Tanzini

Topological field theories

- "operative" def of field theories:
  - theory of maps from Σ domain Riemann mfd (world-volume) to tgt mfd M (target space)
  - set $\dim_{\mathbb{R}} \Sigma := d \underset{\text{course}}{\overset{\text{(in this}}{=}}) \; 0, 1, 2$

    0 ← finite-dim or 0-dim QFT

    1 ← S^1 or $[a,b]$ quantum mechanics

    2 ← strings

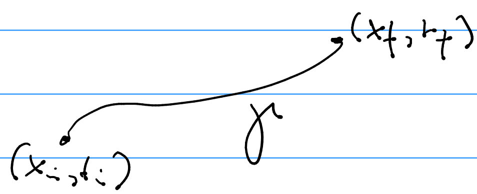
- and "topological"?
  1) one can only compute metric-independent quantities (independent of both Σ, M metrics)
  11) the semi-classical limit is exact

     $\Rightarrow$ TQFT $\rightsquigarrow$ enumerative invariants

- how to study them?
- AXIOMATIC APPROACH (cobordisms) [Atiyah-Segal]
  - $\dim_{\mathbb{R}} M = 2, 3, \ldots$?
- PATH INTEGRAL APPROACH (heuristic!)
  - very flexible
  - gives hints on calculating top. inv.

    $\rightarrow$ and on dualities (e.g. mirror symmetry)

- classical mechanics:



$\gamma : \underset{[t_i, t_f]}{\underset{\shortparallel}{I}} \to M$

$\delta S[\gamma] = 0$

- but in q.m. $\Delta x \Delta p \geq \frac{\hbar}{2}$
  - so look at probability $\langle x_i t_i | x_f t_f \rangle = \int_{(\ldots)} [D\gamma]\, e^{\frac{i}{\hbar} S[\gamma]}$

    $(\ldots) = \left\{ \text{space of maps } \gamma \in \mathrm{Map}(I, M) \;\middle|\; \begin{array}{l} \gamma(t_i) = x_i \\ \gamma(t_f) = x_f \end{array} \right\}$

    - if semi-cl. approx exact, then ∞-dim integral collapses to integration over moduli space of solns. to partial diff. eqns

- main reference: Mirror symmetry, AMS

- stationary phase

$$I(s) := \int_{\mathbb{R}} dx\, e^{isf(x)} g(x), \quad f, g \in C^\infty(\mathbb{R})$$

  - asymptotics? (as $s \to +\infty$)
- suppose $f(x)$ has isolated extremum at x_0 and $f''(x_0) \neq 0$

$$\Rightarrow f(x) = f(x_0) + \tfrac{1}{2} f''(x_0)(x - x_0)^2 + \ldots$$

- leading term:

$$I_0(s) = g(x_0)\, e^{isf(x_0)} \int_{\mathbb{R}} dx\, e^{\frac{i}{2} s f''(x_0)(x-x_0)^2}$$
$$= g(x_0) \exp\left[i\left(s f(x_0) + \mathrm{sign}(f''(x_0)) \frac{\pi}{4} \right) \right] \left(\frac{2\pi}{s\, |f''(x_0)|} \right)^{1/2}$$

exercise: prove it (hint: contour integral)

- n dimensions:

$$I(s) = \int d^n x \; g(x)\, e^{isf(x)}$$

- again, $\nabla f|_{x_0} = 0$, $\mathrm{Hess} f|_{x_0}$ nondeg.
- put $f(x) = f(x_0) + \sum_{ij=1}^{n} f_{ij}(x_0)\,(x-x_0)_i (x-x_0)_j$

$$I_0(s) = g(x_0)\, e^{isf(x)} \left(\frac{2\pi}{s}\right)^{n/2} \frac{e^{i\sigma\frac{\pi}{4}}}{|\det \mathrm{Hess} f|_{x_0}|^{1/2}}$$

where σ = signature of Hess := # of pos. eigenvalues − # of neg. —"—

- example: two-sphere S^2
  - $g(x,y,z) = 1$
    $f(x,y,z) = z \rightarrow$ extrema N & S poles

