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Introduction to NCG

- 40 hrs, Mon & Fri 11am for now

- NCG $\rightsquigarrow$ hybrid field
- we focus on recent "layer", Riemannian (mainly spin) NCG
  - encoded in spec. 3ple $(A, \mathcal{H}, D)$
- describe some props + Connes' reconstr.

- while describing the S.T. we encounter also some old friends, e.g NC topology:

  (loc). cpt top.sp. $\Leftrightarrow$ comm. (non) unital C^*-alg
  loc. trivial vbdls $\Leftrightarrow$ fin.proj modules over $\uparrow$
- K-theory, Hochschild cyclic cohomology

- we describe: coupling to gauge fields, products, quotients, ...

- examples : $\mathcal{H} \simeq L^2(\Omega(M))$, $D = d + d^*$
  $\mathbb{T}^2_\theta$, S_q^2, N.C.S.M.

- <u>won't</u> discuss:
  - index th.
  - symmetries (isometries, diffeos, q-groups & Hopf algs)

- presentation style: $M \cap \phi$ (or $M \cup \phi$?)
  - proofs usually sketched

Spinors

- "implicit" in Euclid's pf. of Pythagoras' thm

⋮

1770 Euler: "cover" of $SO(3)$
1878 Clifford: "geom" algebra
Lipschitz: representation th
1913 É. Cartan: reps of rotational Lie algs. which do ~~not~~ exponentiate
1935 Brauer & Weyl: rep theory for "proj" reps of $O(n)$

- in physics, Dirac introduces (1928) RQM

$$(p_0 - \alpha_1 p_1 - \alpha_2 p_2 - \alpha_3 p_3 - \beta)\psi = 0$$

where $p_\mu = i\hbar \partial_\mu$, α_i, β 4×4 $\mathbb{C}$-matrices s.t. $\alpha_j^2 = \mathbb{1}$, $\{\alpha_i, \alpha_j\} = 0$ for $i \neq j$,
$\beta^2 = m^2c^2\, \mathbb{1}$, $\{\alpha_i, \beta\} = 0$

$$\Rightarrow \quad p_0^2 - p_1^2 - p_2^2 - p_4^2 - m^2c^2 = 0.$$

- set $m = c = \hbar = 1$, write $\gamma^0 := \beta$, $\gamma^i := -\gamma^0\alpha_i$

$$\Rightarrow \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \begin{cases} +2 & \mu = \nu = 0 \\ -2 \quad \text{if} & \mu = \nu \in \{1,2,3\} \\ 0 & \mu \neq \nu \end{cases}$$

and $D\psi = \psi$ in $\mathbb{R}^{3|1}$, where $D = i\gamma^m\partial_m$ Dirac op.

- coupling to em-field minimally,
  $\partial_\mu \mapsto \partial_\mu + eA_\mu$
  was an enormous success
  - described: "spin", antiptcls
- extended to any mfd M → spin mfds
  - math: index theory (M Riemannian, elliptic ops)
  - phys: dim ≥ 4, Kaluza-Klein, susy, str. theory

Alg. preliminaries

- fix V vsp (linear space) / $\mathbb{K} = \mathbb{R}, \mathbb{C}$
- let $\eta : V \times V \to \mathbb{K}$ sym., bilin. form

- claim: we can reconstruct η from $Q : V \to \mathbb{K}$, $Q(v) := \eta(v,v)$
  - since $2\eta(v,w) = Q(v+w) - Q(v) - Q(w)$

Thm (Sylvester) $\exists$ lin. basis $\{\varepsilon_j\}_{j=1,\dots,n}$ of V s.t. $\eta_{jk} := \eta(\varepsilon_j, \varepsilon_k)$

$$= \begin{cases} \mathrm{diag}(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q}, 0, \dots, 0) & \text{if } \mathbb{K} = \mathbb{R} \\ \mathrm{diag}(1, \dots, 1, 0, \dots, 0) & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

(p,q) is called the signature of η (or Q).

- let $T(V) = \mathbb{K} \oplus V \oplus V \otimes V \oplus \dots = \bigoplus_{k \in \mathbb{N}} V^{\otimes k}$
  be full tensor alg. of V
  - unital, $1 \in \mathbb{K}$.
  - multipl. is $\otimes$, i.e. concatenation

Def $\mathcal{C}(V,Q)$ "$=\mathcal{C}(V)$" $:= T(V)/I(V)$

where $I(V) := \langle v\otimes v - Q(v) \mid v\in V\rangle$

$= \langle v\otimes w + w\otimes v - 2q(v,w) \mid v,w\in V\rangle$

$= \langle \varepsilon_j\otimes\varepsilon_k + \varepsilon_k\otimes\varepsilon_j - q_{jk} \mid j,k=1,\dots,n\rangle$

- as notation, we omit $\otimes$ in $\mathcal{C}(V)$

- note: $\mathbb{K} \overset{i}{\hookrightarrow} \mathcal{C}(V) \hookleftarrow V$, $i(v)^2 = Q(v)$

Rmk A ass. unit. alg, $j: V\to A$ s.t. $j(v)^2 = Q(v)\cdot 1_A$,
then

$$\begin{tikzcd} & A \\ V \arrow[ur, "j"] \arrow[r, hook, "i"'] & \mathcal{C}(V) \arrow[u, dashed, "\exists!\,\tilde{j}"'] \end{tikzcd}$$

commutes.

Application $T: V \circlearrowleft$ isometry induces
$\tilde{T}: \mathcal{C}(V) \circlearrowleft$ automorphism,

since

$$\begin{tikzcd} V \arrow[r, hook] & \mathcal{C}(V) \\ V \arrow[u, "T"] \arrow[ur] \arrow[r, hook] & \mathcal{C}(V) \arrow[u, dashed, "\tilde{T}"'] \end{tikzcd}$$

- $\mathcal{C}(V)$ univ. unital ass. alg. gen by ε_i: $\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 2q_{ij}$

$\to$ basis $\{1, \varepsilon_j, \underset{j<k}{\varepsilon_j\varepsilon_k}, \dots, \underset{\substack{j_1\le\dots\le j_k \\ 0\le k\le n}}{\varepsilon_{j_1}\cdots\varepsilon_{j_k}}, \dots\}$

$\to \dim_{\mathbb{K}} \mathcal{C}(V) = 2^n$

- if $q\equiv 0$, $\mathcal{C}(V) = \Lambda V$

- from now on q nondegenerate

$\Rightarrow$ $p=n$, $\mathcal{C}(V)\cong\mathcal{C}(\mathbb{C}^n)$ if $\mathbb{K}=\mathbb{C}$

$p+q=n$, $\mathcal{C}(V)\cong\mathcal{C}(\mathbb{R}^{p|q})$ if $\mathbb{K}=\mathbb{R}$

Def $\mathcal{C}(V,q)$, q nondeg, we call **Clifford algebra**.

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Prop $\mathcal{C}(\mathbb{C}^n) \simeq \mathcal{C}(\mathbb{R}^{p|q}) \otimes_{\mathbb{R}} \mathbb{C}$ as $\mathbb{C}$-algs.

$\mathcal{C}(\mathbb{R}^{p+1,q+1}) \simeq \mathcal{C}(\mathbb{R}^{p,q}) \otimes \mathcal{C}(\mathbb{R}^{1,1})$

$\mathcal{C}(\mathbb{R}^{p+1,q}) \simeq \mathcal{C}(\mathbb{R}^{q+1,p})$

$\mathcal{C}(\mathbb{R}^{p,q+3}) \simeq \mathcal{C}(\mathbb{R}^{q,p+3})$

Pf. Take $\{\varepsilon_1, \ldots, \varepsilon_p, \varepsilon_{p+1}, \ldots, \varepsilon_{p+q}\}$ basis of $\mathcal{C}(\mathbb{R}^{p|q})$, and build $\{\varepsilon_1 \otimes 1, \ldots, \varepsilon_p \otimes 1, \varepsilon_{p+1} \otimes i, \ldots, \varepsilon_{p+q} \otimes i\}$ basis of $\mathcal{C}(\mathbb{R}^{p|q}) \otimes_{\mathbb{R}} \mathbb{C}$. But all of these anticommute and square to 1, so they're a basis of $\mathcal{C}(\mathbb{C}^n)$, universal prop! $\square$

For 2nd isom,

$\{\varepsilon_1 \otimes \varepsilon_1\varepsilon_2, \ldots, \varepsilon_p \otimes \varepsilon_1\varepsilon_2, 1 \otimes \varepsilon_1,$
$\varepsilon_{p+1} \otimes \varepsilon_1\varepsilon_2, \ldots, \varepsilon_{p+q} \otimes \varepsilon_1\varepsilon_2, 1 \otimes \varepsilon_2\}$

Since $(\varepsilon_1\varepsilon_2)^2 = \varepsilon_1\varepsilon_2\varepsilon_1\varepsilon_2 = -\varepsilon_1\varepsilon_2\varepsilon_2\varepsilon_1 = +1$ (remember $\varepsilon_1^2 = 1, \varepsilon_2^2 = -1$), we get the claim.

3rd, $\{\varepsilon_{q+2}\varepsilon_{q+1}, \ldots, \varepsilon_{q+p+1}\varepsilon_{p+1}, \varepsilon_{q+1};$
$\varepsilon_1\varepsilon_{q+1}, \ldots, \varepsilon_q\varepsilon_{q+1}\}$

4th, $\{\varepsilon_{q+1}\varepsilon, \ldots, \varepsilon_{q+p}\varepsilon; \varepsilon_1\varepsilon, \ldots, \varepsilon_q\varepsilon, \varepsilon_{p+q+1}, \varepsilon_{p+q+2}, \varepsilon_{p+q+3}\}$
where $\varepsilon = \varepsilon_{p+q+1} \cdot \varepsilon_{p+q+2} \cdot \varepsilon_{p+q+3}$, $\varepsilon^2 = 1$.

Of course, we need to check that the dimensions of both sides add up.

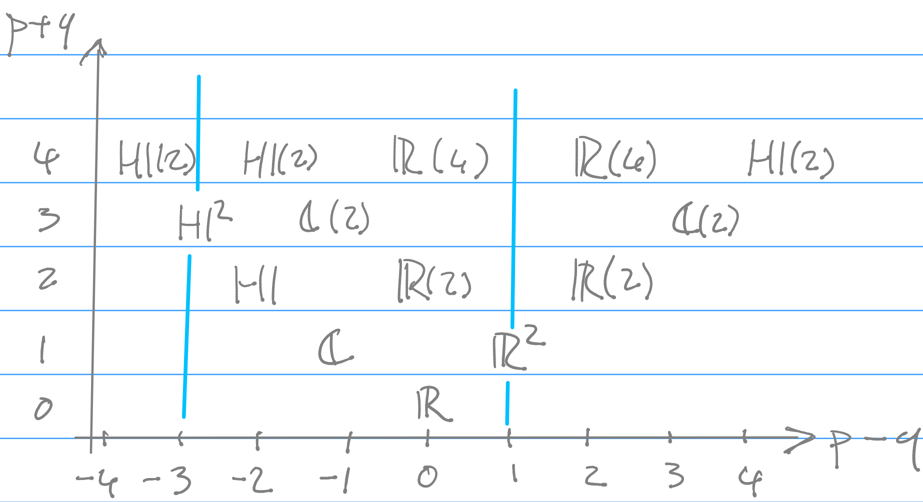
- focusing on low dim. $\mathcal{C}(\mathbb{R}^{p,q})$, $\mathcal{C}(\mathbb{C}^n)$...

(p,q)	$(1,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,1)$
$\mathcal{C}(\mathbb{R}^{p,q})$	$\overbrace{\mathbb{R}\oplus\mathbb{R}}^{\mathbb{R}^2}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\underset{\text{// notation}}{\mathbb{R}(2)}$ $M_2(\mathbb{R})$

$\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$,
$\mathbb{D}(n) := M_n(\mathbb{D})$

all $\otimes = \otimes_{\mathbb{R}}$

- recall: $\mathbb{R}\otimes_{\mathbb{R}}\mathbb{R} \simeq \mathbb{R}$, $\mathbb{R}(n)\otimes\mathbb{D} \simeq \mathbb{D}(n)$,
$\mathbb{R}(n)\otimes\mathbb{R}(m) = \mathbb{R}(n\cdot m)$, $\mathbb{C}\otimes\mathbb{C} \simeq \mathbb{C}\oplus\mathbb{C}$
$\mathbb{C}\otimes\mathbb{H} \simeq \mathbb{C}(2)$



- symmetry around $1, -3$ by Prop,
going up by $p,q \mapsto p+1, q+1$ gives you
$-\otimes_{\mathbb{R}}\mathcal{C}(\mathbb{R}^{1,1}) = -\otimes_{\mathbb{R}}\mathbb{R}(2) = -(\times 2)$

- note a funny thing,
$\mathcal{C}(\mathbb{R}^{1,3}) = \mathbb{H}(2) \not\simeq \mathbb{R}(4) = \mathcal{C}(\mathbb{R}^{3,1})$,
so we can "distinguish" (anti) Minkowski
- but this amounts to the choice of $\pm$ in defining Clifford ideal

- mod-8 periodicity, $\mathcal{C}(\mathbb{R}^{p+4,q}) \simeq \mathcal{C}(\mathbb{R}^{q+4,p})$
gives $\mathcal{C}(\mathbb{R}^{p+8,q}) \simeq \mathcal{C}(\mathbb{R}^{p+4,q+4}) \simeq \underset{\text{is } \mathcal{C}(\mathbb{R}^{p,q+8})}{\mathcal{C}(\mathbb{R}^{p,q})\otimes\mathbb{R}(16)}$

- for $\mathcal{C}(\mathbb{C}^n)$, mod 2 periodic

Def Volume element $\omega := \varepsilon_1 \cdots \varepsilon_n$

- ω (anti) commutes with $v \in V$ if n (even) odd.

