Fantechi

- Recall: $\mathcal{A}$ ab. cat, $M \in \mathrm{ob}\,\mathcal{A}$ proj if
  $\forall\ N \overset{\varphi}{\twoheadrightarrow} N'$ surj, given $M \to N'$ $\exists$ lift making commute

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[dr] & N \arrow[d] \\ & N' \end{tikzcd}$$

Rmk In Mod_A every free is proj

Cor In Mod_A, every obj is quot of a proj

Cor Given any cpx $M^\bullet \in \mathcal{C}^-(\mathrm{Mod}_A)$ $\exists$ $F^\bullet \to M^\bullet$ quiso s.t. each F^i is free, therefore proj.

Examples $A = \mathbb{C}[x,y]$, $M^\bullet = [\cdots 0 \to \mathbb{C} \to 0 \to \cdots] = [\mathbb{C}]$

$A/(x,y)$ mod. structure

- $\mathbb{A}^2$, $\mathrm{mod}_A = \mathrm{Qcoh}(\mathbb{A}^2)$
- $\mathcal{O}_p$, $p \in \mathbb{A}^2$ origin

$$0 \leftarrow \mathbb{C} \leftarrow A \leftarrow (x,y) \leftarrow 0$$

$(x,y) \cong I_p$, $m_p = I_p$

$$0 \leftarrow \mathcal{O}_p \leftarrow \mathcal{O}_{\mathbb{A}^2} \leftarrow \mathcal{I}_p \leftarrow 0$$

$$\dim_{\mathbb{C}}(I_p/I_p^2) = 2$$

$I_p/I_p^2 = I_p \otimes \mathcal{O}_p/m_p$

$$0 \leftarrow \mathbb{C} \xleftarrow{\alpha} A^{\oplus 2} \leftarrow A \leftarrow 0$$

$(xf + yg) \mapsfrom (f,g)$, $(y,-x)$

$xf + yg = 0 \iff xyf_1 = -xyg_1 \overset{\text{domain}}{\iff} f_1 = -g_1$ $\Big\} \Rightarrow (f,g) \in \ker\alpha$

$f,g \in \mathbb{C}[x,y]$ UFD $\Rightarrow$ $g = xg_1$, $f = yf_1$ $\quad (f,g) = (yf_1, -xf_1)$

Rmk i) special case of Koszul
ii) resol of len 2

Thm (Hilbert) A comm. ring, noeth, any A-mod. M has resln of length $\equiv \dim \operatorname{Spec} A$ (nonsing.)

- $0 \leftarrow I_p \leftarrow A^{\oplus 2} \leftarrow A \leftarrow 0$ also a resn.
- I_p has no torsion on nonsing. affine sfc, a coh sh. has len 1 free resl $\Leftarrow\Rightarrow$ torsion free

- $\mathrm{Tor}^i_A(M,M) = ?$ $\quad \mathrm{Tor}^A_i(-,M) = L^i(- \otimes_A M)$
$M = A/I_p$
$h^{-i}\left([F^{-2} \to F^{-1} \to F^0] \otimes_A M\right) = \mathrm{Tor}^A_i(\ ,M)$

$A^{\oplus s} \otimes_A N = N^{\oplus s}$, $[F] \otimes_A M = [M \xrightarrow[(-y,x)]{} M^{\oplus 2} \xrightarrow[(x,y)]{} M]$

$= [\mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C}]$, on M, mult by x,y is zero.
$\Rightarrow \mathrm{Tor}^i_A(M,M) = \begin{cases} \mathbb{C}, & i = 0,1,2 \\ 0, & \text{otherwise} \end{cases}$

- what about $\mathbb{P}^2$? can I construct loc. free resoln similarly of $\mathcal{O}_p \in \mathrm{Coh}_{\mathbb{P}^2}$
$0 \leftarrow \mathcal{O}_p \leftarrow \mathcal{O}_{\mathbb{P}^2} \leftarrow \mathcal{J}_p \leftarrow 0$
- $\mathrm{Hom}(\mathcal{O}_p, -)$:
$0 \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{J}_p) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_p)$
$0 \to \underset{\substack{\| \\ 0}}{\Gamma(\overset{\|}{\mathcal{J}_p})} \longrightarrow \underset{\substack{\| \\ \mathbb{C}}}{\Gamma(\mathcal{O}_{\mathbb{P}^2})} \to \Gamma(\mathcal{O}_p)$
$\mathbb{C} \xrightarrow{\sim} \mathbb{C}$

- want map $\underset{\text{loc. free}}{\mathcal{F}} \to \mathcal{O}_{\mathbb{P}^2}$ whose img is (x,y)

