Tikhomirov

- conventions. M smooth mfd, $TM = \bigcup TM_x$ tgt bdl $\ni v : C^\infty(M) \to \mathbb{R}$, $f \mapsto v(f)$ satisfying Leibniz

- we are given X top.sp (paracpt, e.g. cpt), B top.sp., G Lie gp (structure gp of pbdl)

Def. A top.sp. P w/ surj cont. map $\pi : P \to B$, and a cont. map $P \times G \to P$ called right G-action, $(p,g) \mapsto pg$ s.t. $(pg_1)g_2 = p(g_1 g_2)$ satisfying the following properties:

i) π is G-invariant, i.e. $\pi(pg) = \pi(p)$ $\forall g \in G$
ii) G-action is free: $pg = p \Rightarrow g = id_G$

Further, $\exists$ open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of B and

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}_\alpha) \arrow[r, "\varphi_\alpha", "\text{homeo}"'] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{U}_\alpha \times G \arrow[d, "pr_1"] \\ \mathcal{U}_\alpha \arrow[r, equal] & \mathcal{U}_\alpha \end{tikzcd}$$

$\to$ transitivity follows from $(x,g)h := (x,gh)$ on $\mathcal{U}_\alpha \times G$.

G-equivariant homeomorphisms

- B, P smooth mfds, π should be smooth submersion so $\pi^{-1}(b)$ will be smooth mfds $\forall b \in B$
- recall, $f : X \to Y$ is submersion if smooth and $f_*|_{TX_x} : TX_x \to TY_{f(x)}$ is surj $\forall x \in X$, where $f_* : TX \xrightarrow{df} TY$, $\tau \mapsto f_*(\tau)$, $f_*(\tau)(\varphi) = \tau(\varphi \circ f)$.

Examples

- B sm. mfd, $\dim B = n$,

$$p \in P = \{ (\underbrace{b, v_1, \dots, v_n}_{= p}) \mid b \in B,\ v_1, \dots, v_n \text{ basis of } TB_b \}$$

$$\begin{tikzcd} p \arrow[d, mapsto] \\ b \in B \end{tikzcd}$$

- $G = GL(n,\mathbb{R})$-action as $pA = (b, w_1, \dots, w_n)$
 where $w_i = v_j A_{ji}$
- P is called the frame bdl

- $\mathbb{C}P^n = \mathbb{P}(V^{n+1}) = B$,
 $(z_1, \dots, z_{n+1}) \in V^{n+1}$

$$\begin{tikzcd} V^{n+1} \arrow[dr, "\text{taut.}"'] & \supset_{\text{restr.}} S^{2n+1} = \{ (z_1, \dots, z_{n+1}) \mid \sum |z_i|^2 = 1 \} \arrow[d, "\pi"] \\ & \mathbb{C}P^n \end{tikzcd}$$

$G = U(1) = \{ z \mapsto z e^{i\varphi} \mid \varphi \in [0, 2\pi) \}$

So set $P = S^{2n+1}$. Hopf bdl

- transition funcs for pbdl $P \to B$, $\{\mathcal{U}_\alpha\}$
 are $(\varphi_\alpha|_{P_{\alpha\beta}}) \circ (\varphi_\beta|_{P_{\alpha\beta}})^{-1} =: g_{\alpha\beta}$

 where

$$\begin{tikzcd} \mathcal{U}_{\alpha\beta} \times G \arrow[rr, bend right, "g_{\alpha\beta}", "\sim"'] & \pi^{-1}(\mathcal{U}_{\alpha\beta}) = P_{\alpha\beta} \arrow[l, "\varphi_\beta|_{P_{\alpha\beta}}"', "\sim"] \arrow[r, "\varphi_\alpha|_{P_{\alpha\beta}}", "\sim"'] & \mathcal{U}_{\alpha\beta} \times G \end{tikzcd}$$

$g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to G$, equivariance saves us from dependence of $- \times G$.
- cocycle condition on $\mathcal{U}_{\alpha\beta\gamma}$, $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{id}_{\mathcal{U}_{\alpha\beta\gamma}}$
- classified by $H^1(B, G)$

$$\begin{tikzcd} P \arrow[r, "\pi"] & B \\ f^*P \arrow[u, "\tilde{f}"] \arrow[r, "\tilde{\pi}"'] & A \arrow[u, "f"'] \end{tikzcd}$$

G-pbdl

$$f^*P = P \times_B A = \{(p,a) \in P \times A \mid \pi(p) = f(a)\}$$

pullback bdl

- $P \xrightarrow{\pi} B$ G-pbdl, F top sp with G-left action
  $G \times F \to F$, $(g,v) \mapsto gv$

- define $P \times_G F = P \times F/\sim$,
  $(p,v) \sim (pg, g^{-1}v)$
  so $P \times_G F$, $[p,v]$ pbdl w fiber F

$$\begin{tikzcd} P \times_G F \arrow[d, "\lambda"', two heads] & {[p,v]} \arrow[d, mapsto] \\ B & \pi(b) \end{tikzcd}$$

associated bdl

- e.g. $F = V$ vsp., $S: G \to \mathrm{Aut}\, V$ exact

  $\mathcal{V} := P \times_G V \xrightarrow{\lambda} B$ is
  G-vbdl assoc. to P

  - taking its frame bdl gives back P!
  - denote it by $P(\mathcal{V})$

Universal bdls.

- $G = GL(n, \mathbb{R})$, $Gr(n, n+k)$ Grassmannian of n-dim subspaces in $\mathbb{R}^{n+k}$, smooth, $\dim = nk$

- we have
$$\ldots \subset \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1} \subset \ldots \subset \bigcup_{k \geq 0} \mathbb{R}^{n+k} =: \mathbb{R}^{\infty}$$
$$(x_1, \ldots, x_{n+k}) \mapsto (x_1, \ldots, x_{n+k}, 0)$$
inducing
$$\ldots \subset Gr(n, n+k) \subset Gr(n, n+k+1) \subset \ldots$$
$$\ldots \subset \bigcup_{k \geq 0} Gr(n, n+k) =: Gr(n, \infty)$$
as inductive or direct limit

- now, for every k $\exists$ tautological vbdl

$$\begin{tikzcd} \tau(n, n+k) \arrow[d] & V \arrow[l, hook'] \arrow[d] \\ Gr(n, n+k) & \{V\} \arrow[l, phantom, "\ni"] \end{tikzcd}$$
$, \dim V = n, V \subset \mathbb{R}^{n+k}$

$\mathrm{rk}\, \tau(n, n+k) = n$

- so we also get
$$\begin{tikzcd} \tau(n) \arrow[d] \\ Gr(n, \infty) \end{tikzcd}$$

such that $\tau(n)|_{Gr(n, n+k)} = \tau(n, n+k)$

- B paracpt w countable covers $\{\mathcal{U}_\alpha\}_{\alpha\in\mathbb{Z}_+}$
in particular cpt., $\exists$ part of 1 $\{\lambda_\alpha\}$ subord to $\{\mathcal{U}_\alpha\}$

$V \xrightarrow{f} B$ vbdl of rk n, $\varphi_\alpha : V|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{R}^n$

$$\begin{tikzcd} \mathcal{U}_\alpha \times \mathbb{R}^n \arrow[d, "pr_2"] \\ \mathbb{R}^n \end{tikzcd}$$

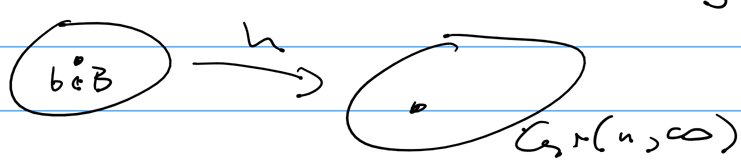
$$\mu_\alpha := pr_2 \circ (\lambda_\alpha \varphi_\alpha) : V|_{\mathcal{U}_\alpha} \longrightarrow \mathbb{R}^n =: \mathbb{R}^n_\alpha$$

$$\mu : V \longrightarrow \bigoplus_{\alpha\in\mathbb{Z}_+} \mathbb{R}^n_\alpha, \quad \mu := \sum_\alpha \mu_\alpha$$

$\exists\, h : B \to Gr(n,\infty)$ s.t.

$$\begin{tikzcd} \underline{\underline{V}} = h^*\tau(n) \arrow[r] \arrow[d, "f"'] & \tau(n) \arrow[d] \\ B \arrow[r, "h"] & Gr(n,\infty) \end{tikzcd}$$

$V_b \longrightarrow \tau(n)$, via μ only finitely many coordinates $\neq 0$.



$$\begin{array}{l} P = P(v) \text{ frame pbdl of } v \\ \downarrow \pi \\ B \qquad \longrightarrow \text{then } \exists h \text{ s.t. } P = h^* P(\tau(n)) \text{ ?} \end{array}$$

- let $I = [0,1] \subseteq \mathbb{R}$, we naively get $\begin{array}{c} P \times I \\ \downarrow \pi \times id \\ B \times I \end{array}$

Prop For B paracpt., $\exists$ an isom. of G-bdls
$$\begin{array}{c} P \\ \downarrow \pi \\ B \times I \end{array} \quad \text{and} \quad \begin{array}{c} (P|_{B \times \{1\}}) \times I \\ \downarrow \\ B \times I \end{array}$$

Cor $P|_{B \times \{1\}} \simeq P|_{B \times \{0\}}$

- $B \xrightarrow{f_1, f_0} Gr(n, \infty)$ homotopic maps via $B \times I \to Gr(n, \infty)$
  - then $f_1^* P(\tau(n)) \simeq f_0^* P(\tau(n))$

- so, homotopy classes of $[B, Gr(n, \infty)]$
$$\downarrow$$
isom. classes of G-pbdls.
where $G = GL(n, \mathbb{R})$

- in fact, for B paracpt w count. basis, it injects.

Tikhomirov

- M mfd, $\Omega^k(M) = \Gamma(M, \Lambda^k T^* M)$ space of diff k-forms on M
- E vbdl on M, $\Omega^k(M, E) := \Gamma(M, \Lambda^k T^* M \otimes E)$ sp. of E-valued k forms
- $\Gamma(TM) \otimes \Omega^1(E) \xrightarrow{ev} \Gamma(E)$

$$\begin{matrix} \Gamma(TM) & \otimes & \Omega^1(E) & \xrightarrow{ev} & \Gamma(E) \\ \cup & & \cup & & \\ X & \otimes & s & \longmapsto & s(X) \end{matrix}$$

- note that any $f : \Gamma(TM) \to \Gamma(E)$ satisfying $f(hX) = h f(X)$ for any $h \in C^\infty(M)$ can be realised using an evaluation

---

- G Lie gp, $\mathfrak{g}$ Lie alg. of G, $\mathfrak{g} = TG_{id}$
- def $L_g, R_g : G \circlearrowleft$ by $h \mapsto gh$, $h \mapsto hg$ $\forall g \in G$ giving us $(L_g)_*, (R_g)_* : TG \circlearrowleft$, $\underset{TG_h}{\tau_h} \mapsto \underset{TG_{gh}}{g_* \tau_h}$, ...
- define $Ad(g) : \underset{TG_{id}}{\tau_{id}} \mapsto \underset{TG_{id}}{g_* \tau_{id} g_*^{-1}}$

$\Rightarrow Ad(g) : \mathfrak{g} \circlearrowleft$, $Ad : G \to Aut\, G$

- $\exists$ distinguished $\omega \in \Omega^1(G, \underline{\mathfrak{g}})$, Maurer-Cartan where by underline we mean $\underline{V} = M \times V$ if V vsp
- $\underset{TG_h}{\omega(\tau_h)} := \underset{TG_{\underline{id}}}{h_*^{-1} \tau_h}$

$$(L_g^* \omega)(\tau_h) = \omega(L_{g*} \tau_h) = \omega(\underset{TG_{gh}}{g_* \tau_h})$$
$$= (gh)_*^{-1} g_* \tau_h = h_*^{-1} \tau_h = \omega(\tau_h).$$

so ω left invariant

$$(R_g^* \omega)(\tau_h) = \omega(\tau_h g_*) = (hg)_*^{-1} \tau_h g_* = g_*^{-1} h_*^{-1} \tau_h g_*$$
$$= (Ad(g^{-1}) \omega)(\tau_h)$$

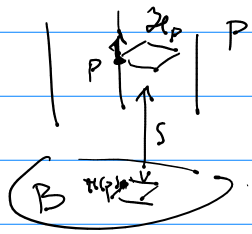
so $R_g^* \omega = Ad(g^{-1}) \omega$.

Connections on smooth pbdls

$$\begin{array}{c} P \\ \downarrow\pi \\ B \end{array}\qquad G\text{-pbdl},\ \dim B = n$$

Def. A connection on P is a smooth family
$$\{\mathcal{H}_p \subset TP_p \mid p \in P,\ (R_g)_*\mathcal{H}_p = \mathcal{H}_{pg}\}$$
s.t. $\pi_* : \mathcal{H}_p \xrightarrow{\sim} TB_{\pi(p)}$.

