G + P seminar

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- we call $\tau(x)$ a τ-function if it satisfies the Hirota equations
$$[b \pm c]\,\tau(x \pm \delta a) + [c \pm a]\,\tau(x \pm \delta b) + [a \pm b]\,\tau(x \pm \delta c) = 0$$
where a, b, c vectors on Fcc root lattice i.e. c_3-frame, $\delta \in \mathbb{C}$ constant, $f(x \pm y)$ shorthand for $f(x+y)f(x-y)$ and $[z] = \vartheta(z;p)$, $|p|<1$

- special functions:
  1. theta functions
  $$\vartheta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty \text{ where}$$
  $$(z;p)_\infty = \prod_{j=0}^{\infty} (1-p^j z), \quad |p|<1$$
  $\rightarrow$ satisfies $\vartheta(pz;p) = -z^{-1}\vartheta(z;p) = \vartheta(z^{-1};p)$
  2. elliptic gamma funcs
  $$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^j q^k}, \quad |p|,|q|<1$$
  3. 3-ple gamma func $\Gamma(z;p,q,r) = \prod_{ijk=0}^{\infty} (1-zp^iq^jr^k)(1-z^{-1}p^{i+1}q^{j+1}r^{k+1})$

- consider hyperplane $H_{c+n\delta} = \{x \in \mathbb{C}^\infty \mid (\varphi, x) = c + n\delta\}$ with $\varphi = \frac{1}{2}(v_0 + \dots + v_7)$
$\{v_i\}$ basis of $\mathbb{C}^\infty$, $n \in \mathbb{Z}$, $c, \delta \in \mathbb{C}$ const.
- put $D = \bigcup_{n \in \mathbb{Z}} H_{c+n\delta}$
- set $p = e^{2\pi i \tau}$, $q = e^{2\pi i \delta}$, $c = \tau$, $\mathrm{Im}(\tau), \mathrm{Im}(\delta) > 0$

- the τ-function $T(x)$ is an infinite collection of τ-functions with $\tau^{(n)}(x) := \mathfrak{T}(x)|_{H_{\tau+n\delta}}$
- set $T^{(n<0)}(x) = 0$

$$T^{(0)}(x) = \prod_{0\le i<j\le 7} \Gamma(q\, u_i u_j; p, q, q), \quad u_i = e^{2\pi i x_i}, \; x_i \in H_\tau$$

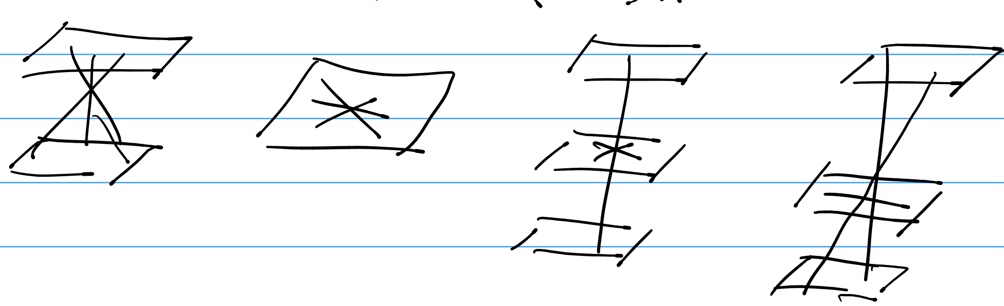
$$\tau^{(1)}(x) = I(\underline{u}; p, q) \prod_{0\le i<j\le 7} \Gamma(u_i u_j; p, q, q),$$

$$I(\underline{u}; p, q) = \oint_{|z|=1} \frac{\prod_{i=0}^{7} \Gamma(a_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}$$

is an elliptic hypergeom. function

- $\exists$ 4 types of C_3-frames:

$(\mathrm{I})_n$ $\quad(\mathrm{II}_0)_n$ $\quad(\mathrm{II}_1)_n$ $\quad(\mathrm{II}_2)_n$



- $W(E_7)$ is generated by S_8 (coordinate permutations on $\mathbb{C}^8$) + reflections wrt $\alpha = \varphi - v_0 - v_1 - v_2 - v_3$:

$$S_{\alpha_0}(v) := v - (\alpha_0|v)\alpha_0, \; v \in \mathbb{C}^8, \; v = \sum v_i \ldots$$

$$\underset{S_{\alpha_0}}{S_0}(x_i + x_j) = \begin{cases} x_i + x_j, & i \in \{0, \ldots, 3\}, j \in \{4, \ldots, 7\} \\ x_i + x_j + (\varphi|x) - x_0 - x_1 - x_2 - x_3 \end{cases}$$

- for $T^{(0)}(x)$, $x \in H_\tau$, $(\varphi|x) = \tau$

$S_0(x_i + x_j) = -x_k - x_l + \tau$

$S_0(u_i u_j) = p/u_k u_l$

$$S_0\left(\Gamma(q u_i u_j; p\, q\, q)\right) = \Gamma\left(\frac{pq}{u_k u_l}; p\, q\, q\right)$$
$$= \Gamma(q u_k u_l; p\, q\, q)$$

- since T^0 sums over (ij), $S_0 T^0 = T^0$

- now we can show $T^{(0)}$ satisfies Hirota