Gauge @ IGAP

6D invariants cont'd. – Putrov

- VOA $(V, \omega, \mathbb{1}, Y : V \to \mathrm{End}\, V[[z, z^{-1}]])$

  V: $\mathbb{C}$-vsp, $\mathbb{Z}\times\mathbb{Z}_2$-gr.; $\mathbb{1} \in V$

- modules (M, Y_M)

Example 1: Heisenberg VOA

- $\mathcal{H}$ – $\mathbb{R}$-vsp w nondeg sym pairing $(\cdot,\cdot)$
- $\mathrm{Heis}(\mathcal{H}) := (\mathcal{H} \otimes \mathbb{C}[[t, t^{-1}]]) \oplus \mathbb{C}\cdot c$ (c central)

  $\forall a \in \mathcal{H}$, $a_n := a \otimes t^n$

  s.t. $\forall a, b \in \mathcal{H}$, $[a_n, b_m] = n\,\delta_{n,-m}(a,b)\cdot c$
- for $\lambda \in \mathcal{H}$, build

  $\mathrm{Fock}_\lambda(\mathcal{H})$, highest weight module,

  $a_0|\lambda\rangle = (a,\lambda)|\lambda\rangle$

  $a_n|\lambda\rangle = 0 \quad \forall n > 0$

  $c|\lambda\rangle = |\lambda\rangle$

$$\Rightarrow \mathrm{Fock}_\lambda(\mathcal{H}) \cong \langle a^{(1)}_{-n_1-1} a^{(2)}_{-n_2-1} \cdots a^{(k)}_{-n_k-1} |\lambda\rangle \rangle_{\forall n_i \geq 0}$$
$$\cong \mathrm{Sym}(H \otimes \mathbb{C}[[t^{-1}]]\cdot t^{-1})$$

- "2d free chiral bosons"
- $V := \mathrm{Fock}_0(\mathcal{H})$ has VOA structure
  - $\forall a \in \mathcal{H}$, let $a(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$,

$$Y(a^{(1)}_{-n_1-1} \cdots a^{(k)}_{-n_k-1}|0\rangle, z) := :\prod_{i=1}^{k} \frac{d^{n_i} a^{(i)}(z)}{dz^{n_i}}:$$

$$\mathbb{1} = |0\rangle, \quad \omega = \frac{1}{2}\sum_{i,j} (e_i, e_j)\, e^i_{-1} e^j_{-1}|0\rangle$$

where $\{e^i\}, \{e_j\}$ bases for $\mathcal{H}, \mathcal{H}^\vee$

- $\mathrm{Fock}_\lambda(\mathcal{H})$ is a V-module
- also, we have

$$U_\lambda : \mathrm{Fock}_\mu(\mathcal{H}) \to \mathrm{Fock}_{\mu+\lambda}(\mathcal{H})[[z,z^{-1}]]\cdot z^{(\lambda,\mu)}$$

given as

$$U_\lambda(z) = :e^{\int \lambda(z)}: = z^{\lambda_0} e^{-\sum_{n<0} \frac{\lambda_n}{n z^n}} e^{-\sum_{n>0} \frac{\lambda_n}{n z^n}} \cdot \varepsilon_\lambda$$

where $\varepsilon_\lambda |\mu\rangle = |\lambda+\mu\rangle$ is a ladder op.

- $\langle \mu | U_{\lambda_k}(z_k) \cdots U_{\lambda_1}(z_1) | 0 \rangle$

$$= \begin{cases} \prod_{i<j} (z_i - z_j)^{(\lambda_i,\lambda_j)} & \text{if } \sum_i \lambda_i = \mu \\ 0 & \text{otherwise} \end{cases}$$

Example 2. Lattice VOA

- pick $\Lambda \subset \mathcal{H}$, $\Lambda \simeq \mathbb{Z}^{\dim \mathcal{H}}$, $\Lambda^\vee$ dual
- $V_\Lambda := \bigoplus_{\lambda \in \Lambda} \mathrm{Fock}_\lambda(\mathcal{H})$

$$Y(\text{same}, z) = : \prod_{i=1}^{k} \frac{1}{n_i!} \frac{d^{n_i} a^{(k)}(z)}{d z^{n_i}} U_{\lambda_k}(z_k) :$$

VOAs associated to 4-mfds

Physics:

$$\begin{matrix} \text{6d } (2,0) \\ \text{SCFT } T_g \end{matrix} \xrightarrow[\text{on } M^4]{\text{top. tw. cpctfy}} \begin{matrix} \text{2d } \mathcal{N}=(0,2) \\ \text{QFT} \\ T_g[M^4] \end{matrix} \xrightarrow{\text{BPS spectrum}} \mathrm{VOA}_g[M^4]$$

- e.g. NLSM w tgt X $\rightsquigarrow$ $H^0(X, \text{sheaf of chiral diff operators})$

$$\mathrm{Lie}\, G = \mathfrak{g}$$

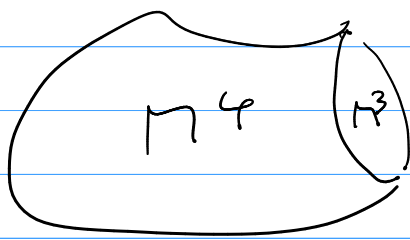
$$- \mathrm{VOA}_{\mathfrak{g}}[M^4] \overset{(\text{naive})}{=\!=} H^{*}\Big(\bigsqcup_{c_2 \geq 0} \mathcal{M}_{\mathrm{inst}}(M^4, G, c_2)\Big)$$

$\mathbb{Z}_2$-grading $\qquad\qquad$ $\mathbb{Z}$-grading

- if no "monopole" contributions
- c_2 not always defined
- ambiguity in G choice

$$\mathrm{Tr}_{\mathrm{VOA}_{\mathfrak{g}}[M^4]} (-)^F q^{L_0} = \sum_{c_2} q^{c_2} \chi^{(v,s)}\big(\mathcal{M}_{\mathrm{inst}}(\dots)\big)$$

- e.g. $\mathfrak{g} = \mathfrak{u}(1)$, M^4 smooth cpt w/ $\pi_1(M^4) = 0$ and $M^3 := \partial M^4$ possibly $\neq \emptyset$



$$\mathrm{VOA}_{\mathfrak{u}(1)}[M^4] = \bigoplus_{\lambda \in \Lambda} \mathrm{Fock}_0\big(H^0(M^4;\mathbb{R}) \oplus H^4(M^4;\mathbb{R})\big) \otimes \mathrm{Fock}_{\lambda}\big(H^2(M^4;\mathbb{R})\big)$$

$$\mathcal{H} = H^*(M^4, \mathbb{R}), \quad \Lambda = H_2(M^4; \mathbb{Z}) \simeq H^2(M^4, M^3, \mathbb{Z}) \subset H^2(M^4, \mathbb{Z}) := \Lambda^{\vee}$$

- if $M^4 \cong S$ cpx proj. sfc, by Grojnowski-Nakajima

$$\mathrm{VOA}_{\mathfrak{u}(1)}[S] \cong H^*\Big(\bigsqcup_{\substack{ch_2 \in \mathbb{Z} \\ c_1 \in \Lambda}} \mathcal{M}_{ch_2, c_1}\Big), \quad \mathcal{M}_{ch_2, c_1} \cong \mathrm{Hilb}_{ch_2}(S)$$