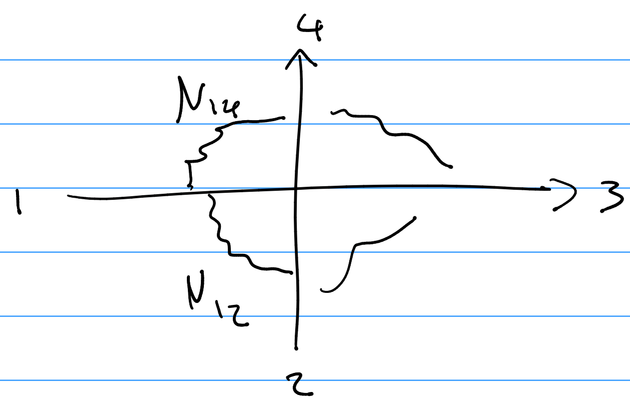
Gauge @ IGAP

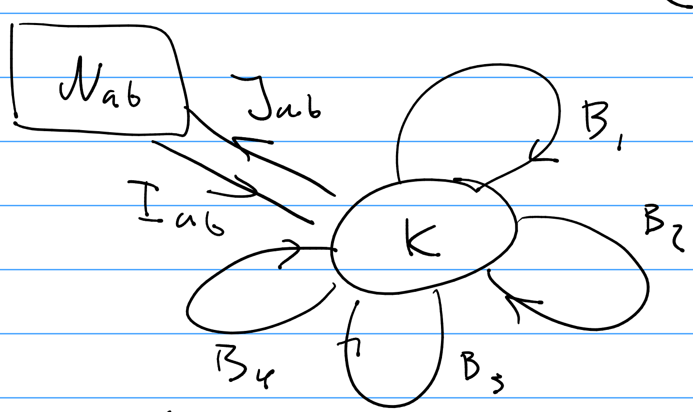
Nekrasov, cont'd.

Gauge origami

- generalized gauge theory
  = locally (in a CY4 Z) a collection of $\mathcal{N}=2^*$ 4d gauge theories
  - "integrates Euler cl. of $T^*\mathcal{M}_k^{t,N}$"
  - extra parameter, adjoint mass
    $\longrightarrow$ equiv. parameter for $\mathbb{C}^\times \circlearrowright$ fibers
  - in a toric symm. case we have up to 6
  - $U(N_{ab})$, $1 \leq a < b \leq 4$
    - ranks N_{ab} individual, but level k is shared: $\mathcal{M}_k^{\vec{N}}$



- ADHM description, quiver $\begin{cases} K = \mathbb{C}^k \\ N_{ab} = \mathbb{C}^{N_{ab}} \end{cases}$



- letting $\mu^{\mathbb{C}}_{ab} := [B_a, B_b] + I_{ab} J_{ab}$,

$$\begin{cases} \mu^{\mathbb{C}}_{ab} + \frac{1}{2}\varepsilon_{abcd}(\mu^{\mathbb{C}}_{cd})^{\dagger} = 0 \\ \sum_{a=1}^{4}[B_a, B_a^{\dagger}] + \sum_{a<b} I_{ab} I_{ab}^{\dagger} - J_{ab}^{\dagger} J_{ab} = \zeta \cdot \mathbb{1}_k \end{cases}$$

$$\begin{cases} B_a I_{bc} + \sum_{d} \varepsilon_{abcd} B_d^{\dagger} J_{bc}^{\dagger} = 0 & a \neq b \neq c \\ J_{ab} I_{cd} + \text{perm.} = 0 & a \neq b \neq c \neq d \end{cases} \Big/ U(k)$$

- exactly # equns = # "variables", although packaged strangely
  → collection of pts (w/ some redundancy)
- $\mathcal{M}^{\vec{N}}_{k}$

$$\downarrow \qquad \mu^{\mathbb{C}}_{ab} = 0 \text{ individually}$$

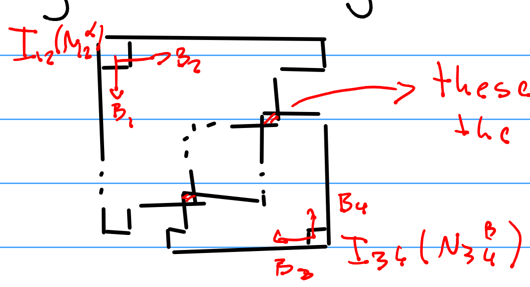
$$\bigcup \mathcal{M}^{+, N_{ab}}_{k_{ab}} \qquad k_{ab} = \dim \mathbb{C}[B_a, B_b]\, I_{ab}(N_{ab})$$

$$\sum_{a>c} k_{ab} \geq k$$

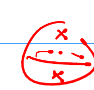
- toric $U(1)^3 \subset SU(4) \curvearrowright \mathbb{C}^4$
  by $(z_\alpha) \mapsto (q_\alpha z_\alpha)$, $\prod_\alpha q_\alpha = 1$

