Bruzzo

Lie algebra cohomology

- L Lie algebra over R, M rep. of L

$M \otimes_R \Lambda^\bullet L^\vee \qquad \mathrm{Hom}_R(\Lambda^\bullet L, M)$

omit x_i, x_j

$\xi \in C^p(L, M)$, $d\xi(x_1, \dots, x_{p+1}) = \sum_{i<j} (-)^{i+j} \xi([x_i, x_j], \dots, x_p)$
$+ \sum (-)^{i-1} \delta(x_i) \xi(x_1, \dots, \hat{x}_i, \dots, x_p)$

- X scheme, noetherian separated,
$\mathcal{F}$ q.c. sheaf on X, $\mathcal{U}$ open affine cover

$$H^p(\mathcal{U}, \mathcal{F}) \overset{\varphi}{\not\to} H^p(X, \mathcal{F})$$

$$\begin{array}{ccccccc} 0 \to & \mathcal{F} & \to & \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) & \to & \cdots \\ & \| & & \downarrow & & \\ 0 \to & \mathcal{F} & \to & \mathcal{Y}^0 & \to & \cdots \end{array}$$

Lemma. X noeth scheme, $\mathcal{F} \in \mathrm{QCoh}_X$.
$\mathcal{F}$ embeds into a flabby q.c. sheaf.

Pf. $\mathcal{U} = \{U_i\}$ aff open cover. $\mathcal{F}|_{U_i} = \tilde{M}_i$, M_i an A_i-module (where $X|_{U_i} = \mathrm{Spec}\, A_i$).
Put $0 \to M_i \to I_i \Rightarrow 0 \to \tilde{M}_i \to \tilde{I}_i$
Denote $f_i : U_i \hookrightarrow X$ and put $\mathcal{G} := \oplus f_{i*}(\tilde{I}_i)$.
For $s \in \mathcal{F}(U)$, $s|_{U \cap U_i} \in \mathcal{F}(U_i \cap U) \longmapsto f_{i*}(\tilde{I}_i)$.
So by construction, $\mathcal{F} \hookrightarrow \mathcal{G}$, and $\mathcal{G}$ is flabby, q.c.

- so now
$$0 \to \mathcal{F} \to \underset{\text{q.c. flabby}}{\mathcal{G}} \to \underset{\text{q.c}}{\mathcal{R}} \to 0$$

$$\Rightarrow 0 \to \mathcal{F}(U_{i_0 \dots i_p}) \to \mathcal{G}(U_{i_0 \dots i_p}) \to \mathcal{R}(U_{i_0 \dots i_p}) \to H^1(U_{i_0 \dots i_p}, \mathcal{F}) = 0$$

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{R}) \to 0$$

$$\Rightarrow \begin{array}{ccccccccc} 0 \to & H^0(\mathcal{U}, \mathcal{F}) & \to & H^0(\mathcal{U}, \mathcal{G}) & \to & H^0(\mathcal{U}, \mathcal{R}) & \to & H^1(\mathcal{U}, \mathcal{F}) & \to 0 \\ & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \varphi & \\ 0 \to & H^0(X, \mathcal{F}) & \to & H^0(X, \mathcal{G}) & \to & H^0(X, \mathcal{R}) & \to & H^1(X, \mathcal{F}) & \to 0 \end{array}$$

$\Rightarrow \varphi$ iso $\Rightarrow H^1(\mathcal{U}, \mathcal{F}) \simeq H^1(X, \mathcal{F})$

and
$$\begin{array}{ccccc} 0 \to & H^p(\mathcal{U}, \mathcal{R}) & \xrightarrow{\sim} & H^{p+1}(\mathcal{U}, \mathcal{F}) & \to 0 \\ & \wr & & \wr & \\ 0 \to & H^p(X, \mathcal{R}) & \xrightarrow{\sim} & H^{p+1}(X, \mathcal{F}) & \to 0 \end{array} \quad p \geq 1$$

Thm. (Leray) $X \in Top$, $\mathcal{F} \in Sh_X$, $\mathcal{U} = \{U_i\}$ open cover and $H^q(U_{i_0 \dots i_p}, \mathcal{F}) = 0$ $\forall q > 0$, for all intersections.
Then $H^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$, $p \geq 0$.

