Тихомиров

- $\mathcal{G} = \{ \varphi : P \to SU(2) \mid \varphi(pg) = g^{-1}\varphi(p) g \}$
- $\xi = P \times_{SU(2)} \mathbb{C}^2$

$\Rightarrow \varphi \in \mathcal{G} \Leftrightarrow \varphi \in \mathrm{Aut}_h \xi$, smooth

- $\mathcal{E}$ holom. str. on $E = P \times_{SU(2)} \mathbb{C}^2$

$\Leftrightarrow \bar\partial_{\mathcal{E}} : \Omega^0(E) \to \Omega^{0,1}(E)$,

$\bar\partial_{\xi}(f \cdot s) = (\bar\partial f)\cdot s + f(\bar\partial_{\xi} s)$ Leibnitz

- $\mathcal{E}_1 \xrightarrow[\sim]{\psi} \mathcal{E}_2$ gives $(E \xrightarrow[\sim]{\varphi_\psi} E) \in \mathcal{G}^{\mathbb{C}}$

$\to \bar\partial_{\mathcal{E}_2} \in \mathcal{G}^{\mathbb{C}}$-orbit of $\bar\partial_{\mathcal{E}_1}$,

which simply comes from

$$\begin{array}{ccc} \Omega^0(E) & \xrightarrow[\sim]{\varphi_\psi} & \Omega^0(E) \\ \downarrow \bar\partial_{\mathcal{E}_1} & & \downarrow \bar\partial_{\mathcal{E}_2} \\ \Omega^{0,1}(E) & \xrightarrow[\sim]{\varphi_\psi} & \Omega^{0,1}(E) \end{array}$$

- $\mathcal{A}^{1,1}_h = \{ A \in \mathcal{A} \mid F_A \in \Omega^{1,1}(\mathcal{E}nd\, E), h \text{ herm. form on } E \}$

- $\tilde{\mathcal{M}}(P, g) = \tilde{\mathcal{M}}^* = \{ A \in \mathcal{A}^{1,1}_h \mid \hat{F}_A = 0 \Leftrightarrow A \text{ is } {}^{ASD}_{\text{irred}} \}$

- $\mathcal{M}^* = \tilde{\mathcal{M}}^* / \mathcal{G} \ni [A] \longmapsto [A]_{\mathbb{C}} \in \mathcal{M}_{[\omega]}$

- $\mathcal{A}^{1,1}_{\mathbb{C}} / \mathcal{G}^{\mathbb{C}}$ = space of holom. str's on E

$\cup$

$\mathcal{M}_{[\omega]} = \{ [A]_{\mathbb{C}} \mid \mathcal{E}$ defined by $\bar\partial_A$ is $[\omega]$-stable $\}$

from $\nabla_A = \partial_A + \bar\partial_A$

- Donaldson: $[A] \mapsto [A]_{\mathbb{C}}$ is isom.

- $\gamma_d(X)$ is a smooth (orient. pres.) diffeom. invariant, $\gamma_d(X): H_2(X,\mathbb{Z})\times \dots \times H_2(X,\mathbb{Z}) \to \mathbb{Z}$

Seiberg-Witten

simply conn., $b_2^+(X)>1$ and odd

- $\mathcal{C}_X = \{ K \in H_2(X,\mathbb{Z}) \mid K$ is characteristic, i.e. $\omega_X(K,C) \equiv \omega_X(C,C) \pmod 2$, $\forall C \in H_2(X,\mathbb{Z}) \}$.

- Seiberg-Witten is $\mathcal{C}_X \to \mathbb{Z}$
  1) for any $K \in \mathcal{C}_X$ there is defined (for generic metric g on X and generic $\delta \in \Omega^2_{+,g}(X)$) smooth $\mathcal{M}_K^\delta(g)$ mfd of dimension

$$\dim \mathcal{M}^\delta_K(g) = \underbrace{\frac{1}{4}\left[K^2 - (3\sigma(X) + 2\chi_{top.}(X))\right]}_{=:2m}$$