- $Z^{N=2^*}_{N_{1,2},k}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \int_{\mathcal{M}_k^{N_{12},0,0,0,0,0}} \text{"}1\text{"}$
  ε_3: adj. mass

- the intersections are described by
  $\lambda B_1^{i-1} B_2^{j-1} I_{12}(N_{12}^{\alpha}) + \mu B_3^{k-1} B_4^{l-1} I_{34}(N_{34}^{\beta}) = 0$
  $(\lambda, \mu) \in \mathbb{P}^1$ fixed set,
  or by "kissing" Young diagrams



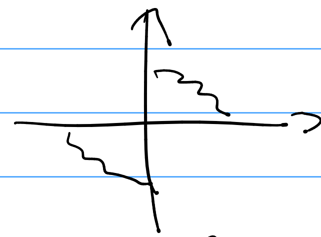
these poles are like the two poles on $\mathbb{P}^1$



- $Z^{\vec{N}}_{k} = \sum_{(\lambda^{(ab,\alpha)})}$ explicit rational function of $(\vec{a}, \vec{\varepsilon})$

- 2 important cases:

① $N_{12}, N_{34} \neq 0$, rest $= 0$



$$Z^{N_{12},N_{34}}_{k} = \sum_{\substack{(\lambda^{(12,\alpha)}) \\ (\lambda^{(34,\beta)})}} \mathbb{E}\left[-\frac{P_3 S_{12} S_{12}^*}{P_{12}^*} - \frac{P_1 S_{34} S_{34}^*}{P_{34}}\right] \times \mathbb{E}\left[-q_{12} S_{34} S_{12}^*\right]$$

where $\mathbb{E}\left[\sum e^{x_i} - \sum e^{y_i}\right] = \dfrac{\prod y_i}{\prod x_i}$,

explicitly, $\mathbb{E}\left[f(t)\right] = \exp \dfrac{d}{ds}\Big|_{s=0} \dfrac{1}{\Gamma(s)} \displaystyle\int_0^\infty \frac{dt}{t}\, t^s f(-t)$

is a ζ-regularisation and

$S_{12} = N_{12} - P_{12} K_{12}$ where

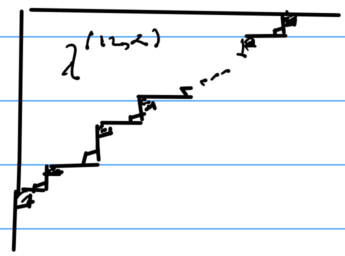
$$N_{ab} = \sum_{\alpha=1}^{n_{ab}} e^{t a_\alpha^{(ab)}}, \quad K_{ab} = \sum_\alpha e^{a_\alpha^{(ab)}} \sum_{(i,j)\in\lambda^{(ab,\alpha)}} q_a^{i-1} q_b^{j-1}$$

- now, $Z^{N_{12},N_{34}}(\vec{a}^{(12)}, \vec{a}^{(34)}, \vec{\varepsilon}; q) = \begin{pmatrix}\text{convenient} \\ q\text{-dep.} \\ \text{prefactors}\end{pmatrix} \cdot \sum_{k=1}^{\infty} q^k Z_k^{N_{12},N_{34}}$

$$= \Big\langle \underbrace{\chi^{N_{34}, \vec{a}^{(34)}}}_{\text{local observable}} \Big\rangle_{\substack{\mathcal{N}=2^* \\ U(N_{12}) \\ \mathbb{C}^2_{12}}}$$

- let $Y_{12}(x) := \mathbb{E}\left[-e^x\, s_{12}^*\right]$,

$$Y_{12}(x)\Big|_{\lambda^{(12,\alpha)}} = \prod_{\alpha=1}^{n_{12}} \frac{\prod_{\blacksquare\in\partial_+\lambda^{(12,\alpha)}}\left(x - a_\alpha^{(12)} - c_\blacksquare^{(12)}\right)}{\prod_{\square\in\partial_-\lambda^{(12,\alpha)}}\left(x - a_\alpha^{(12)} - c_\square^{(12)} - \varepsilon_1 - \varepsilon_2\right)}$$



