A. G. exercises.

Exercise. $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ morphisms of S-schemes

Prove $f=g$ if
1) $\exists\, U \subseteq X$ open, dense, $f|_U = g|_U$
2) X reduced
3) Y separated $/S$

Preliminaries:

① $X \xrightarrow{f} Y$, $Z \overset{i}{\hookrightarrow} Y$ closed subscheme
$\Rightarrow$ scheme-theoretic preimage of Z under f is $f^{-1}(Z) := Z \times_{i, Y, f} X$

- being separated over S means $Y^2 := Y \times_S Y$ closed imbedding under $\Delta_{Y/S} := (id_Y, id_Y)$
- look at:

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "(f,g)"] \\ Y \arrow[r, "\Delta"] & Y^2 \end{tikzcd}$$

1) $E \xrightarrow{i} X$ closed imb
2) $E \xrightarrow{i} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ commutes

$$\begin{tikzcd} & Y \\ Y^2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & Y \end{tikzcd}$$

$(f,g)i = \Delta\pi$

$\Downarrow$

$\pi_1(f,g)i = \pi_1\Delta\pi$

$\Downarrow$

$f \circ i = \pi$

3) given Z, $Z \xrightarrow{h} X$ s.t.
$Z \to X \overset{f}{\underset{g}{\rightrightarrows}} Y$ commutes
$\Rightarrow \exists!\, Z \xrightarrow{\xi} X$, etc.

- back to exercise:

$\to$ let $E \xrightarrow{i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be the equaliser,

$j: U \to X$ open imbedding

$$\Rightarrow \begin{tikzcd} U \arrow[d, dashed, "\exists!"'] \arrow[dr, "j"] & & \\ E \arrow[r, "i"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

commutes since $f|_U = g|_U \Leftrightarrow f \cdot j = g \cdot j$

$\Rightarrow$ topologically, $U \subseteq E \Rightarrow \overline{U} \subseteq \overline{E}$

$\Rightarrow X \subseteq E$

Rmk. X reduced, $Z \hookrightarrow X$ closed subscheme s.t. $Z = X$ as sets. Then $Z = X$ as schemes.

Pf. Affine case. Let $X = \operatorname{Spec} A$, A reduced

$\Rightarrow Z \simeq \operatorname{Spec} A/I$ for some $I \subseteq A$ since closed subscheme.

Topologically, $Z = V(I) = V(0) = X$

$\Rightarrow \sqrt{I} = \sqrt{0} \overset{A \text{ reduced}}{=\!=} 0 \Rightarrow I = (0)$

$\Rightarrow Z \simeq \operatorname{Spec} A/(0) = \operatorname{Spec} A = X$.

Counterexamples.

① X nonreduced

② Y nonseparated

① Take $\mathbb{k}$ alg. cl., $S = \operatorname{Spec} \mathbb{k}$, $X = \operatorname{Spec} A = \operatorname{Spec} \frac{\mathbb{k}[x,y]}{(y^2, xy)}$,
$Y = \mathbb{A}^1_{\mathbb{k}}$, X, Y $\mathbb{k}$-schemes.

If $y = 0$ in A, $y \in (y^2, xy) \Leftrightarrow y = ay^2 + bxy$

$\Rightarrow 1 = ay + bx$ $\to$ $1 = 0$ for $(y, x) = (0, 0)$.

- stopped writing for a while

- now $f, g \in \operatorname{Map}_{\mathbb{k}}(\underset{\operatorname{Spec} A}{X}, Y = \operatorname{Spec} \mathbb{k}[t]) \simeq \operatorname{Map}_{\mathbb{k}\text{-alg}}(\mathbb{k}[t], A) \simeq A$

Take $y, 0 \in A$

- $y \neq 0$, but $\exists$ open dense $U \subseteq X$ s.t. $\overbrace{y|_U = 0|_U}^{f|_U = y|_U}$
  $\Rightarrow U = X_x \Rightarrow f|_U = y|_U = y|_{y=0} = 0 = 0|_U$.

② (Y not separated)

- look at
$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, hook, "i"'] & \mathbb{A}^n_{\mathbb{k}} \arrow[d, "f"] \\ \mathbb{A}^n_{\mathbb{k}} \arrow[r, "g"'] & Y \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}, \quad Y = \mathbb{A}^n \coprod_{i, U, i} \mathbb{A}^n$$

$\to Y$ affine n-space $/\mathbb{k}$ with doubled origin

$\to$ not separated, $X := \mathbb{A}^n_{\mathbb{k}}$ reduced $\mathbb{k}$-scheme,

$\to$ but, $f|_U = f \circ i = g \circ i = g|_U$ by definition

get $f \neq g$ since $f(0) = 1^{st}$ origin

$g(0) = 2^{nd}$ origin $\Rightarrow$ ↯.

Exercise. S affine, X S-scheme, $U, V \overset{\text{open}}{\subseteq} X$.

(U, V affine & X separated) $\Rightarrow$ ($U \cap V$ affine)

- first show $U \cap V$ closed subscheme of an affine scheme

$\to U, V, S$ affine $\Rightarrow U \underset{z_i, S, z_j}{\times} V$ affine,

$U \overset{i}{\hookrightarrow} X$, $V \overset{j}{\hookrightarrow} X$, $z: X \to S$

$$\begin{tikzcd} U \cap V \arrow[d, "{(j,i)e}"'] \arrow[dr, bend left] \arrow[dd, bend right=60] & & \\ U \underset{S}{\times} V \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & V \arrow[d, "z_j"] \arrow[r, bend left, "j"] & X \arrow[dl, "z"] \\ U \arrow[r, "z_i"'] & S & \end{tikzcd} \qquad U \underset{S}{\times} V \overset{i\pi_1}{\underset{j\pi_2}{\rightrightarrows}} X$$

Exercise: Prove $U \cap V \xrightarrow{e} U \underset{S}{\times} V \overset{i\pi_1}{\underset{j\pi_2}{\rightrightarrows}} X$ equaliser in (Sch),

which implies e is closed embedding.

$$\left(\hookrightarrow \text{from} \quad \begin{tikzcd} & X \arrow[d, "{(f,g)}"] \\ Y \arrow[r, "\Delta"'] & Y^2 \end{tikzcd} \right)$$