Dąbrowski

- on $\mathbb{C}^{2^m} = \mathbb{C}(2^{\overset{=2m}{\uparrow}{n}})$,

$$\gamma(e_j) =: \gamma_j = \begin{cases} \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2 \otimes \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 & 1 \leq j \leq m \\ \quad '' \quad \cdot \otimes \sigma_2 \otimes \quad '' & m < j \leq 2m \end{cases}$$

- for $n = 2m+1$,

$\exists!!$ $\gamma_n = \gamma_{2m+1} := \pm (i)^m \gamma_1 \cdots \gamma_{2m}$

- irred., inequivalent reps, but not faithful

- while $\gamma|_{Pin(n,\mathbb{C})}$ remains irreducible,

for $\gamma|_{\mathcal{C}\ell^+(\mathbb{C}^n) \supset Spin(n,\mathbb{C})}$ → n odd, irred.

→ n even, $S = S_+ \oplus S_-$,

$S_\pm$ eigensp. of γ_{2n+1}

→ Weyl-, half- or semispinors.

→ $\gamma_i \gamma_{2m+1} = -\gamma_{2m+1} \gamma_i$, $S_\pm \mapsto S_\mp$ by a reflection

- notice γ_i are hermitian, with $\gamma_j \begin{cases} \text{real}, & j \in 1, \ldots, m \\ \text{imaginary}, & j \in m+1, \ldots, 2n \end{cases}$

↳ always have σ_2

- so for $\gamma|_{\mathcal{C}\ell(\mathbb{R}^{p,q})}$, $\gamma_j' := \begin{cases} \gamma_j, & j = 1, \ldots, p \\ i\gamma_j, & j = p+1, \ldots, n \end{cases}$

$$\rightarrow p - q = \left\{ \begin{matrix} 1, 5 \text{ mod } 8 \\ 2, 3, 4 \text{ mod } 8 \\ 0, 6, 7 \text{ mod } 8 \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \oplus \text{ of } 2 \text{ irreps} \\ \text{quaternionic} \\ \text{real} \end{matrix} \right.$$

q	0	1	2	3	4	5	6	7
$\mathcal{C}(\mathbb{R}^{0,q})$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}^2$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)^2$
$(\mathcal{C}(\mathbb{R}^{0,q}))^{\text{simple}}$	·	·	·	$\mathbb{H}$	·	·	·	$\mathbb{R}(8)$
$\dim_{\mathbb{R}}\text{irrep}$	1	2	4	4	8	8	8	8
$\chi(q)$	0	1	2	2	3	3	3	3

- $\chi(q) = \log_2(\dim_{\mathbb{R}}\text{irrep}) =$ Radon-Hurwitz #, satisfying $\chi(q+8) = \chi(q) + 4$

- γ also induces reps of $\text{spin}(p,q)$
  - also on $\text{spin}(n) = \langle \frac{1}{2}\gamma_j\gamma_k \mid j<k \rangle$

  - but on $\text{spin}(n)$, $(\gamma_j\gamma_k)^{\dagger} = \gamma_k^{\dagger}\gamma_j^{\dagger} = \gamma_k\gamma_j = -\gamma_j\gamma_k$ so all generators antihermitean
    $\Rightarrow$ exponentials are unitary

$$\begin{tikzcd}
\text{Spin}^{\mathbb{C}}(n) = \frac{\text{Spin}(n)\times U(1)}{\mathbb{Z}_2} \arrow[r, hook, "\gamma"] \arrow[d, "\rho"] & \mathcal{U}(2^m) \arrow[d, "\pi"] \\
SO(n) \arrow[r, dashed, "\exists\ \text{mono}"] & P\mathcal{U}(2^m) = \mathcal{U}(2^m)/\langle \mathcal{U}(1)\cdot \mathbb{1}_{2^m\times 2^m} \rangle
\end{tikzcd}$$

