Antonini

- E invariant closed, $\pi \Rightarrow \pi \sim \pi_E \oplus \pi_{E^\perp}$
- $T \in Hom(\pi_1, \pi_2)$, polar decomp. $T = u|T|$
  with $u \in Hom(\pi_1, \pi_2)$, $|T| \in End(\pi_1)$

$$\begin{tikzcd} H_1 \arrow[r, bend left, "T"] & H_2 \arrow[l, bend left, "T^*"] \end{tikzcd}$$

- consequence: $\pi_1 |_{\overline{T^* H_2}} \sim \pi_2 |_{\overline{T H_1}}$

Def π_1, π_2 are *disjoint* if they don't have equivalent subrepresentations

Thm π_1, π_2 are disjoint iff $Hom(\pi_1, \pi_2) = \{0\}$.

- take π, H_π and look at, $S \subset H_\pi \rightsquigarrow$
  $\{\pi(a) S \mid a \in A\} \subset H_\pi$, stable by π

Def. S is *totalising* for π if $\overline{\{\pi(a) S\}} = H$

Def. If $S = \{x\}$ a singlet, $x \in H_\pi$, we say x is *cyclic* for π, if $\overline{\pi(a) x} = H_\pi$.

Def If H_π is totalising for π, π is called *nondegenerate*.

- facts:
  - sum of disj. cyclic representations is cyclic $\rightarrow$ so a cyclic rep need not be irred
  - any nondeg. rep of $*$-alg is a direct sum of cyclic reps.

GNS

- any $f: A \to \mathbb{C}$ linear we call a linear form
- it is positive if $f(A_+) \subseteq \mathbb{R}_+$
  - it can be shown pos. forms are continuous
- the form $\mathbb{B}(H) \to \mathbb{C}$ given by $x \in H$ and $T \mapsto \langle Tx, x \rangle =: f_x(T)$ is positive

- Claim:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, bend right, "\text{positive form}"'] & \mathbb{B}(H) \arrow[d, "f_x"] \\ & \mathbb{C} \end{tikzcd}$$

, where $\pi(a^* a) = \pi(a)^* \pi(a)$

- define sesquilinear nonneg. form

$$A \times A \to \mathbb{C}$$
$$(a, b) \mapsto f(a^* b) =: \langle a | b \rangle_f$$

- $\langle a | a \rangle_f \geq 0$
- $\ker \langle \cdot | \cdot \rangle_f = \{ a \in A \mid \langle a | b \rangle = 0 \ \forall b \in A \}$
- completion:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi_f"] & A / \ker \langle \cdot | \cdot \rangle_f \arrow[dl, bend left] \\ H_f & \end{tikzcd}$$

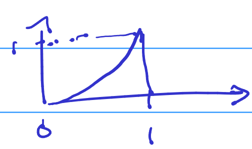
$$\langle \psi_f(a) | \psi_f(b) \rangle = f(a^* b)$$

Thm Let $f: A \to \mathbb{C}$ positive form.
$\exists!$ involutive rep. $\pi_f : A \to \mathbb{B}(H_f)$
such that $\forall\, a, b \in A$
$$\eta_f(ab) = \pi_f(a)\,\eta_f(b)$$

Rmk. Think of this as a left A-mod map ${}_A A \to {}_A \mathbb{B}(H_f)$, $a \cdot v = \pi_f(a)\, v$.

Pf. $a \in A$, $\|a\| \leq 1$. $\mathrm{Spec}(a^* a) \subset [0,1]$ implies we can use continuous functional calculus to define $c := h(a^* a)$, where h is the map

$$s \overset{h}{\longmapsto} 1 - (1-s)^{1/2}$$



and $c \in A$. The following holds:
$$\begin{cases} c = c^* \in A \\ (1-c)^2 + a^* a = 1 \text{ in } \tilde{A} \end{cases}$$

$\forall\, b \in A$, $(b - cb)^*(b - cb) + b^* a^* a b = b^* b$ follows, and so

$$\|\eta_f(b)\|^2 = \|\eta_f(ab)\|^2 + \|\eta_f(b - cb)\|^2$$

This tells us that for a fixed, the map $\underset{A}{\underset{\in}{b}} \mapsto \eta_f(ab) \in H_f$

satisfies $\|\eta_f(ab)\| \leq \|\eta_f(b)\|$.
So it extends to the statement of thm. $\square$

Prop A unital. Any pos. form f is bounded, with $\|f\| = f(1)$

Pf. $|f(a)| = |f(1^* a)| = |\langle \eta_f(1) | \eta_f(a) \rangle_{H_f}|$

$\leq \underbrace{\|\eta_f(1)\|}_{=\|\xi\|} \cdot \|\eta_f(a \cdot 1)\|$

$\overset{GNS}{=} \|\xi\| \cdot \|\pi_f(a)\| \, \|\eta_f(1)\|$

$= \underbrace{\|\xi\|^2}_{f(1)} \cdot \|a\| \qquad \square$

- recall we defined approx unit as directed set $\Lambda := \{ x \in A_+ | \ \|x\| < 1 \}$ under increasing order, so that $\lim_{x \in \Lambda} xa = \lim_{x \in \Lambda} ax = a$

Thm $a_\lambda \leftarrow \lambda \in I$, $z = \lim_{\lambda \in I} a_\lambda$ if $\forall \mathcal{U} \ni z$

Prop A C^*-alg.