Prop The center of $\mathcal{C}(V)$ is

$$Z(\mathcal{C}(V)) = \begin{cases} \mathbb{k} \\ \mathbb{k} \oplus \mathbb{k}\omega \end{cases} \text{for } n \begin{cases} \text{even} \\ \text{odd} \end{cases}$$

- $\omega^2 = (-)^{(p-q)(p-q-1)/2} = \begin{cases} +1 \\ -1 \end{cases}$ if $p-q = \begin{cases} 0,1 \\ 2,3 \end{cases}$ mod 4

- means $A = \mathcal{C}(\mathbb{R}^{p,q})$ is $\oplus$ of 2 simple algs (± 1 eigensp. of ω) if $p-q = 1$ mod 4 & otherwise simple. $A = \frac{1+\omega}{2} A \left(\frac{1+\omega}{2}\right) \oplus \frac{1-\omega}{2} A \left(\frac{1-\omega}{2}\right)$

- $\mathcal{C}(\mathbb{C}^n)$ is $\begin{cases} \oplus \text{ of 2 simpl. algs.} \\ \text{simple} \end{cases}$ when $n = \begin{cases} \text{odd} \\ \text{even} \end{cases}$

$\omega = (-i)^m \varepsilon_1 \cdots \varepsilon_n$, $n = 2m+1$, $m = \left[\frac{n}{2}\right]$

- $\mathcal{C}(V)$ is $\mathbb{Z}_2$ graded under main involution $a \mapsto \bar{a}$, $v \mapsto -v$ antiautom., so sends evens to evens, changes sgn of odd

$\mathcal{C}^{\pm}(V) = \pm 1$ eigensp. of main invol.

- can be shown: $\mathcal{C}^+(\mathbb{R}^{p,q}) \overset{\text{alg.}}{\cong} \mathcal{C}(\mathbb{R}^{p,q-1})$, take $\{\varepsilon_1 \varepsilon_{p+q}, \ldots, \varepsilon_p \cdot \varepsilon_{p+q} ; \varepsilon_{p+1} \varepsilon_{p+q}, \ldots, \varepsilon_{p+q-1} \varepsilon_{p+q}\}$

- $\mathcal{C}^+(\mathbb{R}^{p,q})$ is $\begin{cases} \oplus \text{ of 2 simple} \\ \text{simple} \end{cases}$ if $p-q = \begin{cases} 1 \text{ mod 4} \\ \end{cases}$

$\mathcal{C}^+(\mathbb{R}^{q,p}) \cong \mathcal{C}(\mathbb{R}^{q,p-1})$

- If we use graded tensor product,
$A \hat{\otimes} B$, $(a \otimes b)\cdot(a' \otimes b') = (-)^{|a'||b|}(a\cdot a' \otimes b b')$

$\Rightarrow \mathcal{C}(V_1, Q_1) \hat{\otimes} \mathcal{C}(V_2, Q_2) = \mathcal{C}(V_1 \oplus V_1, Q_1 \oplus Q_2)$

$v_1 \otimes 1 + 1 \otimes v_2$

$V_1 \oplus V_2 \ni v_1 \oplus v_2$

$(-)^2 = Q_1 \otimes 1 + v_2 \otimes v_2 - v_1 \otimes v_2 + 1 \otimes Q_2$ ✓

$\Rightarrow \mathcal{C}(\mathbb{R}^{p,q}) \simeq \mathcal{C}(\mathbb{R}^{1,0})^{\hat{\otimes} p} \hat{\otimes} \mathcal{C}(\mathbb{R}^{0,1})^{\hat{\otimes} q}$

$\Rightarrow \mathcal{C}(\mathbb{R}^{0,q}) \simeq \mathbb{C}^{\hat{\otimes} q}$, $\mathcal{C}(\mathbb{C}^n) \simeq \mathcal{C}(\mathbb{C})^{\hat{\otimes} n} \simeq (\mathbb{C}^2)^{\hat{\otimes} n}$

Spin groups

$\mathcal{C}^*(V) := \{ a \in \mathcal{C}(V) \mid \exists a^{-1} \}$

Def. Twisted adjoint rep. of $\mathcal{C}^*(V)$ on $\mathcal{C}(V)$,
$S: \mathcal{C}^*(V) \to \mathcal{C}(V)$, $a \mapsto S(a)$,

$$S(a)\, b := \bar{a}\, b\, a^{-1} \text{ for any } b \in \mathcal{C}(V).$$

Prop $\ker S = \mathbb{k}^*$

Pf. $\bar{a} b = b a \ \forall b \Leftrightarrow (a_+ - a_-) b = b(a_+ + a_-) \ \forall b$.
Take $b = 1 \Rightarrow a_- = 0$. So $a_+ b = b a_+ \ \forall b$
$\Leftrightarrow a \in Z \cap (\mathcal{C}^+)^* = \mathbb{k}^*$, since we have even dim. □

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- recall, $s: \mathcal{C}^*(V) \to L(\mathcal{C}(V))$, $s(a) = \bar{a}ba^{-1}$
- $\ker s = \mathbb{k}^*$
- in particular $\forall v \in V \; \exists v^{-1} = \frac{v}{\eta(v,v)}$

so $\forall w \in V$,

$$s(v)w = \bar{v}wv^{-1} = -\overbrace{vw}^{-wv+\{w,v\}}v^{-1}$$
$$= w - 2\frac{\eta(v,w)}{\eta(v,v)}v \leftarrow \text{reflection wrt } \perp v \text{ hyperplane}$$

- we can't always get all reflections, e.g. in $\mathbb{R}^{1,0}$
- say: v covers or v is lift of reflection

Def. $\Gamma(\{0\}) := \mathbb{k}^*$ and for $|V| \geq 1$,
$\Gamma(V) := \{v_1 \cdots v_r \mid r \in \mathbb{N}, v_k \in V, \eta(v_k, v_k) \neq 0\} \subseteq \mathcal{C}^*(V)$

Rmk $\Gamma(V)$ is gp
Further, $s(\Gamma(V)) \hookrightarrow O(V) :=$ isometries of V
Cartan–Dieudonné: any $a \in O(V)$
is prod of such reflctns

- indeed, looking at inclusion as $\Gamma(V) \xrightarrow{s} O(V)$
also means $\ker s|_{\Gamma(V)} = \mathbb{k}^*$
- to confirm, $a \in \Gamma(V)$, $a = \prod_{i=1}^{r} v_i$, r odd
$a = a a a^{-1} = -s(a)a = \prod(-s(a)v_i) = -\prod s(a)v_i = -\prod v_i = -a$
so no such odd elements, while for even reduce
to $\mathbb{k}^*$ using Eli Hord alg.

$$1 \to \mathbb{k}^* \to G(V) \to s(\Gamma(V)) \to 1$$

Rmks $G(V)$:

- not conn.
- when $(p=0)\vee(q=0)$, $G(\mathbb{K}^{n,0}) \cong G(\mathbb{R}^{0,n}) \cong G(n)$
- conn. comp. is $SO(n)$

$$\mathcal{E}^*(V) \supset \Gamma(V) \supset$$

Defs $Pin(V) := \{ a \in \Gamma(V) \mid q(v_k, v_k) = \pm 1 \}$

$\cup$

$Spin(V) := Pin(V) \cap \mathcal{E}^+(V)$

$\cup$

$Spin_0(V) := \left\{ a \in Spin(V) \;\middle|\; \begin{matrix} \# v_k \text{ with } q(v_k,v_k)=+1 \text{ even,} \\ \# v_k \text{ with } q(v_k,v_k)=-1 \text{ even} \end{matrix} \right\}$

- unfortunately, $Spin_0(V)$ is not always conn.

Rmk
$$1 \to H \to Pin(V) \xrightarrow{s} O(V) \to 1$$
$$1 \to H \to Spin(V) \xrightarrow{s} SO(V) \to 1$$
$$1 \to \mathbb{Z}_2 \to Spin_0(V) \xrightarrow{s} SO_0(V) \to 1$$
where $H := \mathbb{Z}_2$ for $\mathbb{K} = \mathbb{R}$,

$H := \mathbb{Z}_4 = \{\pm 1, \pm i\}$ for $\mathbb{K} = \mathbb{C}$

Prop $\Gamma(V) \subseteq \{ a \in \mathcal{E}^*(V) \mid s(a) V \subset V \}$

Pf. If $s(a)v \in V$ for some $v \in V$, using $v = -\bar{v}$ gives
$\| \bar{a} v a^{-1} \| = -\overline{\bar{a} v a^{-1}}\, \bar{a} v a^{-1} = a \overline{(-v)}\, \bar{a}^{-1} \bar{a} v a^{-1} = v \cdot v = Q(v)$, so $\Rightarrow \square$.

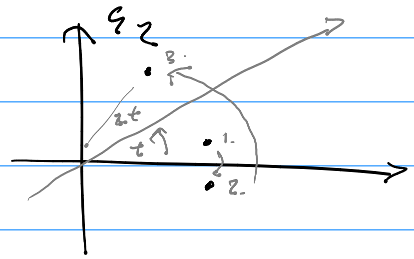
Def Spinor norm $N(a) := \bar{a}^\tau a$ where $(v_1 \cdots v_k)^\tau = v_k \cdots v_1$ main anti-involution

Rmks
- $N(v) = -\eta(v,v)$ for $v \in V$
- $N(S(a)v) = N(v)$ for $a \in \Gamma(V)$
- unfortunately, $N(a) \notin \mathbb{K}$ always!
- alternate definitions possible:

$$\mathrm{Pin}(V) = \{ a \in \Gamma(V) \mid N(a) = \pm 1 \}$$
$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathcal{C}^+(V)$$
$$\mathrm{Spin}_0(V) = \{ a \in \mathrm{Spin}(V) \mid N(a) = +1 \}$$

- examples:
  - $SO(1) = 1$, $\mathrm{Spin}(1) = \mathbb{Z}_2$
  - $SO(1,1) = \mathbb{R}$, rapidity, $\mathrm{Spin}(1,1) = \mathbb{R} \sqcup \mathbb{R}$
  - $p > 1$, nontrivial, e.g.

    $a(t) = e_1 (\cos t \, e_1 + \sin t \, e_2)$, $t \in [0, \pi)$



$S(a(t)) =$ rot by $2t$
$\to$ so $S(a(\pi)) = \mathrm{id}$,
but $a(\pi) = -1$

- topological reason: $SO(p,q)$ not conn.,

$SO_0(p,q) \supset SO(p) \times SO(q)$

$\Rightarrow \pi_1(SO_0(p,q)) = \pi_1(SO(p) \times SO(q))$
$\qquad \simeq \pi_1(SO(p)) \times \pi_1(SO(q))$

and $\pi_1(SO_0(n)) = \begin{cases} 1 & , n = 1 \\ \mathbb{Z} & , n = 2 \\ \mathbb{Z}_2 & , n \geq 3 \end{cases}$

- exercise: $SO(2) \simeq \mathbb{R}P^1 \simeq S^1/\mathbb{Z}_2 \simeq S^1$
$SO(3) \simeq \mathbb{R}P^3 \simeq S^3/\mathbb{Z}_2$

- complexification:

$$\mathrm{Spin}_{\mathbb{C}}(n) := \frac{\mathrm{Spin}(n) \times U(1)}{\{(1,1),(-1,-1)\}} \xrightarrow{2:1} SO(n) \times U(1)$$

$$\overset{\Psi}{[a, \lambda]}, \longmapsto (S(a), \lambda^2)$$

$$(a,\lambda) \sim (-a,-\lambda)$$

$i_1 : \mathrm{Spin} \hookrightarrow \mathrm{Spin}_{\mathbb{C}}$, $\mathrm{Spin}_{\mathbb{C}} \xrightarrow{p_1} SO$, $[a,\lambda] \mapsto S(a)$
$i_2 : U(1) \hookrightarrow \mathrm{Spin}_{\mathbb{C}}$, $\mathrm{Spin}_{\mathbb{C}} \xrightarrow{p_2} U(1)$, $[a,\lambda] \mapsto \lambda^2$

Lie algebras

- $\mathrm{Lie}(\mathcal{C}^*(V)) = \mathcal{C}(V)$ with $[a,b] := ab - ba$.

