Dąbrowski

Prop $\mathcal{C}(\mathbb{C}^n) \simeq \mathcal{C}(\mathbb{R}^{p|q}) \otimes_{\mathbb{R}} \mathbb{C}$ as $\mathbb{C}$-algs.

$\mathcal{C}(\mathbb{R}^{p+1,q+1}) \simeq \mathcal{C}(\mathbb{R}^{p,q}) \otimes \mathcal{C}(\mathbb{R}^{1,1})$

$\mathcal{C}(\mathbb{R}^{p+1,q}) \simeq \mathcal{C}(\mathbb{R}^{q+1,p})$

$\mathcal{C}(\mathbb{R}^{p,q+3}) \simeq \mathcal{C}(\mathbb{R}^{q,p+3})$

Pf. Take $\{\varepsilon_1, \ldots, \varepsilon_p, \varepsilon_{p+1}, \ldots, \varepsilon_{p+q}\}$ basis of $\mathcal{C}(\mathbb{R}^{p|q})$, and build $\{\varepsilon_1 \otimes 1, \ldots, \varepsilon_p \otimes 1, \varepsilon_{p+1} \otimes i, \ldots, \varepsilon_{p+q} \otimes i\}$ basis of $\mathcal{C}(\mathbb{R}^{p|q}) \otimes_{\mathbb{R}} \mathbb{C}$. But all of these anticommute and square to 1, so they're a basis of $\mathcal{C}(\mathbb{C}^n)$, universal prop!

For 2nd isom,

$\{\varepsilon_1 \otimes \varepsilon_1\varepsilon_2, \ldots, \varepsilon_p \otimes \varepsilon_1\varepsilon_2, 1 \otimes \varepsilon_1,$
$\varepsilon_{p+1} \otimes \varepsilon_1\varepsilon_2, \ldots, \varepsilon_{p+q} \otimes \varepsilon_1\varepsilon_2, 1 \otimes \varepsilon_2\}$

Since $(\varepsilon_1\varepsilon_2)^2 = \varepsilon_1\varepsilon_2\varepsilon_1\varepsilon_2 = -\varepsilon_1\varepsilon_2\varepsilon_2\varepsilon_1 = +1$ (remember $\varepsilon_1^2 = 1, \varepsilon_2^2 = -1$), we get the claim.

3rd, $\{\varepsilon_{q+2}\varepsilon_{q+1}, \ldots, \varepsilon_{q+p+1}\varepsilon_{p+1}, \varepsilon_{q+1};$
$\varepsilon_1\varepsilon_{q+1}, \ldots, \varepsilon_q\varepsilon_{q+1}\}$

4th, $\{\varepsilon_{q+1}\varepsilon, \ldots, \varepsilon_{q+p}\varepsilon; \varepsilon_1\varepsilon, \ldots, \varepsilon_q\varepsilon, \varepsilon_{p+q+1}, \varepsilon_{p+q+2}, \varepsilon_{p+q+3}\}$
where $\varepsilon = \varepsilon_{p+q+1} \cdot \varepsilon_{p+q+2} \cdot \varepsilon_{p+q+3}$, $\varepsilon^2 = 1$.

Of course, we need to check that the dimensions of both sides add up.

- focusing on low dim. $\mathcal{C}(\mathbb{R}^{p,q})$, $\mathcal{C}(\mathbb{C}^n)$...

(p,q)	$(1,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,1)$
$\mathcal{C}(\mathbb{R}^{p,q})$	$\overbrace{\mathbb{R}\oplus\mathbb{R}}^{\mathbb{R}^2}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{R}(2)$ // notation $M_2(\mathbb{R})$

$\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$,
$\mathbb{D}(n) := M_n(\mathbb{D})$

all $\otimes = \otimes_{\mathbb{R}}$

- recall: $\mathbb{R}\otimes_{\mathbb{R}}\mathbb{R} \simeq \mathbb{R}$, $\mathbb{R}(n)\otimes\mathbb{D} \simeq \mathbb{D}(n)$,
$\mathbb{R}(n)\otimes\mathbb{R}(m) = \mathbb{R}(n\cdot m)$, $\mathbb{C}\otimes\mathbb{C} \simeq \mathbb{C}\oplus\mathbb{C}$
$\mathbb{C}\otimes\mathbb{H} \simeq \mathbb{C}(2)$

$p+q$ \ $p-q$	-4	-3	-2	-1	0	1	2	3	4
4	$\mathbb{H}(2)$		$\mathbb{H}(2)$		$\mathbb{R}(4)$		$\mathbb{R}(4)$		$\mathbb{H}(2)$
3		$\mathbb{H}^2$		$\mathbb{C}(2)$				$\mathbb{C}(2)$	
2			$\mathbb{H}$		$\mathbb{R}(2)$		$\mathbb{R}(2)$		
1				$\mathbb{C}$		$\mathbb{R}^2$			
0					$\mathbb{R}$				

- symmetry around $1, -3$
by Prop,
going up by
$p,q \mapsto p+1, q+1$
gives you
$-\otimes_{\mathbb{R}}\mathcal{C}(\mathbb{R}^{1,1}) = -\otimes_{\mathbb{R}}\mathbb{R}(2) = -(\times 2)$

- note a funny thing,
$$\mathcal{C}(\mathbb{R}^{1,3}) = \mathbb{H}(2) \not\simeq \mathbb{R}(4) = \mathcal{C}(\mathbb{R}^{3,1}),$$
so we can "distinguish" (anti) Minkowski
- but this amounts to the choice of $\pm$ in defining Clifford ideal

- mod-8 periodicity, $\mathcal{C}(\mathbb{R}^{p+4,q}) \simeq \mathcal{C}(\mathbb{R}^{q+4,p})$
gives $\mathcal{C}(\mathbb{R}^{p+8,q}) \simeq \underset{\substack{\text{is} \\ \mathcal{C}(\mathbb{R}^{p,q+8})}}{\mathcal{C}(\mathbb{R}^{p+4,q+4})} \simeq \mathcal{C}(\mathbb{R}^{p,q})\otimes\mathbb{R}(16)$

- for $\mathcal{C}(\mathbb{C}^n)$, mod 2 periodic

Def Volume element $\omega := \varepsilon_1 \cdots \varepsilon_n$

- ω (anti) commutes with $v \in V$ if n (even) odd.

