFAE

  i) $\exists h : T \to X$ in S s.t. $f \circ h = g \circ h$

  ii) $\exists l : Y \to W$ in S s.t. $l \circ f = l \circ g$

- informally, a mor in the localisation of $\mathcal{C}$ at S looks like
$$f = f_1 \circ s_1^{-1} \circ f_2 \circ s_2^{-1} \circ \cdots \qquad f_i \in \text{mor } \mathcal{C}, \quad s_i \in \text{mor } S$$

but iii) tells us we can write $\tilde{s}_1^{-1} \circ f_1 \circ \cdots$ etc

Thm If S mult sys, then the local. of $\mathcal{C}$ at S, $S^{-1}\mathcal{C}$, will be defined as
$$\mathrm{Mor}(X,Y) = \{ \overbrace{X \xleftarrow{s} Z \xrightarrow{f} Y}^{f \circ s^{-1}} \mid s \in S, f \in \mathrm{Hom}(Z,Y) \} / \sim$$
where $X \xleftarrow{s} Z \to Y \sim X \xleftarrow{s} Z_1 \to Y$ iff $\exists$ comm. diag.
$$\begin{tikzcd} & Z \arrow[dl, "s_0"'] \arrow[dr] & \\ X & Z_2 \arrow[l, "s"'] \arrow[u, "\in S"] \arrow[d, "\in S"'] \arrow[r] & Y \\ & Z \arrow[ul, "\hat{s}"] \arrow[ur] & \end{tikzcd}$$

Rmk We can also define it using $X \to Z \leftarrow_s Y/\sim$

Rmk Let $A \xrightarrow{\varphi} B$ in $C(\mathcal{A})$.
Then φ qiso $\iff h^i(M(\varphi)) = 0$
$\forall i \in \mathbb{Z}$ $\Rightarrow A \xrightarrow[\varphi]{} B \to C \xrightarrow{+1}$ d.t. in $K(\mathcal{A})$
so, φ iso $\iff \forall i \in \mathbb{Z}$ $h^i(C) = 0$

Def. Let $\mathcal{T}$ triag cat. A **null system** $\mathcal{N}$ is a subset of $\mathrm{ob}\,\mathcal{T}$ s.t.

(N1) $0 \in \mathcal{N}$
(N2) for $X \in \mathrm{ob}\,\mathcal{T}$, $X \in \mathcal{N} \iff X[1] \in \mathcal{N}$
(N3) given $X \to Y \to Z \xrightarrow{+1}$ d.t., $(X, Y \in \mathcal{N}) \Rightarrow Z \in \mathcal{N}$

Rmk. If $\mathcal{T} = K(\mathcal{A})$, $\mathcal{N} = \{ A \mid h^i(A) = 0 \ \forall i \}$

Def. For $\mathcal{N}$ null system, define $S(\mathcal{N}) \subset \mathrm{mor}\,\mathcal{T}$ by $\varphi \in S(\mathcal{N}) \iff \exists$ d.t. $X \xrightarrow[\varphi]{} Y \to Z \xrightarrow{+1}$ where $Z \in \mathcal{N}$

Prop $\mathcal{N}$ null sys $\Rightarrow S(\mathcal{N})$ mult. sys

Exercise If $X \xrightarrow[\sim]{\varphi} Y$, $\varphi \in S(\mathcal{N})$ [use TR5?]

Notation $\mathcal{T}$ Δcat, $\mathcal{N}$ null sys, then $\mathcal{T}/\mathcal{N} := S(\mathcal{N})^{-1}\mathcal{T}$

Prop. i) $\mathcal{T}/\mathcal{N}$ is Δcat where d.t. are images of d.t. in $\mathcal{T}$
ii) the image of an object in $\mathcal{N}$ is (isom to) zero in $\mathcal{T}/\mathcal{N}$.
iii) for any other Δcat $\mathcal{T}'$, any Δcat functor $F: \mathcal{T} \to \mathcal{T}'$ factors uniquely via $\mathcal{T}/\mathcal{N}$ iff $F(X) = 0$ $\forall X \in \mathcal{N}$

Cor For any $\mathcal{A}$ ab. cat, $D(\mathcal{A}) = K(\mathcal{A}) / {\scriptstyle\text{cpxs w/ zero coh}}$ has a natural Δ cat structure, same for $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$

Cor let $n \in \mathbb{Z}$, recall $\tau_{\geq n}, \tau_{\leq n} : C(\mathcal{A}) \circlearrowleft$.

i) $\tau_{\geq n}, \tau_{\leq n}$ send homotopic mor. into homotopic mor, i.e. induce fctors on $K(\mathcal{A})$

ii) by iii) in Prop., since if $A \in C(A)$ has $h^i(A) = 0$ $\forall i$, same holds for $\tau_{\geq n} A, \tau_{\leq n} A$, so $\tau_{\geq n}, \tau_{\leq n}$ induce functors on $D(A)$

– so

$$\begin{tikzcd}
C(A) \arrow[r, "\tau_{\geq n}"] \arrow[d] & C(A) \arrow[d] \\
K(A) \arrow[r, "\tau_{\geq n}"] \arrow[d] & K(A) \arrow[d] \\
D(A) \arrow[r] & D(A)
\end{tikzcd}$$

Prop. Let $\mathcal{A}$ ab. cat w/ enough injectives, let $K^+(\mathcal{J}) \subseteq K^+(A)$ full subcat of cpx of inj. objects.

Then
$$K^+(\mathcal{J}) \to K^+(\mathcal{A}) \to D^+(\mathcal{A})$$
is eq. of cats.

Similarly with enough projs P,
$$K^-(P) \to K^-(A) \to D^-(A)$$

– so we don't need $D(A)$ in these cases

Fantechi

- big idea: given $\begin{cases}\text{right}\\ \text{left}\end{cases}$ ex. ftor $F: \mathcal{A} \to \mathcal{B}$
we can extend to $\begin{cases} \mathbb{L}F : D^-(\mathcal{A}) \to D^-(\mathcal{B}) \\ \mathbb{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B}) \end{cases}$

if the ab. cat $\mathcal{A}$ has enough $\begin{cases} F\text{-proj} \\ F\text{-inj} \end{cases}$

i.e. every object in $\mathcal{A}$ is $\begin{cases} \text{quot of an } F\text{-proj} \\ \text{subobj} \; - \, " \, - \; F\text{-inj} \end{cases}$

Lemma let X proj sch $/\mathbb{k} = \bar{\mathbb{k}}$. let $\mathcal{F} \in Coh_X$, choose $\mathcal{O}_X(1)$ very ample. Then $\exists n_0$ s.t. $\forall n \geq n_0$ $\mathcal{F}(n)$ is gen by global secns

Pf. (hint) $X \hookrightarrow \mathbb{P}^N$. $M = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$ is grad. mod. over $S = \mathbb{k}[x_0 \ldots x_N]$ and $\mathcal{F}$ coh, $\Rightarrow M$ fin. gen. Pick $s \in \Gamma(X, \mathcal{F}(n_0))$, then $x_0 s, \ldots, x_N s \in \Gamma(X, \mathcal{F}(n_0+1))$. ◻

Cor. With same assumptions, $\forall n \geq n_0$ $\exists r > 0$ and $\mathcal{O}_X(-n)^{\oplus r} \twoheadrightarrow \mathcal{F}$

