# Tanzini

## Supermanifold

$\longrightarrow (M, A) =: M^{m|n}$, where $M$ mfd, $\dim_{\mathbb{R}} M = m$; $A$ graded algebra, $\dim A = n$

- consider $E \to M$ vector bdl, $A := \Lambda^\bullet E$
- • $E = TM$. $\Pi TM$ tautological smfd ($\pi$ parity reversal)
- • $A = A^+ \oplus A^-$, $\mathbb{Z}_2$-graded comm. alg.

  $[\alpha, \beta] = \alpha\beta - (-)^{\deg\alpha \deg\beta} \beta\alpha$

- local coords: $(x_1, \dots, x_m | \underbrace{\psi_1 \dots \psi_n}_{\text{Graßmann}})$

- superfields:

$$f(x, \psi) = f_0(x) + f^a(x)\psi_a + f^{ab}\psi_a\psi_b + \dots + f^{1\dots n}\psi_1 \cdots \psi_n$$

- Berezin integration:

$$\int \psi \, d\psi = 1, \quad \int d\psi = 1$$

$$\int \psi_1 \cdots \psi_n \, d\psi_1 \cdots d\psi_n = 1, \quad \int \psi_1 \cdots \hat{\psi}_i \cdots \psi_n \, d\psi_1 \cdots d\psi_n = 0$$

## Gaussian integration

$A$ $n \times n$ sym. real. (positive?)

$$\int_{\mathbb{R}^n} d^n x \, e^{-\frac{1}{2}(x, A x)} = (2\pi)^{N/2} (\det A)^{-1/2}$$

- if 0 modes present, use $\det' A = \prod(\text{nonzero eigenvals})$

$H$ $n \times n$ hermitian:

$$\int_{\mathbb{C}^n} \prod_i \frac{dz_i d\bar{z}_i}{2\pi i} e^{-z^t H z} = (\det H)^{-1}$$

- for fermions, integrate $\omega_{ij}\psi_i\psi_j$ skew-sym.

$$\int \exp\left[\tfrac{1}{2}\psi^t\omega\psi\right] d\psi_1 \cdots d\psi_{2m} = \mathrm{Pf}(\omega)$$

where $\frac{1}{m!}\left(\frac{1}{2}\psi^t\omega\psi\right)^m = \psi_1\cdots\psi_{2m}\cdot \mathrm{Pf}(\omega)$

- more explicitly:

$$\mathrm{Pf}(\omega) = \frac{1}{2^m m!}\sum_{\sigma\in S_{2m}} \mathrm{sign}(\sigma)\prod_{i=1}^{m}\omega_{\sigma(2i-1)\sigma(2i)}$$

$$\rightarrow \mathrm{Pf}(\omega)^2 = \det\omega$$

$$\int e^{\bar\psi^t\omega\psi}\prod_a d\psi_a d\bar\psi_a = \det\omega$$

## Localisation on smfds.

- Supergroup: Lie grp w/ $\mathbb{Z}_2$-graded generators

- $Q$ odd: $v\cdot y = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{-\varepsilon Q}y$, $\varepsilon^2 = 0$

**Prop.** Given a $Q$-inv sfunc $f$, only fixed pts contribute to the integral over $E$.

**Pf.** Suppose action free. Then $E/F$ smooth. So $\int_E f = \int_F d\varepsilon \int_{E/F} f = 0$, because $\int d\varepsilon = 0$.

