Kucerovski

C^*-algs and their many K-th gps

- cpx C^*-algs
  - axiomatic view
  - concrete view
- real C^*-algs
  - ax. view
  - conc. view
  - antiautomorphism view
- foundation of K-th.: Bott periodicity (homotopy gps of $GL_n(A)$)
- cpx K-th: homotopy of
  - projections
  - unitaries
- real K-th: — " —
  - — " —
  - — " —
  - but with some flavors
- algebraic K-th
- KK-theory:
  - homotopy classes of 'absorbing' Busby maps (absorbing extensions)
  - homotopy classes of Fredholm 3ples
    - bdd
    - unbdd
- Kasparov products:
  - Kasparov's original, very nonconstructive "NC part. of unity"
  - Connes–Skandalis bdd conn's
  - unbdd version

Rel. between real & cpx case

- very simple ... or not, depending on how simple the respective K-th gps are
- UCT ... refinement of connecting map picture

C^*-algs (cpx)

- view pts:
- algebra/analysis: a norm-closed *-subalg of the algebra of bdd ops on a Hilb. sp.
- top./algebra: NC analogue of $\mathcal{C}$(top. sp.)
- algebra: rings + axioms
  -> this one led to K-th ...

Hilb. sp

- Hilbert originally wanted to axiomatize geometry

- inner prod $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ generalising properties of dot prod.
- e.g. $\mathbb{C}^n$, ℓ^2
- but funny things happen in ∞-dim case

- $\mathcal{U}(\ell^2)$ is contractible (Kuiper's thm)
- $\mathcal{U}(\mathbb{C}^n)$ is complicated
- finite: $VV^* = \mathrm{Id} \Rightarrow$ row. rk = col. rk $\Rightarrow V^*V$ invertible
- infinite: $VV^* = \mathrm{Id} \not\Rightarrow V^*V$ invertible

C^*-alg axioms

- Banach alg / $\mathbb{C}$ w involution s.t. $\|x^* x\| = \|x\|^2$

B^*-algebra

- Banach alg / $\mathbb{C}$ with $*$
→ will be C^*-alg if $sp(a^* a) \subseteq \mathbb{R}$

- holomorphically closed subalgebras
⟺ $f(a) \in A$ if $a \in A$ and f analytic ($f(0) = 0$)

- von Neumann alg
  - strongly closed $*$-subalg of $\mathbb{B}(\mathcal{H})$
  - $\exists$ path connecting unitaries
    → K-theory trivial

Real C^*-algs

- Banach alg w involution $*$
  $\|x^* x\| = \|x\|^2$
  $\|x^* x\| \leq \|x^* x + y^* y\|$ $\forall x, y \in$ alg.

- A real C^*-alg
  ⟹ $A \oplus iA$ with $A \oplus iA \xrightarrow{\tau} A \oplus (-i)A$
  τ antiautom. is cpx C^*-alg
  ⟹ for B cpx C^*-alg, $\tau : A \otimes_B \mathbb{C}$ ↻
  antiautom where $A = \{ b \in B \mid \tau(b) = b^* \}$,
  A is real C^*-alg

- A real C^*-alg $\subset \mathbb{B}(\mathcal{H}_{real})$