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Why do some K-th gps of C^*-algs have Bott periodicity, and some not?

- alg. K-th, no ✗
- usual K-th, yes ✓
- KK-th, yes ✓
- ..., yes ✓

} homotopy.

- Bott:

	π_1	π_2	π_3	π_4	...
S^0					
S^1	$\mathbb{Z}$	$\mathbb{Z}_2$			
S^2					
S^3	0				

- consider $GL_n \longrightarrow GL_{n+1}$

$$\begin{pmatrix} & \end{pmatrix} \mapsto \begin{pmatrix} \ast & \vdots \\ & 0 \\ \cdots 0 & 1 \end{pmatrix}$$

- Bott's 2nd result: $\pi_p(Gl_n) \simeq \pi_{p+2}(Gl_n)$
$n > \frac{p+2}{2}$

- $GL_\infty(\mathbb{C}) = \varinjlim GL_n(\mathbb{C})$

$$GL_1 \to GL_2 \to \cdots$$

- $M_\infty(\mathbb{C}) = \varinjlim M_n(\mathbb{C})$ where we extend by zeroes

- unital ring A, $M_\infty(A)$

- projections, $P = P^2$ ($= P^*$, not talking abt Banach now)

- look at homotopy classes of projections in $M_\infty(A)$
  - semigroup under direct sum
  - given semigp $(M,+)$, Groth. gp $G(M)$ $= \{[(a,b)] \in M \times M/\sim \mid (a,b) \sim (c,d)$ if $\exists e$ s.t. $a+d+e = b+c+e\}$
- $K_0(A) = G($projs in $M_\infty(A))$

Suspensions, higher K-th gps

- A C^*-alg, suspension $SA = C_0(\mathbb{R}) \otimes A$
- $K_n(A) = K_0(S^n A)$

$\Rightarrow K_2(A) = K_0(A)$ from def involved in homotopy and Bott's $\pi_p(GL_\infty) = \pi_{p+2}(GL_\infty)$

- note, A cpx: $SA = C_0(\mathbb{R}, \mathbb{C}) \otimes A$
  A real: $SA = C_0(\mathbb{R}, \mathbb{R}) \otimes A$
  - in this case,
    $K_0(S^8 A) = K(A)$

Rmk $K_0(\tilde{A}) = K_0(A) \oplus \mathbb{Z}$, $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$
unitization
- rmk by Landi: $K_0(\mathbb{R}^2 \simeq \mathbb{C}) = \mathbb{Z}$, trivial vbdls, $K_0(\tilde{\mathbb{C}} \simeq S^2) = \underset{\text{triv.?}}{\mathbb{Z}} \oplus \underset{\text{Bott?}}{\mathbb{Z}}$

Clifford alg.

- $KK(A,B)$, $KK^n(A,B) = KK(C_n \otimes A, B)$

Real K-th as in literature

- A real C^*-alg, $A_{\mathbb{C}} = A \otimes \mathbb{C}$, τ antiautomorph. invol.
- define K-th gps through unitaries in $A_{\mathbb{C}}$
  - condition on unitaries: $u^* = u$
  - $KU_0(A)$, $KU_1(A)$
  - $\#$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ adjugation
  - KO_0 $\quad u = u^*$, $u^\tau = u^*$

most natural $\}$ $KO_1 = \{u^\tau = u^*\}$ = unit. classes in real part

KO_2 $\quad u = u$, $u^\tau = -u$

KO_3 $\quad u^{\tau \otimes \#} = u$

KO_4 $\quad u = u^*$, $u^{\tau \otimes \#} = u^*$

KO_5 $\quad u^{\tau \otimes \#} = u^*$

KO_6 $\quad u = u^*$, $u^{\tau \otimes \#} = -u$

KO_7 $\quad u^\tau = u$