Cecotti

- we will be exploring only the lowest energy regime ($<<< 1$ eV) of all consistent quantum theories
- naively, we give description in terms of $\mathcal{L}_{eff}$

Rmk (sociology) Is there reeeally something fundamental hiding in $\mathcal{L}_{eff}$, having been studied for a century?

Fundamental fact: 4d sugra (any $\mathcal{N}$, matter content, couplings):

$$\tau(\varphi) = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2} \text{ is HARMONIC.}$$

(tamed by bosonic symmetry)

Rmk. Should be on first pg of any susy book. But isn't.

* $\mathcal{N}=1$ $\quad \int d^2\vartheta \, \tau(X) W_a W^a$
  - holomorphic $\Rightarrow$ harmonic
* $\mathcal{N}=2$ $\quad tt^* \Rightarrow$ pluriharmonic $\Rightarrow$ harmonic
* $\mathcal{N} \geq 3$ $\quad$ totally geodesic $\Rightarrow$ harmonic

$\hookrightarrow$ differences: the taming by bosonic symmetries, $U(N)_R$

- simply: the question has never been posed?
  - it makes little sense in QFT
- but we are now interested in it

Fundamental principle (Vafa)
"Going from QFT to QG requires a radical change of paradigm: QFT is like Diff. Geometry while QG is like Number theory."

Meta-physics (as in meta-mathematics)

- 2 categories: $\mathbb{QFT}$, $\mathbb{QG}$ (nonempty)

- $\exists$ a functor $m : \mathbb{QG} \to \mathbb{QFT}$, just sending $m_{\text{Planck}} \to \infty$

  - e.g. m(M-theory) = 11d sugra

- Q1) Describe $m(\mathbb{QG}) \subset \mathbb{QFT}$.
  - swampland
- Q2) Understand m.
  - clearly, information is lost
    - but $m_{Pl} \to \infty$ is a QFT perspective
  - $\to$ understand it in terms of QG?

- usual QFT book talk abt Lagrangian mechanics, dual to Hamiltonian, which corresponds just to symplectic geometry (quantization just small deform.)

- wishful thinking: $QG \xrightarrow{m} QFT$, $QG \xrightarrow{m'} \mathcal{C} \rightarrow QFT$

- $\mathcal{L}_{eff} = \dots + \frac{1}{2}\sqrt{g}\, G(\varphi)_{ij}\, \partial^\mu \varphi_i \partial_\mu \varphi_j + \dots$

susy, $U(1)_R$-symmetry,

and $\mathcal{M}$ is Kähler

- by our considerations $\exists$ a cat. $\mathcal{C}$ s.t.

number-theoretical flavor $\left\{ \begin{array}{l} \mathcal{C} \xrightarrow{f} \left(\begin{array}{c}\text{Kähler} \\ \text{mfds}\end{array}\right) \\ \cup \\ \mathbb{M} \end{array} \right.$

- and $\mathcal{M} = f(\mathbb{M})$

- we have

$\begin{array}{c}\text{alg. var.'s} \\ \text{over } \mathbb{Q}\end{array} \longrightarrow \begin{array}{c}\text{alg. var.'s} \\ \text{over } \mathbb{C}\end{array} \longrightarrow \begin{array}{c}\text{Kähler} \\ \text{mfds}\end{array}$

$\|$ ← naive guess!

$\mathcal{C}$

- take the consistent QG 10 Het $\xrightarrow[\substack{T^6\\ \text{cpt. on}}]{}$ 4d 16 supercharges,

$$\mathcal{M} = \Gamma \backslash SL(2,\mathbb{R}) / U(1) \times (\dots)$$

↳ finite $SL(2,\mathbb{Z})$ subgp

- here we have modularity, which QFT translates as S-duality
  → but this language is foreign to QG, where the analoguous fact is simply grammatical ...

- symm. in QFT ... group object G in (Smooth mfds)

- we guess ... G is a gp. object in ($\mathbb{Q}$-schemes)

Type II 10d $\xrightarrow{T^6}$ 32 superchanges

$\mathcal{M} = \Gamma \backslash \underbrace{\tilde{\mathcal{M}}}_{\text{univ cover}}$, $\Gamma \subset E_{7(7)}(\mathbb{R})$

Claim: $E_{7(7)}$ admits the structure of a Mumford-Tate gp.

