Gauge (9) | GAP

Quantum solved from crossed and folded - Nekrasov

- relation between instanton counting and quantum mechanics
- main example:
  pure $\mathcal{N}=2$ $U(N)$ on $\mathbb{R}^4$ w sfc defect along $\mathbb{R}^2$

$\Updownarrow$

periodic N-ptcl Toda chain

$$\hat{H} = -\frac{\hbar^2}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \Lambda^2 \sum_{i=1}^{N} e^{x_i - x_{i+1}}$$

- $x_{N+1} \equiv x_1$
- $x_i \in \mathbb{R}$, or $\sqrt{-1}\, x_i \in \mathbb{R}/2\pi\mathbb{Z}$

$A = A_\mu dx^\mu$ $U(N)$-conn. on $P \downarrow X = \mathbb{R}^4 \cup \infty = S^4$; $\sigma \in \Gamma(\mathfrak{g}_P^{\mathbb{C}})$, $\mathfrak{g} = \mathrm{Lie}\, G$

- on SYM side we have $A_\mu, \sigma, \bar{\sigma}$ + fermions $(\psi_\mu, \chi^+_{\mu\nu}, \eta)$, $\psi_\mu \in \Gamma(\Omega^1_X \otimes \Pi \mathfrak{g}_P)$, $\chi^+_{\mu\nu} \in \Gamma(\Omega^{2,+}_X \otimes \Pi \mathfrak{g})$, $\eta \in \Gamma(\Pi \mathfrak{g})$
- exact calculations thanks to localisation to fixed pts of Q_ε, inherited from $U(2) \circlearrowright (\mathbb{C}^2, 0) \simeq (\mathbb{R}^4, 0)$ action, $U(2) \subset Spin(4)$

where $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{C} \times \mathbb{C} = \mathrm{Lie}(U(1)^2) \otimes \mathbb{C}^2 \xhookrightarrow{\text{diag}} U(2)$

- $Q_\varepsilon A = \psi$

  $Q_\varepsilon \psi = D_A \sigma + \iota_{V(\varepsilon)} F_A$, $\quad V(\varepsilon) = \varepsilon_1 (x^2\partial_1 - x^1\partial_2) + \varepsilon_2 (x^3\partial_4 - x^4\partial_3)$

  $Q_\varepsilon \sigma = \iota_{V(\varepsilon)} \psi$

  $\rightarrow$ useful to fix cpx str $\mathbb{R}^4 \simeq \mathbb{C}^2$: $z_1 = x^1 + \sqrt{-1}\, x^2$, $z_2 = x^3 + \sqrt{-1}\, x^4$

- $\mathcal{N}=2$ $d=4$ descends from $\mathcal{N}=1$ $d=6$ on $M^6 = \mathbb{T}^2 \times_{\text{twist}} \mathbb{R}^4$ = v6dl over $\mathbb{T}^2$ w flat $Spin(4) \overset{\text{reduction}}{\supset} U(1)^2$-conn

  $\rightarrow$ given by (commuting due to flatness) A, B cycles on $\mathbb{T}^2$,

$$\Omega_A = \left(\begin{array}{cc|cc} \cos\alpha_1 & \sin\alpha_1 & & 0 \\ -\sin\alpha_1 & \cos\alpha_1 & & \\ \hline & 0 & \cos\alpha_2 & \sin\alpha_2 \\ & & -\sin\alpha_2 & \cos\alpha_2 \end{array}\right), \quad \Omega_B = \left(\begin{array}{cc|cc} \cos\alpha_3 & \sin\alpha_3 & & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & & \\ \hline & 0 & \cos\alpha_4 & \sin\alpha_4 \\ & & -\sin\alpha_4 & \cos\alpha_4 \end{array}\right)$$

- action:

$$S^{b.s} = \frac{1}{g^2}\int_X \mathrm{Tr}\, F_A \wedge * F_A + \mathrm{Tr}\left(D_A z + \iota_{V(\varepsilon)} F_A\right)\wedge *\left(D_A \bar z + \iota_{V(\bar\varepsilon)} F_A\right) + \mathrm{Tr}\left([z,\bar z] + \iota_{V(\varepsilon)} D_A \bar z - \iota_{V(\bar\varepsilon)} D_A z + \iota_{V(\varepsilon)}\iota_{V(\bar\varepsilon)} F_A\right)^2$$

$$+ \underbrace{\frac{\vartheta}{2\pi}\int_X \mathrm{Tr}\, F_A \wedge F_A}_{\sim \frac{1}{\varepsilon_1\varepsilon_2}\mathrm{Tr}\, z^2(0)}, \text{ up to } Q_\varepsilon\text{-exact terms}$$

