Bertola

- $T = T_{A,B}$, $A, B : \gamma \longrightarrow GL_n$, $\mathcal{H} = L^2(\gamma, \cdot) \otimes \mathbb{C}^k$
  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $Tf = A\Pi_+ f - B\Pi_- f$,
  $\Pi_+ = C_+$, $\Pi_- = -C$

- let $T = \Pi_+ A - \Pi_- B$, $AB^{-1} = R_- \underset{\text{diag}(z^{k_1}, \dots, z^{k_n})}{D} R_+$

- $\dim \ker T = \sum_{k_i < 0} -k_i$, $\dim \operatorname{coker} T = \sum_{k_i > 0} k_i$

  $\operatorname{ind} AB^{-1} = \operatorname{ind}_\gamma \det AB^{-1} = \sum_j k_j$

$\to T$ inv iff $D = \mathbb{1}$

- write $T = (\underbrace{\Pi_+ A B^{-1} - \Pi_-}_{=\tilde{T}}) B$

Claim: $\tilde{T}$ inv iff ind $= 0$ and
$\Pi_+ A B^{-1} \Pi_+ \in \operatorname{Aut} \mathcal{H}_+$

Pf. ind $= 0$, enough to show $\dim \ker \tilde{T} = 0$.
$\ker \tilde{T} = \{ \vec{f} \mid \Pi_+ A \underset{\shortparallel}{B^{-1} \vec{f} - f_-} = 0 \}$

$\underbrace{\Pi_+ AB^{-1} f_+ + \Pi_+ AB^{-1} f_-}_{\mathcal{H}_+} - f_-$

$\Leftrightarrow f_- = 0 \Rightarrow \Pi_+ A B^{-1} \Pi_+ f = 0$

- standard Töplitz ops $M : \gamma \to GL_n(\mathbb{C})$ (ind $M = 0$)
- $T_M : \mathcal{H}_+ \to \mathcal{H}_+$, $\vec{f} \mapsto \Pi_+(Mf)$

- $M = R_- R_+$, $R_+ = R_-^{-1} M$ ($R_\pm$ bdd, analytic, inv (w/ bdd inv) in $D_\pm$)
- so study (RHP) $\begin{cases} \Gamma_+ = \Gamma_- M \\ \Gamma(\infty) = \mathbb{1} \end{cases}$ $\left(\begin{array}{l} \Gamma_-(z) = R_-(\infty) R_-^{-1}(z), \\ \Gamma_+(z) = R_-(\infty) R_+(z) \end{array} \right)$

Prop (RHP) has (unique) soln iff the equ. in $L^2(\gamma,(dz))\otimes\mathbb{C}^n$ has soln

$$\vec{F}(z)=\vec{c}+\int_\gamma \frac{F(w)(M-\mathbb{1})}{(w-z)_-}\frac{dw}{2\pi i}$$

$$=\vec{c}+\Pi_-\left(\vec{F}(M-\mathbb{1})\right),\ \forall \vec{c}\in\mathbb{C}^n.$$

Pf $\Leftarrow$ Supp $\overset{n\times n}{F(z)}$ is a fund. soln

$F(z)=\mathbb{1}+C_-(FM)(z),\ z\in\gamma$

Now define

$$\Gamma(z)=\mathbb{1}+\int_\gamma \frac{F(w)(M-\mathbb{1})}{w-z}\frac{dw}{2\pi i},\ z\notin\gamma$$

By Sokhotsky-Plemelj

$\Gamma_+(z)-\Gamma_-(z)=F(z)(M(z)-\mathbb{1})$

OTOH,

$\Gamma_-=\mathbb{1}+C_-(F(M-\mathbb{1}))=F(z)$,

$\Rightarrow \Gamma_-=F$, and $\Gamma_+=\Gamma_-M$.

$\Gamma(\infty)=0$ clear.

$\Rightarrow$ Supp. Γ solves (RHP).

$\Gamma_+-\Gamma_-=\Gamma_-(M-\mathbb{1})$

Consider

$$G(z)=\mathbb{1}+\oint_\gamma \frac{\Gamma_-(w)(M(w)-\mathbb{1})}{(w-z)}\frac{dw}{2\pi i},\ z\notin\gamma.$$

It solves $G_+-G_-=\Gamma_-(M-\mathbb{1})=\Gamma_+-\Gamma_-$.

Let $R=G-\Gamma$

$\Rightarrow R_+=R_-\Rightarrow R$ is entire

But $R(\infty)=0\Rightarrow R\equiv 0\Rightarrow G=\Gamma$.

