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Axioms

1) Dimension, for now

- continuing...

Finiteness & abs. continuity

- $f \in C^\infty(M, \mathbb{C})$ induces $f\cdot$ on $L^2(\Sigma)$,
 $\|f\cdot\| = \|f\|_\infty = \|f\|$, $\overline{\mathcal{C}^\infty(M, \mathbb{C})}^{\|\cdot\|} = \mathcal{C}(M)$
 is Banach $*$-alg, $*f = \bar{f}$
 - but $\|f^* f\| = \|f\|^2 \Rightarrow C^*$-alg
- GN, I: $M \underset{\text{homeo}}{\simeq} \widehat{\mathcal{C}(M)}$, $*$-isom., $x \mapsto \chi_x$, $\chi_x(a) := a(x)$
 so $\mathcal{C}(M) \simeq \mathcal{C}(\widehat{\mathcal{C}(M)})$, $a \mapsto \tilde{a}$, $\tilde{a}(\chi) := \chi(a)$
 - inj + surj by Stone-Weierstraß

- GN, II: any C^*-alg is (sub)alg in $\mathbb{B}(\mathcal{H})$
 for some $\mathcal{H}$.

- on $\mathbb{B}(\mathcal{H})_{s.a.}$ $\exists$ (and only then) a cont.
 func. calc on $\mathcal{C}(\sigma(T))$ for $T = T^* \in \mathbb{B}(\mathcal{H})$
 s.t. $\forall F \in \mathcal{C}(\sigma(T))$ $\exists F(T) \in \mathbb{B}(\mathcal{H})$,
 $\sigma(F(T)) = F(\sigma(T))$, etc.

- building $A =$ "$\mathcal{C}^\infty(M)$" $\xrightarrow{\text{dense}}$ $\mathcal{C}(M)$
 is not very successful, need to consider
 all norms $\|f\|_D := \|Df\|$, D diff op on M

- however, A is stable under holo-calculus,
 i.e. if $f^{-1} \in C(M) \Rightarrow f \in \mathcal{C}^\infty(M)$
 $\forall x \in M$, $\exists \frac{1}{f(x)}$ smooth

- recall: if $T \in \mathbb{B}(\mathcal{H})$ (or even $\in \mathbb{B}$(Ban. sp.))
  and F holomorphic func. $\mathcal{U}$, $\sigma(T) \subset \mathcal{U} \subset \mathbb{C}$,

$$\Rightarrow F(T) := \frac{1}{2\pi i} \int_{\sigma(T) \subset \text{pcwise sm} \subset \mathcal{U}} \frac{F(\xi)}{\xi - T} \, d\xi \qquad \left(\begin{matrix}\text{Dunford-}\\ \text{Riesz}\end{matrix}\right)$$

$\Rightarrow$ A is pre-C^*-alg (i.e. dense in C^*-alg

- recall:
  - faithful bdd $*$-rep of $A \in \mathcal{C}^\infty(M)$ on $\mathcal{H} = L^2(\mathcal{E})$ extends to $C(M)$
  - $D = D^*$, $\operatorname{Dom} D = \mathcal{H}^1$, $\operatorname{Ran} D^k = \mathcal{H}^k$,
    then $\mathcal{H}^\infty := \bigcap_{k \in \mathbb{N}} \mathcal{H}^k = \Gamma(\mathcal{E})$.
  - $\Gamma(\mathcal{E})$ is proj & fin. rk $C^\infty(M)$-mod ($\mathcal{C}^\infty(M)$-mod)

- recall:
  - $\exists \langle -,- \rangle : \Gamma(\mathcal{E})^2 \to \mathcal{C}^\infty(M)$ sesquil. s.t.

$$\langle \psi, \varphi \rangle_M := \int_M \langle \psi, \varphi \rangle \, d\mathrm{Vol}_g$$

$$= W_{res}\left((\psi, \varphi) |D|^{-n} \right)$$

$$= \mathrm{Tr}^+\left((\psi, \varphi) |D|^{-n} \right)$$

$$= ⨍ (\psi, \varphi) |D|^{-n}$$

- which leads us to...