Pf $\mathcal{F}(n)$ gen by glob secns $\Leftrightarrow$

$$\Gamma(X, \mathcal{F}(n)) \otimes_{\mathbb{k}} \mathcal{O}_X \twoheadrightarrow \mathcal{F}(n) \quad / - \otimes_{\mathcal{O}_X} \mathcal{O}_X(-n)$$

$$\Rightarrow \underbrace{\Gamma(X, \mathcal{F}(n))}_{\simeq \mathbb{k}^{\oplus r} \text{ for some } r} \otimes_{\mathbb{k}} \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{F}$$

Rmk Fix $n > 0$. Then $H^i(\mathbb{P}^n, \mathcal{O}(m)) = 0$,
except $\begin{cases} i = 0 & m \geq 0 & H^0(\mathbb{P}^n, \mathcal{O}(m)) \simeq \mathbb{k}[x_0, \ldots x_n]_m \\ i = n & m \leq -n-1 & H^n(\mathbb{P}^n, \mathcal{O}(-n-1-m)) \simeq H^0(\mathbb{P}^n, \mathcal{O}(m))^\vee \end{cases}$

Rmk On a sep. sch., cohom can be computed using Čech coh. on any open affine $\Rightarrow \forall \mathcal{F} \in Qcoh_{\mathbb{P}^n}$ $\forall i > n$, $H^i(\mathbb{P}^n, \mathcal{F}) = 0$

Cor If X proj. sch $\mathcal{F} \in Qcoh_X \Rightarrow H^i(X, \mathcal{F}) = 0$ $\forall i > \dim X$

Cor If $X = \mathbb{P}^n$ then $H^n : Qcoh_X \to Vect$ is right-exact

Prop Let $\mathcal{A} = \mathrm{Coh}_{\mathbb{P}^n}$. Then $\mathcal{A}$ has enough H^n-proj.

Pf. consider subcat of sheaves which "only have H^n", by which we mean $\mathcal{F}$ s.t. $H^i(\mathcal{F}) = 0$ $\forall i \neq n$.
If $\mathcal{F}_1, \mathcal{F}_2$ are such, so is $\mathcal{F}_1 \oplus \mathcal{F}_2$.
If $\mathcal{F}, \mathcal{F}''$ — " — and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, so is $\mathcal{F}'$ by long ex. seqn. $\square$

- given $\mathcal{F} \in \mathrm{Coh}_{\mathbb{P}^n}$, we can find a resn s.t. $0 < N_0 \leq N_1 \leq \dots$,

$$\dots \longrightarrow \mathcal{O}(-N_i)^{\oplus r_i} \longrightarrow \mathcal{O}(-N_0)^{\oplus r_0} \longrightarrow \mathcal{F} \to 0 \quad (*)$$

- if $0 \to \xi \to \mathcal{O}(-N_{n-1})^{\oplus r_{n-1}} \to \dots \to \mathcal{O}(-N_0)^{\oplus r_0} \to \mathcal{F} \to 0$ exact, then $H^i(\xi) = 0$ $\forall i \neq n$?
By Hilbert syzygy thm applied to open $x_i \neq 0$ in $\mathbb{P}^n$, $\Gamma(x_i \neq 0, \xi)$ is proj $\Leftrightarrow$ $\xi|_{x_i \neq 0}$ loc. free $\Rightarrow$ ξ loc. free

Aim $R\Gamma : D^b(\mathrm{Coh}_{\mathbb{P}^n}) \to D^b(\mathrm{Vect}_{\mathbb{K}})$ can be computed in terms of $LH^n : D^b(\mathrm{Coh}_{\mathbb{P}^n}) \to D^b(\mathrm{Vect}_{\mathbb{K}})$

Prop Let $0 \to \xi_{-n} \to \xi_{-(n-1)} \to \dots \to \xi_0 \to \mathcal{F} \to 0$ ex. seqn in $\mathrm{Coh}_{\mathbb{P}^n}$ s.t. $H^i(\xi_j) = 0$ $\forall i \neq 0$.
Then $\forall i = 0, \dots, n$, $H^i(\mathbb{P}^n, \mathcal{F}) \underset{\text{can}}{\simeq} h^{i-n}\big(H^n(\xi_{-n}) \to \dots \to H^n(\xi_0)\big)$

Pf. i) if $n = 0$, $\xi_0 \xrightarrow{\sim} \mathcal{F}$

ii) if $i = n$, $h^0(\dots) = \mathrm{coker}\big(H^n(\xi_{-1}) \to H^0(\xi_0)\big) \underset{\text{can}}{\simeq} H^n(\mathcal{F})$

iii) $i < n$, use induction on m. Let $\mathcal{G} = \ker(\xi_0 \to \mathcal{F})$
i.e. $0 \to \mathcal{G} \to \xi_0 \to \mathcal{F} \to 0$, ...

Cor $H^i(\mathbb{P}^n, \mathcal{F})$ is fin. dim. $\forall \mathcal{F} \in \mathrm{Coh}_{\mathbb{P}^n}$.

- we also have nice results like Serre vanishing, $\exists n_0$ s.t. $H^i(\mathcal{F}(n))=0$ for $\mathcal{F}\in Coh_X$, X proj, for $\forall n\geq n_0$.

- let $F: Coh_X^{op} \to Vect_{\mathbb{k}}$, $F = Hom_{\mathcal{O}_X}(-,\mathcal{F})$ is left exact on Coh_X^{op}
- want to show that $\{\xi\in Coh_X \mid Ext^i(\xi,\mathcal{F})=0\ \forall i>0\}$
  1) is closed under $\oplus$ in Coh_X^{op} (and therefore in Coh_X)
  ii) given $0\to\xi'\to\xi\to\xi''\to 0$ ex. in Coh_X^{op}
  $\Leftrightarrow$ $0\to\xi''\to\xi\to\xi'\to 0$ in Coh_X, if ξ',ξ ok
  then also ξ'', using $Ext^i(\xi',\mathcal{F})\to Ext^i(\xi,\mathcal{F})\to Ext^i(\xi'',\mathcal{F})\to\cdots$
  iii) in Coh_X^{op} $\forall\xi\ \exists\tilde{\xi}$ F-proj and $0\to\xi\to\tilde{\xi}$ in Coh_X^{op}
  $\Leftrightarrow$ $\forall\xi\ \exists\tilde{\xi}\to\xi$ in Coh_X s.t. $Ext^i(\tilde{\xi},\mathcal{F})=0\ \forall i>0$
  - choose $n_0\geq 0$ from Serre vanishing.
  $\forall n\geq n_0\ \exists r\geq 0$, $\mathcal{O}_X(-n)^{\oplus r}\to\xi\to 0$.
  Then

$$
\begin{aligned}
&Ext^i(\mathcal{O}_X(-n)^{\oplus r},\mathcal{F})\\
&\quad = Ext^i(\mathcal{O}_X(-n),\mathcal{F})^{\oplus r}\\
&\quad \cong H^i(X,\mathcal{F}\otimes(\mathcal{O}_X(-n)^\vee))^{\oplus r}\\
&\quad \cong H^i(X,\mathcal{F}(n))^{\oplus r} = 0.
\end{aligned}
$$

Cohomology & base change

- X,Y loc. of finite type $/\mathbb{k}=\bar{\mathbb{k}}$
- $f: X\to Y$ proj mor (usually we take proper, but this case can be reduced to proj), $\mathcal{F}\in Coh_X$ flat over Y, e.g. f flat, $\mathcal{F}$ loc. free
  f smooth (in alg. geom sense, in diff. geom étale)
  $\Rightarrow$ flat $A[x_1,\dots,x_{n+r}]/(f_1,\dots,f_r)$, rk $\frac{\partial f_i}{\partial x_j}<r$