Matrix forms

$$\begin{tikzcd}
\mathbb{C}^4 \ni (z_1, \dots, z_4) \arrow[r, mapsto, "\text{lin.}", "\sim"', "\text{isom.}"'] \arrow[d, "\rho"] & \hat{z} = z_4 \mathbb{1}_2 + i\sum_{j=1}^{3} z_j \sigma_j = \begin{pmatrix} z_4 + i z_3 & i z_1 + z_2 \\ i z_1 - z_2 & z_4 - i z_4 \end{pmatrix} \\
SO(4,\mathbb{C}) &
\end{tikzcd}$$

- for $U, V \in SL(2,\mathbb{C})$, $\det \hat{z} = \det U \hat{z} V^{-1} = \sum z_i^2$

- claim: ρ 2ble cover, $\text{Spin}(4,\mathbb{C}) = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$

- take $z_i \in \mathbb{R} \iff \overline{(\hat{z})}\,\sigma_2 = \sigma_2 \hat{z}$

$U, V \in SU(2)$

$\Rightarrow \mathrm{Spin}(4) = SU(2) \times SU(2)$

- if $z_4 \in i\mathbb{R}$, $z_i \in \mathbb{R} \iff \hat{z}^{\dagger} = \hat{z}$
  $\Rightarrow U^{\dagger} = V^{-1}$
  $\Rightarrow \mathrm{Spin}_0(3,1) = \mathrm{Spin}_0(1,3) = SL(2,\mathbb{C})$
- $z_4 = 0 \iff \mathrm{Tr}\,\hat{z} = 0 \iff U = V \Rightarrow \mathrm{Spin}(3,\mathbb{C}) = SL(2,\mathbb{C})$ also

Charge conj.

- in physics: $J\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) = \gamma^{\mu}(\partial_{\mu} - ieA_{\mu})J$
- $J_{\pm}$ should $\underset{\text{anticommute}}{\overset{\text{commute}}{\vee}}$ w all γ, $\mathbb{C}$-antilinear
- let $J_{\pm} = C_{\pm} \circ (\text{cpx. conj.})$
  $\Rightarrow C_{+}$ should commute w real, anticom. w imag.
- n even. On $\mathcal{E}(\mathbb{R}^{0,n})$, $\gamma_j = \begin{cases} \text{real} & 1 \leq j \leq m \\ \text{pure im.} & m < j \leq 2m \end{cases}$
  - C_{+}?
  - $C_{+} \sim \begin{cases} \gamma_1 \cdots \gamma_m \\ \gamma_{m+1} \cdots \gamma_{2m} \end{cases}$ if $m \begin{cases} \text{even} \\ \text{odd} \end{cases}$
- $n = \text{odd} = 2m+1$, $\gamma_{2m+1} = (\ )i\gamma_1 \text{—} \gamma_{2m}$ imaginary $\Rightarrow$ J_{+} if $\substack{m = 1,3 \bmod 4 \\ (n = 3,7 \bmod 8)}$
  J_{-} if $\substack{m = 0,2 \bmod 4 \\ (n = 1,5 \bmod 8)}$

- $J_\pm J_\pm^\dagger = 1$. But adjoint needs to be defined as $\langle \psi, J\varphi\rangle = \langle J^\dagger \varphi, \psi\rangle$ for antiunitary op. ($\langle J\varphi, J\psi\rangle = \overline{\langle \varphi, \psi\rangle}$)

- further: $J_\pm^2 = \varepsilon \cdot \mathbb{1}_2$

$$J_\pm D = \varepsilon' D J_\pm$$

$$J_\pm \gamma(\omega) = \varepsilon'' \gamma(\omega) J_\pm$$

$\varepsilon, \varepsilon', \varepsilon''$ signs:

	n	0	1	2	3	4	5	6	7	mod 8
	ε	+	×	−	−	−	×	+	+	
J_+	ε'	+		+	+	+		+	+	
	ε''	+		−	×	+		−	×	
	ε	+	+	+	×	−	−	−	×	
J_-	ε'	−	−	−		−		−		
	ε''	+	×	−		+		×		

- we say more $\gamma(\mathrm{Spin}(n)) < \gamma(\mathrm{Spin}_c(n))$ is precisely the $\mathrm{Ad}_{J_\pm}$ inv subgroups
- physically these have no charge