- claim: $\mathrm{Lie}(\text{conn. comp of } \mathrm{Spin}(V))$
$:= \mathfrak{spin}(V) = \{[v,w] \mid v, w \in V\}$
- has good dim, $n(n-1)/2$

- define $\mathring{S}(\overset{\mathrm{Lie}\,\mathcal{C}^*(V)}{\overset{\Psi}{\alpha}}) = (ev_{t=0} \circ \frac{\partial}{\partial t} \circ \exp)(t\alpha)$

$\Rightarrow \mathring{S}(\alpha) \overset{\mathrm{Lie}\,\mathcal{C}^*(V)}{\overset{\Psi}{u}} = \frac{\partial}{\partial t} e^{t\alpha} u e^{t\alpha}\Big|_{t=0} = [\alpha, u]$

- $\mathring{s}([v,w])u = 4\left(v\eta(w,u) - w\eta(v,u)\right)$

- introduce basis $\epsilon^{jk} := \frac{1}{4}[\epsilon^j, \epsilon^k] = \frac{1}{2}\epsilon^j\epsilon^k$, $j<k$
  then $\mathring{s}(\epsilon^{jk}) = E^{(jk)}$
  where $E^{(jk)}_{jk} = \eta_{jj}$, $E^{(jk)}_{kk} = \eta_{kk}$, other entries zero

- note that $[\mathring{s}(M), u] = Mu$ for $M \in so(p,q)$

Representations

- we focus on cpx reps, $\mathbb{D} = \mathbb{C}$, $T(iu) = iT(u)$
  and $\mathcal{C}(\mathbb{C}^{n=p+q}) = \mathcal{C}(\mathbb{R}^{p,q}) \otimes_{\mathbb{R}} \mathbb{C}$
- think of $\mathcal{C}^{2n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

Prop $\exists!$ irred. faithful cpx rep γ (for n even)

$1 \le j \le \frac{n}{2}$: $\gamma(\epsilon_j) =: \gamma_j = \underbrace{\mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{j-1} \otimes \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3$

$\frac{n}{2}+1 \le j \le n$: $\gamma(\epsilon_j) =: \gamma_j = \underbrace{\mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{j-1} \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3$

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- on $\mathbb{C}^{2^m} = \mathbb{C}(2^n)$, $\overset{\uparrow}{=2m}$

$$\gamma(e_j) =: \gamma_j = \begin{cases} \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2 \otimes \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 & 1 \leq j \leq m \\ \quad - \text{''} - \ \otimes \sigma_2 \otimes \ - \text{''} - & m < j \leq 2m \end{cases}$$

- for $n = 2m+1$,
  $\exists!!$ $\gamma_n = \gamma_{2m+1} := \pm (i)^m \gamma_1 \cdots \gamma_{2m}$

- irred., inequivalent reps, but not faithful

- while $\gamma|_{Pin(n,\mathbb{C})}$ remains irreducible,
  for $\gamma|_{\mathcal{C}\ell^+(\mathbb{C}^n) \supset Spin(n,\mathbb{C})}$ → n odd, irred.
  → n even, $S = S_+ \oplus S_-$,
  $S_\pm$ eigensp. of γ_{2n+1}
  → Weyl-, half- or semispinors.

→ $\gamma_i \gamma_{2m+1} = -\gamma_{2m+1} \gamma_i$, $S_\pm \mapsto S_\mp$ by a reflection

- notice γ_i are hermitian, with $\gamma_j \begin{cases} \text{real}, & j \in 1, \ldots, m \\ \text{imaginary}, & j \in m+1, \ldots, 2n \end{cases}$
  ↳ always have σ_2

- so for $\gamma|_{\mathcal{C}\ell(\mathbb{R}^{p,q})}$, $\gamma_j := \begin{cases} \gamma_j, & j = 1, \ldots, p \\ i\gamma_j, & j = p+1, \ldots, n \end{cases}$

$$\to p - q = \begin{cases} 1,5 \mod 8 \\ 2,3,4 \mod 8 \\ 0,6,7 \mod 8 \end{cases} \Rightarrow \begin{cases} \oplus \text{ of 2 irreps} \\ \text{quaternionic} \\ \text{real} \end{cases}$$

q	0	1	2	3	4	5	6	7
$\mathcal{C}(\mathbb{R}^{0,q})$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}^2$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)^2$
$(\mathcal{C}(\mathbb{R}^{0,q}))^{\text{simple}}$	$\cdot$	$\cdot$	$\cdot$	$\mathbb{H}$	$\cdot$	$\cdot$	$\cdot$	$\mathbb{R}(8)$
$\dim_{\mathbb{R}}$ irrep	1	2	4	4	8	8	8	8
$\chi(q)$	0	1	2	2	3	3	3	3

- $\chi(q) = \log_2(\dim_{\mathbb{R}} \text{irrep})$ = Radon–Hurwitz #, satisfying $\chi(q+8) = \chi(q)+4$

- γ also induces reps of $\mathrm{spin}(p,q)$
  - also on $\mathrm{spin}(n) = \langle \frac{1}{2}\gamma_j\gamma_k \mid j<k \rangle$

  - but on $\mathrm{spin}(n)$, $(\gamma_j\gamma_k)^\dagger = \gamma_k^\dagger\gamma_j^\dagger = \gamma_k\gamma_j = -\gamma_j\gamma_k$
    so all generators antihermitean
    $\Rightarrow$ exponentials are <u>unitary</u>

$$\begin{array}{ccc}
\mathrm{Spin}^{\mathbb{C}}(n) = \dfrac{\mathrm{Spin}(n)\times U(1)}{\mathbb{Z}_2} & \xrightarrow{\gamma} & \mathcal{U}(2^m) \\
\Big\downarrow S & & \Big\downarrow \pi \\
SO(n) & \color{red}{\overset{\exists \text{ mono}}{\dashrightarrow}} & P\mathcal{U}(2^m) = \mathcal{U}(2^m)/\langle \mathcal{U}(1)\cdot \mathbb{1}_{2^m\times 2^m}\rangle
\end{array}$$

Matrix forms

$$\mathbb{C}^4 \ni (z_1,\dots,z_4) \xmapsto[\text{isom.}]{\text{lin.}} \hat{z} = z_4 \mathbb{1}_2 + i\sum_{j=1}^{3} z_j\,\sigma_j = \begin{pmatrix} z_4 + iz_3 & iz_1 + z_2 \\ iz_1 - z_2 & z_4 - iz_3 \end{pmatrix}$$

$\Big\downarrow S$ — for $U, V \in SL(2,\mathbb{C})$, $\det \hat{z} = \det U\hat{z}V^{-1} = \sum z_i^2$

$SO(4,\mathbb{C})$

- claim: S 2-ble cover, $\mathrm{Spin}(4,\mathbb{C}) = SL(2,\mathbb{C})\times SL(2,\mathbb{C})$

- take $z_i \in \mathbb{R} \iff \overline{(\hat{z})}\,\sigma_2 = \sigma_2 \hat{z}$

$U, V \in SU(2)$

$\Rightarrow \mathrm{Spin}(4) = SU(2) \times SU(2)$

- if $z_4 \in i\mathbb{R}$, $z_i \in \mathbb{R} \iff \hat{z}^\dagger = \hat{z}$

$\Rightarrow U^\dagger = V^{-1}$

$\Rightarrow \mathrm{Spin}_0(3,1) = \mathrm{Spin}_0(1,3) = SL(2,\mathbb{C})$

- $z_4 = 0 \iff \mathrm{Tr}\,\hat{z} = 0 \iff U = V \Rightarrow \mathrm{Spin}(3,\mathbb{C}) = SL(2,\mathbb{C})$ also

Charge conj.

- in physics: $J\gamma^\mu(\partial_\mu + ieA_\mu) = \gamma^\mu(\partial_\mu - ieA_\mu)J$

- $J_\pm$ should $\underset{\text{anticommute}}{\overset{\text{commute}}{\vee}}$ w all γ, $\mathbb{C}$-antilinear

- let $J_\pm = C_\pm \circ (\text{cpx. conj.})$

$\Rightarrow C_\pm$ should commute w real, anticom. w imag.

- n even. On $\mathcal{E}(\mathbb{R}^{0,n})$, $\gamma_j = \begin{cases} \text{real} & 1 \le j \le m \\ \text{pure im.} & m < j \le 2m \end{cases}$

- $C_\pm$?

- $C_\pm \sim \begin{cases} \gamma_1 \cdots \gamma_m \\ \gamma_{m+1} \cdots \gamma_{2m} \end{cases}$ if $m \begin{cases} \text{even} \\ \text{odd} \end{cases}$

- $n = \text{odd} = 2m+1$, $\gamma_{2m+1} = (\ )i\gamma_1 \cdots \gamma_{2m}$ imaginary $\Rightarrow$ J_+ if $m = 1,3 \bmod 4$ $(n = 3,7 \bmod 8)$; J_- if $m = 0,2 \bmod 4$ $(n = 1,5 \bmod 8)$

- $J_\pm J_\pm^{\dagger} = 1$. But adjoint needs to be defined as $\langle \psi, J\varphi \rangle = \langle J^{\dagger}\varphi, \psi \rangle$ for antiunitary op. $\left(\langle J\varphi, J\psi \rangle = \overline{\langle \varphi, \psi \rangle} \right)$

- further: $$J_\pm^{\,2} = \varepsilon \cdot \mathbb{1}_2$$

$$J_\pm D = \varepsilon' D J_\pm$$

$$J_\pm \gamma(\omega) = \varepsilon'' \gamma(\omega) J_\pm$$

$\varepsilon, \varepsilon', \varepsilon''$ signs:

	n	0	1	2	3	4	5	6	7	mod 8
J_+	ε	+	×	−	−	−	×	+	+	
	ε'	+	×	+	+	+	×	+	+	
	ε''	+	×	−	×	+	×	−	×	
J_-	ε	+	+	+	×	−	−	−	×	
	ε'	−	−	−	×	−		−	×	
	ε''	+	×	−		+		×		

- we say more $\gamma(\mathrm{Spin}(n)) \subset \gamma(\mathrm{Spin}^{\mathbb{C}}(n))$ is precisely the $\mathrm{Ad}_{J_\pm}$ inv subgroups
- physically these have no charge

Dąbrowski

Geometric rudiments

- spinors carry reps of $\mathcal{E}(V)$ and Spin, and either can be globalized
  - Spin is more traditional, but $\mathcal{E}(V)$ is better for NCG due to links w/ vbdls
- we start with Spin, and (anti)Euclidean case

- we take bdl of frames F with setus $\epsilon_i \in V$, $i=1,\dots,n$
- it carries a right $SO(n)$-action
  $F \times SO(n) \to F$, $(\epsilon, g) \mapsto \epsilon g$
  which is clearly free & transitive
- we can think of a real vector as $v = \sum v_i \epsilon_i$, $v_i \in \mathbb{R}$, and this gives us an equivalent characterisation $t: F \to W \subseteq \mathbb{R}^n$ which gives us coordinates
- we see that
  1) a (p,s)-tensor $\rightsquigarrow (\mathbb{R}^n)^{\otimes p} \otimes (\mathbb{R}^{n*})^{\otimes s}$
  2) $t(\epsilon g) = R(g^{-1})\, t(\epsilon)$ where R rep. of $SO(n)$

- equivalently, work with assoc. vbdl to F with rep. R,

$$[\epsilon, w] \in F \times_R W := \frac{F \times W}{\sim},$$
$$(\epsilon, w) \sim (\epsilon g, R(g^{-1}) w)$$

- so we got $t \leftrightarrow [\epsilon, t(\epsilon)]$.

