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Introduction

- study top inv. of moduli spaces
- what is a moduli space?
  alg. var. M which naturally parametrises certain objects in alg. geom.
- Example: Hilb. scheme of pts $S^{[n]} = Hilb^n(S)$ (nonsing. alg. sfc. S)
  $S^{[n]} = \{[Z] \subset S \mid$ 0-dim subschemes of degree n on $S\}$
- related to $S^{(n)} := Sym^n S$
  $S^{(n)} = \{\sum_i n_i p_i \mid p_i \in S$ distinct, $n_i > 0$, $\sum n_i = n\}$
- Hilbert-Chow morphism $\pi : S^{[n]} \to S^{(n)}$
  $Z \mapsto supp(Z)$
- Fogarty: $S^{[n]}$ nonsing. of dim $2n$
  $\pi : S^{[n]} \to S^{(n)}$ is resol. of singularities
- topology: $e(X) = \sum_{i=0}^{\dim X} (-)^i \dim H^i(X, \mathbb{Q})$
- Thm $\displaystyle\sum_{n\geq 0} e(S^{[n]}) t^n = \frac{1}{\prod_{n>0}(1-t^n)^{e(S)}}$
- Aim: show similar formulas for moduli of sheaves

Moduli of sheaves on proj. sfc.

- let S now projective sfc, nonsing., over $\mathbb{C}$, H ample
- fix $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S, \mathbb{Z})$, $c_2 \in H^4(S, \mathbb{Z}) \simeq \mathbb{Z}$
- study moduli of rk r torsion free sheaves on S w fixed $c_1(E) = c_1$, $c_2(E) = c_2$
  → too many such sheaves!
  → restrict to semistable sheaves to get a projective mod. sp.

Def. $\mathcal{E} \in Coh_X$, hol. Euler char of $\mathcal{E}$

$$\chi(X, \mathcal{E}) = \sum_{i=0}^{\dim X} (-)^i \dim H^i(X, \mathcal{E})$$

- If H ample, let $P_H(\mathcal{E}, m) := \chi(X, \mathcal{E} \otimes H^{\otimes m})$
  $\to$ polynomial in m! Hilb. polynomial.

Def. $\mathcal{E}$ on S is called semistable if $\forall$ subsheaves of $\mathcal{F} \subsetneq \mathcal{E}$

$$\frac{P_H(\mathcal{F}, m)}{rk(\mathcal{F})} \leq \frac{P_H(\mathcal{E}, m)}{rk(\mathcal{E})} \text{ for all } m >> 0.$$

$\mathcal{E}$ is called stable if ineq. is strict.

- trivially: $\exists$ a coarse moduli sp. $\mathcal{M}_S^H(r, c_1, c_2)$ for H semistable coh. sh. on S rk r, Ch. cl. c_1, c_2.
  - iso. classes of stable sh. are parametrised by dense Zariski open $\mathcal{M}^H_S(r, c_1, c_2)^s$

- expected dimension: assume $b_1(S) = 0$, then we expect

$$vd(M) := 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_S)$$

- (Kuranishi) locally in anal. top. $\mathcal{M}^H_S(r, c_1, c_2)$ is the iso set of holom map $\mathbb{C}^m \xrightarrow{k} \mathbb{C}^k$ s.t. $vd(M) = m - k$

Vafa-Witten formula

$r = 2$, S smooth proj sfc, H ample on S, $b_1(S) = 0$, $P_g(S) = \dim H^0(S, K_S) > 0$

- e.g. $S = K3$ or elliptic or sfc of gen. type

- assume $\mathcal{M}^H_S(c_1, c_2) = \mathcal{M}^{Gi}_S(c_1, c_2)^S$
- assume $\exists$ smooth conn. curve in $|K_S|$