- since π submersion, $\pi_* : TP_p \to TB_{\pi(p)}$ surjects:
$$0 \to TP^v \to TP \to \pi^* TB \to 0 \qquad (1)$$
where $TP \supset TP^v := \{ v \in TP_p \mid p \in T,\ v \text{ is tang. to } \pi^{-1}(\pi(p)) \}$



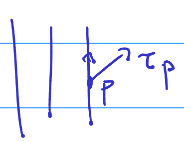
- $\underline{\mathcal{H}} = \{\mathcal{H}_p\}$ yields the splitting of (1)

Prop The existence of a connection A (on a smooth G-pbdl $P \xrightarrow{\pi} B$) is equivalent to the existence of a 1-form $\omega_A \in \Omega^1(P, \mathfrak{g})$ satisfying

i) $\omega_A(\tau_h g^*) = Ad(g^{-1}) \circ \omega_A(\tau_h)$ $\qquad G$ is

ii) $\forall\, b \in B$ $\quad (\varphi_{\alpha, b}^{-1})^* \omega_A = \omega$, where $\varphi_{\alpha,b} := \varphi_\alpha|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \{b\} \times G$

Pf. Let ω_A satisfy i) $\wedge$ ii), $\omega_A : \tau \mapsto a \in \mathfrak{g}$.



For τ_p vertical, $\omega_A|_{TP^v_p} : TP^v_p \xrightarrow{\sim} \mathfrak{g}$, so $TP_p = TP^v_p \oplus \mathcal{H}_p$, where $\mathcal{H}_p := \ker(\omega_A|_{TP_p})$.

Equivariance follows from $\mathcal{H}_{pg} = \ker(\omega_A|_{TP_{pg}}) \overset{(1)}{=} \mathcal{H}_p g_*$.

Conversely, given $A = \{\mathcal{H}_p\}_{p \in P}$, define $\omega_A \in \Omega^1(P, \mathfrak{g})$ by

$$\omega_A : TP_p \xrightarrow[pr_{\mathcal{H}_p}]{} TP_p^v \xrightarrow[(\varphi_\alpha, \pi(p))^{-1}_*]{} TG \xrightarrow[\omega]{\sim} \mathfrak{g}$$

$\hookrightarrow$ proj. w/ kernel $\mathcal{H}_p$

- existence?
- note that it exists for trivial
  $P = B \times G \xrightarrow[\pi = pr_1]{} B$, $A = \{\mathcal{H}_p = T_{pr_2^{-1}(pr_2(p))}\}$
  - so take $\varphi_\alpha : P|_{\mathcal{U}_\alpha} \xrightarrow{\sim} \mathcal{U}_\alpha \times G$ local trivs and $\{\lambda_\alpha\}$ part. of unity subordinate to a cover $\{\mathcal{U}_\alpha\}$.
  - then $\omega = \sum_\alpha \lambda_\alpha \omega_\alpha$, where ω_α prod. conn's

~

- notice that $\vartheta = \omega_A - \omega_{A'}$ restricts to 0 along fibers & satisfies $\vartheta(\tau g_*) = Ad(g^{-1})\vartheta(\tau)$
so look at $\Omega^1(P, \mathfrak{g})$
$\cup$
$\Omega^1_{Ad}(P, \mathfrak{g}) = \{\vartheta \in \Omega^1(P, \mathfrak{g}) \mid \vartheta \text{ satisfies above}\}$

Rmk The set of connections on P is an affine space over $\Omega^1_{Ad}(P, \mathfrak{g})$

- now let $S = Ad : G \to \mathrm{Aut}\,\mathfrak{g}$, look at

$$P \times_S G = P \times_{Ad} G =: \mathrm{ad}\, P$$

$$\overset{\Psi}{[(p,v)]} = (p,v) \bmod \sim \;\; | \;\; (p,v) \sim (pg, Ad(g^{-1})v)$$

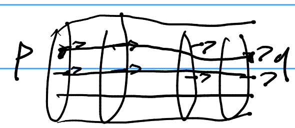
adjoint vbdl

- claim: $\Omega^1_{Ad}(P, \mathfrak{g}) \xrightarrow{\sim} \Omega^1(B, \mathrm{ad}\,P)$

- note that connections survive pullbacks, just pull the 1-form back

$$\begin{array}{ccc} P & \hookleftarrow & f^*P \\ \downarrow & & \downarrow \\ B & \xleftarrow{f} & A \end{array}$$

- take $A = [0,1] \overset{f}{\hookrightarrow} B$



$\frac{\partial}{\partial t} \longrightarrow \widetilde{\frac{\partial}{\partial t}}$

so we get, integrating along $\widetilde{\frac{\partial}{\partial t}}$ g-equiv maps $p \mapsto q$, $\tilde{\pi}^{-1}(0) \to \tilde{\pi}^{-1}(1)$

Tikhomirov

- given $P \xrightarrow{\pi} B$ G-pbdl, V vsp. s.t. $G \xrightarrow{S} \mathrm{Aut}\,V$ rep., build assoc. vbdl $W = P \times_S V$.
- what is its connection?
- recall on pbdl, $A = \{\mathcal{H}_p\}$, $\omega_A \in \Omega^1(P, \mathfrak{g})$, $\mathcal{H}_p = \ker(\omega_{A,p} : TP_p \twoheadrightarrow \mathfrak{g})$
- on $W = P \times_S V$ we build a diff. operator

$$\nabla_A : \Omega^0(B, W) \longrightarrow \Omega^1(B, W)$$
$$s \longmapsto \nabla_A s, \quad (\nabla_A s)(\overset{TB}{\overset{\in}{\tau}}) \in W|_B$$

by working locally (since diff. op. is local)

- let $s = \{[p(b), v(b)] \mid b \in \mathcal{U} \subseteq B\}$ where $p : \mathcal{U} \to \pi^{-1}(\mathcal{U})$, $v : \mathcal{U} \to V$

(don't take $\mathcal{U} = B$ unless P trivial!!)

and let

$$(\nabla_A s)(\tau_b) := [p(b), S_*(\omega_A(p_* \tau_b))(v(b)) + \overbrace{\tau_b(v(b))}^{\frac{\partial v(b)}{\partial \tau_b}}]$$

where $S_* : \mathfrak{g} \to \mathrm{Gl}(V) = \mathrm{End}\, V$

- covariant differential

- for $f \in \Omega^0(B)$, $s \in \Omega^0(B, W)$ we check $\nabla_A(fs) = f \nabla_A s + df \otimes s$, Leibnitz rule
- extend by multilinearity to $\nabla_S : \Omega^k(B, W) \to \Omega^{k+1}(B, W)$ by taking $\eta \in \Omega^r(B)$, $\vartheta \in \Omega^{k-r}(B, W)$ and letting

$$\nabla_A(\eta \wedge \vartheta) = d\eta \wedge \vartheta + (-)^r \eta \wedge (\nabla_S \vartheta)$$

- we get $\Omega^0(B,W) \xrightarrow{\nabla_A} \Omega^1(B,W) \xrightarrow{\nabla_A} \Omega^2(B,W) \xrightarrow{\nabla_A} \dots$
but this is no cpx ?

$$\nabla_A^2(fs) = \nabla_A(f\nabla_A s + df\otimes s)$$
$$= f\cdot\nabla_A^2 s + df\otimes\nabla_A s + d^2f\otimes s - df\otimes\nabla_A s$$
$$= f\cdot\nabla_A^2 s$$

so we define tensor $\nabla_A^2 s = s\otimes F_A \xrightarrow{ev} \Omega^2(B,W)$
where F_A is called curvature of ∇_A,
$F_A \in \Omega^2(B, \mathrm{End}\, W)$

~

- let N fixed mfd, $\varphi\in\Omega^i(N,\mathfrak{g})$, $\psi\in\Omega^j(N,\mathfrak{g})$
- using Lie brckt $[\,,\,]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ we build $[\varphi,\psi]\in\Omega^{i+j}(N,\mathfrak{g})$ s.t.

$$[\varphi,\psi](X_1,\dots,X_{i+j}) := \frac{1}{i!\,j!}\sum_{\sigma\in S_{i+j}} [\varphi(X_{\sigma(1)},\dots,X_{\sigma(i)}),\ \psi(X_{\sigma(i+1)},\dots,X_{\sigma(i+j)})]$$

- properties:
 i) $[\psi,\varphi] = -(-)^{i\cdot j}[\varphi,\psi]$

 ii) for $\vartheta\in\Omega^k(N,\mathfrak{g})$,

$$(-)^{ik}[[\varphi,\psi],\vartheta] + (-)^{kj}[[\vartheta,\varphi],\psi] + (-)^{j\cdot i}[[\psi,\vartheta],\varphi] = 0$$

- in particular, $[[\omega,\omega],\omega]=0$

- for particular uses we take, e.g. $\mathrm{Aut}\,V = SO(2)$
  so we're interested in matrix groups
  - always think like this
- so, $\psi \wedge \varphi$ = combination of matrix product
  and wedge product
  - in particular, $[\psi, \varphi] = \psi \wedge \varphi - (-)^{i \cdot j} \varphi \wedge \psi$

Curvature of conn. on a pbdl.

Def. For $\varphi \in \Omega^k(P, \mathfrak{g})$ and connection A,
define $(D_A \varphi)(\tau_1, \dots, \tau_{k+1}) := d\varphi(\tau_1^H, \dots, \tau_{k+1}^H)$,
where $\tau = \tau^H + \tau^V \in TP$ is the splitting.

The curvature 2-form is $\Omega_A := D_A \omega_A \in \Omega^2(P, \mathfrak{g})$.

Prop $\Omega_A = d\omega_A + \frac{1}{2}[\omega_A, \omega_A]$ $(*)$

- first recall some things. For $A \in \mathfrak{g}$,
  $A^* \in \Omega^0(TP)$ given by $A^* = \frac{\partial}{\partial t} p \exp(tA)|_{t=0}$
  satisfies $[A^*, B^*] = [A, B]^*$ (fundamental field)
  - further, for $X \in \Omega^0(TU)$ denote by $\tilde{X} \in \Omega^0(T\pi^{-1}U)$
    its horizontal lift, $R_{g*}\tilde{X} = \tilde{X}$
    - then $[A^*, \tilde{X}] = 0$

Pf. $1°$ τ_1, τ_2 horizontal. By def, $\Omega(\tau_1, \tau_2) = d\omega(\tau_1, \tau_2)$
and $\frac{1}{2}[\omega, \omega](\tau_1, \tau_2) \equiv 0$.
$11°$ τ_1 vert, τ_2 horiz.
$\exists A^*$ s.t. $A^*_p = \tau_1$, $\exists \tilde{X}$ for some X s.t.
$\tilde{X}_p = \tau_2$.

Then $\Omega(\tau_1,\tau_2)=0$. Also,

$$d\omega(\tau_1,\tau_2) = \underbrace{A^*(\underbrace{\omega(\tilde{X})}_{=0\text{ by horizontality}})_p}_{=0} - \tilde{X}\overbrace{(\overbrace{\omega(A^*)}^{=A})_p}^{=0} - \omega(\overbrace{[A^*,\tilde{X}]}^{=0})_p$$

and $[\omega,\omega](\tau_1,\tau_2)=0$

III° τ_1,τ_2 vertical.

Now by def $\Omega(\tau_1,\tau_2) = d\omega(0,0)=0$,

and $d\omega(\tau_1,\tau_2) = A^* B - B^* A - [A,B]$,

and $\frac{1}{2}[\omega,\omega](A^*,B^*) = [A,B]$. $\square$

- given $\{\mathcal{U}_\alpha\}$, we get sections $\mathcal{U}_\alpha \xhookrightarrow{\sigma_\alpha} P|_{\mathcal{U}_\alpha}$ from local trivialisations $\varphi_\alpha : P|_{\mathcal{U}_\alpha} \xrightarrow{\sim} \mathcal{U}_\alpha \times G \hookleftarrow \mathcal{U}_\alpha \times \{1\} \simeq \mathcal{U}_\alpha$

- we have also $\omega_\alpha = \sigma_\alpha^* \omega$, $\Omega_\alpha = \sigma_\alpha^* \Omega$, forms on $\mathcal{U}_\alpha$.
- what happens on overlaps $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$?
  - with local trivs, $\varphi_\alpha = g_{\alpha\beta}\varphi_\beta$, while for $\{\omega_\alpha\},\{\Omega_\alpha\}$

$$\omega_\beta = g_{\alpha\beta}^{-1}\, d g_{\alpha\beta} + g_{\alpha\beta}^{-1}\,\omega_\alpha\, g_{\alpha\beta}$$

$$\Omega_\beta = g_{\alpha\beta}^{-1}\,\Omega_\alpha\, g_{\alpha\beta} \implies F_A = \{\Omega_\alpha\} \in \Omega^2(\mathrm{ad}\, P)$$

- we have $\pi^* F_A = \Omega$.

