Tanzini.

$\mathcal{N}=(2,2)$ nonlinear σ model

- maps $\Sigma_d \xrightarrow{\varphi} M$. general comments on spin in this case
- spinors: Dirac modules of $Spin(V)$
  - Spin acting on $SL(2,\mathbb{F})$ are
    $\mathbb{F}=\mathbb{R}$ (for $d=3$), $\mathbb{C}$ (for $d=4$),
    $\mathbb{H}$ (for $d=6$), $\mathbb{O}$ (for $d=10$)

- fields of (minimal) susy nonlin. σ. model:
  $\varphi \in \mathcal{C}^\infty(\Sigma_d, M)$
  $\psi \in \mathcal{C}^\infty(\Sigma_d, \pi S \otimes_{\mathbb{F}} \varphi^*(TM))$
  $\hookrightarrow$ in $d=1$, $\Sigma_1 = S^1$
  $\varphi \in LM$
  $\psi \in \mathcal{C}^\infty(S^1, \varphi^*(TM))$

- so, susy twists the geometry! $D_t \psi^I = \partial_t \psi^I + \Gamma^I{}_{JK} \partial^J \varphi \, \psi^K$
  s.t. $\varphi^*(TM)$

- TM endowed w left multiplication by $\mathbb{F}$
  - trivial only for $\mathbb{F}=\mathbb{R}$
  - for $\mathbb{F}=\mathbb{C}$ we need almost cpx. structure
    $J: TM \circlearrowleft$, $J^2 = -id_{TM}$
  - for others, some other tensor controls it
  $\rightarrow$ we want its covariant derivative to vanish, which is $\Longleftrightarrow$ to ∇ being Levi-Civita
    $\rightarrow$ so for $\mathbb{F}=\mathbb{C}$, M is Kähler,
    for $\mathbb{F}=\mathbb{H}$, M is Hyperkähler

$\dim_{\mathbb{R}} M = 2n$, holonomy group $SO(2n) \xrightarrow{\text{reduces to}} U(n)$
$\dim_{\mathbb{R}} M = 4k$, — " — $Sp(k)$

Thm minimal susy nonlin. σ models $\exists$ iff

$$\begin{cases} d=3 \ldots & M \text{ Riemannian} \\ d=4 \ldots & M \text{ Kähler} \\ d=6 \ldots & M \text{ Hyperkähler} \end{cases}$$

look at Freed, Deligne?

- for $\mathcal{N}=(2,2)$ susy we need $\dim_{\mathbb{R}} \Sigma = 4$, since $\dim S = 2^{[\frac{d}{2}]}$

- assume Σ closed orientable Riem. sfc.

- $Q_+, \overline{Q}_+$ sections of $K_\Sigma^{1/2}$ $\quad (K_\Sigma = T^{\vee(1,0)}\Sigma)$
  $Q_-, \overline{Q}_-$ —— " —— $\overline{K}^{1/2}_\Sigma$

- $\Sigma = \mathbb{C}$, $\mathbb{R}^{2|4}$
  $$S = \int d^2z \, d^2\vartheta \, d^2\overline{\vartheta} \, K(\varphi^i, \overline{\varphi}^{\bar{\imath}}) + \int d^2z \, d\vartheta^+ d\vartheta^- \, W(\varphi^i)\big|_{\overline{\vartheta}^{\pm}=0} + \text{c.c.}$$
  - K real, called D-term
  - W holom., called superpotential or F-term

- since M Kähler, write
  $$ds^2 = g_{i\bar{\jmath}} \, d\varphi^i \odot d\varphi^{\bar{\jmath}}$$
  $$\omega = \tfrac{1}{2} i \, g_{i\bar{\jmath}} \, d\varphi^i \wedge d\varphi^{\bar{\jmath}}, \quad d\omega = 0$$

$$d\omega = 0 \Rightarrow \begin{matrix} \partial_k g_{i\bar{\jmath}} = \partial_i g_{k\bar{\jmath}} \\ \partial_{\bar{k}} g_{i\bar{\jmath}} = \partial_{\bar{\jmath}} g_{i\bar{k}} \end{matrix} \Rightarrow g_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial\varphi^i \partial\varphi^{\bar{\jmath}}}$$

