Automini

- $\overset{\substack{\text{unital} \\ \text{comm}}}{}\mathrm{Sp}\, A = \{\text{nonzero characters}\}$
- note that top. of pt-wise convergence is the weakest top. making $\{ev_a : \mathrm{Sp} A \to \mathbb{C},\ a \mapsto f(a)\}$ continuous

Thm (Gel'fand transform)

a) $\forall x \in A$, $\mathrm{Sp}_A x = \{\chi(x) \mid \chi \in \mathrm{Sp} A\}$

b) $\mathrm{Sp}\, A$ is cpt.

c) $\exists$ natural continuous algebra morphism

$$A \to C(\mathrm{Sp}\, A)$$
$$x \mapsto G(x) = (\chi \mapsto \chi(x))$$
$$G(x)(\chi) = \chi(x)$$

Pf. a) $\subseteq$. $\lambda \in \mathrm{Sp}_A x \implies ((x-\lambda)A$ is ideal$)$ $\wedge ((x-\lambda)A \neq A)$. Zorn implies $\exists\, J$ s.t. $(J \supseteq (x-\lambda)A) \wedge (J$ maximal in $A)$.
$\exists!\ \chi : A \to \mathbb{C}$ s.t. $\ker \chi = J$, so $\chi(x-\lambda) = 0 \implies \chi(x) = \lambda$.

$\supseteq$. $\chi \in \mathrm{Sp} A$. $x - \chi(x)\cdot 1_A \in \ker \chi$, maximal ideal $\implies x - \chi(x)\cdot 1_A \notin A^{-1}$.

b) $\to$ b$_1$) $\mathrm{Sp}\, A \subset \underbrace{B_1(0, A')}_{\text{cpt. by Alaoglu}}$

b) $\to$ b$_2$) closed

b$_1$) $\chi \in \mathrm{Sp}\, A$, $x \in A \implies |\chi(x)| \leq \|x\| \implies \|\chi\| \leq 1$

b$_2$) $\mathrm{Sp}\, A = \underbrace{\{\ell \in A' \mid \ell(xy) = \ell(x)\ell(y)\}}_{\text{closed (use nets?)}} \cap \underbrace{\{\ell \in A' \mid \ell(1) = 1\}}_{\text{closed}}$

c) $\forall x \in A$, $G(x)$ is cont.

$G(xy)(\chi) = \chi(xy) = \chi(x)\chi(y) = (G(x)\cdot G(y))(\chi)$

Now apply a) to show boundedness.

Cor. If $a, b \in A$ commute, then
$Sp_A(a+b) \subset Sp(a) + Sp(b)$, $Sp_A(ab) \subset Sp(a)\,Sp(b)$

- if A is not unital,
  $Sp\, A = \{ \varphi \in Sp\, \tilde{A} \mid \varphi|_A \neq 0 \}$
- $Sp\, \tilde{A}$ is Alexandroff cptification of $Sp\, A$

Naturality

- let $\pi : A \to B$ morph. of unital, comm. Banach algs
- let $\pi^* : Sp\, B \to Sp\, A$, $\chi \mapsto \chi \circ \pi$
  $\pi_* : C(Sp\, A) \to C(Sp\, B)$, $\pi_*(f) = f \circ \pi^*$
- the following commutes

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[d, "G_A"'] & B \arrow[d, "G_B"] \\
C(Sp\, A) \arrow[r, "\pi_*"] & C(Sp\, B)
\end{tikzcd}$$

Def. A Banach alg A is rationally generated by z if A is the smallest closed subalgebra containing z and $\{ (z - \lambda)^{-1} \mid \lambda \notin Sp_A z \}$

- facts: A comm., chars determined by values at this subalg.
- $Sp\, A \to Sp_A z \subset \mathbb{C}$, $\chi \mapsto \chi(z)$

