Antonini.

Quick recap.

- let H be Hilbert space, separated (i.e. $\cong \ell^2(\mathbb{N})$),
  our inner product is $\langle \cdot | \cdot \rangle$ ($\mathbb{C}$-linear, antilinear)

Thm (Riesz-Frechet) φ bounded lin. functional,
$\exists! \, \xi \in H$ s.t. $\varphi(\eta) = \langle \eta | \xi \rangle$, $\|\xi\| = \|\varphi\|$

- so we get $H \to H^*$
  1) norm-preserving
  11) antilinear

- our goal will be to construct the adjoint

Def. sesquilinear forms $(\cdot | \cdot) : H \times H \to \mathbb{C}$ (lin., antilin.)

- bounded if $\exists k$ s.t. $|(x, y)| \leq k \|x\| \cdot \|y\| \quad \forall x, y \in H$
- its norm is the smallest such k

- there is a 1-1 correspondence:

bounded sesquil. forms $\longleftrightarrow$ bounded lin. operators
$(\cdot | \cdot) : X \times Y \to \mathbb{C}$. $\quad T : X \to Y$, $T \in \mathcal{L}(X, Y)$

given by $(\xi, \eta) = \langle T\xi, \eta \rangle$ (inner product on Y)
such that $\|T\| = \|(\cdot | \cdot)\|$

- fix $T \in \mathcal{L}(X, Y)$. We construct $T \mapsto T^* \in \mathcal{L}(Y, X)$ such that $\langle T^* \eta, \xi \rangle_X = \langle \eta | T\xi \rangle_Y$

- given just H, $\mathcal{L}(H, H) = \mathbb{B}(H)$, we have a map $\mathbb{B}(H) \longrightarrow \mathbb{B}(H)$

$$T \longmapsto T^*$$

such that

i) $\|T\| = \|T^*\|$ isometric
ii) $(\alpha T)^* = \bar{\alpha} T^*$
iii) $T^{**} = T$
iv) $\|T^* T\| = \|T\|^2$ C^*-identity

- $\operatorname{Ker} T^* = (\operatorname{Range} T)^\perp$

- C^*-id follows from $\|T\| = \sup_{\|\xi\| = \|\eta\| = 1} \langle T\xi, \eta \rangle$

- now let $P, T \in \mathbb{B}(H)$ s.t.:
 i) T selfadjoint, $T = T^*$
 ii) P projection, $P = P^* = P$
 iii) isometry $T^* T = I$
 iv) unitary $T^* T = T T^* = I$
 v) normal

Principles of func. analysis

① uniform boundedness.
- X, Y Banach spaces
- $\mathcal{T} \subset \mathcal{L}(X, Y)$. If $\{\|T\xi\| \mid T \in \mathcal{T}\}$ is bounded $\forall \xi$, then $\mathcal{T}$ is bounded

② open mapping theorem

- X, Y Banach, $T \in \mathcal{L}(X, Y)$
- if T maps X <u>onto</u> Y, then it is open

③ closed graph.

- X, Y Banach, $T : X \to Y$ linear map
- then, if $\mathrm{graph}(T) := \{ (\xi, T\xi) \in X \times Y \mid \forall \xi \in X \}$ is closed $\Rightarrow$ T is bounded

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- H, $T \in \mathbb{B}(H)$
- $P \in \mathbb{C}[x] \rightsquigarrow P(T) \in \mathbb{B}(H)$
- recall spectrum $\sigma(T) : \{ \lambda \in \mathbb{C} \mid \lambda - T \text{ not invertible} \}$
- then $\sigma(P(T)) = \{ P(\lambda) \mid \lambda \in \sigma(T) \}$
  - just use fund. thm. of algebra to write
    $$P(T) - \lambda I = \alpha (T - \mu_1 I) \cdots (T - \mu_r I)$$

- for $T \in \mathbb{B}(H)$, define numeric range
  $$W(T) := \{ (T\xi, \xi) \mid \|\xi\| = 1 \}$$
  - then $\sigma(T) \subset \overline{W(T)}$

- if T selfadj., $\sigma(T) \subset \mathbb{R}$, so define
  $$\mathbb{R} \ni \begin{matrix} m = \inf W(T) \\ M = \max W(T) \end{matrix}$$
  - then, $\sigma(T) \subset [m, M]$, $m, M \in \sigma(T)$
  - also note $\|T\| = \sup \{ |\lambda| \mid \lambda \in W(T) \}$
- recall (real) Stone-Weierstraß:
  polynomials w/ real coefficients are dense in the (Banach) space $\mathcal{C}([m, M], \mathbb{R})$ in the sup norm

polyn w coeff $\in \mathbb{R}$

uniformly by S-W

- take $g_n \xrightarrow{} g \in \mathcal{C}([m,M],\mathbb{R})$,
  then $g_n(T) \to g(T)$ uniformly in $\mathbb{B}(H)$