Rmk $K(\varphi,\overline{\varphi}) \mapsto K(\varphi,\overline{\varphi}) + f(\varphi) + \overline{f}(\overline{\varphi})$
does not change the metric
$\to$ in the Lagrangian it gives
total derivatives

Exercise. show that
$$\mathcal{L} = \int d^4\vartheta \, K(\varphi^i, \overline{\varphi}^{\bar{\imath}}) = g_{i\bar{\jmath}} \partial_z \varphi^i \partial_{\bar{z}} \varphi^{\bar{\jmath}} + i g_{i\bar{\jmath}} \overline{\psi}^{\bar{\jmath}}_- D_z \psi^i_-$$
$$+ i g_{i\bar{\jmath}} \overline{\psi}_+^{\bar{\jmath}} D_{\bar{z}} \psi_+^i + R_{i\bar{\jmath}k\bar{l}} \psi_+^i \psi_+^{\bar{\jmath}} \psi_-^k \overline{\psi}_-^{\bar{l}}$$
$$+ g_{i\bar{\jmath}} (F^i - \Gamma^i_{jk} \psi_+^j \psi_-^k)(F^{\bar{\jmath}} - \Gamma^{\bar{\jmath}}_{\bar{k}\bar{l}} \overline{\psi}_-^{\bar{k}} \psi_+^{\bar{l}})$$

$$D_{\bar{z}} \psi^i := \partial_{\bar{z}} \psi^i + \Gamma^i_{jk} \partial_{\bar{z}} \varphi^j \psi^k$$

- however, it turns out only $\Sigma \simeq \mathbb{T}^2$ admits global sections !
  → we solve this by top. twist

- R-symmetries
  - vector: $Q_\pm \mapsto e^{-i\alpha} Q_\pm$
  - axial: $Q_\pm \mapsto e^{\mp i\beta} Q_\pm$

- topological twist:

$$U(1)'_E = \begin{cases} \mathrm{diag}\left(U(1)_E \times U(1)_V \right) & \text{A-twist} \\ \mathrm{diag}\left(U(1)_E \times U(1)_A \right) & \text{B-twist} \end{cases}$$

- redefinition of the spin connection

$$\partial_z + \underset{\substack{\uparrow \\ \text{spin}}}{\omega_z} + \underset{\substack{\uparrow \\ U(1)_R}}{A_z}$$

	Σ model $U(1)_V$	$U(1)_A$	$U(1)_E$	A model	B model
φ	0	0	0	φ	φ
ψ_-^i	-1	1	1	$\chi^i \in \Gamma(\varphi^* T^{1,0} M)$	$\rho_z^i \in \Gamma(\varphi^*(T^{1,0}M) \otimes K_\Sigma)$
$\overline{\psi}_+^{\bar{i}}$	1	1	-1	$\overline{\chi}^{\bar{i}} \in \Gamma(\varphi^* T^{0,1} M)$	$-\frac{1}{2}(\overline{v} + \overline{\eta})^{\bar{i}} \in \Gamma(\varphi^* T^{0,1} M)$
$\overline{\psi}_-^{\bar{i}}$	1	-1	-1	$\rho_z^{\bar{i}} \in \Gamma(\varphi^*(T^{0,1}M) \otimes K_\Sigma)$	$\frac{1}{2}(\overline{v} - \overline{\eta})^{\bar{i}} \in \Gamma(\varphi^* T^{0,1} M)$
ψ_+^i	-1	-1	-1	$\rho_{\bar{z}}^i \in \Gamma(\varphi^*(T^{1,0}M) \otimes \overline{K}_\Sigma)$	$\rho_{\bar{z}}^i \in \Gamma(\varphi^*(T^{1,0}M) \otimes \overline{K}_\Sigma)$

- $\begin{cases} \text{A twist: } Q_-, \overline{Q}_+ \text{ are scalars} \\ \text{B twist: } \overline{Q}_+, \overline{Q}_- \text{ are scalars} \end{cases}$

- define $\begin{array}{l} Q_A := \overline{Q}_+ + Q_- \\ Q_B := \overline{Q}_+ + \overline{Q}_- \end{array} \rightsquigarrow \varphi : \Sigma \to M$ (symplectic) (Calabi-Yau)

Rmk. $\mathbb{Z}_2$ automorphism $Q_- \leftrightarrow \overline{Q}$, $F_V \leftrightarrow F_A$ ⇒ exchanges A & B models