  - N: $z = 1 - \frac{1}{2}(x^2+y^2) + \dots$
  - S: $z = -1 + \frac{1}{2}(x^2+y^2) + \dots$

$$I_0(s) = \frac{2\pi}{s}\Big(\underbrace{e^{i(-2)\frac{\pi}{4}} e^{is}}_{N} + \underbrace{e^{i(2)\frac{\pi}{4}} e^{-is}}_{S}\Big)$$

$$= 4\pi \frac{\sin(s)}{s}$$

- note that $\int_{S^2} dA\; e^{is\cdot z}\Big|_{s=0} = 4\pi$ ✓
- full integration?

$$I(s) = \int \left(-d(\cos\vartheta)\, d\varphi\right) e^{is\cos\vartheta}$$

$$= -2\pi \int_{+1}^{-1} d(\cos\vartheta)\, e^{is\cos\vartheta}$$

$$= -2\pi \frac{1}{is}\left(e^{is} - e^{is}\right) = 4\pi \frac{\sin(s)}{s} \;✓$$

Equivariant cohomology [Berline–Vergne
B. Getzler: Heat kernel & Dirac op. §VII
Atiyah–Bott: Moment maps & equiv. coh.]

- $G \circlearrowright M$:
  i) if action free, M/G is smooth mfd, we set $H^\bullet_G(M) = H^\bullet(M/G)$
  ii) action not free $\Rightarrow$ stacky points (pt, stab_{pt})

$\Rightarrow$ introduce univ. bdl EG:
  i) contractible space (so we don't change coh. of M)
  ii) G-action free

$$\Rightarrow H^\bullet_G(M) := H^\bullet(M \times_G EG) = H^\bullet\left(\frac{M \times EG}{G}\right)$$

- example: $M = pt$, $H^n(pt, \mathbb{R}) = \begin{cases} \mathbb{R} & , n = 0 \\ 0 & , n > 0 \end{cases}$

- but equiv. coh.:

$$H^\bullet_G(pt) = H^\bullet(EG/G) = H^\bullet(BG)$$

- example: $G = U(1) \simeq S^1$
  $ES^1 \simeq S^{2n+1}$
  $BS^1 \simeq \mathbb{C}P^n$ as $n \to \infty$

$$H_{S^1}(pt) = \mathbb{C}[t], \quad t \in H^2(\mathbb{C}P^\infty, \mathbb{C})$$

Cartan model for equiv. coh.

- $G \circlearrowright M$ acts on $f \in C^\infty(M)$ as $h \cdot f(x) = f(h^{-1}x)$
- for $L \in g$ denote by v the generating v.f.

$$(v \cdot f)(x) := \frac{d}{d\varepsilon} f(e^{-\varepsilon L} x)\Big|_{\varepsilon = 0}, \quad v = v^a T_a^i \partial_i$$

- $\mathbb{C}[g]$ alg. of cpx val. fns on g
- consider $\alpha \in \mathbb{C}[g] \otimes \Omega(M)$
  $\rightarrow G$-action $(h \cdot \alpha)(x) = h\, \alpha(h^{-1}x)$

Def. Equivariant dif. forms satisfy $\alpha(hx) = h\alpha(x)$.

$\rightarrow$ invariant under G-action!

Def. Equiv. exterior differential

$$d_v \alpha := d\alpha + i \xi\, i_v \alpha$$

$\nwarrow$ formal param

- need to check!
- first define grading:

$$\deg(P \otimes \beta) = 2 \deg P + \deg_{dR} \beta$$

$$d_v : \Omega^\bullet(M, g) \rightarrow \Omega^{\bullet+1}(M, g)$$

$$d_v^2 = i\xi(d\, i_v + i_v d) = i\xi\, \mathcal{L}_v$$

$\rightarrow$ vanishes on equivariant forms

- example S^2: sympl. form $\omega = d\cos\vartheta\, d\varphi$.
  $\rightarrow S^1 \circlearrowright S^2$ generated by $v = \xi \frac{\partial}{\partial \varphi}$
  $\rightarrow \omega(\xi) = \omega + \xi R(\vartheta)$

$$0 = d_v \omega(\xi) = (d + i\xi\, i_{\frac{\partial}{\partial\varphi}})(\omega + \xi R(\vartheta)) = i\xi(i_{\frac{\partial}{\partial\varphi}} \omega - i\, dR(\vartheta))$$

$\Rightarrow R(\vartheta) = i \cos \vartheta \rightarrow$ the height function!