Bruzzo

Derived functors.

- suppose $F: \mathcal{A} \to \mathcal{B}$ left exact.
- given $0 \to A' \to A \to A'' \to 0$ exact seq.
  we have $0 \to FA' \to FA \to FA'' \to$
  $\to R^1FA' \to R^1FA \to R^1FA''$
  $\to R^2FA' \to R^2FA \to R^2FA''$
  $\to \dots$ long exact. $(R^0F \simeq F)$
- suppose $(K^\bullet, d)$ cochain cplx
- natural notion of cohomology, $H^n(K^\bullet, d) = \dfrac{\ker d: K^n \to K^{n+1}}{\operatorname{Im} d: K^{n-1} \to K^n}$.
- a morphism $f: K^\bullet \to L^\bullet$ induces
  a morphism $H(f): H(K) \to H(L)$.
  - explicitly, $\xi \in H^n(K)$, $\xi = [x] \Rightarrow H(f)(\xi) = [f(x)]$

- given a cochain exact seqn. $0 \to K' \xrightarrow{f} K \xrightarrow{g} K'' \to 0$,
  we get a long exact seqn

$$0 \to H^0(K') \xrightarrow{H^0(f)} H^0(K) \to H^0(K'')$$

$$\xrightarrow{\partial_0} H^1(K') \xrightarrow{H^1(f)} H^1(K) \to H^1(K'') \xrightarrow{\partial_1} \dots$$

where ∂_n is the connecting morphism
- let's define it.
  $\xi \in H^n(K'')$, $\partial\xi \in H^{n+1}(K')$?
  - $\xi = [x'']$, $g(x) = x''$ s.t. $g(dx) = d''g(x) = d''x'' = 0$
  - now $dx = f(x')$, $f(d'x') = d^2x = 0 \Rightarrow d'x' = 0$
  $\Rightarrow \partial_n(\xi) = [x']$.

- when do maps $(K^\bullet, d) \overset{f}{\underset{g}{\rightrightarrows}} (L^\bullet, d')$ induce the same cohomological morphism?
- define homotopy $s_{n+1}: K^{n+1} \to L^n \ \forall n$,
  i.e.

$$\begin{tikzcd}
\arrow[r] & K^{n-1} \arrow[r, "d_{n-1}"] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & K^n \arrow[r, "d_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "s_n"'] & K^{n+1} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, "s_{n+1}"'] & \cdots \arrow[dl] \\
\arrow[r] & L^{n-1} \arrow[r, "d'_{n-1}"] & L^n \arrow[r, "d'_n"] & L^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

s.t. $s_{n+1} \circ d_n - d'_{n-1} \circ s_n = f_n - g_n, \ \forall n$

- now $H(f) = H(g)$, since

$$H(f)([x]) - H(g)([x])$$
$$= [f(x)] - [g(x)]$$
$$= [s \circ d(x) - d' \circ s(x)] = 0.$$

- as a special case, $f = id, g = 0 \Rightarrow H^n(K) = 0, \ \forall n$.

- we say $(K^\bullet, d)$ and $(L^\bullet, d')$ are homotopy equivalent if $\exists$ morphisms $(K^\bullet, d) \underset{g}{\overset{f}{\rightleftarrows}} (L^\bullet, d')$ s.t. $g \circ f \sim id_K$, $f \circ g \sim id_L$.

Rmk. this means $H(g) \circ H(f) = H(f) \circ H(g) = id$,
so $H(f) = H(g)$.

Abelian categories.

- $\mathcal{A} \in (Cat)_0$ is abelian if, for any $A, B \in \mathcal{A}_0$,
  1) $Hom(A, B)$ is an abelian group.
  11) $\mathcal{A}$ has sums & products
  111) $\mathcal{A}$ has (co)kernels and every mono is a kernel / epi is a cokernel

$$\begin{tikzcd}
 & C \arrow[dl, dashed] \arrow[d, "g"] & \\
Ker f \arrow[r] & A \arrow[r, "f"] & B
\end{tikzcd}$$

Def. A **resolution** of $A \in \mathcal{A}$ is a pair $((L,d),\varepsilon)$
$\varepsilon: A \to L^0$ and the sequence
$$0 \to A \xrightarrow{\varepsilon} L^0 \xrightarrow{d_0} L^1 \xrightarrow{d_1} L^2 \xrightarrow{d_2} \cdots$$
is exact.

- now take $F: \mathcal{A} \to \mathcal{B}$ functor of ab. cats, additive & left-exact.
- take an **injective** resolution of $A \in \mathcal{A}_0$
$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$
and apply F to get the exact sequ.
$$FA \to FI^0 \to FI^1 \to FI^2 \to \cdots$$

Def. $H^n(FI^\bullet) =: R^nFA$ is the **n-th right derived functor of F.**

Rmk. Injective resolns aren't unique.

- what is an injective object?

Def. $I \in \mathcal{A}_0$ is **injective** if $\mathrm{Hom}_{\mathcal{A}}(-,I)$ is an exact functor.

Rmk. h_A is only left-exact for any $A \in \mathcal{A}_0$

- e.g. in (Ab), injectives are divisible groups, i.e. $G \in (Ab)_0$ s.t. $\forall g \in G\ \exists h$ s.t. $g = nh$
  $\to$ e.g. $\mathbb{Q}$.

Prop. TFAE:

i) $I \in \mathcal{A}$ is injective

ii)
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[dl, dashed, "g"] \\ & I & \end{tikzcd}$$
s.t. $g \circ f = id$

iii) every s.e.s. $0 \to I \to A \to B \to 0$ splits.

Def. $\mathcal{A}$ has **enough injectives** if every object can be embeded into an injective object, $0 \to A \dashrightarrow I$

- examples: Ab, A-mod, Sh_X, $\mathcal{O}_X$-mod
- note that in this case any object has an injective resolu

$$\begin{tikzcd} & & & 0 \arrow[d] & & \\ 0 \arrow[r] & A \arrow[r] & I_0 \arrow[r] \arrow[dr, dotted] & Q_0 \arrow[r] \arrow[d] & 0 & \\ & & & I_1 \arrow[d] \arrow[drr, dotted] & & \\ & & 0 \arrow[r] & Q_1 \arrow[r] \arrow[d] & I_2 \arrow[r] \arrow[dr, dotted] & \cdots \\ & & & 0 & & \end{tikzcd}$$

- note that for
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & L^0 \arrow[r] & L^1 \arrow[r] & \cdots \\ 0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd}$$

where $I^\bullet$ is injective ($L^\bullet$ not necessarily), f "lifts", ie $\exists\, g_n : L^n \to I^n$, $n = 0, 1, \dots$

These morphisms aren't unique, but any 2 such lifts f & g are homotopic.