- it can be shown that for $\varphi \in \Omega^k(P, \mathfrak{g})$ s.t.
$\varphi(\tau_1 g_*, \ldots, \tau_k g_*) = g^{-1}\varphi(\tau_1, \ldots, \tau_k)\, g$
and $\varphi(\tau_1, \ldots, \tau_k) = 0$ if τ_1 vertical,
then $D_A \varphi = d\varphi + [\omega_A, \varphi]$

Prop $D_A \Omega = d\Omega + [\omega, \Omega] = 0$. (Bianchi id)

$D_A F_A = 0$.

- let G matrix gp

- notice $\underbrace{tr(F_A \wedge \ldots \wedge F_A)}_{\in\, \Omega^{2k}(B, \mathbb{R})}$ (k times) $= tr(g^{-1}F_A g \wedge \ldots \wedge g^{-1}F_A g)$

- also $d\, tr(F_A \wedge \ldots) = tr(D_A F_A \wedge \ldots) = 0$

- further, if A' another conn, the difference of these traces will be exact

- so $[tr\, F_A \wedge \ldots \wedge F_A] \in H^{2k}(B, \mathbb{R})$

Тихомиров

Sobolev completions

- B cpt mfd, $\dim B = n$, (B,g) Riemannian
- $W \to B$ vbdl, A any conn'n on W

$$s \in \Omega^k(B,W) = \Gamma(B, \Lambda^k T B^* \otimes W)$$

conn'n A

conn'n A_{LC}, Levi-Civita connection

- $\nabla(f \otimes g) = \nabla_A f \otimes g + f \otimes \nabla_{A'} g$ is extension
- we also want a scalar product (any pos.def.) on W, so we may talk abt $|s(x)| = \sqrt{\langle s(x), s(x)\rangle}$
- on $\Omega^\ell(B,W)$ define norms

$$\|s\|_{L^p_k} := \left[\int_B \left(|s(x)|^p + |\nabla_A(s)(x)|^p + \dots + |(\nabla_A)^k(s)(x)|^p\right) dVol\right]^{\frac{1}{p}}$$

with $p \geq 1$, $k \geq 0$

- this is a norm on $\Omega^\ell(B,W)$
- we denote by $L^p_k(\Omega^\ell(B,W))$ the completion of $\Omega^\ell(B,W)$ wrt $\|\cdot\|_{L^p_k}$
- notation: $L^p_k(W) := L^p_k(\Omega^0(W))$
- let $w(p,k) := k - \frac{n}{p}$
- let $W, W' \to B$ 2 vbdls
- clearly $\Omega^0(W) \otimes \Omega^0(W') \xrightarrow{\text{mult.}} \Omega^0(W \otimes W')$

Thm (Sobolev multiplication thm)
$L^p_k(W) \otimes L^{p'}_{k'}(W') \xrightarrow{\text{mult.}} L^q(W \otimes W')$, if $w(p,k) + w(p',k') > w(q,r)$, $\min(k,k') \geq r$, is cont. (bounded) operation

Thm (Sobolev embedding thm)
$L^p_k(W) \xrightarrow{i} L^{p'}_{k'}(W)$ is bounded if $w(k,p) > w(k',p')$, $k \geq k'$.
Moreover, if $k > k'$, i is compact.

- letting $\mathcal{C}^r(W)$ = space of all sectn's of W r times differentable, we get $L^p_k(W) \xrightarrow[\text{cpt.}]{} C^r(W)$ if $w(k,p) \geq r$.

$\sim$

- armed with this ...
- recall $\mathcal{A}(P)$ = aff. space over $\Omega^1(\mathrm{ad}P)$, where $P \xrightarrow{\pi} B$ G-pbdl, G = cpt mtrx gp, B cpt mfd

- def. $\mathcal{A}_{L^p_k}(P) = \{A + a \mid a \in L^p_k(\Omega^1(\mathrm{ad}P))\}$

- Banach mfd. (with 1 chart ?)

- recall $\mathrm{Ad}P = P \times_{\mathrm{Ad}} G$,
$\mathcal{G}(P) = \Gamma(\mathrm{Ad}P)$ = gp of gauge transformation
↳ terminology note: G is called the "gauge gp", $\mathcal{G}(P)$ gp of gauge tr's

- recall, $\Gamma(\mathrm{Ad}\,P) \ni s : P \longrightarrow G$, $s(pg) = g^{-1} s(p) g$ and for $G = SU(2)$, $Z(SU(2)) = \{\pm 1\}$,

$$\mathrm{Ad}\,P \;\substack{\xleftarrow{\{-1\}} \\ \longrightarrow \\ \xleftarrow{\{1\}}}\; B$$

- want to extend to Banach (mfd) Lie gp $\mathcal{G}_{L^p_k}(P) = L^p_k(\mathrm{Ad}\,P)$

- now, we Rmk. this is Hilbert sp. if $p=2$
- however, we are interested in $k=2$, $p>2$, $n=4$ (in $p=2$, $\mathcal{G}_{L^2_2}$ not compact?) 1st good case.
- notice $L^3_2(W) \xhookrightarrow{\text{S.E.Th.}} C^0(W)$
- so $\mathcal{G}_{L^3_2}(P)$ will be gp under mult. of (cont.) sectn's

- let $T\mathcal{G}_{L^3_2}(P) := L^3_2(\mathrm{ad}\,P)$ Lie alg. w/ fibrewise bracket

- $\exp : L^3_2(\mathrm{ad}\,P) \longrightarrow \mathcal{G}_{L^3_2}(P)$ defined fibrewise is bijection in nbhd of $\{1\}$-section

- we stop writing k, e.g. $\mathcal{A}_{L^2_2}(P) := \mathcal{A}_2(P)$

- action $\mathcal{A}_2(P) \times \mathcal{G}_3(P) \to \mathcal{A}_2(P)$
  - check: $w(2,2) > 0$ $\quad w(3,2) > 0 \quad w(2,2) > 0$
    - this works, and even is smooth (won't prove)

- note in passing that $E \xrightarrow{f} F$, E, F Banach, f is called diff. at $x \in E$ if $\exists$ bounded lin op $d_x f : E \to F$ s.t. $\lim_{\|h\| \to 0} \frac{\| f(x+h) - f(x) - (d_x f)(h) \|}{\|h\|} = 0$

- $\mathcal{A}_2(P) \twoheadrightarrow \mathcal{A}_2(P)/\mathcal{G}_3(P) =: \mathcal{B}(P)$,

moduli sp. of conn's

- q: is $\mathcal{B}(P)$ Hausdorff?
  - if $\{A_n\}, \{B_n\} \in \mathcal{A}_2(P)$ are s.t.

  i) $\exists\, g_n \in \mathcal{G}_3(P)$ with $g_n^* A_n = B_n$,

  ii) $A_n \to A$, $B_n \to B$, $A, B \in \mathcal{A}_2(P)$ as $n \to \infty$

  iii) $\exists\, g = \lim_{n\to\infty} g_n \in \mathcal{G}(P)$

  then $\mathcal{B}(P)$ Hsdff if $g^* A = B$

- take $A_n = A + a_n$, $B_n = A + b_n$, $a_n, b_n \in L^2_2$

$$\Rightarrow B_n = A + b_n = g_n^{-1} d g_n + g_n^{-1}(A + a_n) g_n$$

$$\Rightarrow \underbrace{d g_n - g_n A + A g_n}_{\nabla_A(g_n) = d g_n + [A, g_n]} = g_n b_n - a_n g_n$$

Шхомиров

- $\begin{matrix} P \\ \downarrow \pi \\ B \end{matrix}$, $G = SU(2)$, $g \in \mathcal{G} = \Gamma(\mathrm{Ad}\, P)$ gauge tr. gp.

- $\mathcal{A}(P) =: \mathcal{A}$ sp. of conn. on P, affine over $\Omega^1(\mathrm{ad}\, P)$
  $\mathcal{A} \times \mathcal{G} \longrightarrow \mathcal{A}$, $(A, g) \mapsto g^* A$
- $\mathcal{A}_{L^p_k}$, $p = 2$, $k = l - 1$, $l \geq 3$

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathcal{A}/\mathcal{G} =: \mathcal{B} \\ A &\mapsto A \bmod \mathcal{G} =: [A] \end{aligned}$$

- elaboration on gp action,

$$\begin{tikzcd} \mathrm{Ad}\, P \arrow[r, hook, "\mathrm{cpt}", "\mathrm{subbundle}"'] \arrow[d, "\mathrm{fibre}\, G"] & P \times_{G, \mathrm{ad}} M_{2\times 2}(\mathbb{R}) =: \xi \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

commutes. We have $(A, B) := -\mathrm{tr}(A \cdot \bar{B})$ inner prod.,
$\underline{G} \hookrightarrow M_{2\times 2}(\mathbb{C}) \subset M_{4\times 4}(\mathbb{R})$,

$$\mathcal{G}_{L^p_k} \hookrightarrow L^p_k(\Gamma(\xi))$$

- $\mathcal{A}_{L^p_k} \times \mathcal{G}_{L^p_{k+1}} \longrightarrow \mathcal{A}_{L^p_k}$ smooth action

- $\mathcal{A}_{L^p_k} / \mathcal{G}_{L^p_{k+1}} = \mathcal{B}_{L^p_k}$

- $\mathcal{A}_{l-1} / \mathcal{G}_l = \mathcal{B}_{l-1}$

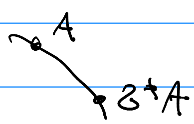
Thm $\mathcal{B}_{l-1}$, $l \geq 3$ is Hsdff iff
$\Gamma_{\mathcal{G}_l} := \{(A, B) \mid A = g^* B,\ g \in \mathcal{G}_l\} \overset{\text{closed}}{\subset} \mathcal{A}_{l-1} \times \mathcal{A}_{l-1}$.

- didn't write pf. down

- recall that every pt in $\mathcal{A}_2$ has stabilisers $\mathbb{Z}_2$, $U(1)$ or entire $G \subset SU(2)$
- we call A with $\mathrm{Stab}\,A = U(1)$ reducible
  - $\xi = P \times \mathbb{C}^2$ splits, $\xi = L \oplus L'$
  - if $\mathrm{Stab}\,A = \mathbb{Z}_2$, called irreducible.
- $\mathcal{A}^* := \{ A \in \mathcal{A} \mid A \text{ irred.} \} \xhookrightarrow{\text{open}} \mathcal{A}$

Thm (on slices) Consider action $\mathcal{A}_2 \times \mathcal{G}_3 \to \mathcal{A}_2$, let $A \in \mathcal{A}_2$. $\exists$ a $\mathrm{Stab}\,A$-invariant subset $\mathcal{U} \subset \mathcal{A}_2$ s.t. $A \in \mathcal{U}$ and

$$\mathcal{U} \times_{\mathrm{Stab}\,A} \mathcal{G}_3 \xrightarrow[\text{diff. onto its image}]{\cong} V \xhookrightarrow{\text{open}} \mathcal{A}.$$

- now, we have inn. prod $(-,-)$ on $L^2_3(\Omega^0(\mathrm{ad}P))$ and $L^2_2(\Omega^1(\mathrm{ad}P))$
- look at 

, $g \in \tilde{\mathcal{G}} := \mathcal{G}/\mathbb{Z}_2$

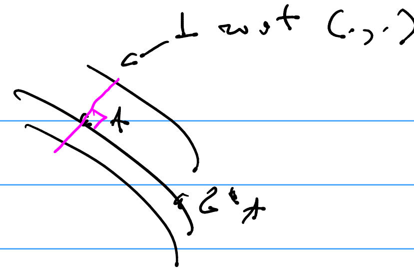
$f : \mathcal{G}_3 \to \mathcal{A}_2$, $g \mapsto g^* A$

- the derivative of $A \mapsto g^* A$ is

$$\nabla_A : \underbrace{L^2_3(\Omega^0(\mathrm{ad}P))}_{= T_{id}\mathcal{G}_3} \to \underbrace{L^2_2(\Omega^1(\mathrm{ad}P))}_{= T_A \mathcal{A}_2} = (f_*)_{id}$$

$\Rightarrow$ if $\mathrm{Stab}\,A = \mathbb{Z}_2$, $\ker \nabla_A = \{0\} = T_{id}\,\mathrm{Stab}\,A$

- now, we want to look at



- $\chi_A := \{ A + a \mid (a, u) = 0 \quad \forall u \in \operatorname{Im}(\nabla_A)$ and
$u = \nabla_A(v)$ where $v \in L^2_3(\Omega^0(\operatorname{ad} P))\}$
$= \{ A + a \mid (\nabla_A^* a, v) = 0 \}$
$= \{ A + a \mid \nabla_A^* a = 0 \}$
<u>Coulomb gauge</u>

by nondegeneracy of $(\cdot, \cdot)$,
and $\nabla_A^* : L^2_2(\Omega^1(\operatorname{ad} P)) \longrightarrow L^2_1(\Omega^0(\operatorname{ad} P))$.

