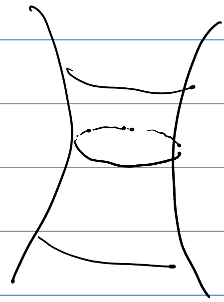
Stoppa.

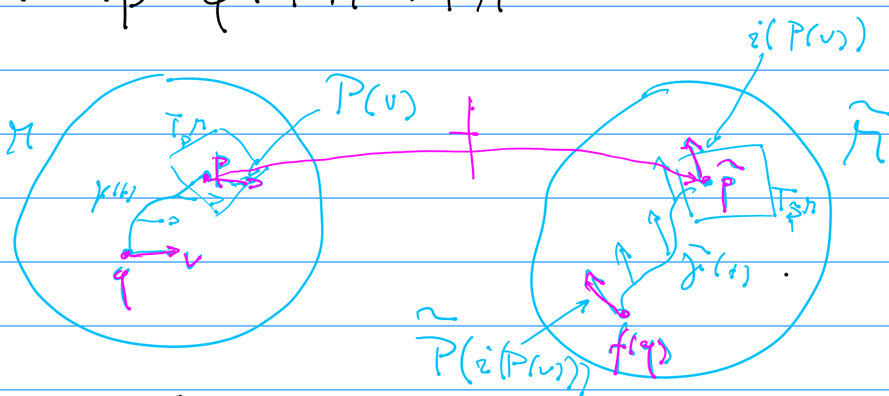
- e.g. Surface of revolution $S \subset \mathbb{R}^3$



- even if $K < 0$, $\pi_1(S, \{pt\}) \neq \{1\}$
  $\rightarrow S \not\cong \mathbb{R}^2$

Cartan's construction.

- take $p \in (M^n, g)$, $\tilde{p} \in (\tilde{M}^n, \tilde{g})$
- define a linear isometry $i : T_p M \to T_{\tilde{p}} \tilde{M}$
- suppose $\exp_p : T_p M \to M$ is a diffeo (?), $\exp_{\tilde{p}}$ defined on all $T_{\tilde{p}} \tilde{M}$
- define $f : M \to \tilde{M}$ by $f(q) = \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(q)$
- we can extend this by parallel transport to a map $\varphi : TM \to T\tilde{M}$



$\Rightarrow \varphi(v) := (\tilde{P} \circ i \circ P)(v)$, $\gamma(t), \tilde{\gamma}(t)$ unique, normalised geodesic

Thm (Cartan) Assume everything as above and suppose $\langle R(X,Y)U, V \rangle_M = \langle \tilde{R}(\varphi X, \varphi Y)\varphi U, \varphi V \rangle_{\tilde{M}}$.
Then f is a <u>local isometry</u> & $df_p = i$.

Pf. We need to compute df_q.

By construction, $q = \exp_p(\tilde{t}\,\dot{\gamma}(0))$

so $df_q(v) = (d\exp_{\tilde{p}})_{\tilde{t}\,\underbrace{i(\dot{\gamma}(0))}_{\dot{\tilde{\gamma}}(0)}} \circ i \circ (d\exp^{-1}_p)_{\tilde{t}\,\dot{\gamma}(0)}(v)$

To compute it, we use Jacobi fields.

Since $\exp_p$ is diffeo, $d\exp_p$ is invertible

$\Rightarrow \exists!$ Jacobi field along $\gamma(t)$ s.t.

$J(0)=0$, $J(\tilde{t}) = v \in T_qM$, ...

Now define $\tilde{J}(t) := \varphi(J(t))$ along $\tilde{\gamma}(t)$

Note: $\tilde{J}(0)=0$, $\|\tilde{J}(t)\|_{\tilde{M}} = \|J(t)\|_M \;\; \forall t$

because we parallel transport using Levi-Civita.

Claim: $\tilde{J}(t)$ is a Jacobi field.

If this is true,

$\|df_q(v)\|_{\tilde{M}} = \|\tilde{J}(\tilde{t})\|_{\tilde{M}} = \|J(\tilde{t})\|_M = \|v\|_M$.

Sketch: - write $J(t) = J_i e_i(t)$ using an ON frame.

$\tilde{J}(t) = \varphi(J(t)) = J_i(t)\varphi(e_i(t)) \doteq J_i(t)\tilde{e}_i(t)$

- write $\ddot{J}^j + \langle R(e_k, e_i)e_k, e_j\rangle = 0$

- use assumption.

Thm ("Fundamental thm on spaces of constant curvature")

Let (M^n, g) be complete w/ const. sect. curvature.

Up to rescaling, the universal cover of M endowed with the pullback metric, $(\tilde{M}, \pi^*g)$, is isometric to either:

$(\mathbb{R}^n, g^{eucl})$, $\quad K \equiv 0$

$(\mathbb{H}^n, g^{hyp})$, $\quad K \equiv -1$

(S^n, g^{round}), $\quad K \equiv +1$.

Pf. $\pi: \tilde{M} \to M$ univ. cover $\Rightarrow (\tilde{M}, \pi^* g)$ is still complete, and $\pi_1(\tilde{M}, \{pt\}) = \{1\}$. Assume, after rescaling, $K(g) \in \{0, \pm 1\} \Rightarrow K(\pi^* g) = K(g)$, local isometry.

Now fix $K(g) \in \{0, -1\}$.

Hadamard: $\forall p \in \tilde{M}$, $\exp_p$ is well-defined & diffeo.

Define $(D, g_D) = \begin{cases} (\mathbb{R}^n, g^{euc.}), & \text{if } K = 0 \\ (\mathbb{H}^n, g^{hyp.}), & \text{if } K = -1 \end{cases}$,

do Cartan with it. ...