Fantechi

- missing topics: Grothendieck - Serre duality for proj. mos, cotangent cpx
- Serre $\mathrm{Ext}^i(\mathcal{F}, \omega_X)^\vee = H^{n-i}(X, \mathcal{F})$

- simplest case, X sm proj curve $/\mathbb{C}$
  $\Rightarrow H^1(\Omega^1_X) \xrightarrow[\int]{\sim} \mathbb{C}$
  in complex geometry
- need more work for alg. geom
- L line bdl, E vb.

$$H^0(X, E) \otimes H^1(X, E^\vee \otimes \Omega^1_X)$$
$$\downarrow$$
$$H^1(X, \underbrace{E \otimes E^\vee}_{Hom(E,E)} \otimes \Omega^1_X) \longrightarrow H^1(X, \Omega^1_X) \to \mathbb{C}$$

Duality This is a perfect pairing.
Which means: V, W f.d. vsp$/\mathbb{K}$, $V \otimes W \xrightarrow{\varphi} \mathbb{K}$ is perfect pairing if induced maps $W \to V^\vee$ and $V \to W^\vee$ are isomorphisms, i.e. $\exists$ bases $v_1 - v_n \in V$, $w_1 - w_n \in W$ s.t. $\varphi(v_i \otimes w_j) = \delta_{ij}$.

- X proj. sm. of dim n:
- Serre duality: $H^i(X, E) \otimes H^{n-i}(X, E^\vee \otimes \Omega^n_X) \to H^n(X, \Omega^n_X) \to \mathbb{C}$
- Grothendieck $\to$ relative, works in der. cats.

- work over A Noetherian ring

Lemma If A is local $\wedge$ M is f.g. flat A-mod then M is free, $M \simeq A^{\oplus d}$.

Pf. $M \otimes_A k(A)$ f.d. $k(A)$ vsp, basis $\bar{e}_1, \dots, \bar{e}_d$.
$M \to M \otimes_A k(A) \to 0$ lifts $\bar{e}_i$ to $e_i \in M$

$\rightsquigarrow \quad 0 \to N \to A^{\oplus d} \xrightarrow{\text{Nak.}} M \to 0 \quad | - \otimes_A k(A)$

$\underset{\overset{\|}{0}}{\mathrm{Tor}^1(M, k(A))} \to N \otimes_A k(A) \to \underbrace{A^{\oplus d} \otimes_A k(A)}_{k(A)^{\oplus d}} \xrightarrow{\sim} M \otimes_A k(A) \to 0$

$\Rightarrow$. $N \otimes_A k(A) = 0$, N f.g. $\overset{\text{Nak.}}{\Longrightarrow}$ $N = 0$. $\square$

Lemma $\mathcal{F} \in \mathrm{Coh}_X$, X (loc) Noeth. sch. Let $p \in X$. Assume stalk $\mathcal{F}_p$ is free $\mathcal{O}_{X,p}$-mod of rk r. Then $\exists$ $U \subseteq X$ open nbhd of p s.t. $\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus r}$.

Lemma $\mathcal{F} \in \mathrm{Coh}_X$, X loc Noeth sch. $\mathrm{Supp}\, \mathcal{F} = \{ p \in X \mid \mathcal{F}_p \neq 0 \}$ is closed in X.

Summing up X loc. Noeth. $\mathcal{F} \in \mathrm{Coh}_X$. TFAE

i) $\mathcal{F}$ is flat as $\mathcal{O}_X$-mod
ii) $\mathcal{F}$ is loc. free
iii) $\mathcal{F}_p$ is free $\forall p \in X$.

Rmk i) X sch loc. f.t. $/k = \bar{k}$ $\Rightarrow$ $X(k)$ dense in X

ii) X — " — , but seen as top. sp.,
$Z \subseteq X$ all k-valued pts.
Then $\underset{\text{all pts closed}}{Z} \longrightarrow X$ gives $\mathrm{Top}(X) \underset{\text{iso}}{\overset{\text{stuut}}{\xrightarrow{\sim}}} \mathrm{Top}(Z)$

$\rightarrow$ therefore these define equivalent sheaf datum.

- fix A Noeth. ring, let $D(\mathrm{Mod}_A) \underset{\substack{\text{full}\\ \text{subcat}}}{\supseteq} D_{coh}(\mathrm{Mod}_A)$ of coherent cohomology

Exercise $E \to B \to C \xrightarrow{+1}$ dist $\Delta \in D(\mathrm{Mod}_A)$. If $E, B \in D_{coh}(\mathrm{Mod}_A)$, so is C.

Def. $E \in D(A)$ perfect in $[a,b]$ if isom to some $[\tilde{E}_a \to \cdots \to \tilde{E}_b]$ of free finite rank A-mods.

Rmk. E perfect in $[a,b] \Rightarrow E \in D_{coh}^{[a,b]}(\cdot\,\cdot)$

- $\_ \otimes_A M$ is rexact in Mod_A, which has enough projs
$\Rightarrow \_ \overset{L}{\underset{A}{\otimes}} M$ is $D^-(\mathrm{Mod}_A)$ endofunctor
- if $N \in \mathrm{Mod}_A$, view it as cpx. conc. in 0 and then $\mathrm{Tor}_{-i}^A(N, M) = h^i(N \overset{L}{\underset{A}{\otimes}} M) \simeq h^i(M \overset{L}{\otimes} N)$

Cor E pfct in $[a,b] \Rightarrow \forall i \notin [a,b], \forall M \in \mathrm{Mod}_A$, $h^i(E \overset{L}{\underset{A}{\otimes}} M) = 0$

Thm Let A Noeth. ring, $\mathcal{E} \in D^{[a,b]}_{coh}(Mod_A)$.

TFAE

1) $\mathcal{E}$ perfect in $[a,b]$

ii) $\forall M \in Mod_A$, $i \notin [a,b]$, $h^i(\mathcal{E} \overset{L}{\otimes} M) = 0$

iii) for any cpx $\cdots \to F^{b-1} \to F^b \to 0 \cdots$

where F^i loc. free of fin. rank

isom. to $\mathcal{E}$ in $D^b(Mod_A)$,

$G := \operatorname{coker}(F^{a-1} \to F^a)$ is loc. free of f.r.