- let $y \in Y(\mathbb{k})$

Thm 1) $\exists$ nat. map $R^i f_* \mathcal{F} \otimes \underset{\substack{\uparrow \\ \text{res. fld at } y}}{\mathbb{k}(y)} \xrightarrow[\pi_{i,y}]{} H^i(X_y, \mathcal{F}|_{X_y})$

$$\begin{matrix} \wr\!\wr & & \wr\!\wr \\ \alpha^* R^i f_* \mathcal{F} & & R^i g_*(\beta^* \mathcal{F}) \end{matrix}$$

in
$$\begin{tikzcd} X_y \arrow[r, "\beta"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathbb{k}(y) \arrow[r, "\alpha"] & Y \end{tikzcd}$$

ii) if map surjects for some $y_0 \in Y$, then $\exists U$ open nbhd of y_0 in Y s.t. it surjects $\forall y \in U$, and bijects.

iii) assuming $\pi_{i,y}$ does surject, TFAE

$\pi_{i-1,y}$ surj $\Leftrightarrow R^i f_* \mathcal{F}$ is loc. free near y

Rmks

- assuming $\dim X_y \le n$ $\forall y \in Y$, $\underset{\overset{\shortparallel}{0}}{R^{n+1} f_* \mathcal{F} \otimes \mathbb{k}(y)} \to \underset{\overset{\shortparallel}{0}}{H^{n+1}(X_y, \mathcal{F}|_{X_y})}$
- $\pi_{n,y}$ surj $\Leftrightarrow R^{n+1} f_* \mathcal{F}$ loc. free

  $\to$ of course, since it is zero!
- $i = 0$ says $f_* \mathcal{F} \otimes \mathbb{k}(y) \xrightarrow[\pi_{0,y}]{} H^0(X_y, \mathcal{F}|_{X_y})$
- if $\pi_{0,y}$ surj., TFAE $\pi_{-1,y}$ surj $\Leftrightarrow f$ ...

Fantechi

- $\mathcal{A}, \mathcal{A}'$ ab. cats, $F : \mathcal{A} \to \mathcal{A}'$ left exact
- $RF : D^+(\mathcal{A}) \to D^+(\mathcal{A}')$ "unique" (if exists)

- $\mathcal{A}$ ab. cat, $C(\mathcal{A})$ ab. cat, $K(\mathcal{A})$ Δ cat,
  $D(\mathcal{A}) = K(\mathcal{A})/\mathcal{N}$ Δ cat

Def. Let $\mathcal{T}_1, \mathcal{T}_2$ Δ cats. A functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ is a functor of Δ cats if it is additive $\wedge$ commutes with $[1]$ $\wedge$ sends dist. Δ to dist. Δ.

Lemma $F : \mathcal{A} \to \mathcal{A}'$ induces $C(F), K(F)$.
Pf. Define $F(A^i, d^i) = (F(A^i), F(d^i)), \ldots$

- what do we want from RF?
  - If $\mathcal{A}$ has enough injectives, let $\mathcal{J}$ be subcat of $\mathcal{A}$ of inj objects
  - $K^+(\mathcal{J}) \subseteq K^+(\mathcal{A})$ full subcat of cpx A^i s.t. $A^i \in \mathcal{J}$ $\forall i$, $A^i = 0$ $\forall i << 0$.

Lemma $\dfrac{K^+(\mathcal{J})}{\mathcal{N} \cap K^+(\mathcal{J})} \xrightarrow{\alpha} \dfrac{K^+(\mathcal{A})}{\mathcal{N} \cap K^+(\mathcal{A})} = D^+(\mathcal{A})$ equiv.

- goal. Find univ prop. for $RF : D^+(\mathcal{A}) \to D^+(\mathcal{A}')$

$$\begin{tikzcd}
D^+(\mathcal{A}) \arrow[r, "\alpha^{-1}"] & \dfrac{K^+(\mathcal{J})}{\mathcal{N} \cap K^+(\mathcal{J})} \arrow[r] & D^+(\mathcal{A}') \\
& K^+(\mathcal{J}) \arrow[u] \arrow[r, "K^+(F)"] & K^+(\mathcal{A}') \arrow[u]
\end{tikzcd}$$

not really unique? it is only an equiv. of cats

Lemma $K^+(F)(\mathcal{N}\cap K^+(\mathcal{J}))\subseteq \mathcal{N}_{K^+(A')}$.

Recall Enough inj. $\Rightarrow$ $\forall A^\bullet \in \text{ob}\, K^+(A)$ $\exists$ qis $A^\bullet \xrightarrow{\varphi} I^\bullet$ with $I^\bullet \in K^+(\mathcal{J})$

$$K^+(F)(A) \longrightarrow K^+F(I) \underset{\text{can.}}{\simeq} RF(A^\bullet)$$

$$\begin{tikzcd} K^+F(I) \arrow[d, equal] \arrow[r, "\simeq", "\text{can.}"'] & RF(A^\bullet) \arrow[dl, "\sim"', "RF(\varphi)"] \\ RF(I) & \end{tikzcd}$$

Def. Let $F: A \to A'$ be left exact fctor of ab cats. A **right derived functor** for F is a pair (T, φ) such that

i) $T: D^+(A) \to D^+(A')$ is a fctor of Δ cats.

ii) φ is a nat transformation

$$\varphi: Q' \circ K^+(F) \Rightarrow T \circ Q$$

$$\begin{tikzcd} K^+(A) \arrow[r, "K^+(F)"] \arrow[d, "Q"'] & K^+(A') \arrow[d, "Q'"] \arrow[dl, phantom, "\Leftarrow \varphi"] \\ D^+(A) \arrow[r, "T"'] & D^+(A') \end{tikzcd}$$

Such that for any $G: D^+(A) \to D^+(A')$ Δcat fctor, the natural map

$$\text{Hom}(T, G) \longrightarrow \text{Hom}(Q' \circ K^+(F), G \circ Q)$$
$$\eta: T \Rightarrow G \longmapsto \eta \circ \varphi \qquad \text{(whiskering)}$$

is bijective.

Cor A right der. fctor is unique up to a can. eq., if it exists.

- so if it exists, we call it *the* r.d.fctor RF.

Thm Let $F: \mathcal{A} \to \mathcal{A}'$ left exact fctor, assume $\mathcal{A}$ has enough F-inj., let $\mathcal{J}_F \subseteq \mathcal{A}$ be choice of full subcat of F-inj.
Then the functor

$$\begin{tikzcd} D^+(\mathcal{A}) & D^+(\mathcal{J}_F) \arrow[l, "\sim"'] \arrow[r, dashed, "\exists!"] & D^+(\mathcal{A}') \\ & K^+(\mathcal{J}_F) \arrow[u] \arrow[r, "K^+F"'] & K^+(\mathcal{A}') \arrow[u] \end{tikzcd}$$

is a right der fctor.

Rmk Choosing different $\mathcal{J}_F' \neq \mathcal{J}_F$ gives you different functors.

Example X top.sp., $\underline{Ab_X}$ sh. of ab. grp on X,
$\Gamma: \underline{Ab_X} \to Ab$ left exact.
$R\Gamma$ can be computed using injective $\underline{Mod_{\mathbb{Z}}}$
$\mathcal{F} \in \underline{Ab_X}$ is F-acyclic if $H^i(\mathcal{F}) = 0 \ \forall i$,
$\mathcal{J}_\Gamma$ = all acyclics.

- we define left derived fctors the same way, but F right exact., $T: D^-(\mathcal{A}) \to D^-(\mathcal{A}')$,
$\varphi: T \circ Q \Rightarrow Q' \circ K^+F$.