Шилзоматиров

- $\nabla_A : L^2_3(\Omega^0(\mathrm{ad}P)) \longrightarrow L^2_2(\Omega^1(\mathrm{ad}P))$
  $\nabla_A^* : L^2_2(\Omega^1(\mathrm{ad}P)) \longrightarrow L^2_1(\Omega^0(\mathrm{ad}P))$
  $\Delta_A := \nabla^*_A \circ \nabla_A : L^2_3(\Omega^0(\mathrm{ad}P)) \longrightarrow L^2_1(\Omega^0(\mathrm{ad}P))$
  is elliptic.
- $A \in \mathcal{A}_2^*$ ($\mathrm{Stab}\, A = \mathbb{Z}_2$), $f : \mathcal{G}_3 \to \mathcal{A}_2, g \mapsto g^* A$
$\Rightarrow (f_*)_{id} = \nabla_A$
- $\ker \nabla_A = \{0\} \Rightarrow \ker \Delta_A = \{0\}$
- further, $\begin{cases} \mathrm{coker}\, \nabla_A^* = \ker \nabla_A & \text{means} \\ \mathrm{coker}\, \nabla_A = \ker \nabla_A^* \end{cases}$

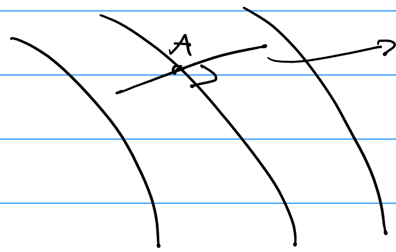
  Δ_A bijects
  - not that surprising ... L^2_3 and L^2_1 are separable Hilbert spaces, and there is only one such space up to iso (ℓ^2)

- let $H = L^2_3(\Omega^0(\mathrm{ad}P)) = \mathrm{Lie}(\mathcal{G})$, then
  $\exp : H \to \mathcal{G}$ gives $(f_*)_{id}(s) = \frac{d}{dt}(f(\exp ts))\big|_{t=0}$
  $= ds + [\omega_A, s] = \nabla_A s$

Prop (Implicit func thm for Banach mfds)
Let $\hat{E}_1, \hat{E}_2, \hat{F}$ Banach mfds, $f : \hat{E}_1 \times \hat{E}_2 \to \hat{F}$ smooth with differential at point
$x^0 = (x_1^0, x_2^0) \in \hat{E}_1 \times \hat{E}_2$ $(f_*)_{x^0} = (f_{1*}, f_{2*}) : E_1 \oplus E_2 \to F$
where $E_i = T_{x_0}\hat{E}_i$, $i = 1, 2$, $F = T_{f(x_0)}\hat{F}$.
Assume $f_{2*} : E_2 \to F$ is invertible.
Then $\exists\, \mathcal{U}_i \subset \hat{E}_i$ open nbhds of x_i^0, $i = 1, 2$,
and $\exists$ smooth map $h : \mathcal{U}_1 \to \mathcal{U}_2$ s.t.
$f(x_1, h(x_2)) = f(x_1^0, x_2^0)$ for $x_i \in \mathcal{U}_i$.

$$\begin{array}{c}\mathcal{U} \times \tilde{\mathcal{G}}_3 \\ \parallel\end{array}$$

- $\tilde{\mathcal{G}}_3 = \mathcal{G}_3/\mathbb{Z}_2$, $\mathcal{U} \times_{\text{Stab}} \mathcal{G}_3 \xrightarrow{\sim} \mathcal{O} \overset{\text{open}}{\hookrightarrow} \mathcal{A}_2^*$



$X_A = \{A + a \mid \nabla_A^* a = 0\}$
$\overset{?}{\subset} \mathcal{U}$?

$f: \mathcal{A}_2^* \times \tilde{\mathcal{G}}_3 \longrightarrow L^2_1(\Omega^0)$, $(A+a, \tilde{g}) \mapsto \nabla_A^*(g^{-1})^* a$

- $(x_1^0, x_2^0) = (A, id)$
- $(f_*)_{(A,id)}: L^2_2(\Omega^1) \oplus L^2_3(\Omega^0) \longrightarrow L^2_1(\Omega^0)$

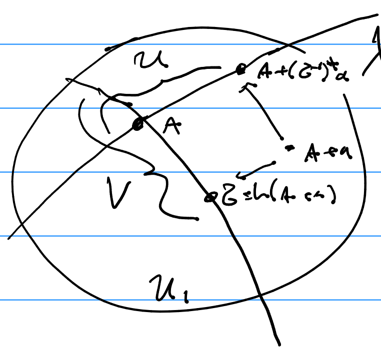
$(a, u) = \nabla_A^*(-\nabla_A u + a)$

$f_{*2} = -\nabla_A^* \circ \nabla_A = -\Delta_A$ invertible

$\overset{\text{IFT}}{\Longrightarrow} \exists\, \mathcal{U}_1 \overset{\text{open}}{\hookrightarrow} \mathcal{A}_2^*$, $\mathcal{U}_2 \overset{\text{open}}{\hookrightarrow} \tilde{\mathcal{G}}_3$
and $h: \mathcal{U}_1 \longrightarrow \mathcal{U}_3$, $A + a \mapsto g$

- $b = (g^{-1})^* a$, $\nabla_A^*(g^{-1})^* a = 0$, $b \in X_A \cap \mathcal{U}_1 =: \mathcal{U}$

$V := \mathcal{U}_2$



$X_A : \{A + a \mid \nabla_A^* a = 0\}$

$\varphi: \mathcal{U} \times V \xrightarrow{\sim} \operatorname{im} \varphi =: \mathcal{O} \overset{\text{open}}{\hookrightarrow} \mathcal{U}_1 \subset \mathcal{A}_2^*$

$(A + b, g) \mapsto A + g^* b$

$\varphi^{-1}: \mathcal{O} \longrightarrow \mathcal{U} \times V$

$A + a \longmapsto \left(A + (g^{-1})^* b,\ g \circ h(A + a)\right)$

- $\mathcal{A}_2 \supset \tilde{\mathcal{M}} = \tilde{\mathcal{M}}(P) = \{A \in \mathcal{A}_2 \mid * \underset{\Omega^2(\mathrm{ad}\,P)}{F_A} = -F_A\}$

$*: \Omega^i \to \Omega^{4-i}$, $\alpha \wedge *\beta = (\alpha, \beta)\, d\mathrm{Vol}$,

$*^2 = (-)^{i(4-i)}\,\mathrm{id}$

- so for 2-forms on 4-mfd we have splitting

$\Omega_x^2 B = \Lambda^2 T_x^* B = \Omega^2_{+,x} \oplus \Omega^2_{-,x}$

$\to$ bases $\Omega^2_{\pm,x} = \begin{cases} e_1 \wedge e_2 \pm e_3 \wedge e_4 \\ e_1 \wedge e_3 \pm e_4 \wedge e_2 \\ e_1 \wedge e_4 \pm e_2 \wedge e_3 \end{cases}$

$$\Omega^2(\mathrm{ad}\,B) = \Gamma(\Lambda^2 T^* B \otimes \mathrm{ad}\,P)$$
$$= \Omega^2_+(\mathrm{ad}\,P) \oplus \Omega^2_-(\mathrm{ad}\,P)$$
$$\underset{\cup}{F_A} = (F_A^+, F_A^-)$$

with $* F_A^\pm = \pm F_A^\pm$.

Def A is called anti-self-dual (ASD) (respectively SD) if $F_A = F_A^-$ ($F_A = F_A^+$).

- $\tilde{\mathcal{M}}$ is $\mathcal{G}$-invariant, elements of $\mathcal{M} = \tilde{\mathcal{M}}/\mathcal{G}$ are called instantons

- $F_{A+a} = F_A + \nabla_A a$

$$\begin{tikzcd}
& \Omega^2(\mathrm{ad}\,P) \arrow[dr, "P_+"] & \\
\mathcal{A} \arrow[ur, "A \mapsto F_A"] \arrow[rr, "\mathcal{F}"'] & & \Omega^2_+(\mathrm{ad}\,P) \\
A \arrow[rr, mapsto] & & P_+(F_A)
\end{tikzcd}$$

- $\mathcal{F}_* = P_+ \nabla_A : \Omega^1(\mathrm{ad}\,P) \to \Omega^2_+(\mathrm{ad}\,P)$

- if A is ASD, $P_+ F_A = 0$, so $\mathcal{U}_*$ is not Fredholm (kernel not fin. dim)
- however, it is, when restricted to X_A?
- $\Omega^1(\mathrm{ad}\,P) = \mathrm{im}\,\nabla_A \oplus \ker \nabla_A^*$

- now, take $\begin{array}{c} T^*B \\ \downarrow p \\ B \end{array}$, look at

$$0 \to \Gamma(E_1) \xrightarrow{L_1} \Gamma(E_2) \xrightarrow{L_2} \Gamma(E_3) \to 0$$

and lift to

$$0 \to p^*\Gamma(E_1) \xrightarrow{\mathrm{Symb}_{L_1}} p^*\Gamma(E_2) \xrightarrow{\mathrm{Symb}_{L_2}} \Gamma(E_3) \to 0 \qquad (*)$$

where $\mathrm{Symb}_{L_1}(s) := L_1\left(\frac{1}{k!}(g - g(x))^k \tilde{s}(x)\right)\Big|_x$

if L_1 diff op. of order k, where $\tilde{s} \in \Gamma(E_1)$ any s.t. $\tilde{s}(x) = s$, $g \in C^\infty(B)$ any s.t. $dg|_x = v$.

- we call this an elliptic cpx if $(*)$ is exact.

Инстантоны

- $A \in \mathcal{A}_2$, $0 \to \Omega^0(\mathrm{ad}\,P) \xrightarrow{\nabla_A} \Omega^1(\mathrm{ad}\,P) \xrightarrow{P_+ \circ \nabla_A} \Omega^2(\mathrm{ad}\,P) \to 0$
- on $\begin{smallmatrix} P \\ \downarrow \\ B \end{smallmatrix}$, $\dim B = 4$, (B,g) orientable, $*: \Omega^i(B) \xrightarrow{\simeq} \Omega^{4-i}(B)$
- for $F_A^+ = 0$ i.e. A ASD, this is a cpx

- from $0 \to \Omega^0(B) \xrightarrow{d} \Omega^1(B) \xrightarrow{d} \Omega^2(B)$

$$\begin{tikzcd} \Omega^0(B) \arrow[r, "d"] & \Omega^1(B) \arrow[r, "d"] \arrow[dr] & \Omega^2(B) \arrow[d, "P_+"] \\ & & \Omega_+^2(B) \arrow[r] & 0 \end{tikzcd}$$

we get for $\begin{smallmatrix} T^*B \ni (x,v) \\ p \downarrow \\ B \end{smallmatrix}$

$$0 \to \underset{\substack{\mathbb{1} \\ \mathbb{R} \cong}}{\Lambda^0 T^*B} \xrightarrow{\mathrm{Symbol}} \Lambda^1 T^*B \xrightarrow{\mathrm{Symbol}^+} \Lambda_+^2 T^*B \to 0$$

$$1 \mapsto v$$

$$w \mapsto (v \wedge w)_+ = P_+(v \wedge w)$$

- $\Lambda_\pm^2 = \mathrm{span}(a_1^\pm, a_2^\pm, a_3^\pm)$

where $a_1^\pm = e_1 \wedge e_2 \pm e_3 \wedge e_4$

$a_2^\pm = e_1 \wedge e_3 \pm e_4 \wedge e_2$

$a_3^\pm = e_1 \wedge e_4 \pm e_2 \wedge e_3$

for some g-orth. basis $e_1, \dots, e_4$
- note that in sequence $\dim \Lambda^1 T^*B = \dim \Lambda^0 T^*B + \dim \Lambda_+^2 T^*B$

so only check exactness in middle

- $v = e_1$, $w = \sum_i \lambda_i e_i \Rightarrow (v \wedge w)_+ = \lambda_2 (e_1 \wedge e_2)_+ + \lambda_3 (e_1 \wedge e_3)_+ + \lambda_4 (e_1 \wedge e_4)_+$

$= \frac{1}{2}(\lambda_2 a_1^+ + \lambda_3 a_2^+ + \lambda_4 a_3^+) = 0$

$\Rightarrow \lambda_2 = \lambda_3 = \lambda_4 = 0$ since a_1^+, a_2^+, a_3^+ lin. indep

$\Rightarrow v = \lambda \cdot w$, $\lambda \in \mathbb{R}$, so it's exact.