  which is even (if $b_2(X)>1$ and odd). Orientation on $\mathcal{M}^\delta_K(g)$ is given by that of $H^0(X,\mathbb{R}) \oplus H^2_+(X,\mathbb{R})$
  ii) $\mathcal{M}^\delta_K(g) \subset \mathcal{B}_K$ config. sp., ∞-dim, and $\mathcal{B}_K \simeq \mathbb{CP}^\infty$, $H^*(\mathcal{B}_K,\mathbb{Z}) = \mathbb{Z}[h]$

$$\exists\, [\mathcal{M}^\delta_K(g)] \in H_{2m}(\mathcal{M}^\delta_K(g), \mathbb{Z})$$

$$\mathcal{C}_X(K) := \langle h^m, [\mathcal{M}^\delta_K(g)] \rangle$$

- $SW: \mathcal{C}_X \to \mathbb{Z}$ are (orient. pres.) diff. invariants of simply conn. cpt oriented 4-mfd X with $b^+(X)>1$, odd.

Thm (vanishing) If X as above and

i) $X = Y \# Z$, $b_2^+(Y) > 0$, $b_2^+(Z) > 0$, then $SW_X = 0$!

ii) If X has a metric of positive scalar curv., then $SW_X = 0$

iii) If $\exists\, S^2 \hookrightarrow X$ w. $\langle S^2, S^2 \rangle \geq 0$, $[S^2] \neq 0$, also $SW_X = 0$

Thm (nonvanishing) If $X = S$ cpx. sfc, $b_+^2(S)$ odd > 1, then $SW_X(\pm c_1(K_S)) \neq 0$

- $0 \to \mathbb{Z}_2 \to Spin(4) \to SO(4) \to 1$
  $\Rightarrow P_{Spin(4)} \xrightarrow{2:1} P_{SO(4)}$ if 2nd Stiefel-Whitney class $= 0$.
- $Spin^{\mathbb{C}}(4) = \{(A, B) \in \mathcal{U}(2) \times \mathcal{U}(2) \mid \det A = \det B\}$
  $\subseteq \mathcal{U}(2) \times \mathcal{U}(2)$
- $\det : \mathcal{U}(2) \to \mathcal{U}(1)$, $A \mapsto \det A$
- $\mathcal{U}(2) = S^1 \times SU(2) / \{\pm(1, \mathbb{1})\}$

  $\underset{zA}{\mathcal{U}(2)} \quad \underset{z}{S^1} \quad \underset{A}{SU(2)}$
- so $0 \to \mathbb{Z}_2 \to S^1 \times SU(2) \times SU(2) \to Spin^{\mathbb{C}}(4) \to 0$
  $\{\pm 1\} \mapsto \{\pm(1, \mathbb{1}, \mathbb{1})\}$

  $0 \to \mathbb{Z}_2 \to Spin^{\mathbb{C}}(4) \xrightarrow{2:1} S^1 \times SO(4) \to 1$
- P_{S^1} pbdl w $c_1(P_{S^1}) = K$, $\mathcal{G} = Maps(P, S^1)$

- $\rho_+ : Spin^{\mathbb{C}}(4) \to \mathrm{Aut}\, \mathbb{H}$

$\rho_+([\lambda, q_1, q_2])(h) = q_1 h \lambda$

$\rho_-($ —''— $)(h)$

$W^{\pm} := P_{Spin^{\mathbb{C}}(4)} \times_{\rho_\pm} \mathbb{H}$ spinor bdls rk 2

$\rho_0([\lambda, q_1, q_2])(h) = q_1 h \overline{q_2}$

$P_{Spin^{\mathbb{C}}(4)} \times_{\rho_0} \mathbb{H}$ real rk 4 bdl TX

$TX \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$

$c : \Gamma(W^+ \otimes T^*X) \longrightarrow \Gamma(W^-)$ Cliff. mult.

$$\begin{tikzcd} \Gamma(W^+) \arrow[r, "\nabla_A"] \arrow[dr, "\not{D}_A"'] & \Gamma(W^+ \otimes T^*X) \arrow[d, "c"] \\ & \Gamma(W^-) \end{tikzcd}$$

- monopole equn.'s $\begin{cases} \not{D}_A \psi = 0 \\ i\,\tau(\psi) = F_A^+, \end{cases}$

$\tau : \mathbb{H} \to \mathrm{Im}\, \mathbb{H},$

$h \mapsto -h i \bar{h}$