Tanzini

Localisation formulae

- poly forms $\alpha(\xi) \doteq \alpha_n + \alpha_{n-2} + \dots + \alpha_0$ ($\leftarrow$ even case)
- also $0 = d_v \alpha(\xi) = (d + i\xi\partial_v)(\alpha_n + \dots)$

$$= \underbrace{d\alpha_n}_{=0,\text{ say}} + \underbrace{i\xi i_v \alpha_n + d\alpha_{n-2}}_{\text{relates them!}} + \dots$$

$\rightarrow$ so the α_i are not independent

$$\int_M \alpha(M) = \left(\frac{-2\pi}{i\xi}\right)^{\ell} \sum_{\substack{\{p\} \\ \text{fixed pts}}} \frac{\alpha_0(\xi)(p)}{(\det L_p)^{1/2}},$$

$\left(\begin{matrix}\partial M = \phi, \\ M \text{ cpt.}\end{matrix}\right)$

$L_p \subset \mathfrak{G}(T_pM)$. If we split $T_pM = \bigoplus_{k=1}^{\ell} V_k$,

$$(\det L_p)^{1/2} = e(N_p) = (-)^{\frac{\lambda(p)}{2} \leftarrow \#\text{neg } v_k} \prod_k |v_k|$$

e: equiv. Euler class of normal bdl

- Example. $G = S^1$, $v = \frac{\partial}{\partial\varphi}$

$$x^i(\varphi) = x_0^i + R^i_j(\varphi)(x - x_0)^j$$

$$v = \sum_{i=1}^{\ell} v_i(x_i\partial_{y_i} - y_i\partial_{x_i}), \quad g = \sum_{i=1}^{\ell}(dx_i^2 + dy_i^2)$$

Pf. of localisation formula

- pick S^1-inv metric as above
- let $\psi := \frac{1}{2} g(v, -)$, and define

$$\beta(\xi) = d_v\psi = d\psi + i\frac{\xi}{2}\|v\|^2$$

- locally, $\psi = \frac{1}{2}\sum_{k=1}^{\ell} v_k(x_k dy_k - y_k dx_k)$

so $\beta(\xi) = \sum_{k=1}^{\ell} v_k \, dx_k \wedge dy_k + i\frac{\xi}{2}\sum_{k=1}^{\ell} v_k^2(x_k^2 + y_k^2)$

- now note that $\int_M \alpha(\xi) = \int_M \alpha(\xi) e^{isβ(\xi)}$,
since we are adding d_v - exact terms.
- so take $s \to \infty$ limit:

$$\int \alpha(\xi) \underset{s\to\infty}{\simeq} \int_M \alpha(M) e^{isβ(\xi)}$$

$$= \alpha_0(\xi)(x_p) \prod_{k=1}^{\ell} i s v_k \underbrace{\int dx_k \wedge dy_k \, e^{-s\xi \|v\|^2}}_{\frac{2\pi}{s\xi (v_k^{(p)})^2}}$$

$$= \frac{\alpha_0(\xi)(x_p)}{v_1^{(p)} \cdots v_\ell^{(p)}} \left(-\frac{2\pi}{i\xi}\right)^\ell. \quad \square$$

- another take:

Lemma if $\alpha(\xi)$ d_v-closed, then outside the 0-set of v its top dR component is closed.

$\to$ so $\int_M \alpha(\xi) = \lim_{\varepsilon\to 0} \int_{M \setminus \bigsqcup_{\text{fixed pts}} B(pt,\varepsilon)} \cdots = \cdots$

Duistermaat-Heckmann thm

- symplectic mfd (M, ω).
- let v Hamiltonian, i.e. $\exists H$, $i_v\omega + dH = 0$
- $\mathcal{L} := \frac{\omega^\ell}{\ell!} = [e^\omega]_{top}$ Liouville form
- then $I(\xi) = \int_M \mathcal{L} e^{i\xi H} = \left(\frac{-2\pi}{i\xi}\right)^\ell \sum_p \frac{e^{i\xi H(x_p)}}{\lambda_1^{(p)} \cdots \lambda_\ell^{(p)}}$

- this follows if we define $\omega(\xi) := \omega + i\xi H$,
  - $\to$ then $d_v \omega(\xi) = i\xi(i_v\omega + dH) = 0$
- so $I(\xi) = \int_M \alpha(\xi) = \int_M [e^{\omega(\xi)}]_{top} = \int_M \frac{\omega^k}{k!} e^{i\xi H}$

- example: S^2, $\omega = d\cos\theta\, d\varphi$, $H = \cos\theta$, $v = \partial_\varphi$

$$I(\xi) = \int \omega e^{i\xi z} = \left(\frac{-2\pi}{i\xi}\right)\left(\frac{e^{i\xi z_N}}{v^N} + \frac{e^{i\xi z_S}}{v^S}\right)$$

$$= 4\pi \frac{\sin\xi}{\xi}$$

- noncpt. example: $\mathbb{R}^2$
  $\omega = dx\,dy$, $v = x\partial_y - y\partial_x$, $\omega(\xi) = dx\,dy + i\xi \frac{x^2+y^2}{2}$

$$\Rightarrow \int_{\mathbb{R}^2} e^{i\omega(\xi)} = i\int_{\mathbb{R}^2} dx\,dy\, e^{-\frac{x^2+y^2}{2}\xi} = \frac{2\pi i}{\xi}$$

- Exercise. $S^2 \simeq \mathbb{CP}^1$

$$\omega_{FS} = \frac{i}{2\pi}\frac{dz\, d\bar{z}}{(1+|z|^2)^2}. \quad \text{v.field} \quad \begin{matrix} z \mapsto e^{i\varepsilon} z \\ \bar{z} \mapsto e^{-i\varepsilon}\bar{z} \end{matrix}$$

i) construct equiv. extension, compute equiv. volume
ii) $\mathbb{CP}^n$, $\omega_{FS} = i\partial\bar{\partial}\log|z|^2$, $z_j \mapsto e^{i\varepsilon_j} z_j$ (look at charts)
$\to$ note that each affine chart has a fixed pt!