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Pf (of Gelfand thm)

- statement: $A \xrightarrow{G} C(S_pA)$,
  1. G is $*$-preserving
  2. G isometric $\Rightarrow$ injective — $G(A)$ closed
  - so by Stone-Weierstraß $G(A) = C(S_pA)$

1) $h = h^* \Rightarrow G(h)^* = G(h)$, want.
$G(h)(\chi) = \chi(h) \in S_{p_A} h \underbrace{\subset \mathbb{R}}_{h \text{ s.a.}}$, so $G(h)(\chi)^* = G(h)(\chi)$.

ii) $\forall h \in A$, $\|G(h)\| = \sup_{\chi \in S_pA} \{ |\chi(h)| \}$

$= \sup \{ |\lambda| \mid \lambda \in S_{PA} x \}$

$= \|h\|$ for C^*-alg. $\square$

Cor A comm. $\Rightarrow$ any χ is s.a., $\chi(h^*) = \overline{\chi(h)}$

Pf. $\chi(x^*) = G(x^*)(\chi) = G(x)^*(\chi) = \overline{\chi(h)}$

- U unital C^*-alg
- state on U: lin, cont. functional $f: U \to \mathbb{C}$
  - $f(x^* x) \geq 0$, positive
  - $f(\mathbb{1}) = 1$, unital
- the space of states on U is convex
  - $\exists$ extremal pts

Thm (Krein-Milman) Every cpt convex set S in a locally convex v.sp. is the convex hull of its extremal pts.

- A comm. unital, $\underset{A}{\text{chars of}}$ = pure states

Conj. (NC Stone-Weierstraß) $\mathfrak{U}$ unital, $B \subset \mathfrak{U}$ C^*-subalg. which separates pure states. Then $\mathfrak{U} = B$.

Continuous functional calc.

- A unital C^*-alg, $x \in A$ normal

Claim $\exists!$ $*$-morph of unital algebras (our funct. calc.)

i) $\Phi_x : C(Sp_A x) \longrightarrow A$, $z \mapsto \Phi_x(z) = x$

ii) $Sp_A x \subset \mathbb{R} \Rightarrow x = x^*$

Pf. $A \supset B := \overline{\{ P(x,x^*) \mid P \in \mathbb{C}[x,x^*] \}}^A$

comm. due to normality

$$Sp\, B \ni \chi \longmapsto \chi(x) \in Sp_B x = Sp_A x$$

ψ homeo

$$\chi(x^*) = \overline{\chi(x)}, \quad \chi(P(x,x^*)) = P(\chi(x), \overline{\chi(x)})$$

- now

$$\begin{tikzcd} C(Sp_A x) \arrow[r, "\psi^*"] \arrow[ddr, "\Phi_x"'] & C(Sp\, B) \arrow[d, "G_B^{-1}"] \\ & B \arrow[d, phantom, "\subseteq" rotate=-90] \\ & A \end{tikzcd}$$

- uniqueness can be shown from denseness

C^*-alg

Prop i) B involutive Banach alg, $\pi : B \to A$ involutive morphism is contractive, $\|\pi(x)\|_A \le \|x\|_B$

ii) any involutive injective morphism $\pi : A_1 \to A_2$ between C^*-algs is isometric

Prop (Spectral mapping) $x \in A$ normal

i) $\forall f \in C(Sp_A x)$, $Sp_A f(x) = f(Sp_A x)$

Since $f(x)$ normal, for $g \in C(Sp_A f(x))$,

$(g \circ f)(x) = g(f(x))$

ii) $\forall \pi : A \to B$, $f(\pi(x)) = \pi(f(x))$

- A Banach $*$-alg, nonunital

$\longrightarrow \tilde{A} := \{ a + \lambda, \lambda \in \mathbb{C} \}$

- $(a+\lambda)^* = a^* + \bar{\lambda}$
- $A \hookrightarrow \tilde{A}$ involutive closed
- recall $Sp_A x = Sp_{\tilde{A}} x$

Prop i) $a \in A$ unitary (i.e. it's unitalisation is unitary)

$\Rightarrow Sp\, x \subset U(1)$

ii) $Sp_A x^* = \{ \bar{\lambda} \mid \lambda \in Sp_A x \}$

iii) $x \in A$ normal $\Rightarrow \|x\| = r(x)$

iv) for $x \in A$ normal, $f \in C(Sp_A x)$, $f(x) \in \tilde{A}$.

If $f(0) = 0$. Then $f(x) \in A$

- iv) follows from $\tilde{A} \ni x + \lambda \overset{\pi}{\longmapsto} \lambda$ and functorality $\pi(f(x)) = f(\underbrace{\pi(x)}_{0})$

- for nonunital commutative case,

$$A \xrightarrow[G]{\sim} C_0(\operatorname{Sp} A),$$

where for any X loc. cpt Hsdff, $C_0(X) :=$ cont. funcs. on X vanishing at infty

Enveloping C^*-algs

- $\mathcal{A}$ involutive alg., p C^*-seminorm, i.e. seminorm with $p(x^* x) = p(x)^2$
- C^*-seminorms $\longleftrightarrow$ seminorms on $\mathcal{A}$ s.t. for a $*$-morphism $i: \mathcal{A} \to A$, where A C^*-alg, $\|a\| = \|i(a)\|$

- let Λ = set of seminorms on $\mathcal{A}$,
- let $\mathcal{A}_\Lambda := \{ x \in \mathcal{A} \mid \sup p(x) < \infty, p \in \Lambda \}$
  - involutive subalg. with C^*-seminorm
    $x \mapsto \sup \{ p(x) \mid p \in \Lambda \}$

Def If $\mathcal{A}$ has maximal seminorm p_{max}, then we say $\mathcal{A}$ has enveloping C^*-algeb

$$C^*(\mathcal{A}) := \text{Hausdorff completion of } \mathcal{A} \text{ wrt } p_{max}$$

- note univ. property

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] \arrow[d, "f"'] & C^*(\mathcal{A}) \arrow[dl, dashed, "\exists! g"] \\ B & \end{tikzcd} \quad f = g \circ i$$

- take locally cpt group
  - take left Haar measure $\mu(\overset{\overset{G}{\cup}}{x}\overset{\overset{\text{Borel}}{\downarrow}}{E}) = \mu(E)$

- define left action $L_y f(x) = f \circ y^{-1}(x)$ for $f \in C_c^+(G)$, note $L_x L_y = L_{xy}$

- if μ_1, μ_2 left Haar measures, $\exists! \alpha > 0$ s.t. $\mu_1 = \lambda \mu_2$

Modular function

- fix λ left Haar m., $x \in G$.
- then $\lambda_x(E) := \lambda(E x)$ is another left m.
  $\Rightarrow \exists\, \Delta(x) > 0$, $\lambda_x = \Delta(x)\lambda$
  - measures discrepancy between L & R

- look at $L^1(G) = \{\text{int. funcs wrt Haar}\}$

$$(f * g)(x) := \int_G f(y)\, g(y^{-1}x)\, dx$$

$$\left(\text{"} = \int_{y \cdot z = x} f(y)\, g(z) \text{"} \right)$$

$$f^*(x) := \Delta(x^{-1})\, \overline{f(x^{-1})}$$

$\to (L^1(G), *)$ is $*$-Banach alg