Bertola

- $L(\lambda) = \frac{L_{-n}}{\lambda^n} + \dots + \frac{L_{-1}}{\lambda}$, L_{-n} reg. semisimple

- $L(\lambda) = g(\lambda)\overset{\text{diag.}}{A}(\lambda)\, g^{-1}(\lambda)$, $g(\lambda) = g_0 + g_1\lambda + \dots$, $\det g_0 \neq 0$

- $A(\lambda) = \underbrace{\frac{A_{-n}}{\lambda^n} + \dots + \frac{A_{-1}}{\lambda}}_{\text{Casimirs}} + A_0 + \underbrace{\lambda A_1 + \dots}_{\text{Hamiltonians}}$

Prop We find $M(\lambda) = \frac{M_{-r}}{\lambda^r} + \dots + \frac{M_{-1}}{\lambda}$ s.t. $\mathring{L} = [M, L]$ and then

i) $M = (gBg^{-1})_-$, $B = \frac{B_{-r}}{\lambda^r} + \dots + \frac{B_{-1}}{\lambda}$ diag

ii) $\mathring{g}g^{-1} = -(gBg^{-1})_+$

Pf. $L = gAg^{-1}$ with $\mathring{A}_\ell = 0\ \forall \ell$

$\Rightarrow \mathring{L} = [\mathring{g}g^{-1}, L]$
$\quad \overset{\shortparallel}{[M, L]}$

$\Rightarrow [\mathring{g}g^{-1} - M, L] = 0$, Since $\operatorname{Spec} L(\lambda)$ is simple in nbhd $d\lambda = 0$, $\mathring{g}g^{-1} - M = -gBg^{-1}$ with B diag. Now take pos (neg) parts. □

Hamiltonian str.

- $\mathcal{G} = \{ g(\lambda) = g_0 + \lambda g_1 + \dots,\ \det g_0 \neq 0,\ g_\ell \in \mathrm{Mat}_{n\times n}\}$ (loop gp)
- $\mathrm{Lie}\,\mathcal{G} = \mathfrak{g} = \{\eta(\lambda) = \eta_0 + \lambda\eta_1 + \dots\}$ (loop alg)

Lemma $\mathfrak{g}^* \simeq \{ X(\lambda) = \frac{X_{-1}}{\lambda} + \frac{X_{-2}}{\lambda^2} + \dots \}$
with $\langle X, \eta\rangle := \operatorname{res}_{\lambda=0} \operatorname{tr} X(\lambda)\eta(\lambda) = \sum_{\ell=1}^{\infty} \operatorname{tr} X_{-\ell}\,\eta_{\ell-1}$

- exercise: nondegeneracy

- think of $L \in \mathfrak{g}^*$

Def $Ad : \mathcal{G} \times \mathfrak{g} \to \mathfrak{g}$, $(g, y) \to g^{-1} y g$ action.

Lemma $Ad^* : \mathcal{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$, $Ad_g^*(X) = (gXg^{-1})_-$

Pf. $\langle Ad_g^* X, y \rangle = \langle X, Ad_g y \rangle$
$= \langle X, g^{-1} y g \rangle$
$= \text{res tr } X g^{-1} y g$
$= \text{res tr } g X g^{-1} y$
$= \text{res tr } (gXg^{-1})_- \cdot y$ $\square$

- L itself is $\in \text{Im } Ad_g^*$:
$L = g(A_- + A_+)g^{-1}$, but $L = (L)_-$ so $L = (gA_-g^{-1})_-$
$\Rightarrow L \in \mathcal{O}^*(A_-) := \{ Ad_h^* A_- \mid h \in \mathcal{G} \}$

Cor The ZS of L lives in the coadj. orbit.

Kostant-Kirillov-Souriau (KKS) Poisson str on $\mathfrak{g}^*$

Def. Sm. mfd M is Poisson mfd if
$\exists \{\cdot, \cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$
i) bilin
ii) $\{f, g\} = -\{g, f\}$
iii) Leibnitz
iv) Jacobi
A func. $c \in \mathcal{C}^\infty(M)$ is called Casimir if $\{f, c\} = 0$ $\forall f \in \mathcal{C}^\infty(M)$.

A Poiss. mfd is symplectic iff only Casimirs are constants.

- In practice, pick chart $\{x_j\}_{j=1,\dots,n}$,
$\{x_i, x_j\} = \mathbb{P}_{ij}(\vec{x})$, $\{g, f\} = \sum \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\mathbb{P}_{ij}(\vec{x})$

- $\{\cdot,\cdot\}$ symplectic $\Leftrightarrow$ $\det \mathbb{P} \neq 0$
$\Rightarrow$ $\omega = (\mathbb{P})^{-1}_{jk}\, dx_j \wedge dx_k$

Lemma $\mathbb{P}$ solves Schouten bracket (i.e. Jacobi id)
$\Leftrightarrow$ ω closed.

Def KKS bracket on $\mathfrak{g}^*$. Given $f, g \in C^\infty(\mathfrak{g}^*)$
$\{f, g\}(L) := \langle L, [\nabla f, \nabla g]\rangle$
where $\nabla f(L) \in \mathfrak{g}$ def. by
$\langle X, \nabla f(L)\rangle := \frac{d}{d\varepsilon} f(L + \varepsilon X)\big|_{\varepsilon=0}$

- fix $\eta \in \mathfrak{g}$, let $f_\eta(L) := \langle L, \eta\rangle = \mathrm{res\,tr}\, L\cdot\eta$

- Jacobi check: $\{f_\xi, \{f_\eta, f_s\}\}(L) = \langle L, [\xi, [\eta, s]]\rangle$.

Lemma $C \in C^\infty(\mathfrak{g}^*)$ is constant on $\mathcal{O}^*$
iff $\mathrm{ad}^*_{\nabla C}(L) := [\nabla C, L]_- = 0$.

