Dabrowski

1st order

- $a \in A = C^\infty(M)$, $[D, a] = i\, da$.
- $B = \Gamma(\mathcal{Cl}(M))$, $A \subset Z(B)$, B gen. by $A, [D, A]$

$\Rightarrow$ $\mathcal{H}$ is A-B-bimodule

$\Rightarrow$ thus, $[[D,a], b] = 0 \quad \forall a, b \in A$

$\hookrightarrow$ all of this in comm. case.

- can be formulated NC
- we know $Ad_J : A \to A' \hookrightarrow \mathbb{B}(\mathcal{H})$,
  $[a, JbJ^{-1}] = 0$

- first order cond'n $[[D,a], JbJ^{-1}] = 0$

Orientability

- in NCG diff calculus is done "algebraically", [Hochschild-Kostant-Connes thm]
- e.g. $vol_\alpha = e^1_\alpha \wedge \cdots \wedge e^n_\alpha$, $\dim M = n$, indep. of frame
  - in terms of coords c^j_α, $j = 1, \ldots, n$,

$$vol_\alpha = \det(\vartheta_\alpha)\, dc^1_\alpha \wedge \cdots \wedge dc^n_\alpha$$

  where $e_\alpha^j = \sum_k (\vartheta_\alpha)^j{}_k\, dc^k_\alpha$
  - pick p.o.1. $\sum f_\alpha = 1$, define

$$c^0_\alpha := i^{n-m} f_\alpha \cdot \det \vartheta_\alpha$$

- let $c := \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma \sum_\alpha c^0_\alpha \otimes c^{\sigma_1}_\alpha \otimes \cdots \otimes c^{\sigma_n}_\alpha \in A^{\otimes(n+1)}$

  Hochschild n-chains w/ coeffs in A

- $bc := \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma \sum_\alpha b(\cdot \cdots)$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma \sum_\alpha \Big[c^0_\alpha c^{\sigma_1}_\alpha \otimes \cdots \otimes c^{\sigma_n}_\alpha + \sum_{j=1}^n (-)^j c^0_\alpha \otimes \cdots \otimes c^{\sigma_j}_\alpha \cdot c^{\sigma_{j+1}}_\alpha \otimes \cdots \otimes c^{\sigma_n}_\alpha$$

$$+ (-)^n c_\alpha^{\sigma_n} c_\alpha^0 \otimes \cdots \otimes c_\alpha^{\sigma_{n-1}}] \in A^{\otimes n}$$

$\Rightarrow$ $bc = 0$, and $b^2 = 0$

- define Hochschild homology

- given $(A, \pi, \mathcal{H}, D)$ S.T., emphasising rep π, define for any $c^0 \otimes \cdots \otimes c^n \in A^{\otimes(n+1)}$

$$\pi_D(c^0 \otimes \cdots \otimes c^n) := \pi(c^0)[D, \pi(c^1)] \cdots [D, \pi(c^n)]$$

omitting writing π

$$\Rightarrow \pi_D(c_{vol}) = \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma \sum_\alpha c_\alpha^0 \underbrace{[D, c_\alpha^{\sigma_1}]}_{\gamma(dc_\alpha^{\sigma_1})} \cdots \underbrace{[D, c_\alpha^{\sigma_n}]}_{\gamma(dc_\alpha^{\sigma_n})}$$

$$\gamma(dc_\alpha^{\sigma_1}) = \gamma\Big(\sum_{i_1} (x^{-1})^{\sigma_1}_{i_1} e_\alpha^{i_1}\Big) \qquad \gamma(dc_\alpha^{\sigma_n}) = \cdots$$

$$= \cdots$$

$$= i^{-m} \sum_\alpha f_\alpha \gamma(e_\alpha^1 \cdots e_\alpha^n)$$

$$= \gamma(\omega) = \begin{cases} \chi & \text{for } n = \text{even} \\ 1 & \text{for } n = \text{odd} \end{cases}$$

Axiom (Orientability) $\exists$ Hochschild n-cycle $c \in Z_n(A, A)$ s.t. $\pi_D(c) = \begin{cases} \chi \\ 1 \end{cases}$ for $n \begin{cases} \text{even} \\ \text{odd} \end{cases}$.

Rmk $[a, Jb J^{-1}] = 0 \rightarrow 0^{th}$ order cond.

$[[D, a], J[D, b]J^{-1}] = 0 \rightarrow 2^{nd}$ ord. cond, satisfied by Hodge-de Rham S.T.

Poincaré duality ($/\mathbb{R}$ or $/\mathbb{C}$)

- classically, $H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{C}$
perfect pairing $\langle [\alpha], [\beta] \rangle := \int_M \bar{\alpha} \wedge \beta$
hermitian
- well-def. on classes
- nondeg. due to existence of volume form i.e. Hodge duality

$$\forall \alpha \neq 0, \int_M \bar{\alpha} \wedge * \alpha = \int_M \langle \alpha, \alpha \rangle_g \, vol_g > 0$$

where $\langle \alpha_1 \wedge \dots \wedge \alpha_k, \beta_1 \wedge \dots \wedge \beta_l \rangle_g = \delta_{kl} \det \langle \alpha_i, \beta_j \rangle_K$
where $\langle \alpha, \beta \rangle_g = g^{-1}(\bar{\alpha}, \beta)$ for $\alpha, \beta \in \Omega^1(M)$

- duality comes from isomorphism $K^\bullet(C(M))$ and $K_\bullet(C(M)) = K^\bullet(M)$ in K-th and Chern isomorphism $K^\bullet(M) \otimes \mathbb{C} \simeq H^\bullet_{dR}(M) \otimes \mathbb{C}$

Axiom (Poincaré) The pairing

$$K_0(A) \otimes K_0(A) \to \mathbb{Z},$$
$$([p], [q]) \mapsto \mathrm{ind}(pJqJ^{-1} \underset{D|_{\mathcal{H}^+}}{\underset{\shortparallel}{D_+}} pJqJ^{-1})$$

and

$$K_1(A) \otimes K_1(A) \to \mathbb{Z}$$
$$([u], [v]) \mapsto \frac{1}{4} \mathrm{ind}\left(\left(1 + \frac{D}{|D|}\right) u J v J^{-1} \left(1 + \frac{D}{|D|}\right)\right)$$

are nondegenerate.

- note, we use the fact that if D Fredholm, $P(D \otimes id_N)P'$ is also Fredholm, P, P' projs.