Gauge @ IGAP

q-Virasoro & partition functions - M. Zabzine

- 1810.00761, 1909.10352

- partition functions
  - Nekrasov on $\mathbb{R}^d \times S^1$
  - on S^d

$\rightsquigarrow$ generating functions
$Z(t_1, t_2, \ldots)$
- due to BPS/CFT, look like formal matrix models
- constraints:

$$\left[\begin{array}{c} \mathbb{R}^2 \times S^1 \\ S^3 \end{array}\right] \to q\text{-Virasoro}$$

difference op.

$$T_n Z(t) = 0, \quad n \geq 1$$

Q.)
- can we solve constraints?
- when/why can we do so?

"I don't mean in a bullshit way, I mean really solve and get numbers" :)

Hermitian mtx models

$$Z_N(t) = \int_{u(N)} d\varphi \, e^{\sum_{s=0}^{\infty} t_s \mathrm{Tr}(\varphi^s)} = \int_{\underset{\text{Cartan}}{\mathbb{R}^N}} d^N\lambda \, \overbrace{\prod_{i\neq j}(\lambda_i - \lambda_j)}^{e^{\sum_{i\neq j} \log(\lambda_i - \lambda_j)}} e^{\sum_{s=0}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s}$$

$$\to \sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \lambda_i^{n+1} \rightsquigarrow L_n Z(t) = 0, \; n \geq -1$$

$$L_n = \sum_{s=0}^{n} \frac{\partial^2}{\partial t_s \partial t_{n-s}} + \sum_{s=0}^{\infty} s t_s \frac{\partial}{\partial t_{s+n}}$$

- free field realisation

$$\int_C :e^{\varphi(\lambda)}: d\lambda = J, \quad [L_n, J] = 0$$

$$Z_N(t) \sim J^N |0\rangle$$

- e.g. $Z_N(t) = \int d\varphi \, e^{-\frac{1}{2}\mathrm{tr}\,\varphi^2 + \sum_{s=1}^{\infty} t_s \mathrm{tr}\,\varphi^s}$

$$= e^{\hat{W}_{-2}} e^{N t_0}$$

and $\int d\varphi \, e^{-\frac{1}{2}\mathrm{tr}\,\varphi^2} = 1$, $\frac{\partial}{\partial t_0} Z_N = N Z_N$

- solution around such Gauß pt $\exists!$ for Virasoro constraints
  - $\to$ but for e.g. $\int d\varphi \, e^{-\frac{1}{2}\mathrm{tr}\,\varphi^4 + \sum t_s \mathrm{Tr}(\varphi^s)}$??
    - not really, blind spot here

- but φ - Vir $\Rightarrow$ yes!, at least larger class of soln's

Q. (Tanzini) Usually difference eqns give solutions up to periodic ones

A. (Zabzine) Unique here! We specify algebraic conditions.

q-Virasoro:

$$[T_n, T_m] = \sum_{l>0} f_l \left(T_{n-l} T_{m+l} - T_{m-l} T_{n+l}\right) - \frac{(1-q)(1-t^{-1})}{(1-p)} \left(p^n - p^{-n}\right) \delta_{n,-m}$$

$q, t \in \mathbb{C}$, $t = q^\beta$, $p = q t^{-1}$

• if $q = e^{\hbar}$, β fixed:

$$T_n = 2\delta_{n,0} + \hbar^2 \beta \left(L_n + \frac{Q_\beta^2}{4} \delta_{n,0}\right) + O(\hbar^4)$$

$\xrightarrow{\text{free-field realisation}}$ $[a_n, a_m] = \frac{1}{n} \delta_{n+m} \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}}\right)(q \leftrightarrow t)(q \leftrightarrow p)$

$S(w)$, $T(z) = \sum T_n z^{-n}$

$$[S(w), T_n] = \frac{O_n(qw) - O_n(w)}{w}$$

• further:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1) z}$$

$$\int dz \frac{d}{dz}(\ldots) = 0 \rightsquigarrow \int_{\mathbb{R}^n} d_q x \, D_q(\ldots) = 0$$

$\Rightarrow$ but also, $\int d_q x \underbrace{\lambda_q(x)}_{\text{any } q\text{-constant}} D_q(\ldots) = 0$

e.g. ϑ-funct ratios

- Jackson integral: $\int d_q x \, f(x) = \sum_{n \in \mathbb{Z}} q^n f(q^n x)$

• $J = \int S(w)$, $Z = J^N |0\rangle$

$$Z_{D^2\times S^1} = \oint_{\mathcal{C}} \prod_{i=1}^{N} \frac{dx_i}{x_i} \prod_{k\neq l}^{N} \frac{(x_k/x_l; q)_\infty}{(t\, x_k/x_l; q)_\infty} \prod_{j=1}^{N} \prod_{f=1}^{N_f} \frac{(q x_j \bar{m}_f; q)_\infty}{(x_j m_f; q)_\infty}\, e^{\sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} x_i^s}$$

$\mathcal{N}=2$ vector + adj. matter

$t \sim e^{M_{adj}}$

Answer $N_f = 2$, $\bar{m}_f \sim f(q)\; q(1-q)^{1/2}$

$$D_q e_q[x] = e_q[x] \;,\quad e_q[x] = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \prod_{n=0}^{\infty} \left(1-(1-q)q^k x\right)^{-1}$$