- 0-dim QFT w. susy:

$$Z = \int_{\mathbb{R}^{1|2}} dx\, d\psi_1\, d\psi_2 \exp\left[-S(x,\psi_1,\psi_2)\right]$$

$$S(x,\psi_1,\psi_2) = S_0(x) + \psi_1\psi_2 S_1(x)$$

- let $S_0(x) = \frac{1}{2}(h')^2$, $S_1(x) = -h''$ for $h \in C^\infty(\mathbb{R})$

$$\delta_\varepsilon x = \varepsilon^1\psi_1 + \varepsilon^2\psi_2$$
$$\delta_\varepsilon \psi_1 = \varepsilon^2 h'$$
$$\delta_\varepsilon \psi_2 = -\varepsilon^1 h'$$

$$\Rightarrow \delta_\varepsilon S = 0 \ ?$$

- suppose $\delta_\varepsilon$ has no fixed pts.
$\Leftrightarrow h' \neq 0$
- pick substitution

$$\left.\begin{aligned} \hat{x} &= x - \frac{\psi_1\psi_2}{h'} \\ \hat{\psi}_1 &= \psi_1 - h'\frac{\psi_1}{h'} = 0 \\ \hat{\psi}_2 &= \psi_1 + \psi_2 \end{aligned}\right\} \quad \varepsilon^1 = \varepsilon^2 = -\psi_1/h'$$

- Exercise: show the Jacobian of is 1.

- but now $Z = \int d\hat{x}\, d\hat{\psi}_1\, d\hat{\psi}_2 \exp\left[-S(\hat{x},0,\hat{\psi}_1)\right] = 0.$

- $Z$ is counting crit. pts:

$$h(x) = h(x_c) + \frac{1}{2} h''(x_c)(x-x_c)^2 + \dots$$
$$h'(x) = h''(x_c)(x-x_c) + \dots$$

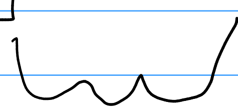
$$Z = \sum_{\{x_c\}} \frac{1}{\sqrt{2\pi}} \int dx\, d\psi_1\, d\psi_2 \exp\left[-\frac{1}{2} h''(x_c)^2 (x-x_c)^2 + h''(x_c)\psi_1\psi_2\right]$$

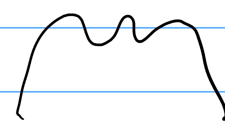
redefinition

$$= \sum_{\{x_c\}} \frac{h''(x_c)}{|h''(x_c)|} = \sum_{\{x_c\}} \mathrm{Sign}(\det \mathrm{Hess}\, h)$$

- supp. $h(x) \in \mathbb{R}[x]$

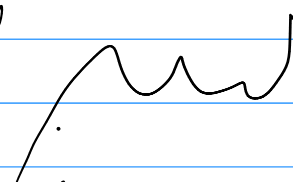
$\deg h(x) \to$ even





$Z = +1$ $\qquad$ $Z = -1$

$\to$ odd



$Z = 0$

- Invariant under deformations, as long as we don't change sign of highest power

- explicitly:

$$Z = \frac{1}{\sqrt{2\pi}} \int dx\, d\psi_1\, d\psi_2 \exp\left[-\frac{1}{2}(h')^2 + h''\psi_1\psi_2\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int \frac{dx\, h''}{d(h')} \exp^{-\frac{1}{2}(h')^2} = \{y = h'\}$$

$$= \frac{D}{\sqrt{2\pi}} \int_{\mathbb{R}} dy\, e^{-y^2} = D, \text{ where } D \text{ counts preimages of } y = h'(x)$$

## Deformation invariance

- $f = \delta_\varepsilon g \Rightarrow \int f e^{-S} = \int \delta_\varepsilon g \cdot e^{-S} = \int \delta_\varepsilon (g e^{-S}) = 0$

- $h \mapsto h + s$, $s$ small

$$S(h+s) = S(h) + \delta_s S, \quad \delta_s S = s' h' - s'' \psi_1 \psi_2$$
$$= \delta_\varepsilon \left( s'(x) \psi_1 \right), \quad \varepsilon^1 = \varepsilon^2 = 2$$

Rmk. classical soln $\Longleftrightarrow$ susy fixed pt !?

## Duistermaat–Heckmann.

- $(M, \omega) \rightsquigarrow$ taut. smfd. $\pi(TM)$ $M^{n|n}$