Fantechi

- $\mathcal{A}, \mathcal{A}'$ ab. cats, $F : \mathcal{A} \to \mathcal{A}'$ left exact
- $RF : D^+(\mathcal{A}) \to D^+(\mathcal{A}')$ "unique" (if exists)

- $\mathcal{A}$ ab. cat, $C(\mathcal{A})$ ab. cat, $K(\mathcal{A})$ Δ cat,
  $D(\mathcal{A}) = K(\mathcal{A})/\mathcal{N}$ Δ cat

Def. Let $\mathcal{T}_1, \mathcal{T}_2$ Δ cats. A functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ is a functor of Δcats if it is additive $\wedge$ commutes with $[1]$ $\wedge$ sends dist. Δ to dist. Δ.

Lemma $F : \mathcal{A} \to \mathcal{A}'$ induces $C(F), K(F)$.
Pf. Define $F(A^i, d^i) = (F(A^i), F(d^i)), \ldots$

- what do we want from RF?
  - If $\mathcal{A}$ has enough injectives, let $\mathcal{J}$ be subcat of $\mathcal{A}$ of inj objects
  - $K^+(\mathcal{J}) \subseteq K^+(\mathcal{A})$ full subcat of cpx A^i s.t. $A^i \in \mathcal{J}$ $\forall i$, $A^i = 0$ $\forall i \ll 0$.

Lemma $\dfrac{K^+(\mathcal{J})}{\mathcal{N} \cap K^+(\mathcal{J})} \xrightarrow{\alpha} \dfrac{K^+(\mathcal{A})}{\mathcal{N} \cap K^+(\mathcal{A})} = D^+(\mathcal{A})$ equiv.

- goal. Find univ prop. for $RF : D^+(\mathcal{A}) \to D^+(\mathcal{A}')$

$$\begin{tikzcd}
D^+(\mathcal{A}) \arrow[r, "\alpha^{-1}"] & \dfrac{K^+(\mathcal{J})}{\mathcal{N} \cap K^+(\mathcal{J})} \arrow[r] & D^+(\mathcal{A}') \\
& K^+(\mathcal{J}) \arrow[u] \arrow[r, "K^+(F)"] & K^+(\mathcal{A}') \arrow[u]
\end{tikzcd}$$

not really unique? it is only an equiv. of cats

Lemma $K^+(F)(\mathcal{N} \cap K^+(J)) \subseteq \mathcal{N}_{K^+(A')}$.

Recall Enough inj. $\Rightarrow$ $\forall A^\bullet \in \mathrm{ob}\, K^+(A)$ $\exists$ qis $A^\bullet \xrightarrow{\varphi} I^\bullet$ with $I^\bullet \in K^+(J)$

$$\begin{tikzcd} K^+(F)(A) \arrow[r] & K^+F(I) \arrow[r, phantom, "\overset{\simeq}{\underset{\text{can.}}{}}"] \arrow[d, equal] & RF(A^\bullet) \arrow[dl, "\sim"', "RF(\varphi)"] \\ & RF(I) & \end{tikzcd}$$

Def. Let $F: A \to A'$ be left exact fctor of ab cats. A **right derived functor** for F is a pair (T, φ) such that

i) $T: D^+(A) \to D^+(A')$ is a fctor of Δ cats.

ii) φ is a nat transformation

$$\varphi: Q' \circ K^+(F) \Rightarrow T \circ Q$$

$$\begin{tikzcd} K^+(A) \arrow[r, "K^+(F)"] \arrow[d, "Q"'] \arrow[dr, phantom, "\Swarrow \varphi"] & K^+(A') \arrow[d, "Q'"] \\ D^+(A) \arrow[r, "T"'] & D^+(A') \end{tikzcd}$$

Such that for any $G: D^+(A) \to D^+(A')$ Δcat fctor, the natural map

$$\mathrm{Hom}(T, G) \longrightarrow \mathrm{Hom}(Q' \circ K^+(F), G \circ Q)$$
$$\eta: T \Rightarrow G \longmapsto \eta \circ \varphi \qquad \text{(whiskering)}$$

is bijective.

Cor A right der. fctor is unique up to a can. eq., if it exists.

- so if it exists, we call it *the* r.d.fctor RF.

