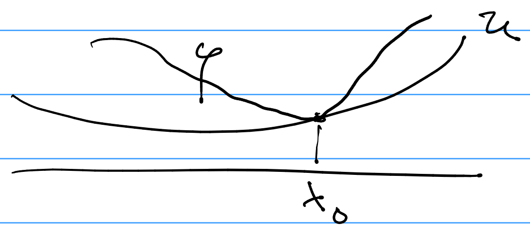
Mosini

Viscosity sol'n's

- consider $u \in \mathcal{C}^2$ subharmonic, i.e. $-\Delta u \leq 0$
  and assume φ touches u from above
  at a pt x_0



- x_0 is a maximum for $u - \varphi$
  so $\Delta(u-\varphi)(x_0) \leq 0$
  $\Rightarrow -\Delta\varphi(x_0) \leq -\Delta u(x_0) \leq 0$

- so $-\Delta\varphi(x_0) \leq 0 \rightarrow$ viscosity

- envelope $\mathcal{E} : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^{N \times N}_{sym} \longrightarrow \mathbb{R}$
  and
$$\mathcal{E}(x, u, \nabla u, \nabla^2 u) = 0 \quad (*)$$

Def. a) we say that an u.s.c. u is a viscosity subsol'n of $(*)$
if
$$\mathcal{E}_*(x_0, u(x_0), \nabla\varphi(x_0), \nabla^2\varphi(x_0)) \leq 0$$
whenever $x_0 \in \Omega$, $\varphi \in \mathcal{C}^2(U_{x_0})$ s.t.
$u - \varphi$ has loc. max at x_0

Rmk. due to derivative-only dependence on φ, we may shift it to touch u!

b) Similarly, define viscosity supersol'n
if $\mathcal{E}^*(x_0, u(x_0), \nabla\varphi(x_0), \nabla^2\varphi(x_0)) \geq 0$
when $x_0 \in \Omega$, $\varphi \in \mathcal{C}^2(U_{x_0})$, $u - \varphi$ has loc. min. at x_0

c) u is viscosity sol'n if
u^* is v. supersol'n and u_* v subsol'n

Rmks
- u VS of $-\mathcal{E}=0$
  $\not\Rightarrow$ u VS of $\mathcal{E}=0$
- If u is VS of $(*)$ and $\mathcal{C}^2$,
  then u is classical sol'n
  if $\begin{cases} \mathcal{E}_*(x,u,\nabla u,\nabla^2 u) \le 0 \\ \mathcal{E}^*(x,u,\nabla u,\nabla^2 u) \ge 0 \end{cases}$
- we may always take loc. min/max
  to be isolated by $\varphi \mapsto \varphi \pm |x-x_0|^4$.

- let $u:\Omega\to\mathbb{R}$ be u.s.c., then for $x_0\in\Omega$
define

$$J^{2,+}u(x_0) := \{(p,X)\in\mathbb{R}^N\times\mathbb{M}^{N\times N}_{sym} \mid u(x)\le u(x_0)+p\cdot(x-x_0) + \tfrac{1}{2}X\cdot(x-x_0)(x-x_0) + o(|x-x_0|^2)\}$$

2nd order superjet of u at x_0

- needn't be nonempty

Rmk If $\varphi\in\mathcal{C}^2$ touches u from above at x_0,
then $(\nabla\varphi(x_0),\nabla^2\varphi(x_0))\in J^{2,+}u(x_0)$

Prop If u is VSB for $(*)$ then
$\mathcal{E}_*(x_0,u(x_0),p,X)\le 0$.