1) any pos. form f is continuous, with $\lim_{a \in \Lambda} f(a) = \|f\|$.

11) f is positive iff (f cont.) $\wedge$ $\left(\sup\{ \operatorname{Re} f(a) : a \in \Lambda \} = \|f\| \right)$

Thm. $f : A \to \mathbb{C}$ positive form.
$\exists$ a vector $\xi_f \in H_f$, normalised as $\|\xi_f\|^2 = \|f\|^2$, such that

- $f(x) = \langle \xi_f | \pi_f(x) \xi_f \rangle$
- $\forall x \in A$, $\eta_f(x) = \pi_f(x) \xi_f$.

Pf. $a, b \in \Lambda$, $b \geq a \Rightarrow \underbrace{(b-a)^2 \leq b-a}$

$$\|\eta_f(b) - \eta_f(a)\|^2 = f((b-a)^2) \leq f(b) - f(a)$$
$$\leq \|f\| - f(a)$$

So $\underset{\Lambda}{a} \mapsto \eta_f(a) \in H_f$ converges,

let $\xi_f := \lim_{a \in \Lambda} \eta_f(a)$.

- $\forall x \in A$, $\eta_f(x) = \lim_{a \in \Lambda} \eta_f(xa)$
$$= \lim_{a \in \Lambda} \pi_f(x) \eta(a) = \pi_f(x) \xi_f$$
- $\forall x \in A$, $f(x) \overset{f \text{ cont.}}{=} \lim_{a \in \Lambda} f(ax) = \lim_{a \in \Lambda} \langle \eta_f(a) | \eta_f(x) \rangle$
$$= \lim_{a \in \Lambda} \langle \eta_f(a) | \pi_f(x) \xi_f \rangle = \langle \xi_f | \pi(x) \xi_f \rangle.$$

It follows that $\|f\| \leq \|\xi_f\|^2$, but
$\|\xi_f\|^2 = \lim_{a \in \Lambda} f(a^2) \leq \|f\|$ $\square$

- we completed the GNS construction

$$f \rightsquigarrow (H_f, \pi_f, \xi_f)$$

- start with A invol. algebra with cyclic rep. $\pi : A \to \mathbb{B}(H)$, x cyclic vect.

- we define positive form f by $a \mapsto \langle x, \pi(a)x \rangle$

$$\begin{array}{ccc} A & \xrightarrow{\pi} & \mathbb{B}(H) \\ {\scriptstyle \eta_f}\big\downarrow & & \\ H_f & & \end{array}$$

$H_f :=$ Hsdff completion of A,

$$\langle \eta_f(a) | \eta_f(b) \rangle = f(a^* b) = \langle \pi(a)x, \pi(b)x \rangle$$

- both $\{\pi(A)x\} \subset H$ & $\{\eta_f(a)\} \subset H_f$ are dense

- $\exists !\ U \in \mathcal{U}(H, H_f)$ s.t. $\left\{ \begin{array}{l} U(\pi(a)x) = \eta_f(a) \\ \forall a \in A \end{array} \right.$

- call $a \mapsto U\pi(a)U^*$ $\pi_f : A \to \mathbb{B}(H_f)$

- then (H_f, π_f, Ux) is $GNS(f)$

- we more or less have

Prop i) a rep. is cyclic iff equiv to a GNS
ii) any nondeg. rep is sum of GNS's.

- let A C^*-alg

$$X := \left\{ A \xrightarrow{f} \mathbb{C} \text{ positive lin forms } \|f\| \leq 1 \right\}$$

- X is cpt in weak convergence

$C(X, \mathbb{R})$ = real ordered Banach space

$$\left[\begin{array}{l} A_h = \text{real ordered} \\ \text{Ban. sp. of} \\ \text{self adj. elements} \\ \text{of } A \end{array}\right] \xhookrightarrow[\text{isometry}]{\text{Embedding}} C(X, \mathbb{R})$$

$$a \longmapsto \left(f \mapsto f(a) \right)$$

- tensor products:

- Wegge-Olsen: K-theory & C^*-algebras
  - appendix (C) tensor products
- define min, max, show properties.