- analoguously for spinors
  - $\tilde{F}$ equipped w/ free & trans. $Spin(n)$-action
  - given rep $\tilde{R}: Spin \to L(S)$, S being e.g. $\mathbb{C}^{2^m}$, $m = \lfloor \frac{n}{2} \rfloor$
    call a spinor of type $\tilde{R}$
    an $\tilde{R}$-equivariant $\psi: \tilde{F} \to S$, $\psi(\tilde{e}\tilde{g}) = \tilde{R}(\tilde{g}^{-1})\psi(\tilde{e})$

- we link this to $SO(n)$ and F:

$$\begin{array}{ccc} \tilde{F} & \circlearrowleft & Spin(n) \\ \exists \eta \downarrow & & \downarrow s \\ F & \circlearrowleft & SO(n) \end{array} \quad \text{s.t. } \eta(\tilde{e}\tilde{g}) = \eta(\tilde{e})s(\tilde{g})$$

Rmk $t \circ \eta: \tilde{F} \to W$ is $R \circ s$-equivariant by construction $\Rightarrow$ tensors of type R are therefore spinors of type $\tilde{R} = R \circ s$.
- in particular, $R = id_V \Rightarrow \tilde{R} = s$.
  $\Rightarrow$ vectors are s-type spinors.

Spin structures

- we generalise to M oriented smooth Riem. mfld

Def. Spin structure is loc. trivial bdle of frames satisfying

$$\begin{array}{ccc} \tilde{F} & \leftarrow & Spin(n) \\ \eta \downarrow & \circlearrowleft & \downarrow s \\ \circlearrowleft\, F & \leftarrow & SO(n) \\ \pi \downarrow & & \\ M & & \end{array} \qquad \tilde{\pi}: \tilde{F} \to M$$

where the fibers have free, transitive $SO(n)$ / $Spin(n)$ - right action.

- we call such an M a spin mfd.
- not all are spin! $\mathbb{CP}^2$ isn't.
- where is the obstruction?
- $\{\mathcal{U}_\alpha\}$ cover of M, F has transition functs $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \xrightarrow[\varphi_{\alpha\beta}]{} SO(n)$ s.t. $e_\beta = e_\alpha \varphi_{\alpha\beta}$
- the spin mfd condition means this lifts to $\tilde{\varphi}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \longrightarrow Spin(n)$

- we have the Čech cocycle
$K_{\alpha\beta\gamma} := \tilde{\varphi}_{\beta\gamma} \tilde{\varphi}_{\alpha\gamma}^{-1} \varphi_{\alpha\beta} : \mathcal{U}_{\alpha\beta\gamma} \longrightarrow Spin(n)$

and always $(\partial K)_{\alpha\beta\gamma\delta} = K_{\beta\gamma\delta} K^{-1}_{\alpha\gamma\delta} K_{\alpha\beta\delta} K^{-1}_{\alpha\beta\gamma}$
$= 1 \in \mathbb{Z}_2 \subset Spin(n)$
so it's true that $[K] \in H^2(M, \underline{\underline{\mathbb{Z}_2}})$

- note, since $S(K_{\alpha\beta\gamma}) = +1$, $K_{\alpha\beta\gamma} \in \mathbb{Z}_2 \subset Spin(n)$

Prop i) $[K]$ is indep. of choice of lifts $\varphi_{\alpha\beta} \mapsto \tilde{\varphi}_{\alpha\beta}$, and indep. of choice of frames $e_\alpha \mapsto \tilde{e}_\alpha$.
ii) M is spin iff $[K] = 1$

Rmk $[K]$ is called the Stiefel-Whitney class of M

- examples:
  - $M \times SO(n) \longleftarrow \overbrace{M \times Spin(n)}^{=\tilde{F}}$
  - $S^n \cong SO(n+1)/SO(n) \Rightarrow F = SO(n+1)$, $\tilde{F} = Spin(n+1)$

Rmk $Spin_{\mathbb{C}}$-structure exists iff $[K]$ is mod 2 reduction of some class in $H^2(M, \mathbb{Z})$

- $\rho(Spin^c) \subset U(\overbrace{\mathbb{C}^{2^m}}^{=S})$, but S has $\langle \cdot, \cdot \rangle_S$, so

$$\langle \underbrace{\psi \circ \tilde{e}_\alpha \tilde{g}}_{\tilde{R}(\tilde{g}^{-1})\psi(\tilde{e}_\alpha)}, \underbrace{\psi' \circ \tilde{e}_\alpha \tilde{g}}_{\tilde{R}(\tilde{g}^{-1})\psi'(\tilde{e}_\alpha)} \rangle_S$$

$$= \langle \psi(\tilde{e}_\alpha), \underbrace{\tilde{R}(\tilde{g}^{-1})^{+}\tilde{R}(\tilde{g}^{-1})}_{=1} \psi'(\tilde{e}_\alpha) \rangle_S$$

$$\Rightarrow \langle \psi, \psi' \rangle := \int_M \langle \psi(\tilde{e}_\alpha), \psi'(\tilde{e}_\alpha) \rangle_S \, Vol_g \in \mathbb{C}$$

is well def.

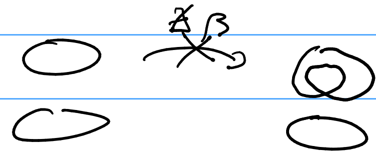
Rmk - spin sts. may not be unique.
- equivalent $\tilde{F}_1, \tilde{F}_2$ if

$$\begin{tikzcd} \tilde{F}_1 \arrow[rr, dashed, "\exists \beta"] \arrow[dr, "q_1"'] & \circlearrowleft & \tilde{F}_2 \arrow[dl, "q_2"] \\ & F \arrow[d] & \\ & M & \end{tikzcd}$$

Prop ∃ free & transitive action $H^1(M,\mathbb{Z}_2)$ on $\Pi(M) = \{\text{spin strs on } M\}/\sim$

– and since $H^1(M,\mathbb{Z}_2) = \mathrm{Hom}(\pi_1(M),\mathbb{Z}_2)$, we can think geometrically

– e.g. $\Pi = S^1$, $SO(1) = 1$, $Spin(1) = \pm 1$



$$M = T^2 = S^1 \times S^1, \quad F = T^2 \times SO(2)$$

$$\tilde{F} = T^2 \times \underset{U(1)}{Spin(2)} \ni (x, y, u)$$

$$\begin{array}{ccc} \tilde{F} & & (x,y,u) \\ \downarrow \eta_{jk} & & \downarrow \\ F & & (x, y, R(jx+ky)S(u)) \end{array}$$

– claim ∄β s.t.

$$\begin{tikzcd} \tilde{F} \arrow[rr, "\beta"] \arrow[dr, "\eta_{jk}"'] & & \tilde{F} \arrow[dl, "\eta'_{j'k'}"] \\ & F \arrow[d] & \\ & M & \end{tikzcd}$$

– show: $\eta'_{j'k'} \circ \beta = \eta_{jk}$ only if $j' = j, k = k'$

$M = T^n \Rightarrow \exists\ 2^n$ inequiv. spin strs.

→ corresponds to (anti) periodic bdry conditions

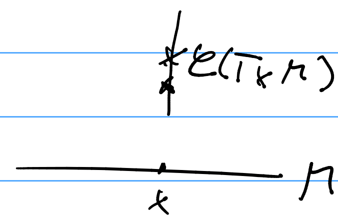
Dąbrowski

- note we get intertwines from $\begin{array}{ccc} id : SO(n) & \to & SO(n) \\ \uparrow s & & \| \\ \varrho : Spin(n) & \to & SO(n) \end{array}$
  which gives us:
  $\exists$ s.s. $\Rightarrow \tilde{F} \times_{\varrho} \mathbb{R}^n \simeq F \times_{id} \mathbb{R}^n \simeq TM$

Clifford bdls & spinor fields

- let $\mathbb{C}(M) = \coprod_{x \in M} \mathcal{C}(T_xM) \otimes \mathbb{C}$ vbdl of algebras

$$\mathbb{C}(M) = \tilde{F} \times_{\bigoplus_{k=0} \varrho^{\wedge k}} (\Lambda \mathbb{R}^n)$$



- $(\mathbb{C}(M))_x = End_{\mathbb{R}}(\Sigma_x)$, $\quad \Sigma = \tilde{F} \times_{\gamma} \mathbb{C}^{2^m}$

- fibrewise multiplication $\underset{\tilde{F} \ni}{[\tilde{e}}, \underset{\mathcal{C}(\mathbb{R}^n) \ni}{a]} \cdot [\tilde{e}, s] = [\tilde{e}, \gamma(a) s]$

- reformulate in terms of modules ..
  - $\Gamma(\mathbb{C}(M))$ is an algebra and defines left action $\Gamma(\mathbb{C}(M)) \times \Gamma(\Sigma) \to \Gamma(\Sigma)$

- we also insist on $\Gamma(\Sigma)$ being a $C^\infty(M)$-module on the right

- after completion we have $\Gamma_0(\Sigma)$ a $\Gamma_0(\mathbb{C}(M))$ – $\mathcal{C}_0(M)$-bimodule and $\Gamma_0(\mathbb{C}(M))$ and $\mathcal{C}_0(M)$ are strongly Morita equivalent

Obs. $\exists$ s.s. $\iff \Gamma(\mathbb{C}(M)) \overset{\text{Mos.}}{\sim} \mathcal{C}(M)$

Digression on Serre-Swan

- M cpt, $E \to M$ finite rk vbdl.
  $\Rightarrow \Gamma^\infty(E)$ is $\underbrace{\text{fin. gen.}}_{(=\text{finite})}$ proj. module over $C^\infty(M) =: A$.

- we construct a projector

- let $\{\mathcal{U}_\alpha\}_{\alpha=1,\dots,r}$ open covering of M, $\{f_\alpha\} \subset C^\infty(M)$ s.t.
  $f_\alpha \geq 0$, $\operatorname{supp} f_\alpha \subset \mathcal{U}_\alpha$, $\sum_\alpha f_\alpha^2 = 1$ "part. of unity"

- write sections as $s|_{\mathcal{U}_\alpha} = \sum_{j=1}^N \overset{s_\alpha}{\sigma^j_\alpha} \, s_{\alpha j}$ local coordinates

  and let $\breve{s}_\alpha := \begin{cases} f_\alpha s_\alpha & \text{in } \mathcal{U}_\alpha \\ 0 & \text{in } M \setminus \mathcal{U}_\alpha \end{cases}$

  so $\breve{s}_\alpha \in \mathcal{C}^\infty(M, \mathbb{C}^N)$

  $\to$ we got a map $\Gamma(E) \xrightarrow{K} A^{\oplus r \cdot N}$, of A-modules

- define $\mathcal{H}: A^{\oplus r\cdot N} \to \Gamma(E)$
  $$t \equiv \{t_\alpha\} \mapsto \hat{t}_\alpha := \sum_\beta \varphi_{\alpha\beta} f_\beta t_\beta$$

- check: $\hat{t}_\alpha = \sum_\beta \varphi_{\alpha\beta} f_\beta t_\beta$
  $$= \sum_\beta \varphi_{\alpha\gamma} \varphi_{\gamma\beta} f_\beta t_\beta = \varphi_{\alpha\gamma} \hat{t}_\alpha$$

  so truly $\hat{t}_\alpha \in \Gamma(E|_{\mathcal{U}_\alpha})$

- compute $\mathcal{H} \circ \kappa(s)|_{U_\alpha} = b_\alpha \sum_\beta \underbrace{\varphi_{\alpha\beta} f_\beta f_\beta s_\beta}_{f_\beta^2 \overbrace{\varphi_{\alpha\beta} s_\beta}^{= s_\alpha}}$
$= s_\alpha b_\alpha \cdot 1 = s$

$\Rightarrow \mathcal{H} \circ \kappa = id_{\Gamma(E)}$

- for $p = \kappa \circ \mathcal{H}$ we get

$p^2 = \kappa \circ \overbrace{\mathcal{H} \circ \kappa}^{= id_{\Gamma(E)}} \circ \mathcal{H} = p$

- can be shown to be projection

$\Rightarrow$ $\Gamma(E) \simeq p A^{\oplus N}$

- explicitly, $p_{\alpha i, \beta j} = f_\alpha (\varphi_{\alpha\beta})_{ij} f_\beta \in A$, $p \in Mat_N(A)$

- vice-versa let $\mathcal{E} = pA^N$
where $p = p^2 \in Mat_N(\underset{C^\infty(M)}{\underset{\shortparallel}{A}}) \simeq C^\infty(M, Mat_N(\mathbb{C}))$

and $E_x := \mathcal{E} / \ker ev_x$
$= p(x)\, \mathbb{C}^N$
$\rightarrow p(x) = tr \circ p \in C^\infty(M, \mathbb{N})$
locally constant
- so $\dim E_x$ loc. constant
- we need to see $E := \coprod_{x \in M} E_x$ is loc. triv.
- this is basically automatic
- pick $e_1, \dots, e_n \in E_x \subseteq \mathbb{C}^N$
let $s_1, \dots, s_n$ sect'n's of E s.t. $s_i(x) = e_i$,
$i \in 1, \dots, \dim E_x$, collections of $k \times N$ matrices
with a $k \times k$ minor with $\det \neq 0$ (??)
so we can define trivialisations

Thm (Serre – Swan) $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ vbdl of fin. rk

$\Longleftrightarrow$ finite proj. modules over $C^\infty(M)$.

$$\begin{tikzcd} Spin^{\mathbb{C}} \arrow[r, hook] \arrow[d, "s"] & U \arrow[d, "\pi"] \\ SO \arrow[r, hook, "i"] & PU = U/U(1)\cdot\mathbb{1} \end{tikzcd}$$

– gives us $Spin^{\mathbb{C}}$ – structures $\begin{tikzcd} Spin^{\mathbb{C}} \arrow[r, dashed] & \tilde{F} \arrow[d, "\eta"'] \\ & F \end{tikzcd}$

Dabrowski

- recalling

Thm (Serre–Swan) smooth vbdls on M
$\longleftrightarrow$ proj fin rk $C^\infty(M)$-modules

- $E \longrightarrow \Gamma(E)$ directly
- $\bigsqcup_x p(x)\mathbb{C}^N =: E \longleftarrow \mathcal{E} = pA^N$ in other dir.

