Fautech 1

- today: Kashiwara Schapira § 1.

Lemma $\mathcal{A}$ ab. cat $\Rightarrow C(\mathcal{A})$ ab. cat.

Pf sketch. Let $A^\bullet \xrightarrow{\varphi} B^\bullet$ in $C(\mathcal{A})$.
Define kernel as $\ker\varphi = (K^\bullet, d_A|_{K^\bullet})$
where $K^n := \ker(A^n \xrightarrow{\varphi^n} B^n)$.
Define direct sum $(A \oplus B)^\bullet \rightsquigarrow (A^n \oplus B^n, d^n = \begin{pmatrix} d_A^n & \\ & d_B^n \end{pmatrix})$

Prop To every s.e.s $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$
in $C(\mathcal{A})$ we can associate natural
maps $H^n(C) \to H^{n+1}(A)$ s.t.

i) $\dots \to H^n(A) \to H^n(B) \to H^n(C) \to H^{n+1}(A) \to \dots$
is exact

ii) the definition is functorial, i.e.
given comm. diag in $C(\mathcal{A})$ w
exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \tilde{A} \arrow[r] & \tilde{B} \arrow[r] & \tilde{C} \arrow[r] & 0 \end{tikzcd}$$

then $\forall n \in \mathbb{Z}$

$$\begin{tikzcd} H^n(C) \arrow[r] \arrow[d, "h^n(\gamma)"] & H^{n+1}(A) \arrow[d, "h^{n+1}(\alpha)"] \\ H^n(\tilde{C}) \arrow[r] & H^{n+1}(\tilde{A}) \end{tikzcd}$$

commutes

Def Let $A^\bullet \xrightarrow{\varphi} B^\bullet$ in $C(\mathcal{A})$. We associate to it $M(\varphi) \in ob\, C(\mathcal{A})$ and morphisms

$$A^\bullet \xrightarrow[\varphi]{} B^\bullet \xrightarrow[\alpha(\varphi)]{} M(\varphi) \xrightarrow[\beta(\alpha)]{} A[1]$$

We call $M(\varphi)$ the mapping cone of φ.

Warning Convention used: in $A[k]$, $d^n[k] := (-)^k d^{n+k}$

We let $M(\varphi)^n := A[1]^n \oplus B^n$,

$$d^n_{M(\varphi)} = \begin{pmatrix} d^n_{A[1]} & \\ \varphi^{n+1} & d^n_B \end{pmatrix}, \text{ and}$$

$$\alpha(\varphi) = \begin{pmatrix} 0 \\ id_{B^n} \end{pmatrix}, \quad \beta(\varphi) = \begin{pmatrix} id_{A[1]^n} & 0 \end{pmatrix}$$

Lemma (informal) The triangles

$$\begin{tikzcd} A \arrow[rr, "\varphi"] & & B \arrow[dl, "\alpha(\varphi)"] \\ & M(\varphi) \arrow[ul, "+1"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M(\alpha(\varphi)) \arrow[rr, "\tau'"] & & B \arrow[dl, "\alpha(\varphi)"] \\ & M(\varphi) \arrow[ul] & \end{tikzcd}$$

"are the same" in $K(\mathcal{A})$ (but not in $C(\mathcal{A})$).

Lemma Given $A^\bullet \xrightarrow{\varphi} B^\bullet$ in $C(\mathcal{A})$ $\exists$ $A[1] \xrightarrow{\psi} M(\varphi)$ s.t.

i) ψ is iso in $K(\mathcal{A})$

ii) the following commutes in $K(\mathcal{A})$

$$\begin{tikzcd} B \arrow[r, "\alpha(\varphi)"] \arrow[d, equal] & M(\varphi) \arrow[r, "\beta(\varphi)"] \arrow[d, equal] & A[1] \arrow[r, "-\varphi[1]"] \arrow[d, "\psi"] & B[1] \arrow[d, equal] \\ B \arrow[r, "\alpha(\varphi)"'] & M(\varphi) \arrow[r, "\alpha(\alpha(\varphi))"'] & M(\alpha(\varphi)) \arrow[r, "\beta(\alpha(\varphi))"'] & B[1] \end{tikzcd}$$

Pf sketch. $M(\alpha(\varphi))^n = B[1]^n \oplus M(\varphi)^n$
$= B[1]^n \oplus A[1]^n \oplus B^n$

So define $\psi^n : A[1]^n \to M(\alpha(\varphi))^n$
by $\psi^n = (-\varphi[1]^n, \mathrm{id}_{A[1]^n}, 0)$.
Also define $\tilde{\psi}^n : M(\alpha(\varphi))^n \to A[1]^n$
by $\tilde{\psi}^n = (0, \mathrm{id}_{A[1]}, 0)$ and note

$$\tilde{\psi} \circ \psi = \mathrm{id}_{A[1]}, \quad \psi \circ \tilde{\psi} = \begin{pmatrix} 0 & -\varphi[1] & 0 \\ 0 & \mathrm{id}_{A[1]} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and show iso in $K(\mathcal{A})$ $\square$