$x, y \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))$

$$\| $$

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2})$$

$\Rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow[(x,y)]{} \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_p \to 0$

extend $\Downarrow$ to

$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_p \to 0$

<u>Example</u> $C \hookrightarrow \mathbb{P}^3$ rat. normal / twisted cubic

image of 3rd deg Veronese $\mathbb{P}^1 \to \mathbb{P}^3$

$(s,t) \mapsto (s^3, s^2t, st^2, t^3)$

- compute loc. free resn of $\mathcal{O}_C \in \mathrm{Coh}_{\mathbb{P}^3}$

$I_C = \{ \mathrm{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 1 \} = (Q_1, Q_2, Q_3)$

where $Q_1 = \det\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}$, $Q_2 = \det\begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \end{pmatrix}$, $Q_3 = \ldots$

$Q_i \in \mathbb{C}[x_0, \ldots, x_3]$ hom. poly. of deg 2

$0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_{\mathbb{P}^3} \xleftarrow[(Q_1,Q_2,Q_3)]{\alpha} \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 3}$

- what is $\ker \alpha$?

- f s.t. $(Q_2 f, -Q_1 f, 0) \in \ker \alpha$ (koszul)

$\det\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix} = 0 \ \forall x_0, \ldots, x_3$

$\Rightarrow x_1 Q_3 + x_2 Q_2 + x_3 Q_1 = 0$

$x_0 Q_3 + x_1 Q_2 + x_2 Q_1 = 0$ relations

$$0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_{\mathbb{P}^3} \xleftarrow{(Q_1,Q_2,Q_3)} \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \xleftarrow{\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}} \mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 2} \leftarrow 0$$

$$f, g \mapsto \begin{pmatrix} x_0 f + x_1 g \\ x_1 f + x_2 g \\ x_2 f + x_3 g \end{pmatrix}$$

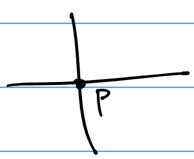
- check!
- useful "criterion":
  - pick pt $\notin C$ in $\mathbb{P}^3$
    - fiber in $\mathcal{O}_C$ is zero,

if $f = x_1 h$, $g = -x_0 h$
$\Rightarrow (x_1^2 - x_0 x_2) h = 0$

so we get $0 \leftarrow \mathcal{O}_{\mathbb{P}^3} \leftarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \leftarrow \mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 2} \leftarrow 0$
finite sqn of vector spaces
$\Rightarrow \sum (-1)^i \operatorname{rk} V_i = 0$, which is true here

- $\exists$ finite free resn in nonsing affine case by Hilbert syzygy thm

$C = \operatorname{Spec} \mathbb{C}[x,y]/xy$, $A = \dfrac{\mathbb{C}[x,y]}{xy}$



$M = A/(x,y)$, $\tilde{M} = \mathcal{O}_P$

- rmk: as v.sp., $A = \mathbb{C} \oplus x\mathbb{C}[x] \oplus y\mathbb{C}[y]$

$$0 \leftarrow M \leftarrow A \xleftarrow[(x,y)]{\alpha} A^{\oplus 2}$$

- $\ker \alpha$? f, g s.t. $xf + yg = 0$ in A?
  $\to f, g \in \mathbb{C}[x,y]$, so $(xf + yg)$ multiple of $xy \Rightarrow (y \mid f) \wedge (x \mid g)$
  $\to f = y f_1$, $g = x g_1$, $f_1 \in \mathbb{C}[y]$, $g_1 \in \mathbb{C}[x]$ unique
  so $xf + yg = xy f_1 + xy g_1 = 0$ in A
- $\ker \alpha$ is gen by $\begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}$

$$0 \leftarrow M \leftarrow A \xleftarrow[(x,y)]{} A^{\oplus 2} \xleftarrow[\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}]{\beta} A^{\oplus 2} \xleftarrow[\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}]{} A^{\oplus 2} \leftarrow \cdots \text{ to } \infty$$

and beyond

- taking $- \otimes_A M$ gives

$$0 \leftarrow \mathbb{C} \xleftarrow[0]{} \mathbb{C}^2 \xleftarrow[0]{} \mathbb{C}^2 \leftarrow \cdots$$

so taking homology gives

Cor $\mathrm{Tor}^i_*(M, M) = \begin{cases} \mathbb{C}, & i = 0 \\ \mathbb{C}^{\oplus 2}, & i > 0 \end{cases}$

- so M does not have finite free resn as A-module, we can't get rid of Tors
- we are doomed without Hilbert's protection

Fact X aff sing sch $\Rightarrow \exists$ objects in Coh_X without finite free resns
In fact, you can always find one of the form $\mathcal{O}_p$ if $\Gamma(X, \mathcal{O}_X)$ f.g. $/ \mathbb{k} = \bar{\mathbb{k}}$, $p \in X(\mathbb{k})$

