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Def. An additive cat is abel. if
i) every mor has ker. & coker
ii) finite direct sums exist
iii) every mor $X \xrightarrow{f} Y$ induces
$$K \xrightarrow{i} X \xrightarrow{\pi} P \xrightarrow{j} Y \xrightarrow{p} C$$
st. $i = \ker f$, $j = \operatorname{coker} f$, $\pi = \operatorname{coker} i$,
$p = \ker p$, $j \circ \pi = f$

- exercise. Show iii) functorial, i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \text{ comm.} \Rightarrow \begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C \arrow[d] \\ K' \arrow[r] & X' \arrow[r] & P' \arrow[r] & Y' \arrow[r] & C' \end{tikzcd} \text{ comm., unique}$$

Def. Let $\mathcal{C}$ any cat, $X, Y \in \mathrm{Ob}\,\mathcal{C}$. A product for X and Y is an object $P = X \times Y$ which represents $h_X \times h_Y : \mathcal{C}^{op} \to \mathrm{Set}$,
$h_X \times h_Y(S) = h_X(S) \times h_Y(S)$

- in other words, to give mor $S \to X \times Y$ is the same as giving $S \to X$ and $S \to Y$
- exercise: see it in Top, Grp, Mfd, ...
- exercise: $\exists$ canonical projs. $X \leftarrow X \times Y \to Y$ (use Yon.).

Def. $\mathcal{C}$ cat, $X \in \mathrm{Ob}\,\mathcal{C}$. Put $h^X : \mathcal{C} \to \mathrm{Set}$, $h^X(Y) = \mathrm{Mor}(X, Y)$.
- exercise: define corepresentable functors, state co-Yoneda.

Def. Let $\mathcal{C}$ cat, $X, Y \in \mathrm{Ob}\,\mathcal{C}$. A coproduct Z of X and Y corepresents $h^X \times h^Y$.
- exercise: as before. What about comm. rings?

Prop. Let $\mathcal{A}$ add. cat satisfying I) and II). Then $\forall X, Y \in \mathrm{Ob}\,\mathcal{A}$, a coproduct exists if and only if a product exists. Further, they are equal.

Pf. Assume $\exists$ product P. Then

$$\begin{tikzcd} & X \arrow[d, "(\mathrm{id}_X, 0)"'] \arrow[dr, equal] & \\ Y \arrow[r, "(0, \mathrm{id}_Y)"'] \arrow[dr, equal] & P \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \\ & Y & \end{tikzcd}$$

- exercise: $\pi_X = \mathrm{coker}\, i_Y$, $i_Y = \mathrm{coker}\, \pi_X$ and $(X \Leftrightarrow Y)$

Given $Z \in \mathrm{Ob}\,\mathcal{C}$,

$$\mathrm{Hom}(P, Z) \longrightarrow \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)$$
$$f \longmapsto (f \circ i_X, f \circ i_Y)$$

is bijection, so P direct sum.

Check converse.

Rmk. $\mathcal{A}$ abel. $\Longleftrightarrow$ $\mathcal{A}^{op}$ abel.

Def. Let $\mathcal{A}$ abel. cat. A (possibly inf.) sequence of objects and morphisms indexed by morphisms indexed by $\mathbb{Z}$

$$\cdots \longrightarrow A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} A_{i+2} \longrightarrow \cdots$$

is called a *complex* if $\forall i$, $d_{i+1} \circ d_i = 0$, if makes sense. Its i-th cohomology (group) object is $\ker d_{i+1} / \mathrm{Im}\, d_i$

$$
\begin{tikzcd}
A_1 \arrow[r, "d_1"] & A_2 \arrow[r, "d_2"] \arrow[dl, hook] \arrow[dr] & A_3 \\
\operatorname{Im} d_1 = P_1 & & K_2 = \ker d_2
\end{tikzcd}
, \quad \ker d_1 = K_1 \xrightarrow{i_1} A_1 \to P_1 = \operatorname{coker} i_1
$$

$$
d_2 \circ d_1 = 0 \qquad
\begin{tikzcd}
A_1 \arrow[rr, "d_1"] \arrow[dr] & & A_2 \\
& K_2 \arrow[ur] &
\end{tikzcd}
, \quad
\begin{tikzcd}
K_1 \arrow[r] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & A_1 \arrow[r] \arrow[d, dashed] & A_2 \\
& K_2 \arrow[ur] &
\end{tikzcd}
$$

$\Rightarrow P_1 \to K_2$. Def. $H_1(A_\bullet) = \operatorname{coker}(P_1 \to K_2)$

Notation. we use cohom. indexing A^i.

