Bruzzo.

- S graded ring, $S = \bigoplus_{d \in \mathbb{N}} S_d$, $S_i S_j \subseteq S_{i+j}$,
S generated by S_0, S_1
- we also defined $\operatorname{Proj} S$, $S_{(f)} = (S_f)_{(0)}$ for $f \in S$
and $D_+(f) = \operatorname{Spec} S_{(f)}$
- M graded S-module, $\tilde{M}$ an $\mathcal{O}_X$-module.
- $S(n) = \bigoplus_{d \geq n} S_d$, $\widetilde{S(n)} =: \mathcal{O}_X(n)$
  - fact: $\mathcal{O}_X(n)$ is locally free of rank 1.
    - to check: i) $\mathcal{O}_X(n)|_{D_+(f)} \simeq \widetilde{S(n)_{(f)}}$
      ii) $S(n)_{(f)} \simeq S_{(f)}$
  - e.g. $X = \operatorname{Proj} \mathbb{k}[x_0, \dots, x_n] = \mathbb{P}^n_{\mathbb{k}}$, $\overline{\mathbb{k}} = \mathbb{k}$
    $\to U_i := D_+(x_i)$
    $\to \mathcal{O}_X(1)|_{U_i}$ generated by x_i
    so $g_{ij} \in \mathcal{O}_X^*(U_i \times_X U_j)$, $g_{ij} = \frac{x_j}{x_i}$

- look at $H_i = \{x_i = 0\}$ prime divisors, $i = 0, \dots, n$,
but not principal since rational fns have 0 degree.
$\to H_i - H_j = \left(\frac{x_i}{x_j}\right) \Rightarrow H_i \sim H_j \Rightarrow$ write $H = [H_i]$ in $\operatorname{Cl}(\mathbb{P}^n_{\mathbb{k}})$
- recall $D = \sum n_i Y_i$ divisor where $\{Y_i\}$ integral
codimension 1 subschemes, $\deg D = \sum n_i \deg D_i$

Prop. a) $[D] = dH$, $d = \deg(D)$
b) $\deg(f) = 0 \quad \forall f \in K^*$
c) $\deg : \operatorname{Cl}(\mathbb{P}^n_{\mathbb{k}}) \to \mathbb{Z}$ iso

Pf. a) $D = D_1 - D_2$, $D_{1,2}$ effective, $D_{1,2} = (g_{1,2})$
Since irred. hypersurfaces in $\mathbb{P}$ correspond to prime ideals
of height 1, principal. So $D - dH = \left(\frac{g_1}{x_0^d g_2}\right) = 0$, $d = \deg D_1 - \deg D_2$

- $\mathcal{O}_{\mathbb{P}^n_k}(H) = \mathcal{O}_{\mathbb{P}^n_k}(1)$, $\mathcal{O}_{\mathbb{P}^n_k}(D) = \mathcal{O}_{\mathbb{P}^n_k}(d)$, $d = \deg D$,
  $\mathcal{O}_{\mathbb{P}^n_k}(-1) = \mathcal{O}_{\mathbb{P}^n_k}(1)^\vee$

Kähler differentials.

- set A ring, B A-algebra, M B-module,
  A, B commutative with unit

Def. An *A-derivation of B into M* is a map
$d: B \to M$ satisfying
i) $d(bb') = b\,db' + b'\,db$, $b, b' \in B$
ii) $da = 0$, $a \in A$

Def. A *module of relative differentials* $\Omega_{B/A}$
is a B-module with an A-derivation $d: B \to \Omega_{B/A}$
such that any $d': B \to M$ factors through $\Omega_{B/A}$
uniquely.

- construct it as $\Omega_{B/A} = B\langle db \rangle / \tilde{\Omega}_{B/A}$,
  $\tilde{\Omega}_{B/A}$ generated by $d(b+b') - db - db'$, $d(bb') - b\,db' - b'\,db$,
  da, put $f(db) = d'b$.

- now consider $B \otimes_A B \xrightarrow{m} B$
  $\to$ obviously surjective $(1 \otimes b \to b)$:
  $$0 \to I \to B \otimes_A B \xrightarrow{m} B \to 0$$

Claim: now $(I/I^2, d)$, $d: B \to I/I^2$ will be
a module of relative differentials
- $d: B \xrightarrow{\tilde{d}} I \twoheadrightarrow I/I^2$, $b \overset{\tilde{d}}{\mapsto} 1 \otimes b - b \otimes 1$
- ...

Example: M diff. mfd, $p \in M$, $m_p \subset C^\infty(M)$

$\Rightarrow m_p/m_p^2 \simeq T_p^* M$

$\rightarrow A \rightsquigarrow \mathbb{R}$, $B \rightsquigarrow C_p^\infty(M)$, $M \rightsquigarrow \mathbb{R} \Rightarrow \mathrm{Der}_{\mathbb{R}}(C_p^\infty, \mathbb{R}) \simeq T_p M$

Example: $f: X \to Y$ morphism of schemes

- not separated in general, but $\Delta(X)$ is locally a closed $\rightsquigarrow$ sheaf of ideals $\mathcal{J}$

$\Rightarrow \Delta^* \mathcal{J}/\mathcal{J}^2 \simeq \Omega_{X/Y}$

Example: $B = A[x_1, \dots, x_n] \rightsquigarrow \Omega_{B/A}$ free of rank n, generated by $dx_1, \dots, dx_n$

$\mathrm{Der}_A(B, B) \ni \frac{\partial}{\partial x_j} : x_i \longmapsto \delta_{ij}$

$\exists f_j : \Omega_{B/A} \to B$, $f_j(dx_i) = \delta_{ij}$

$0 = \sum P_i dx_i \Rightarrow f_j(\sum P_i dx_i) = P_j = 0$

Properties.

- A', B A-algebras, let $B' = A' \otimes_A B$
  Then $\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_B B'$.
  $\Omega_{B/A} \otimes_B B' \to \Omega_{B'/A'}$ given by $db \otimes 1 \mapsto d'(b \otimes 1)$
- B A-algebra, $I \subset B$ ideal, $C = B/I$. $\exists$ maps:
  i) $\Omega_{B/A} \otimes_B C \to \Omega_{C/A}$
  $db \otimes 1 \mapsto \bar{d}[b]$
  ii) $\delta : I/I^2 \to \Omega_{B/A} \otimes_B C$
  $[b] \longmapsto db \otimes 1$

  The sqn $I/I^2 \to \Omega_{B/A} \otimes C \to \Omega_{C/A} \to 0$ is exact.