Rmk Let $F: \mathcal{A} \to \mathcal{A}'$ exact fctor, then $K(F)$ sends qiso to qiso, induces $D(F)$ which restricts to RF on $D^+(\mathcal{A})$ and LF on $D^-(\mathcal{A})$, since $\mathcal{A}$ is made of F-inj / F-proj.

Rmk Composition of left exact fctors is left exact.

Lemma Assume RF, RG, RH exist, where $H = G \circ F$. Then there is an induced nat transf $RH \Rightarrow RG \circ RF$

Pf.

$$\begin{tikzcd}
K^+(\mathcal{A}) \arrow[r, "K^+F"] \arrow[d] \arrow[rr, bend left, "K^+H"] & K^+(\mathcal{A}') \arrow[r, "K^+G"] \arrow[d] \arrow[dl, phantom, "\swArrow \varphi"] & K^+(\mathcal{A}'') \arrow[d] \arrow[dl, phantom, "\psi \swArrow"] \\
D^+(\mathcal{A}) \arrow[r, "RF"'] & D^+(\mathcal{A}') \arrow[r, "RG"'] & D^+(\mathcal{A}'')
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
K^+(\mathcal{A}) \arrow[r] \arrow[d] & K^+(\mathcal{A}'') \arrow[d] \arrow[dl, phantom, "\swArrow \psi\circ\varphi"] \\
D^+(\mathcal{A}) \arrow[r] & D^+(\mathcal{A}'')
\end{tikzcd}$$

Prop Same assumptions, assume $\exists\, \mathcal{J} \subseteq \mathcal{A}$ full subcat of F-inj and $\mathcal{J}' \subseteq \mathcal{A}'$ — " — G-inj so that

i) $\mathcal{J}, \mathcal{J}'$ can be used to def RF, RG

ii) $\forall I \in \mathrm{ob}\, \mathcal{J}$, $F(I) \in \mathrm{ob}\, \mathcal{J}'$.

Then RH exists $\wedge$ $RH \Rightarrow RG \circ RF$ is equiv.

Cor Assume $F: \mathcal{A} \to \mathcal{A}'$ exact, $G: \mathcal{A}' \to \mathcal{A}''$ left exact, $\mathcal{A}, \mathcal{A}'$ have enough inj. Then $R(G \circ F) \Rightarrow RG \circ D^+(F)$ is equiv.

Application $X \xrightarrow[f]{} Y \xrightarrow[g]{} Z$ mor of sch, $h = g \circ f$, f affine, which gives $f_*: Qcoh_X \to Qcoh_Y$ exact.

$\forall \mathcal{F} \in Qcoh_X \subseteq C(Qcoh_X) \longrightarrow K^+(Qcoh_Y)$

$$Rh_*(\mathcal{F}) = Rg_*(K^+(f_*)\mathcal{F}) = Rg_*(f_*\mathcal{F}).$$

$$\Rightarrow \forall i \quad R^i h_* \mathcal{F} \xrightarrow[\text{can.}]{\sim} R^i g_*(f_*\mathcal{F})$$

$$H^i(X, \mathcal{F}) \xrightarrow[\text{can}]{\sim} H^i(Y, f_*\mathcal{F}).$$

Fantechi

- cohom & base change.
  X, Y loc.f.t. $/ k = \bar{k}$, $f: X \to Y$ proj mor,
  $\mathcal{F} \in Coh_X$ flat over Y
  0) $\forall y \in Y(\bar{k})$ $\forall i \in \mathbb{Z}$ $\exists$ nat map $R^i f_* \mathcal{F} \otimes k(y) \xrightarrow{\varphi_{i,y}} H^i(X_y, \mathcal{F}|_{X_y})$
  1) if $\varphi_{i,y}$ surj then iso & same holds $\forall$ pts in nbhd of y in Y
  ii) $\varphi_{i,y}$ iso then
  $\varphi_{i-1,y}$ surj $\Leftrightarrow$ $R^i f_* \mathcal{F}$ loc.free near y

Rmk thm is local in Y, so all proj versions can be reduced to $X \overset{i}{\hookrightarrow} \mathbb{P}^N \times Y \xrightarrow{\pi} Y$, with f the composite,
i cl. emb, π projection

Rmk Grothendieck spec.seqn $R^q \pi_* R^p i_* \Rightarrow R^n f_*$
But i_* exact $\Rightarrow$ $R^n \pi_* \circ i_* \Rightarrow R^n f_*$ is eq,
i.e. $R^n f_* \mathcal{F} \underset{can}{\xleftarrow{\sim}} R^n \pi_* i_* \mathcal{F}$

Rmk $\mathcal{F}$ flat over Y $\overset{def.}{\Leftrightarrow}$ $\forall x \in X$, $\mathcal{F}_x$ flat as $\mathcal{O}_{Y,f(x)}$-mod

Exercise $\exists$ nat iso $\mathcal{F}_x \xrightarrow{\sim} (i_* \mathcal{F})_x$, $\mathcal{O}_{Y,f(x)}$-linear
- hint: local, $X = \operatorname{Spec} R$, $\mathbb{P}^N \times Y = \operatorname{Spec} P$, $Y = \operatorname{Spec} A$
- R is an A-alg, $P = R/I$, $m = m_x \subseteq P$,

$$\begin{array}{ccccccc} k & \to & A & \to & R & \to & P \\ & & \cup\!\vert & & \cup\!\vert & & \cup\!\vert \\ & & m_A & \to & m_R & \to & m \end{array} \Rightarrow k \hookrightarrow A/m_A \hookrightarrow R/m_R \hookrightarrow P/m \cong k$$

A/m_A maximal, so all isos

$i_* \mathcal{F} \hookrightarrow \tilde{M}^R$ as R-mod, $\mathcal{F}_x = M_m \simeq M^R_{m_R}$ as R_{m_R} mod
- reduce thm to case $X = \mathbb{P}^N \times Y \to Y$, Y affine

Rmk. Let $\mathcal{F} \in Coh_{\mathbb{P}^N \times Y}$, Y affine.
Then Groth spec. sequ. $\Gamma(X, -) = \Gamma(Y, -) \circ \pi_*$
gives $H^p(Y, R^q \pi_* \mathcal{F}) \Rightarrow H^n(X, \mathcal{F})$.
But Y affine and $\underbrace{R^q \pi_* \mathcal{F}}_{\text{coherent}}$ so we have
a vanishing thm, $\forall\, \mathcal{G} \in Qcoh_Y$, $H^p(Y, \mathcal{G}) = 0$
$\forall p > 0$. So $\Gamma(Y, R^n \pi_* \mathcal{F}) \underset{\text{canot}}{\cong} H^i(X, \mathcal{F})$

- fix $y \in Y$. Serre vanishing to $\mathcal{F}|_{X_y}$ says
$\exists\, n_0$ s.t. $\forall\, n > n_0$ $H^i(X_y, \mathcal{F}(n)|_{X_y}) = 0$ $\forall i > 0$

Rmk. [Restriction, tensoring] = 0.

means $\mathcal{F}(n)|_{X_y}$ gen. by global sections
- apply thm with $i = 1$.
$\varphi_{1,y}$ surj. because $H^1(X_y, \mathcal{F}(n)|_{X_y}) = 0$

$\overset{\text{thm}}{\Rightarrow}$ $\varphi_{1,y}$ iso $\Rightarrow$ $\underbrace{R^1 \pi_* \mathcal{F}(n)}_{coh} \otimes k(y) = 0$