$\quad \overset{\shortparallel}{\mathcal{E}/\mathcal{E}\cdot\ker(ev_x)}$

- this is actually an equivalence of categories, with morphisms being bdle homs (induced by diffeos) $\longleftrightarrow$ module homs

- now, $\mathrm{Spin}_{\mathbb{C}}$-structure $\longrightarrow \Gamma^\infty(\Sigma = \tilde{P} \times_\chi \mathbb{C}^{2^{[\frac{n}{2}]}})$
$(\tilde{F} \times_S \mathbb{R}^n = TM)$ Morita eq.
$\mathcal{E}\quad \Gamma^\infty(\mathbb{C}(M)) - C^\infty(M,\mathbb{C})$-bimod
$\overset{?}{\longleftarrow}$

$$\begin{array}{ccc} \mathrm{Spin}_{\mathbb{C}}(n) & \overset{\chi}{\longrightarrow} & U(2^{[n/2]}) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle \pi = Ad} \\ SO(n) & \overset{i}{\hookrightarrow} & PU(2^{[n/2]}) \end{array}$$

- by S.S., $\mathcal{E} = \Gamma(\Sigma)$ with Σ a $\mathbb{C}$-v.bdl $/M$

$\Gamma^\infty(\mathbb{C}(M)) = \mathrm{End}_{C^\infty(M,\mathbb{C})} \Gamma^\infty(\Sigma) = \Gamma^\infty(\mathrm{End}(\Sigma))$

- same transition functions, global sctns
$\pi(\tilde{g}_{ab}) = Ad\, g_{ab}$

- note $J_I : \mathbb{C}^{2^{[n/2]}} \circlearrowleft$ induces

$$(\mathcal{J}_I \psi)(\hat{e}) := \underset{C_I \circ c.c.}{\overset{\shortparallel}{J_I}}(\psi(\hat{e}))$$

- this, if it exists, selects the real $Spin \subset Spin_{\mathbb{C}}$
- $\mathcal{J}_I$ shares properties of J_I, $\varepsilon, \varepsilon', \varepsilon''$

- for functions $Ad_I f = \mathcal{J}_I f \mathcal{J}_I^{-1}$

$$= (C_I \circ \underbrace{c.c.) f (c.c.}_{I} \circ C_I)$$

$$= \bar{f}$$

- this is the $*$-antiinvolution seen from C^*-theoretic p.o.v.
  - but Ad_I is an involution...
    - so this works for commutative setting

- we will instead define reality cond. as $[J A J^{-1}, A] = 0$, i.e. $J a J^{-1} b = b J a J^{-1}$ $\forall a, b \in A$

Spin-connection & cov. derivatives of spinor fields

- by connection on $\mathbb{K}$-lin vbdl $E \to M$ we mean $C^\infty(M, \mathbb{R})$-lin map $\nabla : \underset{\Gamma(TM)}{\overset{\rotatebox{90}{$\in$}}{X}} \mapsto \nabla_X$,

$\nabla_X : \Gamma^\infty(E) \to \Gamma^\infty(E)$
satisfying Leibniz rule

$$\nabla_X (f s) = (\overset{df(X)}{\overset{\shortparallel}{\mathcal{L}_X f}}) s + f \nabla_X s$$

- we use instead $\hat{\nabla}: \underset{\Gamma(\Sigma)}{\Gamma^\infty(\Sigma)} \to \Gamma^\infty(\Sigma) \otimes_{C^\infty(M,\mathbb{R})} \Gamma^\infty(T^*M)$

$$(\hat{\nabla} \overset{\omega}{\sigma})(X) := \nabla_X \sigma \qquad\qquad \overset{\text{G.N.}}{\|} \ \Gamma^\infty(\Sigma \otimes T^*M)$$

- conn.'s form an affine sp. over $\Gamma^\infty(\mathrm{End}\,\Sigma \otimes T^*M)$
- we ask to preserve hermitian structure h on E, $\mathcal{L}_X h(\sigma, \sigma') = h(\nabla_X \sigma, \sigma') + h(\sigma, \nabla_X \sigma')$

- for $E = TM$, we get metric connection which can preserve g,

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

- extend it to arbitrary tensors by

$$\nabla_X (A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B$$

- note that the ideal $\{X \otimes Y + Y \otimes X - g(X,Y)\}$ is preserved by ∇.
  - so it descends to quotient by it, and becomes $\nabla_X(\alpha \cdot \beta) = \nabla_X \alpha \cdot \beta + \alpha \cdot \nabla_X \beta$ wrt Clifford multiplication

- we also ask for $\nabla_X Y - \nabla_Y X = [X, Y]$

$\to$ Poop $\exists!$ Levi-Civita conn.

- for $\{e_n\}$ o.n.b., $[e_i, e_j] = c_{ijk} e_k$,
  $\nabla_{e_i} e_j = \alpha_{ijk} e_k$

$\xrightarrow{\text{claim}}$ $\alpha_{ijk} = \frac{1}{2}(c_{ijk} + c_{kij} + c_{kji})$
  - check antisymmetry in $(i \leftrightarrow j)$
- for $Y = Y_i e_i \Rightarrow (\nabla_{e_i} Y) = (\mathcal{L}_{e_i} Y_j + \alpha_{ijk} Y_k) e_j$

- $\mathring{\varsigma} : spin(n) \to so(n)$ lets us def.

$\tilde{\alpha}^{(\tilde{e})} := \mathring{\varsigma}^{-1} \circ \alpha^{(e)}$, where $\eta(\tilde{e}) = e$

and

$$(\nabla_X \psi) \circ (\tilde{e}) = \left(\mathcal{L}_X + \gamma \circ \tilde{\alpha}^{(\tilde{e})}(X)\right)(\psi \circ \tilde{e})$$

or equiv. $(\nabla \psi) \circ \tilde{e} = \left(d + \gamma \circ \mathring{\varsigma}^{-1} \alpha^{(\tilde{e})}\right) \tilde{\psi}(\tilde{e})$

- $\nabla_X (\underbrace{Y \cdot \psi}_{= \gamma(Y)\psi}) = \nabla_X Y \cdot \psi + Y \cdot \nabla_X \psi$

$$\mathcal{L}_X \langle \psi, \varphi \rangle = \langle \nabla_X \psi, \varphi \rangle + \langle \psi, \nabla_X \varphi \rangle$$

where $\langle \psi, \varphi \rangle = \int_M \langle \psi(\tilde{e}), \varphi(\tilde{e}) \rangle_{\mathbb{C}^{2^{[n/2]}}} \cdot vol$

- for different spin structs,

$\tilde{F} \ni \tilde{e}$

$\eta' \quad \eta$

$e' = e \circ g, \; g \in SO(n)$

$\alpha^{(eg)} = g^{-1} \circ \alpha^{(e)} \circ g - g^{-1} dg$

for same but with "rotated source"

$\tilde{e}' = \tilde{e} \tilde{g}$

$\eta' \; e \; \eta \Rightarrow \alpha^{(\tilde{e}\tilde{g})} = \tilde{g}^{-1} \alpha^{(\tilde{e})} \tilde{g} - \tilde{g}\, d\tilde{g}$

where $\varsigma(\tilde{g}) = g$

Def. (Curvature) $R_{ijkl} e_l = \nabla_{e_i} \nabla_{e_j} e_k - \nabla_{e_j} \nabla_{e_i} e_k - \nabla_{[e_i, e_j]} e_k$

- $R_{ijij} =: R$ scalar curvature

Def (Dirac operator) $D = \mu \circ \nabla : \Gamma(B) \to \Gamma(T^*M) \otimes \Gamma(B) \xrightarrow{\mu} \Gamma(B)$

where $\nabla = \sum_{j=1}^{n} e_j \otimes \nabla_{e_j}$ where

$\{e_j\}$ dual basis.

Define $D\psi = \sum e_j \cdot \nabla_j \psi = \sum \gamma_j \nabla_{e_j} \psi$.

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- $\beta \in \Gamma(\Lambda^1 M)$, g metric $\Rightarrow \exists\, \beta^\#$ $g(\beta^\#, v) = \beta(v)$
  - likewise $w^\flat(v) = g(w, v)$, $T^*M \underset{\flat}{\overset{\#}{\rightleftarrows}} TM$

- $D\!\!\!/ = \tilde{\gamma} \circ \nabla : \Gamma(\Sigma) \to \Gamma(T^*M) \otimes \Gamma(\Sigma) \longrightarrow \Gamma(\Sigma)$
  where for $\gamma : TM \to \mathrm{End}\,\Sigma$, $\tilde{\gamma} : TM \times \Sigma \to \Sigma$
  is $\tilde{\gamma}(v, \psi) := \gamma(v) \cdot \psi$

- locally, $D\!\!\!/ \psi = \sum e_i \cdot \nabla_{e_i} \psi$

- $D\!\!\!/ (f\psi) = \sum e_i \left((\mathcal{L}_{e_i} f)\psi + f \nabla_{e_i} \psi \right)$
  $= \underbrace{\mathrm{grad}\, f}_{=(df)^\#} \cdot \psi + f D\!\!\!/ \psi$

- principal symbol $\sigma_{D\!\!\!/}(df) = -i\,[D\!\!\!/, f]$
  $= -i\, df \circ$

  so $\sigma_{D\!\!\!/}(\xi) = -i\,\xi \cdot -$, and $(\sigma_{D\!\!\!/}(\xi))^2 = -\|\xi\|^2$

  $\longrightarrow D\!\!\!/$ is elliptic

- let $\dim M = n$, n even

- consider $\chi = \omega \cdot$, Clifford mult. w $\omega = e_1 \cdots e_n$
  $\chi = \omega \cdot = \gamma(\omega)$
  - $(e_i \cdot \nabla_{e_i}) \circ \chi(\psi) = e_i \cdot \left(\cancel{\nabla_{e_i} \omega} \cdot \psi + \omega \cdot \nabla_{e_i} \psi \right)$
  $= e_i \cdot \omega \cdot \nabla_{e_i} \psi = -\chi \circ (e_i \nabla_{e_i})(\psi)$

  $\Rightarrow \chi D\!\!\!/ = - D\!\!\!/ \chi$
  $\Rightarrow D\!\!\!/ : \Gamma(\Sigma_\pm) \longrightarrow \Gamma(\Sigma_\mp)$

Self-adjointness

- let $\psi, \varphi \in \Gamma_c(\Sigma)$, $\langle \psi, \varphi \rangle_M = \int \langle \psi, \varphi \rangle \, vol$

$$\langle \psi, \not{D}\varphi \rangle = \sum_j \langle \psi, e_j \cdot \nabla_{e_j} \varphi \rangle$$
$$= \sum_j \left(\mathcal{L}_{e_j} \langle \psi, e_j \varphi \rangle - \langle \nabla_{e_j} \psi, e_j \varphi \rangle - \langle \psi, \nabla_{e_j} e_j \varphi \rangle \right)$$
$$= \sum_j \left(\mathcal{L}_{e_j} \beta(e_j) - \beta(\nabla_{e_j} e_j) \right) + \langle \not{D} \psi, \varphi \rangle$$
$$= \operatorname{div} \beta^\# + \langle \not{D} \psi, \varphi \rangle$$

where $\beta(e_i) := \langle \psi, e_i \varphi \rangle$

- now, $\langle \psi, \not{D} \varphi \rangle_M = \langle \not{D} \psi, \varphi \rangle_M + \int_M \underbrace{\operatorname{div} \beta^\# \, vol}$,

$$\operatorname{div} \beta^\# \, vol = \mathcal{L}_{\beta^\#} vol = \underbrace{(d \circ \iota_{\beta^\#} + \iota_\beta \circ d) \, vol}_{= 0 \text{ if } \partial M = 0}$$

so

Prop $\not{D}$ is hermitean on $\Gamma_c(\Sigma)$.

