Gravity @ ICTP

Chrusciel - Intro to mass & energy in GR.

Mass in AF spacetimes

- class. mech $\longrightarrow$ $E = \frac{m\dot{x}^2}{2} + V(x)$ conserved
  - so qualitatively there are oscillations between turning pts, or scattering if E large enough and $V(x)$ bdd
  - similar in 3D with radial motion

- scalar field: $\Box\varphi = m^2\varphi$ implies $E = \frac{1}{2}\int_{\mathbb{R}^3} \dot{\varphi}^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi$ is conserved

- YM: $\mathcal{E} = \frac{1}{2}\int |E|^2 + |B|^2 \xRightarrow[\text{or YM}]{\text{Maxwell}} \frac{d}{dt}\mathcal{E} = 0$
  - enough to have $\frac{d\mathcal{E}}{dt} \le c\mathcal{E}$, $c > 0$

- so energy is useful

- Problem: given $\mathcal{L}(\varphi^A, \partial\varphi^A)$, find $E = \int \text{expression}(\varphi^{\cdot}, \partial\varphi)$

$\longrightarrow$ in GR, $\varphi \mapsto g_{\mu\nu}$, $\partial\varphi \mapsto \partial_\alpha g_{\mu\nu}$

  $\rightarrow$ but in normal coords $g_{\mu\nu}\big|_{x=0} = \eta_{\mu\nu} + O(x^2)$

- answer: ADM energy momentum
  - positive energy thm.
  $\Rightarrow$ Sol'n of Yamabe problem; static BH uniqueness

- but doesn't imply stability
  - e.g. asympt. AdS has pos. energy, but is generically unstable!

Newtonian & post-Newtonian mass

- Newton's theory: $\Delta\varphi = -4\pi G\mu$, $G \equiv 1$, μ mass density

gives $m\frac{d^2\vec{x}}{dt^2} = -mD\varphi$

- if $\mu = 0$ outside $B(R)$, $\varphi = \frac{M}{r} + O(r)$, $r > R$
  where $M = \int_{B(R)} \mu$

$$= -\frac{1}{4\pi}\int_{S(R)} \nabla^i\varphi \, dS_i$$

$$= \lim_{R\to\infty} -\frac{1}{4\pi}\int_{S(R)} \nabla^i\varphi \, dS_i$$

$$= -\frac{1}{4\pi}\int_{S_\infty} \nabla^i\varphi \, dS_i$$

- M is positive since μ is, and $M = 0$ in vacuum.

- weak grav field:

$$d^2s = \underbrace{-(1-2\varphi)}_{g_{00} = g_{tt}} dt^2 + \underbrace{(1+2\varphi)\delta_{ij}}_{g_{ij}} dx^i dx^j$$

use this to get Komar mass

use this to define ADM mass

- these masses are equal in stationary case

Cauchy problem in GR

- problem: give initial data at $t=0$,

which? a "well-behaved" spacelike sfc $= \Sigma$

find a sol'n

$\hookrightarrow$ topologically $M = I \times \Sigma$, $I \subseteq \mathbb{R}$

- initial data?
  - $G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ is 2^{nd} order
  - but if $g_{\mu\nu}$ is soln, so is $g_{\mu\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}$
  - $\rightarrow$ nonuniqueness.
  - impose some useful coordinate conditions
  - bad idea: $g = -dt^2 + g_{ij}(x,t)\, dx^i dx^j$
    with $g_{ij}(x,0)$, $\partial_k g_{ij}(x,0)$ fixed
    - ok, and use Cauchy-Kowalevska if this data is analytic
      - but there are no results for these special coords.
  - good idea: impose (generalised) harmonic coords., $\Box x^\mu = 0$.

- but $\Box x^\mu = 0 = F^\mu(\underbrace{g, \partial g, x}_{\text{no } \partial^2 g})$ (1)

- then (1) + G.G. give

$$g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g, t) \quad (2)$$

$\Downarrow$

not $\nabla_\mu \nabla_\nu$?, covariance broken

- well-posed wave-eqn ...
  what guarantees that sol'n of (2) satisfies (1)
  - ans. constraint eqn.