Autonomi.

- $\pi: A \to B$ morphism of unital algs, $\pi(1_A) = 1_B$,
- $Sp_B\, \pi(x) \subset Sp_A\, x$, since $\pi(\lambda - x) = \lambda - \pi(x)$

- fix $\mathbb{k} = \overline{\mathbb{k}}$ ($\mathbb{C}$), let $\mathbb{k}(x)$ = rational funcs
- Poles. for $T \in \mathbb{k}(x)$ written as
  $T = \frac{a_0(a_1-\lambda_1)\cdots(a_n-\lambda_n)}{b_0(b_1-\mu_1)\cdots(b_m-\mu_m)}$, $P(T)$ set of poles

- for $R \subset \mathbb{k}$ any subset, define
  $\mathbb{k}_R(x) := \{ T \in \mathbb{k}(x) \mid Pol(T) \cap R = \phi \}$

- for $x \in \mathbb{k}(x)$, $Sp_{\mathbb{k}_R(x)}\, x = R$, since $(\lambda - x)$ is not invertible for any $\lambda \in R$ in $\mathbb{k}_R(x)$

- let A any unital alg, $y \in A$ s.t. $Sp_A y \subset \mathbb{k}$
- define $\mathbb{k}(x)_{Sp_A y} := \{$ rat. funcs. without poles on $Sp_A y\}$

  - for $\mathbb{k}_{Sp_A y}(x) \ni T = \frac{P}{Q}$, $Q(y) \in A^{-1}$
    $\Rightarrow T(y) = P(y)\, Q(y)^{-1}$

Prop. given $y \in A$, the map $T \mapsto T(y)$ is the unique morphism of unital algebras
$\varphi: \mathbb{k}_{Sp_A y}(x) \to A$, $\varphi(x) = y$.

Rmk. $P \in \mathbb{k}(x) \Rightarrow Sp_A(P(y)) = P(Sp_A y)$

Banach algebra.

- Banach space B (complete wrt norm $\|\cdot\|_B$)
- B will be a cpx algebra w bounded multiplication
  $$\|xy\|_B \leq \|x\|_B \cdot \|y\|_B$$
- if unital, $\|1_B\|_B = 1$

- example: $(E, \|\cdot\|)$ Banach space, then $\mathcal{L}(E, E) = \mathbb{B}(E)$ Banach alg

- if $1 \notin A$, embed $A \hookrightarrow \tilde{A} = A \times \mathbb{C}$ and set $\|(a, \lambda)\|_{\tilde{A}} = \|a\|_A + |\lambda|$

Prop. Let $(E, \|\cdot\|_E)$ Banach space w bounded multips i.e. $\|x \cdot y\|_E \leq c\|x\|_E \|y\|_E$ and with unit. Then there is an equivalent norm making E into a unital Banach alg

($\|\cdot\|$ equiv to $|||\cdot|||$ iff $\exists c_1, c_2$ s.t. $(\|x\| \leq c_1 |||x|||) \wedge (|||x||| \leq c_2 \|x\|)$)

Pf. $E \xrightarrow{L} \mathbb{B}(E)$ injective

$b \longmapsto L_b$ where $L_b(a) = ba$

Then $|||x||| := \|L_x\|_{\mathbb{B}(E)}$.

Lemma. Let A unital Banach alg.

i) $a \in A$, $\|a\| < 1 \Rightarrow 1 - a \in A^{-1}$

and $(1-a)^{-1} = \sum_{n \in \mathbb{N}} a^n$ is abs. convergent

ii) Set $x \in A$. Then $\forall \lambda \in \mathbb{C}$ s.t. $\|x\| < |\lambda|$, $\lambda \notin \mathrm{Sp}_A x$.

