Stoppa

Hopf-Rinow Thm.

Thm. Fix a pt $p \in (M, g)$. TFAE

i) $\exp_p$ is defined on all of T_pM

ii) closed & bounded of M are compact

iii) (M, d) is a complete metric space

iv) $\exp_q$ is defined on all of T_qM, $\forall q \in M$

Moreover, any of these implies:

v) $\forall p, q \in M$ $\exists \gamma(t)$ geodesic, $\gamma(0) = p$, $\gamma(1) = q$ s.t. $d(p,q) = \operatorname{len} \gamma$.

Pf. For now assume we know i) $\Rightarrow$ v).

i) $\Rightarrow$ ii) $A \subset M$ closed & bounded. uses i) $\Rightarrow$ v)

$\overset{v)}{\Rightarrow}$ $A \subset \exp_p(B_R(0))$ for some $R > 0$, $p \in A$.

$\Rightarrow$ $A \subset \exp_p(\underbrace{\overline{B}_R(0)}_{\text{cpt}})$ $\Rightarrow$ A cpt.

ii) $\Rightarrow$ iii) Basic analysis

iii) $\Rightarrow$ iv) Pick $\gamma : [0, s_0) \to M$ geodesic, $\gamma(0) = q$.

Pick $\{s_m\}_{m \in \mathbb{N}} \subset [0, s_0)$, $\lim s_m = s_0$.

$\Rightarrow \{\gamma(s_m)\}_m$ is a Cauchy sequ. on (M, d).

$\Rightarrow$ $\gamma(s_0) := \lim \gamma(s_m)$

- we only need to show $\exists \dot\gamma(s_0) \in T_{\gamma(s_0)}M$ which extends $\gamma(t)$ by $\exp_{\gamma(s_0)}(t\dot\gamma(s_0))$.

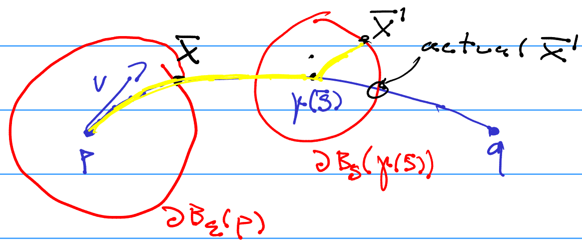
$\Rightarrow$ $(\gamma(s_m), \dot\gamma(s_m))_m$ is a squ. lying on a cpt subset of TM ($\{\dot\gamma(s_m)\}_m$ has fixed lengths)

$\Rightarrow$ Pick a convergent subsequence and take limits.

Main point: i) $\Rightarrow$ v)

- fix any $p, q \in M$, let $d := d(p,q)$.
- Claim: p, q can be joined by minimising geodesic

- choose initial velocity by picking a point $\bar{x}$ on the boundary of a normal ball $B_\varepsilon(p)$ at p which minimises distance from $\partial B_\varepsilon(p)$ to q
  cpt.



- claim: geodesic extends, minimises & hits q
- write $\gamma(t,p,v)$, $\gamma(0,p,v)=p$, $\gamma(\varepsilon,p,v)=\bar{x}$
- stopped writing.

<u>Thm</u> (Hadamard) Let (M,g) be a complete Riemannian mfd (i.e. (M,d) is complete; or $\exp_q$ defined on T_qM $\forall q\in M$). Suppose:

1) $\pi_1(M,\{pt\})=\{1\}$
2) $\forall q\in M$, $\forall \sigma < T_qM$, $K(\sigma)\le 0$

$\Rightarrow \exp_q : T_qM \longrightarrow M$ is diffeo., $\forall q$

Pf. Claim: $\forall q$, $\exp_q : T_qM \to M$ is local diffeo.
This means $(d\exp_q)_v$ is an invertible linear map i.e. $\exp_q$ has no critical pts.
But we know those pts correspond to vanishing of Jacobi fields.
$\Rightarrow \frac{d^2}{dt^2}\langle J(t),J(t)\rangle = 2\|J'(t)\|^2 - 2K(\dot\gamma(t),J(t))A^2(\dot\gamma(t),J(t))$
$\to$ RHS ≥ 0 since $K\le 0$. Evaluating at 0 gives >0.
Now look at $(T_qM, \exp_q^* g)$. By Hopf-Rinow it is complete in both senses, since its geodesics are just straight lines.
Also, $\|(d\exp_q)_v(w)\|_M = \|w\|_{T_qM}$, isometry.

Claim: all of this implies $\exp_q$ is a covering map.

Criterion: $X \xrightarrow{f} Y$ covering map iff lifting property for paths in Y holds.

→ by local diff we can lift locally.

→ but it extends due to local isometry prop.

Finally: it is a trivial covering due to

$\pi_1(M, \{pt\}) = \{1\}$

$\Rightarrow$ homeomorphism

$\Rightarrow$ diffeomorphism.