Gravity @ ICTP

Schoen - Some Geom. Properties of Spacetime

- initial data for G.R $\rightsquigarrow (M^3, g, k)$
  - scal. constraint eqn:

$$8\pi\mu = \frac{1}{2}\left(R_g - \|k\|_g^2 + (\mathrm{Tr}_g k)^2\right)$$

$$8\pi J_i = D^j\left(k_{ij} - \mathrm{Tr}_g(k)\, g_{ij}\right)$$

- D.E.C. $\mu \geq \|J\|_g$

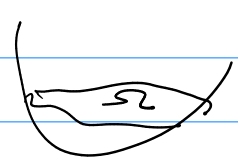
- special case $k = 0$ (initial velocity zero)
- (M^3, g) $R_g \geq 0$
  - one function constructed from (locally) 6 metric components
    $\rightarrow$ quite a weak constraint

- for starters look at $\underbrace{(M^2, g)}_{\pi_1(M^2) = \{1\}}$ $K_g \geq 0$
- a few cases:

① A.F $\rightsquigarrow M \setminus \overset{\text{cpt.}}{K} \cong$ cone
  - Cones are classified by cone angles
  - $d\rho^2 + a^2\rho^2 d\theta^2$, $a > 0$
    $\rightarrow$ cone angle $2\pi a$
  - Gauß-Bonnet: $0 \leq \int_{S_\rho} K = 2\pi - \int_{\partial S_\rho} k_g \, ds$

    $\Rightarrow a \leq 1$ since $\int_{\partial S_\rho} k_g \, ds = 2\pi a$, with "=" only if $K_g = 0$

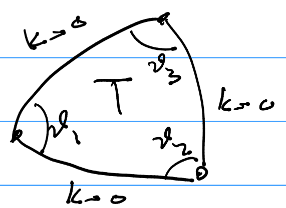
② $\Omega \subset M$ region, $\pi_1(\Omega) = \{1\}$



$$0 \leq \int_\Omega K \, da = 2\pi - \int_{\partial\Omega} k \, ds$$

- isometrically embed $\Gamma := \partial\Omega$ in $\mathbb{R}^2$
- In this case we can view G. B. as a comparison thm $0 \le \int_\Gamma (k_0 \circ \varphi - \varphi)\,ds$

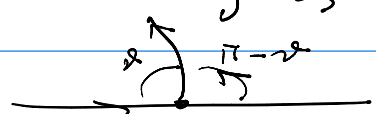
③ polygons, Δ



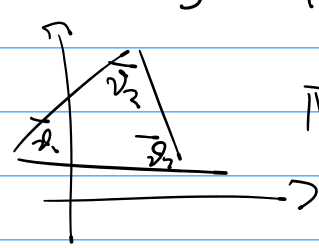
← geodesic arms, so $k=0$

$$\int_T K\,da = \sum_{i=1}^{3} \vartheta_i - \pi$$

Since changes in geodesic curvature are measured by exterior angles,



- clearly $\sum \vartheta_i \ge \pi$, moreover, embedding isom. in $\mathbb{R}^2$ gives, by Toponogov, $\vartheta_i \ge \bar{\vartheta}_i \ \forall i$, for $K \ge 0$.



$\mathbb{R}^2$

Rmk. • We need $k \ge 0$ on edges for the same comparison thm.
• Toponogov works in any $n \ge 0$ using sectional curvature bdd from below, which is much stronger than scalar curv. conditions

3 dim mflds

- special case (M^3, g) $R_g \geq 0$ s.t. $g_{ij} = u^4 \delta_{ij}$ outside K cpt, so $R_g \geq 0$ in that region forces $u(|x|) = 1 - \frac{m}{2|x|} + O(|x|^{-2})$

- PMT: if $R_g \geq 0$ then $m \geq 0$, and $= 0$ only if (M, g) isometric to $\mathbb{R}^3$

- cone metric in conf. flat:

$$d^2 s = r^{2\alpha}(dr^2 + r^2 d\theta) = d\rho^2 + a^2 \rho^2 d\theta^2$$

$$\Rightarrow r^{2\alpha+2} = a^2 \rho^2 \Rightarrow r^{\alpha+1} = a\rho$$

$$\Rightarrow (\alpha+1) r^{\alpha} dr = a\, d\rho$$

but $d\rho^2 = r^{2\alpha} dr^2$

$$\overset{\shortparallel}{\frac{1}{a^2}(\alpha+1)^2 r^{2\alpha} dr^2} \Rightarrow \alpha = a - 1, \quad -1 < \alpha < \infty$$

- generally $g = |x|^{2\alpha}(\underbrace{dx_1^2 + dx_2^2}_{\bar{g}}) = v^2 \bar{g}$

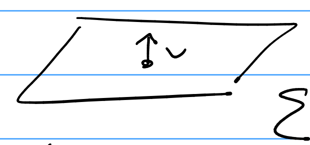
$x_2 = \lambda \gg 1$

$\bar{k} = 0$

$k = |x|^{-2\alpha} \frac{\partial}{\partial x_2} |x|^{\alpha} > 0$

$\uparrow k > 0$

$x_2 = -\lambda$



$$H = v^{-1}\left(\bar{H} - (n-1)\frac{D_\nu v}{v}\right)$$