Dąbrowski

- $\beta \in \Gamma(\Lambda^1 M)$, g metric $\Rightarrow \exists\, \beta^\#$ $g(\beta^\#, v) = \beta(v)$
 - likewise $w^\flat(v) = g(w, v)$, $T^*M \underset{\flat}{\overset{\#}{\rightleftarrows}} TM$

- $\not{D} = \tilde{\gamma} \circ \nabla : \Gamma(\Sigma) \to \Gamma(T^*M) \otimes \Gamma(\Sigma) \to \Gamma(\Sigma)$
 where for $\gamma : TM \to \mathrm{End}\,\Sigma$, $\tilde{\gamma} : TM \times \Sigma \to \Sigma$
 is $\tilde{\gamma}(v, \psi) := \gamma(v) \cdot \psi$

- locally, $\not{D}\psi = \sum e_i \cdot \nabla_{e_i} \psi$

- $\not{D}(f\psi) = \sum e_i ((\mathcal{L}_{e_i} f)\psi + f \nabla_{e_i}\psi)$
 $= \underbrace{\mathrm{grad}\, f}_{=(df)^\#} \cdot \psi + f \not{D}\psi$

- principal symbol $\sigma_{\not{D}}(df) = -i[\not{D}, f]$
 $= -i\, df \circ$
 so $\sigma_{\not{D}}(\xi) = -i\xi \cdot -$, and $(\sigma_{\not{D}}(\xi))^2 = -\|\xi\|^2$

 $\to \not{D}$ is elliptic

- let $\dim M = n$, n even

- consider $\chi = \omega\cdot$, Clifford mult. w $\omega = e_1 \cdots e_n$
 $\chi = \omega \cdot = \gamma(\omega)$
 - $(e_i \cdot \nabla_{e_i}) \circ \chi(\psi) = e_i \cdot (\cancel{\nabla_{e_i} \omega} \cdot \psi + \omega \cdot \nabla_{e_i} \psi)$
 $= e_i \cdot \omega \cdot \nabla_{e_i} \psi = -\chi \circ (e_i \nabla_{e_i})(\psi)$
 $\Rightarrow \chi \not{D} = -\not{D} \chi$
 $\Rightarrow \not{D} : \Gamma(\Sigma_\pm) \to \Gamma(\Sigma_\mp)$

Self-adjointness

- let $\psi, \varphi \in \Gamma_c(\Sigma)$, $\langle \psi, \varphi \rangle_M = \int \langle \psi, \varphi \rangle \, vol$

$$\langle \psi, \not{D}\varphi \rangle = \sum_j \langle \psi, e_j \cdot \nabla_{e_j} \varphi \rangle$$
$$= \sum_j \left(\mathcal{L}_{e_j} \langle \psi, e_j \cdot \varphi \rangle - \langle \nabla_{e_j} \psi, e_j \cdot \varphi \rangle - \langle \psi, \nabla_{e_j} e_j \cdot \varphi \rangle \right)$$
$$= \sum_j \left(\mathcal{L}_{e_j} \beta(e_j) - \beta(\nabla_{e_j} e_j) \right) + \langle \not{D} \psi, \varphi \rangle$$
$$= \operatorname{div} \beta^{\#} + \langle \not{D} \psi, \varphi \rangle$$

where $\beta(e_i) := \langle \psi, e_i \varphi \rangle$

- now, $\langle \psi, \not{D}\varphi \rangle_M = \langle \not{D}\psi, \varphi \rangle_M + \underbrace{\int_M \operatorname{div} \beta^{\#} \, vol}_{=0 \text{ if } \partial M = 0}$,

$$\operatorname{div} \beta^{\#} \, vol = \mathcal{L}_{\beta^{\#}} vol = (d \circ \iota_{\beta^{\#}} + \iota_{\beta} \circ d) \, vol$$

so

Prop $\not{D}$ is hermitean on $\Gamma_c(\Sigma)$.

Rmk $\ker \not{D} = \ker \not{D}^2$ since $\|\not{D}\psi\|^2 = \langle \not{D}\psi, \not{D}\psi \rangle = \langle \psi, \not{D}^2 \psi \rangle$

- now take $H = L^2(\Sigma) = \overline{\Gamma_c(\Sigma)}^{\|\cdot\|_M}$

- $\not{D}$ is defined on dense open, but not bounded
- so cannot extend to H by continuity
- however, it is closeable, i.e. it has a closure $\bar{\not{D}}$ whose graph is closed, where $\psi \in \operatorname{Dom} \bar{\not{D}} \subset L^2(\Sigma)$ if $\psi_n \to \psi$ means $\lim_n \not{D}\psi_n$ exists.

- define also adjoint $\not{D}^*$ s.t. $\not{D}^*|_{\Gamma_c(\Sigma)} = \not{D}$
  - Dom $\not{D} \subseteq$ Dom $\overline{\not{D}} \subseteq$ Dom $\not{D}^*$

    with Dom $\not{D}^* := \{\psi \in H \mid \langle \psi, \not{D}\varphi \rangle$ is cont. in $\varphi\}$

Def T is self-adjoint if Dom T = Dom T^* and $T = T^*$.
T is essentially s.a. if Dom $\overline{T}$ = Dom $(\overline{T})^*$ and $\overline{T} = (\overline{T})^*$

Prop M cpt $\Rightarrow$ $\not{D}$ is ess. s.a. (works for M geod. complete)

Lemma 1 If $\Gamma(M)$ is closed subset of Dom $\not{D}^*$ in the graph norm ($\|\psi\|_{\not{D}^*} = (\|\psi\|^2 + \|\not{D}\psi\|^2)^{1/2}$) then $\not{D}$ is ess. s.a.

Pf. The conditional means $\Gamma(M) \ni \psi_n \to \psi \in$ Dom $\not{D}^*$, giving $\psi \in$ Dom $\overline{\not{D}}$.

Lemma 2. $\Gamma(\Sigma)$ are $\|\cdot\|_{\not{D}^*}$-dense in Dom $\not{D}^*$.

Pf. Let $(\mathcal{U}_\alpha)$ be open cover, (f_α) p.o.u. subordinate to it. Note that $f_\alpha \psi \in$ Dom $\not{D}^*$ if $\psi \in$ Dom $\not{D}^*$, so use local coords. to write $\Sigma|_{\mathcal{U}_\alpha} = \mathbb{R}^n \times \mathbb{C}^{2^{[n/2]}}$