Antonini

- X cpt case
→ $C(X)$ separable iff X metrisable
- take $L : C(X) \to \mathbb{C}$ pos. form
($L(f) \geq 0$ $\forall f \in C(X)$ s.t. $f(t) \geq 0$ $\forall t$)

Thm (Riesz repr.) $\exists!$ Borel measure μ
s.t. $L(f) = \int f \, d\mu$

- now look at $\pi : C(X) \to L^2(X, \mu)$, $\pi(f)g = f \cdot g$.
- notice that $L^\infty(X, \mu)$ is a von Neumann algebra, meaning a weakly closed, nondegenerate as a representation, $*$-subalgebra of $\mathbb{B}(H)$

- L extends to $L^\infty(X, \mu)$

$$\tilde{L}\left(\sum_{\alpha \in I} e_\alpha\right) = \sum_{\alpha \in I} \tilde{L}(e_\alpha)$$

for any family $(e_\alpha)_{\alpha \in I}$ of mutually $\perp$ projs

- so we get $\overline{\pi_f(A)}^{W.O.T.}$ ← weak operator topology in $\mathbb{B}(H_f)$

Def. A unital. **States** $S := \left\{ \begin{array}{l} \varphi \in A^*_+ \\ \varphi(1) = 1 \end{array} \right\}$

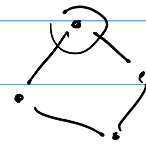
- properties:
  - $S \neq \emptyset$
  - convex
  - separates pts.
  - the corresponding GNS is isometric

- last prop: $\|\pi_f(a)\|^2 \geq \|\pi_f(a)\xi_f\|^2 = f(a^*a) \geq \|a\|^2$

but $\|\pi_f(a)\|^2 \leq \|a\|^2$ since $*$-morphisms are contractive

- cptness in weak $*$-top, i.e. $\sigma(A^*, A)$
- Klein-Milman says

  S = closure of the cnvx envelope of its $\underbrace{\text{extremal pts}}_{=\text{irreducible reps.}}$



- note: weak op. top. is the weakest topology making continuous

$$\forall x \in H, \quad \mathbb{B}(H) \longrightarrow (H, \text{weak top})$$
$$T \longmapsto Tx$$

or, equiv., $\forall x, y \in H$, $T \mapsto \langle y, Tx \rangle$.

- If we had put $(H, \text{norm top})$, this would be strong op. top.

- $X = \{\text{pos lin forms on } A, \|f\| \leq 1\}$
- compact (weak top.) ptwise convgnc

$$A_h \xleftrightarrow{F} C(X, \mathbb{R})$$

self adj $\in A$, real ordered Ban. sp.

$$F(a)(f) = f(a)$$

- note we induce order, since
  $a \le b$ in A_h means $\forall x \in X$,
  $F(a)(x) = x(a) \le x(b) = F(b)(x)$
- F is also isometric (no pf. given)

- recall $f: A \to \mathbb{C}$ Hermitian if $f(a^*) = \overline{f(a)}$

Prop $f: A \to \mathbb{C}$ Hermitian cont. lin. form.
Then $\exists$ pos. forms $f^+, f^-: A \to \mathbb{C}$
s.t. $(f = f^+ - f^-) \wedge (\|f\| = \|f^+\| + \|f^-\|)$.

Cor $(\forall x \in A)$ $\exists$ a representation
$$\pi: A \to \mathbb{B}(H_\pi)$$
s.t. $\|\pi(x)\| = \|x\|$ and π is a GNS.

Pf: $\exists$ pos. form $f: A \to \mathbb{C}$ s.t.
$\begin{cases} \|f\| \le 1 \\ f(x^* x) = \|x\|^2 \end{cases}$
Then (H_f, π_f, ξ_f) GNS, $\|\xi_f\|^2 = \|f\| \le 1$
and $\|\pi_f(x)\|^2 \ge \|\pi_f(x)\xi_f\| = f(x^* x) = \|x\|^2$.
The obverse always holds. $\square$

- so we get many representations, but

Thm (Gel'fand-Naimark) Any C^*-alg has an isometric repr. on a Hilb. sp.
Pf. $\forall a \in A$ $\exists \pi_a$. But $\bigoplus_{a \in A} \pi_a$ is isometric,
since $\forall b \in A$, $\|\bigoplus_{a \in A} \pi_a(b)\| = \sup_{a \in A} \|\pi_a(b)\| = \|\pi_b(b)\| = \|b\|$.