- note that locally $\nabla_A = d +$ 0-order diff op,

so $\mathcal{E}^\bullet(A)$: $0 \to \Omega^0(\mathrm{ad}\,P) \xrightarrow{\nabla_A} \Omega^1(\mathrm{ad}\,P) \xrightarrow{P_+ \circ \nabla_A} \Omega_+^2(\mathrm{ad}\,P) \to 0$

is an elliptic cpx. (since symbols are exact)

- we will want to calculate $\chi(\mathcal{E}^\bullet(A)) = h^0 - h^1 + h^2$

- recall that $* : \Omega^2(B) \circlearrowleft$ induces $\delta : \Omega^i \to \Omega^{i-1}$
as $(d\alpha, \beta) = (\alpha, \delta\beta)$, $\delta = (-)^i *^{-1} \circ d \circ *$
- form Laplace-Beltrami $\Delta = d \circ \delta + \delta \circ d$
- define $\mathbb{H}^p := \{ \alpha \in \Omega^p \mid \Delta\alpha = 0 \}$

- $\forall \alpha \in \Omega^p$ $\exists!$ decomp. $\alpha = \underset{\mathbb{H}^p}{h(\alpha)} \overset{\perp}{\oplus} \underset{d\Omega^{p-1}}{d\beta} \overset{\perp}{\oplus} \underset{\delta\Omega^{p+1}}{\delta\gamma}$

- Hodge theory: $H^p_{dR}(B, \mathbb{R}) \xrightarrow{\sim} \mathbb{H}^p$
$[\alpha] \longmapsto h(\alpha)$

- further, $\mathbb{H}^p_\pm := \{ \alpha \in \mathbb{H}^p \mid *\alpha = \pm\alpha \}$
so $\mathbb{H}^p = \mathbb{H}^p_+ \overset{\perp}{\oplus} \mathbb{H}^p_-$

- on H^2 $\exists$ pairing $H^2 \otimes H^2 \longrightarrow H^4(B, \mathbb{R}) \cong \mathbb{R}$
$\langle [\alpha], [\beta] \rangle \longmapsto \alpha \cup \beta := \int_B \alpha \wedge \beta$
symmetric, nondegenerate

- now, for $\mathcal{E}^\bullet(A)$: $0 \to \Omega^0(B) \otimes \underline{g} \to \Omega^1(B) \otimes \underline{g} \to \Omega^2_+(B) \otimes \underline{g} \to 0$
"trivial", product connection
$\chi(\mathcal{E}^\bullet(A)) = \underset{\dim g,\ g = SU(2)}{3} \big(b_0(B) - b_1(B) + b_2(B) \big)$

- $D_A = (\nabla_A^*, P_+ \circ \nabla_A) : \Omega^1(\mathrm{ad}\,P)_{L^2_2} \longrightarrow \Omega^0(\mathrm{ad}\,P)_{L^2_1} \oplus \Omega^2_+(\mathrm{ad}\,P)_{L^2_1}$

$\to$ $\mathcal{E}^\bullet(A)$ elliptic $\Longleftrightarrow$ D_A is elliptic

- here it is Fredholm
- further, it turns out,

$$-\chi(\mathcal{E}^\bullet(A)) = \operatorname{ind} D_A = \dim \ker D_A - \dim \operatorname{coker} D_A$$

$$= \text{topological index} = f(c_2(P), b_i(B), \dim)$$

$$\overset{\text{in our}}{\underset{\text{case}}{=}} 8c_2(P) - \underbrace{3(b_0 - b_1 + b_2)}_{\chi(\text{trivial } \mathcal{E}^\bullet(A))}$$

- $\left\{ \begin{array}{c} P \\ \downarrow SU(2) \\ B \end{array} \right\}$ are classified by $c_2(P) = \frac{1}{8\pi^2}\int tr(F_A \wedge F_A) \in H^4(B,\mathbb{Z}) \cong \mathbb{Z}$
for A any conn.

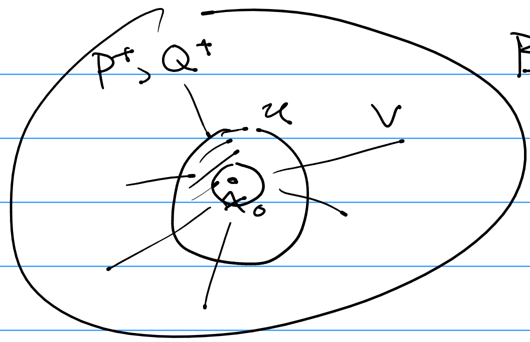
- excision principle:

1) let X^+, X^- smooth cpt var
$\quad \cup \quad \cup$
$\mathcal{U} = \mathcal{U}^+ \simeq \mathcal{U}^-$

ii) given 2 pairs of ell. ops $P^\pm, Q^\pm$ on $X^\pm$,
$P^\pm : \Gamma(E_1^\pm) \to \Gamma(F_1^\pm)$, $Q^\pm : \Gamma(E_2^\pm) \to \Gamma(F_2^\pm)$
s.t. $P^\pm = Q^\pm$ outside $\mathcal{U}^\pm$,
meaning (in P^+ coinc. w Q^+ on V^+ where
$X^\pm = \mathcal{U}^\pm_{\cong \mathcal{U}} \cup V^\pm$, $V^\pm \overset{\text{open}}{\hookrightarrow} X^\pm$)
$\exists$ isos α, β $\quad E_1^+|_{V^+} \xrightarrow{\alpha} E_2^+|_{V^+}$, $F_1^+|_{V^+} \xrightarrow{\beta} F_2^+|_{V^+}$
s.t.
$$\begin{array}{ccc} \Gamma(E_1^+|_{V^+}) & \xrightarrow{P^+} & \Gamma(F_1^+|_{V^+}) \\ \alpha \downarrow & & \downarrow \beta \\ \Gamma(E_2^+|_{V^+}) & \xrightarrow{Q^+} & \Gamma(F_2^+|_{V^+}) \end{array}$$

iii) $P^+|_\mathcal{U}$ coincides with $P^-|_\mathcal{U}$
$Q^+|_\mathcal{U}$ — " — $Q^-|_\mathcal{U}$

$$\overset{\text{i, ii, iii}}{\Longrightarrow} \quad \operatorname{ind} P^+ - \operatorname{ind} Q^+ = \operatorname{ind} P^- - \operatorname{ind} Q^-$$



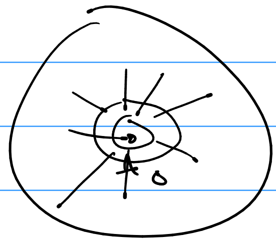
$B = X^+$

$V \simeq B \setminus \{pt\}$ (with $pt = A_0$)

$P^+ = D_{A_1}$ on X

$Q^+ = D_{A_2}$ on X

$c_2(A_1) = c$, $c_2(A_2) = c+1$



$S^4 = X^-$

$\mathcal{U} = D^4$ open disk

$P^- = D_{A_1'}$ on S^4

$Q^- = D_{A_2'}$

$c_2(A_1') = c$, $c_2(A_2') = c+1$

- for Y open mfd (= not geodesically closed), $Y = X \setminus \{pt\}$ all $\begin{array}{c} P \\ \downarrow SU(2) \\ Y \end{array}$ pbdls <u>are trivial</u>

- follows from 3-connectedness of $BSU(2)$ i.e. $\pi_i(BSU(2)) = 0$ for $i \leq 3$ which follows from 2-conn. of $SU(2)$ and fibration $SU(2) \hookrightarrow \overset{\text{contractible}}{ESU(2)}$
$$\downarrow$$
$$BSU(2)$$

long exact sqnc.

→ so excision postulates are satisfied and we only look at S^4 !

Шлихонпров

- $P_+F : \mathcal{A}_2 \longrightarrow \Omega^2_+(\mathrm{ad}\,P)_{L^2_1}$, $A \longmapsto P_+F_A$
- $(P_+F)_{*,A} : T_A\mathcal{A}_2 \to \Omega^2_+(\mathrm{ad}\,P)_{L^2_1}$

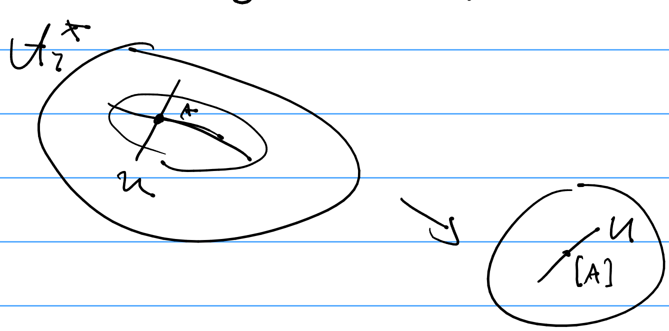
$$T_A\mathcal{A}_2 = \Omega^1(\mathrm{ad}\,P)_{L^2_2}$$

- $F_{A+a} = F_A + \nabla_A a + [a,a]$
  $\longrightarrow \ker (P_+F)_{*,A} = \mathrm{Im}\,\nabla_A$
- P_+A not Fredholm
- $\mathcal{U} \subset \ker \nabla_A^*$ slice, $\Omega^1(\mathrm{ad}\,P)_{L^2_2} = \mathrm{Im}\,\nabla_A \overset{\perp}{\oplus} \ker \nabla_A^*$
  - A – irred $\Rightarrow \mathcal{U} \overset{\text{open}}{\hookrightarrow} \mathcal{B}_2(P) = \mathcal{A}_2/\mathcal{G}_3$
    – red. $\Rightarrow \mathcal{U}/S^1$
      - for V, $\dim V = m_1 + 2m_2$,
        $V/S^1 \simeq \mathbb{R}^{m_1} \oplus \mathbb{R}^{2m_2}/S^1$

- $\ker(P_+F)_{*,A}|_{\ker \nabla_A^*} = \overbrace{\ker D_A}^{\dim < \infty} = H^1(\mathcal{E}^\bullet(A))$ (A is ASD; D_A Fredholm)

$\widetilde{\mathcal{M}}_2^*(P) := \{A \in \mathcal{A}_2^* \mid A \text{ is ASD}\}$ (irred: $\mathcal{A}_2^*$)
$\hookrightarrow \mathcal{A}_2^* \underset{\text{dense}}{\overset{\text{open}}{\hookrightarrow}} \mathcal{A}_2$

- assuming $(P_+F)_{*,A}$ surj $\Rightarrow D_A$ surj.



- notice $\widetilde{\mathcal{M}}^* = (P_+F)^{-1}(0)$
- associate to P_+F

$$\begin{array}{c} \mathcal{A}_2^* \times \Omega^2_+(\mathrm{ad}\,P) \\ s \uparrow \;\downarrow \\ \mathcal{A}_2^* \end{array}$$

$-\ s(A) = (A, P_+ F_A)$, so $\tilde{M}^* = s^{-1}(0)$

$-$ action $\mathcal{A}_2^{\perp} \times \mathcal{G}_3 \to \Omega_+^2(\mathrm{ad}P)_{2,}$, $(A, g) \mapsto P_+ F_{g^*A}$

$$\begin{tikzcd} E \times \mathcal{G} \arrow[r] \arrow[d, "p\times id"'] & E \arrow[d] \\ X \times \mathcal{G} \arrow[r] & X \end{tikzcd}$$

$-$ does it descend?

$$\begin{tikzcd} \hat{E} = E = p^*\mathcal{E} \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{A}_2^* = X \arrow[r, "p"] & X/\mathcal{G} \end{tikzcd}$$

$-\ \mathcal{E} := $ "$\hat{E}/\mathcal{G}_3$" is vbdl on $\mathcal{B}_2^*$, $s \circlearrowright \mathcal{E} \to \mathcal{B}_2^*$

$M^* = \tilde{M}/\mathcal{G} = s^{-1}(0)$

$\phi : \mathcal{B}_2^* \xrightarrow{s} \mathcal{E}$, $[A] \mapsto D_A$

$(\phi_*)_{[A]} : \mathrm{Ker}\, D_A^* \to \Omega^2_{+,A}$

$\quad \parallel$

$D_A|_{\mathrm{Ker}\, D_A^*}$ $\quad \Rightarrow$ A is ASD $\Leftrightarrow [A] \in \phi^{-1}(0) = M^*$

so $\dim M^* = \dim \mathrm{Ker}\, D_A|_{\mathrm{Ker}\, D_A^*}$

Thm Let $\phi : \mathcal{H}_1 \supset \mathcal{U} \to \mathcal{V} \subset \mathcal{H}_2$ be Fredholm map of Hilbert mflds, $x \in \mathcal{H}_1$ and $\phi : T_{x_0}\mathcal{H}_1 \to T_{\phi(x_0)}\mathcal{H}_2$ Fredholm. Then around x (resp. y),
$\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ s.t. $\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_1, x_2))$
where $\phi_1 : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{V}_1$ is isom., $\phi_2 : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{V}_2$
and $\phi^{-1}(y_0) = \phi_2^{-1}(y_2^0) \cap \{x_1^0\} \times \mathcal{U}_2$, where $y_0 = (y_1^0, y_2^0)$, $x_0 = (x_1^0, x_2^0)$

and $\mathcal{U}_2, \mathcal{V}_2$ are <u>finite</u> dimensional.

$$\begin{array}{ccc} T_{x_1^\circ}\mathcal{U}_1 \oplus T_{x_2^\circ}\mathcal{U}_2 & & \\ \downarrow \phi_1 \cong \quad \searrow \phi_2|_{T_{x_1^\circ}\mathcal{U}_1} \quad \downarrow \phi_2|_{T_{x_2^\circ}\mathcal{U}_2} & & \downarrow (\phi_*)_{x_0} \\ T_{y_1^\circ}\mathcal{V}_1 \oplus T_{y_2^\circ}\mathcal{V}_2 & & \end{array}$$

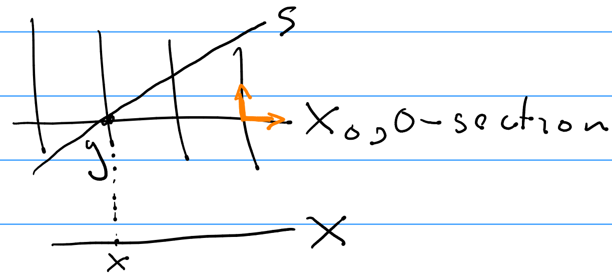
- assume $(\phi_*)_{x_0}$ surjects $\Rightarrow \mathcal{V}_2 = \{0\}$

Шихонипров

- $A \in \mathcal{A}_2^*$, $\ker D_A = T_{[A]}\mathcal{M}$.