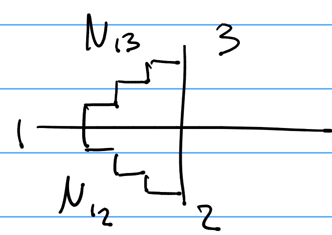
$$c^{(ab)}_{i,j} = \varepsilon_a(i-1) + \varepsilon_b(j-1)$$

$$\left\langle Y_{12}(x+\varepsilon_1+\varepsilon_2) + q\,\frac{Y_{12}(x-\varepsilon_3)\,Y_{12}(x-\varepsilon_4)}{Y_{12}(x)} + q^2\cdots\right\rangle$$

$$= \sum_{\lambda\text{-single partition}} \mu_\lambda(\vec{\varepsilon})\, q^{|\lambda|}\, \frac{\prod_{\square\in\partial_+\lambda} Y_{12}\left(x+\varepsilon_1+\varepsilon_2+c_\square^{(34)}\right)}{\prod_{\blacksquare\in\partial_-\lambda} Y_{12}\left(x + c_\blacksquare^{(34)}\right)}$$

$\to$ qq-characters $\hat{A}_0$

2 folded instantons



- now $\sum \mathbb{E}\left[-\frac{P_3 S_{12} S_{12}^*}{P_{12}^*} - \frac{P_2 S_{13} S_{13}^*}{P_{13}^*}\right] \times \mathbb{E}\left[+\frac{q_2 S_{13} S_{12}^* + q_3 S_{12} S_{13}^*}{P_1^*}\right]$

- $\mathbb{E}\left[\frac{e^x}{P}\right] \sim \Gamma\left(\frac{x}{\varepsilon_1}\right)$ Barnes Γ_2

- $Y(x) = \frac{Q(x)}{Q(x-\varepsilon_1)} = \mathbb{E}\left[-e^x\left(1-q_1^{-1}\right)\frac{S_{12}^*}{1-q_1^{-1}}\right]$

"Baxter operator"

- new trick: orbifolding

- finite subgp $\Gamma \subset SU(4)$ respecting the framing
- today $\Gamma = \mathbb{Z}/p \hookrightarrow \mathbb{T}$. (or, product of cyclic subgps)
- $\mathcal{M}^{\vec{N}}_{\vec{k}}$ = component in $(\mathcal{M}^{\vec{N}}_{k})^{\Gamma}$,

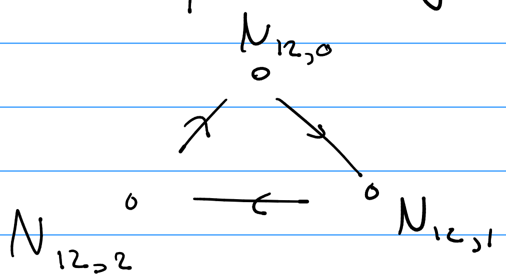
  $\vec{N} = (N_{ab,\omega})_{\omega=0}^{p-1}$, $N_{ab} = \bigoplus_{\omega=0}^{p-1} N_{ab,\omega} \otimes \underbrace{\mathcal{R}_\omega}_{\Gamma\text{ irrep}}$.

  $k = \bigoplus_{\omega=0}^{p-1} \underline{k_\omega} \otimes \mathcal{R}_\omega$ $q^k \mapsto \prod_{\omega=0}^{p-1} q_\omega^{k_\omega}$

  fractional instanton charges

① $\mathcal{N}=2^*$, $\Gamma = \mathbb{Z}/3 \curvearrowright \mathbb{C}^2_{34}$ by $(z_3, z_4) \mapsto (\omega z_3, \omega^{-1} z_4)$

