Bruzzo

- recalling: f separated if Δ_f closed immersion.

$$\begin{tikzcd}
X \arrow[dr, "\Delta_f"] \arrow[drr, bend left, "\sim"] \arrow[ddr, bend right, "2"] & & \\
& X\times_Z X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Properties:

(i) open & closed immersions are separated

(ii) composition respects separatedness

(iii) so does base change

$$\begin{tikzcd}
X\times_Y Y' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

(iv) f is separated iff Y has an open cover $\{U_i\}$ s.t. all restrictions $f^{-1}(U_i) \xrightarrow{f} U_i$ are separated

Proper morphisms

Def. $f: X \to Y$ is proper if it is $\begin{cases} \text{(i) of finite type} \\ \text{(ii) separated} \\ \text{(iii) universally closed} \end{cases}$

(closed after any base change)

$X\times_Y Y' \xrightarrow[\text{closed}]{f'} Y'$

We call a scheme X over field k **complete** if $X \to \operatorname{Spec} k$ proper.

examples: - $\mathbb{P}^n_k$ is complete

- $\mathbb{A}^n_k$ is of finite type, separated,

but $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not universally closed:

$$\left.\begin{tikzcd}
\mathbb{A}^2_k = \mathbb{A}^1_k\times_k \mathbb{A}^1_k \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\
\mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k
\end{tikzcd}\right\}\ \text{base change}$$

not closed!? $\{xy-1=0\}$ closed but image is not.

Towards QCoh.

$(X, \mathcal{O}_X)$ ringed space.

A sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is a sheaf of ab. grps s.t.

(i) $\forall U \subset X$ open, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module

(ii) restrictions $\mathcal{S}_{UV}: \mathcal{M}(U) \to \mathcal{M}(V)$ are morphisms of $\mathcal{O}_X(U)$-modules.

Consider $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules & $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$.

We have a hom-set $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ which we turn into a sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \doteq \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$

not $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$

We give these sheaves a monoidal structure:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \doteq [U \rightsquigarrow \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)]^{\#}$$

Rmk. $x \in X$, $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \neq (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x$

$\mathcal{F}$ is free if $\mathcal{F} \simeq \bigoplus_{i \in I} \mathcal{O}_X$, $|I| < \infty$

$\mathcal{F}$ is locally free if $\forall x \in X$ $\exists$ open nbhd $U \ni x$ s.t. $\mathcal{F}|_U = \bigoplus_{i \in I} \mathcal{O}_U$

For $X \xrightarrow{f} Y$, the pushforward construction gives a functor $f_*: \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}$.

$\to$ f_* is left-exact.

- we also define $(f^{-1}\mathcal{G})(U) = \lim\limits_{U \supset f(U)} \mathcal{G}(V)$, + sheafification

$\to$ sheaf on X !

$\to$ $f^{-1}: Sh_Y \to Sh_X$

① f^{-1} exact. (follows from $(f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$)

② $$\begin{tikzcd} X \times_Y \underline{\mathcal{G}} \arrow[r] \arrow[d] & \underline{\mathcal{G}} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$Sh_X \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} Sh_Y$ are an adjoint pair of functors.

Pf. Construct $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ on opens, $\varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$

For
$$\begin{tikzcd} E \times_Y X \arrow[r] \arrow[d] & E \arrow[d, "\pi"'] \\ X \arrow[r, "f"] \arrow[u, bend left, "s'"] & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$
where $\pi \circ s = id$,
$s'(x) = ((s \circ f)(x), x)$

$\mathcal{G}(U) \ni s \mapsto (f^{-1}\mathcal{G})(f^{-1}(U))$
$= (f_* f^{-1}\mathcal{G})(U)$. $\square$