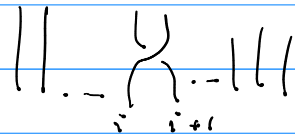
Srní

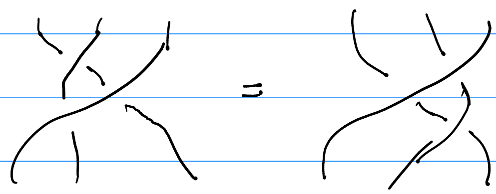
Ševera - From braids to quantization.

Braid groups. B_n = braids w n strands
$= \pi_1((\mathbb{C}^n/\Delta)/S_n)$

- generators: $s_i =$ 

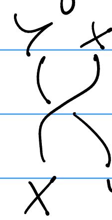
- relations $s_i s_j = s_j s_i$, $|i-j| \geq 2$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$



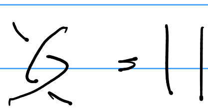
Monoidal cats.

- $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, $1_c \in \mathcal{C}$
- associativity nat. isos satisfying pentagram
- Braided MC-s: $\exists$ natural iso

$\beta_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, $\beta_{X,Y} =$ 

s.t. $X \xrightarrow{\beta} Y \xrightarrow{\beta} Z = X\, Y\, Z$ $\xleftarrow{}$ hexagon rels.

- Sym. monoidal cat:

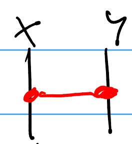
BMC s.t. 

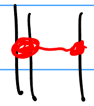
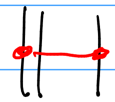
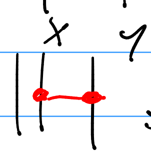
$\to$ so we stop drawing over/under crossings

Monoidal functors.

- $F: \mathcal{C} \to D$ monoidal if $F(X) \otimes F(Y) \underset{(*)}{\xrightarrow{\sim}} F(X \otimes Y)$ s.t. (associativity diagram) holds and $F(1_{\mathcal{C}}) \underset{(*)}{\xrightarrow{\sim}} 1_D$
- F is lax monoidal if $(*)$ not necessarily isos
- similarly braided m.f.-s "commute" w. β

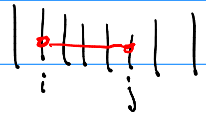
Infinitesimal braids or chord diagrams.

- an inf. braided cat = linear SMC $\mathcal{C}$ w. nat. transf. $t_{X,Y}: X \otimes Y \to X \otimes Y$ s.t. $\beta_{X,Y} := \sigma_{X,Y} \circ (1 + \varepsilon t_{X,Y})$, $\varepsilon^2 = 0$ is a braiding in $\mathcal{C}$, and $t_{X,Y} = t_{Y,X}$
- drawing: $t_{X,Y} =$ 

s.t.  $=$  $+$ , Leibniz rule

- e.g. $\mathfrak{g}$ Lie alg, $t \in (S^2\mathfrak{g})^{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}$, $\mathcal{C} = U\mathfrak{g}$-mod
  $t_{X,Y} = S_X \otimes S_Y(t) \in \mathrm{End}(X \otimes Y)$.

- algebra of inf. pure braids $A_n = \langle t_{ij} \rangle$, $1 \leq i,j \leq n$, $i \neq j$, $t_{ij} = t_{ji}$ with relations $[t_{ij}, t_{kl}] = 0$ if $ijkl$ distinct, $[t_{ij}, t_{ik} + t_{jk}] = 0$

- $t_{ij} =$ 

- A_n is a cocommutative Hopf algebra (t_{ij} primitive)

Drinfeld associators.

- problem: extend $\overline{1^{st}}$ order deformation (iBMC) to true deformation

Thm (Drinfeld) $\exists\, \Phi \in \mathbb{Q}\langle\langle x, y\rangle\rangle$ such that

$$\beta^{new}_{X,Y} := \tau_{X,Y} \circ \exp\left(\frac{\hbar}{2} t_{X,Y}\right) \text{ and}$$

$$\gamma^{new}_{XYZ} := \gamma^{old}_{XYZ} \circ \Phi(\hbar t_{X,Y}, \hbar t_{Y,Z})$$

make any SMC into a BMC.

- recall, $\gamma_{XYZ} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$
- Φ is called a Drinfeld associator if satisfies $\Delta \Phi = \Phi \times \Phi$ for x, y primitive

Where do they come from?

- KZ-connection $A_n \in \Omega^1(\mathbb{C}^n - \Delta s) \otimes \mathfrak{t}_n$,

$$A_n = \sum_{k<l} t_{kl} \frac{d(z_k - z_l)}{z_k - z_l} \text{ is flat}$$

- $\mathrm{hol}_{\circlearrowleft} A_2 = \exp(2\pi i\, t_{12})$

- $\Phi_{KZ}(t_{12}, t_{23}) := \lim_{\varepsilon \to 0^+} \varepsilon^{-t_{23}}\, \mathrm{hol}_{\circlearrowleft} A_3 \cdot \varepsilon^{t_{12}}$

$\to$ then $\Phi(x, y) := \Phi_{KZ}\left(\frac{x}{2\pi i}, \frac{y}{2\pi i}\right) \in \mathbb{C}\langle\langle x, y\rangle\rangle$ is an associator (but over $\mathbb{C}$).