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B. Kruglikov – Overdetermined systems of ODE-s

- coord-free jet spaces, $j^k_x s = [s]^k_x$, $s \sim s'$ if $s - s' \in m_x^{k+1} \subset C^\infty(E)$

- Cartan distribution, $C(a_k) = \langle L(a_{k+1}) \mid a_{k+1} \in \pi^{-1}_{k+1,k}(a_k) \rangle$
$\subset T_{a_k} J^k \pi$
for $a_{k+1} = j^{k+1}_x s$, $a_k = j^k_x s$

- geometrically, a dif. eqn. of order k is a submanifold $\mathcal{E}_k \subset J^k\pi$ which submerses on $J^{k-1}\pi$.
  -> prolongations of $\mathcal{E}_k = \{f = 0\}$ have the defining eqns $D_\tau f = 0$, $|\tau| = s$ = order of prolongation
- dif. operators -> module
  -> Koszul complex $0 \leftarrow M \leftarrow T \otimes M \leftarrow \Lambda T \otimes M \leftarrow \dots$
  -> dualizing over $\mathbb{R}$ => Spencer δ-cpx
    -> in each gradation $0 \to g_k \to g_{k-1} \otimes T \to \dots$
    with $g_i = \ker(\pi_{i,i-1} : T\mathcal{E}_i \to T\mathcal{E}_{i-1}) \subset S^i T \otimes F$

- interpretation of cohomology
  -> $H^{\bullet,0}(g)$ supported in grade 0, $H^{0,0}(g) = F$
  -> $H^{\bullet,1}(g)$ counts generators of module $g^\bullet$
    - for $\mathcal{E}$, $H^{i,1}(\mathcal{E})$ = # of defining dif. eqn. of order i
  -> $H^{\bullet,2}(g)$ counts compatibility conditions of $\mathcal{E}$
    - there is a sense in which compatibility = involutivity

- for non-compatible systems (involutive after proj.) we do prolongation-projection
- e.g. $L_X(g) = 0$ for geometric structure g.

- linearisation operator on $F[u] = F(j^\infty u)$,
$$\ell_{F,u}(v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F[u + \varepsilon v]$$

- write $\ell(f^i)(v) = \sum_{|\sigma| \le k} \sum_j P^{i,\sigma}_j v^j_\sigma$

put $P^i_j(p = (p_1, \dots, p_n) \in T^\vee M) = \sum P^{i,k}_j (p_k)$,
$$\mathrm{smbl}_f(p) = \begin{pmatrix} P^1_1(p) & \cdots & P^1_m(p) \\ \vdots & & \vdots \\ P^r_1(p) & \cdots & P^r_m(p) \end{pmatrix}$$
- ξ is (over) determined if $\dim F = m < r$

$$\mathrm{Char}(\Sigma; a_k) = \{ p \in T^\vee_x M \mid \mathrm{rk}(\mathrm{smbl}_f(p)) < m \}$$

- $M_\xi = g^\bullet$ and $\mathrm{Char}(\xi) = \mathrm{supp}\, M_\xi$
- finite type ($\dim g < \infty$) if $\mathrm{Char}^{\mathbb{C}}_{\mathrm{aff}}(\xi) = 0$ ($= \emptyset$ in proj. case)

Def. Cartan genre $d = \dim \mathrm{Char}^{\mathbb{C}}_{\mathrm{aff}}(\xi)$.
Cartan integer $z = \sum d_\xi \cdot \deg \Sigma_\xi$.
where $\mathrm{Char}^{\mathbb{C}}_{\mathrm{proj}}(\xi) = \cup_\xi \Sigma_\xi$, decomp into irred. comps.

Thm The local soln of Σ depends on d funcs of z parameters.