Porta,

Quantum transport

- we need to introduce time dependence in order to discuss transport
  $H(t) = H + e^{\eta t} Q$, for $t \leq 0$, $\eta > 0$
- e.g. el. field, $Q = -\vec{E} \cdot \vec{x}$

Goal: evolution of $P = \chi(H \leq \mu)$

- remember: $\psi \in \ell^2(\Lambda)$ evolves as $i\partial_t \psi(t) = H(t)\psi(t)$
  and $\psi(-\infty) = \psi$

$\rightarrow$ Liouville eq. $i\partial_t P(t) = [H(t), P(t)]$, $P(-\infty) = \chi(H \leq \mu)$

- adiabatic limit: small $\vec{E}$, as $\eta \searrow 0^+$

$$P(0) - P = \int_{-\infty}^{0} dt \, \frac{d}{dt} e^{iHt} P(t) e^{-iHt}$$

$$= \int_{-\infty}^{0} dt \, i \, e^{iHt} [\underbrace{H - H(t)}_{+e^{\eta t} E \cdot x}, P(t)] e^{-iHt}$$

$$= i \int_{-\infty}^{0} dt \, e^{\eta t} e^{iHt} [E \cdot x, P(t)] e^{-iHt}$$

$$= i \int_{-\infty}^{0} dt \, e^{\eta t} e^{iHt} [E \cdot x, P] e^{-iHt} + \text{"}O(E^2)\text{"}$$

- velocity $v_i = i[H(t), x_i] = i[H, x_i]$

- current: $J_i = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \mathrm{Tr}_y \, \mathbb{1}_{\Lambda_L} v_i (P(0) - P)$

$$\doteq \mathcal{T}(v_i (P_0 - P)) = \sum \sigma_{ij} E_j$$

where $|\Lambda_L| < \infty$ but we want $L \to \infty$ limit.

$$\Rightarrow \sigma_{ij} = \lim_{\eta \searrow 0^+} \tau\left(\int_{-\infty}^{0} dt \, v_i e^{itH} [X_j, P] e^{-itH} e^{\eta t} \right)$$

Thm Let $d=2$. Then $|P(x,y)| \leq C e^{-c\|x-y\|}$

Thm $\sigma_{12} \in \frac{1}{2\pi}\mathbb{Z}$, $\sigma_{11} = 0$

$K_2 \hookrightarrow \mathcal{L}(h)$, $K_2 \doteq \{ M \in \mathcal{L}(h) \mid \tau(M^* M) < +\infty \}$

- e.g., $|M(x,y)| \leq C e^{-c\|x-y\|} \Rightarrow M \in K_2$

- define $\langle A, B \rangle = \tau(A^* B)$ on K_2,
  so $(K_2, \langle \cdot, \cdot \rangle)$ Hilb. sp.
  (τ possible to show $\tau(AB) = \tau(BA)$).

- let H s.a. on $\ell^2(\Lambda)$ finite ranged
& let $\mathcal{L}_H \circlearrowright K_2$, $\mathcal{L}_H(M) = [H, M]$.
 → fact: $\mathcal{L}_H$ s.a. on K_2
 → so $t \mapsto e^{it\mathcal{L}_H}$ is unitary

$$e^{it\mathcal{L}_H} = \sum_{n \geq 0} \frac{(it)^n}{n!} ad_H^n(M)$$

$$e^{it\mathcal{L}_H}(M) = e^{iHt} M e^{-iHt},$$

$$(\mathcal{L}_H - i\eta)^{-1} = i \int_{-\infty}^{0} dt \, e^{it\mathcal{L}_H} e^{\eta t}$$

- now $\sigma_{ij} = \lim_{\eta \downarrow 0^+} i\,\tau\left(\int_{-\infty}^{0} dt\; v_i e^{\eta t} e^{itH} [x_j, P] e^{-itH} \right)$

$$= \lim_{\eta \downarrow 0^+} \tau\left(v_i \left(\mathcal{L}_H - i\eta\right)^{-1} \left(\underbrace{[x_j, P]}_{\in \mathcal{K}_2} \right) \right)$$

Thm. Suppose $|P(x,y)| \le C e^{-c\|x-y\|}$. Then $\sigma_{ij} = i\,\tau\, P\left[[x_i, P], [x_j, P] \right]$

Pf. Note $[x_i, P] = P[x_i, P]P^{\perp} + P^{\perp}[x_i, P]P$, where of course $P^{\perp} = 1 - P$, and note $\mathcal{L}_H(PM) = P\mathcal{L}_H(M)$.

$$\left(\mathcal{L}_H - i\eta\right)^{-1}\left(P[x_i, P]P^{\perp} + P^{\perp}[x_i, P]P \right)$$

$$= P\left(\mathcal{L}_H - i\eta\right)^{-1}[x_i P]P^{\perp} + P^{\perp}(\cdots)^{-1}[x_i, P]P$$

- using cyclicty of τ,

$$\tau\left(v_i \left(\mathcal{L}_H - i\eta\right)^{-1} [x_i P] \right)$$

$$= \tau\left(\underbrace{\left(P^{\perp} v_i P + P v_i P^{\perp}\right)}_{i\mathcal{L}_H\left([[x_i, P], P] \right)} \left(\mathcal{L}_H - i\eta \right)^{-1} \left([x_j ; P] \right) \right)$$