$\overset{\text{NAK}}{\Rightarrow}$ $(R^1 \pi_* \mathcal{F}(n))_y = 0 \Rightarrow \exists$ open nbhd U of y in Y

where $R^1 \pi_* \mathcal{F}(n)|_U = 0$

2) $\varphi_{0,y}$ surj $\Leftarrow$ $R^1 \pi_* \mathcal{F}(n)$ loc. free near y,
which is true. But $\varphi_{0,y}$ surj $\Rightarrow$ $\pi_* \mathcal{F}(n)$ loc. free
- replace Y by affine open nbhd of $y \in Y$ s.t. $\pi_* \mathcal{F}(n)$
free on U, $\mathcal{G} := \pi_* \mathcal{F}(n)$, by adj, $\pi^* \mathcal{G} \to \mathcal{F}(n)$
is surjection at every pt of X_y

Exercise Use Nakayama + π proper to show
$\exists\, V \subseteq Y$ open nbhd of y s.t. $\pi^* \mathcal{G} \to \mathcal{F}(n)$ surj on $\mathbb{P}^N \times V$

- hint. Let $A = \mathrm{coker}(\pi^* \mathcal{G} \to \mathcal{F}(n))$
coh $\Rightarrow$ supp A = supp A in $\mathbb{P}^N \times Y$ & disjoint
from $X_y = \mathbb{P}^N \times \{y\}$ by Nak. properness
Replace Y by smaller open affine
$\Rightarrow \pi^*(\pi_* \mathcal{F}(n)) \twoheadrightarrow \mathcal{F}(n) \quad | \otimes \mathcal{O}(-n)$
$\Rightarrow \underbrace{\pi^* \mathcal{G}(-n)}_{\downarrow} \twoheadrightarrow \mathcal{F}$

flat over Y since loc. free and $\mathbb{P}^N \times Y$ flat over Y.

- repeating this, we get

$$0 \to \mathcal{H} \to \pi^* \mathcal{G}_{N-1}(-n_{N-1}) \to \cdots \to \pi^* \mathcal{G}_0(-n_0) \to \mathcal{F} \to 0$$

where $\mathcal{G}_i$ loc free $/Y$, by induction $\mathcal{H}$ flat $/Y$ coh
So apply coh. & base change thm

$$R^i \pi_*(\pi^* \mathcal{G}_a(-n_a)) \simeq \underbrace{R^i \pi_* \mathcal{O}(-n_a)}_{\text{is}} \otimes \mathcal{G}_a$$

$R^N \pi_* \mathcal{G}_a(-n_a)$ loc. free, all other $i \neq N$ are zero, $H^i(\mathbb{P}^N \times Y, \mathcal{O}_{\mathbb{P}^N \times Y}(-n_a)) = 0$
- exercise: $R^i \pi_* \mathcal{H} = 0 \ \forall i \neq N$ (build resn. of $\mathcal{F}$ by sheaves s.t. $R^i \pi_* = 0 \ \forall i \neq N$)
- exercise: $\forall m \geq 0$

$$R^m \pi_* \mathcal{F} = h^{m-N}(R^N \pi_* \mathcal{H} \to R^N_{\pi_*} \pi^* \mathcal{G}_{N-1}(-n_{N-1}) \to \cdots \to R^N \pi_* \pi^* \mathcal{G}_0(-n_0))$$

$$\begin{array}{ccc} X_y & \xrightarrow{j} & \mathbb{P}^N \times Y = X \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \pi} \\ y & \xrightarrow{i} & Y \end{array}$$

$\pi^* \mathcal{G}_a(-n_a)$ has vanishing $L^i j^*$.
j^* has van. $L^i(R^N \alpha_*)$,
$R^N \pi_*$ has van. $LP(i^*)$,
same for $\mathcal{H}$

$$\begin{tikzcd} \mathrm{Coh}_{\mathbb{P}^N \times Y} \arrow[r, "j^*"] \arrow[d, "R^N\pi_*"'] \arrow[dr, "\beta"] & \mathrm{Coh}\, \mathbb{P}^N \times \{y\} \arrow[d, "R^N \alpha_* = H^N(\,\_\_\,)"] \\ \mathrm{Coh}_Y \arrow[r, "i^*"'] & \mathrm{Coh}_y = k\text{-f.d.vsp} \end{tikzcd}$$

- commutes by coh. & base ch.
- $L^i\beta(\mathcal{F})$ computable by given resn

- why is coh. & b.ch. like it is?

- look at

$$0 \to \mathcal{H}_{-N} \to \mathcal{H}_{-N-1} \to \cdots \to \mathcal{H}_0 \to \mathcal{F} \to 0$$

s.t. $\mathcal{H}_i$ loc free, coh, $R^i\pi_*\mathcal{H}_j = 0$
$\forall\, i \neq N \Rightarrow R^N\pi_*\mathcal{H}_j = M_j$ ((loc) free mod

$Y = \mathrm{Spec}\, A$, $y \leftrightarrow m_y$ max ideal

$\Rightarrow M_{-N} \to \cdots \to M_0$, M_i free mod

$$\begin{array}{c} (h^i(\,\_\_\,)) \otimes k(y) \\ \big\downarrow \exists \text{ nat map} \\ h^i(\,\_\_ \otimes_\pi k(y)) \end{array}$$

this is (0) of coh & b.ch. thm, which is clear since we just mod out some things

$$A^{\oplus a} \xrightarrow[(\ \ )]{d_2} A^{\oplus b} \xrightarrow[(\ \ )]{d_1} A^{\oplus c}$$

matrices

, A domain, rks of d_1, d_2 loc. const. wrt y

$$\text{cohom} = \frac{\ker d_1}{\operatorname{coker} d_2}$$

$\vdots$

$$\begin{array}{ccc} \ker d_1 \otimes A/m_y & \xrightarrow{\cdot} & \ker(d_1 \otimes A/m_y) \\ \uparrow & & \uparrow \\ (\operatorname{coker} d_2) \otimes A/m_y & \xrightarrow{\sim} & \operatorname{coker}(d_2 \otimes A/m_y) \end{array}$$

- so it all boils down to matrices, algebra

Rmk Assuming $\pi: X \to Y$ is flat proj of rel dim N, we don't need to pass to $\mathbb{P}^N \times Y$, $R^N \pi_*$ is okay. Recall, we used flatness to get $\pi^* \xi(n) \twoheadrightarrow \mathcal{F}$, with $\pi^* \xi(n)$ becoming flat. We used these to build the resoln, with now everything flat. But if $\pi: X \to Y$ flat, then this is automatic and we don't need to be passing to $\mathbb{P}^N \times Y$ etc.

Lemma Let $f: X \to Y$ be proj, $\mathcal{F} \in \mathrm{Coh}_X$ flat$/Y$.
Consider cartesian diag

$$\begin{tikzcd} \tilde{X} \arrow[r, "\varphi"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ \tilde{Y} \arrow[r, "\psi"'] & Y \end{tikzcd}$$

[Ha § III] $\forall n \geq$ $\exists$ nat map $\psi^* R^n f_* \mathcal{F} \xrightarrow{\alpha} R^n \tilde{f}_* \varphi^* \mathcal{F}$

Assume $y \in Y$ s.t. $\varphi_{n,y}$ surj.

Let $\tilde{y} \in \tilde{Y}$ st. $\psi(\tilde{y}) = y$.

Then α also iso near $\tilde{y}$.