Rmk $\ker \not{D} = \ker \not{D}^2$ since $\|\not{D}\psi\|^2 = \langle \not{D}\psi, \not{D}\psi \rangle = \langle \psi, \not{D}^2 \psi \rangle$

- now take $H = L^2(\Sigma) = \overline{\Gamma_c(\Sigma)}^{\|\cdot\|_M}$

- $\not{D}$ is defined on dense open, but not bounded
- so cannot extend to H by continuity
- however, it is closeable, i.e. it has a closure $\overline{\not{D}}$ whose graph is closed, where $\psi \in \operatorname{Dom} \overline{\not{D}} \subset L^2(\Sigma)$ if $\psi_n \to \psi$ means $\lim_n \not{D} \psi_n$ exists.

- define also adjoint $D\!\!\!/^*$ s.t. $D\!\!\!/^*|_{\Gamma_c(\Sigma)} = D\!\!\!/$
  - $\mathrm{Dom}\, D\!\!\!/ \subseteq \mathrm{Dom}\, \overline{D\!\!\!/} \subseteq \mathrm{Dom}\, D\!\!\!/^*$

    with $\mathrm{Dom}\, D\!\!\!/^* := \{ \psi \in H \mid \langle \psi, D\!\!\!/ \varphi \rangle$ is cont. in $\varphi \}$

Def T is self-adjoint if $\mathrm{Dom}\, T = \mathrm{Dom}\, T^*$ and $T = T^*$.

T is essentially s.a. if $\mathrm{Dom}\, \overline{T} = \mathrm{Dom}\, (\overline{T})^*$ and $\overline{T} = (\overline{T})^*$

Prop M cpt $\Rightarrow$ $D\!\!\!/$ is ess. s.a. (works for M geod. complete)

Lemma 1 If $\Gamma(M)$ is closed subset of $\mathrm{Dom}\, D\!\!\!/^*$ in the graph norm $\left(\|\psi\|_{D\!\!\!/^*} = (\|\psi\|^2 + \|D\!\!\!/\psi\|^2)^{1/2} \right)$ then $D\!\!\!/$ is ess. s.a.

Pf. The conditional means $\Gamma(M) \ni \psi_n \to \psi \in \mathrm{Dom}\, D\!\!\!/^*$, giving $\psi \in \mathrm{Dom}\, \overline{D\!\!\!/}$.

Lemma 2. $\Gamma(\Sigma)$ are $\|\cdot\|_{D\!\!\!/^*}$-dense in $\mathrm{Dom}\, D\!\!\!/^*$.

Pf. Let $(\mathcal{U}_\alpha)$ be open cover, (f_α) p.o.u. subordinate to it. Note that $f_\alpha \psi \in \mathrm{Dom}\, D\!\!\!/^*$ if $\psi \in \mathrm{Dom}\, D\!\!\!/^*$, so use local coords. to write $\Sigma|_{\mathcal{U}_\alpha} = \mathbb{R}^n \times \mathbb{C}^{2^{[n/2]}}$

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$\|\cdot\|_T^2 = \|\cdot\|^2 + \|T\cdot\|^2$

- $D \subseteq \overline{D} \subseteq D^*$, $\operatorname{Dom} D = \Gamma_c(\Sigma)$

Lemma $\Gamma_c(\Sigma)$ dense in $\|\cdot\|_{D^*}$-norm in $\operatorname{Dom} D^*$
$\Rightarrow \overline{D} = D^*$. ($D$ is ess. s.a.)

Lemma The antecedent is true.

Pf. Using $\{f_\alpha\}_\alpha$ p.o.1 sub. to $\{U_\alpha\}_\alpha$, $\Sigma|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^{2^{n/2}}$.
Pick $h \in C^\infty(\mathbb{R}_+)$, $h: \mathbb{R}_+ \to \mathbb{R}_+$, $\int_{\mathbb{R}^n} h(x)\,dx = 1$,
$\varepsilon > 0$, let $h_\varepsilon(x \in \mathbb{R}^n) := \frac{1}{\varepsilon^n} h\left(\frac{\|x\|}{\varepsilon}\right)$.
Then $h_\varepsilon \to$ "δ_x" as $\varepsilon \to 0$.
$\operatorname{Dom} D^* \ni \psi = \sum \psi_\alpha$, $\psi_\alpha := \psi f_\alpha \in \operatorname{Dom} D^*$.
which can be seen by looking at
$\langle \varphi, D(f\psi)\rangle = \langle \varphi, f\cdot D\psi + \underbrace{df}_{\text{bounded}}\cdot\psi\rangle$.

$\underset{\cup}{C^\infty(\mathbb{R}^n)}$
Now $(\psi_\alpha * h_\varepsilon)(x) = \int_{\mathbb{R}} dy\, \psi_\alpha(y) h_\varepsilon(x-y) \to \psi_\alpha(x)$

$D(\psi_\alpha * h_\varepsilon)(x) = \int dy\, \psi(y)\, D h_\varepsilon(x-y)$
$= (D^*\psi_\alpha) * h_\varepsilon(x) \longrightarrow D^*\psi_\alpha$
$\Rightarrow \psi_\alpha * h_\varepsilon \xrightarrow{\|\cdot\|_{D^*}} \psi_\alpha$, so
$\psi_\varepsilon = \sum(\psi_\alpha * h_\varepsilon) \to \sum \psi_\alpha = \psi \in \operatorname{Dom} D^*$. □

- from now on M cpt, $\partial M = \emptyset$, $D = \overline{D} = D^* \Rightarrow \operatorname{Spec} D \subseteq \mathbb{R}$

- let $\|\psi\|_{H^1}^2 = \|\psi\|^2 + \underbrace{\|\nabla\psi\|^2}_{\text{norm on } \Gamma(\Lambda^1 M)\otimes\Gamma(\Sigma)}$, 1^{st} Sobolev norm

Thm (Gårding) The norms $\|\cdot\|_D$ and $\|\cdot\|_{H^1}$ are equivalent.

- follows: $D: \operatorname{Dom} D \to L^2$ is bdd (= cont.)
on $H_1 \xrightarrow{\text{completion}} L^2$, $\overline{\Gamma_c(\Sigma)}^{\|\cdot\|_{H^1}}$

- check: $\|D\psi\|^2 = \sum_{j,k} \langle e_j \cdot \nabla_{e_j}\psi, e_k \cdot \nabla_{e_k}\psi \rangle$

Schwartz

$\leq \sum \|e_j \cdot \nabla_{e_j}\psi\| \cdot \|e_k \cdot \nabla_{e_k}\psi\|$

$\leq \sum \|\nabla_{e_j}\psi\| \cdot \|\nabla_{e_k}\psi\|$

$0 \leq a^2+b^2-2ab$

$\leq \frac{1}{2} \sum_{j,k} \left(\|\nabla_{e_j}\psi\|^2 + \|\nabla_{e_k}\psi\|^2 \right)$

$= \frac{1}{2} \cdot 2 \sum_{j,k} \|\nabla_{e_j}\psi\|^2$

$= n \cdot \sum_j \underbrace{\|\nabla_{e_j}\psi\|^2}_{= \|e^j \otimes \nabla_{e_j}\psi\|^2}$

$= n \|\nabla\psi\|^2 \leq n \|\psi\|^2_{H^1}$ by definition

- J, χ are bdd, extend to $H = L^2(\Sigma)$,
  $J f^* J^{-1} = f$, $J^2 = \varepsilon$, $\chi^2 = id$, $\chi^* = \chi$, $\chi D = -D\chi$

- Spinor laplacian $\Delta := \nabla^* \circ \nabla$
  - we can show $\sigma_\Delta^{princ.} = \sigma_{D^2}^{princ} = (\xi \cdot)^2 = -\|\xi\|^2$
    so $D^2 = \Delta$ + lower order diff. op.

Thm (Schrödinger-Lichnerowicz-Weitzenberg)

$D^2 = \Delta + \frac{1}{4} R + \underbrace{\frac{1}{2}\, dA \cdot}_{\text{we ignore this, it's bdd}}$

$\to$ asymptotically eigenvalues same

- on $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $(\Delta - \lambda)^{-1}, (D^2 - \lambda)^{-1}$ exist and are bdd

- further, they are compact

$\to$ eigenvalues converge to zero as a sequence ? (no "endless" repetitions)

- check $(D^{n\text{-}"2"} - \lambda)^{-1}$ is cpt, we need

$\|D\psi\|^2 = \langle D^2\psi, \psi\rangle = \|\nabla\psi\|^2 + \int_M d\mathrm{vol}\, \langle\psi,\psi\rangle \frac{1}{4}R$

$\Rightarrow \|\psi\|^2_{H^1} + \left(\frac{R_{min}}{4} - 1\right)\|\psi\|^2 \leq \|D\psi\|^2 \leq \|\psi\|^2_{H^1} + \left(\frac{R_{max}}{4} - 1\right)\|\psi\|^2$

Prop $(D-\lambda)^{-1}$ is cpt for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Pf. $\|\psi\|^2_{H^1} \leq \|(D-\lambda)\psi\|^2 + \left(|\lambda|^2 + 1 - \frac{R_{min}}{4}\right)\|\psi\|^2$

Set $\varphi = (D-\lambda)\psi \in \mathrm{Ran}\,(D-\lambda) = H = L^2(\Sigma)$.

So, substituting,

$\|(D-\lambda)^{-1}\varphi\|^2_{H^1} \leq \|\varphi\|^2 + \mathrm{const}_1 \underbrace{\|(D-\lambda)^{-1}\varphi\|^2}_{\leq\, \mathrm{const}_2 \cdot \|\varphi\|^2}$ (bdd.)

$\leq \mathrm{const}_3 \|\varphi\|^2$

so cpt by Rellich thm.

Rmks

- $\exists$ complete O.N.B $\{\psi_n\}$ of D-eigenv. in $L^2(\Sigma) = H$
- $D\psi_n = \lambda_n\psi_n$ and $|\lambda_n| \nearrow \infty$ as $n \to \infty$
- $\varphi = \sum \alpha_n \psi_n \in H^k$ iff $\sum |\alpha_n|(1 + |\lambda_n|^k) < \infty$ ($\alpha \in \ell^2$)
  $\hookrightarrow \|\cdot\|^2_{H^k} = \|\cdot\|^2 + \|\nabla\cdot\|^2 + \dots + \|\nabla^k\cdot\|^2$
- $D^k\varphi = \sum \alpha_n (\lambda_n)^k \psi_n$

Boundedness of commutators

- $C^\infty(M,\mathbb{C})$ action on $\Gamma(\mathcal{E})$ extends to $L^2(\mathcal{E}) = H$ by bounded operators, as a $*$-representation

 - meaning multiplicative, $*$-preserving, lin. to scalar mult., e.g. $(z\, f^* + f^1 \cdot f^2)\triangleright\_ = \bar{z}\cdot(f \triangleright \_)^* + (f^1 \triangleright\_)(f^2 \triangleright\_)$

- exercise: $\| f \triangleright \_ \| = \| f \|_\infty$ on $L^2(\mathcal{E}, vol_g)$

$\| [D, f\triangleright] \| = \| df \cdot \| = \| \gamma(df) \| \leq const \cdot \sup_M |df|$

- so commutators bdd even though D is not.
- unfortunately, f cannot only be continuous, but...

Rmk $C^\infty(M,\mathbb{C})$ is not the biggest subalg. of $C(M,\mathbb{C})$ s.t. $\| [D, f] \| < \infty$. Indeed, this holds for Lipschitz functions, i.e. $| f(x) - f(y) | < C \cdot dist(x,y)$

Def A **spectral triple** $(A, \mathcal{H}, D)$ consists of

- a $*$-algebra A
- a Hilb. sp. $\mathcal{H}$ (separable), carrying a faithful, bounded $*$-rep π of A
- $D = D^*$ with bdd comm. with $\pi(a), \forall a \in A$ with a cpt. resolvent

A s.t. is **even** if $\exists \chi = \chi^*, \chi^2 = id, [\chi, \pi(a)] = 0$ $\forall a \in A$, $\{\chi, D\}_+ = 0$. If not, s.t. is **odd**.

A s.t. is real if $\exists$ antiunitary J s.t. that
$J\pi(a^*)J^{-1}$ commutes w/ $\pi(b)$ $\forall b \in A$
and $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, $J\chi = \varepsilon'' \chi J$
where $\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$.

A s.t. is finite if $\dim \mathcal{H} < \infty$
— " — commutative if A is.

Prop Let M spin, cpt w/o boundary, $\Sigma \to M$ Dirac spinor vbdl.
Then $(C^\infty(M), L^2(\Sigma), \not{D})$ is
a comm. s.t., even if $\dim M$ = even,
and real if M is spin.

