Fantechi

$0 \to H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{F})) \to Ext^1(\mathcal{E}, \mathcal{F})$
$\to H^0(X, \mathcal{E}xt^1(\mathcal{E}, \mathcal{F})) \to H^2(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$

- today: deformation th in finite study of moduli
- moduli study of flat families $\to$ var, sch, vsp, cohsh, ...
- X proj sch family of cl. subsch. par by B

$$\begin{array}{ccc} Z & \hookrightarrow & X \times B \\ \downarrow \text{flat} & & \\ B & & \end{array}$$

- e.g. Graßmannian, univ fam. of lin subsp.
- hypersfcs of deg d in $\mathbb{P}^N$

- $X \ni p \rightsquigarrow T_pX$?
- moduli $\rightsquigarrow B$ fat point, ie $B_{red} = Spec(k)$, say affine so $B_{red} = Spec(A/N_A)$

Lemma $\exists$ bijection between T_pX & $\{\varphi \in Mor(Spec \frac{\mathbb{C}[t]}{t^2}, X)\}$

Pf. Assume $p = 0$. $p \in X = Spec\, R = Spec \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_s)}$
$\Leftrightarrow f_1(0) = \dots = f_s(0) = 0$.
Write $f_i = f_{i,0} + \dots + f_{i,m_i}$, $f_{i,a}$ hom. of deg a
$\Rightarrow \frac{\partial f_i}{\partial x_j}(0) = \frac{\partial f_{i,0}}{\partial x_j}(0)$

Let $l_i := f_{i,1}$. Then $T_pX = \ker(\mathbb{C}^n \xrightarrow{(l_1, \dots, l_s)} \mathbb{C}^s)$

On the other hand, $\varphi : Spec \frac{\mathbb{C}[t]}{t^2} \to Spec\, R$
$\Leftrightarrow \varphi^\# : R \to \mathbb{C}[t]/t^2$ s.t. $\varphi^\#(m_0) \subseteq (t)$
$\Leftrightarrow (\varphi^\#)^{-1}((t)) = m_0$, ...

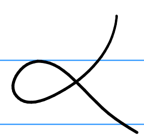
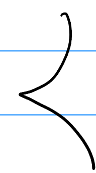
Def. Let $X = \operatorname{Spec} \dfrac{\mathbb{C}[x_1,\dots,x_n]}{(f_1,\dots,f_r)}$. X is nonsing/smooth at zero $\iff$ X red. at 0 and $\dim X|_0 = \dim T_0 X$

Fact X smooth at 0 $\iff$ $\forall n \geq 1$ $\forall v \in T_0 X$

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C}[t]/t^2 \arrow[r, "v"] \arrow[d] & X \\ \operatorname{Spec} \mathbb{C}[t]/t^{n+1} \arrow[ur, "\exists"'] & \end{tikzcd}$$

-example.

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C}[t]/t^2 \arrow[r, "(at, bt)"] \arrow[d] & \operatorname{Spec} \dfrac{\mathbb{C}[X,Y]}{Y^2 - X^3 - X^2} \\ \operatorname{Spec} \mathbb{C}[t]/t^3 \arrow[ur, "?"'] & \end{tikzcd}$$

$\to$  works but try the cusp  ...

Def. Let X scheme, $B = \operatorname{Spec} A$ fat point.
A deformation of X over B is
i) a flat mor $X_B \xrightarrow{f} B$
ii) a cl. embedding $u : X \hookrightarrow X_B$
such that

$$\begin{tikzcd} X \arrow[r, hook, "u"] \arrow[d] & X_B \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, "A/m_A \leftarrow A"'] & B \end{tikzcd}$$

is comm & cartesian.

Given another deformation, $X \overset{\tilde{u}}{\hookrightarrow} \tilde{X}_B \xrightarrow{\tilde{\pi}} B$,
an isom. is the comm diag.

$$\begin{tikzcd}
 & X \arrow[dl, hook', "\tilde{u}"'] \arrow[dr, hook, "u"] & \\
\tilde{X}_B \arrow[rr, "\sim"] \arrow[dr, "\tilde{\pi}"'] & \circlearrowleft & X_B \arrow[dl, "\tilde{\pi}"] \\
 & B &
\end{tikzcd}$$

Def. A 1^{st} order (infinit) defn of X is defn$/\mathrm{Spec}\,\frac{\mathbb{C}[t]}{t^2}$.

Thm Assume X proj. var$/\mathbb{C}$ (sch of f.t./$\mathbb{C}$, gener. smooth)
Then

$$\{1^{st}\text{-ord defns}\} \longleftrightarrow \{\text{extensions } 0 \to \mathcal{O}_X \to \mathcal{E} \to \Omega^1_X \to 0\}$$

is an equiv of categories.
In particular, $\{1^{st}\text{-ord defns}\}/_{\text{isom}} \overset{1:1}{\hookrightarrow} \mathrm{Ext}^1(\mathcal{O}_X, \Omega^1_X)$.