Axiom: Finiteness & A.C. prop

$\longrightarrow$ A is a pre-C^*-alg, $H^\infty = \bigcap_{k \in \mathbb{N}} \mathrm{Dom}(D^k)$ is a fin. gk. proj. module $/A$, and $\exists$ hermitian A-valued form $\langle \cdot, \cdot \rangle$ on H^∞ s.t.

$$\langle \psi, \varphi \rangle_H = \int (\psi, \varphi) \, |D|^{-n}$$

- last analytic axiom...

Regularity (smoothness)

operator	order	$\mathcal{B}^{\text{prime.}}_{\text{op.}}$	
D	1	matrix	
$\lvert D \rvert$	1	scalar	
$[\lvert D\rvert, a], a \in A$	0	scalar	$\}$ bdd
$[\lvert D\rvert, [D, a]]$	0	matrix	

$\Longrightarrow$ iterating $\underbrace{[|D|, \dots, [|D|}_{k \text{ times}}, \overset{a \text{ or } [D,a]}{b}]] \in \mathbb{B}(\mathcal{H}) \quad \forall k \geq 1$

- so $A \cup [D, A] \subset \mathrm{Dom}^\infty \delta := \bigcap_{k \in \mathbb{N}} \mathrm{Dom}\, \delta^k$

where $\mathrm{Dom}\, \delta^k := \{ \beta \in \mathbb{B}(\mathcal{H}) \mid \delta^k(\beta) \in \mathbb{B}(\mathcal{H}) \}$

and $\delta(\beta) := [|D|, \beta]$.

- equivalently:

$\mathbb{R} \ni t \longmapsto \| e^{it|D|} \beta e^{-it|D|} \|$ should be C^∞ $\forall \beta \in A \cup [D, A]$ $(*)$

$\Downarrow$

Axiom: $(\mathcal{A}, \mathcal{H}, D)$ is regular iff $(*)$

- exercise: why $[D,[D,a]] \cancel{\in} \mathbb{B}(\mathcal{H})$

Rmk
- possible to see $\mathcal{A}$ is largest subalg of cont. funcs. s.t. $(*)$
- can be char. as Frechet alg. complete in $\|a\|_k := \|\delta^k a\|$, $\|a\|'_k := \|\delta^k([D,a])\|$ $\forall k \in \mathbb{N}$

- last one...

Reality

- a S.T. is also called (unbdd) KR-cycle; in our case we have $\mathcal{A}, D, J, \chi$ (n = even) on $\mathcal{H} = L^2(\Sigma)$ s.t. Ad_J implements involution τ on $\mathcal{A}$

- if $\mathcal{A}$ N.C. alg, $JaJ^{-1} \neq a^*$ since $*$ is anti involution

- but $J\mathcal{A}J^{-1} \subset \mathcal{A}' \, (\subset \mathbb{B}(\mathcal{H}))$ permits to define $*$-rep of $\mathcal{A}^{op}$ which commutes w rep of $\mathcal{A}$
$\Rightarrow$ $*$-rep of $\mathcal{A} \otimes \mathcal{A}^{op}$ on $\mathcal{H}$,
$$\pi_{\mathcal{A}\otimes\mathcal{A}^{op}}(a \otimes b^o)\psi := \underset{\substack{\shortparallel \\ (\pi_{\mathcal{A}}(a),\text{ we don't write it})}}{a}\, J b^* J^{-1} \psi$$

- equivalently, $\mathcal{H}$ becomes an A-bimod by

$$a \psi b := \pi_{A \otimes A^{op}}(a \otimes b^{o}) \psi$$

- moreover, by $\mathrm{Ad}\, J$ we get an invol. on $A \otimes A^{op}$ by $a \otimes b^{op} \mapsto b^* \otimes a^{o}$

- the signs $\epsilon, \epsilon', \epsilon'' \equiv n = \dim(A, \mathcal{H}, D) \bmod 8$