Prop The center of $\mathcal{C}(V)$ is

$$Z(\mathcal{C}(V)) = \begin{cases} \mathbb{k} \\ \mathbb{k} \oplus \mathbb{k}\omega \end{cases} \text{for } n \begin{cases} \text{even} \\ \text{odd} \end{cases}$$

- $\omega^2 = (-)^{(p-q)(p-q-1)/2} = \begin{cases} +1 \\ -1 \end{cases}$ if $p-q = \begin{cases} 0,1 \\ 2,3 \end{cases}$ mod 4

- means $\mathcal{A} = \mathcal{C}(\mathbb{R}^{p,q})$ is $\oplus$ of 2 simple algs (± 1 eigensp. of ω) if $p-q = 1$ mod 4 & otherwise simple. $\mathcal{A} = \frac{1+\omega}{2}\mathcal{A}\left(\frac{1+\omega}{2}\right) \oplus \frac{1-\omega}{2}\mathcal{A}\left(\frac{1-\omega}{2}\right)$

- $\mathcal{C}(\mathbb{C}^n)$ is $\begin{cases} \oplus \text{ of 2 simpl. algs.} \\ \text{simple} \end{cases}$ when $n = \begin{cases} \text{odd} \\ \text{even} \end{cases}$

$\omega = (-i)^m \varepsilon_1 \cdots \varepsilon_n$, $n = 2m+1$, $m = \left[\frac{n}{2}\right]$

- $\mathcal{C}(V)$ is $\mathbb{Z}_2$ graded under main involution $a \mapsto \bar{a}$, $v \mapsto -v$ antiautom., so sends evens to evens, changes sgn of odd
$\mathcal{C}^{\pm}(V) = \pm 1$ eigensp. of main invol.

- can be shown: $\mathcal{C}^+(\mathbb{R}^{p,q}) \overset{\text{alg.}}{\cong} \mathcal{C}(\mathbb{R}^{p,q-1})$,
take $\{\varepsilon_1\varepsilon_{p+q}, \ldots, \varepsilon_p\varepsilon_{p+q}; \varepsilon_{p+1}\varepsilon_{p+q}, \ldots, \varepsilon_{p+q-1}\varepsilon_{p+q}\}$

- $\mathcal{C}^+(\mathbb{R}^{p,q})$ is $\begin{cases} \oplus \text{ of 2 simple} \\ \text{simple} \end{cases}$ if $p-q = \begin{cases} 1 \text{ mod } 4 \\ \end{cases}$
$\mathcal{C}^+(\mathbb{R}^{q,p}) \cong \mathcal{C}(\mathbb{R}^{q,p-1})$

- If we use graded tensor product, $A \hat{\otimes} B$, $(a \otimes b)\cdot(a' \otimes b') = (-)^{|a'||b|}(a \cdot a' \otimes b b')$

$$\Rightarrow \mathcal{C}(V_1, Q_1) \hat{\otimes} \mathcal{C}(V_2, Q_2) = \mathcal{C}(V_1 \oplus V_1, Q_1 \oplus Q_2)$$

$V_1 \oplus V_2 \ni v_1 \oplus v_2 \mapsto v_1 \otimes 1 + 1 \otimes v_2$

$(-)^2 = Q_1 \otimes 1 + v_1 \otimes v_2 - v_1 \otimes v_2 + 1 \otimes Q_2$ ✓

$$\Rightarrow \mathcal{C}(\mathbb{R}^{p,q}) \simeq \mathcal{C}(\mathbb{R}^{1,0})^{\hat{\otimes} p} \hat{\otimes} \mathcal{C}(\mathbb{R}^{0,1})^{\hat{\otimes} q}$$

$$\Rightarrow \mathcal{C}(\mathbb{R}^{0,q}) \simeq \mathbb{C}^{\hat{\otimes} q}, \quad \mathcal{C}(\mathbb{C}^n) \simeq \mathcal{C}(\mathbb{C})^{\hat{\otimes} n} \simeq (\mathbb{C}^2)^{\hat{\otimes} n}$$

Spin groups

$$\mathcal{C}^*(V) := \{ a \in \mathcal{C}(V) \mid \exists\, a^{-1} \}$$

Def. Twisted adjoint rep. of $\mathcal{C}^*(V)$ on $\mathcal{C}(V)$, $\varsigma : \mathcal{C}^*(V) \to \mathcal{C}(V)$, $a \mapsto \varsigma(a)$,

$$\varsigma(a)\, b := \bar{a}\, b\, a^{-1} \text{ for any } b \in \mathcal{C}(V).$$

Prop $\ker \varsigma = \mathbb{k}^*$

Pf. $\bar{a} b = b a \ \forall b \Leftrightarrow (a_+ - a_-) b = b(a_+ + a_-) \ \forall b$.
Take $b = 1 \Rightarrow a_- = 0$. So $a_+ b = b a_+ \ \forall b$
$\Leftrightarrow a \in Z \cap (\mathcal{C}^+)^* = \mathbb{k}^*$, since we have even dim. □