- we constructed an algebra morphism
  $\mathcal{C}(\sigma(T)) \xrightarrow{\mathcal{F}} \mathbb{B}(H)$
  i) can be extended to $\mathcal{C}(\sigma(T),\mathbb{C})$ (and from now on we do this)
  ii) $\mathcal{F}(\mathcal{C}(\sigma(T))) \subset \mathbb{B}(H)$
  $\bigcap$
  $\{T\}''$ ⟵ double commutant

  iii) $\forall f \in \mathcal{C}(\sigma(T))$, $\sigma(f(T)) = f(\sigma(T))$
  spectral mapping theorem

Positive operators

- write $T \geqslant 0$ if $\langle Tx, x\rangle \geqslant 0 \ \forall x \in H$ $(T \in \mathbb{B}(H))$
- $T \geqslant 0 \Rightarrow T = T^*$ (use polarization identity
  $$\langle x,y\rangle = \frac{1}{4}\sum_{\substack{z\in\mathbb{C}\\ z^4=1}} z\|x+zy\|^2 \ )$$

- for such T $\exists$ continuous map
  $$\mathbb{R}_+ \longrightarrow \mathbb{B}(H)$$
  $$\alpha \longmapsto T^\alpha$$
  such that $T^\alpha \cdot T^\beta = T^{\alpha+\beta}$, $T^1 = T$, T^α positive
- in particular $T^{1/2}$ is the unique positive square root of T
- a bit abstract, but e.g. if $\|T\| < 1$, $\sigma(T) \subset [0,1]$
  and let $\{P_n\}$ polynomials s.t. $P_0 = 0$, $P_{n+1}(t) = P_n(t) + \frac{1}{2}(t - P_n(t))^2$
  $\to$ then $P_n \to t^{1/2}$

- an operator U is a partial isometry if $\exists\, N \subset H$ closed s.t. $\begin{cases} U|_N \text{ is norm-preserving} \\ U|_{N^\perp} = 0 \end{cases}$

- it follows that $N = \overline{\text{range } U^*}$, $\operatorname{Ker} U = N^\perp$

- now $P := U^* U \Rightarrow P : H \to H$ is orth. proj to N

- conversely if $U^* U := P$ is a projection then U is partial isom. with $N = U(H)$
- how to show?
- for $x \in N$, $\langle Px | x \rangle = \langle Ux, Ux \rangle = \|Ux\|^2 = \|x\|^2$ and clearly $Px = 0$ for $x \notin N$

Thm (Polar decomposition, von Neumann)

1) $T \in \mathbb{B}(H)$. $\exists!$ positive $|T| \in \mathbb{B}(H)$ s.t. $\|Tx\| = \|\,|T|x\|$, furthermore $|T| = (T^* T)^{1/2}$
ii) $\exists!$ partial isometry U s.t.
   - $\operatorname{Ker} U = \operatorname{Ker} T$
   - $T = U|T|$
   - $U^* U |T| = |T|$

     $U^* T = |T|$

     $U U^* T = T$

- take field $K = \overline{K}$ (in particular $K = \mathbb{C}$),
  let A algebra over K, unital ($1 \in A$)
- for $\lambda \in K$, write $\lambda \cdot 1 =: \lambda \in A$
- label $A^{-1} = \{$ inv. elements $\in A \}$

- for $x \in A$, define spectrum $Sp_A x := \{\lambda \in K \mid (x - \lambda) \notin A^{-1}\}$

- if A not unital, $\exists$ an algebra $\tilde{A}$ called
  the unitalization, such that, only as vect. sp.
  $\tilde{A} = A \times K$, and $(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda \mu)$
  $\rightarrow$ its unit is $(0, 1) = 1_{\tilde{A}}$
  - furthermore $\exists$ embedding of algebras

$$A \xhookrightarrow{i} \tilde{A}$$
$$a \longmapsto (a, 0)$$

  - identify $A \sim i(A)$
  $\tilde{A} \ni (a, \lambda) = i(a) + \lambda (0, 1) =: a + \lambda$
- we define $Sp'_A a := Sp_{\tilde{A}} a$

- note that $0 \in Sp'_A a$ for any $a \in A$
  because $(a, 0)(a', \lambda) = (aa' + \lambda a, 0) \neq (0, 1)$

- if A already unital w. $e \in A$ unit,
  then $\exists$ canonical iso $\tilde{A} \xrightarrow{\pi} \underbrace{A \times K}_{\text{as algebras}}$

- in this case $Sp'_A a = Sp_A a \cup \{0\}$

- $\pi(a, \lambda) = (a + \lambda e, \lambda)$