- but if A separable, we can do better

Def A pos. form f is *faithful* if
$$\{a \in A \mid f(a^* a) = 0\} = \{0\}.$$

Prop A separable. Then:

i) $\exists$ pos. form on A which is faithful

ii) $\exists$ isometric reps. on a separable Hilbert space.

Pf. $\overline{(a_n)_n} = A$. $\forall n$ $\exists f_n : A \to \mathbb{C}$ positive

and $\begin{cases} f_n(a_n^* a_n) = \|a_n\|^2 \\ \|f_n\|^2 \leq 1 \end{cases}$

and all f_n have $(H_{f_n}, \pi_{f_n}, \xi_{f_n})$.

Define $f := \sum 2^{-n} f_n$, so that a nonzero, $\exists n$ s.t. $\|a - a_n\| < \frac{\|a\|}{2}$, $\|a - a_n\| < \|a_n\|$ gives
$$\|\pi_{f_n}(a)\xi_{f_n}\| \geq \|\pi_f(a_n)\xi_f n\| - \|a - a_n\|$$
$$= \|a_n\| - \|a - a_n\| > 0.$$
But $2^n f(a^* a) \geq f_n(a^* a) = \|\pi_{f_n}(a)\xi_{f_n}\|^2 > 0$.

H will be separable since $(a_n)_n$ dense by $A \to A/Ann^{\{0\}} \to$ completion.

π inj. since $\|\pi_f(a)\xi_f\|^2 = f(a^* a) > 0$ for $a \neq 0$. $\square$

- let $U \subset \mathbb{B}(H)$ any subset

Def. $U' := \{ T \in \mathbb{B}(H) \mid TS = ST \ \forall S \in U \}$
is called commutant.

- always a subalgebra, regardless of U
- always weakly closed, since we can see U' as $\forall x, y \in H$, all T s.t.
$$\langle S^* x, Ty \rangle = \langle Tx, S^* y \rangle$$
$\forall S \in U$, so take net and it works due to continuity

- if U is $*$-symmetric (i.e. $x^* \in U$ if $x \in U$) then U is a unital C^*-algebra

- $A \subset B \Rightarrow B' \subset A'$
- $A \subset A'' =: (A')'$

$\rightarrow$ these give $A''' = A'$, so we only really have A, A' and A''.

Def. A subalg $B \subset \mathbb{B}(H)$ is called nondeg. if its nat. repr $B \hookrightarrow \mathbb{B}(H)$ is nondeg.

Lemma. A nondeg. subalg., $T \in A'$ fixed. Then

1) $T \in \overline{A}^{\text{strong}}$
2) $T = T^* \Rightarrow T \in \overline{\{ S \in A \mid S^* = S \}}^{\text{strong}}$
3) $\|T\| \le 1 \Rightarrow T \in \overline{\{ S \in A \mid \|S\| \le 1 \}}^{\text{strong}}$

Thm (Double commutant, von Neumann)

H Hilbert, A involutive nondeg subalg of $\mathbb{B}(H)$. TFAE

1) $A = A''$
2) A is weakly closed in $\mathbb{B}(H)$
3) — " — strongly — " —
4) the unit ball of A is weakly cl. in $\mathbb{B}(H)$
5) — " — strongly — " —

- von Neumann algs are (almost) never separable
- in some properties they resemble measure theory
- also Borel functional calculus

Def. A repr. of a $*$-alg is irreducible if the only invariant closed subspaces are $\{0\}$ and H_π.

Thm For a nontrivial representation of an involutive algebra A TFAE

1) π is irred.
2) any $H_\pi \ni v \neq 0$ is cyclic (or totalising)
3) $\pi(A)' = \mathbb{C}$
4) $\pi(A) \subset \mathbb{B}(H)$ is strongly dense

Cor For a C^*-alg, π is irred. $\Rightarrow$ π algebraically irreducible.