- first, in general let $s \subset \begin{matrix} E \\ \downarrow \\ X \end{matrix}$ v6 dl,

fiber of any rk, with section s



$Z = \{0\text{-set of } s\}$
$= s \cap X_0$

$y = (x, 0) \in Z$

$$(s_*)_x : T_x M \xrightarrow{s_*} T_y X_0 \oplus E_x \xrightarrow{\ pr_2\ } E_x$$

- s transversal on $y \in Z \iff pr_2 \circ s_*$ is epi
- — " — on $Z \Rightarrow Z \underset{\text{sbmfd}}{\overset{\text{smooth}}{\hookrightarrow}} X$

$$0 \to \underbrace{TZ \to TX|_Z}_{\text{from } Z \hookrightarrow X} \to N_{Z/X} \to 0$$

- if $Z = (s)_0$, $N_{Z/X} = E|_Z$

- we had $\Omega^1(\mathrm{ad}\,P)_{L^2_2} \xrightarrow{P_+ \circ D_A} \Omega^2_+(\mathrm{ad}\,P)_{L^2_1}$?

$$P_+F : \mathcal{A}_2^* \to \Omega^2_+$$
$$\cup \qquad\qquad \cup$$
$$\text{also} = \tilde{\mathcal{M}}^* \to \{0\}$$

$\tilde{\mathcal{M}}^* = (P_+F)^{-1}(0)$

$$\begin{tikzcd} \phi: \mathcal{A}_2^* \arrow[r, "A \mapsto (A, P_+F_A)"] \arrow[dr] & \mathcal{A}^* \times \Omega_+^2 \arrow[d] \\ & \mathcal{A}_2^* \end{tikzcd}$$

$S_* = \phi_* = (id, P_+\nabla_A)$

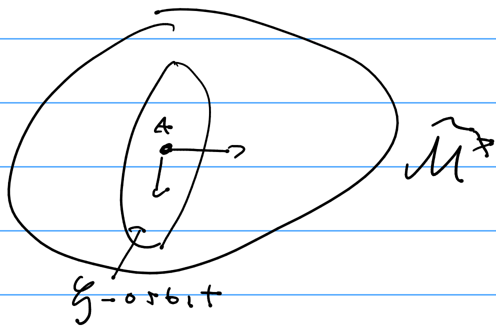
$A \overset{\phi}{\mapsto} y$

$T_A\mathcal{A}_2^* \overset{P_+\nabla_A}{\twoheadrightarrow} \Omega_+^2$

$\Rightarrow$ ϕ is transversal at $\tilde{\mathcal{M}}^*$

$\Rightarrow$ $\tilde{\mathcal{M}}^*$ smooth Banach (sub)mfd

$-\ \mathcal{G}_3 \circlearrowright \tilde{\mathcal{M}}^*$



$\mathcal{M}^* = \{A \in \mathcal{A}_2^* \mid ASD\}$

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & T_{[A]}\mathcal{G}_3 \arrow[r] \arrow[d, equal] & T_A\tilde{\mathcal{M}}^* \arrow[r] \arrow[d] & T_{[A]}\mathcal{M}^* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T_{[A]}\mathcal{G}_3 = \Omega_0(\mathrm{ad}P)_{L^2_3} \arrow[r, "\nabla_A"] & T_A\mathcal{A}_2^* = \Omega_1 \arrow[r] \arrow[d, "P_+\circ\nabla_A"] & \mathrm{coker}\,\nabla_A = \ker\nabla_A^* \arrow[r] \arrow[d, "P_+\circ\nabla_A"] & 0 \\ & & N_{A,\tilde{\mathcal{M}}^*/\mathcal{A}_2^*} \arrow[r, equal] \arrow[d] & \Omega_2 \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

by 5-lemma, or indeed previous considerations
$\rightarrow$ but here everything commutes, more info

$\Rightarrow \ker\{D_A = (\nabla_A^*, P_+\circ\nabla_A)\} = \ker(P_+\circ\nabla_A|_{\ker\nabla_A^*}) = T_{[A]}\mathcal{M}^*$

- deep result by Uhlenbeck

- now, it's true that $(B, g) \Rightarrow \mathcal{M}^*(P, g)$
  - clear metric dependence
    - can we get rid of it?

$$\begin{array}{ccl} T^*B & F & \text{- frame bdl for } T^*B \\ \downarrow & \downarrow & = GL(4,\mathbb{R})\text{-pbdl} \\ B & B & \text{- it has } \mathcal{G}_F\text{, its gp of gauge tr.} \end{array}$$

- fix metric g_0

$$\{ s(x) \mid x \in B \} =: s \in \Gamma(\mathrm{Sym}^2 T^*B)$$

- $\underset{\{\varphi_x \mid x \in B\}}{\varphi} \in \mathcal{G}_F$ acts on s by

$$\varphi(s) = \{ \varphi_x \, s(x) \, \varphi_x^\tau \mid x \in B \}$$

$$\Rightarrow \mathcal{C} = \{ \varphi(g_0) \mid \varphi \in \mathcal{G}_F \}$$

- however, completion of <u>smooth</u> metrics might not give metrics $\Rightarrow$ we lose pos. definiteness
  - UNLESS, pick $g_0 \in \mathcal{C}^k(\mathrm{Sym}^2 T^*B)$, $k < \infty$
    - has canonical norm
      - over cpt B, this is already complete
        - look at cont. funcs over interval, e.g.
    - of course, $\mathcal{G}_F$ should probably $\in \mathcal{C}^{k+1}$

$\mathcal{P}_+ F : \mathcal{A}^*_2 \times \mathcal{C} \longrightarrow \Omega^2$

$(A, g) \longmapsto P_{+,g} F_A$

$-\ (\mathcal{P}_+ F)^{-1}(0) = \bigcup_{g \in \mathcal{C}} \tilde{\mathcal{M}}^*(P, g) =: \tilde{\mathcal{M}}^*(P, \mathcal{C})$

- Uhlenbeck:

  i) 0 is a regular pt of $\mathcal{P}_+ F$

  $\overset{\text{cor.}}{\Longrightarrow} \tilde{\mathcal{M}}^*(P, \mathcal{C})$ is smooth Banach mfd

  ii) $\pi : \tilde{\mathcal{M}}^*(P, \mathcal{C}) \longrightarrow \mathcal{C}$ is a sm. map of Banach mfds and, by Sard-Smale, for almost all $g \in \mathcal{C}$, $\tilde{\mathcal{M}}^*(P, g)$ satisfies ($\Omega^1(\mathrm{ad}\,P)_{L^2_2} \xrightarrow{P_+ \circ D_A} \Omega^2_+(\mathrm{ad}\,P)_{L^2_1}$ surjects)

Rmk $SU(2)$ conn. A is reducible, on P nontrivial, if $\mathrm{Stab}_{\mathcal{G}}(A) = S^1$

- so $\exists$ S^1-pbdl Q s.t.

  $P = Q \times_{S^1} SU(2)$

- in fact, these are the same statement

$\Leftrightarrow$ $\xi = P \times_{SU(2)} \mathbb{C}^2$ decomposes to $L \oplus L^\vee$

Тихомиров

- $\begin{matrix}P\\\downarrow\\B\end{matrix}, g$, assume $\Omega^1_{L^2_2}(\mathrm{ad}P) \xrightarrow{P_{+,g}\circ\nabla_A} \Omega^2_{+,g}(\mathrm{ad}P)_{L^2_1}$ surjects (*)
- $A \in \mathcal{A}_2^*$
- $D_A = (\nabla_A^*, P_{+,g}\circ\nabla_A) : \Omega^1 \to \Omega^0 \oplus \Omega^2_{+,g}$

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & 0 \arrow[d] & \ker D_{A,g} \arrow[d] \arrow[r, phantom, "="] & \ker P_{+,g}\circ\nabla_A|_{\ker \nabla_A^{*,g}} \arrow[d] \\
\ker\nabla_A = 0 \arrow[r] & \Omega^0 \arrow[r, "\nabla_A"] \arrow[d, "s"', "\Delta_A = \nabla_A^*\nabla_A"] & \Omega^1 \arrow[r] \arrow[d, "D_{A,g}"] & \ker\nabla_A^{*,g} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^0 \arrow[r] \arrow[d] & \Omega^0\oplus\Omega^2_{+,g} \arrow[r, "pr_2"] \arrow[d] & \Omega^2_+ \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0
\end{tikzcd}$$

$\subset$ if A irred, since $\ker \Delta_A = \{0\} \Leftrightarrow \mathrm{Stab}_A = \mathbb{Z}_2$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
 & & \mathbb{R} \arrow[d] \arrow[dll, equal] & \ker D_{A,g} \arrow[d] \arrow[r, phantom, "="] & \ker P_{+,g}\circ\nabla_A|_{\ker \nabla_A^{*,g}} \arrow[d] \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Omega^0 \arrow[r, "\nabla_A"] \arrow[d, "\Delta_A = \nabla_A^*\nabla_A"] & \Omega^1 \arrow[r] \arrow[d, "D_{A,g}"] & \ker\nabla_A^{*,g} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \Omega^0 \arrow[r] \arrow[d] & \Omega^0\oplus\Omega^2_{+,g} \arrow[r, "pr_2"] \arrow[d] & \Omega^2_+ \arrow[r] \arrow[d] & 0 \\
 & & \mathbb{R} \arrow[r, equal] \arrow[d] & \mathbb{R} \arrow[d] & 0 \\
 & & 0 & 0
\end{tikzcd}$$

$\subset$ for A red., $\ker \Delta_A = \mathbb{R}$

- recall: $\mathrm{ind}\, D_{A,g} = 8c_2(P) - 3(1 - b_1 + b_2^+)$

- $\mathcal{C}$ param sp. of metrics (Banach mfd)
- $\underline{\Omega^2} := \Omega^2(\mathrm{ad}\,P)_{L^2_1} \times \mathcal{C}$

$$\begin{tikzcd} \Omega^2(\mathrm{ad}\,P)_{L^2_1} \times \mathcal{C} \\ \bigcup_{g\in\mathcal{C}} \{\Omega^2_{+,g}(\mathrm{ad}\,P)_{L^2_1}, g\} =: \Omega^2_+ \arrow[u, hook, "i"] \end{tikzcd}$$

$$\begin{tikzcd} & \Omega^2 & \underline{\Omega^2} \arrow[l, "pr_1"'] \\ \mathcal{A}^*_2 \times \mathcal{C} \arrow[ur] \arrow[r, "\phi"] & \Omega^2_+ \arrow[ur, hook, "i"] & \\ (A,g) \arrow[r, mapsto] & P_{+,g} F_A & \end{tikzcd}$$

$\widetilde{\mathcal{M}}^*(P,\mathcal{C}) : \phi^{-1}(0\text{-sect'n}) = \bar{\phi}^{-1}(0)$

Uhlenbeck:

1) $\forall (A,g) \in \widetilde{\mathcal{M}}^*(P,\mathcal{C})$, $(\phi_*)_{(A,g)} : \Omega^1_{L^2_2} \oplus T_g\mathcal{C} \twoheadrightarrow \Omega^2_{+,g}$ surjects

   - hence, $\widetilde{\mathcal{M}}^*(P,\mathcal{C})$ is smooth Banach mfd

ii) for generic $g \in \mathcal{C}$, $P_{+,g} \circ \nabla_A$ surjects, i.e. (*) holds

iii) $\overline{\pi} : \mathcal{M}^*(P,\mathcal{C}) \to \mathcal{C}$ is a Fredholm map and $(\overline{\pi}_*)$ has at any pt $y = ([A],g)$ index $=$ ind $D_{A,g}$, and $\mathcal{M}^*(P,\mathcal{C})$ is sm. Ban. mfd.

$$\begin{tikzcd} \widetilde{\mathcal{M}}^*(P,\mathcal{C}) \arrow[r, hook] \arrow[d, "/\mathcal{G}_3"] & \mathcal{A}^*_2 \times \mathcal{C} \arrow[d, "/\mathcal{G}_3"] \\ y = ([A],g) \in \mathcal{M}^*(P,\mathcal{C}) \arrow[r, hook] \arrow[d, "\overline{\pi}"] & \mathcal{B}^*_2 \times \mathcal{C} \arrow[d, "pr_2"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

A reducible

$\Updownarrow$

$\exists\ \$^1$-pbdl $Q \to B$ with $c_1(Q) = x \in H^2(B, \mathbb{Z})$

s.t. $P = Q \times_{\$^1} SU(2)$

$\Updownarrow$

$$\xi = P \times_{SU(2)} \mathbb{C}^2 = L \oplus L^{\vee}$$ cpx. dual

where $L = Q \times_{\$^1} \mathbb{C}^1$ line bdl with $c_1(L) = c_1(Q) = x$

$$-c_2(\xi) = c_2(P) = -(x, x)$$

Prop Fix $g \in \mathcal{C}$. $A \in \mathcal{A}_2$ is reducible ASD-conn. with $P = Q \times_{\$^1} SU(2)$ where $x = c_1(Q)$, $(x^2) = -c_2(P)$

.

