Fantechi

Yesterday

- $\mathcal{A}$ ab. cat, $A \oplus B$ also $A \times B$.
- $A \oplus B$ means $\forall C \in \mathrm{ob}\,\mathcal{A}$,
  $A \xrightarrow{i} A \oplus B \xleftarrow{j} B$ s.t. $\mathrm{Hom}(A \oplus B, C) \to \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C)$
  sending $\lambda \mapsto (\lambda \circ i, \lambda \circ j)$ bijects
- we want dual statement with $A \xleftarrow{p} A \times B \xrightarrow{q} B$
- step 1

$$A \xrightarrow{i} A \oplus B \xrightarrow{\pi} \mathrm{coker}(i) =: B',$$

$$\beta : B \to B', \quad \beta = \pi \circ j$$

- pick $C \in \mathrm{ob}\,\mathcal{A}$, then

$$\begin{array}{c} \mathrm{Hom}(B', C) = \{\lambda \in \mathrm{Hom}(A \oplus B, C) \mid \lambda \circ i = 0\} \\ \Big\downarrow {-\circ \beta} \\ \mathrm{Hom}(B, C) \end{array}$$

- since $\lambda \mapsto (0, \lambda \circ j)$ bijects, $-\circ\beta$ bijects, so by coYoneda, equivalent functors $\Rightarrow \beta$ bijects
- step 2
  - fix $C \in \mathrm{ob}\,\mathcal{A}$, want $\gamma \mapsto (p \circ \gamma, q \circ \gamma)$ bijects.
  - explicit inverse:
    - for $C \begin{array}{c} \xrightarrow{\varphi} A \\ \xrightarrow[\psi]{} B \end{array} \xrightarrow{\gamma} A \oplus B$, $\gamma = i \circ \varphi + j \circ \psi$.
  - exercise: $\gamma^{-1} = (\varphi, \psi)$.

Rmk. By induction $A_1 \oplus \cdots \oplus A_n \simeq A_1 \times \cdots \times A_n$.
But untrue for infinite I s.t. $\forall i \in I$, $A_i \in \mathrm{ob}\,\mathcal{A}$,
even if $\bigoplus_{i \in I} A_i$ and $\prod_{i \in I} A_i$ both exist.
E.g. in $\mathcal{A} = \mathrm{Mod}_R$,

$$\prod_{i \in I} A_i = \left\{ (a_i)_{i \in I} \;\middle|\; \begin{array}{l} a_i \in A_i \\ \forall i \in I \end{array} \right\} \supsetneq \bigoplus_{i \in I} A_i = \left\{ \begin{array}{l} \text{same, but} \\ \text{fin. many } a_i\text{'s} \neq 0 \end{array} \right\}$$

- R noeth rings, $\mathcal{A}$ subcat of Mod_R of f.g. R-mod.

$\rightarrow$ show arb $\oplus$'s or Π's don't exist

- A comm. ring, $P = A[x_1 \dots x_n]$, $B = P/I$,

$Q = P[y]$, $B = Q/J$,
$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, hook] & B \\ Q \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

- claim
$$\begin{tikzcd} I/I^2 \arrow[r] \arrow[d] & \Omega_{P/A} \otimes_P B \\ J/J^2 \arrow[r] & \Omega_{Q/A} \otimes_Q B \end{tikzcd}$$
is qiso

- Step 1
  - $\exists f_0 \in P$ s.t. $\tilde{\pi}(y) = \pi(f_0)$, $f_0 = 0 \Leftrightarrow \tilde{\pi}(y) = 0$.
  - what can be said of J/J^2 wrt I/I^2 in that case?
  - $\Omega_{P/A} = \bigoplus_{i=1}^{n} P\,dx^i$, $\Omega_{Q/A} = \bigoplus_{i=1}^{n} Q\,dx^i \oplus Q\,dy$

$$\begin{aligned} Q &= P \oplus Py \oplus Py^2 \oplus \dots \\ &\cup \\ J &= I \oplus Py \oplus Py^2 \oplus \dots \\ &\cup \\ J^2 &= I^2 \oplus Iy \oplus Py^2 \oplus \dots \end{aligned}$$

$$\Rightarrow J/J^2 = I/I^2 \oplus By$$

- Step 2
  - gen. case, let $\tilde{Q} = P[z]$, $\alpha(z) = y - f_0$, $\alpha: \tilde{Q} \xrightarrow{\sim} Q$
  - then
$$\left.\begin{array}{l} I/I^2 \to \Omega_{P/A} \otimes_P B \\ \downarrow \\ \tilde{J}/\tilde{J}^2 \to \Omega_{\tilde{Q}/A} \otimes_{\tilde{Q}} B \end{array}\right\} \text{qiso}$$
$$\left.\begin{array}{l} \downarrow \\ J/J^2 \to \Omega_{Q/A} \otimes_Q B \end{array}\right\} \text{iso} \Rightarrow \text{qiso} \quad \square$$