- same X, A. look at $\Omega_X = \Omega_{A/\mathbb{C}}$
  - $X \hookrightarrow \mathbb{A}$ cl. emb

$$\Rightarrow I_X / I_X^2 \xrightarrow[\alpha]{} \Omega_{\mathbb{A}^2}|_X \to \Omega_X \to 0$$

$$I_X \leq \underset{\substack{\| \\ R}}{\mathbb{C}[x,y]}, \quad I_X = R \cdot xy, \quad I_X / I_X^2 = A \cdot (xy)$$

- α induced by $d : R \to \Omega_R$, $d(xy) = dx\, y + x\, dy$
  $\Omega_{\mathbb{A}^2} = R\,dx \oplus R\,dy$, $\Omega_{\mathbb{A}^2}|_X = A\,dx \oplus A\,dy$

$$A \xrightarrow[(x\cdot,\, y\cdot)]{\alpha} A\,dx \oplus A\,dy \to \Omega_X \to 0$$

$\mathbb{C}[x]$ $\quad$ $\mathbb{C}[y]$

- $\ker \alpha$? $f \in A$, $f = c + x f_1 + y f_2$

$\alpha(f) = (yf, xf) = (y(c + y f_2), x(c + x f_1))$

$= 0 \iff c + y f_2 = 0,\ c + x f_1 = 0 \iff c = f_1 = f_2 = f = 0$

- so $\ker\alpha$ empty, $0 \to A \xrightarrow{\alpha} A\,dx \oplus A\,dy \to \Omega_X \to 0$

- recall, $X \to Y$ cl. emb. of schemes, $\mathcal{J} = \mathcal{J}_{X/Y}$

$(*)\ \Rightarrow \mathcal{J}/\mathcal{J}^2 \to \Omega_{Y|X} \to \Omega_X \to 0$ in Coh_X

[Ha] If X, Y nonsing then α injects and $(*)$ sh. ex. seq of loc free sh

Def $X \hookrightarrow Y$ cl. emb. is **regular** of cod r if locally near each pt of X, $I_X \subseteq \mathcal{O}_Y$ is gen by len r reg seq.

Fact If $X \hookrightarrow Y$ reg, $\mathcal{J}/\mathcal{J}^2$ is loc. fr. of rk r.
Moreover, if Y nonsing $\wedge$ X reduced then $(*)$ is exact on left ($\iff \ker\alpha = 0$
$\Rightarrow \Omega_X$ has a loc free resn of len 1)

Cor Let $X = \operatorname{Spec} \mathbb{C}[x,y]/x\cdot y$. Then $\operatorname{Ext}^1(\Omega_X, \mathcal{O}_X) = \mathbb{C}$, $\operatorname{Ext}^{i>1}(\Omega_X, \mathcal{O}_X) = 0$.

Pf $\operatorname{Hom}(-, A) : \mathrm{Mod}_A^{op} \to \mathrm{Mod}_A$ right exact, Mod_A has enough proj.
Apply to

$$\underset{-1}{A} \xrightarrow{\binom{y}{x}} \underset{0}{A^{\oplus 2}} \rightsquigarrow \underset{0}{A^{\oplus 2}} \xrightarrow[(y,\,x)]{\alpha} \underset{1}{A}$$

- $\ker \alpha = \{(f,g) \in A^{\oplus 2} \mid yf + xg = 0\}$
$= x\,\mathbb{C}[x] \oplus y\,\mathbb{C}[y]$
- so $\operatorname{coker} \alpha = \mathbb{C} = \operatorname{Ext}^1(\Omega_X, \mathcal{O}_X)$
- note $\operatorname{Hom}(\Omega_X, \mathcal{O}_X) = \Gamma(X, \underline{\underline{T_X}})$

where $T_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X)\ (= \vartheta_X)$

Exercise For any scheme X (loc. fin. / $\mathbb{k} = \bar{\mathbb{k}}$), $\Gamma(T_X)$ are tgt sp. to $\operatorname{Aut}(X)$.

- $\operatorname{Ext}^1(\Omega_X, \mathcal{O}_X)$ are 1^{st} order deformations of X
- informally, $X = \{xy = 0\} \subseteq \mathbb{A}^2$
so $\{xy + \varepsilon f = 0\}_{\varepsilon^2 \to 0} \subseteq \mathbb{A}^2 \times \mathbb{C}[\varepsilon]/\varepsilon^2$
is deformation
- but if $f = gx + hy$,
$xy + \varepsilon f = (x + \varepsilon f)(y + h\varepsilon) + O(\varepsilon^2)$
so nothing changes
$\rightarrow$ so $f \in \mathbb{C}$