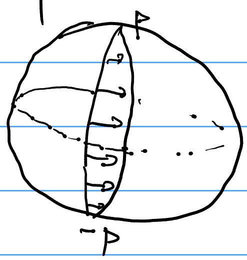
Stoppa.

- let's recall some definitions
  - $\Rightarrow \mathcal{J} := \{ \text{Jacobi fields along } \gamma(t),\ J(0)=0 \}$
  - $\mathcal{J}^\perp := \{ J(t) \in \mathcal{J} \mid \langle \dot\gamma(t), J(t) \rangle = 0 \ \forall t \}$
- recall also: $J \in \mathcal{J} \Rightarrow (d\exp_{\gamma(0)})_{t\dot\gamma(0)} \left(t \frac{DJ}{dt}(0) \right)$ (*)

Rmk Pick $y(s) \in T_{\gamma(0)}M$ s.t. $\frac{dy}{ds}(0) = \frac{DJ}{dt}(0)$
Construct a pencil $f(t,s) := \exp_p(t\, y(s))$.
By (*), $J(t)$ is the Jacobi field attached to f.

Def. Let $\gamma(t)$ be a geodesic. We call a point $\gamma(\bar t)$ a point conjugate to $\gamma(0)$ if $\exists J \in \mathcal{J}$ s.t. $J(\gamma(\bar t)) = 0$.

- e.g. antipodes in S^n are conjugate.



Def. The multiplicity of a pt $\gamma(\bar t)$ conj. to $\gamma(0)$ is the max # of lin. indep elements of $\mathcal{J}$ vanishing at $\gamma(\bar t)$

- e.g. $\mathrm{mult}_{S^n}(-p) = n-1$.

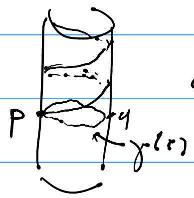
Lemma. $\bar t \dot\gamma(0) \in T_{\gamma(0)}M$ is a critical pt of $d\exp_p$ iff $\gamma(\bar t)$ is a conj. pt. to $\gamma(0)$ along $\gamma(t)$
Moreover, $\dim \ker d\exp_{\gamma(0)}|_{\bar t \dot\gamma(0)} = \mathrm{mult}\, \gamma(\bar t)$

Pf. $0 = J(\bar t) \overset{(*)}{\iff} (d\exp_p)_{\bar t \dot\gamma(0)} \left(\bar t \frac{DJ}{dt}(0) \right) = 0$
$\iff \frac{DJ}{dt}(0) \in \ker d\exp_{\gamma(0)}|_{\bar t \dot\gamma(0)}$

Def. $p\in M$. $\mathrm{Conj}(p) := \{1^{st}$ conj pts to p along all geodesics from $p\}$

$\mathrm{Cut}(p) := \{$pts after which geodesics from p stop minimising length$\}$

- e.g. i) $p\in S^n \Rightarrow \mathrm{Conj}(p) = -p = \mathrm{Cut}(p)$

ii) $\mathbb{R}\times S^1$ 

$\subset \mathbb{R}^3$. $\left.\begin{array}{l}\mathrm{Conj}(p) = \emptyset \\ \mathrm{Cut}(p) = q\end{array}\right\}$ along $\gamma(t)$

Rmk. $\gamma(T)\in \mathrm{Conj}\,\gamma(0) \Rightarrow \mathrm{mult}_\gamma\, \gamma(t) \le n-1$, because $t\dot\gamma(t)\in\mathcal{J}$ and $\neq 0$ at other pts.

Lemma. $\dim \mathcal{J}^\perp = n-1$

Pf. Claim: $\mathcal{J}^\perp$ is given by $J(t)$ s.t. $\langle \frac{DJ}{dt}(0), \dot\gamma(0)\rangle = 0$.
This is 1 condition on the initial conditions that define $\mathcal{J}^\perp$.
Compute the following:

$$\frac{d^2}{dt^2}\langle J(t),\dot\gamma(t)\rangle = \frac{d}{dt}\Big(\langle \tfrac{DJ}{dt}(t),\dot\gamma(t)\rangle + \langle J(t), \overset{0}{\overset{\shortparallel}{\tfrac{D\dot\gamma}{dt}(t)}}\rangle\Big)$$
$$= \langle \frac{D^2J}{dt^2}(t),\dot\gamma(t)\rangle$$
$$= \langle R(\dot\gamma(t), J(t))\dot\gamma(t), \dot\gamma(t)\rangle$$
$$= 0$$

$$\Rightarrow \langle J(t),\dot\gamma(t)\rangle = \langle \frac{DJ}{dt}(t),\dot\gamma(t)\rangle t + \langle J(0),\dot\gamma(0)\rangle.$$

Jacobi fields & sectional curvature.

Prop. Let $J(t) \in \mathcal{J}$, write $\dot{J}(0) = v$, $\frac{DJ}{dt}(0) = w$ and normalise $\|w\| = 1$.
Then

$$\|J(t)\|^2 = t^2 - \frac{1}{3}\underbrace{\langle R(v,w)v, w\rangle}_{\|v \wedge w\|^2 \cdot K(v,w)} t^4 + O(t^5)$$

$\|v\wedge w\|^2 \cdot K(v,w) \rightsquigarrow$ sectional curvature.

Rmk. Morally, $K \leq 0$ means geodesics "spread-out", and vice-versa.

Pf. Examine coefficients of the Taylor expansion of $\langle J(t), J(t)\rangle$ at $t=0$, use $J(0)=0$, Jacobi eqn., 2nd Bianchi id...

Theorem of Hopf-Rinow.

Def. Let (M,g) Riem. For $p,q \in M$ define

$$d(p,q) := \inf_{\{c:[0,1]\to M \text{ pcwise } C^1,\ c(0)=p,\ c(1)=q\}} \left(\operatorname{len}(c(t))\right)$$

Lemma. $d(-,-)$ is a distance fun, i.e. (M,d) metric space.

Lemma. (M,d) with topology induced by d is the same top. sp. as M.

Cor. $d: M\times M \to \mathbb{R}$ is continuous.