Example $n=1$ $\begin{cases} \Gamma_+(z) = \Gamma_-(z)\, M(z) \\ \Gamma(\infty) = 1 \end{cases}$,

- $M(z): S^1 \to \mathbb{C}^*$, $\mathrm{ind}_S M = 0 \Rightarrow m(z) = \ln M(z) \in C(S)$
- pt of problem: $\Gamma(z), \Gamma^{-1}(z)$ must be bdd in $\mathbb{CP}^1$
- letting $y(z) = \ln \Gamma(z)$

$$\Rightarrow \begin{cases} y_+ = y_- + m \\ y(\infty) = 0 \end{cases} \Rightarrow y_+ - y_- = m$$

!! Cauchy

$$y(z) = \underbrace{h(z)}_{\text{entire}} + \oint \frac{m(w)}{w - z} \frac{dw}{z}$$

$\Rightarrow h \equiv 0$ due to boundary cond.

$$\Rightarrow \Gamma(z) = \exp \oint \frac{\ln M(z)}{w - z} \frac{dw}{2\pi i}$$

- but take $\gamma =$, e.g. $[0,1] \hookrightarrow \mathbb{C}$,
  $M = \text{const.} =: c$

$$\Rightarrow y = \int_0^1 \frac{\ln c}{w - z} \frac{dw}{2\pi i} = \frac{\ln c}{2\pi i} \ln \frac{z-1}{z}$$

$$\Gamma(z) = \left(\frac{z-1}{z} \right)^{\ln c / 2\pi i}$$

$\longrightarrow$ not bdd at endpts! (depending on c, but in general)

- worse, $\Gamma_n(z) = \Gamma(z) \cdot \left(\frac{z-1}{z} \right)^n$, $n \in \mathbb{Z}$,
  also satisfies jump cond'n $\Gamma_{n,+} = \Gamma_{n,-} \cdot c$

- so we need to further specify meaning of "solution"
  a) specify growth at endpts
  b) (in tomography) insist it's e.g. $\in L^2$

Nontrivial class of RHP's

- γ closed simple Jordan curve
- $M : \gamma \to GL_n(\mathbb{C})$, admitting meromorphic extension inside γ
- find sol'n to $\begin{cases} \Gamma_+ = \Gamma_- M \\ \Gamma(\infty) = \mathbb{1} \end{cases}$, Γ, Γ^{-1} analytic in $\mathbb{C} \setminus \gamma$, bdd in $\mathbb{C}$

Prop i) if $\exists$, $\exists!$

ii) $\Gamma_-(z) = \Gamma(z)|_{D_-}$ admits merom. ext to D_+ and has poles coinciding with $\mathcal{B} :=$ set of poles of M^{-1}

iii) $\Gamma_-^{-1}(z)$ — " — inside, — " — poles at $\mathcal{A} :=$ poles of M

Pf.

i) suppose $\Gamma, \hat{\Gamma}$ sol'ns. $R(z) := \hat{\Gamma} \cdot \Gamma^{-1}$,

$R_+ = \hat{\Gamma}_+ \Gamma_+ = \hat{\Gamma}_- M M^{-1} \Gamma_- = R_-$, so R anal. across γ

$\Rightarrow$ actually R entire (Γ^{-1} has no poles)

and since $R \longrightarrow \mathbb{1}$ as $z \to \infty$, $R(z) = \mathbb{1}$.

ii) - let $\Gamma_{\mp} = \Gamma|_{D_{\mp}}$

- we know $\Gamma_+ = \Gamma_- M$, $\Gamma_- = \Gamma_+ M^{-1} \Rightarrow$ claim.

iii) similar

- let $V = C_-[\mathcal{H}_+ \cdot M^{-1}]$, $W = C_-[\mathcal{H}_+ M]$,

$\mathcal{H}$ = vectors of anal. funcs in D

- V, W are spaces of rat. vect func. w poles at $\mathcal{A} = (M)_-$, $\mathcal{B} = (M^{-1})_-$

Prop RHP solvable iff $\mathcal{G}: V \to W$, $\vec{v} \mapsto C_-[vM]$ is invertible,
in which case $\mathcal{G}^{-1}: W \to V$, $\vec{w} \mapsto C_-[wM^{-1}\Gamma_-^{-1}]\Gamma_-$
Moreover, $\Gamma_- = \mathbb{1} - \mathcal{G}^{-1}(C_-[M])$.

Pf. $\mathcal{G}$ is well-defined: Let $v \in C_-[\vec{h}M^{-1}]$.
Then $C_-[vM] = C_-[C_-[\vec{h}M^{-1}]M]$
$= C_-[C_+[hM^{-1}]M - \vec{h}M^{-1}M]$
$= C_-[\underbrace{C_+[hM^{-1}]}_{\in \mathcal{H}_+} \cdot M] \in W$

$\boxed{\Leftarrow}$ Supp. $\mathcal{G}$ inverts, define $L(z) := \mathcal{G}^{-1}[C_-[M]]$.
We claim $\Gamma_- \doteq \mathbb{1} - L$, $\Gamma_+ = \Gamma_- M$
- verify $\Gamma_+ = (1-L)\cdot M$ is analytic inside D

$$C_-[\Gamma_+] = C_-((1-L)\cdot M) = C_-[M] - C_-[LM]$$
$$= C_-[M] - \mathcal{G}(L) = 0$$