Cor Let $\mathcal{F} \in \mathrm{Coh}_X$, $X \xrightarrow{f} Y$ proj, $\mathcal{F}$ flat$/Y$
locally on Y $Rf_*\mathcal{F} = [R^N \pi_* \mathcal{H}_{-N} \to \cdots \to R^N \pi_* \mathcal{H}_0][N]$

$Rf_*\mathcal{F} \in D^b(\mathrm{Coh}_Y)$ is loc. isom to a finite cpx of loc. free sheaves

Def. Let X sch, $A \in D^b(\mathrm{coh}_Y)$ or $\in D^b_{\mathrm{coh}}(Y\text{-mod})$,
$a, b \in \mathbb{Z}$, $a \leq b$. We call A perfect of
perfect amplitude contained in $[a,b]$
if loc on X, A isom in the der. cat.
to $[\mathcal{E}^a \to \cdots \to \mathcal{E}^b]$, $\mathcal{E}^i$ loc free of fin. rk.

Thm $X \xrightarrow{f} Y$ proj (f.t./$k = \bar{k}$). Let $\mathcal{F} \in \mathrm{Coh}_X$.
Then $\mathcal{F}$ is flat $/Y$ $\Leftrightarrow$ $Rf_*\mathcal{F}$ perfect

[Ha III flatness] Y red ~~& irred~~ not necess. $\mathcal{F}$ flat $\Leftrightarrow$ $\forall n \gg 0$ $f_*\mathcal{F}(n)$ is loc. free on Y.

Fantechi

$$0 \to H^1(X, \mathcal{H}om(\xi, \mathcal{F})) \to Ext^1(\xi, \mathcal{F})$$
$$\to H^0(X, \mathcal{E}xt^1(\xi, \mathcal{F})) \to H^2(X, \mathcal{H}om(\xi, \mathcal{F}))$$

- today: deformation th in finite study of moduli
- moduli study of flat families → var, sch, vsp, cohsh, ...
- X proj sch family of cl. subsch. par by B

$$\begin{array}{ccc} Z & \overset{\varphi}{\hookrightarrow} & X \times B \\ \downarrow \text{flat} & & \\ B & & \end{array}$$

- e.g. Graßmannian, univ fam. of lin subsp.
- hypersfcs of deg d in $\mathbb{P}^N$

- $X \ni p \rightsquigarrow T_pX$?
- moduli $\rightsquigarrow B$ fat point, ie $B_{red} = Spec(k)$, say affine so $B_{red} = Spec(A/N_A)$

Lemma $\exists$ bijection between T_pX & $\{\varphi \in Mor(Spec\frac{\mathbb{C}[t]}{t^2}, X)\}$

Pf. Assume $p=0$. $p \in X = Spec\, R = Spec \dfrac{k[x_1, \dots, x_n]}{(f_1, \dots, f_r)}$
$\Leftrightarrow f_1(0) = \dots = f_r(0) = 0$.
Write $f_i = f_{i,0} + \dots + f_{i,m_i}$, $f_{i,a}$ hom. of deg a
$\Rightarrow \dfrac{\partial f_i}{\partial x_j}(0) = \dfrac{\partial f_{i,0}}{\partial x_j}(0)$

Let $l_i := f_{i,1}$. Then $T_pX = \ker(\mathbb{C}^n \xrightarrow{(l_1, \dots, l_r)} \mathbb{C}^r)$

On the other hand, $\varphi : Spec\frac{\mathbb{C}[t]}{t^2} \to Spec\, R$
$\Leftrightarrow \varphi^\# : R \to \mathbb{C}[t]/t^2$ s.t. $\varphi^\#(m_0) \subseteq (t)$
$\Leftrightarrow (\varphi^\#)^{-1}((t)) = m_0, \dots$

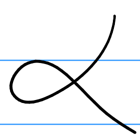
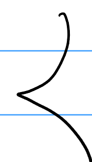
Def. Let $X = \mathrm{Spec}\,\dfrac{\mathbb{C}[x_1,\dots,x_n]}{(f_1,\dots,f_r)}$. X is nonsing/smooth at zero $\iff$ X red. at 0 and $\dim X|_0 = \dim T_0 X$

Fact X smooth at $0 \iff \forall n \geq 1\ \forall v \in T_0 X$

$$\begin{tikzcd}
\mathrm{Spec}\,\mathbb{C}[t]/t^2 \arrow[r, "v"] \arrow[d, hook] & X \\
\mathrm{Spec}\,\mathbb{C}[t]/t^{n+1} \arrow[ur, "\exists"'] &
\end{tikzcd}$$

- example.

$$\begin{tikzcd}
\mathrm{Spec}\,\mathbb{C}[t]/t^2 \arrow[r, "{(at,bt)}"] \arrow[d, hook] & \mathrm{Spec}\,\dfrac{\mathbb{C}[X,Y]}{Y^2-X^3-X^2} \\
\mathrm{Spec}\,\mathbb{C}[t]/t^3 \arrow[ur, "?"'] &
\end{tikzcd}$$

$\to$  works but try the cusp  ...

Def. Let X scheme, $B = \mathrm{Spec}\,A$ fat point.
A deformation of X over B is
i) a flat mor $X_B \xrightarrow{f} B$
ii) a cl. embedding $u : X \hookrightarrow X_B$
such that

$$\begin{tikzcd}
X \arrow[r, hook, "u"] \arrow[d] & X_B \arrow[d, "f"] \\
\mathrm{Spec}\,k \arrow[r, "A/m_A \leftarrow A"'] & B
\end{tikzcd}$$

is comm. & cartesian.

Given another deformation, $X \overset{\tilde{u}}{\hookrightarrow} \tilde{X}_B \xrightarrow{\tilde{\pi}} B$,
an isom. is the comm diag.

$$\begin{tikzcd}
& X \arrow[dl, hook', "\tilde{u}"'] \arrow[dr, hook, "u"] & \\
\tilde{X}_B \arrow[rr, "\sim"] \arrow[dr, "\tilde{\pi}"'] & \circlearrowleft & X_B \arrow[dl, "\tilde{\pi}"] \\
& B &
\end{tikzcd}$$

Def. A 1^{st} order (infinit.) defn of X is defn / Spec $\frac{\mathbb{C}[t]}{t^2}$.

Thm Assume X proj. var / $\mathbb{C}$ (sch of f.t. / $\mathbb{C}$, gener. smooth)
Then

$$\{1^{st}\text{-ord defns}\} \longleftrightarrow \{\text{extensions } 0 \to \mathcal{O}_X \to \mathcal{E} \to \Omega^1_X \to 0\}$$

is an equiv of categories.
In particular, $\{1^{st}\text{-ord defns}\}/\text{isom} \overset{1:1}{\hookrightarrow} \mathrm{Ext}^1(\mathcal{O}_X, \Omega^1_X)$.

- before proving, let's remember something abt flatness

Lemma Let M be $\mathbb{C}[t]/t^2$-mod, $M_0 = M \otimes_{\mathbb{C}[t]/t^2} \mathbb{C}$.

i) $\exists$ nat ex. seq. $M_0 \xrightarrow{\cdot t} M \xrightarrow{t \mapsto 0} M_0 \to 0$

ii) M flat over $\mathbb{C}[t]/t^2$ iff $0 \to M \xrightarrow{t} M_0$ ex.