Recall $X \to Y \in \mathrm{mor}\, Sch/_{\mathrm{Spec}\, A}$, A noeth,
X, Y f.g. $/\mathrm{Spec}\, A$ $\rightsquigarrow [I/I^2 \to \Omega_{P/\mathbb{Z}} \otimes_P B]$
"unique up to qiso" in $C(\mathrm{Mod}_B) = C(\mathrm{QCoh}_X)$

Def. Let $\mathcal{C}$ be cat, $Q \subseteq \mathrm{Mor}\,\mathcal{C}$. We say that the **localization** of $\mathcal{C}$ at Q is a functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ s.t.

i) $\forall \varphi \in Q$, $F(\varphi)$ iso

ii) $\forall G : \mathcal{C}' \to \mathcal{C}''$ s.t. $\forall \varphi \in Q$, $G(\varphi)$ is iso $\wedge$ $\exists!\, H : \mathcal{C}' \to \mathcal{C}''$ s.t. $G = H \circ F$

iii) $F|_{ob} : ob\,\mathcal{C} \to ob\,\mathcal{C}'$ is bijective

- exercise: if localisation exists, it is unique up to canonical iso,

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "\hat{F}"] & \\ \mathcal{C}' & & \hat{\mathcal{C}}' \arrow[ll, dashed, "\exists!"] \end{tikzcd}$$

Problems

q i) does $F : \mathcal{C} \to \mathcal{C}'$ exist?

q ii) if yes, how explicit is $\mathcal{C}'$?

a i) not in general due to size issues.

a ii) usualy not at all.

Thm Let $\mathcal{A}$ abel. cat. Then the localisation of $K(\mathcal{A})$ at $Q = \{\text{q-isom}\}$ exists, and we call it the **derived cat.** $D(\mathcal{A})$ of $\mathcal{A}$.
Given $A^\bullet, B^\bullet \in ob\, C(\mathcal{A}) = ob\, K(\mathcal{A}) = ob\, D(\mathcal{A})$, any mor in $D(\mathcal{A})$ can be written as

"roof"
$$\begin{tikzcd} & \tilde{A} \arrow[dl, "\substack{\text{qiso}\\ \text{in } C(\mathcal{A})}"'] \arrow[dr, "\substack{\text{mor in}\\ C(\mathcal{A})}"] & \\ A & & B \end{tikzcd}$$
or "sink"
$$\begin{tikzcd} A \arrow[dr, "\substack{\text{mor}\\ \text{in } C(\mathcal{A})}"'] & & B \arrow[dl, "\substack{\text{qiso in}\\ C(\mathcal{A})}"] \\ & \tilde{B} & \end{tikzcd}$$

- sketch of "roof" composition

$$\begin{tikzcd} & \tilde{A} \arrow[dl, "\varphi"', "q.is"] \arrow[dr, "\alpha"] & & \tilde{B} \arrow[dl, "\psi"', "q.is"] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

- idea: define $\hat{A} := \tilde{A} \times_B \tilde{B}$ by

$$\hat{A}^n = \ker\left(\tilde{A}^n \oplus \tilde{B}^n \xrightarrow{(\varphi, -\psi)} B^n\right),$$
$$\hat{d} = (\tilde{d}_A, \tilde{d}_B) : \tilde{A}^n \oplus \tilde{B}^n \to \tilde{A}^{n+1} \oplus \tilde{B}^{n+1}$$

Rmk. Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}'$ additive functor.
$\mathcal{F}$ induces $C(\mathcal{F}) : C(\mathcal{A}) \to C(\mathcal{A}')$, $K(\mathcal{F}) : \ldots$
by $C(\mathcal{F})(A^\bullet) = \left(\cdots \to F(A^n) \xrightarrow{F(d^n)} F(A^{n+1}) \to \cdots\right)$
But in general, does not induce $D(\mathcal{F})$.

- e.g. $\mathcal{A} = \mathrm{Mod}_{\mathbb{C}[t]}$, $\mathcal{A}' = \mathrm{Mod}_{\mathbb{C}}$, $F(M) := M \otimes_{\mathbb{C}[t]} \mathbb{C}$.

- $\mathbb{C}$ is $\mathbb{C}[t]$-alg via $1 . t = 0$,
- $\{0\} \xhookrightarrow{i} \mathbb{A}^1_{\mathbb{C}}$, $F = i^* : \mathrm{QCoh}(\mathbb{A}^1) \to \mathrm{QCoh}(pt)$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}[t] \arrow[r, "\cdot t"] \arrow[d] & \mathbb{C}[t] \arrow[r] \arrow[d, "t \mapsto 0"] & 0 \\ & 0 \arrow[r] & \mathbb{C} & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[r, "\cdot 0"] \arrow[d] & \mathbb{C} \arrow[d, equal] \\ 0 \arrow[r] & \mathbb{C} \end{tikzcd} \quad \text{not q.iso}$$

Rmk. If $\mathcal{F} : \mathcal{A} \to \mathcal{A}'$ exact, then "obvious" $D(\mathcal{F})$ really works

