Bruzzo.

- the proof that $Cl(\text{double cone}) \simeq \mathbb{Z}_2$ was fleshed out → unfortunately I arrived late.

Cartier divisors.

- X integral, $U \subseteq X$ nonempty open
- $\mathcal{K}_X$ (constant) sheaf of rational functions, $\mathcal{K}_X^* \subset \mathcal{K}_X$ the nonzero — " —

→ $\mathcal{O}_X$ and $\mathcal{O}_X^*$ are their respective subsheaves

→ Cartier divisors are elements of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$

- for $\{U_i\}_i$ cover of X, these are given by $\{(U_i, f_i)\}$ with $f_i/f_j \in \mathcal{O}_X^*(U_{ij})$

- noting that $1 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 1$ s.e.s.:

Def. A Cartier divisor is principal if it lies in the image of $\mathcal{K}_X^* \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$

Put $CaCl(X) = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)/\sim$

Def. A scheme is locally factorial if all loc. rings UFD.

Thm. Let X integral separated noetherian loc. fact. scheme. Then $\exists$ a 1-1 correspondence between Weil & Cartier divisors.

Pf. Let $D \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ be represented by $\{(U_i, f_i)\}$ so write $D = \sum v_Y(f_i)\, Y$.

Now take $D \in Div(X)$, $x \in X$. We are interested in local behaviour, so take $X = Spec\, A$.

Write $\mathcal{O}_{X,x} =: A_x$. $A \to A_x^{\text{UFD}} \Rightarrow Spec\, A_x \to Spec\, A$.

Now look at
$$\begin{tikzcd} D_x \arrow[r] \arrow[d] & \operatorname{Spec} A_x \arrow[d] \\ D_x \arrow[r] & \operatorname{Spec} A \end{tikzcd} \quad D_x = (f_x).$$

- now look at $(X, \mathcal{O}_X)$ ringed space, X conn. of rk r.
- let $\mathcal{L} \in \mathcal{O}_X$-mod.
  - we say $\mathcal{L}$ is free if $\mathcal{L} \simeq \overbrace{\mathcal{O}_X \otimes \cdots \otimes \mathcal{O}_X}^{r}$,
  and $\mathcal{L}$ is locally free if every pt $\in X$ has a nbhd U s.t. $\mathcal{L}(U)$ is free of rank r over $\mathcal{O}_X(U)$.
- if $rk = 1$, we call locally free $\mathcal{O}_X$-modules line bdls or invertible sheaves.

$\rightarrow$ given $\mathcal{L}_1, \mathcal{L}_2$ line bdls, so is $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$

$\rightarrow$ given $\mathcal{L}$, $\exists\, \mathcal{L}^{-1}$ s.t. $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$

$\rightarrow$ given open cover $\mathfrak{U} = \{U_i\}$, $\mathcal{L}(U_i) \simeq \mathcal{O}_X(U_i)$
and $g_{ij} \in \mathcal{O}_X^*(U_{ij})$ transition functions,,
$g_{ij} g_{jk} g_{ki} = 1$, which is just the Čech 1-cocycle condition written in multiplikative notation.
$\Rightarrow \operatorname{Pic}(X) = H^1(\mathfrak{U}, \mathcal{O}_X^*) \simeq H^1(X, \mathcal{O}_X^*)$,
noting that iso. line bdles are related by coboundaries

- back to divisors.

$\rightarrow$ take Cartier div. $\{(U_i, f_i)\}$ and let $f_i/f_j \in \mathcal{O}_X^*(U_{ij})$ be transition fns.

$\rightsquigarrow$ line bdle $\mathcal{O}_X(D)$.

$\rightarrow$ subsheaf of K_X generated by f_i^{-1} on U_i over $\mathcal{O}_{U_i}$

Props. 1) $\mathcal{O}_X(D)$ line bundle & 1-1 corr. $D \Leftrightarrow \mathcal{O}_X(D)$

ii) $D_1 \sim D_2 \iff \mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$. This gives $\operatorname{CaCl}(X) \to \operatorname{Pic}(X)$.

iii) $\mathcal{O}_X(D_1 - D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$