Pf. $\tilde{L} \in T\mathcal{O}^*(\Lambda_-)$ $\Rightarrow$ $\tilde{L} = (e^{\varepsilon\eta} L e^{-\varepsilon\eta})_-$
$= L + \varepsilon[\eta, L]_-$. C is const. on $\mathcal{O}^*$ iff $\forall \eta \in \mathfrak{g}$;
$0 = \langle \nabla C, [\eta, L]_-\rangle = \mathrm{res\,tr}\, \nabla C \cdot [\eta, L]_-$
$= \mathrm{res\,tr}\, \nabla C\, [\eta, L] = \mathrm{res\,tr}\, \eta \cdot [L, \nabla C]_-$ $\Rightarrow$ $[L, \nabla C] = 0$.

Rmk $L = L_{-n}/\lambda^n + \ldots$, L_{-n} reg. semis.
$\Rightarrow L \in$ orbit of $G^*(A_-)$, $A_- = \frac{A_{-n}}{\lambda^n} + \ldots + \frac{A_{-1}}{\lambda}$, $(A_{-\ell})_{jj}$ const.

Prop $G^*(A_-)$ is sympl. mfd with KKS.

Pf. Show $C \in \mathcal{C}^\infty(\mathfrak{g}^*)$, $\{f, C\}(L) = 0 \quad \forall f, L$
$\Leftrightarrow$ C const. on G^*.
Supp. $\{f, C\}(L) = \operatorname{res} \operatorname{tr} L \cdot [\nabla f, \nabla C] = 0$
Pick $f(L) = f_\eta(L) = \langle L, \eta \rangle$.
$\Rightarrow \{f_\eta, C\} = \operatorname{res} \operatorname{tr} L[\eta, \nabla C]$
$= \operatorname{res} \operatorname{tr} \eta [\nabla C, L]$
wherefrom the claim follows.

Classical rational r-matrix

Prop $\{L(\lambda) \overset{\otimes}{,} L(\mu)\} = \left[L(\lambda) \otimes 1\!\!1 + 1\!\!1 \otimes L(\mu), \frac{\Pi}{\lambda - \mu}\right]$
is KKS.
In coords, $\{L_a{}^b(\lambda), L_c{}^d(\mu)\} = \frac{L^d{}_a(\lambda) - L^d{}_a(\mu)}{\lambda - \mu} \delta^c_b - \binom{d \leftrightarrow c}{a \leftrightarrow b}$

and $\Pi(v \otimes w) = w \otimes v$ is flip.

N.B. Makes sense for arb. rat. matrices
Call $L^1 := L \otimes 1\!\!1$, $L^2 := 1\!\!1 \otimes L$

Cor $H_r^{(\lambda)} := \operatorname{tr} L^r(\lambda)$ Poisson commute.

Rmk We would be in a position to discuss Darboux coords on symp. leaves now, but more on that another time.

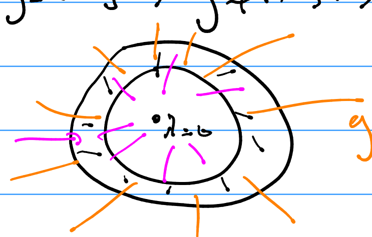
Prop Let $B(\lambda), \tilde{B}(\lambda)$ constant Laurent polys.
Consider vfields on $\mathcal{G}^*$
$$\partial_t L = [M, L], \quad M = (g B g^{-1})_-$$
$$\partial_s L = [\tilde{M}, L], \quad \tilde{M} = (g \tilde{B} g^{-1})_-$$
they commute.

Pf. Know $\overset{\partial_t g}{g_t}\, g^{-1} = -(g B g^{-1})_+$,
$$\partial_s \partial_t L = [\partial_s M, L] + [M, [\tilde{M}, L]].$$
$$\partial_s M = \partial_s (g B g^{-1})_- = \dots$$
$$= [\tilde{M}, M] - [\tilde{M}, g_t g^{-1}]_-$$
$$= [\tilde{M}, M] - [g \tilde{B} g^{-1}, g_t g^{-1}]$$
$\Rightarrow$ claim follows by inspection.

- elementary flows: $B_{s,j}(\lambda) = \dfrac{E_{jj}}{\lambda^s} \mapsto t_{s,j}$,
other flows being lin. combs.

Soln of ZS system

Thm. Let $B(\lambda)$ const. diag. Then ZS of $\partial_t L(\lambda, t) = [(g B g^{-1})_-, L]$
are given as follows:
- $g(\lambda, t) = g_+(\lambda, t)$ solves the Birkhoff factorisation problem (aka Riemann-Hilbert)
$$g_-^{-1}(\lambda, t)\, g_+(\lambda, t) = e^{tB(\lambda)} g(\lambda, 0)\, e^{-tB(\lambda)} =: K(\lambda, t)$$



g_- is analytic here

g_+ analytic here

$\to$ we are given trivialisation on overlap

Pf. take deriv: $-g_-^{-1}\dot{g}_- g_-^{-1} g_+ + g^{-1}\dot{g}_+ = [B(\lambda), K(\lambda, t)]$

$$\Rightarrow \underbrace{-\dot{g}_- g_-^{-1}}_{<0} + \dot{g}_+ g_+^{-1} = \underbrace{g_- B g_-^{-1}}_{<0} - g_+ B g_+^{-1}$$

$\Rightarrow$ positive part: $\dot{g}_+ g_+^{-1} = -(g_+ B g_+^{-1})_+$

Rmk As t goes on, we wrap around the Riemann sphere, potentially hitting problematic pts $\rightarrow$ these are isolated by Fredholm alternative, and this is in essence the Painlevé property.