Stoppa

Rmk. (about last time) we used the property
$$\forall X, Y \quad [Y,X] - J[X,JY] - J[JX,Y] + [JX,JY] = 0$$
– in general:

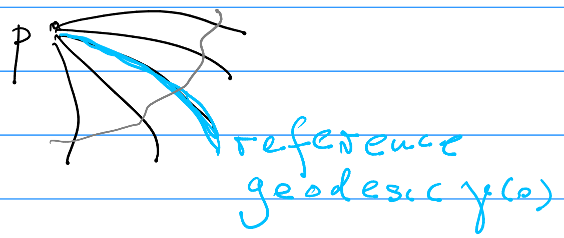
Def. An *almost cpx structure* on M^{2n} is a smooth section of $\mathrm{End}(TM) \simeq T^*M \otimes TM$ which squares to $-\mathrm{Id}$ everywhere.

Rmk. Clearly, a cpx str. is an almost cpx. str.

Th. (Newlander-Nirenberg)
An almost cpx str comes from a cpx str iff $N_J(X,Y) = 0$, $\forall X, Y$.

First glimpse of curvature.

Start w geodesic pencil $f(t,s) \doteq \exp_p(t\,v(s))$.



Def. Associate to the pencil a v.f. along $\exp_p(t\,v(0)) =: \gamma(t)$,
$$J(t) = \frac{\partial f}{\partial s}(t,0) = df\left(\frac{\partial}{\partial s}\right)\Big|_{(t,0)}.$$

Rmk. $J(t) = (d\exp_p)_{tv}(t\,\dot v(0))$.

Lemma. $J(t)$ satisfies $\frac{D^2}{dt^2} J(t) + \mathcal{L}(\dot\gamma(t)) = 0$, where the operator $\mathcal{L}$ acts on v.f.s along $f(t,s)$ by $\mathcal{L}(V) \doteq \left[\frac{D}{ds}, \frac{D}{dt}\right](V)$.

Pf. By definition, $\frac{D}{dt}\frac{\partial f}{\partial t} = 0$.
$$\Rightarrow 0 = \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \frac{D}{dt}\underbrace{\frac{D}{ds}\frac{\partial f}{\partial t}}_{= \frac{D}{dt}\frac{\partial}{\partial s} f, \ \mathrm{Tor}=0} + \mathcal{L}\left(\frac{\partial f}{\partial t}\right).$$

Claim. $\mathcal{L}$ is induced by a global object on M.

Def. The Riemann curvature tensor on (M, g).
is given as
$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

Rmk i) This is doCarmo's convention.
ii) C^∞-linear in all args.
$\Rightarrow$ tensor $(R(X,Y)Z)_p = R(X_p, Y_p) Z_p$.
$\rightarrow$ $\mathrm{End}(TM)$-valued 2-form, $R \in \Gamma(M, \Lambda^2 T^*M \otimes \mathrm{End}(TM))$

Lemma. Let $f(s,t)$ be any param. sfc in M.
Let V be a v.f. along $f(s,t)$.
Then

$$\left(\frac{D}{dt}\frac{D}{ds} - \frac{D}{ds}\frac{D}{dt}\right)V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V.$$

Pf. Fix notation $(-)' := \frac{\partial}{\partial s}(-)$, $(\mathring{-}) := \frac{\partial}{\partial t}(-)$.
In local coordinates write $V = V^i \frac{\partial}{\partial x^i}$,
$f(t,s) = (x_1(t,s), \dots, x_n(t,s))$.
$$\frac{D}{dt}\frac{D}{ds}V = v^i \frac{D}{dt}\frac{D}{ds}\frac{\partial}{\partial x_i} + \mathring{v}^i \frac{D}{dt}\frac{\partial}{\partial x_i} + (V^i)' \frac{D}{ds}\frac{\partial}{\partial x_i} + (\mathring{V}^i)' \frac{\partial}{\partial x_i}.$$

Clearly, $\left[\frac{D}{dt}, \frac{D}{ds}\right](V) = V^i \left[\frac{D}{dt}, \frac{D}{ds}\right]\frac{\partial}{\partial x_i}$.

$d(f)\left(\frac{\partial}{\partial s}\right) = (x^k)' \frac{\partial}{\partial x^k}$
$d(f)\left(\frac{\partial}{\partial t}\right) = \mathring{x}^k \frac{\partial}{\partial x^k}$

Now $\frac{D}{ds}\frac{D}{dt}\frac{\partial}{\partial x_i} = \nabla_{df(\frac{\partial}{\partial s})} \nabla_{df(\frac{\partial}{\partial t})} \frac{\partial}{\partial x_i}$
$$= \mathring{x}_j' \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} + \mathring{x}_k x_j' \nabla_{\frac{\partial}{\partial x_k}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}.$$

And so $\left[\frac{D}{dt}, \frac{D}{ds}\right]\frac{\partial}{\partial x_i} = \mathring{x}_k x_j' \left(-\nabla_{\frac{\partial}{\partial x_k}} \nabla_{\frac{\partial}{\partial x_j}} + \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_k}}\right)\frac{\partial}{\partial x_i}$

And since $\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j}$, $\left[\frac{D}{dt}, \frac{D}{ds}\right]V = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V$.

Corollary. $\frac{D^2}{dt^2}J(t) + R(\dot{\gamma}(t), J(t)) = 0.$

Rmk. We call $J(t)$ the Jacobi field of the pencil.

Motivation. $p, q \in M$. Let $\Omega(p,q) = \{$paths from p to $q\}$
Take $\gamma(t) \in \Omega(p,q)$.
Define $T_\gamma \Omega(p,q) = \{$v.f.s along $\gamma(t)$ vanishing at $p, q\}$

We have the energy functional $E : \Omega(p,q) \to \mathbb{R}$:

$$E(\gamma(t)) := \int_0^1 (\dot{\gamma}(s))^2 \, ds$$

Geodesics are its critical points.
$\to$ in the directions given by the Jacobi field $\Rightarrow$ degeneracy.

Properties of R

Prop. (1st Bianchi identity)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$