Alg geom seminar

$B \subset \mathbb{C}^m$. $\{M_\tau \mid \tau \in B\}$ an analytic family if $\exists$ a pair $\{M, \varpi\}$ s.t.

1) M cpx mfd
ii) $\varpi : M \to B$ holomorphic, s.t. $\varpi^{-1}(\tau) = M_\tau$ & $\operatorname{rk} J(\varpi) = m$.

- tori: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{C} \to \mathbb{C}$
  $(m, n, z) \mapsto m + n\omega + z$, $\omega \in \mathbb{H}^+$

  $\to \mathbb{C}_\omega \doteq \dfrac{\mathbb{C}}{\mathbb{Z} \times \mathbb{Z}} \Rightarrow \mathbb{C}_\omega \overset{b.h.}{\simeq} \mathbb{C}_{\omega'}$ iff ω, ω' on same $PSL(2, \mathbb{C})$ orbit

  $\to [x, y, z]$, $z \neq 0$, E ... $y^2 = x^3 + c_2 x + c_3$

  $\to$ j-invariant, $j = 1728 \dfrac{4 c_2^3}{4 c_2^3 + 27 c_3}$

  $\to$ Weierstraß $\wp(z) = \dfrac{1}{z^2} + \sum_{m,n} \left(\dfrac{1}{(z - \omega_{mn})^2} - \dfrac{1}{\omega_{mn}^2} \right)$

  $\Rightarrow E_\omega$ ... $y^2 = x^3 + \underset{60 \sum \frac{1}{\omega_{mn}^4}}{\underset{\shortparallel}{c_2(\omega)}} x + \underset{140 \sum \frac{1}{\omega_{mn}^6}}{\underset{\shortparallel}{c_3(\omega)}}$

  $z \mapsto [\wp(z), \wp'(z), 1]$

- $W = \mathbb{C}^2_\times$, $n \in \mathbb{N}$, $\mathbb{Z} \times W \to W$, Hopf surface
  $(m, z_1, z_2) \mapsto (z_1 \alpha_1^m + m \alpha_1^{m-1} \lambda z_2^n, \alpha_2^m z_2)$, $0 < |\alpha_{1,2}| < 1$
  $\lambda(\alpha_1 - \alpha_2^n) = 0$

  1° $\lambda = 0$, $W \simeq W$, $(z_1 z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$
  2° $\lambda \neq 0$, – " –, – " – $(\alpha_1 z_1 + \lambda z_2^n, \alpha_2 z_2)$

  $\Rightarrow \mathbb{Z} \times W \times \mathbb{C} \to W \times \mathbb{C}$
  $(m, (z_1 z_2), \lambda) \mapsto (\text{—}, \text{—}, \lambda) \Rightarrow \dfrac{W \times \mathbb{C}}{\mathbb{Z}} = M$

Rmk: KODAIRA (1966): cpx surface $\underset{\text{diff}}{\simeq} S^1 \times S^3$ iff Hopf.

$\to$ pick $n = 1$, $(m, (u_1, u_2), \lambda) \mapsto (u_1 \alpha^m + m \alpha^{m-1} u_2, \alpha u_2, \lambda)$

- construct invariant global holom. v.f. $v = v_1(z_1, z_2)\partial_1 + v_2(z_1, z_2)\partial_2$

$\Rightarrow \dots \Rightarrow$

$\lambda = 0: \; v = (a z_1 + b z_2)\partial_1 + (c z_1 + d z_2)\partial_2$

$\lambda \neq 0: \; v = a(z_1 \partial_1 + z_2 \partial_2) + b z_2 \partial_1$

$\rightarrow$ unequal # of global v.f.s !

Rmk: this is a general property...