, using $[H, P] = 0$

$$\sigma_{ij} = \lim_{\eta \downarrow 0^+} i\,\tau\left(\mathcal{L}_H\left([[x_i, P], P] \right) \left(\mathcal{L}_H - i\eta \right)^{-1}\left([x_j, P] \right) \right)$$

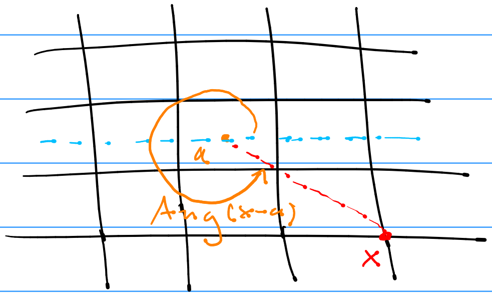
$$= \lim_{\eta \downarrow 0^+} i \left\langle \mathcal{L}_H\left([[x_i P] P] \right)^{*}, \left(\mathcal{L}_H - i\eta\right)^{-1} [x_i P] \right\rangle$$

$$= \lim_{\eta \downarrow 0^+} i \left\langle \underbrace{\left(\mathcal{L}_H + i\eta\right)^{-1} \mathcal{L}_H}_{= \mathbb{1} \text{ in the limit.}} \left([[x_i P] P] \right), [x_i, P] \right\rangle$$

$$= i\,\tau\left([[x_i, P], P] [x_j, P] \right). \qquad \square$$

$d = 2$.

$a \in \mathbb{Z}^2 + \{1/2, 1/2\}$



$\psi \mapsto U_a \psi$,

$(U_a\psi)(x) = e^{i\vartheta_a(x)}\psi(x)$,

$\vartheta_a(x) = \mathrm{Arg}(x-a)$

– magnetic flux. $\Delta Q = \mathrm{Tr}(\overbrace{U_a P U_a^*}^{Q} - P)$

$\neq$ well-defined.

Def. let P, Q be 2 orth. proj. such that $P-Q$ is compact ($P-Q = \sum a_n |f_n\rangle\langle f_n|$, $a_n \to 0$ as $n \to \infty$)

Define: $\mathrm{Ind}(P,Q) = \dim\ker(P-Q-1) - \dim\ker(P-Q+1)$.

Rmk. • $\ker(P-Q-1) = \{\psi \in \mathcal{H} \mid P\psi = \psi, Q\psi = 0\}$

• $\ker(P-Q+1) = \{\psi \in \mathcal{H} \mid P\psi = 0, Q\psi = \psi\}$

• by cptness of $P-Q$, $\dim\ker(P-Q-1)$ is finite, since in ($\sum (a_n - 1)|f_n\rangle\langle f_n|$, only finitely many a_n can $=1$)

Thm (quantization of σ_{12}) let $d=2$, $|P(x,y)| \le Ce^{-c|x-y|}$.

Then $\sigma_{12} = \frac{1}{2\pi}\mathrm{Ind}(P, U_a P U_a^*)$

Prop. let P, Q orth. proj. on $\mathcal{H}$. Supp $(P-Q)^{2n+1}$ is trace class for some n.

Then $\mathrm{Ind}(P,Q) = \mathrm{Tr}_{\mathcal{H}}(P-Q)^{2n+1}$

Pf. Show that $\sigma(P-Q)$ is given by pairs $(-\lambda, \lambda)$ with same multiplicities if $|\lambda| < 1$.

$\dim\ker(P-Q+1)$

So $\mathrm{Tr}(P-Q)^{2n+1} = \sum (b_m)^{2n+1} = (+1)^{2n+1}(\text{mult of } +1)$ $\dim\ker(P-Q-1)$

$+ (-1)^{2n+1}(\text{mult. of } -1)$

- so, check that if λ eigenv. of $P-Q$, so is $-\lambda$;
- def $C = P-Q$, $S = P+Q$, check $S^2+C^2 = 1$, $SC+CS = 0$.
- let $\lambda \in \sigma(C)\setminus\{0\}$, $\varphi \in h$ its eigenvector
  → claim, $S\varphi$ eigenvector w e.v. $S\varphi$
- $CS\varphi = -SC\varphi = -\lambda S\varphi$.
- for $S: \ker(C-1) \to \ker(C+1)$ we claim

Injects Supp. $\ker(C-1)\setminus\{0\} \ni \psi$.
$S\psi = 0 \Rightarrow S^2\psi = 0 \Rightarrow (1-C^2)\psi = 0$
$(1-\underbrace{\lambda^2}_{<1})\psi = 0 \Rightarrow \psi = 0$ ↯

Surjects given $\psi \in \ker(C+1)$, $\exists \tilde\psi \in \ker(C-1)$:
$$S\tilde\psi = \frac{S^2\psi}{(1-\lambda^2)} = \frac{(1-C^2)}{(1-\lambda)^2}\psi = \psi$$

so multiplicities same. □

- next we'll take $P = \chi(h \leq \mu)$, $Q = U_a P U_a^*$
  → $(P - U_a P U_a^*)^3$ will be tr. class...