Gravity @ ICTP

Wald

- ptcl mech. Lagrangian, $\delta L = \left[\partial_q L - \frac{d}{dt}\partial_{\dot q} L\right]\delta q + \frac{d}{dt}\left[\partial_{\dot q} L \cdot \delta q\right]$
  for $L = L(q, \dot q)$
  $\Rightarrow \quad E = \partial_q L - \frac{d}{dt}\partial_{\dot q} L \equiv 0$
  $\qquad \Theta = \frac{\partial L}{\partial \dot q}\delta q = p\,\delta q$
- now consider this $\delta q := \delta_1 q$ and take another variation $q \mapsto q + \delta_1 q + \delta_2 q$ and look at the antisymmetrized 2nd variation of Θ.

$$\Omega(p, \delta_1 q, \delta_2 q) = \left[\delta_1 \Theta(p, \delta_2 q) - \delta_2 \Theta(p, \delta_1 q)\right]_{t_0}$$
$$= \left[\delta_1 p\, \delta_2 q - \delta_2 p\, \delta_1 q\right]_{t_0}$$

which is conserved if $E \equiv 0$ from δL

- Ω is highly degenerate, so quotient out by degeneracy locus Ω of space of paths $\mathcal{F}$ to get a fin. dim. phase space $\mathcal{F}/\Omega =: \Gamma$

- in classical field theory with $\{\varphi\}$ denoting dynamical fields, we view the Lagrangian as n-form on n-dim spacetime
  $\Rightarrow \quad \delta L = E\,\delta\varphi + d\Theta$
  with Θ an $(n-1)$-form
  and consider $\omega = \delta_1 \Theta(\varphi, \delta_2\varphi) - \delta_2 \Theta(\varphi, \delta_1\varphi)$.
  so that
$$\Omega = \int_{\mathcal{C}} \omega(\varphi, \delta_1\varphi, \delta_2\varphi)$$
  is conserved, where $\mathcal{C}$ is Cauchy
- remove deg. locus and define for ξ^a
$$\delta H_\xi = \Omega(\varphi; \delta\varphi, \mathcal{L}_\xi \varphi)$$

- for a diff. inv. theory, every ξ^a generates a symmetry $\to$ the current can be written as $J_\xi = \xi^a C_a + dQ_\xi$ with $C_a = 0$ constraints and Q_ξ $(n-2)$-form
- It may be shown that, locally,

$$Q_\xi = W_c(\varphi)\xi^c + X^{cd}(\varphi)\nabla_{[c}\xi_{d]} + Y(\varphi, \mathcal{L}_\xi\varphi) + dZ(\varphi, \xi)$$

- also, we can write

$$\omega(\varphi, \delta\varphi, \mathcal{L}_\xi\varphi) = \delta J_\xi - d(\xi \cdot \Theta)$$

so $\Omega = \int_\Sigma [\xi^a \delta C_a + \delta d Q_\xi - d(\xi \cdot \Theta)]$

$$= \int_\Sigma \xi^a \delta C_a + \int_{\partial\Sigma} [\delta Q_\xi - \xi \cdot \Theta]$$

- for AF spacetimes, we can find B s.t. $\delta \int_{\partial\Sigma} \xi \cdot B = \int_{\partial\Sigma} \xi \cdot \Theta$

so

$$H_\xi = \int_\Sigma \xi^a C_a + \int_{\partial\Sigma} (Q_\xi - \xi \cdot B)$$

since $\Omega = \delta H_\xi$

- for a stationary bh, $\xi|_\Sigma = 0$ at Killing Horizon Σ, $\xi^a = t^a + \Omega_H \varphi^a$

$$0 = \int_\infty (\delta Q_\xi - \xi \cdot \Theta) - \int_\Sigma \delta Q_\xi$$

$$\Rightarrow \delta \int_\Sigma \delta Q_\xi = \delta \mathcal{E} - \Omega_H \delta J$$

and from K.H. properties $\delta \int_\Sigma \delta Q_\xi = \frac{\kappa}{2\pi} \delta S$

$$\Rightarrow \frac{\kappa}{2\pi} \delta S = \delta E - \Omega_H \delta J$$

for any diff. inv theory