Stoppa:

- we can check that $g_{i\bar{j}} = \overline{g_{j\bar{i}}}$, which means that positive definiteness of g implies the hermitian matrix $(g_{i\bar{j}})$ is also pos.def.

Rmk. Previously we showed $d\omega = 0 \iff \nabla^g J = 0$ on a cpx mfd.

Claim: $d\omega = 0 \iff \dfrac{\partial g_{i\bar{j}}}{\partial z_k} = \dfrac{\partial g_{k\bar{j}}}{\partial z_i}$ (just write out in local coords.)

- now extend $\nabla = \nabla^g$ to act on $TM \otimes \mathbb{C}$ by $\mathbb{C}$-linearity
- write $\nabla_{\frac{\partial}{\partial z_j}} \dfrac{\partial}{\partial z_k} = \Gamma^i_{jk} \dfrac{\partial}{\partial z_i} + \Gamma^{\bar{i}}_{jk} \dfrac{\partial}{\partial \bar{z}_i}$
- consider the Kähler case

Lemma. The only nonvanishing Christoffels are Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$. Also, $\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^i_{jk}}$.

Pf. Recall $(\Gamma^k_{ij})_{\mathbb{R}} = \frac{1}{2} g^{kl}(g_{il,k} + g_{jl,i} - g_{ij,l})$

The wholly analogueous formula holds in the complexified case, except we must allow both kinds of indices. But:

$\Gamma^k_{i\bar{j}} = \frac{1}{2} g^{k\bar{l}}(g_{i\bar{l},\bar{j}} + \cancel{g_{\bar{j}\bar{l},i}} - g_{i\bar{j},\bar{l}}) + \frac{1}{2}\cancel{g^{kl}}(\ldots$

$=0$ by Kähler

$= 0.$

Similarly other cases. The last claim follows from $\overline{g_{i\bar{j}}} = g_{j\bar{i}}$.

Cor. Explicitly, $\Gamma^i_{jk} = g^{i\bar{l}} \partial_j g_{k\bar{l}}$.

Curvature tensor of a Kähler manifold.

N.B. From now on we use the opposite sign convention, so $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

Lemma. $J(R(X,Y)Z) = R(X,Y)(JZ)$, $\forall\, X,Y,Z$

Cor. $g(R(X,Y)Z,W) = g(R(X,Y)JZ, JW)$.

Cor. Upon complexification, $g(R(X,Y)Z,W)=0$ $\forall X,Y$ if Z,W are of the same type.

Def. $R_{i\bar{j}k\bar{l}} := g\left(R\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right)\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$

Lemma. The only nonvanishing symbols R_{abcd} are $R_{i\bar{j}k\bar{l}}$, upon complexification.

Lemma. $R_{i\bar{j}k\bar{l}} = -\partial_k\partial_{\bar{l}}\, g_{i\bar{j}} + g^{p\bar{q}}(\partial_k g_{i\bar{q}})(\partial_{\bar{l}}\, g_{p\bar{j}})$

Pf. It's a computation.

Ricci curvature.

Def. Define 2-form $R(X,Y) := Ric(JX,Y)$
$\to$ locally, $R = \sqrt{-1}\, R_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$

Lemma. $R_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log\det(g_{p\bar{q}})$.