QFT:

Nature of perturbation theory

- take series expansion $f(\lambda) = \sum_{n\in\mathbb{N}} c_n (\lambda-\lambda_0)^n$
  → analysis: rad. of convergence $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| > 0$

- Dyson (1952): radius of convergence of QED is 0.
  - any observable in pert. theory can be written as $G(\alpha) = \sum_{n\in\mathbb{N}} \alpha^n G_n$
    → but any value $\alpha < 0$ is nonsensical

- euclidean field theory: $G^{(n)}(\hbar) = \int D\varphi\, \varphi_1 \cdots \varphi_n e^{-\frac{1}{\hbar} S[\varphi]}$
  $= \sum_{p\in\mathbb{N}} \hbar^{p-1} G_p^{(n)}$
  → never makes sense for $\hbar < 0$, since exponent becomes divergent

  → also, for $G^{(n)} = \int D\varphi\, \underbrace{e^{-(S_0+\lambda S_{int})}}_{e^{-S_0} \sum (-\lambda)^p S_{int}} \varphi_1 \cdots \varphi_n$
  $\neq \sum \lambda^p G_p^{(n)}$ → calculable,
  since the sum is not uniformly conv.

- asymptotic series: $Z(\lambda) - \sum_{n=0}^{N-1} Z_n \lambda^n = O(\lambda^N)$, $\lambda \to 0$

  - for QED, $Z_n = c \cdot a^n n!\, n^b (1 + O(\frac{1}{n}))$
  - take $Z_n = a^n n!$
    $\Rightarrow Z(\lambda) - \sum_{n=0}^{N-1} Z_n \lambda^n = Z_N \lambda^N + \cdots$,
    so let's minimize $Z_N \lambda^N \sim a^N N^N e^{-N} \lambda^N$
    $= e^{N(-1+\log Na\lambda)}$

    $\Rightarrow N_{best} = \frac{1}{a\lambda}$, error $\sim e^{-\frac{1}{a\lambda}}$
  - in QED, $a = O(1)$, $\lambda \sim 10^{-2}$

- In QED, we have no realistic worry, but in QCD we could reach this within several loops

Borel summation

- given $Z(\lambda) = \sum \lambda^n Z_n$, $BZ(t) := \sum_{n=0}^{\infty} \frac{t^n Z_n}{n!}$

- now put $Z_B(\lambda) = \int_0^\infty dt\, e^{-t} BZ(t) = \ldots = \sum \lambda^n Z_n$ asymptotic

if we exchange sum and integral, which is illegal, since $BZ(t)$ has a finite radius of conv.

$\rightarrow$ e.g. $Z_n = a^n \cdot n!$

$\rightarrow BZ(t) = \frac{1}{1-at}$,

$a > 0$: $Z(\lambda)$ not Borel ressumable

$a < 0$ Borel resommable

$\longrightarrow$ $d=4$ qft, usually not Borel resummable, q.m. in $d=1,2,3$ might be