Porta.

Conductivity quantization, cont'd.

- we saw if $(P-Q)^{2n+1}$ tr. class,
  $\mathrm{Ind}(P,Q) = \mathrm{Tr}(P-Q)^{2n+1}$

→ take $P = \chi(H \le \mu)$, $Q = U_a P U_a^*$
- claim: $(P - U_a P U_a^*)^3$ is tr. cl.

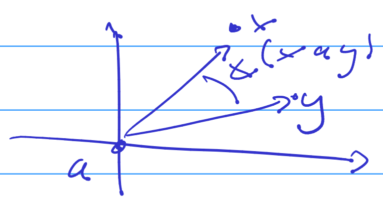
$$(P - U_a P U_a^*)(x+\alpha, x) = \underbrace{P(x+\alpha, x)}_{\le C e^{-c\|\alpha\|}} \underbrace{\left(1 - e^{i\vartheta_a(x)} e^{-i\vartheta_a(x+\alpha)}\right)}_{\le \frac{c}{\|x\|}}$$

- fact: $\|T\|_1 = \mathrm{Tr}|T| \le \sum_a \sum_x |T(x+a, x)|$
→ so $(P - \ldots)^3$ is tr class because the bound becomes summable

Prop $\sigma_{12} = \frac{1}{2\pi} \mathrm{Tr}(P - U_a P U_a^*)^3$

Pf.
$$\mathrm{Tr}(P - U_a P U_a^*)^3 = \sum_{xyz} P(x,y) P(y,z) P(z,x) \left.\begin{array}{l} \cdot\left(1 - e^{i(\vartheta_a(x) - \vartheta_a(y))}\right) \\ \cdot\left(1 - e^{i(\vartheta_a(y) - \vartheta_a(z))}\right) \\ \cdot\left(1 - e^{i(\vartheta_a(z) - \vartheta_a(x))}\right) \end{array}\right\} (*)$$

$$(*) = 2i\left(\sin \measuredangle(z,a,x) + \sin \measuredangle(x,a,y) + \sin \measuredangle(y,a,z)\right)$$



$$\mathrm{Tr}(P - U_a P U_a^*)^3 = \frac{2i}{L^2} \sum_{a \in \Lambda_L^*} \sum_{xyz} P(x,y) P(y,z) P(z,x)\, S_a(x,y,z)$$

$$= \frac{2i}{L^2} \sum_{a\in\Lambda_L^*} \sum_{x\in\Lambda_L} \sum_{y,z} P(x,y)\,P(y,z)\,P(z,x)\, S_a(x,y,z)$$

$$+ \underbrace{\frac{2i}{L^2} \sum_{a\in\Lambda_L^c} \sum_{x\in\mathbb{Z}^2\setminus\Lambda_L}}_{\to 0 \text{ as } L\to\infty} \sum_{y,z} (\;)\; \underbrace{S_a(x,y,z)}_{\le \max\left(|a-x|^{-3},\, |a-y|^{-3},\, |a-z|^{-3}\right)}$$

$$= \frac{2i}{L^2} \sum_{x\in\Lambda_L} \sum_{y,z} \sum_{a\in\mathbb{Z}^{2*}} (\cdots) + (\text{previous error})$$

$$- \frac{2i}{L^2} \sum_{x\in\Lambda_L} \sum_{y,z} \sum_{a\in\mathbb{Z}^{2*}\setminus\Lambda_L^*} (\cdots)$$

↪ again $O(L^{-2})$

Claim: $\frac{1}{2\pi} \sum_{a\in\mathbb{Z}^{2*}} S_a(x,y,z) = \mathrm{Area}(x,y,z)$

$\frac{1}{2}(x-y)\wedge(y-z)$

→ assuming that, we get

$$\mathrm{Tr}(P - U_a P U_a^*)^3 = \frac{2\pi\cdot 2i}{L^2} \sum_{x\in\Lambda_L} \sum_{y,z} P(x,y)\,P(y,z)\,P(z,x) \cdot \left[(x_1-y_1)(y_2-z_2) - (1\leftrightarrow 2)\right]$$

– note: $P(x,y)(x_1-y_1) = [X_1, P](x,y)$ ...

$$= \frac{2\pi i}{L^2} \sum_{x\in\Lambda_L} \Big(\mathrm{Tr}\,[X_1,P][X_2,P]P - \mathrm{Tr}\,[X_2,P][X_1,P]P + O(1) \Big).$$

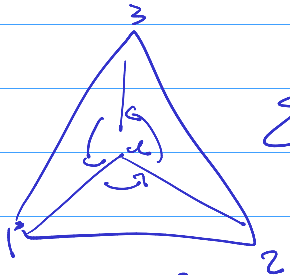
Prop let $g(\alpha)$ be bounded func., $g(\alpha) = -g(-\alpha)$, $g(\alpha) = \alpha + O(\alpha^3)$ as $\alpha \to 0$. Let $u_1, u_2, u_3 \in \mathbb{Z}^2$, $a \in \mathbb{Z}^{2*}$, let $\alpha_i(a) = \measuredangle(u_{i+1}, a, u_{i+2}) \in (-\pi, \pi]$ with convention $\measuredangle(\text{collinear}) = 0$.

