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Further properties of canonical S.T.s

- 7 properties selected by Connes
  → reconstruction thm
- in general, we would like to extract properties of spaces

Dimension

- we study $\mathbb{T}^n = M = \mathbb{R}^n / 2\pi\mathbb{Z}^n$
  - $N_\ell := \#\{ \Lambda_t \cap B_\ell \}$

    $\Lambda_t = \mathbb{Z}^n + t/2, \quad t = \{ \{t_j\} \mid j = 1, \dots, n, \ t_j = 0, 1 \}$

    $B_\ell = \{ \lambda \in \mathbb{R}^n \mid |\lambda| \le \ell \}$

→ $N_\ell \sim \mathrm{vol}\, B_\ell \sim n^{-1} \underbrace{V_{n-1}}_{2\pi^{n/2}/\Gamma(\frac{n}{2})} \ell^n$ as $\ell \to \infty$

where $a_\ell \sim b_\ell$ iff $\lim \frac{a_\ell}{b_\ell} = 1$, $b_\ell = 0$ for finitely many ℓ.

- wlog $\exists\, |D|^{-1}$, $(|D| + \varepsilon)^{-1}$, $\varepsilon > 0$
  - let $s > 0$, consider Σ of first N_ℓ eigenvals

$$\sigma_{N_\ell}(|D|^{-s}) := 2^m \sum_{1 \le |\lambda| \le \ell} |\lambda|^{-s}$$

$$\overset{?}{\sim} 2^m \int_0^\ell r^{-s} \underbrace{(N_{r+dr} - N_r)}_{\frac{dN}{dr} dr}$$

$$= 2^m V_{n-1} \int_0^\ell r^{-s+n-1} dr = 2^m V_{n-1} \begin{cases} \frac{\ell^{n-s}}{n-s}, & s \ne n \\ \log \ell, & s = n \end{cases}$$

- since $\ell \sim \left(n \frac{N_\ell}{V_{n-1}} \right)^{1/n}$,

$$\sigma_{N_\ell}(|D|^{-s}) \sim 2^m \begin{cases} c N_\ell^{\frac{n-s}{n}} & s \ne n \\ c' + \frac{V_{n-1}}{n} \log N_\ell & s = n \end{cases}$$

$$\Rightarrow \frac{\zeta_N(|D|^{-s})}{\log N} \overset{N\to\infty}{\sim} \begin{cases} c \cdot \frac{N^{\frac{n-s}{n}}}{\log N} \nearrow \infty & , s \lessgtr n \\ V_{n-1}/n \searrow 0 & , s = n \end{cases}$$

$\Rightarrow \exists! s$ s.t. $\zeta_N(|D|^{-s}) \sim \frac{V_{n-1}}{n} \log N$

$\to s = n$, dimension

- independent of shift t
- holds for $T = \mathbb{R}^n / k_1\mathbb{Z} \times k_2\mathbb{Z} \times \dots \times k_n\mathbb{Z}$
- holds for any (spin) mfd due to Weyl thm $\ell_k(|D| \text{ or } \Delta^{1/2}) \sim k^{1/n}$

- more modern: **Dixmier ideal** $\mathcal{L}^{1\infty} = \mathcal{L}^{1+} = \{T \in \mathbb{B}(\mathcal{H}) \mid \zeta_N(T) = O(\log N)\}$ (part. sum of sing. eigenvals of $\sqrt{T^*T}$)

$\subseteq \mathcal{L}^p \subseteq K(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$

$\hookrightarrow$ p-summable, $\sum |\lambda|^p < \infty$

- recall, $f_n = O(g_n)$ iff $\frac{f_n}{g_n}$ bdd $\forall n$

$f_n = o(g_n)$ iff $f_n/g_n \to 0$

- $\|T\|_{1+} := \sup \left\{ \frac{\zeta_N(T)}{\log N} \right\}$ is norm

$\leftarrow \exists \infty$ family of ≥ 0 tracial states, but never constructively shown

- all coincide and give ≥ 0 tracial state on closed subspace $\subseteq \mathcal{L}^{1+}$ of **measurable** T, i.e. T s.t. $\exists \lim_{\ell\to\infty} \tau_\ell(T)$, where

$$\tau_\ell(T) := \frac{1}{\log \ell} \int_{\text{any } a>0}^{\ell} \frac{\zeta_u(|T|)}{\log u} \frac{du}{u}$$

$\leftarrow$ interpolation of $\zeta_N(|T|)$

Def. $\mathrm{Tr}^+(T) := \lim_{\ell\to\infty} \tau_\ell(T)$, $T \geq 0$

- main example, $|D|^{-n} \in \mathcal{L}^{1,\infty}$ is measurable and $\mathrm{Tr}^+(|D|^{-n}) = 2^m v_{n-1}/n$
  - $\to$ the coeff we got before