- check $\Gamma_\pm$ analytic in $D_\pm$:
$\Gamma_+ = \Gamma_- M \Rightarrow \operatorname{ind}\det\Gamma_+ = \operatorname{ind}\det\Gamma_- + 0$
$\#$ 0's of $\det\Gamma_- = -\#\ldots\Gamma_-$ outside γ

$\boxed{\Rightarrow}$ Supp. RHP has sol'n, define
$g: W \to V$, $g[\vec{w}] = C_-[\vec{w}M^{-1}\Gamma_-^{-1}]\Gamma_-$
Then
$$(\mathcal{G}\circ g)(\vec{w}) = C_-[\underbrace{C_-[\vec{w}M^{-1}\Gamma_-^{-1}]}_{\in\mathcal{H}_+}\underbrace{\Gamma_-\cdot M}_{\Gamma_+}]$$
$$= C_-[\overbrace{C_+[\vec{w}\Gamma_+^{-1}]}\Gamma_+ - w\Gamma_+^{-1}\Gamma_+]$$
$$= C_-[-C_-[hM^{-1}]] = \underset{\substack{\uparrow \\ C_-^2 = -C_-}}{C_-}[hM^{-1}] = \vec{w}.$$

Example $M(z) = \begin{bmatrix} 1 & \frac{2}{2z-1} \\ \frac{2z-1}{2z+1} & \frac{2z+3}{2z+1} \end{bmatrix}$, $M^{-1}(z) = \tilde{M}(z)$

$$V = C_-[\mathcal{H}_+ M^{-1}] = \operatorname{span}\left\{ \frac{e_1}{z+1/2}, \frac{e_2}{z-1/2} \right\}$$
$$=: \operatorname{span}\{v_1, v_2\}$$

$$W = \operatorname{span}\left\{ \frac{e_2}{z-1/2}, \frac{e_1 - e_2}{z+1/2} \right\} = \operatorname{span}\{w_1, w_2\}$$

$$\xi(v_1) = w_1 + w_2, \quad \xi(v_2) = 2w_1 + w_2$$

$$\Gamma_- = \left[\; \right], \quad \Gamma_+ = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Deformations & τ-functions

- we know RHP soluble iff $T_{M^{-1}} : \mathcal{H}_+ \to \mathcal{H}_+$ inverts

- is there $\det T_{M^{-1}}$? (Toeplitz)

- scalar: $M^{-1}(z) = \sum_{j \in \mathbb{Z}} c_j z^j$

$$T_{M^{-1}}(z^\ell) = \left(\sum_{j \in \mathbb{Z}} c_j z^{j+\ell} \right)$$
$$= c_{-\ell} z^0 + c_{-\ell+1} z + \dots$$
$$\Rightarrow (T_{M^{-1}})_{ij} = c_{i-j}$$

- doesn't work... even if $M^{-1}(z) = \lambda$,
$\det \lambda \cdot \mathrm{id} \stackrel{?}{=} \lambda^\infty$?

- Widom (Szegö):

Thm φ symbol on S^1, $\varphi(z) = \sum_{j\in\mathbb{Z}} \varphi_j z^j$

$T_n(\varphi) := \det(\varphi_{j-i})_{i,j=0}^{n-1}$

1st Szegö thm:

$$\frac{\det T_n}{\det T_{n-1}} \xrightarrow[n\to\infty]{} G = \exp \frac{1}{2\pi i}\oint \ln\varphi(z)\frac{dz}{z}$$

2nd Szegö: Fourier coeffs

$$\lim_{n\to\infty} \frac{\det T_n}{G^n} = \exp\sum_{k=1}^{\infty} k \cdot \widehat{\ln\varphi}(k)\cdot\widehat{\ln\varphi}(-k)$$

$$= \det\left[T_\varphi \cdot T_{\varphi^{-1}}\right]$$

- if φ any symbol, $T_\varphi \circ T_{\varphi^{-1}} = \mathrm{Id}_{\mathcal{H}_+} + K$ where K is trace-class
  - recall: - operator A is Hilbert-Schmidt if $\operatorname{tr} AA^* < \infty$
    - for A bdd, $\operatorname{tr} A := \operatorname{tr}|A| = \operatorname{tr}\sqrt{A^+A}$
      - tr. class $\Longleftrightarrow$ prod. of 2 H.S.
      - it's an ideal
- if K tr. cl., $\det(\mathrm{Id}+K) = 1 + \operatorname{tr} K + \frac{1}{2!}\operatorname{tr} K^{\wedge 2} + \frac{1}{3!}\operatorname{tr} K^{\wedge 3} \dots$
  $= 1 + \sum_a k_{aa} + \frac{1}{2!}\sum_{a,b}\det\begin{pmatrix}k_{aa} & k_{ab}\\ k_{ba} & k_{bb}\end{pmatrix} + \dots$

  $\to$ Fredholm det. properties:
- $\det(\mathrm{Id}+K) = 0 \iff \mathrm{Id}+K$ is not invertible
- $\det[(\mathrm{Id}+K)(\mathrm{Id}+\tilde{K})] = \det(\mathrm{Id}+K)\cdot\det(\mathrm{Id}+\tilde{K})$
- det is cont. in trace norm