Def. A cpx $A^\bullet = (\cdots \to A^i \to A^{i+1} \to \cdots)$ is called exact at i if $h^i(A^\bullet) = 0$. It is called exact if $\forall i$ $h^i(A^\bullet) = 0$.

Def. Let $A^\bullet, B^\bullet$ complexes with same index interval I. A morphism $\varphi : A^\bullet \to B^\bullet$ is the datum: $\forall i \in I$ $\varphi_i : A^i \to B^i$ s.t. if $i, i+1 \in I$, then

$$
\begin{tikzcd}
A^i \arrow[r, "d_A^i"] \arrow[d, "\varphi_i"'] & A^{i+1} \arrow[d, "\varphi_{i+1}"] \\
B^i \arrow[r, "d_B^i"] & B^{i+1}
\end{tikzcd}
$$

commutes.

We call φ a (co)chain map.

-exercise: show I-complexes constitute a cat.

Lemma. $\varphi : A^\bullet \to B^\bullet$ mor $\Rightarrow h^i(\varphi) : h^i(A) \to h^i(B)$ whenever $i-1, i, i+1 \in I$.

Def. The cat $C(\mathcal{A})$ has as objects $\mathbb{Z}$-cpxs in $\mathcal{A}$ and chain maps as morphisms.

- exercise: $\forall i \in \mathbb{Z}$, $h^i : C(\mathcal{A}) \to \mathcal{A}$, $A^\bullet \mapsto h^i(\cdot)$, $\varphi \mapsto h^i(\varphi)$ is an additive functor (respects abel. structure on mor)

Def. Let $\varphi, \psi : A^\bullet \to B^\bullet \in \operatorname{Mor} \mathcal{C}(\mathcal{A})$.
We say φ is homotopic to ψ if $\exists\ \forall i \in \mathbb{Z}$ $\alpha_i : A^i \to B^{i-1}$ mor in $\mathcal{A}$ s.t.
$\forall i \in \mathbb{Z}$ $\psi^i - \varphi^i = \alpha^{i+1} \circ d_A^i + d_B^{i-1} \circ \alpha^i$.
In this case we call $\alpha = (\alpha_i)_{i \in \mathbb{Z}}$ a **homotopy** from φ to ψ.

$$
\begin{tikzcd}
\arrow[r] & A^{i-1} \arrow[r, "d_A^i"] \arrow[dl] \arrow[d, shift left] \arrow[d, shift right] & A^i \arrow[r, "d_A^i"] \arrow[dl, "\alpha^i"'] \arrow[d, shift left] \arrow[d, shift right] & A^{i+1} \arrow[r] \arrow[dl, "\alpha^{i+1}"'] \arrow[d, shift left] \arrow[d, shift right] & {} \arrow[dl] \\
\arrow[r] & B^{i-1} \arrow[r, "d_B^i"'] & B^i \arrow[r] & B^{i+1} \arrow[r] & {}
\end{tikzcd}
$$

- philosophical q.: what is the natural structure on $C(\mathcal{A})$?
- higher category theory, due to homotopies.

Lemma. Assume $\varphi, \psi : A^\bullet \to B^\bullet$ homotopic mors in $C(\mathcal{A})$. Then $\forall i \in \mathbb{Z}$, $h^i(\varphi) = h^i(\psi)$.

Pf. (We use objects since it is simpler. Diagram chasing CAN BE DONE, tho.)
Let $[x] \in h^i(A)$, meaning $x \in Z^i(A) = \ker d_A^i$.
$h^i(\psi)[x] = [\psi_i(x)]$. Now
$\psi_i(x) - \varphi_i(x) = \alpha^{i+1}(d_i x) + d^i(\alpha^{i-1} x)$
so $[\psi_i(x)] = [\varphi_i(x)]$.

Def. A morphism φ in $C(\mathcal{A})$ is called a quasi-isomorphism if $\forall i \in \mathbb{Z}$, $h^i(\varphi)$ is isom.

Rmk. Analoguous to weak-equivalence in alg. top.

Rmk. Like top. spaces, $C(\mathcal{A})$ also has a natural structure: model cat.

- why quisos? example:

- let A (Noeth) comm. ring, B fin.gen A-alg. Define module of kähler differentials $\Omega_{B/A}$ by saying it represents a functor $(\mathrm{Mod}_B)^{op} \to (\mathrm{Sets})$, $M \mapsto \mathrm{Der}_A(B, M)$, or by saying $B = A[x_1 \dots x_n]/(f_1 \dots f_r)$ and

$$\bigoplus_{j=1}^{r} Be_j \xrightarrow{\frac{\partial f_j}{\partial x_i}} \bigoplus_{i=1}^{n} B \cdot dx_i \to \Omega_{B/A} \to 0 \text{ exact.}$$

- the first approach is great, but the 2^{nd} possibly depends on basis!