Don Zagier

K knot $\to J_n^K(q) \in \mathbb{Z}(q^{\pm 1})$ colored Jones polyn.

$q = \zeta$ = root of 1 $\Rightarrow$ 1) $n \mapsto J_n^K(\zeta)$ periodic, period N

$\zeta^N = 1$

2) very small, $n \not\equiv 0$
very big, $n \equiv 0$

$\Rightarrow J_0^K : \mu_\infty \to \overline{\mathbb{Q}}$

$J^K : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}$

$x \mapsto J_0^K(e^{2\pi i x})$

$\underbrace{J_0^K(\zeta_N)}_{\parallel \; J^K(\frac{1}{N})} = \langle K \rangle_N$ Kashaev invariant

$\underbrace{\text{Volume conjecture}}$

$= e^{v_c(k) N + O(N)}$ $M = S^3 - K$, knot complem.

where $v(K) = \frac{1}{2\pi}\left(Vol(M) - i\,CS(M)\right)$

($v_c(K)$ is the complexification)

$J^K(\frac{1}{N}) \sim \mu_0 N^{3/2} \delta(K) e^{v(K)N}\left(1 + K_1 \frac{2\pi i}{N} + K_2 \left(\frac{2\pi i}{N}\right)^2 + \ldots\right)$, $\delta(K) \in F^\times$, $K_i \in F$

$\hookrightarrow$ arithmetic conjecture.

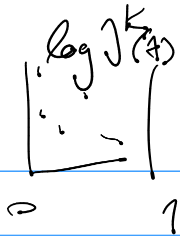
Q: How to compute $J_n(q)$?

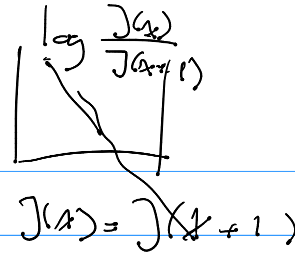
<u>Th</u>. $N \mapsto J_N(q)$ is "q-holonomic",

i.e. $\exists$ a recurrence rel $* J_n(q) = * J_{N-1}(q) + \ldots + J_{n-p}(q)$

$* \in \mathbb{Z}[q, q^N]$

Modularity conject.





Fix K, $\alpha\in\mathbb{Q}$, $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$, $\alpha = \frac{a}{c} = \gamma|_\infty$

Then as $X\to\infty$ through r#'s w/ bdd denom,

$$\underset{\downarrow\atop\alpha}{J^K\left(\frac{aX+b}{cX+d}\right)} \sim (cX+d)^{3/2}\, J^K(X) \cdot e^{v_K(X+\frac{d}{c})}\,\Phi_\alpha^K\left(\frac{2\pi i}{cX+d}\right),\quad \Phi^K_\alpha(h)\in\mathbb{C}((h))$$

Moreover, $\Phi_\alpha^K(h)\in\overline{\mathbb{Q}}((h))$ & in fact

$$" = "\ \mu_\alpha\cdot \delta(K)^{-1/2}\sqrt[c]{\varepsilon_\alpha}\,(c_0+c_1h+c_2h^2+\dots)$$

$\mu_\alpha^{8c}=1$ where all $c_i\in F_K(\zeta)$, ε=unit

Q. (D.Z). Given ① a number field F

② $\xi\in B(F)$

③ ζ n^{th} rt of 1

can one canonically define an element $\varepsilon\in\mathcal{U}_n$!! units of F_n, $F_n := F(\zeta)$

A. (Frank Calegari + DZ + SG) Yes & yes.

Th. F = number field, ζ = prime rt of 1.

Then $\exists$ a canonical map $\chi^{-1} \to G = \mathrm{Gal}(F_n/F)\xrightarrow[\sim]{\chi}(\mathbb{Z}/n\mathbb{Z})^\times$

$$R_\zeta : B(F)/nB(F) \longrightarrow \left(\mathcal{O}_{S,n}^\times/(\mathcal{O}_{S,n}^{\times n})\right)_G^{\chi^{-1}} \hookrightarrow \mathcal{U}_n^\times/F_n^{\times n},\ S \text{ finite}$$

(+ a lot more, I got distracted...)

Th2 Same but replace "B" w "K_3"

$$c_\zeta : K_3(F)/nK_3(F) \xrightarrow{\sim} (\mathcal{U}_p/\mathcal{U}_p^{\,p})^{\chi^{-1}}$$

Th3 $R_\zeta = c_\zeta^{\gamma}$ for some $\gamma\in(\mathbb{Z}/p\mathbb{Z})^\times$

Th4 $\varepsilon = R_\zeta(\xi)$ describes rad. asymptotics $F_{\xi BC}(q)$ as $q\to\zeta$