- let $\bar\eta(x) = \prod_{n>0} (1 - x^n) = 1 - x - x^2 + \ldots$
$\vartheta(x) = \sum_{n \in \mathbb{Z}} x^{(n^2)} = 1 + 2x + 2x^4 + \ldots$

$$\Psi_S(x) := 2^{3 - \chi(\mathcal{O}_S) + K_S^2} \left(\frac{1}{\bar\eta(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)} \cdot \left(\frac{\bar\eta(x^4)}{\vartheta(x)}\right)^{K_S^2}$$

- Vafa-Witten: $e(\mathcal{M}^H_S(2, c_1, c_2)) = \mathrm{coeff}_{x^{vd}}(\Psi_S(x))$

- not really, actual V-W formula computes invariants of a Higgs moduli sp
$\mathcal{M}^{Higgs}_S(c_1, c_2) = \{ E \xrightarrow{\varphi} E \otimes K_S \}$

- but also, it is false, unless we reinterpret $e(\ldots)$ somehow

Virtual top. invariants

- $M = \mathcal{M}^H_S(c_1, c_2)$ is "virtually smooth" of virt. dim $vd(M)$, i.e. it has a perfect obstruction theory

Def. Set M scheme w $M \overset{i}{\hookrightarrow} X$ emb. into smooth scheme M, $I = I_{M/X}$ ideal sh.
A perfect obstruction theory on M is $E^\bullet = [E^{-1} \xrightarrow{d} E^0]$ of vect bdls on M w a complex morphism

$$\begin{array}{ccc} E^{-1} & \xrightarrow{d} & E^0 \\ \downarrow \varphi & & \downarrow \varphi \\ I/I^2 & \xrightarrow{d} & \Omega_X|_M \end{array}$$

s.t.: i) φ: Coker d iso
ii) φ: Ker d surj.

Thm (Behrend–Fantechi, Li–Tian)

Let M be sch. w perf. obs. th.,

i) M has a vir. fund. class $[M]^{vir} \in H_{2vd(M)}(M, \mathbb{Z})$

If $\alpha \in H^\bullet(M, \mathbb{Z})$, $\int_{[M]^{vir}} \alpha \in \mathbb{Z}$.

virtual intersection number

ii) M has virt struc sheaf $\mathcal{O}_M^{vir} \in K_0(M)$

- if $E \to M$ vect bdl,

$\chi^{vir}(M, E) := \chi(X, E \otimes \mathcal{O}_M^{vir})$ behaves well

- $M_S^H(c_1, c_2)$ has perf. obst. th.
- let $\mathcal{E}/S \times M$ be univ. sheaf, $\mathcal{E}|_{S \times [E]} = E$, $\pi: S \times M \to M$ projection.
- dual of obstr. theory is $R\pi_* R\mathcal{H}om(\mathcal{E}, \mathcal{E})_0[1]$
  - represent it as cpx $\mathcal{E}_0 \to \mathcal{E}_1$

  ends of trace 0

$\Rightarrow$ i) M has virt. fund. class

ii) M has virt tangent bdl (put $\mathcal{E}_i := (\mathcal{E}^{-i})^\vee$)

$T_M^{vir} = \mathcal{E}_0 - \mathcal{E}_1 \in K_0(M)$

- define virtual Euler number of M

$$e^{vir}(M) = \int_{[M]^{vir}} c_{vd(M)}(T_M^{vir}) \in \mathbb{Z}$$

Conjecture (Vafa–Witten):

Assume $b_1(S) = 0$, $p_g(S) > 0$, $|K_S| \ni$ nonsing curve, and $M_S^H(c_1, c_2) = M_S^H(c_1, c_2)^S$.

Then VW formula holds for $e^{vir}(M)$

$$e^{vir}(M_S^H(c_1, c_2)) = \mathrm{Coeff}_{x^{vd}} 2^{3 - \chi(\mathcal{O}_S) + K_S^2} \left(\frac{1}{\overline{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{\overline{\eta}(x^4)}{\vartheta(x)}\right)^{K_S^2}$$