Gauge @ IGAP.

Hard Lefschetz for Char. vars - A. Mellit

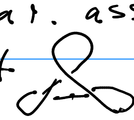
Context (lvl 1)

- X smooth proj$/\mathbb{C}$ dim d, $\omega \in H^2(X)$ hyperplane class, $\omega = c_1(L)$
- $(\wedge\omega)^k : H^{d-k} \to H^{d+k}$ is iso.
- Rmk: similar thing in sympl. hol. affine vars.

Example 1 $\quad T = (\mathbb{C}^\times)^{2d}$, $\omega = \sum a_{ij} \frac{dt_i}{t_i} \wedge \frac{dt_j}{t_j}$,
log-canonical, $a_{ij} = -a_{ji}$ nondeg.
- then $(\wedge\omega)^k : H^{d-k}(T) \to H^{d+k}(T)$ iso
  - $\to$ but here d is half of dim!

Example 2 $\quad \mathbb{C}^\times \times \mathbb{C}^\times$, $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$
- $X = \overline{Bl_{(1,0)}} \, \mathbb{C}^\times \times \mathbb{C}^\times$
  .... $x - 1 = y \cdot z$, $x \in \mathbb{C}^\times$
  $\iff$ complement of $\{yz+1=0\}$ in $\mathbb{C}^2$
  i) ω extends to X
  ii) as a set, $X = \mathbb{C}^\times \times \mathbb{C}^\times \cup \overbrace{\mathbb{C}}^{\text{exceptional}}$
- ω iso $H^0(X) = \mathbb{Z} \cong H^2(X)$

Example 3
- same, but another Bl at $(0,1)$

$\Rightarrow$ $x - 1 = y z_1$
$\quad\;\;\; y - 1 = x z_2$ $\quad \Rightarrow \quad y z_1 z_2 + z_2 - y + 1 = 0$
("augmentation var." assoc. to trefoil knot )

$H^0(X) = \mathbb{Z} \simeq H^2(X)$
- 3ply graded Khovanov homology.

Example 4

- Blowup at 4 pts
  - recall $\mathrm{Bl}_{6pts}\mathbb{P}^2$ = cubic
- our blowup ⇒ affine cubic
  ⇝ char. var. of $\mathbb{P}^1 \setminus 4$ pts, rk 2

q^2 .. $H^0(T) = \mathbb{Z}$

$2q$ .. $H^1(T) = \mathbb{Z}^2 \rightsquigarrow H^0(\mathbb{C} \cup \mathbb{C}, \mathbb{C}) = \mathbb{Z}^6$

1 .. $H^2(T) = \mathbb{Z}$

Thm Fix d. Suppose X filtered by closed subvarieties $X_i \subset X_{i+1}, \dots$, $X_i \setminus X_{i-1} \simeq (\mathbb{C}^\times)^{2d-2a_i} \times \mathbb{C}^{a_i}$.
Suppose X has a closed hol. 2-form ω s.t. its restriction to $(\mathbb{C}^\times)^{2d-2a_i} \times \mathbb{C}^{a_i}$ is a pullback via projection of log-canonical nondeg form on $(\mathbb{C}^\times)^{2d-2a_i}$.
Then curious hard Lefschetz holds:

$$\mathrm{Gr}^{\omega}_{2d-2i} H^j \xrightarrow[\sim]{(\wedge\omega)^i} \mathrm{Gr}^{\omega}_{2d+2i} H^{j+2i}$$

Context (lvl 2)

$$\begin{array}{ccc} \text{char. var of } \Sigma_{g,k} & \xleftrightarrow[\text{Hodge}]{\text{nonab.}} & \mathcal{M}_{\text{stable Higgs}} \\ \text{(parabolic)} & & X^{2d} \\ \text{affine var.} & & \downarrow \\ & & \mathbb{C}^d \end{array}$$

Observation 1 Hausel–Letellier, Rodriguez-Villegas conjectured explicit formula for refined Poincaré polyn. $\in \mathbb{Z}[q,t]$ of char. var. w/ mysterious $q \leftrightarrow t$ symmetry

Observation 2 (Follow up de Cataldo, Migliorini)

$P = W$ conjecture : $W_{2i} = P_i$,

where P perverse Leray filtration

on $H(X)$ induced by Hitchin map $X \to \mathbb{C}^d$