Pf. (of Thm) Step 1: from defns to exts.
$B_{red} = \mathrm{Spec}\, \mathbb{C} \to B$ homeo of top. sp $\Rightarrow u : X \to X_B$ homeom.
so $\mathcal{A} := u^{-1}\mathcal{O}_{X_B}$ sheaf of flat $\mathbb{C}[t]/t^2$-alg.
$I/I_2 \to u^* \Omega_{X_B} \to \Omega_X \to$ exact, since
u closed emb., with $I = \mathcal{J}_{X/X_B}$.
$0 \to u^{-1} I \to u^{-1}\mathcal{O}_{X_B} \to \mathcal{O}_X \to 0$, $\mathcal{O}_X = (u^{-1}\mathcal{O}_{X_B}) \otimes_{\mathbb{C}[t]/t^2} \mathbb{C}$
$0 \to \mathcal{O}_X \to A \to \mathcal{O}_X \to 0$, $I \underset{\text{can}}{\simeq} u_* \mathcal{O}_X$, $I^2 = 0$
$\mathcal{J}/\mathcal{J}^2 = u^* \mathcal{J} \underset{\text{can}}{\simeq} \mathcal{O}_X$, $\mathcal{O}_X \xrightarrow{\alpha} u^* \mathcal{O}_{X_B} \to \Omega_X \to 0$

We only need α injects. On locus $X^{sm} \subseteq X$ it is [won't show]. $\forall U \underset{open}{\hookrightarrow} X$, $f \in \mathcal{O}_X(U)$, $\alpha(f) = 0 \Rightarrow f|_{\underset{nonempty}{U \times X^{sm}}} = 0 \Rightarrow f = 0$.

Step 2.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\tilde{\alpha}"] \arrow[d, equal] & \mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_X \arrow[r] \arrow[d, "d"] & 0 \quad \text{exact.} \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\alpha"] & \Sigma \arrow[r, "\beta"] & \Omega_X \arrow[r] & 0 \quad \text{exact}
\end{tikzcd}$$

where $\mathcal{A} := \{(e,f) \in \Sigma \oplus \mathcal{O}_X \mid \beta(e) = df\}$ pullback. $\tilde{\alpha}(f) := (\alpha(f), 0)$, d $\mathbb{C}$-lin not $\mathcal{O}_X$-lin. $\mathcal{A}$ is sheaf of $\mathbb{C}$-mod vsp, but also a $\mathbb{C}$-alg, $(e_1,f_1)(e_2,f_2) = (f_2 e_1 + f_1 e_2, f_1 f_2)$ since $\beta(e_i) = df_i$ gives $d(f_1 f_2) = f_2 df_1 + f_1 df_2$ $= f_2 \beta(e_1) + f_1 \beta(e_2) = \beta(\dots)$.

Now, $X_B = X$ as top. sp., so $u: X \to X_B$ is actually the identity. Let $\mathcal{O}_{X_B} := u_* \mathcal{A}$. sheaf of $\mathbb{C}[t]/t^2$-alg.

Define $t: \mathcal{A} \to \mathcal{A}$ by $t(e,f) = \tilde{\alpha}(f) = (\alpha(f), 0)$, $t^2 = t(\alpha(f), 0) = (\alpha(0), 0) = 0$.
so $\mathcal{A} \otimes_{\mathbb{C}[t]/t^2} \mathbb{C} \xrightarrow{\sim} \mathcal{O}_X$, $(e,f) \mapsto f$.

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{A} \xrightarrow{\otimes_{\mathbb{C}[t]/t^2} \mathbb{C}} \mathcal{O}_X \to 0 \Rightarrow \mathcal{A} \text{ flat.}$$

$\forall U \subseteq X$ affine, $U_B = \operatorname{Spec} \mathcal{A}(U)$.

$$\Rightarrow \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X_B \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & B \end{tikzcd}. \quad \square$$

- so we can interpret

$$0 \to H^1(X, T_X) \to \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to H^2(X, T_X)$$

1st ord defs, trivial on every open affine

1st ord defs of X / iso

compatible 1st ord. defns on each open / iso

- $T_X := \mathcal{H}om(\Omega_X, \mathcal{O}_X)$

- $0 \in \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \leftrightarrow$ split ext $\leftrightarrow$ triv def $X_B = X \times B$

- if X nonsing $\Rightarrow$ all defs loc. triv.
- If X is red proj curve, $H^2(X, T_X) = \{0\}$.

Fantechi

- missing topics: Grothendieck - Serre duality for proj. mos, cotangent cpx

- Serre $\operatorname{Ext}^i(\mathcal{F}, \omega_X)^\vee = H^{n-i}(X, \mathcal{F})$

- simplest case, X sm proj curve $/\mathbb{C}$
$\Rightarrow H^1(\Omega^1_X) \xrightarrow[\int]{\sim} \mathbb{C}$
in complex geometry
- need more work for alg. geom
- L line bdl, E vb.

$$H^0(X, E) \otimes H^1(X, E^\vee \otimes \Omega^1_X)$$
$$\downarrow$$
$$H^1(X, \underbrace{E \otimes E^\vee}_{Hom(E,E)} \otimes \Omega^1_X) \longrightarrow H^1(X, \Omega^1_X) \longrightarrow \mathbb{C}$$

Duality This is a perfect pairing.
Which means: V, W f.d. vsp$/\mathbb{K}$, $V \otimes W \xrightarrow{\varphi} \mathbb{K}$ is perfect pairing if induced maps $W \to V^\vee$ and $V \to W^\vee$ are isomorphisms, i.e. $\exists$ bases $v_1 - v_n \in V$, $w_1 - w_n \in W$ s.t. $\varphi(v_i \otimes w_j) = \delta_{ij}$.

- X proj. sm. of dim n:
- Serre duality: $H^i(X, E) \otimes H^{n-i}(X, E^\vee \otimes \Omega^n_X) \to H^n(X, \Omega^n_X) \to \mathbb{C}$
- Grothendieck $\to$ relative, works in der. cats.

– work over A Noetherian ring –

Lemma If A is local $\wedge$ M is f.g. flat A-mod then M is free, $M \simeq A^{\oplus d}$.

Pf. $M \otimes_A k(A)$ f.d. $k(A)$ vsp, basis $\bar{e}_1, \ldots, \bar{e}_d$.

$M \to M \otimes_A k(A) \to 0$ lifts $\bar{e}_i$ to $e_i \in M$

$\rightsquigarrow \; 0 \to N \to A^{\oplus d} \xrightarrow{\text{Nak.}} M \to 0 \quad | \; - \otimes_A k(A)$

$$\underset{\overset{\|}{0}}{\mathrm{Tor}^1(M, k(A))} \to N \otimes_A k(A) \to \underbrace{A^{\oplus d} \otimes_A k(A)}_{k(A)^{\oplus d}} \xrightarrow{\sim} M \otimes_A k(A) \to 0$$

$\Rightarrow$. $N \otimes_A k(A) = 0$, N f.g. $\overset{\text{Nak.}}{\Longrightarrow}$ $N = 0$. $\square$

Lemma $\mathcal{F} \in \mathrm{Coh}_X$, X (loc) Noeth. sch. Let $p \in X$. Assume stalk $\mathcal{F}_p$ is free $\mathcal{O}_{X,p}$-mod of rk r. Then $\exists$ $U \subseteq X$ open nbhd of p s.t: $\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus r}$.

Lemma $\mathcal{F} \in \mathrm{Coh}_X$, X loc Noeth sch.
$\mathrm{Supp}\, \mathcal{F} = \{ p \in X \mid \mathcal{F}_p \neq 0 \}$ is closed in X.

Summing up X loc. Noeth. $\mathcal{F} \in \mathrm{Coh}_X$. TFAE

i) $\mathcal{F}$ is flat as $\mathcal{O}_X$-mod

ii) $\mathcal{F}$ is loc. free

iii) $\mathcal{F}_p$ is free $\forall p \in X$.