$\Updownarrow$

A is obtained from a conn. A_Q on $\$^1$-pbdl $\begin{smallmatrix} Q \\ \downarrow \\ B \end{smallmatrix}$ with $c_1(Q) = x$,

$(x^2) = -c_2(P)$ s.t. $h(x) \in \mathcal{H}_{-, g}$

$[x] = h(x) \oplus d\alpha \oplus \delta\beta$

$\{\alpha \in \Omega^2_{dR}(B, \mathbb{R}) \mid \Delta_g \alpha = 0,\ *\alpha = -\alpha\}$

- how is it obtained?

$$\nabla_{A_Q} : \Gamma(L) \longrightarrow \Gamma(L \otimes T^*B)$$

$$\rightsquigarrow \nabla_A : \Gamma(L \oplus L^{\vee}) \longrightarrow \Gamma((L \oplus L^{\vee}) \otimes T^*B),$$

$$\nabla_A = \nabla_{A_Q} \oplus \nabla_{A_Q}^{\vee},$$

$$d\langle s, t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A^{\vee} t\rangle$$

Pf. $\Rightarrow$ Show $h(x) \in \mathbb{H}_{-,g}$. Since $\xi = L \oplus L^\vee$,

$F_A = \begin{pmatrix} F_{A_Q} & 0 \\ 0 & -F_{A_Q} \end{pmatrix}$ where F_{A_Q} is ASD,

so F_A is also. But $x = c_1(Q) = \frac{i}{2\pi} F_{A_Q}$.

Note Bianchi $0 = \nabla_{A_Q} F_{A_Q} = dF_{A_Q} + [\omega_{A_Q}, F_{A_Q}]$

$= dF_{A_Q}$ since $\mathcal{L}ie(S^1) = \mathbb{R}$

$\Rightarrow \delta F_{A_Q} = 0$.

Prop Take $x \in H^2(B, \mathbb{Z})$ w $(x^2) = -c_2(P)$.

Then $\mathbb{R}x \subset H^2(B, \mathbb{R})$. Consider

$$\begin{array}{ccc} f: \mathcal{C} & \longrightarrow & Gr(b_2^-, H^2(B,\mathbb{R})) \\ \cup & & \cup \\ g & \longmapsto & \mathbb{H}_{-,g} \subset \mathbb{H}_g \end{array}$$

$$N_x := \{ V^{b_2^-} \in Gr \mid V^{b_2^-} \ni \mathbb{R}x \} \subset Gr(b_2^-, H^2(B,\mathbb{R}))$$

$$\begin{array}{cc} \text{is} & \| \\ Gr(b_2^- - 1, b_2 - 1) & Gr(b_2^-, b_2) \end{array}$$

So $\operatorname{codim}_{Gr} N_x = b_2^+ \leftarrow b_2(\underbrace{b_2 - b_2^-}_{b_2^+}) - (b_2 - 1)(\underbrace{b_2 - b_2^-}_{b_2})$

Outcome: if $b_2^+ > 0$, then for

$g \in \mathcal{C} \setminus f^{-1}\left(\bigcup_{(x^2) = -c_2} N_x \right)$ ($\leftarrow$ dense in $\mathcal{C}$)

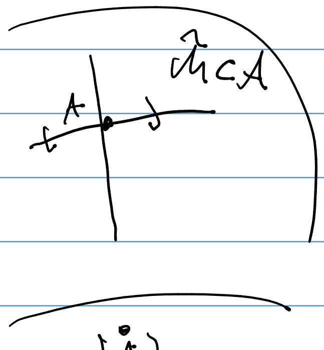
we have $\mathcal{M}^*(A, g) = \mathcal{M}(A, g)$.

Тихомиров

- $b_2^- \neq 0, b_2^+ = 0 \Rightarrow (-,-)_B$ neg. def.
- Uhlenbeck shows: given $([A], g) \in \mathcal{M}(P,g)$, $\exists g'$ perturbation of g in $\mathcal{C}$ s.t. $P_{+,g'} \circ \nabla_A$ surjects
- $\underline{\mathcal{M}(P,g)} := $ closure of $\mathcal{M}^*(P,g)$ in $\mathcal{B}(P)$
- Donaldson: $\mathcal{M}^*(P,g)$ is simply conn $\Rightarrow$ oriented
- in this case $\dim \mathcal{M}(P,g) = 8c_2(P) - 3$, and $c_2(P) = 1 \Rightarrow \dim \mathcal{M}(P,g) = 5$

- $\partial\mathcal{M} := \mathcal{M}(P,g) \setminus \mathcal{M}^*(P,g) = \{[A] \in \mathcal{M}(P,g) \mid A \text{ reducible}\}$
  $\Rightarrow 0 \leq \# \partial\mathcal{M} \leq b_2^-$

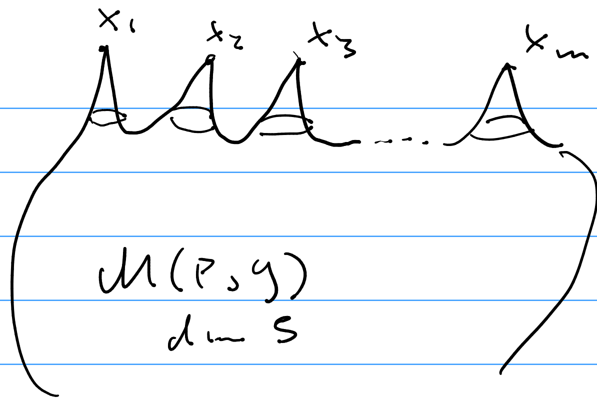
- recall A red. $\Leftrightarrow P = Q \times_{S^1} SU(2)$, $c_1(Q) = x \in H^2(B, \mathbb{Z}) , (x^2) = -1$
$$H^2(B,\mathbb{Z}) \overset{\shortparallel}{} \mathbb{Z}^{b_2^-}$$
  - $\mathrm{rk}(,) = b_2^-$
- $\exists x : (x^2) = -1$, $|\det(,)| = 1$
  $\forall y \in \mathbb{Z}^{b_2^-} = \mathbb{Z}x \overset{\perp}{\oplus} \mathbb{Z}^{b_2^- - 1}$
  - take $\mathbb{Z}$-basis in $\mathbb{Z}^{b_2^-}$, $x = e_1, \ldots, e_{b_2^-}$
  $\Rightarrow y = \alpha e_1 + w$, $w \in (\mathbb{Z}e_1)^\perp$
  $\Rightarrow \alpha = -(y, x)$



$\tilde{M} \cap$ slice $=$ subm'fd $\mathcal{U}$ of slice s.t. $T_A\mathcal{U} = \ker D_A \subset \mathbb{R}^6$

Slice $\times_{S^1} \mathcal{G}_3 = \mathbb{R}^6 \simeq \mathbb{C}^3/S^1$
$=$ cone over $\mathbb{CP}^2$

- nbhd of $[A]$ in $\mathcal{M} \simeq$ slice $\cap \tilde{M} / S^1$



$(x_m^2)_B = -1, \; m \leq b_2$

- compact?

Thm (Uhlenbeck) $P \to B$ has $c_2(P) = k > 0$.

Let $\{A_n\}_{n \geq}$ be sqnc in $\mathcal{A}_2(P)$.

Then

i) $\exists$ $SU(2)$-bdl $P' \to B$ with $0 \leq c_2(P') = k' \leq k$ and ASD-conn. A' on P'

ii) $\exists$ pts $x_1, \ldots, x_t$ and nonnegative integers $m_1, \ldots, m_t$ s.t. $\sum_{i=1}^{t} m_i = k - k'$

iii) $\exists \{ g_n \subset \mathcal{G}_3 \}_{n \geq 1}$ s.t. for any cpt $K \subset B \setminus \{x_1, \ldots, x_t\}$

$$g_n^* A_n|_K \xrightarrow{n \to \infty} A'|_K \text{ in } L^2_2(K)$$

iv) $|F_{A_n}|^2$ weakly converges to $|F_{A'}|^2 + 8\pi^2 \sum_{i=1}^{\infty} m_i \delta_{x_i}$,

by which we mean $\forall f \in \mathcal{C}(B)$,

$$\int_B f\, |F_{A'}|^2 dVol \longrightarrow \int_B f \left(|F_{A'}|^2 + 8\pi^2 \sum m_i \delta_{x_i} \right) dVol$$

- so, labelling $P_r \to B$ $SU(2)$-bdls w/ $c_2(P_r) = r$,

$\mathcal{M}_r := \mathcal{M}(P_r)$, $\mathcal{M}_k \ni [A_n] \xrightarrow{n \to \infty} [A'] \in \mathcal{M}_{k'} \times Sym^{k-k'}(B)$

$$\hat{\mathcal{M}}_k := \bigsqcup_{0 \leq r \leq k} \mathcal{M}_r \times Sym^{k-r} B \supset \mathcal{M}_k$$

$\overline{\mathcal{M}}_k^{UD} :=$ closure of $\mathcal{M}_k$ in $\hat{\mathcal{M}}_k$ $\leftarrow$ Uhlenbeck-Donaldson cptification

- $\overline{\mathcal{M}_2}^{uo} = \mathcal{M}_2 \sqcup \underbrace{\mathcal{M}_1}_{} \overbrace{\times \mathbb{C}\mathbb{P}^1}^{=\emptyset} \sqcup S^2\mathbb{P}^2 = \mathbb{P}^5$

$\to$ Kobayashi - Hitchin correspondence

- $\mathcal{M}_2(S^4) = \dfrac{SO(5,1)}{SO(5)}$ $\quad SO(5,1) \to (5+1)(5+1-1)/2$, $\quad SO(5) \longrightarrow 5\cdot 4/2$

$\hookrightarrow$ conformal gp

$\to *$ is conf. inv on Ω^2

Тихомиров

- topologise $\hat{\mathcal{M}}_k \ni a = ([A_a], \{x_1, \dots, x_t\})$
by picking open geod. ball $B_\varepsilon(x_i)$,
$K_\varepsilon := B \setminus \bigcup_{i=1}^{t} B_\varepsilon(x_i)$ is cpt.
- $\mathcal{U}_\varepsilon := \{ a' \in \hat{\mathcal{M}}_k \mid$ i) $\exists\, \mathcal{G}: P_{a'}|_{K_\varepsilon} \xrightarrow{\sim} P_a|_{K_\varepsilon}$
s.t. $\|A'_{a'} - \mathcal{G}^* A_a\|_{L^2_l(K_a)} < \varepsilon$
ii) $|\mu_a - \mu_{a'}| < \varepsilon$ in weak-$*$ top
iii) $\sup\{ |\int_B f_i \mu_a - \int_B f_i \mu_{a'}| \mid i = 1, \dots, m\} < \varepsilon \}$