Def. A triangle in $K(\mathcal{A})$ is seq. of mor

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

A triangle is called distinguished :
if it is isomorphic to

$$\tilde{A}^\bullet \xrightarrow[\varphi]{} \tilde{B}^\bullet \to M(\varphi) \to \tilde{A}^\bullet[1]$$

where a morphism is
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\gamma"] \arrow[dr, phantom, "\circlearrowleft"] & A[1] \arrow[d, "\alpha"] \\
\tilde{A} \arrow[r] & \tilde{B} \arrow[r] & \tilde{C} \arrow[r] & \tilde{A}[1]
\end{tikzcd}$$

Cor (of lemma) If $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\eta} A[1]$
distinguished then so is
$$B \xrightarrow{\psi} C \xrightarrow{\eta} A[1] \xrightarrow{-\varphi[1]} B[1]$$

- we now formalise dist. triangles

Prop Distinguished triangles obey the following

(TR0) If a triangle isom. to a dist. tri.,
then it is dist.

(TR1) $\forall A \in \mathrm{ob}\, K(\mathcal{A})$, $A \xrightarrow{id} A \to 0 \to A[1]$
is dist.

(TR2) any mor $\varphi : A \to B$ can be included
in a dist tr $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\xi} A[1]$

(TR3) $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\xi} A[1]$ distinguished
$\iff B \xrightarrow{\psi} C \xrightarrow{\xi} A[1] \xrightarrow{-\varphi[1]} B[1]$ distinguished

(TR4) Given comm diag $\exists$ mor of dist tr

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\ \tilde{A} \arrow[r] & \tilde{B} \arrow[r] & \tilde{C} \arrow[r] & \tilde{A}[1] \end{tikzcd}$$

(TR5) Octahedron.

Def A *triangulated category* $\mathcal{T}$ is
an add. cat with autoeq. $[1]$ (and $[k] = [1] \circ \dots \circ [1]$, $[-k] \dots$)
and a collection of triangles called
distinguished which satisfy (TR0)-(TR5).

- key structure on derived cats.: triangulated cat
- today we show this structure on $K(\mathcal{A})$

Philosophy $F : \mathcal{A} \to \mathcal{B}$ half-ex. among ab. cats
$\implies$ derived fctor $RF : D^+(A) \to D^+(B)$ unique, if exists.
$LF : D^-(A) \to D^-(B)$

Lemma If $0 \to A^\bullet \xrightarrow{\varphi} B^\bullet \xrightarrow{\psi} C^\bullet \to 0$ ex. seq. in $C(\mathcal{A})$ then $\exists$ nat map $M(\varphi) \to C$, $A[1] \oplus B \xrightarrow[(0,\psi)]{} C$ on objects, which is iso in $K(\mathcal{A})$.

Def Let $\mathcal{T}$ triang. cat, $\mathcal{A}$ ab. cat.
A *cohomological* functor $F: \mathcal{T} \to \mathcal{A}$ is additive functor s.t. $\forall$ dist. tr.
$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{\eta} X[1]$,
$F(X) \xrightarrow{F(\varphi)} F(Y) \xrightarrow{F(\psi)} F(Z)$ is exact in $\mathcal{A}$.

Exercise Using (TR3) show (F coh. funct)
$\Rightarrow \quad \cdots \to F(X) \to F(Y) \to F(Z) \to F(X[1]) \xrightarrow{-F(\varphi[1])} F(Y[1]) \to \cdots$
is exact

Prop If $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{\eta} X[1]$ is dist. tr.,
then $\psi \circ \varphi = 0$

Pf
$$\begin{array}{ccccccc} X & \xrightarrow{id} & X & \to & 0 & \to & X[1] \\ \| & & \downarrow f & & \downarrow & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \to & X[1] \end{array}$$
by (TR1&4)

Thm let $\mathcal{T}$ tr. cat, $W \in \mathrm{ob}\,\mathcal{T}$. Then
$\mathrm{Hom}_{\mathcal{T}}(W, -): \mathcal{T} \to (Ab)$ is cohomological.

Rmk. Same for representables

Pf: $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{\eta} X[1]$ dist. tr.
$\Rightarrow \forall W \quad h^W(X) \to h^W(Y) \to h^W(Z)$ exact in Ab.
$\alpha \longmapsto \varphi \circ \alpha$
$\beta \longmapsto \psi \circ \beta$

$\beta : W \to Y$ s.t. $\psi \circ \beta = 0$,

$$\begin{tikzcd}
W \arrow[r, "id"] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[r, "0"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & W[1] \arrow[d, dashed] \\
X \arrow[r, "\varphi"'] & Y \arrow[r, "\psi"'] & Z \arrow[r, "\lambda"] & X[1]
\end{tikzcd} \qquad \square$$

Prop $\mathcal{A}$ ab. cat $\Rightarrow$ $H^0 : K(\mathcal{A}) \to \mathcal{A}$ cohom. f.

Pf. Enough to show for dist. tr

$$X \xrightarrow[\varphi]{} Y \xrightarrow[\psi]{} M(\varphi) \xrightarrow[\lambda]{} X[1],$$

$$H^0(Y) \to H^0(M(\varphi)) \to H^0(X[1])$$

exact. Follows from:

Lemma If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ ex in $C(\mathcal{A})$ then $\exists$ nat map $M(\varphi) \to C$ iso in $K(\mathcal{A})$