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- recall:
  - we started with $\overset{\text{Dirac}}{\gamma}: \mathcal{C}(X) \longrightarrow L(\mathbb{C}^{2^{[n/2]}})$
  - passed to (M,g) spin mfd w/ Σ $\mathbb{C}$-vbdl of spinors s.t. for $x \in M$, $\check{\gamma}: \mathcal{C}(T_xM) \times \Sigma_x \longrightarrow \Sigma_x$,

    $\check{\gamma}(\alpha, \psi) := \gamma(\alpha)\psi$.

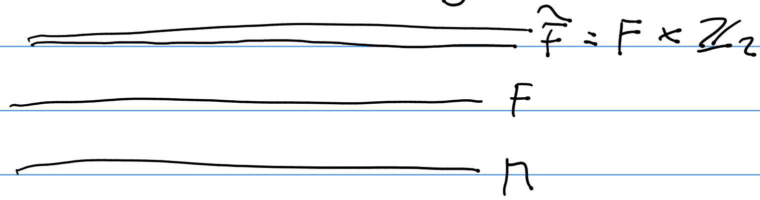
  and built $D\!\!\!/ = \gamma \circ \nabla \in \mathrm{Aut}\,\Gamma(\Sigma)$

$\Rightarrow (\mathcal{C}^\infty(M), L^2(\Sigma, vol_g), D\!\!\!/)$ is S.T.

- dim M = even $\Rightarrow \exists\, \chi = \gamma(vol) = vol\cdot$
- M spin $\Rightarrow \exists\, J$ real structure

Examples

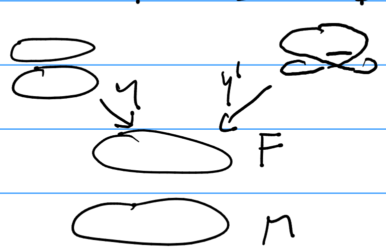
- $M = \mathbb{R}$, "x" = "id", $g = dx^2 = dx \otimes dx$, $g(e,e) = 1$



$\partial_1 = i$, $D = i\partial_x$, J = c.c.

$\Gamma(\Sigma)$ = smooth cpx functions

- $M = \mathbb{R}/2\pi\mathbb{Z} = S^1$, same x mod 2π, g



$- \Gamma(\Sigma) = \begin{cases} \text{periodic} \\ \text{antiperiodic} \end{cases}$

ψ

$\psi_m = \text{const}\cdot e^{imx}$, $m \in \begin{cases} \mathbb{Z} \\ \mathbb{Z} + 1/2 \end{cases}$

$D\!\!\!/ = i\partial_x$, $D\!\!\!/\psi_m = \pm\sqrt{m^2}\,\psi_m$

$\longrightarrow$ spectrum depends on sp.str (e.g., $\ker D\!\!\!/|_{\text{antip.}} = \{0\}$)

• $M = \mathbb{R}^2$, $g = dx \otimes dx + dy \otimes dy$

$\exists$ global frame $(x, y) = e$

$\Rightarrow F = M \times SO(2)$, $\tilde{F} = M \times Spin(2)$ trivial,

so $\Sigma = M \times \mathbb{C}^2$, $\Gamma(\Sigma) = C^\infty(M, \mathbb{C}^2)$

$$D = i \underset{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}{\sigma_1} \partial_x + i \underset{\begin{pmatrix} & -i \\ i & \end{pmatrix}}{\sigma_2} \partial_y = i \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}, \quad z = x + iy$$

$$\chi = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad J_- = \begin{pmatrix} & -i \\ i & \end{pmatrix} \circ c.c., \quad J_+ = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \circ c.c.$$

• $M = \mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{T}^2 = S^1 \times S^1$

same x, y, g

$$\begin{tikzcd} \tilde{F} = M \times Spin(2) \arrow[d, "\eta_{jk}"] \\ F = M \times SO(2) \arrow[d] \\ M \arrow[u, bend right, "e_{jk}"'] \arrow[uu, bend left, "\tilde{e}_{jk}"] \end{tikzcd}$$

- rotation $\begin{pmatrix} \cos(jx + ky) & \sin(\ ) \\ -\sin(\ ) & \cos(\ ) \end{pmatrix}$

- $j, k = 0, 1$

$\Rightarrow$ different sp. str.

$\Rightarrow \psi$ (anti) periodic in x, y

$$\not{D} \psi_{m,n} = \pm \sqrt{m^2 + n^2}\, \psi_{m,n}, \quad m \in \mathbb{Z} + j/2, \quad n \in \mathbb{Z} + k/2$$

$$\psi_{m,n} = e^{imx + iny} \begin{pmatrix} a \\ b \end{pmatrix}$$

[this is for $e_{jk}, \tilde{e}_{jk}$ with $\tilde{j} = j$, $\tilde{k} = k$

- if we take $\tilde{j} \neq j, \tilde{k} \neq k$, the lifts $\tilde{e}_{\tilde{j}\tilde{k}}$ and $\eta_{\tilde{j}\tilde{k}}^{-1} \circ e_{jk}$ differ by rotation, which is "local gauge transf."

- we use $\nabla = d + \alpha^{e\cdot g}$, $\alpha^e = 0$,
  $\alpha^{e\cdot g} = g^{-1}\alpha^e g + g^{-1} d g$
  and $g = R = \begin{pmatrix} \cos(jx+ky) & \sin(\cdot) \\ -\sin(\text{—}) & \cos(\cdot) \end{pmatrix}$

$\to \nabla^{spin} = d + \rho \circ \dot{s}^{-1}(R^{-1} \circ dR)$, $\dot{s}$: $spin(2) \to SO(2)$
$= d + i/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (j\,dx + k\,dy)$

- but spec is the same modulo discrete shift

- $M = \mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n = S^1 \times \text{—} \times S^1$
  - 2^n - sp. str, $t = \{t_j\}$, $j = 1, \ldots, n$, $t_j \in \{0, 1\}$
  - F, $\tilde{F}$, Σ trivial, $\cancel{D}^2 = \Delta \cdot \mathbb{1}_{2^{[n/2]}}$

- eigenspinors labelled by $\lambda = (\lambda_j) \in \Lambda_t := \mathbb{Z}^n + \frac{t}{2}$, $\lambda_j \in \mathbb{Z}^n + \frac{t_j}{2}$
  - t spin index $\in \{1, \ldots, 2^{[n/2]}\}$, $j \in 1, \ldots, n$
  $\sim e^{i(\lambda_1 x_1 + \cdots + \lambda_n x_n)} \begin{pmatrix} \vdots \end{pmatrix}$

  $\to$ eigenvalues $\ell = |\lambda|$, degenerate

- in general, for curved g, impossible to compute exactly
  - perhaps on homogenous spaces, etc.
  - what if g has torsion? etc.

$\cup \Lambda^\bullet(T_x^* M \otimes \mathbb{C})$

• Hodge-deRham S.T.

- (M, g) oriented, $\left(C^\infty(M), L^2(\Lambda M, \mathrm{vol}_g), d + d^*\right)$

- basis $\{e^{j_1} \wedge \dots \wedge e^{j_k} \mid j_1 < \dots < j_k, k = 0, \dots, n\}$

- check: $d + d^* = \lambda \circ \nabla_{\text{Levi-Civita}}$,
where $\lambda(v) \cdot \alpha = (v \wedge - v \lrcorner)\alpha$
$v \in \Gamma(T^*M)$

- $(\lambda(v))^2 = -|v|^2$
- λ rep. of $Cl(-)$ of $\dim = 2^n$ $\to$ reducable, but who cares
- check: $(\sigma_{d+d^*}(\xi))^2 = -|\xi|^2$

- further, we have gradings:
1) $\gamma_\Lambda = (\pm)^k$ on $\alpha \in \Gamma(\Lambda^k M)$
ii) $\gamma_\Lambda' = \text{const} \circ *$, $*$ = Hodge star
s.t. $\gamma_\Lambda'(e^{j_1} \wedge \dots \wedge e^{j_k}) = i^{k(k-1)+n} e^{j_{k+1}} \wedge \dots \wedge e^{j_n}$ $\forall 0 \le k \le n$
where $e^{j_1}, \dots, e^{j_k}, e^{j_{k+1}}, \dots, e^{j_n}$ is
an even perm. of $e^1, \dots, e^n$
- check: $\gamma_\Lambda, \gamma_\Lambda'$ commute w $C^\infty(M, \mathbb{C})$,
anticomm. w $d + d^*$

- index $d + d^* \big|_{\Lambda^{\text{even}}, \Lambda^{SD}} = \begin{cases} \text{Euler ind.} \\ \text{Signature} \end{cases}$ of M
$\to$ +1 eigenspace of $\gamma_\Lambda, \gamma_\Lambda'$

- $\exists J = \text{c.c.}$

- using λ, $\Gamma(\Lambda M)$ is $Cl(V)$-bimod
- $\lambda_R = (v \wedge + v \lrcorner) \circ \gamma_\Lambda$

- as vsp, $\Lambda V = \mathcal{C}(V)$, so we get Morita equiv. over itself
- recalling $[\not{D}, f] = df\cdot$, $(\mathcal{E}^\infty(M), [d + d^*, f]) \simeq \Gamma(\mathcal{C}(M))$

  - $[[J f J, \alpha]] = 0 \quad \forall \alpha \in Cl, f \in \mathcal{E}^\infty(M)$

- $\exists\ J_\Lambda' = (-)^{k(k-1)/2} \circ c.c. \to$ intertwines λ_R, λ_L

- R. Plymen

- so we can build $\overset{NC}{\text{diff. forms}}$ from any $(A, H, D) \longmapsto \mathcal{C}_D(A)$ in this way

- to close, operator D on $\Gamma(\xi)$ is of Dirac-type if $[D, f]^2 = -g(df, df)$
  $\Rightarrow \mathcal{C}(df) := [D, f]$
  $\to$ then $(\mathcal{C}^\infty(M), L^2(\xi), D)$ is a S.T.

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Further properties of canonical s.T.s

- 7 properties selected by Connes
  $\rightarrow$ reconstruction thm
- in general, we would like to extract properties of spaces

Dimension

- we study $\mathbb{T}^n = M = \mathbb{R}^n / 2\pi \mathbb{Z}^n$
  - $N_\ell := \# \{ \Lambda_t \cap B_\ell \}$

$$\Lambda_t = \mathbb{Z}^n + t/2, \quad t = \{ \{t_j\} \mid j = 1, \dots, n, \ t_j = 0, 1 \}$$

$$B_\ell = \{ \lambda \in \mathbb{R}^n \mid |\lambda| \leq \ell \}$$

$\rightarrow N_\ell \sim \operatorname{vol} B_\ell \sim n^{-1} \underbrace{V_{n-1}}_{= 2\pi^{n/2} \Gamma(\frac{n}{2})} \ell^n$ as $\ell \to \infty$

where $a_\ell \sim b_\ell$ iff $\lim \frac{a_\ell}{b_\ell} = 1$, $b_\ell = 0$ for finitely many ℓ.