- before proving, let's remember something abt flatness

Lemma Let M be $\mathbb{C}[t]/t^2$-mod, $M_0 = M \otimes_{\mathbb{C}[t]/t^2} \mathbb{C}$.

i) $\exists$ nat ex. seq. $M_0 \xrightarrow{\cdot t} M \xrightarrow{t\mapsto 0} M_0 \to 0$

ii) M flat over $\mathbb{C}[t]/t^2$ iff $0 \to M \xrightarrow{t} M_0$ ex.

Pf. (of Thm) Step 1: from defns to exts.
$B_{red} = \mathrm{Spec}\,\mathbb{C} \to B$ homeo of top. sp $\leadsto u: X \to X_B$ homeom.
so $A := u^{-1}\mathcal{O}_{X_B}$ sheaf of flat $\mathbb{C}[t]/t^2$-alg.
$I/I_2 \to u^*\Omega_{X_B} \to \Omega_X \to$ exact, since
u closed emb., with $I = \mathcal{J}_{X/X_B}$.
$0 \to u^{-1}I \to u^{-1}\mathcal{O}_{X_B} \to \mathcal{O}_X \to 0$, $\mathcal{O}_X = (u^{-1}\mathcal{O}_{X_B}) \otimes_{\mathbb{C}[t]/t^2} \mathbb{C}$
$0 \to \mathcal{O}_X \to A \to \mathcal{O}_X \to 0$, $I \underset{\text{can}}{\simeq} u_*\mathcal{O}_X$, $I^2 = 0$
$\mathcal{J}/\mathcal{J}^2 = u^*\mathcal{J} \underset{\text{can}}{\simeq} \mathcal{O}_X$, $\mathcal{O}_X \xrightarrow{\alpha} u^*\mathcal{O}_{X_B} \to \Omega_X \to 0$

We only need α injects. On locus $X^{sm} \subseteq X$ it is [won't show]. $\forall U \xhookrightarrow{\text{open}} X$, $f \in \mathcal{O}_X(U)$,
$\alpha(f) = 0 \Rightarrow f|_{\underbrace{U \cap X^{sm}}_{\text{nonempty}}} = 0 \Rightarrow f = 0$.

Step 2.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\tilde{\alpha}"] \arrow[d, equal] & \mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_X \arrow[r] \arrow[d, "d"] & 0 & \text{exact.} \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\alpha"] & \Sigma \arrow[r, "\beta"] & \Omega_X \arrow[r] & 0 & \text{exact}
\end{tikzcd}$$

where $\mathcal{A} := \{(e,f) \in \Sigma \oplus \mathcal{O}_X \mid \beta(e) = df\}$
pullback. $\tilde{\alpha}(f) := (\alpha(f), 0)$, d $\mathbb{C}$-lin not
$\mathcal{O}_X$-lin. $\mathcal{A}$ is sheaf of $\mathbb{C}$-mod vsp,
but also a $\mathbb{C}$-alg, $(e_1,f_1)(e_2,f_2) = (f_2e_1 + f_1e_2, f_1f_2)$
since $\beta(e_i) = df_i$ gives $d(f_1f_2) = f_2 df_1 + f_1 df_2$
$= f_2\beta(e_1) + f_1\beta(e_2) = \beta(\dots)$.

Now, $X_B = X$ as top.sp., so $u: X \to X_B$
is actually the identity. Let $\mathcal{O}_{X_B} := u_* \mathcal{A}$.
sheaf of $\mathbb{C}[t]/t^2$-alg.

Define $t: \mathcal{A} \to \mathcal{A}$ by $t(e,f) = \tilde{\alpha}(f) = (\alpha(f), 0)$,
$t^2 = t(\alpha(f), 0) = (\alpha(0), 0) = 0$.
so $\mathcal{A} \otimes_{\mathbb{C}[t]/t^2} \mathbb{C} \xrightarrow{\sim} \mathcal{O}_X$, $(e,f) \mapsto f$.

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{A} \xrightarrow{\otimes_{\mathbb{C}[t]/t^2}\mathbb{C}} \mathcal{O}_X \to 0 \Rightarrow \mathcal{A} \text{ flat.}$$

$\forall U \subseteq X$ affine, $U_B = \operatorname{Spec} \mathcal{A}(U)$.

$$\Rightarrow \begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X_B \arrow[d] \\
\operatorname{Spec} \mathbb{C} \arrow[r] & B
\end{tikzcd}. \quad \square$$

- so we can interpret

$$0 \to H^1(X, T_X) \to \mathrm{Ext}^1(\Omega_X, O_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, O_X)) \to H^2(X, T_X)$$

1^{st} ord defs, trivial on every open affine

1^{st} ord defs of X / iso

compatible 1^{st} ord. defns on each open / iso

- $T_X := \mathcal{H}om(\Omega_X, O_X)$

- $0 \in \mathrm{Ext}^1(\Omega_X, O_X) \leftrightarrow$ split ext $\leftrightarrow$ triv def $X_B = X \times B$

- if X nonsing $\Rightarrow$ all defs loc. triv.
- If X is red proj curve, $H^2(X, T_X) = \{0\}$.