Umar

Toric varieties

- algebraic torus $T \simeq (\mathbb{C}^*)^n$
- character $\chi^m : T \to \mathbb{C}^*$, $m \in \mathbb{Z}^n$
$t \mapsto t_1^{m_1} \cdots t_n^{m_n}$

- one-parameter homomorphism:
$\lambda^u : \mathbb{C}^* \longrightarrow T$, $u \in \mathbb{Z}^n$
$t \longmapsto (t^{u_1}, \dots, t^{u_n})$

$\to M^\vee \underline{\simeq} N$, where $M = \{\chi^q\}$, $N = \{\lambda^u\}$

Def. Affine toric variety is an alg. variety V s.t. the torus T_N is Zariski open $\hookrightarrow V$, and the action T_N on itself extends algebraically on V, $T_N \times V \to V$.

- ex: $(\mathbb{C}^*)^n$; $\mathbb{C}^n$; $V = V(x^3 - y^2) \Rightarrow V / \{0\} = \{(t^2, t^3)\}$
$V(xz - yw) \hookrightarrow \mathbb{C}^4$

- ex: $\varphi : \mathbb{C}^2 \to \mathbb{C}^{d+1}$
$(s,t) \mapsto (s^d, s^{d-1}t, \dots, t^d)$ Veronese
- let $\mathcal{A} = \{(d,0), (d-1,1), \dots, (0,d)\} \subseteq \mathbb{Z}^2$
$\Rightarrow \varphi_{\mathcal{A}} : (\mathbb{C}^*)^2 \to \mathbb{C}^{d+1}$
$\underline{t} \longmapsto (\chi^{(d,0)}(t), \dots, \chi^{(0,d)}(t))$

Def. CPC $\sigma = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0$, S finite
$S \hookrightarrow N_{\mathbb{R}}$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n\}$

- $\sigma^\vee = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \}$

- SCPC (strongly CPC), if, equivalently:
  - i) $\{0\}$ is a face
  - ii) $\mathcal{Z} \cap (-\mathcal{Z}) = \{0\}$
  - iii) $\dim(\mathcal{Z}) = n$
  - iv) $\nexists$ positive dim. subspace of $\mathbb{R}^n$
- RPC (rational PC), $S \subseteq \mathbb{N}$
- we need SCRPC-s: