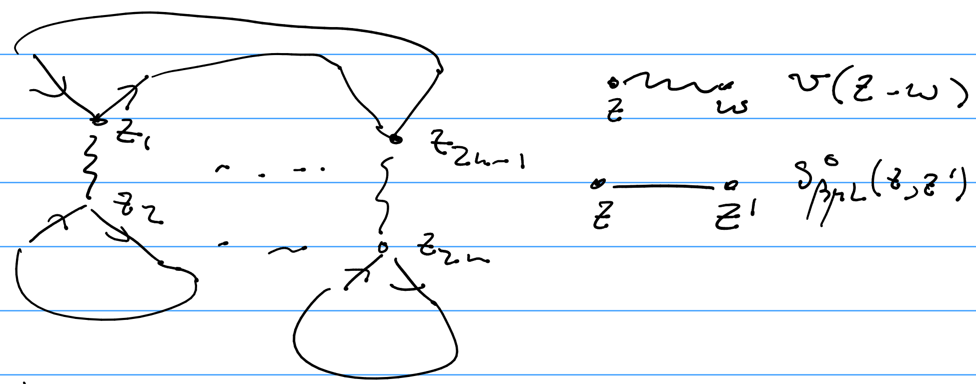
Part a.

$$\frac{Z_{\beta,\mu,L}}{Z^0_{\beta,\mu,L}} = 1 + \sum_{n\geq 1} \frac{(-)^n}{n!} \int_{[0,\beta]^n} d\underline{t} \, \langle \mathsf{T} V_{t_1} \cdots V_{t_n} \rangle^0_{\beta,\mu,L}$$

where $V_t = e^{t(\mathcal{H}_0 - \mu \mathcal{N})} V e^{-t(\mathcal{H}^0 - \mu\mathcal{N})}$

$$= \sum_{n\geq 0} \frac{(-)^n}{n!} \int d\underline{Z}_n \, d\underline{W}_n \sum_{\xi \in G_n(\underline{Z}_n, \underline{W}_n)} \mathrm{Val}(\xi).$$

$\xi \rightsquigarrow$ 

$$\mathrm{Val}(\xi) = \mathrm{sign}(\xi) \prod_{i=1}^n v(z_{2i-1} - z_{2i}) \prod_{e\in\xi} g_e$$

where $g_e = S^0_{\beta\mu L}(z(e), z'(e))$

Goal. Prove analiticity of $f_{\beta\mu L} = -\frac{1}{\beta|\Lambda_L|} \log Z_{\beta\mu L}$

Problems. (*) estimate of $\mathrm{Val}(\xi)$
(*)

- if gapped:

$$\frac{1}{\beta|\Lambda_L|}\int d\underline{Z}\, |\mathrm{Val}(\xi)| \leq \|v\|_1^n \, \|g_e\|_1^{n-1} \underbrace{\|g_e\|_\infty^{n+1}}_{\text{loops}}$$

since $|S^0_{\beta\mu L}(z,z')| \leq C e^{-c\|z-z'\|}$ for gap

- but we're completely ignoring signs

- also, # graphs $\approx n!$ at $O(n)$, and series has only $\frac{1}{n!}$.

- Brydges - Battle - Federbush:

- let $f(n) = \sum_{\xi \in G_c(n)} \mathrm{Val}(\xi)$ $\left[\prod v(z_{2i}-z_{2i-1})\right]$

- then $f(n) = \sum_{T \in T_n} \left[\prod_{e \in T} g_e\right] \times \int d\mu_T(t) \det G_T(t)$

where:

1) T is a tree between n vertices (by int. out $\xi \overset{\int}{\mapsto} \bullet$ interactions)

ii) $G_T(t)$ is a $(2n-(n-1)) \times (2n-(n-1))$ matrix with entries

$$\left[G_T(t)\right]_{(J,i)(J',i')} = t_{J,J'}\, g\big(\underbrace{x(J,i)-x(J',i')}_{\ell}\big)$$

where $1 \le T, T' \le n$, $1 \le i, i' \le 2$,
$x(J,1) = z_{2T-1}$, $x(J,2) = z_{2T}$
and the element is only nonzero provided the vertices are connected by propagator

iii) $t_{J,J'} \in [0,1]$, $d\mu_T(t)$ is a ? measure on $\{t_{JJ'}\}$

Rmk # of $T \sim n!$

Problem. estimate det?