Then $\displaystyle\sum_{a \in \mathbb{Z}^{2*}} \sum_{i=1,2,3} g(\alpha_i(a)) = 2\pi \, \text{Area}(u_1, u_2, u_3)$

Pf. Look first at $g(\alpha) = \alpha$, assume $\overset{u_3}{\triangle}_{u_1 \; u_2}$ positively oriented,

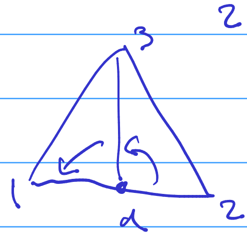
then $\displaystyle\sum_{i=1}^{3} \alpha_i(a) = 2\pi \cdot \begin{Bmatrix} 1 \\ 1/2 \\ 0 \end{Bmatrix}$ for a $\begin{Bmatrix} \text{inside } \Delta \\ \text{on } \partial\Delta \\ \text{outside } \Delta \end{Bmatrix}$

I°



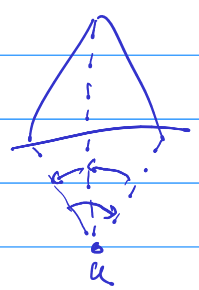
$\sum \alpha_i = 2\pi$

II°



$\alpha_3 = 0$
$\Rightarrow \sum \alpha_i = \pi$

III°



$\sum \alpha_i = 0$

So $\displaystyle\frac{1}{2\pi} \sum_{a \in \mathbb{Z}^{2*}} \sum_{i=1,2,3} \alpha_i(a) = \#$ of lattice pts of $\mathbb{Z}^{2*}$ inside Δ and $\frac{1}{2}\#\ldots$ on $\partial\Delta$

This # does not change if we translate Δ in $\mathbb{Z}^2$ or reflect it wrt sym. axis of $\mathbb{Z}^2$ and $\mathbb{Z}^{2*}$

Triangles we obtain in that manner tile the plane.

Conclude: $\frac{1}{2\pi}\sum_a \sum_{i=1,2,3} \alpha_i(a) = \text{Area}(\Delta)$

If $\neq \text{Area}(\Delta)$, let $\Lambda \subseteq \mathbb{Z}^2$ be a region obtained as tiling by copies of Δ.

$\Rightarrow |\{\#\text{ lattice pts of } \mathbb{Z}^{2*} \text{ in } \Lambda\}|$

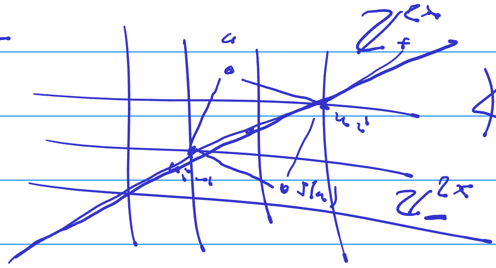
$= \sum_{\Delta \subset \Lambda} |\{\# \cdots \text{ in } \Delta\}|$

as $|\Lambda| \to \infty$, $\frac{|\{\# \cdots\}|}{|\Lambda|} \to 1$, so we can get claim.

Now if $g(\alpha) \neq \alpha$, claim: $\underline{\sum_{a \in \mathbb{Z}^{2*}} \sum_{i=1,2,3} \left(g(\alpha_i(a)) - \alpha_i(a)\right) = 0}$.

Idea: 1-to-1 corr. between $a \in \mathbb{Z}^{2*}$ and $\sigma(a) \in \mathbb{Z}^{2*}$ s.t. $\alpha_i(\sigma(a)) = -\alpha_i(a)$.

- now



$\measuredangle(u_{i+1}, a, u_i)$

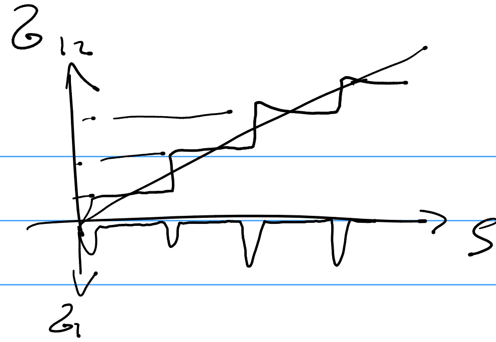
$= -\measuredangle(u_{i+1}, \sigma(a), u_i)$

- so, letting $f(\alpha_i(a)) = g(\alpha_i(a)) - \alpha_i(a)$,

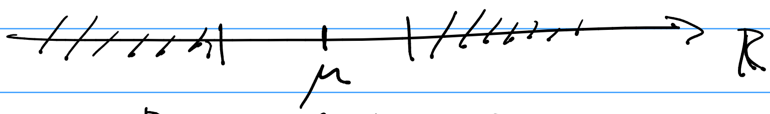
$\sum_{a \in \mathbb{Z}^{2*}} f(\alpha_i(a)) = \sum_{a \in \mathbb{Z}^{2*}_+} \left(f(\alpha_i(a)) + f(\underbrace{\alpha_i(\sigma(a))}_{-\alpha_i(a)})\right)$

$= 0.$ $\square$

Recall IQHE

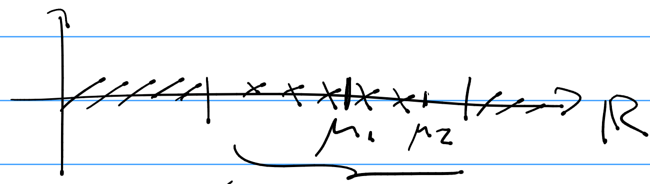


- H gapped:



$\Rightarrow P = \chi(H \leq \mu)$

- for $H_\omega = H + \lambda V_\omega$, $|\lambda| >> 1$,



"mobility gap"

$\Rightarrow \mathbb{E}\,\sigma_{12}\big|_{\mu_1} = \mathbb{E}\,\sigma_{12}\big|_{\mu_2}$

- we get the jumps when μ gets to the continuous spectra