Furthermore, $\lim_{\substack{\lambda \to \infty \\ \lambda \notin \mathrm{Sp}_A x}} \|\lambda \cdot (\lambda - x)^{-1} - 1\| = 0$

Pf. i) $\|a\|<1$ means $\|\sum a^n\| \le \sum \|a\|^n = \frac{1}{1-\|a\|}$,
and also $a \cdot \sum_{n\in\mathbb{N}} a^n = \sum_{n=1}^{\infty} a^n$.

ii) let $a := x\lambda^{-1}$, so $\|a\| = \|x\|/|\lambda| < 1$.
By i), $1-a \in A^{-1}$, and $1-a = \frac{1}{\lambda}(\lambda - x)$
$\Rightarrow \lambda - x \in A^{-1}$.
Also $\lambda(\lambda-x)^{-1} = (1-a)^{-1}$. i) again gives
$$\|\lambda(\lambda-x)^{-1} - 1\| = \|(1-a)^{-1} - 1\| = \|\sum_{n=1}^{\infty} a^n\|$$
$$\le \frac{\|a\|}{1-\|a\|} = \frac{\|x\|}{|\lambda| - \|x\|}$$

Def. V, W normed spaces, $U \subset V$ open.
$f : U \to W$ is Fréchet-differentiable at $x \in U$ if $\exists$ bounded lin. op. $T : V \to W$ s.t.
$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Th\|_W}{\|h\|_W} = 0$$
We call T its Fréchet derivative (write $d_x f$).

Prop A unital Banach alg. Then $A^{-1} \subset A$ is open, and $\varphi : A^{-1} \circlearrowleft$ given by $x \mapsto x^{-1}$ is C^1 and satisfies $d_x\varphi(h) = -x^{-1}hx^{-1}$

Thm. A cpx unital Banach alg. Fix $x \in A$.
Then

$\mathrm{Sp}_A x$
- i) $\neq \emptyset$
- ii) is compact
- iii) the resolvent, i.e. map
$$\mathbb{C} \setminus \mathrm{Sp}_A x \xrightarrow{f} A \quad \text{is holomorphic}^*$$
$$z \longmapsto (x-z)^{-1}$$

* "weakly holom.", meaning $\forall \ell \in A^*$, $\ell \circ \varphi : \mathbb{C} \setminus \mathrm{Sp}_A x \to \mathbb{C}$ is holom.

Pf. ii) $\lambda \xmapsto{\psi} (\lambda - x)$ is continuous so
$\mathbb{C} \setminus Sp_A x = \psi^{-1}(A^{-1})$ is open.
We know $Sp_A x \subseteq B(0, \|x\|)$.
So $Sp_A x$ cpt.

iii) fix any $\ell \in A^*$ and look at

$$\mathbb{C} \setminus Sp_A x \longrightarrow \mathbb{C}$$
$$z \longmapsto \ell((x-z)^{-1})$$

1) assume $Sp_A x = \phi$. Then $\ell((x-z)^{-1})$ is entire for any $\ell \in A^*$.
Since $\lim_{z \to \infty} \ell((x-z)^{-1}) = 0$, $\ell((x-z)^{-1}) = 0$.
But A is not empty.

Corollary. (Gel'fand – Mazur)
A Banach alg and a division ring (skew-field) $\Rightarrow A = \mathbb{C}$.

Pf. let $i : \mathbb{C} \to A$
$\lambda \mapsto \lambda \cdot 1_A$
Clearly i surjects.
Further, for $x \in A$ $\exists \lambda \in \mathbb{C}$ st. $i(\lambda) - x \notin A^{-1}$
since $Sp_A x \neq 0$. Since $A^{-1} = \{0\}$,
$i(\lambda) = x$.

- fix A commutative unital

Def. A **character** is a nonzero continuous morphism of unital algebras $\chi : A \to \mathbb{C}$

- recall that a principal ideal $I \subset A$ is maximal if $(I \subset J \neq A) \Rightarrow J = I$.

Prop. In A unital Banach alg, all maximal ideals are closed.

Pf. I maximal $\Rightarrow$ $I = \overline{I} \neq A$,

since $I \neq A$, $I \cap A^{-1} = 0$

$\overline{I} \overset{\shortparallel}{\cap} A^{-1}$

- recall: A comm. ring. $I \triangleleft A$ maximal $\Longleftrightarrow$ A/I field.

- for A commutative unital Banach alg,

characters $\longleftrightarrow$ maximal ideals

$\chi \longleftrightarrow \ker \chi$

Spectrum.

- A comm. unital Banach alg.

$Sp\, A :=$ space of characters with topology of pointwise convergence

$\searrow$ weakest top making the family of maps $\{ \underset{Sp A}{\overset{\in}{\chi}} \longmapsto \chi(x) \}_{x \in A}$ continuous