Fantechi

- schedule: 3x weekly till Easter, then break
  until May 30

- new timetable:

Wed 9–11	Thu 14–16	Fri 9–11
136	136	136

- plan: review ab. cats., complexes,
  exactness/cohom., (half) exact functors
  - primary exs: Sheaves of modules
    over (not necc.) ringed spaces
- content of course: derived categories,
  derived functors in more general way
  than Hartshorne ("nobody has ever seen an injective object!", also usually lack of projectives)
  Ext, Tor, Rf_*, Lf^*, Spect. sqnc (Leray, local-to-global Ext)
  coh. and base changes, Grothendieck–Serre duality

Examples of NC rings/sheaves of rings

- ex 1 - G finite gp, k field.
  $Rep_k(G)$ cat. of G-representations,
  objects V k-vsp $G \to GL(V)$,
  morphisms $V \to W$ k-lin. G-equiv. maps
  - modules over $k[G]$ NC group alg
  - product induced by G-product $\wedge$ linearity:
    $a, b \in k[G]$, $a = \sum_G a_g [g]$, $b = \sum b_g [g]$
    $\Rightarrow$ $a \cdot b = \sum_{g,h \in G} a_g b_h [g \cdot h]$
- exercise: check that $k[G]$ is k-alg
  check $Mod_{k[G]} \simeq Rep(G)$ canonically

- ex 2. - let X be C^∞-mfd, C^∞_X sheaf of $C^\infty_{\mathbb{R}}$-funcs on X.
  - $OP(C^\infty_X)$ sheaf of <u>lin.</u> maps $C^\infty_X \circlearrowleft$
  - (very big) NC sheaf of NC $\mathbb{R}$-alg where product is composition
  - possesses an interesting subalg: differential operators, subalg gen by $f \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_r}}$ (Kashiwara-Schapira)

- <u>recall</u>: A nc. ring, $[a,b] := ab - ba$.
- let $D^{\leq k}_X$ = $\mathbb{R}$-lin subsheaf of $OP(C^\infty_X)$ gen by $f \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_r}}$ with $r \leq k$
- $D_X = \bigcup_k D^{\leq k}_X$, $D^{\leq k}_X \subset D^{\leq k+1}_X$
- exercise:
  1) $D^0_X = C^\infty_X$
  ii) $D^{\leq k}_X = \{ \varphi \in OP(C^\infty_X) \mid \forall f \in C^\infty_X, [f, \varphi] \in D^{\leq k-1}_X \}$

Def. Let $\mathcal{C}$ cat. We always assume $\forall x, y \in Ob\,\mathcal{C} \Rightarrow Mor_{\mathcal{C}}(x,y)$ is a set.
To every $x \in Ob\,\mathcal{C}$ associate $h_x : \mathcal{C}^{op} \to Set$ functor $h_x(y) := Mor_{\mathcal{C}}(y, x)$, $f: y_1 \to y_2 \rightsquigarrow h_x(f) = Mor_{\mathcal{C}}(y_2, x) \to Mor_{\mathcal{C}}(y_1, x)$, $g \mapsto g \circ f$

- <u>Recall</u>: $\mathcal{C}, \mathcal{C}'$ cats. $Fun(\mathcal{C}, \mathcal{C}')$ cat with objects functors $\mathcal{C} \to \mathcal{C}'$, mor := nat transf $\alpha : F \Rightarrow G$

i.e. datum $\forall x \in Ob\,\mathcal{C}$ of a mor. $\alpha(x)$ in $\mathcal{C}'$
s.t. $\forall f: x_1 \to x_2$,

$$\begin{tikzcd} F(x_1) \arrow[r, "\alpha(x_1)"] \arrow[d, "F(f)"'] & G(x_1) \arrow[d, "G(f)"] \\ F(x_2) \arrow[r, "\alpha(x_2)"] & G(x_2) \end{tikzcd}$$

commutes.

- <u>exercise</u> Show $\alpha: F \Rightarrow G$ iso in $Fun(\mathcal{C}, \mathcal{C}')$
$\iff \forall x \in Ob(\mathcal{C})$, $\alpha(x)$ iso in $\mathcal{C}'$

- <u>example</u>. $V \in Vect_K$. Construct nat trans
$id_V \Rightarrow DD$, $DD: V \to V$
$V \mapsto V^{\vee\vee} = Hom(Hom(V, K), K)$

<u>Def.</u> $F: \mathcal{C}^{op} \to Set$ is *representable* if
$\exists\, x \in Ob\,\mathcal{C} \wedge \exists$ nat transf $h_x \to F$.
We say x *represents* F.

<u>Lemma.</u> (Yoneda) Let $F: \mathcal{C}^{op} \to Set$, $x \in Ob\,\mathcal{C}$.
Then the natural map
$Mor(h_x, F) \to F(x)$
$\alpha: h_x \to F \mapsto \alpha(x)(id_x)$
where $\alpha(x): h_x(x) \to F(x)$,
is a bijection.

