BRUZZO

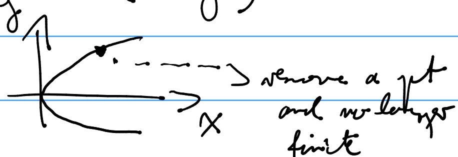
## Morphisms (locally) of finite type

- quasi-finite $\to \{f^{-1}(y)\}$ is finite for any $y \in Y$, $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$
- ex: $\mathbb{A}^n_k = \operatorname{Spec} k[x_1 \dots x_n]$ is of finite type

$\mathbb{A}^1_k \to \mathbb{A}^1_k$, $z \mapsto z^n$ $\to$ $k[t]$-module structure given by raising to $n$-th power $\to$ basis given by $1, x, \dots, x^{n-1}$

- ex: $k^\times \to \mathbb{A}^1_k$, $\mathbb{A}^1_k - \{0\} = D(x) = \operatorname{Spec} k[x, \frac{1}{x}]$ $\to$ quasi finite but not finite

- ex: $X = \operatorname{Spec} \dfrac{k[x,y]}{(x-y^2)} \rightsquigarrow \dfrac{k[x,y]}{(x-y^2)} = k[x] \oplus y\,k[x]$ $\hookrightarrow$ finitely gen. as a module $\Rightarrow$ finite




- ex: $X = \operatorname{Spec} \dfrac{k[x,y,t]}{(ty-x^2)}$, $f \downarrow$, $Y = \operatorname{Spec} k[t] \simeq \mathbb{A}^1_k$

$\to X, Y$ integral schemes of fin. type
$\to$ fibres: $t = a \neq 0$, integral variety
$t = 0 \Rightarrow f^{-1}(0) = \operatorname{Spec} \dfrac{k[x,y]}{(x^2)}$
$\to$ NONREDUCED!

ERRATA from last time: $\operatorname{Proj} S = \{\text{prime ideals } p, S_+ \not\subset p\}$

## Fiber product of top. spaces.

- standard def.

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "q"] \\ & X \arrow[r, "p"] & Z \end{tikzcd}$$

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ X \arrow[r, "g"] & S \end{tikzcd}$$

$X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$, $f \rightsquigarrow R \xrightarrow{f^\#} A$, $g \rightsquigarrow R \xrightarrow{g^\#} B$, $C = A \otimes_R B$
$\hookrightarrow (a \otimes b)(a' \otimes b') = (aa' \otimes bb')$

$\Rightarrow X \times_S Y = \operatorname{Spec} C$

$$\begin{tikzcd} C = A \otimes B & A \arrow[l, "\pi_1^\#"'] \\ B \arrow[u, "\pi_2^\#"] & R \arrow[l, "g^\#"] \arrow[u, "f^\#"'] \end{tikzcd}$$

$\pi_1^\#(a) = a \otimes 1$

$(-)^\#$ is a rather inaccurate notation, $(-)^*$ is better...

- EXAMPLE

$X, Y$ schemes over $k$: $X \times_k Y \equiv X \times_{\operatorname{Spec} k} Y$

- $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \operatorname{Spec} \underbrace{k[x] \otimes_k k[y]}_{k[x,y]} = \mathbb{A}^2_k$

→ note that closed sets in $\mathbb{A}^1_k$ are finite collections of points, but in $\mathbb{A}^2_k$ they could be curves, e.g. $x=y$
→ difference in topology

**Def.**

$$\begin{tikzcd} X \arrow[dr, "\Delta_f"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\ & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

DIAGONAL MORPHISM $\Delta_f$ associated to $X \xrightarrow{f} Y$.

**Def.** $f$ is a <u>separated morphism</u> if $\Delta_f$ closed immersion.

$X$ scheme over $k$ is separated (over $k$) if $X \to \operatorname{Spec} k$ separated.

**Prop.** All morphisms over affine schemes are separated.

**Cor.:** All affine schemes over $k$ are separated over $k$.

→ consider $X =$ —:— line w/ double origin
⇒ $X \times_k X =$ aff plane w/ 2 axes & 4 origins
$\Delta =$ line w/ double origin
⇒ all origins are in the closure of $\Delta$ ⇒ $\Delta$ ¬ closed ⇒ <u>$X$ not affine</u>

**Pf:**
- note that on CRing we have the codiagonal $\Delta^\#: A \otimes A \to A$, $a \otimes a' \mapsto aa'$
- surjective ⇒ $\mathcal{O}_{X \times_Y X} \xrightarrow{\Delta^\#} \Delta_* \mathcal{O}_X$ surjective on stalks
- for $I \subset A$, $A \twoheadrightarrow A/I$, $\operatorname{Spec} A/I \to \operatorname{Spec} A$ morphism of Spec-s
→ take $I = \ker \Delta^\#$ ⇒ $\Delta$ closed immersion □

**Prop.** $X \xrightarrow{f} Y$ separated iff $\Delta_f$ closed in $X \times_Y X$

**Pf** ⇒ obvious, ⇐

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[dr, "\sim"'] & X \times_Y X \arrow[d, "\pi_1"] \\ & X \end{tikzcd}$$

$\Delta_f^{-1}$ may be identified with the identity ⇒ closed im. □