Gauge ∞ IGAP

Mocking the u-plane integral - J. Manschot

- let M be smooth cpt 4-mfd w $(b_1, b_2^+) = (0,1)$, $L = H^2(M,\mathbb{Z})/\text{tor}$ lattice, $B : (L\otimes\mathbb{R})^2 \to \mathbb{R}$ bilin. form, $Q(k) = B(k,k) = k^2$ quad. form
- let J be the period pt., SD 2-form in $H^2(M,\mathbb{R})$ w $Q(J) = 1$
- project $k\in L$ in pos. & neg. def components $k_+ = B(k,J)\,J \in L\otimes\mathbb{R}$, $k_- = k - k_+$

- u-plane integral:

$$\Phi_\mu^J[\mathcal{O}] = \int d\tau \wedge d\bar\tau \ \tilde{V}(\tau)\, \Psi_\mu^J(\tau,\bar\tau)\cdot \mathcal{O}$$

where

- $\tilde{V}(\tau) = \frac{du}{d\tau} A(u)^{\chi(M)} B(u)^{\sigma(M)}$
- $\Psi_\mu^J(\tau,\bar\tau) = \frac{1}{\sqrt{y}} \sum_{k\in L+\mu} (-)^{B(k,K)} B(k,J)\, q^{-k_+^2/2}\, \bar{q}^{-k_-^2/2}$

$$y = \operatorname{Im}\tau,\ \mu \in L/2$$

- $\Gamma^0(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \bmod n \}$
  $\Gamma(n) = \{ \quad - \text{''} - \quad \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \}$
  congruence subgps
  $\to$ Jacobi ϑ are weight ½ modular forms for $\Gamma(2)$

$$\vartheta_2(\tau) = \sum_{r\in\mathbb{Z}+1/2} q^{r^2/2} \qquad \vartheta_3(\tau) = \sum_{r\in\mathbb{Z}} q^{r^2/2}$$

$$\vartheta_4(\tau) = \sum_{r\in\mathbb{Z}} (-)^r q^{r^2/2}$$

- mock modular forms

$$F(\tau) = -\frac{1}{\vartheta_4(\tau)} \sum_{n\in\mathbb{Z}} \frac{(-)^n\, q^{n^2/2 - 1/8}}{1-q^{n-1/2}}$$

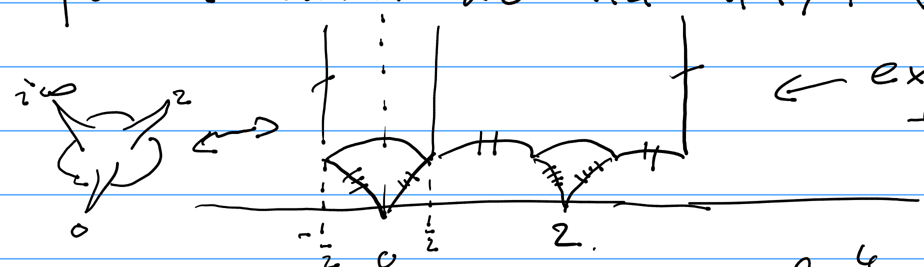
$$= 2q^{3/8}\left(1 + 3q^{1/2} + 7q + \cdots\right)$$

umbral moonshine

then $\hat{F}(\tau,\bar\tau) := F(\tau) - \frac{i}{2}\int_{-\bar\tau}^{i\infty} \frac{\eta(w)^3}{\sqrt{-i(w+\tau)}}\, dw$

transforms as weight $1/2$ m.f. for $\Gamma^0(4)$

- fundamental domain $\mathbb{H}/\Gamma^0(4)$:



← extras + inversions to 0, 2

- SW theory: $u(\tau) = \frac{1}{2}\frac{\vartheta_2^4 + \vartheta_3^4}{\vartheta_2^2\vartheta_3^2}$

  - $\Gamma^0(4)$ - inv

  $$\frac{da}{du} = \frac{1}{2}\vartheta_2(\tau)\vartheta_3(\tau)$$

- classification of $b_2^+ = 1$ lattices:

  L odd: