Stoppa

Rmk. (about last time) we used the property

$$\forall X, Y \quad [Y, X] - J[X, JY] - J[JX, Y] + [JX, JY] = 0$$

- in general:

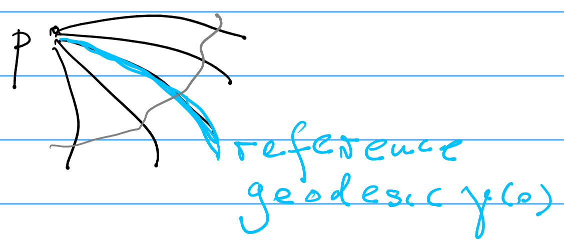
Def. An *almost cpx structure* on M^{2n} is a smooth section of $End(TM) \simeq T^*M \otimes TM$ which squares to $-Id$ everywhere.

Rmk. Clearly, a cpx str. is an almost cpx str.

Th. (Newlander-Nirenberg)
An almost cpx str comes from a cpx str iff $N_J(X, Y) = 0, \forall X, Y$.

First glimpse of curvature.

Start w geodesic pencil $f(t,s) \doteq \exp_p(t\, v(s))$.



Def. Associate to the pencil a v.f. along $\exp_p(t\, v(0)) =: \gamma(t)$,

$$J(t) = \frac{\partial f}{\partial s}(t, 0) = df\left(\frac{\partial}{\partial s}\Big|_{(t,0)}\right).$$

Rmk. $J(t) = (d\exp_p)_{tv}(t\, \dot{v}(0))$.

Lemma. $J(t)$ satisfies $\frac{D^2}{dt^2} J(t) + \mathcal{L}(\dot{\gamma}(t)) = 0$, where the operator $\mathcal{L}$ acts on v.f.s along $f(t,s)$ by $\mathcal{L}(V) \doteq \left[\frac{D}{ds}, \frac{D}{dt}\right](V)$.

Pf. By definition, $\frac{D}{dt}\frac{\partial f}{\partial t} = 0$.

$$\Rightarrow 0 = \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \frac{D}{dt}\underbrace{\frac{D}{ds}\frac{\partial f}{\partial t}}_{= \frac{D}{dt}\frac{\partial}{\partial s} f \text{, Tors} = 0} + \mathcal{L}\left(\frac{\partial f}{\partial t}\right).$$

Claim. $\mathcal{L}$ is induced by a global object on M.

Def. The Riemann curvature tensor on (M, g) is given as

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

Rmk i) This is do Carmo's convention.

ii) C^∞-linear in all args.

$\Rightarrow$ tensor $(R(X,Y)Z)_p = R(X_p, Y_p) Z_p$.

$\Rightarrow$ End(TM)-valued 2-form, $R \in \Gamma(M, \Lambda^2 T^*M \otimes \mathrm{End}(TM))$

Lemma. Let $f(s,t)$ be any param. sfc in M. Let V be a v.f. along $f(s,t)$. Then

$$\left(\frac{D}{dt}\frac{D}{ds} - \frac{D}{ds}\frac{D}{dt}\right)V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V.$$

Pf. Fix notation $(-)' := \frac{\partial}{\partial s}(-)$, $\dot{(-)} := \frac{\partial}{\partial t}(-)$.

In local coordinates write $V = V^i \frac{\partial}{\partial x_i}$, $f(t,s) = (x_1(t,s), \dots, x_n(t,s))$.

$$\frac{D}{dt}\frac{D}{ds}V = V^i \frac{D}{dt}\frac{D}{ds}\frac{\partial}{\partial x_i} + \dot{V}^i \frac{D}{dt}\frac{\partial}{\partial x_i} + (V^i)'\frac{D}{ds}\frac{\partial}{\partial x_i} + (\dot{V}^i)'\frac{\partial}{\partial x_i}.$$

Clearly, $\left[\frac{D}{dt}, \frac{D}{ds}\right](V) = V^i \left[\frac{D}{dt}, \frac{D}{ds}\right]\frac{\partial}{\partial x_i}$.

$d(f)\left(\frac{\partial}{\partial s}\right) = (x^\ell)' \frac{\partial}{\partial x_\ell}$

$d(f)\left(\frac{\partial}{\partial t}\right) = \dot{x}^\ell \frac{\partial}{\partial x_\ell}$

Now $\frac{D}{ds}\frac{D}{dt}\frac{\partial}{\partial x_i} = \nabla_{df(\frac{\partial}{\partial s})} \nabla_{df(\frac{\partial}{\partial t})} \frac{\partial}{\partial x_i}$

$$= \dot{x}_j' \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} + \dot{x}_k x_j' \nabla_{\frac{\partial}{\partial x_k}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}.$$

And so $\left[\frac{D}{dt}, \frac{D}{ds}\right]\frac{\partial}{\partial x_i} = \dot{x}_k x_j' \left(-\nabla_{\frac{\partial}{\partial x_k}} \nabla_{\frac{\partial}{\partial x_j}} + \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_k}}\right)\frac{\partial}{\partial x_i}$

And since $\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j}$, $\left[\frac{D}{dt}, \frac{D}{ds}\right] V = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) V$.

Corollary. $\frac{D^2}{dt^2}J(t) + R(\dot{\gamma}(t), J(t)) = 0.$

Rmk. We call $J(t)$ the Jacobi field of the pencil.

Motivation. $p, q \in M$. Let $\Omega(p,q) = \{\text{paths from } p \text{ to } q\}$
Take $\gamma(t) \in \Omega(p,q)$.
Define $T_\gamma \Omega(p,q) = \left\{ \begin{matrix} \text{v.f.s along } \gamma(t) \\ \text{vanishing at } p, q \end{matrix} \right\}$

We have the energy functional $E : \Omega(p,q) \to \mathbb{R}$:

$$E(\gamma(t)) := \int_0^1 (\dot{\gamma}(s))^2 \, ds$$

Geodesics are its critical points.
$\to$ in the directions given by the Jacobi field $\Rightarrow$ degeneracy.

Properties of R

Prop. (1st Bianchi identity)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Stoppa

Def ("4-index tensor")
$$(X,Y,Z,W) := g\,(R(X,Y)Z, W)$$

Rmk. it is induced by a global section of $(T^*M)^{\otimes 4}$

Prop. $\forall X Y Z W$ we have

a) $\sum_{cycl} (X\, Y\, Z\, W) = 0$

b) $(Y X Z W) = -(X Y Z W)$

c) $(X Y Z W) = -(X Y W Z)$

d) $(Z W X Y) = (X Y Z W)$

Pf. a) Bianchi; b) $R(X,Y) = -R(Y,X)$

c) since char $\neq 2$, we check $(X Y Z Z) = 0$.

$$(X Y Z Z) = \langle \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]} Z, Z \rangle$$
$$= Y \langle \nabla_X Z, Z \rangle - X \langle \nabla_Y, Z \rangle + \tfrac{1}{2} [X,Y] \langle Z, Z \rangle = 0.$$

d) $$\sum^{\circlearrowleft}_{XYZ} (X Y Z T) + \sum^{\circlearrowleft}_{YZT} (Y Z T X) + \sum^{\circlearrowleft}_{ZTX} (Z T X Y) + \sum^{\circlearrowleft}_{TXY} (T X Y Z) = 0 + 0 + 0 + 0$$

Sectional curvature

Def. Pick $p \in (M, g)$. Pick $\sigma \subset T_pM$ a 2-dim subspace.
Write $\sigma = \mathrm{Span}(X, Y)$.

$$K(\sigma) := \frac{(X Y X Y)}{(\mathrm{Area}(X,Y))^2} \qquad \text{where } \underset{\sqrt{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}}{\underset{\shortparallel}{\mathrm{Area}(X,Y)}}$$

Lemma. This is well-posed.

Pf. Check basis (in)dependence.

Lemma. The statement "$\mathcal{K}(z)$ determines R, $\forall z$, at every pt of M^4" makes sense:
If R, R' two 3-linear forms on $(V, \langle -,- \rangle)$,
and if $\mathcal{K}(z) = \mathcal{K}'(z)$

$$\underset{\dfrac{(XYXY)}{Ar(XY)^2}}{\shortparallel} \qquad \underset{\dfrac{(XYXY)'}{Ar(XY)^2}}{\shortparallel}$$

where $(XYZW) := \langle R(XY)Z, W \rangle$
$(XYZW)' := \langle R'(XY)Z, W \rangle$

$\forall z$, then $R = R'$.

Pf. $(XYXY)' - (XYXY) = 0$
$\Rightarrow \langle (R'(XY) - R(XY))X, Y \rangle = 0$
Since $\mathcal{K}(z)$ is basis-indep,
pick $X \perp Y$. $\Rightarrow (R'(X,Y) - R(X,Y))X \perp Y$
$\Rightarrow (R'(X,Y) - R(X,Y))X = \lambda_Y X$, $\lambda_Y \in \mathbb{R}$.
$\Rightarrow \lambda_Y = \frac{1}{\langle X, X \rangle} \langle (R'(X,Y) - R(X,Y))X, X \rangle$
$= \frac{1}{\|X\|^2}\Big((XYXX)' - (XYXX) \Big) = 0$, by antisym. & $\forall Y$.

Prop. (2nd Bianchi id) $\displaystyle\sum_{ZWT}^{\circlearrowleft} \nabla_T (XYZW) = 0.$

Lemma. Let $\gamma(t)$ geodesic, $X(t)$ v.f. along $\gamma(t)$, $X(0) = 0$.
Then

$$\frac{D}{dt} R(\dot\gamma, X)\dot\gamma = \nabla_{\dot\gamma} R(\dot\gamma, X)\dot\gamma \Big|_{t=0} = R(\dot\gamma, \tfrac{DX}{dt})\dot\gamma \Big|_{t=0}$$

Pf. By 2nd Bianchi id, $\forall Z$

$$0 = \nabla_{\dot\gamma}(\dot\gamma X \dot\gamma Z) = \frac{d}{dt}\langle R(\dot\gamma X)\dot\gamma, Z \rangle - \langle R(\dot\gamma, \tfrac{DX}{dt})\dot\gamma, Z \rangle - \langle R(\dot\gamma, X)\dot\gamma, \tfrac{DZ}{dt} \rangle$$

Evaluate at $t = 0$.

- recall the Jacobi equation $\frac{D^2}{dt^2} J + R(\dot{\gamma}, J)\dot{\gamma} = 0$.

Lemma. Jacobi fields along a fixed $\gamma(t)$ form a fin. dim. v.s, with $\dim = 2n = 2\dim M$.

Pf. Pick set of v.f.s $\{e_i(t)\}_{i\in\{1,\dots,n\}}$ along $\gamma(t)$. s.t. they form an onb for $T_{\gamma(t)}M$ at every t. and $\nabla_{\dot{\gamma}(t)} e_i(t) = 0$.
Well-def. since linear ODE given by initial conds.
Now write $J(t) = J^k(t) e_k(t) \Rightarrow n$ 2nd order linear ODEs for $J^k(t)$ by Jacobi equ.

Rmk There exist 2 obvious solns:

$$J(t) = \dot{\gamma}(t), \quad J(t) = t\,\dot{\gamma}(t).$$

- denote by $\mathcal{J}$ the space of Jacobi fields along $\gamma(t)$ with $J(0) = 0$.

Lemma. $\forall J(t) \in \mathcal{J}$, $J(t) = (d\exp_{\gamma(0)})_{t\dot{\gamma}(0)}\left(t \frac{DJ}{dt}(0)\right)$

Pf. By the previous lemma, we need to check that RHS is a Jacobi field w initial conds $(0, \frac{DJ}{dt}(0))$.
First, note that, analogueously to $\frac{\partial}{\partial s}(\exp_p(t\,v(s))\big|_{(t,0)}$, it is a Jacobi field.
Secondly, check vanishing at $t=0$.

Stoppa.

- let's recall some definitions

  $\Rightarrow \mathcal{J} := \{$ Jacobi fields along $\gamma(t)$, $J(0)=0\}$

  $\mathcal{J}^{\perp} := \{ J(t)\in\mathcal{J} \mid \langle \dot\gamma(t), J(t)\rangle = 0 \;\; \forall t\}$

- recall also: $J\in\mathcal{J} \Rightarrow (d\exp_{\gamma(0)})_{t\dot\gamma(0)}\left(t\,\frac{DJ}{dt}(0)\right)$ (*)

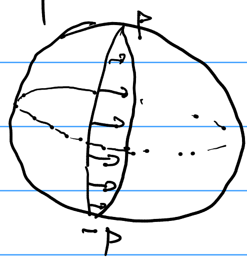
Rmk Pick $y(s)\in T_{\gamma(0)}M$ s.t. $\frac{dy}{ds}(0) = \frac{DJ}{dt}(0)$

Construct a pencil $f(t,s) := \exp_p(t\,y(s))$.

By (*), $J(t)$ is the Jacobi field attached to f.

Def. Let $\gamma(t)$ be a geodesic. We call a point $\gamma(\bar t)$ a point *conjugate* to $\gamma(0)$ if $\exists J\in\mathcal{J}$ s.t. $J(\gamma(\bar t))=0$.

- e.g. antipodes in S^n are conjugate.



Def. The *multiplicity* of a pt $\gamma(\bar t)$ conj. to $\gamma(0)$ is the max # of lin. indep elements of $\mathcal{J}$ vanishing at $\gamma(\bar t)$

- e.g. $\mathrm{mult}_{S^n}(-p) = n-1$.

Lemma. $\bar t\dot\gamma(0)\in T_{\gamma(0)}M$ is a critical pt of $d\exp_p$ iff $\gamma(\bar t)$ is a conj. pt. to $\gamma(0)$ along $\gamma(t)$

Moreover, $\dim\ker d\exp_{\gamma(0)}|_{\bar t\dot\gamma(0)} = \mathrm{mult}\,\gamma(\bar t)$

Pf. $0 = J(\bar t) \overset{(*)}{\iff} (d\exp_p)_{\bar t\dot\gamma(0)}\left(\bar t\,\frac{DJ}{dt}(0)\right) = 0$

$\iff \frac{DJ}{dt}(0) \in \ker d\exp_{\gamma(0)}|_{\bar t\dot\gamma(0)}$

Def. $p\in M$. $\mathrm{Conj}(p) := \{1^{st}$ conj pts to p along all geodesics from $p\}$

$\mathrm{Cut}(p) := \{$ pts after which geodesics from p stop minimising length $\}$

- e.g. i) $p\in S^n \Rightarrow \mathrm{Conj}(p) = -p = \mathrm{Cut}(p)$

ii) $\mathbb{R}\times S^1$

$\subset \mathbb{R}^3$. $\left.\begin{array}{l}\mathrm{Conj}(p) = \emptyset \\ \mathrm{Cut}(p) = q\end{array}\right\}$ along $\gamma(t)$

Rmk. $\gamma(T)\in \mathrm{Conj}\,\gamma(0) \Rightarrow \mathrm{mult}_\gamma\, \gamma(t) \le n-1$, because $t\dot\gamma(t)\in \mathcal{J}$ and $\neq 0$ at other pts.

Lemma. $\dim \mathcal{J}^\perp = n-1$

Pf. Claim: $\mathcal{J}^\perp$ is given by $J(t)$ s.t. $\langle \frac{DJ}{dt}(0), \dot\gamma(0)\rangle = 0$.
This is 1 condition on the initial conditions that define $\mathcal{J}^\perp$.
Compute the following:

$$\frac{d^2}{dt^2}\langle J(t),\dot\gamma(t)\rangle = \frac{d}{dt}\left(\langle \frac{DJ}{dt}(t),\dot\gamma(t)\rangle + \langle J(t), \overset{0}{\overset{\shortparallel}{\frac{D\dot\gamma}{dt}(t)}}\rangle\right)$$

$$= \langle \frac{D^2J}{dt^2}(t),\dot\gamma(t)\rangle$$

$$= \langle R(\dot\gamma(t), J(t))\dot\gamma(t),\dot\gamma(t)\rangle$$

$$= 0$$

$$\Rightarrow \langle J(t),\dot\gamma(t)\rangle = \langle \frac{DJ}{dt}(t),\dot\gamma(t)\rangle t + \langle J(0),\dot\gamma(0)\rangle.$$

Jacobi fields & sectional curvature.

Prop. Let $J(t) \in \mathcal{J}$, write $\dot{J}(0) = v$, $\frac{DJ}{dt}(0) = w$ and normalise $\|w\| = 1$.
Then

$$\|J(t)\|^2 = t^2 - \frac{1}{3} \underbrace{\langle R(v,w)v,w\rangle}_{\|v \wedge w\|^2 \cdot K(v,w)} t^4 + O(t^5)$$

$\|v\wedge w\|^2 \cdot K(v,w) \rightsquigarrow$ sectional curvature.

Rmk. Morally, $K \leq 0$ means geodesics "spread-out", and vice-versa.

Pf. Examine coefficients of the Taylor expansion of $\langle J(t), J(t)\rangle$ at $t=0$, use $J(0)=0$, Jacobi eqn., 2nd Bianchi id...

Theorem of Hopf-Rinow.

Def. Let (M,g) Riem. For $p,q \in M$ define

$$d(p,q) := \inf_{\{c:[0,1]\to M \text{ pcwise } C^1,\ c(0)=p,\ c(1)=q\}} \left(\operatorname{len}(c(t))\right).$$

Lemma. $d(-,-)$ is a distance fun, i.e. (M,d) metric space.

Lemma. (M,d) with topology induced by d is the same top. sp. as M.

Cor. $d: M \times M \to \mathbb{R}$ is continuous.

Stoppa

Hopf - Rinow Thm.

Thm. Fix a pt $p \in (M,g)$. TFAE

i) $\exp_p$ is defined on all of T_pM

ii) closed & bounded of M are compact

iii) (M,d) is a complete metric space

iv) $\exp_q$ is defined on all of T_qM, $\forall q \in M$

Moreover, any of these implies:

v) $\forall p,q \in M$ $\exists \gamma(t)$ geodesic, $\gamma(0)=p$, $\gamma(1)=q$ s.t. $d(p,q) = \text{len}\,\gamma$.

Pf. For now assume we know i) $\Rightarrow$ v).

i) $\Rightarrow$ ii) $A \subset M$ closed & bounded. uses i) $\Rightarrow$ v)

$\overset{v)}{\Rightarrow} A \subset \exp_p(B_R(0))$ for some $R>0$, $p \in A$.

$\Rightarrow A \subset \exp_p(\underbrace{\overline{B}_R(0)}_{\text{cpt}}) \Rightarrow A$ cpt.

ii) $\Rightarrow$ iii) Basic analysis

iii) $\Rightarrow$ iv) Pick $\gamma : [0,s_0) \to M$ geodesic, $\gamma(0)=q$.

Pick $\{s_m\}_{m \in \mathbb{N}} \subset [0,s_0)$, $\lim s_m = s_0$.

$\Rightarrow \{\gamma(s_m)\}_m$ is a Cauchy sequ. on (M,d).

$\Rightarrow \gamma(s_0) := \lim \gamma(s_m)$

- we only need to show $\exists \dot{\gamma}(s_0) \in T_{\gamma(s_0)}M$ which extends $\gamma(t)$ by $\exp_{\gamma(s_0)}(t\dot{\gamma}(s_0))$.

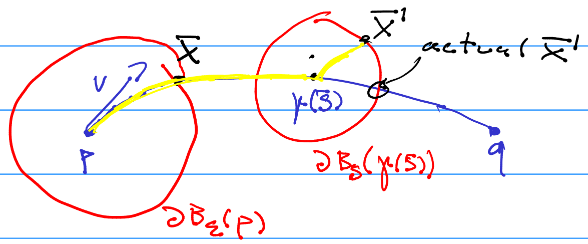
$\Rightarrow (\gamma(s_m), \dot{\gamma}(s_m))_m$ is a squ. lying on a cpt subset of TM ($\{\dot{\gamma}(s_m)\}_m$ has fixed lengths)

$\Rightarrow$ Pick a convergent subsequence and take limits.

Main point: i) $\Rightarrow$ v)

- fix any $p,q \in M$, let $d := d(p,q)$.
- Claim: p, q can be joined by minimising geodesic

- choose initial velocity by picking a point $\bar{x}$ on the boundary of a normal ball $B_\varepsilon(p)$ at p which minimises distance from $\partial B_\varepsilon(p)$ to q



- claim: geodesic extends, minimises & hits q
- write $\gamma(t,p,v)$, $\gamma(0,p,v)=p$, $\gamma(\varepsilon,p,v)=\bar{x}$
- stopped writing.

Thm (Hadamard) Let (M,g) be a complete Riemannian mfd (i.e. (M,d) is complete; or $\exp_q$ defined on T_qM $\forall q\in M$). Suppose:

1) $\pi_1(M,\{pt\})=\{1\}$
2) $\forall q\in M$, $\forall \sigma\subset T_qM$, $K(\sigma)\le 0$

$\Rightarrow \exp_q : T_qM \to M$ is diffeo., $\forall q$

Pf. Claim: $\forall q$, $\exp_q : T_qM\to M$ is local diffeo.
This means $(d\exp_q)_v$ is an invertible linear map i.e. $\exp_q$ has no critical pts.
But we know those pts correspond to vanishing of Jacobi fields.

$$\Rightarrow \frac{d^2}{dt^2}\langle J(t),J(t)\rangle = 2\|J'(t)\|^2 - 2K(\dot\gamma(t),J(t))\,\mathrm{Ar}^2(\dot\gamma(t),J(t))$$

$\to$ RHS ≥ 0 since $K\le 0$. Evaluating at 0 gives >0.
Now look at $(T_qM, \exp_q^* g)$. By Hopf-Rinow it is complete in both senses, since its geodesics are just straight lines.
Also, $\|(d\exp_q)_v(w)\|_M = \|w\|_{T_qM}$, isometry.

Claim: all of this implies $\exp_q$ is a covering map.

Criterion: $X \xrightarrow{f} Y$ covering map iff lifting property for paths in Y holds.

$\rightarrow$ by local diff we can lift locally.

$\rightarrow$ but it extends due to local isometry prop.

Finally: it is a trivial covering due to

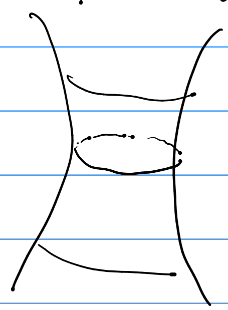
$\pi_1(M, \{pt\}) = \{1\}$

$\Rightarrow$ homeomorphism

$\Rightarrow$ diffeomorphism.

Stoppa.

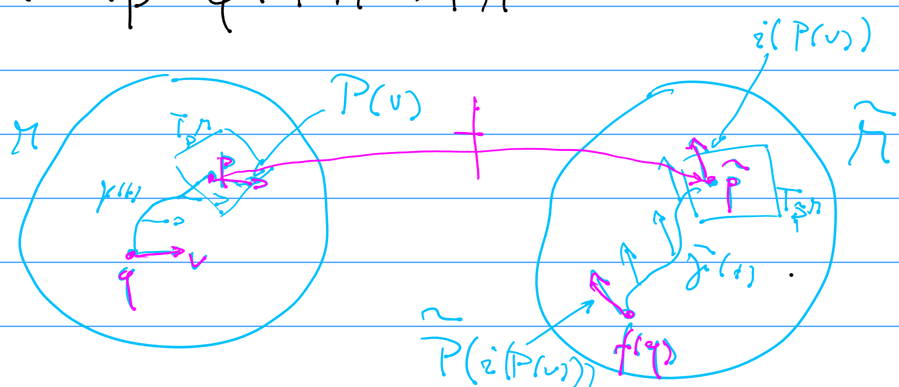
- e.g. Surface of revolution $S \subset \mathbb{R}^3$



- even if $K < 0$, $\pi_1(S, \{pt\}) \neq \{1\}$
  $\rightarrow S \not\cong \mathbb{R}^2$

Cartan's construction.

- take $p \in (M^n, g)$, $\tilde{p} \in (\tilde{M}^n, \tilde{g})$
- define a linear isometry $i: T_pM \to T_{\tilde{p}}\tilde{M}$
- suppose $\exp_p: T_pM \to M$ is a diffeo (!), $\exp_{\tilde{p}}$ defined on all $T_{\tilde{p}}\tilde{M}$
- define $f: M \to \tilde{M}$ by $f(q) = \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(q)$
- we can extend this by parallel transport to a map $\varphi: TM \to T\tilde{M}$



$\Rightarrow \varphi(v) := (\tilde{P} \circ i \circ P)(v)$, $\gamma(t), \tilde{\gamma}(t)$ unique, normalised geodesic

Thm (Cartan) Assume everything as above and suppose $\langle R(X,Y)U, V\rangle_M = \langle \tilde{R}(\varphi X, \varphi Y)\varphi U, \varphi V\rangle_{\tilde{M}}$.
Then f is a local isometry & $df_p = i$.

Pf. We need to compute df_q.

By construction, $q = \exp_p(\tilde t \dot\gamma(0))$

so $df_q(v) = (d\exp_{\tilde p})_{\tilde t \underbrace{i(\dot\gamma(0))}_{\dot{\tilde\gamma}(0)}} \circ i \circ (d\exp_p^{-1})_{\tilde t \dot\gamma(0)}(v)$

To compute it, we use Jacobi fields.

Since $\exp_p$ is diffeo, $d\exp_p$ is invertible

$\Rightarrow \exists!$ Jacobi field along $\gamma(t)$ s.t.

$J(0) = 0$, $J(\tilde t) = v \in T_q M$, ...

Now define $\tilde J(t) := \varphi(J(t))$ along $\tilde\gamma(t)$

Note: $\tilde J(0) = 0$, $\|\tilde J(t)\|_{\tilde M} = \|J(t)\|_M \quad \forall t$

because we parallel transport using Levi-Civita.

Claim: $\tilde J(t)$ is a Jacobi field.

If this is true,

$\|df_q(v)\|_{\tilde M} = \|\tilde J(\tilde t)\|_{\tilde M} = \|J(\tilde t)\|_M = \|v\|_M$.

Sketch: - write $J(t) = J_i e_i(t)$ using an ON frame.

$\tilde J(t) = \varphi(J(t)) = J_i(t)\varphi(e_i(t)) \doteq J_i(t)\tilde e_i(t)$

- write $\ddot J^j + \langle R(e_k, e_i)e_k, e_j\rangle = 0$

- use assumption.

Thm ("Fundamental thm on spaces of constant curvature")

Let (M^n, g) be complete w/ const. sect. curvature.

Up to rescaling, the universal cover of M endowed with the pullback metric, $(\tilde M, \pi^* g)$, is isometric to either:

$(\mathbb{R}^n, g^{eucl})$, $\quad K \equiv 0$

$(\mathbb{H}^n, g^{hyp})$, $\quad K \equiv -1$

(S^n, g^{round}), $\quad K \equiv +1$.

Pf. $\pi: \tilde{M} \to M$ univ. cover $\Rightarrow (\tilde{M}, \pi^* g)$ is still complete, and $\pi_1(\tilde{M}, \{pt\}) = \{1\}$. Assume, after rescaling, $K(g) \in \{0, \pm 1\} \Rightarrow K(\pi^* g) = K(g)$, local isometry.

Now fix $K(g) \in \{0, -1\}$.

Hadamard: $\forall p \in \tilde{M}$, $\exp_p$ is well-defined & diffeo.

Define $(D, g_D) := \begin{cases} (\mathbb{R}^n, g^{euc.}), & \text{if } K = 0 \\ (\mathbb{H}^n, g^{hyp.}), & \text{if } K = -1 \end{cases}$,

do Cartan with it. ...

Stoppa.

Ricci curvature.

- (M,g), $\{e_1,\dots,e_n\}$ ON frame at $p\in M$.
- $\to$ fix $x := e_n$

Def. (Ricci) $\mathrm{Ric}_p(x) := \frac{1}{n-1}\sum_{i=1}^{n-1}\langle R(x,e_i)x, e_i\rangle$

Lemma. $\mathrm{Ric}_p(x)$ is well-defined, independently of the choice of frame.

Pf. Consider $L\in \mathrm{End}(T_pM)$, $L(Z) := R(X,Y)Z$ for $X,Z\in T_pM$ fixed. Trace in (FDVect) is just the usual one; define $Q(X,Y) = \mathrm{Tr}\, L$
Not hard to see it is a symmetric quadratic form. In particular, if $\|X\|=1$, then
$Q(X,X) = \mathrm{Tr}\, L_{X,X} = \sum_{i=1}^{n-1}\langle R(X,e_i)X, e_i\rangle$
$= (n-1)\,\mathrm{Ric}_p(X)$.

Rmk. Usually $Q(X,Y)$ is also called the Ricci curvature, denoted by $\mathrm{Ric}_p(X,Y)$.

Scalar curvature.

Def. (Sc. curv.) $\forall p\in M$ given by $S(p) := \frac{1}{n}\sum_{j=1}^{n}\mathrm{Ric}_p(e_j)$

Lemma. Well-defined.

Pf. $\forall p\in M$, T_pM possesses 2 symmetric quadratic forms, $\langle -,-\rangle_p$ & $Q_p(-,-)$.
By linear algebra $\exists!\, N\in \mathrm{End}(T_pM)$ s.t.
$\langle N(X),Y\rangle_p = \langle X, N(Y)\rangle_p$ & $Q_p(X,Y) = \langle N(X),Y\rangle_p$ $\forall X,Y\in T_pM$.

Consider $\operatorname{Tr} N = \sum_{i=1}^{n} \langle N(e_i), e_i \rangle_p$
$= \sum_{i=1}^{n} Q_p(e_i, e_i) = n(n-1)\, S(p).$

Bonnet–Myers thm.

Thm. (M, g) complete. Suppose Ric uniformly bounded from below by a strictly positive constant;
$\forall p \in M, \forall v \in T_pM, \|v\|=1, \ \operatorname{Ric}_p(v) \geq \frac{1}{r^2} > 0.$
Then M is cpt & $\operatorname{diam}(M, g) \leq \pi r.$

Rmk. Paraboloid of revolution $\subset \mathbb{R}^3$: $\operatorname{Ric}_p(v) \geq 0$ & noncpt.

Energy functional.

$E(c(t)) := \int_a^b \|\dot{c}(s)\|\, ds$

- fix $p, q \in M$ for now.

Def. $\Omega_{p,q} := \{$ piecewise C^1 paths from p to $q \}$

Def $c(t) \in \Omega_{p,q}$. $T_c \Omega_{p,q} := \{$ piecewise C^1 v.f.s along $c(t)$ vanishing at endpoints $\}$

- given $V(t) \in T_c \Omega_{p,q}$, we get a variation (assuming $c:[0,a] \to M$.)
$$f(s,t) = \exp_{c(t)}(s\, V(t)), \quad (s,t) \in (-\varepsilon, \varepsilon) \times (0, a)$$
& conversely, given $f(s,t)$, we get
$$V(t) = \frac{\partial f}{\partial s}\Big|_{s=0}(t) = df_{(s=0,t)}\left(\frac{\partial}{\partial s}\right)$$

→ we study E along these variations

$\Longrightarrow$ $E(s) := \int_0^a \left\| \frac{\partial f}{\partial t}(s,t) \right\|^2 dt$

Lemma. $c(t)$ is a minimum of E iff $c(t)$ is a minimising geodesic parametrised by $\|\dot{c}(t)\| = \text{const}$.

Pf. By Schwarz ineq., $(\mathrm{Len}(c))^2 \le a\,E(c)$, with equality iff $\|\dot{c}(t)\| = \text{const}$.
Take $\gamma(t)$ to be parametrised as such and also a minimising geodesic:

$$\underbrace{L(\gamma)^2}_{a E(\gamma)} \le \underbrace{L(c)^2}_{\le\, a E(c)}$$

Def. $c(t)$ is a critical pt of E if $\forall$ variations $f(s,t)$, $\frac{d}{ds}E(s)\big|_{s=0} = 0$.

Lemma. Suppose given $0 = t_0 < t_1 < \dots < t_{k+1} = a$, $c(t)$ is C^1 on each $[t_i, t_{i+1}]$.
Then
$$\frac{1}{2}\frac{d}{ds}E(s)\Big|_{s=0} = -\int_0^a \left\langle V(t), \frac{D}{dt}\dot{c}(t) \right\rangle dt - \sum_{i=1}^{k} \left\langle V(t_i), \dot{c}(t_i+0) - \dot{c}(t_i-0) \right\rangle$$

Cor. If $\frac{dE}{ds}\big|_{s=0} = 0$ $\forall$ variations $\Rightarrow$ $c(t)$ geodesic.

Lemma. Everything as above. Fix $\gamma(t)$ crit. pt of E (geodesic).
Then
$$\frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_0 = -\int_0^a \left\langle V(t), \frac{D^2}{dt^2}V(t) + R(\dot{\gamma}(t), V(t))\dot{\gamma}(t) \right\rangle dt - \sum_{i=1}^{k} \left\langle V(t_i), \frac{DV}{dt}(t_i+0) - \frac{DV}{dt}(t_i-0) \right\rangle$$

Cor. Elements $J \in \mathcal{J}_{p,q}$ give $\frac{d^2E}{ds^2}(s)\big|_{s=0} = 0$.

Pf of Bonnet-Myers.

Fix $p, q \in M$. Claim: $d(p,q) \le \pi r$.

M complete $\Rightarrow \exists\ \gamma(t)$ minimising geodesic, $\gamma \in \Omega_{p,q}$,

s.t. $\mathrm{len}(\gamma) = d(p,q)$ (Hopf-Rinow)

Normalise $\gamma : [0,1] \to M$, $\|\dot\gamma\| = \mathrm{len}(\gamma)$.

Assume $\mathrm{len}(\gamma) > \underline{\pi r}$.

Fix ON frame $\{e_1, \dots, e_{n-1}, e_n = \dot\gamma(t)/\mathrm{len}\,\gamma\}$

parallel along $\gamma(t)$.

Consider v.f.s $V_k(t) := \sin(\pi t)\, e_k(t)$, $k = 1, \dots, n-1$

$\to V_k(t) \in T_\gamma \Omega_{p,q}$.

$$\Rightarrow \frac{1}{2} \frac{d^2 E_k}{ds^2}\Big|_{s=0} = -\int_0^1 \langle V_k, V_k'' + R(\dot\gamma, V_k)\dot\gamma \rangle\, dt$$

$$= \int_0^1 \sin^2(\pi t)\left(\pi^2 - \mathrm{len}^2\gamma\, K(e_n(t), e_k(t)\right) dt$$

$$\Rightarrow \frac{1}{2} \sum_{k=1}^{n-1} \frac{d^2 E_k}{ds^2}\Big|_{s=0} = \dots = \int_0^1 \sin^2(\pi t)\big((n-1)\pi^2$$

$$- (n-1)\mathrm{len}^2\gamma\, \mathrm{Ric}_{\gamma(t)}(e_n(t))\big)\, dt < 0$$

$\to$ but $\mathrm{Ric}_{\gamma(t)}(v) \ge \frac{1}{r^2}$ $\lightning$.

Stoppa.

Cor (to Bonnet-Myers) (M,g) complete, $Ric(x) \geq \delta > 0$, $\forall x$, $\|x\|=1$ and some δ. Then $\pi_1(M)$ is finite.

Pf. Take $\tilde{M} \xrightarrow{\pi} M$ universal cover. Consider pullback metric $\pi^* g$ on $\tilde{M}$. We know $(\tilde{M}, \pi^* g)$ complete.
Then π becomes an isometry.
So we get $Ric_{(\tilde{M},\pi^* g)}(x) \geq \delta > 0$.
By Bonnet-Myers, $\tilde{M}$ cpt. In particular, π is a finite-to-one map, so $\#(\pi^{-1}(m)) = \#(\pi_1(M, pt))$.

Rmk. $K(\sigma) \geq \delta > 0 \Rightarrow Ric(x) \geq \delta' > 0$ (converse fails).

Rmk. Compact tori $\mathbb{T}^n$ do not admit a metric of strictly positive Ricci curvature, since $\pi_1(\mathbb{T}^n, pt) \simeq \mathbb{Z}^n$.
In 2d we see this directly from Gauß-Bonnet, since $\int_{\mathbb{T}^2} K(p)\, dVol = 0$ means $K(p) \not> 0$ on $\mathbb{T}^2$.

- we have structure thm for spaces of constant curvature $\Rightarrow$ universal cover is isometric to $\begin{cases} \mathbb{H}^n & < 0 \\ \mathbb{R}^n & = 0 \\ \mathbb{S}^n & > 0 \end{cases}$

$\rightarrow$ what about spaces of constant Ricci curvature?
$\rightarrow \forall p \in M, \forall x \in T_pM, \|x\|=1$, $Ric_p(x) = \lambda$, $\lambda \in \mathbb{R}$ fixed
- equivalently, $Ric = \lambda g$ Einstein manifolds
- Q: find an example of a cpt (M,g) w $Ric = 0$.
  $\rightarrow$ solved in 80's by Yau.
  $\rightarrow$ in general, a hopeless task.

Idea: use cpx structure to understand Ricci better.

- on cpx mfd, pick special basis (eigenvectors of J^{dual}, $dz^k, d\bar{z}^k$) to get $\mathcal{A}^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(M)$

→ using natural projections & inclusions, it's enough to define exterior derivative on each summand → $d: \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{k+1}(M) \otimes \mathbb{C} = \bigoplus_{p'+q'=k+1} \mathcal{A}^{p',q'}(M)$

→ in particular $\partial := \pi^{p,q+1} \circ d$, $\bar{\partial} := \pi^{p+1,q} \circ d$

→ claim: $d = \partial + \bar{\partial}$, which follows from $df = \frac{\partial f}{\partial z^k} dz^k + \frac{\partial f}{\partial \bar{z}^k} d\bar{z}^k = (\partial + \bar{\partial}) f$ on functions, since $\alpha = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{I,J}\, dz_I \wedge d\bar{z}_J \mapsto d\alpha = \sum d\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J$

→ we used local coordinates, but this is in fact the type decomposition of J^{dual}, into $\pm\sqrt{-}$ eigenspaces

→ it can be shown that

$$d = \partial + \bar{\partial} \iff J \text{ integrable} \iff N(J) = 0$$

- for the metric, we extend it to $TM \otimes \mathbb{C}$ by $\mathbb{C}$-linearity
- the Hermitian condition: $g(JX, JY) = g(X, Y)$

Lemma $g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = g\left(\frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{z}_l}\right) = 0$

Pf. $g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = g\left(J\left(\frac{\partial}{\partial z_i}\right), J\left(\frac{\partial}{\partial z_j}\right)\right) = -g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right).$

Cor. g can be written as $g = \sum g_{j\bar{k}}\, dz_j \odot d\bar{z}_k$

Cor. $\omega(X,Y) := g(JX, Y)$ as $\omega = \sqrt{-1} \sum g_{j\bar{k}}\, dz_j \wedge d\bar{z}_k$

Stoppa:

- we can check that $g_{i\bar{j}} = \overline{g_{j\bar{i}}}$, which means that positive definiteness of g implies the hermitian matrix $(g_{i\bar{j}})$ is also pos. def.

Rmk. Previously we showed $d\omega = 0 \iff \nabla g J = 0$ on a cpx mfd.

Claim: $d\omega = 0 \iff \frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}$ (just write out in local coords.)

- now extend $\nabla = \nabla^g$ to act on $TM \otimes \mathbb{C}$ by $\mathbb{C}$-linearity
- write $\nabla_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial z_k} = \Gamma^i_{jk} \frac{\partial}{\partial z_i} + \Gamma^{\bar{i}}_{jk} \frac{\partial}{\partial \bar{z}_i}$
- consider the Kähler case

Lemma. The only nonvanishing Christoffels are Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$. Also, $\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^i_{jk}}$.

Pf. Recall $(\Gamma^k_{ij})_{\mathbb{R}} = \frac{1}{2} g^{kl}(g_{il,k} + g_{jl,i} - g_{ij,l})$

The wholly anulogueous formula holds in the complexified case, except we must allow both kinds of indices. But:

$$\Gamma^k_{i\bar{j}} = \frac{1}{2} g^{k\bar{l}}(\underbrace{g_{i\bar{l},\bar{j}} + \cancel{g_{\bar{j}\bar{l},i}} - g_{i\bar{j},\bar{l}}}_{=0 \text{ by Kähler}}) + \frac{1}{2}\cancel{g^{kl}}(\cdots$$

$= 0$.

Similarly other cases. The last claim follows from $\overline{g_{i\bar{j}}} = g_{j\bar{i}}$.

Cor. Explicitly, $\Gamma^i_{jk} = g^{i\bar{l}} \partial_j g_{k\bar{l}}$.

Curvature tensor of a Kähler manifold

N.B. From now on we use the opposite sign convention, so $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Lemma. $J(R(X,Y)Z) = R(X,Y)(JZ)$, $\forall X,Y,Z$

Cor. $g(R(X,Y)Z, W) = g(R(X,Y)JZ, JW)$.

Cor. Upon complexification, $g(R(X,Y)Z,W) = 0$ $\forall X,Y$ if Z,W are of the same type.

Def. $R_{i\bar{j}k\bar{l}} := g\left(R\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right)\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$

Lemma. The only nonvanishing symbols R_{abcd} are $R_{i\bar{j}k\bar{l}}$, upon complexification.

Lemma. $R_{i\bar{j}k\bar{l}} = -\partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{p\bar{q}}(\partial_k g_{i\bar{q}})(\partial_{\bar{l}} g_{p\bar{j}})$

Pf. It's a computation.

Ricci curvature.

Def. Define 2-form $R(X,Y) := Ric(JX,Y)$
$\to$ locally, $R = \sqrt{-1}\, R_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$

Lemma. $R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{p\bar{q}})$.

Stoppa.

Prop. (M, g) Kähler, general type ($c_1(M)<0$).
Fix $\omega_0 \in -2\pi c_1(M)$ as a reference Kähler form. Then $\exists F \in C^\infty(M, \mathbb{R})$ such that for $\omega_0 + i\partial\bar\partial\varphi > 0$, if
$$(\omega_0 + i\partial\bar\partial\varphi)^n = e^{\varphi + F}\omega_0$$
holds, then $Ric(\omega_0 + i\partial\bar\partial\varphi) = -(\omega_0 + i\partial\bar\partial\varphi)$,
i.e. $\omega_\varphi := \omega_0 + i\partial\bar\partial\varphi$ satisfies $Ric(\omega_\varphi) = -\omega_\varphi$
so ω_φ is Einstein with constant $=-1$.

Pf. Suppose the conditional of the claim hold. The condition on top forms implies $\det g_\varphi = e^{\varphi+F}\det g$, locally. Locally we then also have
$-i\partial\bar\partial \log\det g_\varphi = -i\partial\bar\partial(\varphi + F) - i\partial\bar\partial\log\det g$
but we know this expression defines the Ricci form, a global object, i.e. $Ric(\omega_\varphi) = -i\partial\bar\partial(\varphi + F) + Ric(\omega_0)$
The assumption $\omega_0 \in -2\pi i\, c_1(M)$ gives $[Ric(\omega_0)] = -[\omega_0]$
$\Rightarrow$ $Ric(\omega_0) = -\omega_0 + i\partial\bar\partial\tilde F$, $\tilde F \in C^\infty(M, \mathbb{R})$
For $F = \tilde F$,
$$Ric(\omega_\varphi) = -i\partial\bar\partial\varphi - \omega_0 = -\omega_\varphi \quad \square$$

Rmk/exercise. Show converse is true, up to normalisation of φ.

Uniqueness. There is at most one soln to $Ric(\omega) = -\omega$, $[\omega] = -2\pi i c_1(M)$

- pick some reference pt ω_0, $[\omega_0] = -2\pi i c_1(M)$ such that $Ric(\omega_0) = -\omega_0$
- then pick $F \equiv 0$ in previous Prop.
- the $\partial\bar\partial$-lemma says that if ω another solution, ω s.t. $\omega_\varphi = \omega_0 + i\partial\bar\partial\varphi$
  - $\rightarrow$ Rmk/ex said φ can be chosen s.t. $(\omega_0 + i\partial\bar\partial\varphi)^n = e^{\varphi}\omega_0^n$
- now apply maximum principle (remember M is cpt), so pick $p \in M$ s.t. it is a maximum of φ $\Rightarrow$ $Hess_p\varphi \leq 0$
  - $\Rightarrow$ $i\partial\bar\partial\varphi$ is negative semidefinite at p
  - $\rightarrow$ so $\omega_0 + i\partial\bar\partial\varphi$ is "smaller" than ω_0
  - $\rightarrow$ more precisely $\det g_\varphi|_p \leq \det g_0|_p$ and we had $\det g_\varphi|_p = e^{\varphi(p)}\det g_0|_p$
  - $\Rightarrow$ $\varphi(p) \leq 0$ $\Rightarrow$ $\sup\varphi \leq 0$

Repeat argument at minimum of φ to get $\inf\varphi \geq 0$ $\Rightarrow$ $\varphi \equiv 0$. $\square$

Sketch of Pf of Aubin-Yau.

- we want to prove

Thm $\exists$ solution of $(\omega_0 + i\partial\bar\partial\varphi)^n = e^{\varphi + F}\omega_0^n$.

- idea: continuity method

- look at family $(*)_t$ $\begin{cases} (\omega_0 + i\partial\bar\partial\varphi_t)^n = e^{\varphi_t + tF}\omega_0^n \\ \omega_0 + i\partial\bar\partial\varphi_t > 0 \end{cases}$, $t \in [0,1]$

- define $A := \{t \in [0,1] \mid (*)_t \text{ has solution}\}$
  $\to$ now if A nonempty, open, closed $\Rightarrow A = [0,1]$
  so $(*)_1$, the required soln, exists
- nonempty: $t = 0$ solved by $\varphi_0 \equiv 0$.
- open?