<u>Cor.</u> Assume we are given nat eq

$$\begin{tikzcd} h_{x_1} \arrow[r, "\alpha_1"] \arrow[d, dashed, "\beta"'] & F \\ h_{x_2} \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

Then $\exists!\, \varphi: x_1 \xrightarrow{\sim} x_2$ in $\mathcal{C}$
such that $\overset{\beta}{\dashrightarrow}$ defined by $\varphi \in h_{x_2}(x_1) = Mor(x_1, x_2)$
makes diagram commute.

<u>Pf.</u> α_1, α_2 isos in $Fun(C^{op}, Set) \Rightarrow \beta := \alpha_2^{-1} \circ \alpha_1$ well-def.
iso in $Fun(C^{op}, Set) \Rightarrow$ by Yoneda it is defined by $\varphi: x_1 \to x_2$

φ is iso since (by Yoneda) co.rs. to β^{-1}. □

- <u>examples.</u> $\mathcal{C} = (Sch/_k \vee C^\infty \text{ mfds})$, V f.d. vsp, $r \in \mathbb{N}$. $G(r,V)$ represents $G: (Sch/_k)^{op} \to (Set)$, $G(X) = \{rk = r$ subbdls of $V \times X\}$.
- for $S_2 \subseteq V \times X_2 \Rightarrow \varphi^* S_2 = S_2 \times_{X_2} X_1$, $X_1 \xrightarrow{\varphi} X_2$

- <u>exercise.</u> Let $\mathcal{C}$ be $C^\infty_{\mathbb{R}}$-mfds or $Sch/_k$.

Define $F: \mathcal{C}^{op} \to Set$ by $F(X) = \begin{cases} C^\infty(X) \\ \text{or} \\ G_X(X) \end{cases}$

i) define F on morphisms, demonstrate functoriality
ii) show it is representable & find representative ob.

- how to use repr. in alg. geom?
 i) to define schemes: Fix $P \in \mathbb{Q}[t]$, $X \subseteq \mathbb{P}^N$ closed.
 define $Hilb^P(X)$ to be scheme repr. the functor
 $Q(S) = \{ Z \in X \times S \text{ closed} \mid Z \text{ flat over } S,$
 i.e. $\forall s \in S$, $Z_s \subseteq X$ has Hilb. polyn. $= P\}$

<u>Thm</u> (Grothendieck) Q represented by a proj. sch/$_k$.

 ii) describe props of schemes or morph. of sch in terms of Yoneda functors
 - e.g. val criterion: $f: X \to Y$ proper
 $\iff \forall$ comm. diag.

$$\begin{tikzcd} \eta = \operatorname{Spec} K(A) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ c = \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with A valuation domain, $K(A)$ quot field,

$h_X(c)$

$\exists! \xrightarrow{\omega}$ making everything commute
- so $h_X(c) \to h_X(\eta) \times_{h_Y(\eta)} h_Y(c)$ bijects

What is an abel. category?

- Grothendieck, Tohoku is great!
  - has a few superfluous assumptions for us

Def. An additive category $\mathcal{C}$ is a cat s.t.
$\forall x, y \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(x,y)$ has a structure of an abelian gp. & $\mathrm{Mor}(x,y) \times \mathrm{Mor}(y,z) \to \mathrm{Mor}(x,y)$ is bilinear.

Def. If $\mathcal{A}$ add. cat., $\varphi: X \to Y \in \mathrm{Mor}\,\mathcal{A}$, a kernel of φ is a mor $i: K \to X$ st.
$\forall\, Z \in \mathrm{Ob}\,\mathcal{C}$ the following sqn. of ab. gps is exact:

$$0 \to \mathrm{Hom}(Z,K) \xrightarrow{i} \mathrm{Hom}(Z,X) \xrightarrow{\varphi} \mathrm{Hom}(Z,Y),$$

i.e. $\mathrm{Hom}(Z,K) = \{ \psi \in \mathrm{Hom}(Z,K) \mid \varphi \circ \psi = 0 \in \mathrm{Hom}(Z,Y) \}$
i.e. $h_K \subseteq h_X$ is a subfunctor.

- by Yoneda, if kernel exists, then unique up to iso which commutes w $(-) \hookrightarrow X$.

- exercise: define coker of $X \to Y$
  - hint coYoneda

Def. An abelian category is an additive cat. such that

1) finite direct sums exist
ii) ker & coker exist
iii) $\forall f: X \to Y$ mor, $\exists$ morphisms
$$K \xrightarrow{i} X \xrightarrow{\pi} P \xrightarrow{j} Y \xrightarrow{p} C \text{ s.t.}$$
$i = \ker f$, $\pi = \operatorname{coker} i$, $p = \operatorname{coker} f$, $j = \ker p$, $j \circ \pi = f$.

Omissions in course

i) all set-theoretic questions
ii) strict use of def. of ab. cat.
- ∃ embedding thm guaranteeing that every ab. cat is an ab. subcat of cat. of modules ?? (Freyd)