Stoppa.

Prop. (M, g) Kähler, general type $(c_1(M)<1)$. Fix $\omega_0 \in -2\pi c_1(M)$ as a reference Kähler form. Then $\exists F \in C^\infty(M, \mathbb{R})$ such that for $\omega_0 + i\partial\bar\partial\varphi > 0$, if

$$(\omega_0 + i\partial\bar\partial\varphi)^n = e^{\varphi+F}\omega_0$$

holds, then $Ric(\omega_0 + i\partial\bar\partial\varphi) = -(\omega_0 + i\partial\bar\partial\varphi)$, i.e. $\omega_\varphi := \omega_0 + i\partial\bar\partial\varphi$ satisfies $Ric(\omega_\varphi) = -\omega_\varphi$ so ω_φ is Einstein with constant $= -1$.

Pf. Suppose the conditional of the claim hold. The condition on top forms implies $\det g_\varphi = e^{\varphi+F}\det g$, locally. Locally we then also have
$-i\partial\bar\partial \log\det g_\varphi = -i\partial\bar\partial(\varphi+F) - i\partial\bar\partial\log\det g$
but we know this expression defines the Ricci form, a global object, i.e. $Ric(\omega_\varphi) = -i\partial\bar\partial(\varphi+F) + Ric(\omega_0)$
The assumption $\omega_0 \in -2\pi i\, c_1(M)$ gives $[Ric(\omega_0)] = -[\omega_0]$

$$\Rightarrow \quad Ric(\omega_0) = -\omega_0 + i\partial\bar\partial\tilde F, \tilde F \in C^\infty(M, \mathbb{R})$$

For $F = \tilde F$,

$$Ric(\omega_\varphi) = -i\partial\bar\partial\varphi - \omega_0 = -\omega_\varphi \quad \square$$

Rmk/exercise. Show converse is true, up to normalisation of φ.

Uniqueness. There is at most one soln
to $Ric(\omega) = -\omega$, $[\omega] = -2\pi i c_1(M)$

- pick some reference pt ω_0, $[\omega_0] = -2\pi i c_1(M)$
  such that $Ric(\omega_0) = -\omega_0$
- then pick $F \equiv 0$ in previous Prop.
- the $\partial\bar\partial$-lemma says that if ω another
  solution, ω is $\omega_\varphi = \omega_0 + i\partial\bar\partial\varphi$
  → Rmk/ex said φ can be chosen
  s.t. $(\omega_0 + i\partial\bar\partial\varphi)^n = e^{\varphi}\omega_0^n$
- now apply maximum principle (remember
  M is cpt), so pick $p \in M$ s.t. it is
  a maximum of φ ⇒ $Hess_p \varphi \leq 0$
  ⇒ $i\partial\bar\partial\varphi$ is negative semidefinite at p
  → so $\omega_0 + i\partial\bar\partial\varphi$ is "smaller" than ω_0
  → more precisely $\det g_\varphi|_p \leq \det g_0|_p$
  and we had $\det g_\varphi|_p = e^{\varphi(p)} \det g_0|_p$
  ⇒ $\varphi(p) \leq 0$ ⇒ $\sup \varphi \leq 0$
  Repeat argument at minimum of φ to get
  $\inf \varphi \geq 0$ ⇒ $\varphi \equiv 0$. □

Sketch of Pf of Aubin-Yau.

- we want to prove

Thm ∃ solution of $(\omega_0 + i\partial\bar\partial\varphi)^n = e^{\varphi + F}\omega_0^n$.

- idea: continuity method

- look at family $(*)_t$ $\begin{cases} (\omega_0 + i\partial\bar\partial\varphi_t)^n = e^{\varphi_t + tF}\omega_0^n \\ \omega_0 + i\partial\bar\partial\varphi_t > 0 \end{cases}$, $t \in [0,1]$

- define $A := \{t \in [0,1] \mid (*)_t \text{ has solution}\}$
  $\rightarrow$ now if A nonempty, open, closed $\Rightarrow A = [0,1]$
  so $(*)_1$, the required sol'n, exists
- nonempty: $t=0$ solved by $\varphi_0 \equiv 0$.
- open?