Informal deformation theory (Michele?)

- deforming smth like $F(\underline{x}) = c_0 x^{d_0} + a_1 \underline{x}^{d_1} + \dots$
  $\rightarrow$ deforming coefficients is not a good idea, it might no longer be a curve
- infinitesimal family:
  $\Bbbk = \Bbbk(\mathbb{C})$, $A \cong \Bbbk[t_1, \dots, t_r]/\mathfrak{a}$, $\sqrt{\mathfrak{a}} = (t_1, \dots, t_n)$
  $X = \operatorname{Spec} B \hookrightarrow \mathbb{A}^n_{\Bbbk}$, $T = \operatorname{Spec} A$
  - family (Ξ, φ) s.t.
    $\underset{\text{scheme}}{}$

$$
\begin{array}{ccc}
X \cong \operatorname{Spec}\Bbbk \times_T \Xi & \longrightarrow & \Xi \\
\downarrow & & \downarrow \text{flat} \\
\operatorname{Spec}\Bbbk & \longrightarrow & T
\end{array}
$$

Lemma. $Z_0 \hookrightarrow Z$, $N \subseteq \mathcal{O}_Z$ nilpotent.
$\quad\quad Z_0$ affine implies Z affine.
$\rightarrow$ so we can take $\Xi = \operatorname{Spec}\dots$

Rmk.
$$
\begin{array}{ccc}
\Bbbk \otimes_A A_2 & \hookleftarrow & \text{``}A_2\text{''} \\
\uparrow & & \uparrow \quad ? \\
\Bbbk & \leftarrow & A \quad \circ
\end{array}
$$

- why flatness? it "preserves relations":

$$
P^n \longrightarrow P^m \longrightarrow P \longrightarrow P/I \rightarrow 0
$$
with I and 0, $\otimes \Bbbk$

$$
P^u_A \rightarrow P^m_A \rightarrow P_A \rightarrow P_A/I_A \rightarrow 0
$$
with I_A and 0

Def. $X \hookrightarrow \mathbb{A}^n$ is a complete intersection
   if $I_X = (f_1, \dots, f_m)$ where $\operatorname{codim}_{\mathbb{A}^n}(X) \leq m$.

- we would like to deform these?
- start with 1st order deformations: $A = \mathbb{k}[t]/(t^2)$

$$\| \mathbb{k}[\varepsilon]$$

$\Rightarrow \quad I_X = (f_1, \dots, f_m)$

$\Leftrightarrow \quad I_A = ((f_1 + \varepsilon g_1), \dots, (f_m + \varepsilon g_m), \varepsilon g_{m+1})$

forbidden by flatness!

- if $J_A = ((f_1 + \varepsilon \tilde{g}_1), \dots, (f_m + \varepsilon \tilde{g}_m))$
also a deformation - is it the same? when?

CLAIM. $I_A = J_A \iff g_i - \tilde{g}_i \in I_X$.

- we will not define a "change of coordinates"
explicitly beyond this, but use the term:

$x_i \rightsquigarrow x_i' = x_i + \varphi_i(x_1, \dots, x_n) \varepsilon$

$\Rightarrow \quad f_i(\underline{x}) + g_i(\underline{x})\varepsilon \rightsquigarrow f_i(\underline{x}) + \varepsilon \left[\sum_j \frac{\partial f_i}{\partial x_j} \varphi_j(\underline{x}) + g_i \right]$

$\underbrace{\qquad\qquad}_{\text{Jacobian ideal, } J}$

- now look,
$$\overset{\mathbb{k}[x_1, \dots, x_n]}{\downarrow}$$

$$\underbrace{P_{\mathbb{k}} \times \cdots \times P_{\mathbb{k}}}_{n-d \text{ times}} / I_X^{n-r} = \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X / J$$

Rmk. for locally complete intersections,
if X is smooth $\Rightarrow X$ is <u>RIGID</u>.
   $\rightarrow$ because not all derivatives vanish,
     $J \supseteq 1$.

- now look at d-vector spaces ($d = $ length)

in $H^0(\zeta, G_\zeta)$

		dim	length
$\to \Bbbk[\varepsilon]$	$= \langle 1, \varepsilon \rangle$	0	2
$\Bbbk[t]/(t^3)$	$= \langle 1, t, t^2 \rangle$	1	3
$\Bbbk[x,y]/(x^2, y^2, xy) = \langle 1, x, y \rangle$		2	3

$$\zeta \hookrightarrow \mathbb{P}^{n \geq 3}, \quad d \geq 4.$$

- e.g. let's deform $y^2 - x^2(x-1)$

$$\Rightarrow \Bbbk[x,y] / \big(y^2 - x^2(x-1) + \varepsilon P(x,y) \big)$$

$$\Rightarrow (I, J) = \big(y^2 - x^2(x-1), \; y, \; -3x^2 - 2x \big)$$

$$\underset{\substack{\text{chinese} \\ \text{remainder}}}{=} (x, y)\big(x^2(x-1), \; y, \; -3x - 2\big)$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{sing. locus}}}{\mathbb{P}_{\Bbbk/(x,y)}} \times \underset{\underset{\substack{= 0, \; \text{since} \\ -3x-2 \; \& \; x^2(x-1) \\ \text{aren't} = 0 \text{ at the} \\ \text{same time?}}}{\underbrace{\mathbb{P}_{\Bbbk/(\quad)}}} \; \overset{\sim}{\to} \; \mathbb{C}$$

$\to$ so $\mathbb{P}[x,y] = \lambda \in \mathbb{C}$, just a scale.

$$P^{\ell} \xrightarrow{(r_{ij})} P^m \xrightarrow{(f_i)} P \longrightarrow P/I \to 0 \qquad \text{exact}$$

deform
$$\longrightarrow \quad P_A^{\ell} \xrightarrow{(\sigma_{ij}')} P_A^m \xrightarrow{(f_i')} P_A \to P_A/I_A \to 0$$

s.t.
$$\sigma_{ij}' = \sigma_{ij} + \varepsilon s_{ij}$$
$$f_i' = f_i + \varepsilon g_i$$
and exact

$$\Rightarrow \quad r' f' = \underbrace{rf}_{\substack{=0 \text{ by} \\ \text{exactness}}} + \varepsilon (g\sigma + fs) \overset{!}{\underset{0}{=}} 0$$

$$\Rightarrow \quad g\sigma = -fs$$

$$g : I \longrightarrow P/I \quad \in \mathrm{Hom}_P(I, P/I)$$

$-$ note that $\mathrm{Hom}_{P/I}(I/I^2, P/I) \cong N_{X/\mathbb{A}^n}$

$-$ for smooth X, $\quad 0 \longrightarrow T_X \longrightarrow T_{\mathbb{A}^n}|_X \longrightarrow N_{X/\mathbb{A}^n} \longrightarrow 0$

$-$ for nonsmooth, $\cdots \cdots \cdots \cdots \cdots \cdots \cdots \longrightarrow T_X^1 \to 0$

Schlessinger sheaf