- let $\mathcal{A}$ ab. cat, define full subcat of $C(\mathcal{A}), K(\mathcal{A}), D(\mathcal{A})$ s.t. $\forall\, a \leq b$, $a, b \in \mathbb{Z} \cup \{\pm\infty\}$,
$\mathrm{Ob}\, C^{[a,b]}(\mathcal{A}) = \{A \in \mathrm{ob}\, C(\mathcal{A}) \mid h^i = 0 \text{ if } i \notin [a,b]\}$,
similarly with $K^{[a,b]}(\mathcal{A})$, $D^{[a,b]}(\mathcal{A})$

- also, $\mathrm{Ob}\, C^b(\mathcal{A}) = \bigcup_{-\infty < a \leq b < +\infty} \mathrm{Ob}\, C^{[a,b]}(\mathcal{A})$.
bounded,
$\mathrm{Ob}\, C^+(\mathcal{A}) = \bigcup_{-\infty < a} \mathrm{Ob}\, C^{[a,\infty]}(\mathcal{A})$, $\mathrm{Ob}\, C^-(\mathcal{A})$ ...

- define functor $\tau_{\geq n} : C(\mathcal{A}) \longrightarrow C^{[n,\infty]}(\mathcal{A})$
by $A' := \tau_{\geq n} A$, $A'_i = \begin{cases} A_i & \text{if } i > n \\ A_n / d A_{n-1} & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$

so that
$$\begin{tikzcd} A'_n = A_n / d A_{n-1} \arrow[rr, "\bar{d}"] & & A'_{n+1} = A_{n+1} \\ & A_n \arrow[ul, "d_n"] \arrow[ur, "d_n"'] & \end{tikzcd}$$

- exercises: i) $\exists$ nat map $A \xrightarrow{\alpha} \tau_{\geq n} A$ in $C(\mathcal{A})$
ii) $h^i(\alpha)$ is iso $\forall\, i \geq n$
iii) if $\varphi : A \to B$ qiso, $\tau_{\geq}(\varphi)$ qiso
iv) $\tau_{\geq}$ induces functors on $K(\mathcal{A})$, $D(\mathcal{A})$

- let $\overline{C}^{[n,\infty]}(\mathcal{A})$ = full subcat of cpx s.t. $A^i = 0$ $\forall\, i < n$. Then $\tau_{\geq n}$ is left adj of inclusion $\overline{C}^{[n,\infty]}(\mathcal{A}) \to C(\mathcal{A})$.

Recall $\mathcal{A}, \mathcal{B}$ abel cat, $\mathcal{A}$ enough injectives,
$\mathcal{F}: \mathcal{A} \to \mathcal{B}$ left exact. Define $R^i F : \mathcal{A} \to \mathcal{B}$.
$\forall i \geq 0$ s.t. $\forall A \in \mathrm{Ob}\,\mathcal{A}$ $\forall$ $0 \to A \to I_0 \to I_1 \to \cdots$
inj resn of A, $R^i F(A) = h^i(F(I_0 \to I_1 \to \cdots))$

- with assumptions unchanged, we want:
$\exists R\mathcal{F}: D^+(\mathcal{A}) \to D^+(\mathcal{B})$, uniquely determined
by demanding if ($I^\bullet \in D^+(\mathcal{A})$ is cpx) $\wedge$
(I^n is inj $\forall n \in \mathbb{Z}$) $\wedge$ ($\exists n_0$ s.t. $I^n = 0$ $\forall n < n_0$)
then $RF(I^\bullet) = (\cdots \to F(I^n) \to F(I^{n+1}) \to \cdots)$

Rmk. $0 \to A \xrightarrow{i} I^0 \xrightarrow{d} I^1 \to \cdots$ inj resn

$\iff$

$$\begin{array}{ccccccccc} 0 & \to & A & \to & 0 & \to & 0 & \to \\ \downarrow 0 & & \downarrow i & & \downarrow 0 & & \downarrow 0 & \\ 0 & \to & I^0 & \to & I^1 & \to & I^2 & \to \end{array} \quad \text{qiso}$$

Intermediate step. Let $\mathcal{J}$ all injectives in $\mathcal{A}$
$C^+_{\mathcal{J}}(\mathcal{A}) \subseteq C^+(\mathcal{A})$ full subcat
- claim: induces $D^+_{\mathcal{J}}(\mathcal{A}) \to D^+(\mathcal{A})$.
This is equiv. of cats.

- replace "enough injs" by "enough acyclics"
- note that injectives are acyclic for left ex. functors
since $\forall$ $0 \to J' \to J \to J'' \to 0$ inj res in $\mathcal{A}$,
$\forall F : \mathcal{A} \to \mathcal{B}$ lexact $\Rightarrow$ $0 \to F(J') \to F(J) \to F(J'') \to 0$

- look at Gel'fand-Manin, or Ch. on derived cats
from Kashiwara-Shapira Sheaves on mfds