Gozzetti

$z^2 u'' + z u' + (z^2 - \nu^2) z = 0, \nu \in \mathbb{C}$

- $z=0$ Fuchsian; $z=\infty$ sing 2nd kind
  - $a_2(z) = 1 - \frac{\nu^2}{z^2} = \frac{\text{Hol at } \infty}{z^q}$, $q \le 1 \Rightarrow$ 2nd kind

$y = \begin{pmatrix} u \\ u' \end{pmatrix} \Rightarrow \frac{dy}{dz} = \Big[\underbrace{\begin{pmatrix} \cdot & 1 \\ -1 & \cdot \end{pmatrix} + \frac{1}{z}\begin{pmatrix} \cdot & \cdot \\ \cdot & -1 \end{pmatrix}}_{A(z)} + \frac{1}{z^2}\begin{pmatrix} \cdot & \cdot \\ \nu^2 & 0 \end{pmatrix} \Big] y$

1. diagonalize:

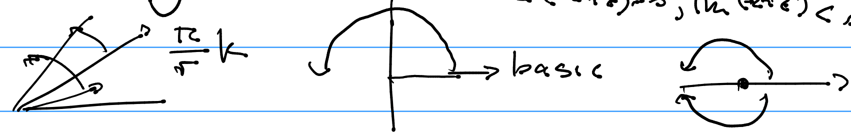
$$G_0^{-1}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} G_0 = \begin{pmatrix} i & \\ & -i \end{pmatrix} \equiv \Lambda, \quad G_0 = \begin{pmatrix} \gamma_1 & \gamma_2 \\ i\gamma_1 & -i\gamma_2 \end{pmatrix}$$

$\xrightarrow[\text{soln}]{\text{formal}}$ $Y_F(z) = G_0 \left(\mathbb{1} + \sum F_k z^{-k} \right) e^{\Lambda z + \Lambda_1 \ln z}$

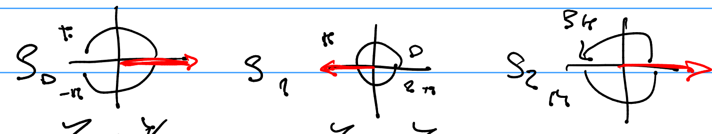
where $T_F^{-1} A(z) T_F = \Lambda + \frac{1}{z}\Lambda_1 + \frac{1}{z^2}\Lambda_2 + \dots$, $T_F = G_0 + \frac{1}{z} T_1 + \dots$

$\xrightarrow[\text{soln}]{\text{actual}}$ $Y(z) = G_0\, y(z)\, z^{-1/2} \begin{pmatrix} e^{iz} & \\ & e^{-iz} \end{pmatrix}$

- Stokes rays: $\lambda_1 = i, \lambda_2 = -i$ $\quad \mathrm{Re}(2iz) = 0, \mathrm{Im}(2iz) < 0$; $\mathrm{Re}(-2iz) = 0, \mathrm{Im}(-2iz) < 0$ $\Rightarrow \arg z = k\pi, k \in \mathbb{Z}$



S: $S_1 \dots S_{2r+1}$
$2r+1 = 3$



$Y_0 \sim Y_F \qquad Y_1 \sim Y_F$

$Y_1 = Y_0 \mathbb{S}_0, \quad Y_2 = Y_1 \mathbb{S}_1$

- write $Y_0 = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}, \; Y_1 = \begin{pmatrix} v_1 & v_2 \\ v_1' & v_2' \end{pmatrix}$

$\Rightarrow [u_1\ u_2]\, \mathbb{S}_0 = [v_1\ v_2]$

- Asymptotics $\begin{cases} u_1(z) = \hat{u}_1(z) z^{-1/2} e^{+iz} \\ u_2(z) = \hat{u}_2(z) z^{-1/2} e^{-iz} \end{cases}$, $\hat{u}_j = \gamma_j\left(1 + \sum u_k^{(j)} z^{-k}\right)$ $\quad z \to \infty$ in S_0

$\begin{cases} v_1(z) = \hat{v}_1(z) z^{-1/2} e^{+iz} \\ v_2(z) = \hat{v}_1(z) z^{-1/2} e^{-iz} \end{cases}$ $\quad \hat{v}_j = \dots$

$\nu \in \mathbb{C}$ $\quad Y_\nu(z) \rightsquigarrow H_\nu^{(1,2)}(z) = J_\nu(z) \pm i Y_\nu(z)$ Hankel fns.

$$H_\nu^{(1)}(z) = \text{const.} \left(\int_0^{+\infty e^{i\beta}} e^{-t} t^{\nu - 1/2} \left(1 + \frac{2t}{z}\right)^{\nu - 1/2} dt \right) z^{-1/2} e^{iz}, \qquad -\frac{\pi}{2} < \beta < \frac{\pi}{2}, \; \mathrm{Re}(\nu - \tfrac{1}{2}) > -1, \; -\frac{\pi}{2} - \beta < \arg z < \frac{3\pi}{2} - \beta$$

contains $S_{\ge 1}$

$$H_\nu^{(2)}(z) = -e^{i\pi\nu} H_\nu^{(1)}(z e^{-i\pi}), \quad -2\pi < \arg z < \pi$$

contains S_0

$$\left[H_\nu^{(1)}(z) \quad -H_\nu^{(2)}(ze^{-2\pi i}) \right] = \left[H_\nu^{(1)}(z) \quad H_\nu^{(2)}(z) \right] \underbrace{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}}_{S_0}, \quad s = e^{2\pi\nu} \frac{\sin 2\pi\nu}{\sin \pi\nu}$$

- from cyclic relns $H_\nu^{(2)}(ze^{-2\pi i}) = -sH_\nu^{(1)}(z) - H_\nu^{(2)}(z)$

- global solns ...