Rmk i) X sch loc. f.t. $/k = \bar{k}$ $\Rightarrow$ $X(k)$ dense in X

ii) X — " — , but seen as top. sp.,
$Z \subseteq X$ all k-valued pts.
Then $\underset{\text{all pts closed}}{Z} \longrightarrow X$ gives $\mathrm{Top}(X) \xrightarrow[\text{iso}]{\text{sturt}\ \sim} \mathrm{Top}(Z)$

→ therefore these define equivalent sheaf datum.

- fix A Noeth. ring, let $D(\mathrm{Mod}_A) \underset{\text{full subcat}}{\supseteq} D_{coh}(\mathrm{Mod}_A)$ of coherent cohomology

Exercise $E \to B \to C \xrightarrow{+1}$ dist $\Delta \in D(\mathrm{Mod}_A)$.
If $E, B \in D_{coh}(\mathrm{Mod}_A)$, so is C.

Def. $E \in D(A)$ perfect in $[a,b]$ if isom to some $[\tilde{E}_a \to \cdots \to \tilde{E}_b]$ of free finite rank A-mods.

Rmk. E perfect in $[a,b] \Rightarrow E \in D^{[a,b]}_{coh}(\cdot\,\cdot)$

- $\_\otimes_A M$ is rexact in Mod_A, which has enough projs
$\Rightarrow \_\overset{L}{\otimes}_A M$ is $D^-(\mathrm{Mod}_A)$ endofunctor
- if $N \in \mathrm{Mod}_A$, view it as cpx. conc. in 0 and then $\mathrm{Tor}^A_{-i}(N, M) = h^i(N \overset{L}{\otimes}_A M) \simeq h^i(M \overset{L}{\otimes} N)$

Cor E pfct in $[a,b] \Rightarrow \forall i \notin [a,b]$, $\forall M \in \mathrm{Mod}_A$,
$h^i(E \overset{L}{\otimes}_A M) = 0$

Thm Let A Noeth. ring, $\mathcal{E} \in D^{[a,b]}_{coh}(\mathrm{Mod}_A)$.
TFAE

i) $\mathcal{E}$ perfect in $[a,b]$

ii) $\forall M \in \mathrm{Mod}_A$, $i \notin [a,b]$, $h^i(\mathcal{E} \overset{L}{\otimes} M) = 0$

iii) for any cpx $\cdots \to F^{b-1} \to F^b \to 0 \cdots$
where F^i loc. free of fin. rank
isom. to $\mathcal{E}$ in $D^b(\mathrm{Mod}_A)$,
$G := \mathrm{coker}(F^{a-1} \to F^a)$ is loc. free of f.r.

Fantechi

- R loc. ring, m max ideal
- $\hat{R} = \varprojlim R/m^n$ its completion along m

$$R/m \leftarrow R/m^2 \leftarrow R/m^3 \leftarrow \cdots$$
$$a_0 \longleftarrow\!\shortmid\ a_1 \longleftarrow\!\shortmid\ a_2 \longleftarrow\!\shortmid \cdots$$

- $\exists$ morphism $R \to \hat{R}$, $a \mapsto ([a],[a],\ldots)$
- $\hat{R}$ has ring structure with product $(a_0,a_1,\ldots)(b_0,b_1,\ldots) = (a_0b_0, a_1b_1,\ldots)$
- $\hat{R}$ local ring?
  - max id. $\hat{m}$ spanned by elements beginning w/ zero, $(0,a_1,a_2,\ldots)$

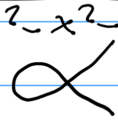
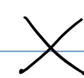
Rmk • This works for any ring A w max.id. m, $\hat{A} = \varprojlim A/m^n$

• Let $u_{n-1} \in A/m^n$ and denote by u_0 its image in A/m. If $u_0 \neq 0$ then u_{n-1} is a unit. (A/m^n is loc. ring w max. ideal mA/m^n)

Cor.
$$\begin{tikzcd} A \arrow[r] \arrow[d] & \hat{A} \\ A_m \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

- wasn't writing for a while

Example $C = \operatorname{Spec} \dfrac{\mathbb{C}[x,y]}{y^2-x^2-x^3}$, $C_1 = \operatorname{Spec} \dfrac{\mathbb{C}[x,y]}{y^2-x^2}$

- clearly "the same" around origin

- the formal completions are $\mathbb{C}[[u,v]]/u^2-v^2-v^3$ and $\mathbb{C}[[x,y]]/(x\cdot y$

Claim: $\exists$ a unit in $\mathbb{C}[[u,v]]$ s.t. $a^2 = 1+v$

$\Rightarrow$ $a = \tilde{a}_0 + \tilde{a}_1 + \tilde{a}_2 + \dots$

$a^2 = 1+v$

$\Rightarrow$ $(a^2)_0 = \tilde{a}_0^2 = 1$, choose $\tilde{a}_0 = 1$.

$(a^2)_n = \tilde{a}_0 \tilde{a}_n + \dots + \tilde{a}_n \tilde{a}_0 = \underline{1} + v$

$\Rightarrow$ $2 \underbrace{\tilde{a}_0}_{\substack{\parallel \\ 1}} \tilde{a}_n = 1 + v - \dots$ per choice

$\Rightarrow$ $\tilde{a}_n = \frac{1}{2}(\dots)$

- ok, so since a is unit, $\begin{array}{l} x \mapsto av \\ y \mapsto u \end{array}$ is isomorphism,

$$\frac{\mathbb{C}[[u,v]]}{u^2 - v^2 a^2} \xleftarrow{\sim} \frac{\mathbb{C}[[x,y]]}{y^2 - x^2}$$

- so completions are isomorphic

- now do an inclusion

$$\frac{\mathbb{C}[u,v]}{u^2 - v^2(1+v)} \hookrightarrow \frac{\mathbb{C}[u,v,a]}{(u^2 - v^2(1+v), a^2 - (1+v))}$$

$$C \longleftarrow C_2$$

Claim $C_2 \longrightarrow C$ étale.

$$\frac{\mathbb{C}[u,v,a]}{u^2 - v^2(1+v), a^2 - (1+v)} \simeq \frac{\mathbb{C}[u,a]}{\underbrace{u^2 - a^2(a^2-1)^2}_{(u - a(a^2-1)) \cdot (u + a(a^2-1))}}$$

- so $C_2 = C_2' \cup C_2''$, $C_2' = \operatorname{Spec} \frac{\mathbb{C}[u,a]}{u - a(a^2-1)} \simeq \operatorname{Spec} \mathbb{C}[a]$

$C_2' \cap C_2'' = \operatorname{Spec} \frac{\mathbb{C}[u,a]}{(u, a(a^2-1))}$.

Fact. (Milne, étale coh.) Let $A \to B$ homomorphism of f.g. algs $/\mathbb{k} = \overline{\mathbb{k}}$. Then $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is étale $\iff$ up to localisation

$$B = \frac{A[u_1, \dots, u_n]}{(f_1, \dots, f_n)} \text{ s.t. } \det\left(\frac{\partial f_i}{\partial u_j}(p)\right) \neq 0$$

in $\mathbb{k}(p)$.

→ If we want smooth of rel dim s instead of étale, same but $B = A[u_1, \dots, u_n, v_1, \dots, v_s]/(f_1, \dots, f_n)$

- $B[a]/a^2 - (1+v)$ is étale at (a_0, u_0, v_0) if $\partial(a^2 - (1+v))/\partial a\,\big|_{(a_0, u_0, v_0)} = 2a \neq 0$
  → so if char $\mathbb{k} \neq 2$, it works if $a \neq 0$
  ⇒ but then $a_0 \neq 0$ so a invertible