Claim: Consider 4d sugra (any $\mathcal{N}$ matter blabla), and assume

$\mathcal{M} \simeq \Gamma_1 \backslash D_1 \times \Gamma_2 \backslash D_2 \times \cdots \times \Gamma_n \backslash D_n$,

$D_i = G_i(\mathbb{R})/H$

Then $G_i(\mathbb{R})$ are MT gps

- this is sugra-indep., but it is dimension-dependent
  → in 5d it is never true

- e.g. only for $d=4$, is $SO(d-1,1)$ actually $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{Aut}(\mathbb{P}^1)$

- Modular curves are constructed from Γ_N, arithmetic, but are only "1 d" cases of Shimura varieties

$\Gamma \backslash \overline{SL(2,\mathbb{R})/U(1)}$

↑
congruence

- classical case: $\Gamma \backslash G(\mathbb{R}) / H$ Hermitian symmetric space

$\Rightarrow$ Shimura varieties

- finally, more generally,

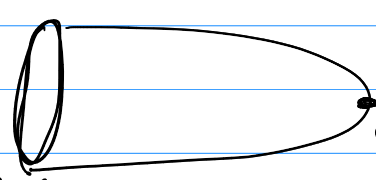
Mumford-Tate domains

$\equiv \Gamma \backslash \underbrace{G / H}_{\text{not neccessarily Herm. symm.}}$

Exercise $\mathcal{N}=3$ 4d susy, strict ($\mathcal{N} \neq 4$), are distinguished from $\mathcal{N}=4$ by having a smaller MT-gp.

- what about strings? We should "see" arithmetic on $\overset{\text{worldsheet,}}{\Gamma}$
- consider $\mathcal{N}=(2,2)$ 2d theory on V
  - lots of susy: $\frac{1}{2}$ BPS are labelled by single $\zeta \in \mathbb{P}^1$, twist param.
  - for branes, $B(\zeta)_a$, construct them as $[B(\zeta)_a] \in H^*(V, \mathbb{Z})$ (more precisely a relative coh., but unimportant)

- if $V \simeq B \times \mathbb{C}$,



$B(\zeta)_a$ φ_0

consider $\langle B_a(\zeta) | \varphi_\sigma \rangle$.

Gaiotto-Moore-Neitzke geometry

$$\begin{tikzcd}
\mathbb{P}^1 & \mathcal{M} \arrow[l] \arrow[d, bend right] & \\
& M \arrow[u, bend right] \arrow[d, "\pi"] & \nabla \arrow[l, "\text{hyperhol.}"'] \arrow[dl, "\sigma"] \\
& S &
\end{tikzcd}$$

M hyperkähler

hyperhol., meaning: holomorphic $\forall \zeta \in \mathbb{P}^1$ cpx. structures

Int. system

S Kahler

- hermitian conn ∇ of $\nabla \xrightarrow{\sigma} S$ is flat, $\nabla^2 = 0$, since $B_a(\zeta)$ is a lattice
  $\Rightarrow \nabla = \partial + i g \partial g^{-1}$
- so, $\langle B_a(\zeta) | \varphi_\sigma \rangle = -g_{ab}(\zeta, \bar{\zeta}) \in SL(n, \mathbb{R})$ due to $|\zeta|^2 = 1$
- $SL(n, \mathbb{R})$ not MT for $n \geq 3$, however.
- nice, but not really deep...
- reason? For Q_a, need $(2,2)$ SCFT

Claim: $(2,2)$ 2d QFT is supercont. $\Longleftrightarrow$ MT group

Lemma (Schur) Let $\mathcal{E}$ be $\mathbb{k}$-linear Krull-Schmidt category, S a simple object. Then $\mathrm{End}_{\mathcal{E}}(S)$ = skew field

- S corresponds to irreps for gps.
- e.g. $\mathbb{k} = \mathbb{C}$, skew field = $\mathbb{C}$ itself

  $\mathbb{k} = \mathbb{R}$, $\mathrm{End}\, S = \begin{cases} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H} \end{cases}$

  $\mathbb{k} = \mathbb{Q}$? ∞-ly many skew-fields