Fantechi

- R loc. ring, m max ideal
- $\hat{R} = \varprojlim R/m^n$ its completion along m

$$\begin{array}{ccccccc} R/m & \longleftarrow & R/m^2 & \longleftarrow & R/m^3 & \longleftarrow & \cdots \\ \Psi & & \Psi & & \Psi & & \\ a_0 & \longleftarrow\!\shortmid & a_1 & \longleftarrow\!\shortmid & a_2 & \longleftarrow\!\shortmid & \cdots \end{array}$$

- $\exists$ morphism $R \to \hat{R}$, $a \mapsto ([a],[a],\ldots)$
- $\hat{R}$ has ring structure with product
  $(a_0, a_1, \ldots)(b_0, b_1, \ldots) = (a_0 b_0, a_1 b_1, \ldots)$
- $\hat{R}$ local ring?
  - max id. $\hat{m}$ spanned by elements beginning w/ zero, $(0, a_1, a_2, \ldots)$

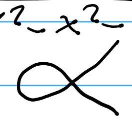
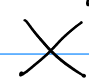
Rmk
- This works for any ring A w max id. m,
  $\hat{A} = \varprojlim A/m^n$
- Let $u_{n-1} \in A/m^n$ and denote by u_0 its image in A/m.
  If $u_0 \neq 0$ then u_{n-1} is a unit.
  (A/m^n is loc. ring w max. ideal mA/m^n)

Cor.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \hat{A} \\ A_m \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

- wasn't writing for a while

Example $C = \operatorname{Spec} \dfrac{\mathbb{C}[x,y]}{y^2 - x^2 - x^3}$, $C_1 = \operatorname{Spec} \dfrac{\mathbb{C}[x,y]}{y^2 - x^2}$

- clearly "the same" around origin

- the formal completions are
  $\mathbb{C}[[u,v]]/u^2 - v^2 - v^3$ and $\mathbb{C}[[x,y]]/(x \cdot y$

Claim: $\exists$ a unit in $\mathbb{C}[[u,v]]$ s.t. $a^2 = 1+v$

$\rightarrow$ $a = \tilde{a}_0 + \tilde{a}_1 + \tilde{a}_2 + \cdots$

$a^2 = 1+v$

$\Rightarrow$ $(a^2)_0 = \tilde{a}_0^{\,2} = 1$, choose $\tilde{a}_0 = 1$.

$(a^2)_n = \tilde{a}_0 \tilde{a}_n + \cdots + \tilde{a}_n \tilde{a}_0 = 1+v$

$\Rightarrow$ $2\underbrace{\tilde{a}_0}_{\substack{\| \\ 1 \text{ per choice}}} \tilde{a}_n = 1+v - \cdots$

$\Rightarrow$ $\tilde{a}_n = \frac{1}{2}(\cdots)$

- ok, so since a is unit, $\begin{matrix} x \mapsto av \\ y \mapsto u \end{matrix}$ is isomorphism,

$$\frac{\mathbb{C}[[u,v]]}{u^2 - v^2 a^2} \xleftarrow{\sim} \frac{\mathbb{C}[[x,y]]}{y^2 - x^2}$$

- so completions are isomorphic

- now do an inclusion

$$\frac{\mathbb{C}[u,v]}{u^2 - v^2(1+v)} \hookrightarrow \frac{\mathbb{C}[u,v,a]}{(u^2 - v^2(1+v),\, a^2 - (1+v))}$$

$$C \longleftarrow C_2$$

Claim: $C_2 \longrightarrow C$ étale.

$$\frac{\mathbb{C}[u,v,a]}{u^2 - v^2(1+v),\, a^2-(1+v)} \simeq \frac{\mathbb{C}[u,a]}{\underbrace{u^2 - a^2(a^2-1)^2}_{(u - a(a^2-1))\cdot(u + a(a^2-1))}}$$

- so $C_2 = C_2' \cup C_2''$, $C_2' = \mathrm{Spec}\, \frac{\mathbb{C}[u,a]}{u - a(a^2-1)} \simeq \mathrm{Spec}\, \mathbb{C}[a]$

$$C_2' \cap C_2'' = \mathrm{Spec}\, \frac{\mathbb{C}[u,a]}{(u, a(a^2-1))}.$$

Fact. (Milne, étale coh.) Let $A \to B$ homomorphism of f.g. algs / $\mathbb{k} = \bar{\mathbb{k}}$. Then $\mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$ is étale $\iff$ up to localisation

$$B = \frac{A[u_1, \dots, u_n]}{(f_1, \dots, f_n)} \text{ s.t. } \det\left(\frac{\partial f_i}{\partial u_j}(p)\right) \neq 0$$

in $\mathbb{k}(p)$.

→ If we want smooth of rel dim s instead of étale, same but $B = A[u_1, \dots, u_n, v_1, \dots, v_s]/(f_1, \dots, f_n)$

- $B[a]/a^2 - (1+v)$ is étale at (a_0, u_0, v_0) if $\partial(a^2 - (1+v))/\partial a\big|_{(a_0, u_0, v_0)} = 2a \neq 0$
→ so if char $\mathbb{k} \neq 2$, it works if $a \neq 0$
⇒ but then $a_0 \neq 0$ so a invertible