$\subseteq$ open $\hat{\mathcal{M}}_k$

- $k = c_2(P) = 1$, $b_2^+ = 0$, $\dim \mathcal{M}_k = 8k - 3(1 + b_2^+) =: d$

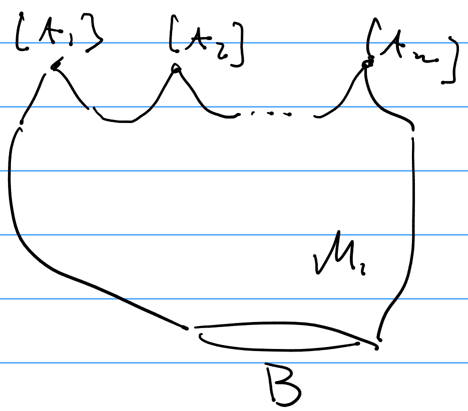
Thm (Donaldson) $\exists$ fund. class $[\bar{\mathcal{M}}_k] \in H_d(\bar{\mathcal{M}}_k, \mathbb{Z})$

- $H_d(\mathcal{M}_k^*, \mathbb{Z}) \to H_d(\mathcal{M}_k^*, \mathcal{M}_k - \{*\}, \mathbb{Z})$

$$[\bar{\mathcal{M}}_k]|_{\mathcal{M}_k^*} \overset{\sim}{\longmapsto} [1]$$

- so $\mathcal{M}_k^*$ is oriented, $\mathcal{M}_k^* \subset \mathcal{M}_k \subset \bar{\mathcal{M}}_k$

- $0 \le m \le b_2^-$

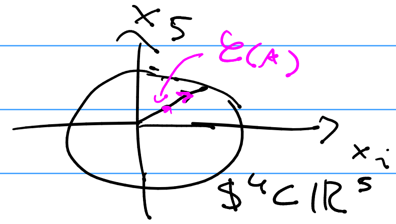


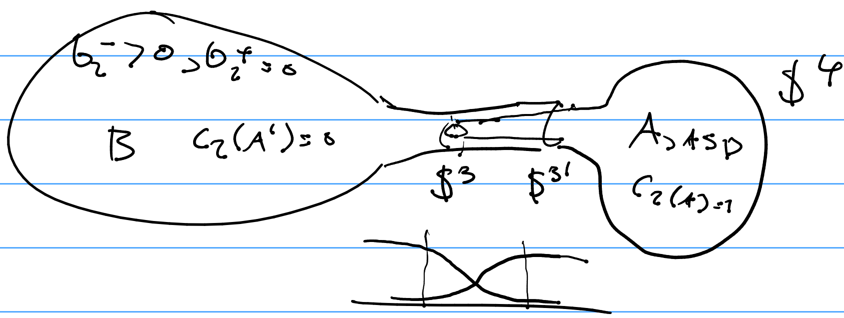
$\bar{\mathcal{M}}_1 - \mathcal{M}_1 = B$

$\hat{\mathcal{M}}_1 = \mathcal{M}_1 \sqcup \underset{\mathcal{M}_0}{\underset{\shortparallel}{\{*\}}} \times B$

- $B = S^4$, $\mathcal{M}_1(S^4) = B^5$, with $S^4 = \partial B^5$
- let $\int_{S^4} x_i |F_A|^2 \, dvol = x_i(A)$, $\mathcal{E}(A) := \{x_1(A), \dots, x_5(A)\}$

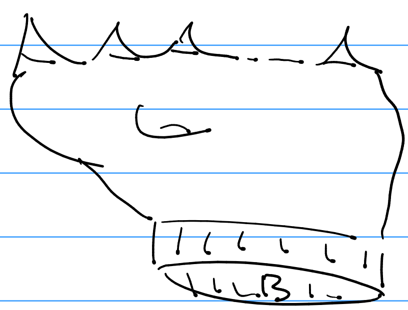
$x \mapsto \lambda x$, $\lambda \to 0$,





$B \# S^4 \overset{\text{diff}}{\cong} B$

- it can be shown that $\exists$ a collar



- $B \cup_i \mathbb{C}P^2_i = \partial(\underbrace{\overline{\mathcal{M}_1 \setminus \text{cones}}}_{\text{sm. 5-dim mfd w bdry}})$
- $\sigma(\omega) = \tau_+ - \tau_-$, signature of int. form
- result: $\sigma(B) + \sum_i \sigma(\mathbb{C}P_i^2) = 0$

- $\sigma(\mathbb{C}P^2) = 1 - 0 = 1$, $H^2(\mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}[\mathbb{C}P^2]$
  $\omega_{\mathbb{C}P^2} = (1)$
- opposite orient., $\sigma(\overline{\mathbb{C}P^2}) = -1$, $\omega_{\overline{\mathbb{C}P^2}} = (-1)$
- $\sigma(B) = \pm b_2^-$

$$\Rightarrow \pm b_2^{-} + \sum_{i=1}^{\ell} (\pm 1) = 0$$

$$\Rightarrow w_B = (-1) \oplus \cdots \oplus (-1)$$

$$w_{\bar{B}} = (1) \oplus \cdots \oplus (1)$$

Тихомиров

- $\mathcal{G} = \{ \varphi : P \to SU(2) \mid \varphi(pg) = g^{-1}\varphi(p)g \}$
- $\mathcal{E} = P \times_{SU(2)} \mathbb{C}^2$

$\Rightarrow \varphi \in \mathcal{G} \iff \varphi \in Aut_h \mathcal{E}$, smooth

- $\mathcal{E}$ holom. str. on $E = P \times_{SU(2)} \mathbb{C}^2$

$\iff \bar\partial_{\mathcal{E}} : \Omega^0(E) \to \Omega^{0,1}(E)$,

$\bar\partial_{\mathcal{E}}(f \cdot s) = (\bar\partial f)\cdot s + f(\bar\partial_{\mathcal{E}} s)$ Leibnitz

- $\mathcal{E}_1 \xrightarrow[\sim]{\psi} \mathcal{E}_2$ gives $(E \xrightarrow[\sim]{\varphi_\psi} E) \in \mathcal{G}^{\mathbb{C}}$

$\to$ $\bar\partial_{\mathcal{E}_2} \in \mathcal{G}^{\mathbb{C}}$-orbit of $\bar\partial_{\mathcal{E}_1}$, which simply comes from

$$\begin{tikzcd}
\Omega^0(E) \arrow[r, "\varphi_\psi", "\sim"'] \arrow[d, "\bar\partial_{\mathcal{E}_1}"] & \Omega^0(E) \arrow[d, "\bar\partial_{\mathcal{E}_2}"] \\
\Omega^{0,1}(E) \arrow[r, "\varphi_\psi", "\sim"'] & \Omega^{0,1}(E)
\end{tikzcd}$$

- $\mathcal{A}^{1,1}_h = \{ A \in \mathcal{A} \mid F_A \in \Omega^{1,1}(\mathcal{E}nd\, E),\ h \text{ herm. form on } E \}$

- $\tilde{\mathcal{M}}(P,g) = \tilde{\mathcal{M}}^* = \{ A \in \mathcal{A}^{1,1}_h \mid \hat{F}_A = 0 \iff A \text{ is ASD, irred} \}$

- $\mathcal{M}^* = \tilde{\mathcal{M}}^*/\mathcal{G} \ni [A] \longmapsto [A]_{\mathbb{C}} \in \mathcal{M}_{[\omega]}$

- $\mathcal{A}^{1,1}_{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}$ = space of holom. str's on E

$\cup$

$\mathcal{M}_{[\omega]} = \{ [A]_{\mathbb{C}} \mid \mathcal{E} \text{ defined by } \bar\partial_A \text{ is } [\omega]\text{-stable} \}$ from $\nabla_A = \partial_A + \bar\partial_A$

- Donaldson: $[A] \mapsto [A]_{\mathbb{C}}$ is isom.

- $\gamma_d(X)$ is a smooth (orient. pres.) diffeom. invariant, $\gamma_d(X): H_2(X,\mathbb{Z}) \times \cdots \times H_2(X,\mathbb{Z}) \to \mathbb{Z}$

Seiberg-Witten

simply conn., $b_2^+(X) > 1$ and odd

- $\mathcal{C}_X = \{ K \in H_2(X, \mathbb{Z}) \mid K$ is characteristic, i.e. $\omega_X(K,C) \equiv \omega_X(C,C) \pmod 2$ $\forall C \in H_2(X, \mathbb{Z}) \}$.

- Seiberg-Witten is $\mathcal{C}_X \to \mathbb{Z}$
  1) for any $K \in \mathcal{C}_X$ there is defined (for generic metric g on X and generic $\delta \in \Omega^2_{+,g}(X)$) smooth $\mathcal{M}^\delta_K(g)$ mfd of dimension

$$\dim \mathcal{M}^\delta_K(g) = \underbrace{\frac{1}{4}\left[K^2 - (3\sigma(X) + 2\chi_{top.}(X))\right]}_{=:\, 2m}$$

  which is even (if $b_2(X) > 1$ and odd). Orientation on $\mathcal{M}^\delta_K(g)$ is given by that of $H^0(X, \mathbb{R}) \oplus H^2_+(X, \mathbb{R})$
  II) $\mathcal{M}^\delta_K(g) \subset \mathcal{B}_K$ config. sp., ∞-dim, and $\mathcal{B}_K \simeq \mathbb{CP}^\infty$, $H^*(\mathcal{B}_K, \mathbb{Z}) = \mathbb{Z}[h]$

$$\exists\, [\mathcal{M}^\delta_K(g)] \in H_{2m}(\mathcal{M}^\delta_K(g), \mathbb{Z})$$

$$\mathcal{E}_X(K) := \langle h^m, [\mathcal{M}^\delta_K(g)] \rangle$$

- $SW: \mathcal{C}_X \to \mathbb{Z}$ are (orient. pres.) diff. invariants of simply conn. cpt oriented 4-mfd X with $b^+(X) > 1$, odd.

Thm (vanishing) If X as above and

i) $X = Y \# Z$, $b_2^+(Y) > 0$, $b_2^+(Z) > 0$, then $SW_X = 0$!

ii) if X has a metric of positive scalar curv., then $SW_X = 0$

iii) If $\exists\, S^2 \hookrightarrow X$ w. $\langle S^2, S^2 \rangle \geq 0$, $[S^2] \neq 0$, also $SW_X = 0$

Thm (nonvanishing) If $X = S$ cpx. sfc, $b_+^2(S)$ odd > 1, then $SW_X(\pm c_1(K_S)) \neq 0$

- $0 \to \mathbb{Z}_2 \to Spin(4) \to SO(4) \to 1$
  $\Rightarrow P_{Spin(4)} \xrightarrow{2:1} P_{SO(4)}$ if 2nd Stiefel-Whitney class $= 0$.
- $Spin^{\mathbb{C}}(4) = \{ (A, B) \in \mathcal{U}(2) \times \mathcal{U}(2) \mid \det A = \det B \} \leq \mathcal{U}(2) \times \mathcal{U}(2)$
- $\det : \mathcal{U}(2) \to \mathcal{U}(1)$, $A \mapsto \det A$
- $\mathcal{U}(2) = S^1 \times SU(2) / \{ \pm (1, \mathbb{1}) \}$
  $\underset{zA}{\ } \quad \underset{z}{\ } \quad \underset{A}{\ }$
- so $0 \to \mathbb{Z}_2 \to S^1 \times SU(2) \times SU(2) \to Spin^{\mathbb{C}}(4) \to 0$
  $\{\pm 1\} \mapsto \{ \pm (1, \mathbb{1}, \mathbb{1}) \}$

$0 \to \mathbb{Z}_2 \to Spin^{\mathbb{C}}(4) \xrightarrow{2:1} S^1 \times SO(4) \to 1$

- P_{S^1} pbdl w $c_1(P_{S^1}) = K$, $\mathcal{G} = Maps(P, S^1)$

- $\rho_+ : Spin^{\mathbb{C}}(4) \to \mathrm{Aut}\, \mathbb{H}$

  $\rho_+([\lambda, q_1, q_2])(h) = q_1 h \lambda$

  $\rho_-($ — " — $)(h)$

$W^\pm := P_{Spin^{\mathbb{C}}(4)} \times_{\rho_\pm} \mathbb{H}$ spinor bdls rk 2

$\rho_0([\lambda, q_1, q_2])(h) = q_1 h \overline{q_2}$

$P_{Spin^{\mathbb{C}}(4)} \times_{\rho_0} \mathbb{H}$ real rk 4 bdl TX

$TX \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$

$c : \Gamma(W^+ \otimes T^* X) \to \Gamma(W^-)$ Cliff. mult.

$$\begin{tikzcd} \Gamma(W^+) \arrow[r, "\nabla_A"] \arrow[dr, "\not{D}_A"'] & \Gamma(W^+ \otimes T^* X) \arrow[d, "c"] \\ & \Gamma(W^-) \end{tikzcd}$$

- monopole equ.'s $\begin{cases} \not{D}_A \psi = 0 \\ i\,\sigma(\psi) = F_A^+ , \end{cases}$

$\sigma : \mathbb{H} \to \mathrm{Im}\,\mathbb{H},$
$h \mapsto -h i \bar{h}$