- put $\check{H}^p(X, \mathcal{F}) := \varinjlim_{\{\mathcal{U}\}} H^p(\mathcal{U}, \mathcal{F})$ where the covers are ordered by refinement

$$\Rightarrow \begin{tikzcd} H^p(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & H^p(X, \mathcal{F}) \\ \check{H}^p(X, \mathcal{F}) \arrow[ur, dashed, "\varphi_p"'] & \end{tikzcd}$$

Thm If X paracompact, φ_p iso $\forall p$.

Def. An open cover is *locally finite* if every pt $\in X$ has a nbhd intersecting only a finite # of sets $\in \mathcal{U}$.

Def. X *paracompact* if Hausdorff & every open cover admits a loc. fin. refin.

- given $s \in \mathcal{F}(X)$, put $\mathrm{supp}(s) = \{x \in X \mid s_x \neq 0\}$ (closed)
- take sheaf of rings R on X
- a partition of unity subordinated to a loc. fin. open cover $\mathcal{U} = \{U_i\}_{i \in I}$ is a collection $\{s_i \in R(X)\}_{i \in I}$ s.t. i) $\mathrm{supp}(s_i) \subset U_i$, ii) $\sum_{i \in I} s_i = 1$.

Def. R is fine if, $\forall$ loc. fin. open cover $\mathcal{U}$, there is a partition of unity $\underbrace{\ }_{\text{of } R}$ subordinated to $\mathcal{U}$.

- e.g. diff. mfds, paracpt. top. sp.

- $\mathcal{M}$ R-module, R fine, $\mathcal{U}$ f.o.c.
  $\Rightarrow H^p(\mathcal{U}, R) = 0$, $p > 0$

Pf. take partition of unity $\{s_i\}$, $\alpha \in C^q(\mathcal{U}, \mathcal{M})$, $\alpha = \{\alpha_{i_0 \dots i_q}\}$.
Define $(K\alpha)_{i_0 \dots i_{q-1}} := \sum_{k=0}^{q} (-)^k \sum_{i_{k-1} < j < i_k} s_j \alpha_{i_0 \dots j \dots i_{q-1}}$

so $K : C^q(\mathcal{U}, \mathcal{M}) \to C^{q-1}(\mathcal{U}, \mathcal{M})$.
Now show $K \circ \delta + \delta \circ K = \mathrm{id} - 0$, i.e. the cohomology is homotopic to that of a point.

Thm If $\mathcal{M}$ a module over a fine sheaf of rings over a paracpt. space $X \Rightarrow \check{H}^p(X, \mathcal{M}) = 0$, $p > 0$.

- note that for $0 \to \mathbb{R} \to \Omega_X^0 \to \Omega_X^1 \to \cdots$, every Ω_X^p is a C_X^∞-module, i.e. it is fine
  $\Rightarrow H^p(X, \Omega_X^q) = 0$, $\forall p > 0$, $\forall q \geq 0$ $\Rightarrow H^p(\mathbb{R}, X) \simeq H^p_{dR}(X)$
- if X, Y homotopic diff mfd $\Rightarrow H^p_{dR}(X) \simeq H^p_{dR}(Y)$

Mayer-Vietoris sequence

- $X = U \cup V$.

$$0 \to H^0(\mathcal{U}, \Omega^k) \to C^0(\mathcal{U}, \Omega^k) \to C^1(\mathcal{U}, \Omega^k) \to H^1(\mathcal{U}, \Omega^k) \to 0$$

$$0 \to \Omega^\bullet(X) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V) \to 0$$

$$\Rightarrow \quad 0 \to H^0_{dR}(X) \to H^0_{dR}(U) \oplus H^0_{dR}(V) \to H^0_{dR}(U \cap V)$$

$$\to H^1_{dR}(X) \to \dots \qquad \text{MV sequence}$$