- $\varepsilon = 0 \Rightarrow Q_0 z = 0$ in usual $\mathcal{N}=2$ SYM

$\Rightarrow Q_0 P(z(x)) = 0$ for any $P \in (\mathrm{Sym}\,\mathfrak{g}^*)^G$, $\forall x \in X$

and $P(z(x_1)) - P(z(x_0)) = Q_0 \int_{x_0}^{x_1} \frac{\partial P}{\partial z^a}\psi^a$

- localisation? $G \circlearrowright M$, $\Omega^*_G(M) = \mathrm{Fun}(\mathfrak{g}, \Omega^*(M))^G$

$d_G = d + \iota_{V(\cdot)}$

- if $d_G \omega = 0$ for $\omega \in \Omega^*_G(M)$, and $A \xrightarrow{R\to\infty}$ flat conn. on S^3, $z \longrightarrow z_\infty = \mathrm{diag}(a_1, \dots, a_N)$

s.t. $z = z_\infty + \delta z$ $\qquad$ $\mathrm{Lie}\,\mathfrak{g}^{\mathbb{C}}$, $a_i \in \mathbb{C}$

$$\int_{\text{"}M/G\text{"}} \omega = \oint_{\mathfrak{g}\subset\mathfrak{g}^{\mathbb{C}}} \frac{dz}{\mathrm{vol}(G)} \int_M \int_{\Pi T\mathfrak{g}} \omega(z)\, d\bar z\, d\eta\, \exp \mathrm{Tr}\left(\eta\,\Theta + \bar z(z + d\Theta + \Theta^2)\right)$$

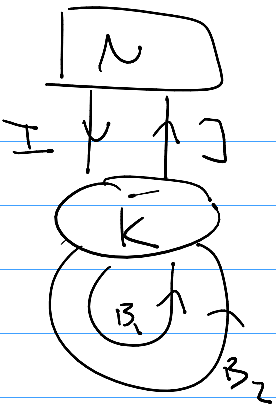
- path. int. in Ω-def. th. on $\mathbb{R}^4$ w fixed z_∞, ε = int. of $G \times U(2)$-equiv. diff. form on $\mathcal{M}^{+,\text{framed}} = \left\{A \,\middle|\, \begin{array}{c} F^+(A) = 0 \\ \text{on } \mathbb{R}^4_\varepsilon \end{array}\right\} / \mathcal{G}_\infty$

- Nakajima–Gieseker $\mathcal{M}_k^{+,\text{fr}}$ has $\overset{\text{isometric}}{\circlearrowright}$ $U(N)\times U(2)$,

a metric g $\qquad$ $U(a,\varepsilon) \to$ V.field representing toric action

$k \in \pi^3(G)$

$= -\frac{1}{8\pi^2}\int_X \mathrm{Tr}\, F_A \wedge F_A$



$N = \mathbb{C}^N$

$K = \mathbb{C}^k$

ADHM description of $\mathcal{M}_k^{t,N}$

$$\simeq \begin{cases} [B_1,B_2]+IJ=0 \\ [B_1,B_1^\dagger]+[B_2,B_2^\dagger]+II^\dagger-J^\dagger J = \begin{cases} \overset{\text{ordinarily}}{0} \\ \underbrace{\zeta\cdot \mathbb{1}_k}_{\text{noncomm}} \end{cases} \end{cases}$$

modulo $U(k)$-action ?

$$\Longleftrightarrow \left\{ \begin{array}{l} \mathbb{C}[B_1,B_2]\, I(N) = K \\ [B_1,B_2]+IJ=0 \end{array} \right\} \Big/ Gl(k)$$

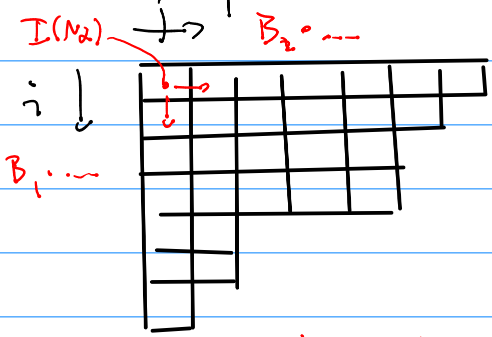
where $h\cdot(B_1, B_2, I, J) = (h^{-1}B_{1,2}h,\ h^{-1}I,\ Jh)$, $h\in Gl(k)$

- $\dim_{\mathbb{C}} \mathcal{M}_k^{t,N} = 2N\cdot k$

$$Z_k(a,\varepsilon) = \int_{\mathcal{M}_k^{t,N}} \exp\ldots$$

$$= \sum_{\substack{(\lambda^{(1)},\ldots,\lambda^{(N)}) \\ \sum_{i=1}^N |\lambda^{(i)}| = k}} \underbrace{M_{\lambda^{(1)}\ldots\lambda^{(N)}}(a,\varepsilon)}_{\text{rat. function of } \deg = -2Nk}$$

$\lambda^{(\alpha)}$ = partitions $(\lambda_1^{(\alpha)} \geq \lambda_2^{(\alpha)} \geq \ldots)$



$(i,j)\in\lambda^{(\alpha)}$

iff $1\leq j\leq \lambda_i^{(\alpha)}$

$1\leq i\leq \lambda_j^{(\alpha)t}$

$K_\alpha = \mathbb{C}[B_1,B_2]\, I(N_\alpha)$, N_α = eigensp. of a w eigenval. a_α, $N = \bigoplus_\alpha N_\alpha$

$$\| $$

$$\bigoplus_{(i,j)\in\lambda^{(\alpha)}} \mathbb{C}\cdot B_1^{i-1} B_2^{j-1} I(N_\alpha)$$

- this comes from fixed pts

$$\begin{cases} q_i B_i = h^{-1}B_i h, \ i=1,2 \\ I t^{-1} = h^{-1} I \\ q_1 q_2 t J = Jh \end{cases} \Longrightarrow \begin{array}{l} J=0 \\ [B_1,B_2]=0 \\ K = \bigoplus K_\alpha \end{array}$$

- once we have fixed pts, we put $q_i = e^{\varepsilon_i}$, $t = e^{a}$ to linearize action on tgt sp.:

$$[B_1, B_2] + I\,J = 0$$

$$\Rightarrow [B_1, \delta B_2] + [\delta B_1, B_2] + \delta I \overset{=0}{\widetilde{J}} + I\,\delta J = 0$$

modulo

$$(\delta B_1, \delta B_2, \delta I, \delta J) = ([B_1, \xi], [B_2, \xi], -\xi I, J\xi)$$

$$T_\lambda \mathcal{M}^{t,N}_{\text{framed}} = \frac{\ker d_2}{\operatorname{Im} d_1} \quad \text{of cpx}$$

$$\begin{array}{ccccccccc} \operatorname{End} K & \xrightarrow{d_1} & \operatorname{End}(K) \otimes \mathbb{C}^2 & \oplus & \operatorname{Hom}(N, K) & \oplus & \operatorname{Hom}(K, N) & \xrightarrow{d_2} & \operatorname{End}(K) \\ \rotatebox{90}{$\in$} & & & & & & & & \\ \xi & & (\delta B_1, \delta B_2) & & \delta I & & \delta J & & \end{array}$$

- then, instead of $M_{\lambda^{(1)} \ldots \lambda^{(N)}}(a, \varepsilon) = \dfrac{1}{\prod_{T_\lambda \mathcal{M}^{t,N}_K} \text{weights}(a,\varepsilon)}$

compute $\displaystyle\sum_{T_\lambda \mathcal{M}^{t,N}_K} e^{\text{weights}(a,\varepsilon)} = \operatorname{Ch}_{\mathbb{T}}(T_\lambda \mathcal{M}^{t,N}_K)$

$$\underset{\text{Euler char}}{=} N K^* + N^* K q_{1,2} - P_{12} K K^*$$

in terms of characters (notation abuse), s.t. $\chi^* = \chi(-a, -\varepsilon)$

- $\operatorname{Ch}_{\mathbb{T}}(K) \text{ "=" } K = \displaystyle\sum_{\alpha=1}^{N} e^{a_\alpha} \sum_{(i,j)\in\lambda^{(\alpha)}} (e^{\varepsilon_1})^{i-1} (e^{\varepsilon_2})^{j-1}$

$\operatorname{Ch}_{\mathbb{T}}(N) \text{ "=" } N = \displaystyle\sum_{\alpha=1}^{N} e^{a_\alpha}$

$q_{1,2} = e^{\varepsilon_1 + \varepsilon_2}$, $P_{1,2} = (1 - e^{\varepsilon_1})(1 - e^{\varepsilon_2})$

- finally, $Z^{\text{inst}}(a, \varepsilon; \Lambda) = \displaystyle\sum_{k=0}^{\infty} \Lambda^{2Nk} Z_k(a, \varepsilon)$

- relevant poles at $a_\alpha - a_\beta + \varepsilon_1 i + \varepsilon_2 j = 0$
  where $\alpha \neq \beta$, $i, j \geq 1$
- if $p\varepsilon_1 + q\varepsilon_2$, $p, q < 0$, individual contributions may diverge
- if ε_1 or $\varepsilon_2 \to 0$, $Z^{inst} \underset{\varepsilon_1 \to 0}{\sim} \exp \frac{1}{\varepsilon_2} \overbrace{W(a, \varepsilon_1; \Lambda)}^{\text{link to QM.}}$
  essential sing.

- tomorrow: inst. counting w defects,
  folding, QM.