Thm Let $F: \mathcal{A} \to \mathcal{A}'$ left exact fctor, assume $\mathcal{A}$ has enough F-inj., let $\mathcal{J}_F \subseteq \mathcal{A}$ be choice of full subcat of F-inj.
Then the functor

$$\begin{tikzcd} D^+(\mathcal{A}) & D^+(\mathcal{J}_F) \arrow[l, "\sim"'] \arrow[r, dashed, "\exists!"] & D^+(\mathcal{A}') \\ & K^+(\mathcal{J}_F) \arrow[u] \arrow[r, "K^+F"'] & K^+(\mathcal{A}') \arrow[u] \end{tikzcd}$$

is a right der fctor.

Rmk Choosing different $\mathcal{J}_F' \neq \mathcal{J}_F$ gives you different functors.

Example X top.sp., $\underline{Ab}_X$ sh. of ab. grp on X, $\Gamma: \underline{Ab}_X \to Ab$ left exact.
$R\Gamma$ can be computed using injective $\underline{Mod}_{\mathbb{Z}}$.
$\mathcal{F} \in \underline{Ab}_X$ is F-acyclic if $H^i(\mathcal{F}) = 0 \ \forall i$,
$\mathcal{J}_\Gamma$ = all acyclics.

- we define left derived fctors the same way, but F right exact, $T: D^-(A) \to D^-(A')$,
$\varphi: T \circ Q \Rightarrow Q' \circ K^+F$.

Rmk Let $F: \mathcal{A} \to \mathcal{A}'$ exact fctor, then $K(F)$ sends qiso to qiso, induces $D(F)$ which restricts to RF on $D^+(\mathcal{A})$ and LF on $D^-(\mathcal{A})$, since $\mathcal{A}$ is made of F-inj / F-proj.

Rmk Composition of left exact fctors is left exact.

Lemma Assume RF, RG, RH exist, where $H = G \circ F$. Then there is an induced nat transf $RH \Rightarrow RG \circ RF$

Pf.

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[r, "K^+F"] \arrow[d] \arrow[rr, bend left, "K^+H"] & K^+(\mathcal{A}') \arrow[r, "K^+G"] \arrow[d] \arrow[dl, phantom, "\Swarrow \varphi"] & K^+(\mathcal{A}'') \arrow[d] \arrow[dl, phantom, "\Swarrow \psi"] \\ D^+(\mathcal{A}) \arrow[r, "RF"'] & D^+(\mathcal{A}') \arrow[r, "RG"'] & D^+(\mathcal{A}'') \end{tikzcd} \Rightarrow \begin{tikzcd} K^+(\mathcal{A}) \arrow[r] \arrow[d] & K^+(\mathcal{A}'') \arrow[d] \arrow[dl, phantom, "\Swarrow \psi\circ\varphi"] \\ D^+(\mathcal{A}) \arrow[r] & D^+(\mathcal{A}'') \end{tikzcd}$$

Prop Same assumptions, assume $\exists\, \mathcal{J} \subseteq \mathcal{A}$ full subcat of F-inj and $\mathcal{J}' \subseteq \mathcal{A}'$ — " — G-inj so that

i) $\mathcal{J}, \mathcal{J}'$ can be used to def RF, RG

ii) $\forall I \in \mathrm{ob}\, \mathcal{J}$, $F(I) \in \mathrm{ob}\, \mathcal{J}'$.

Then RH exists $\wedge$ $RH \Rightarrow RG \circ RF$ is equiv.

Cor Assume $F: \mathcal{A} \to \mathcal{A}'$ exact, $G: \mathcal{A}' \to \mathcal{A}''$ left exact, $\mathcal{A}, \mathcal{A}'$ have enough inj. Then $R(G \circ F) \Rightarrow RG \circ D^+(F)$ is equiv.

Application $X \xrightarrow[f]{} Y \xrightarrow[g]{} Z$ mor of sch, $h = g \circ f$, f affine, which gives $f_*: Qcoh_X \to Qcoh_Y$ exact.

$\forall \mathcal{F} \in Qcoh_X \subseteq C(Qcoh_X) \longrightarrow K^+(Qcoh_Y)$

$$Rh_*(\mathcal{F}) = Rg_*(K^+(f_*)\mathcal{F}) = Rg_*(f_*\mathcal{F}).$$

$\Rightarrow \forall i \quad R^i h_* \mathcal{F} \xrightarrow[\text{can.}]{\sim} R^i g_*(f_*\mathcal{F})$

$$H^i(X, \mathcal{F}) \xrightarrow[\text{can}]{\sim} H^i(Y, f_*\mathcal{F}).$$