Bruzzo.

- take $\mathcal{A}$ abelian cat, $I \in \mathcal{A}$
- equivalent conditions:
  - $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact
  - $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed] \\ & I & \end{tikzcd}$$
  - all exact sequences $0 \to I \to B \to C \to 0$ split

$\to$ we call I an injective object

$\to$ we say $\mathcal{A}$ has enough injectives if every object embeds into an injective object,

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to \cdots$$

- consider $F : \mathcal{A} \to \mathcal{B}$ additive & left exact,

  take $H^i(F(I^\bullet)) = R^iF(A)$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & L^0 \arrow[r] \arrow[d, dashed, "g_0"] & L^1 \arrow[r] \arrow[d, dashed, "g_1"] & \cdots \\ 0 \arrow[r] & B \arrow[r] & M^0 \arrow[r] & M^1 \arrow[r] & \cdots \end{tikzcd}$$

$\Rightarrow$ i) $f : A \to B$ lifts to $g : L^\bullet \to M^\bullet$

ii) g not unique, but all lifts homotopic

Pf (induction)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\varepsilon"] & B \arrow[r] & M^0 \\ & L^0 \arrow[urr, dashed, "g_0"] & & \end{tikzcd}$$

, g_0 not necessarily unique.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varepsilon"] \arrow[d, "f"] & L^0 \arrow[r] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "h_0"] & L^1 \arrow[dl, dashed, "k_1"] \\ & B \arrow[r] & M^0 & \end{tikzcd}$$

, ...

– now consider

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] \arrow[d, shift right, "g_0"'] & I^1 \arrow[r] \arrow[d, shift right, "g_1"'] & \cdots \\ 0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] \arrow[u, shift right, "h_0"'] & J^1 \arrow[r] \arrow[u, shift right, "h_1"'] & \cdots \end{tikzcd}$$

$\Rightarrow$ $h \circ g \sim id_I$, $g \circ h \sim id_J$ $\Rightarrow F(I^\bullet) \sim F(J^\bullet) \Rightarrow H^i(F(I^\bullet))$ is $H^i(F(J^\bullet))$

Elementary properties.

1) $R^0 F \simeq F$

$\rightsquigarrow$ F left-exact : $0 \to F(A) \to F(I^0) \xrightarrow{F(d_0)} F(I^1)$ is exact

$\Rightarrow F(A) = \ker(F(d_0))$

$\Rightarrow H^0 F(A) = \ker(H(F(d_0))) \simeq \ker F(d_0) \simeq F(A)$

II) if A injective, $R^i F(A) = 0$, $i > 0$

examples.

I) Sh_X sheaves of ab. groups on top space X

$\Rightarrow R^i \Gamma(X, \mathcal{F}) \equiv H^i(X, \mathcal{F})$

II) $(X, \mathcal{O}_X)$. $\mathcal{O}_X$-mod has enough injectives $\Rightarrow H^i(X, \mathcal{F})$

III) $Hom_R(M, -) : R\text{-mod} \to R\text{-mod}$

$\Rightarrow R^i Hom(M, N) = Ext^i(M, N)$

IV) $\mathcal{M} \in \mathcal{O}_X$-mod:

$Hom_{\mathcal{O}_X}(\mathcal{M}, -) : \mathcal{O}_X\text{-mod} \to Ab \rightsquigarrow Ext^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{V})$

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, -) : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod} \rightsquigarrow \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{V})$

- a Lie algebra L over a ring R is an R-module with a bilinear operation $[-,-]: L \times L \to L$
  $\to$ skew + Jacobi

Chevalley-Eilenberg cohomology

- direct definition.

$\xi \in C^p(L,M) = \Lambda^p L \otimes M$, $\rho: L \to \mathrm{End}_R(\overset{R\text{-module}}{\overset{\downarrow}{M}})$ representation

$$(d\xi)(x_1, \dots, x_{p+1}) = \sum_{\substack{i,j=1 \\ i<j}}^{p+1} (-)^{i+j} \xi([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1})$$

- indirect definition.

$M^L = \{\text{invariants}\} = \{m \in M \mid \rho(x)(m) = 0 \ \forall x \in L\}$