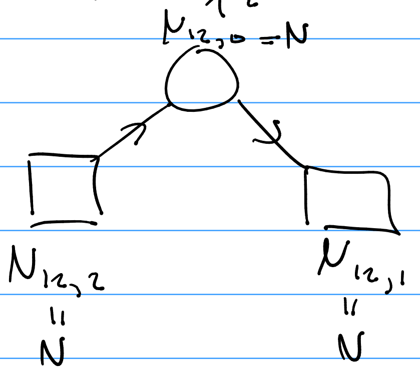
→ result: quiver gauge th



$\mathcal{N}=2$ w bifund. hypers

- let $q_0 = q$, $q_1 \to 0$, $q_2 \to 0$, all ranks $= N$

⇒



A_1-type theory $U(N)$ w $2N$ fundamental hypers

- add crossed instantons, N_{34}

- in $\mathbb{E}[-q_{12} S_{12}^* S_{34}]$, promote

$$S_{12} \rightsquigarrow \hat{S}_{12} = (N_{12,0} - P_{12} K_{12,0}) \mathcal{R}_0 + N_{12,1} \mathcal{R}_1 + N_{12,2} \mathcal{R}_2$$

$$S_{34} \rightsquigarrow \hat{S}_{34} = \hat{N}_{34} - (1 - q_3 \mathcal{R}_1)(1 - q_4 \mathcal{R}_2) \hat{K}_{34}$$

$$= q^{\frac{x}{\varepsilon_1}} Q(x - \varepsilon_1) \sum_{d=0}^{\infty} \frac{\Lambda^{2Nd}}{Q(x + \varepsilon_1 d)\, Q(x + \varepsilon_2(d-2))} \frac{\Gamma(1 + \frac{\varepsilon_2}{\varepsilon_1} + d)}{\Gamma(1 + \frac{\varepsilon_2}{\varepsilon_1}) d!}$$

3 $m_f \to \infty$, $q \to 0$ $\Big|$ $q \prod_f m_f = \Lambda^{2N}$ finite

4 $\varepsilon_2 \to 0$

$$Z_{N_{12}} \sim \exp \frac{1}{\varepsilon_2} W(\vec{a}, \varepsilon, \Lambda)$$

$$\left\langle Y(x + \varepsilon_1 + \varepsilon_2) + \frac{\Lambda^{2N}}{Y(x)} \right\rangle \overset{3}{=} T(x)$$
polynomial

4 $\varepsilon_2 \to 0$, remove $\langle \cdot \rangle$ and define...

$$y(x) = \lim_{\varepsilon_2 \to 0} \frac{\langle Y(x) \rangle}{\langle 1 \rangle} = \frac{Q(x)}{Q(x - \varepsilon)}$$

$$\Rightarrow \quad Q(x + \varepsilon) + \Lambda^{2N} Q(x - \varepsilon) = T(x)\, Q(x)$$

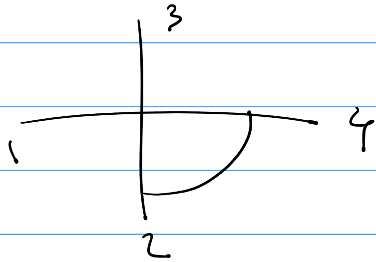
Bethe eqns for $x_{i,\alpha}$'s

$$\tilde{Q}(x) = \Lambda^{2N\frac{x}{\varepsilon}} \sum_{d=0}^{\infty} \frac{\Lambda^{2Nd}}{Q(x+\varepsilon d)\,Q(x+\varepsilon(d-1))}$$

n1) $\hat{Q}$ entire

n2) $\hat{Q}(x+\varepsilon) + \Lambda^{2N}\tilde{Q}(x-\varepsilon) = T(x)\,\tilde{Q}(x)$

- take one more orbifold, $\Gamma = \mathbb{Z}/N$ acting in $\mathbb{C}^2_{24}$



$Z^{\text{Sfc. defect}}$ solves

$$N\varepsilon_1\varepsilon_2\Lambda\frac{d}{d\Lambda} + \hat{H}_{\text{pToda}}(t_i = \varepsilon_1 x)$$

$\mathbb{C}_2 \times \left(\widetilde{\text{curve} \subset \mathbb{C}^2_{24}/\Gamma}\right)$

$\mathbb{P}^1$-s giving fractional fluxes

$\varepsilon_2 \to 0$, $\quad Z^{\text{Sfc}} = e^{\frac{1}{\varepsilon_2}W(a,\varepsilon,\Lambda)}\,\psi_a(x,\Lambda)$

$$\hat{H}_{pT}\psi_a = \underset{\displaystyle \varepsilon_2\Lambda\frac{d}{d\Lambda}W}{\underset{\downarrow}{E}}\,\psi_a$$

$$\frac{1}{\varepsilon_1}\frac{\partial W}{\partial a_i} = 2\pi\sqrt{-1}\,n_i \in 2\pi\sqrt{-1}\,\mathbb{Z}$$