- now fix $X \subset \mathbb{C}$ cpt. Look at
$$R_X = \left\{ \begin{array}{l} \text{rat. functions} \\ \text{with no poles in } X \end{array} \right\} \hookrightarrow C(X)$$

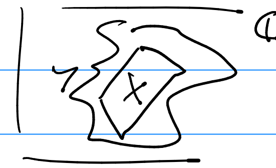
- $Ch(X) := \overline{R_X}^{C(X)}$
$\quad\|$
rat. gen by $z : X \to \mathbb{C}$

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Sp(A) \\ \| & \circlearrowleft & \nearrow \\ Sp_A z & & \end{array} \quad \text{where} \quad \begin{array}{l} X \ni y \to ev_y \\ \xi(z) \leftarrow \xi \end{array}$$

- φ homeo
- we identify Gel'fand transform with $Ch(X) \overset{G}{\hookrightarrow} C(X)$
- why?
  - $X \xrightarrow{\varphi} Sp\,A$ homeomorphism
    $y \mapsto ev_y$
  - here $A = Ch(X) \subset C(X)$
  - now $A = Ch(X) \xrightarrow{G} C(Sp(A))$
$$\downarrow \varphi^*$$
$$C(X)$$

- let's compute it.
- let $y \in X$. $\varphi^*(G(f))(y) \overset{\varphi^*(\cdot) = \cdot \circ \varphi}{=\!=} G(f)(\varphi(y))$
$= G(f)(ev_y) = f(y)$
so indeed it injects.

- In general G does not surject
- if $\mathrm{Int}\,X \neq \emptyset$, take $y \in \mathrm{Int}\,X$,
consider $\underset{\substack{\uparrow \\ Ch(X)}}{f} \mapsto f'(y)$ ← continuous by Cauchy formula
- sometimes, in abstract settings, not injective, e.g. if $x \cdot y = 0$ $\forall x, y \in E$, E Banach



- let $Hol(X) := \left\{ \begin{array}{c} \text{germs of hol. funcs} \\ \text{defined around } X \end{array} \right\}$
- algebra
- let $Y \supseteq X$ cpt neighbourhood

$i_{X,Y} : Ch(Y) \longrightarrow Hol(X)$

rat. func $\longmapsto$ its germ around X

Thm (Hol. functional calc.) A unital Ban. alg.,

fix $x \in A$. $\exists!$ morph. of unital algs

$$\varphi_x : Hol(Sp_A x) \longrightarrow A$$

1) $\varphi_x(z) = x$, $z : Sp_A x \longrightarrow \mathbb{C}$, $z(y) = y$

ii) $\forall\, Y \supset Sp_A x$ cpt nbhd,

$\varphi_x \circ i_{Sp_A x, Y}$ is continuous

Rmk. Suppose we constructed φ_x.

If g rat func w/o poles in $Sp_A x$,

then $\varphi_x(g) = g(x) \in A$.

Prop A i) $\forall$ cpt nbds Y as in theorem,

the map $R_Y \longrightarrow A$, $p \longmapsto p(x)$

is cont. in uniform. conv.

ii) $\exists$ decreasing sqn $\{Y_n\}$ cpt nbds of X s.t.

$$Hol(X) = \bigcup_{n \in \mathbb{N}} i_{X,Y_n} Ch(Y_n)$$

Prop B Let $\varphi \in C^1_c(\mathbb{C})$, $\varphi \equiv 1$ on open $U \supset X$.

$K := \operatorname{supp} \varphi$.

a) $\forall$ rat func P w poles$(P) \cap K \neq 0$,

$$P \in R_K \Rightarrow P(x) = \frac{1}{2\pi i} \int_K P(z)(z - x)^{-1}\, dz \wedge d\varphi$$

Bochner integral

b) $\forall$ holom. f around K and any pt $\lambda \in U$,

$$f(\lambda) = \frac{1}{2\pi i} \int_K \frac{f(z)}{z-\lambda}\, dz \wedge d\varphi$$