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Today: $H_\omega = \underbrace{H + \lambda V_\omega}_{\text{s.a. together}}$, $(V_\omega \psi)(x) = \omega(x)\psi(x)$

not s.a.

$\{\omega(x)\}_{x\in\Lambda}$ i.i.d. random vars., $\omega(x)\in\mathbb{R}$

- also assume $d\mu(\omega(x)) = S(\omega(x))\, d\omega(x)$

  $S(\cdot)$ bounded, cptly supp.

- $H(x,y)=\overline{H(y,x)}$ is finite ranged & bdded

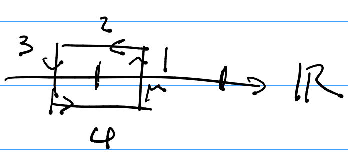
- $\|H_\omega\|\le C$ with $\mathbb{P}=1$

Thm Let $0<\vartheta<1$. $\exists\, \lambda_0$ s.t. $|\lambda|>\lambda_0$, $\forall z\notin\mathbb{R}$

$$\mathbb{E}_\omega\left|\frac{1}{H_\omega - z}(x,y)\right|^{\vartheta} \le C e^{-c\|x-y\|}$$

Rmk for $\vartheta=1$, $\mathbb{E}_\omega|\cdot|$ not bdded uniformly in $\mathrm{Im}\, z$.

Application: decay of $\mathbb{E}_\omega|P_\mu(x,y)|$, $P_\mu=\chi(H_\omega\le\mu)$



1) $$\oint dz\, \frac{1}{H_\omega-\mu}(x,y) = \int_{-\delta}^{+\delta} d\eta\, \frac{1}{H_\omega-\mu-i\eta}(x,y)$$

$$\le \int_{-\delta}^{+\delta} d\eta\, \mathbb{E}_\omega\left|\frac{1}{H_\omega-\mu-i\eta}(x,y)\right|$$

$$\le \int_{-\delta}^{+\delta} d\eta\, \frac{1}{\eta^{1-\vartheta}}\, \mathbb{E}_\omega\left|\frac{1}{H_\omega-\mu-i\eta}(x,y)\right|^{\vartheta} \le C_\vartheta e^{-c\|x-y\|}$$

- now write $H_\omega = tH + V_\omega$ with $|t|\ll 1$ instead of large λ (for easier notation)

- use resolvent identity

$$\frac{1}{H_\omega - z} = \frac{1}{V_\omega - z} - \frac{1}{H_\omega - z} \, tH \, \frac{1}{V_\omega - z}$$

in components:

$$\frac{1}{H_\omega - z}(x,y) = \frac{1}{\omega(x) - z}\delta_{xy} - \sum_a \frac{1}{H_\omega - z}(x,a)\, t H(a,y) \frac{1}{\omega(y) - z}$$

- iterate:

$$\frac{1}{H_\omega - z}(x,y) = \sum_{\gamma : y \to x} (-t)^{|\gamma|} H(x, \gamma(|\gamma|-1)) \cdots H(\gamma(2), \gamma(1)) H(\gamma(1), y) \times \prod_{n=0}^{|\gamma|} \frac{1}{\omega(\gamma(n)) - z}$$

paths $\gamma(0) = y$, $\gamma(|\gamma|) = x$

$\hookrightarrow$ can be arb. large
$\to$ so self ints. can be a problem

$\to$ convergent if $\| tH / \operatorname{Im} z \| < 1$

- we'll use a different expansion

thm (Feenberg expansion)

$$\frac{1}{H_\omega - z}(x,y) = \sum_{\substack{\hat\gamma : y \to x \\ \text{self avoiding}}} (-t)^{|\hat\gamma|} H(x, \hat\gamma(|\hat\gamma|-1)) \cdots H(\hat\gamma(2), \hat\gamma(1)) \cdot H(\hat\gamma(1), y) \cdot \prod_{k=0}^{|\hat\gamma|} \frac{1}{H_{\hat\gamma, k} - z}(\hat\gamma(a), \hat\gamma(a))$$

$$H_{\hat\gamma, k} := H_\omega \Big|_{\ell^2(\Lambda \setminus \bigcup_{\tau < k} \{\hat\gamma(\tau)\})}$$

Pf. We define $\hat{\gamma}$ as

i) $\hat{\gamma}(0) = \gamma(0)$

ii) $\forall \tau \in [0, |\gamma|)$

$$\hat{\gamma}(\tau+1) = \gamma\left(1 + \max\{n \in [0, |\gamma|) \mid \gamma(n) = \hat{\gamma}(\tau)\}\right)$$

$\frac{1}{H_{\hat{\gamma}, k} - z}(\hat{\gamma}(n), \hat{\gamma}(n))$ admits an expansion in terms of loops, avoiding $\hat{\gamma}(\tau < k)$.

Prop. (rank 1 formula) Let $z \in \mathbb{C} / \mathbb{R}$.
Let $H = H_0 + \omega(x)\, |x\rangle\langle x|$ where H_0 s.a. op independent of $\omega(x)$.
Then
$$\frac{1}{H - z}(x, x) = \frac{1}{\omega(x) - \Sigma_0(z; x)}$$
where $\Sigma_0(z; x) = \left(R_z(H_0)(x, x)\right)^{-1}$

Pf. Just use resolvent identity.

Consequence of frac. moment bound: $\sigma(H_\omega) = \sigma_{p.p}(H_\omega)$, $\mathbb{P}$-1