Gram-Hadamard ineq. let M $n \times n$ matrix such that $M_{ij} = \langle A_i, B_j \rangle$.
Then $|\det M| \le \prod_{i=1}^{n} \|A_i\| \cdot \|B_i\|$

- it turns out we get UV divergence

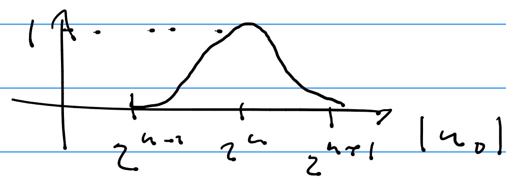
Multiscale analysis (RG)

- write $g^{(\le N)} = g^{(\le 0)} + \sum_{k=0}^{N} g^{(k)}$

$$g^{(k)}(z,z') = \frac{1}{\beta}\sum_{k_0\in M_\beta} \frac{e^{-ik_0(z_0-z_0')}}{-ik_0+H-\mu}(\vec z,\vec z')\, f_k(k_0)$$

$\frac{2\pi}{\beta}(n+\frac{1}{2})$

where $f_k(k_0)=\chi(2^{-k}|k_0|)-\chi(2^{-(k-1)}|k_0|)$



Graßmann variables.

→ finite set $(\psi^+_\alpha,\psi^-_\alpha)_{\alpha\in\Lambda}$ s.t. $\{\psi^\varepsilon_\alpha,\psi^{\varepsilon'}_{\alpha'}\}=0$ $\forall \alpha\alpha'\varepsilon\varepsilon'$

→ let $(d\psi^+_\alpha, d\psi^-_\alpha)_{\alpha\in\Lambda}$ s.t. $\{d\psi^\varepsilon_\alpha, d\psi^{\varepsilon'}_{\alpha'}\}=0$
$\{d\psi^\varepsilon_\alpha,\psi^{\varepsilon'}_{\alpha'}\}=0$

- def. $\int d\psi^\varepsilon_\alpha=0$, $\int d\psi^\varepsilon_\alpha\, \psi^\varepsilon_\alpha = 1$

- $F(\psi)$ polynomial $\Rightarrow$ $F(\psi)=\sum_X f(X)\, \underbrace{\psi(X)}_{\prod_{(\alpha,\varepsilon)\in X}\psi^\varepsilon_\alpha}$

- $\int \psi(X)\prod_{\alpha\in\Lambda} d\psi^+_\alpha d\psi^-_\alpha = \text{sign}(\pi(X))$

$\Rightarrow \psi(X) = \text{sign}(\pi(X))\,\psi^-_{|\Lambda|}\psi^+_{|\Lambda|}\cdots\psi^-_1\psi^+_1$

Prop (Gaussian int.) let M be $|A|\times|A|$ mtx.

Then $$\frac{\int D\psi\, e^{-\sum_{\alpha\beta}\psi_\alpha^+ M_{\alpha\beta}\psi_\beta^-}\psi_{\beta'}^-\psi_{\alpha'}^+}{\int D\psi\, e^{-\sum_{\alpha,\beta}\psi_\alpha^+ M_{\alpha\beta}\psi_\beta^-}} = (M^{-1})_{\beta'\alpha'}$$

- given g $|A|\times|A|$ invertible (& diagonizable) w $\{g_\alpha\}_{\alpha\in|A|}$: eigenvalues, let $C = g^{-1}$ and define

$$P(d\psi) = \left[\prod_{\alpha\in A} d\psi_\alpha^+ d\psi_\alpha^- g_\alpha\right] e^{-\sum_{\alpha,\beta}\psi_\alpha^+ C_{\alpha\beta}\psi_\beta^-}$$

- immediately: $\int P(d\psi) = 1$, $\int P(d\psi)\psi_\alpha^-\psi_\beta^+ = g_{\alpha\beta}$

Prop (Grass. Wick rule)

$$\int P(d\psi)\psi_{\alpha_1}^-\cdots\psi_{\alpha_n}^-\psi_{\beta_1}^+\cdots\psi_{\beta_m}^+$$
$$= \delta_{n,m}\sum_\pi \operatorname{sign}(\pi)\prod_{i=1}^n g_{\alpha_i,\pi(\beta_i)}$$

Rmk. $P_{g_1+g_2}(d\psi) = P_{g_1}(d\psi_1)P_{g_2}(d\psi_2)$
(addition principle)

→ very important & useful!

- we can integrate out things one by one..