Bruzzo.

- A c. ring w/ unity, B A-algebra, $I \subset B$ ideal
  $C = B/I$
- $I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \to 0$ exact,
  $\delta[b] = db \otimes 1$, $\alpha(db \otimes 1) = d[b]$

Corollary. If B fin. gen. over A, $\Omega_{B/A}$ fin. gen. over B.

Pf. Use that α is injective with $C \rightsquigarrow B = \frac{A[x_1, \dots, x_n]}{I}$:
$B \rightsquigarrow A[x_1, \dots, x_n]$, so that
$\Omega_{A[x_1, \dots, x_n]/A} \otimes B \to \Omega_{B/A} \to 0$

Example $A = \mathbb{k}$, $\bar{B}$ $\mathbb{k}$-alg, $X = \operatorname{Spec} \bar{B}$
$B = \mathcal{O}_{X,x}$, $x \in X$ is a $\mathbb{k}$-alg, $\mathcal{O}_{X,x} = \bar{B}_x$

- Suppose we are given rings A, B, C,
  $A \to B \xrightarrow{\chi} C$, not necessarily exact
- we get $\Omega_{B/A} \otimes_B C \xrightarrow{\psi} \Omega_{C/A} \xrightarrow{\varphi} \Omega_{C/B}$
  with $\varphi : dc \mapsto dc$, $\psi : db \otimes 1 \to d\chi(b)$
  $\to$ so it's exact and moreover φ surjects.
- assume B ring, maximal ideal m, $\mathbb{k} = B/m$

Prop Assume $\mathbb{k} \to B$ injects.
Then $\delta : m/m^2 \to \Omega_{B/\mathbb{k}} \otimes_B \mathbb{k}$ is an isomorphism

Pf. Use sequence before with $C \rightsquigarrow \mathbb{k}$ to get surjectivity of δ. Look at
$\delta^* : \operatorname{Hom}_{\mathbb{k}}(\Omega_{B/\mathbb{k}} \otimes_B \mathbb{k}, \mathbb{k}) \to \operatorname{Hom}_{\mathbb{k}}(m/m^2, \mathbb{k})$
is
$\operatorname{Hom}_{\mathbb{k}}(\Omega_{B/\mathbb{k}}, \mathbb{k}) \simeq \operatorname{Der}_{\mathbb{k}}(B, \mathbb{k})$

We claim δ^* surjects. $0 \to m \to B \to \mathbb{k} \to 0$,
so pick $\underset{B}{b} = \underset{\mathbb{k}}{\lambda} + \underset{m}{c}$.

Now let $\varphi \in \mathrm{Hom}(m/m^2, \mathbb{k})$ and put $d b = \varphi([c])$, and check that $\delta^* d = \varphi$ and d derivation.

Thm. B local ring. Assume
i) $\mathbb{k} = B/m \hookrightarrow B$
ii) $\mathbb{k} = \overline{\mathbb{k}}$, $\mathrm{char}\, \mathbb{k} = 0$ (more generally, $\mathbb{k}$ perfect)
iii) B is the localization of a fin. gen. $\mathbb{k}$-alg.
Then B is regular iff $\Omega_{B/\mathbb{k}}$ free of rank $=\dim B$.

- we had $\Omega_{X/Y} = \Delta^* \mathcal{J}/\mathcal{J}^2$, where $\mathcal{J}$ ideal sheaf of $\Delta(X)$.
$\rightsquigarrow$ Put $Y = \mathrm{Spec}\, A$, $X = \mathrm{Spec}\, B$, $X \times_Y X = \mathrm{Spec}\, B \otimes_A B$

Properties
- compatibility w base change
$\rightsquigarrow g: Y \to Y'$, $X' = X \times_Y Y' \xrightarrow{g'} X \implies \Omega_{X'/Y'} \cong g'^* \Omega_{X/Y}$

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

- $X \xrightarrow{f} Y \xrightarrow{g} Z \implies f^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$
- $X \xrightarrow{f} Y$, $Z \subset X$ closed subscheme w ideal sheaf $\mathcal{J}$
$Z \overset{j}{\hookrightarrow} X \to Y$, $\mathcal{O}_X \to j_* \mathcal{O}_Z$,
$\mathcal{J}/\mathcal{J}^2 \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \mathcal{O}_{Z/Y} \to 0$

Def. A variety over $\mathbb{k}$ is an irreducible separated scheme of finite type over any closed field $\mathbb{k}$.

Thm. A variety X over $\mathbb{k}$ is regular (smooth) iff $\Omega_{X/\mathbb{k}}$ is locally free of $rk = \dim X$.

- suppose X smooth variety over $\mathbb{k}$, $Y \subset X$ irred. closed subscheme
  - $\rightarrow$ then Y is nonsingular iff
    - i) $\Omega_{Y/X}$ locally free
    - ii) $0 \rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow \Omega_{X/\mathbb{k}} \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_{Y/\mathbb{k}} \rightarrow 0$ is exact
- when this happens, $\mathcal{J}/\mathcal{J}^2$ is locally free of $rk = \operatorname{codim}_X Y$