Scarpa.

- Q: are there hol. maps $\mathbb{C}P^n \to \mathbb{T}^m = \mathbb{C}^m/\Lambda$?
- answer: only const. maps.
- hol. map $f: \mathbb{C}P^n \to \mathbb{C}$ is constant (*)
→ so let's pass to the covering
- $\exists$ lift $\tilde{\varphi}$ s.t. $\varphi = \pi \circ \tilde{\varphi}$

$$\begin{tikzcd} & \mathbb{C}^m \arrow[d, "\pi"] \\ \mathbb{C}P^n \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & \mathbb{T} \end{tikzcd}$$

π covering map, local biholomorphism

Claim: $\tilde{\varphi}$ is holomorphic

Lemma. $\varphi: (M, J_M) \to (N, J_N)$ is holomorphic iff $\varphi_* \circ J_M = J_N \circ \varphi_*$

→ clear if we write $\mathbb{C} \cong \mathbb{R}^2$, $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$,
$z = x + iy$, $f: \mathbb{C} \to \mathbb{C}$, $f = f_1 + i f_2$
$\Rightarrow f_* = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix}$. C-R equs. $\Leftrightarrow$ lemma

- now $\varphi_* = \pi_* \circ \tilde{\varphi}_*$, so $\varphi_* \circ J_{\mathbb{C}P^n} = J_{\mathbb{T}^m} \circ \varphi_*$
gives $\pi_* \circ \tilde{\varphi}_* \circ J_{\mathbb{C}P^n} = J_{\mathbb{T}^m} \circ \pi_* \circ \tilde{\varphi}_*$
$= \pi_* \circ J_{\mathbb{C}^m} \circ \tilde{\varphi}_*$
$\Rightarrow \tilde{\varphi}$ is holomorphic, so by (*) constant. □

- instead of $\mathbb{C}P^n$ take any cpt. simply. conn. cpx. mfd., and replace $\mathbb{T}^m$ w mfd. whose univ. cover $\subset \mathbb{C}^m$ and repeat proof.
- but also, maps from cpt. Fano vars to Σ_g.

Normal Kähler coordinates.

- recall that on a Riemann (M,g) $\exists$ coords s.t. near $p \in M$, $g_{ij} = \delta_{ij} + O(|\vec{x}|^2)$, $\Gamma_{ij}^{k}(p) = 0$.

- take (M, J, g) cpx mfd w herm. metric, write $\omega = g(J-,-)$ as $\omega = i\, g_{a\bar{b}}\, dz^a \wedge d\bar{z}^b$
- impose $d\omega = 0$

Lemma. $\forall p \in M$ $\exists$ hol. coordinates $\vec{w}$ s.t. $g_{a\bar{b}} = \delta_{ab} + O(|\vec{w}|^2)$ iff g Kähler.

Pf. One direction is easy. If $g_{a\bar{b}} = \delta_{ab} + O(|\vec{z}|^2)$ around $p=0$, $d\omega_p = \partial_{z^c}(i g_{a\bar{b}})|_0\, dz^a \wedge d\bar{z}^b \wedge dz^c = 0$.
Next assume g Kähler and write

$g_{a\bar{b}}(z) = \delta_{ab} + A_{a\bar{b}c} z^c + A_{a\bar{b}\bar{c}} \bar{z}^c + O(|\vec{z}|^2)$

$\rightarrow$ hermitianness $\Rightarrow$ $\overline{A_{a\bar{b}c}} = A_{\bar{a}b\bar{c}} = A_{b\bar{a}\bar{c}}$

$\rightarrow$ Kählerness $\Rightarrow$ $0 = \partial\omega|_p = \cdots = i\, A_{a\bar{b}c}\, dz^c \wedge dz^a \wedge d\bar{z}^b$

$\Rightarrow$ $A_{a\bar{b}c} = A_{c\bar{b}a}$

Write $z^a = w^a + \frac{1}{2} B^a{}_{bc} w^b w^c$, $B^a{}_{bc} = B^a{}_{(bc)}$

and gather terms in $g(w)_{c\bar{d}} = g(z)_{a\bar{b}} \frac{\partial z^a}{\partial w^c} \frac{\partial \bar{z}^b}{\partial \bar{w}^d}\, dw^c \wedge d\bar{w}^d$

$\rightarrow$ constant term: δ_{ab}

$\rightarrow$ lin. term: possible to choose vanishing

Holomorphic sectional curvature.

- for a plane $\pi \hookrightarrow TM$ it may happen that $J_p v \notin \pi$ at some pt $p \in M$ with $v \in \pi$.
- we define hol. sect. curv. K_p^h where we restrict only to cases $J\pi \subseteq \pi$.