- wlog $\exists \, |D|^{-1}$, $(|D| + \varepsilon)^{-1}$, $\varepsilon > 0$
- let $s > 0$, consider Σ of first N_ℓ eigenvals

$$\sigma_{N_\ell}(|D|^{-s}) := 2^m \sum_{1 \leq |\lambda| \leq \ell} |\lambda|^{-s}$$

$$\overset{?}{\sim} 2^m \int_0^\ell r^{-s} \underbrace{(N_{r+dr} - N_r)}_{\frac{dN}{dr} dr}$$

$$= 2^m V_{n-1} \int_0^\ell r^{-s+n-1} \, dr = 2^m V_{n-1} \begin{cases} \frac{\ell^{n-s}}{n-s}, & s \neq n \\ \log \ell, & s = n \end{cases}$$

- since $\ell \sim \left(n \frac{N_\ell}{V_{n-1}} \right)^{1/n}$,

$$\sigma_{N_\ell}(|D|^{-s}) \sim 2^m \begin{cases} c \, N_\ell^{\frac{n-s}{n}} & s \neq n \\ c' + \frac{V_{n-1}}{n} \log N_\ell & s = n \end{cases}$$

$$\Rightarrow \frac{\sigma_N(|D|^{-s})}{\log N} \overset{N\to\infty}{\sim} \begin{cases} c \cdot \frac{N^{\frac{n-s}{n}}}{\log N} \nearrow \infty & , s \lesssim n \\ V_{n-1}/n \searrow 0 & , s = n \end{cases}$$

$\Rightarrow \exists !\, s$ s.t. $\sigma_N(|D|^{-s}) \sim \frac{V_{n-1}}{n} \log N$

$\rightarrow s = n$, dimension

- independent of shift t
- holds for $T = \mathbb{R}^n / k_1\mathbb{Z} \times k_2\mathbb{Z} \times \cdots \times k_n\mathbb{Z}$
- holds for any (spin) mfd due to Weyl thm $\lambda_k(|D| \text{ or } \Delta^{1/2}) \sim k^{1/n}$

- more modern:

part. sum of sing. eigenvals of $\sqrt{T^*T}$

Dixmier ideal $\mathcal{L}^{1\infty} = \mathcal{L}^{1+} = \{ T \in \mathbb{B}(\mathcal{H}) \mid \sigma_N(T) = O(\log N) \}$
$\subseteq \mathcal{L}^p \subseteq K(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$

$\hookrightarrow$ p-summable, $\sum |\lambda|^p < \infty$

- recall, $f_n = O(g_n)$ iff $\frac{f_n}{g_n}$ bdd $\forall n$

$f_n = o(g_n)$ iff $f_n/g_n \to 0$

- $\|T\|_{1+} := \sup \left\{ \frac{\sigma_N(T)}{\log N} \right\}$ is norm

$\leftarrow \exists\, \infty$ family of ≥ 0 tracial states, but never constructively shown

- all coincide and give ≥ 0 tracial state on closed subspace $\subseteq \mathcal{L}^{1+}$ of measurable T, i.e. T s.t. $\exists \lim_{\ell\to\infty} \tau_\ell(T)$, where

$$\tau_\ell(T) := \frac{1}{\log \ell} \int_{\text{any } a>0}^{\ell} \frac{\sigma_u(|T|)}{\log u} \frac{du}{u}$$

$\sigma_u(|T|)$ $\leftarrow$ interpolation of $\sigma_N(|T|)$

Def. $\mathrm{Tr}^+(T) := \lim_{\ell\to\infty} \tau_\ell(T)$, $T \geq 0$

- main example, $|D|^{-n} \in \mathcal{L}^{1,\infty}$ is measurable and $\mathrm{Tr}^+(|D|^{-n}) = 2^m v_{n-1}/n$

$\to$ the coeff we got before

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- $\mathcal{L}^1 \subset \mathcal{L}^{1+} \subset \mathcal{L}^p,\ \forall p>1 \subset \mathbb{K}(\mathcal{H}) \subset \mathbb{B}(\mathcal{H})$
- $\mathrm{Tr}_\omega\big|_{\text{measurable}} =: \mathrm{Tr}^+$ Dixmier trace

  if $\exists \lim \dfrac{\sum_{i=1}^N \lambda_i}{\log N}$.

- $\mathrm{Tr}^+(T) := \lim \tau_e(T)$, smeared versions of

- if $\dim M = n$, $|D|^{-n}$ measurable, $\in \mathcal{L}^{1+}$,
  $\mathrm{Tr}^+ |D|^{-n} = 2^m\, v_{n-1}/n$

- defining $S_\pm = \frac{1}{2}(|S| \pm S)$ which

  takes pos/neg "part", we get a "polarization identity"

$$\mathrm{Tr}^+ T = \frac{1}{2}\Big[\mathrm{Tr}^+(T+T^*)_+ - \mathrm{Tr}^+(T+T^*)_- + i\,\mathrm{Tr}^+\big[(iT-iT^*)_+\big] - i\,\mathrm{Tr}^+\big[(iT-iT^*)_-\big]\Big]$$

- can be related to Wodzicki residue, unique (if $n\geq 2$) tracial state on ΨDO (pseudodiff. op) on $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, M cpt. dim n

$$W_{res}(P) := \int_M \int_{\|\xi\|=1} \mathrm{tr}\, \sigma_P^{-n}(x,\xi)\, \nu_\xi\, d^n x$$

where $\nu_\xi := \sum \xi^j \partial_{\xi_j} \lrcorner\, d^n\xi$
$= \sum (-)^j \xi^j\, d\xi^1 \wedge \dots \wedge \widehat{d\xi^j} \wedge \dots \wedge d\xi^n$

- before showing connection, compute for $P = f|D|^{-n}$, $f \in C^\infty(M)$

$$\sigma^{-n}_{f|D|^{-n}}(\xi) = [f|D|^{-n}, h] \quad \hookrightarrow dh = \xi$$

$$= f \underbrace{|\xi|_g^{-n}}_{g(\xi,\xi)^{-n/2}} \mathbb{1}_{2^m}$$

$$\Rightarrow W_{res} = 2^m \int f(x) \int_{\substack{|\xi| = 1, \\ \text{usual length} \\ \text{on } \mathbb{R}^n}} \underbrace{\left(g(x)^{ij} \xi_i \xi_j\right)^{-n/2} \nu_k}_{\text{check: closed form}} \, d^n x .$$

- so change coords

$y = \psi' \xi$ s.t. $\psi' = \sqrt{g(x)}$

$$\Rightarrow \int_{y \in \text{ellipsoid}} |y|^{-n/2} \nu_y$$

$\quad \parallel$

Sphere by Stokes

$$\Rightarrow W_{res} = 2^m \nu_{n-1} \int_M f(y) \underbrace{(\det g)^{1/2}}_{\text{Jacobian}} d^n y$$

$$= 2^m \nu_{n-1} \int f \, d\mathrm{vol}_g$$

Thm (trace; Connes) Any ΨDO P of order $-n$ on $\mathbb{C}$-vbdl $E \to M$, M cpt, dim n, belongs to $\mathcal{L}^{1+}$ and is measurable if

$$\mathrm{Tr}^+ P = \frac{1}{n(2\pi)^n} W_{res}(P)$$

Pf, conceptual

- change of metric $\overset{\text{on } E}{\checkmark}$ $\Leftrightarrow$ unitary transformatn
  - trace doesn't care
- suffices to do locally using p.o.1,
  since $f \cdot -$ is bdd, so $fP \in \mathcal{L}^{1+}$,
  pick $U \overset{\text{open}}{\subset} M$, $U \cong \mathbb{R}^n \subset \underset{\Pi}{\$^n}\, n$ so $E|_U = U \times \mathbb{C}^{\dim E}$
- any P of order $-n$ write as $P = \underset{(\Psi DO)_0}{\overset{\uparrow}{T}} \Delta^{-n/2}$, $T = P\Delta^{n/2}$

  or something like $\sqrt{1+\Delta}$ if $\operatorname{Ker}\Delta \neq \{0\}$
- $\mathcal{L}^{1+}$ is an ideal, measurable ones maybe... so $P = T\Delta^{-n/2} \in \mathcal{L}^{1+}$
- since $(T\Delta^{-s/2})_{s>n} \in \mathcal{L}^{1+} \Rightarrow \operatorname{Tr}_\omega T\Delta^{-s/2} = 0$
  $\Rightarrow \operatorname{Tr}_\omega$ depends only on $\sigma^{-n}(P)$?

- now $E|_U \cong U \times \mathbb{C}^{\dim E}$
- $\{\sigma_\Delta^{-n}\} = \{\mathcal{C}^\infty(T^*M), (-n)\text{-homog.}\} = \mathcal{C}^\infty(S^*M)$
- $\Delta^{-n/2}$ scalar ΨDO on line bdl
- $\forall\,\omega$, Tr_ω is cont. functional on
- since $P \geq 0$, $\sigma^{-n}(x, \xi) \geq 0$

- now $U \subset \$^n \xrightarrow{\text{emb.}} \mathbb{R}^{n+1}$, sym. gp $SO(n+1)$,
  lift to unitary U_g on $L^2(E, \langle\cdot,\cdot\rangle)$
- trans. action on $\$^n$ extends to
  — " — $S^*\$^n$
- $\Rightarrow \operatorname{Tr}_\omega P = \underset{P\text{-indep}}{\overset{\uparrow}{\text{const.}}} \int_{S^*M} \sigma^{-n}(P)\, \mathrm{vol}_{S^*M}$
  - take $P = |D|^{-n}$ to compute const
  - $\rightarrow c = 1$, indep. of ω
    $\Rightarrow P$ measurable. $\square$

- trace thm applies to $f|D|^{-n}$, since $\mathrm{Tr}^+ f|D|^{-n} \neq 0 \Rightarrow |D|^{-n} \notin \mathcal{L}_0^{1+} = \{T \in \mathbb{B}(H) \mid \frac{\sigma_N(T)}{\log N} \to 0\} \supset \mathcal{L}^1$

$$\mathrm{Tr}^+ f|D|^{-n} = \mathrm{Wres}\, f|D|^{-n} = \int_M f \,\mathrm{vol}_g$$

we generalize

Def When $(A, \mathcal{H}, D)$, A noncomm., we define $⨍ a|D|^{-n}$ as LHS.

$$\begin{aligned}
\mathcal{L}^{p+} &:= \{T \in \mathbb{B}(\mathcal{H}) \mid k\text{-th eigenval.}(|T|) = O(k^{-1/p})\} \\
&= \{ \quad " \quad \mid \sigma_N(|T|) = O(N^{p-1/p})\} \\
&= \{ \quad " \quad \mid N_e = O(e^p)\}
\end{aligned}$$

$\to O(\log N)$ for $p=1$

$$\mathcal{L}_0^{p+} := \overline{\text{finite rank}}^{\|\cdot\|_{p+}}$$

$$\|T\|_{p+} := \lim \frac{\sigma_N(|T|)}{N^{\frac{p-1}{p}}}$$

Axiom (Dimension) $\exists n \in \mathbb{N}$ s.t. $|D|^{-1} \in \mathcal{L}^{n+}$, $|D|^{-n}$ measurable, $D^{-n} \notin \mathcal{L}_0^{1+}$.
We call such n the dimension of the S.T., and it equals the dimension in the canonical case.

Dąbrowski

Axioms

1) Dimension, for now

- continuing...

Finiteness & abs. continuity

- $f \in C^\infty(M, \mathbb{C})$ induces $f\cdot$ on $L^2(\Sigma)$,
  $\|f\cdot\| = \|f\|_\infty = \|f\|$, $\overline{\mathcal{C}^\infty(M, \mathbb{C})}^{\|\cdot\|} = \mathcal{C}(M)$
  is Banach $*$-alg, $*f = \bar{f}$
  - but $\|f^* f\| = \|f\|^2 \Rightarrow C^*$-alg
- GN, I: $M \underset{\text{homeo}}{\cong} \widehat{\mathcal{C}(M)}$, $*$-isom., $x \mapsto \chi_x$, $\chi_x(a) := a(x)$
  so $\mathcal{C}(M) \cong \mathcal{C}(\widehat{\mathcal{C}(M)})$, $a \mapsto \tilde{a}$, $\tilde{a}(\chi) := \chi(a)$
  - inj + surj by Stone-Weierstraß
- GN, II: any C^*-alg is (sub)alg in $\mathbb{B}(\mathcal{H})$ for some $\mathcal{H}$.
- on $\mathbb{B}(\mathcal{H})_{s.a.}$ $\exists$ (and only then) a cont. func. calc on $\mathcal{C}(\sigma(T))$ for $T = T^* \in \mathbb{B}(\mathcal{H})$
  s.t. $\forall F \in \mathcal{C}(\sigma(T))$ $\exists$ $F(T) \in \mathbb{B}(\mathcal{H})$,
  $\sigma(F(T)) = F(\sigma(T))$, etc.
- building $A = $ "$\mathcal{C}^\infty(M)$" $\xrightarrow{\text{dense}} \mathcal{C}(M)$
  is not very successful, need to consider all norms $\|f\|_D := \|Df\|$, D diff op on M
- however, A is stable under holo-calculus,
  i.e. if $f^{-1} \in C(M) \Rightarrow f \in \mathcal{C}^\infty(M)$
  $\forall x \in M$, $\exists \frac{1}{f(x)}$ smooth

- recall: if $T \in \mathbb{B}(\mathcal{H})$ (or even $\in \mathbb{B}$(Ban. sp.)) and F holomorphic func. $\mathcal{U}$, $\sigma(T) \subset \mathcal{U} \in \mathbb{C}$,

$$\Rightarrow F(T) := \frac{1}{2\pi i} \int_{\sigma(T) \subset \text{pcwise sm} \subset \mathcal{U}} \frac{F(\xi)}{\xi - T} \, d\xi \qquad \left(\begin{matrix}\text{Dunford-} \\ \text{Riesz}\end{matrix}\right)$$

$\Rightarrow$ A is pre-C^*-alg (i.e. dense in C^*-alg

- recall:
  - faithful bdd $*$-rep of $A \in \mathcal{C}^\infty(M)$ on $\mathcal{H} = L^2(\mathcal{E})$ extends to $C(M)$
  - $D = D^*$, $\mathrm{Dom}\, D = \mathcal{H}^1$, $\mathrm{Ran}\, D^k = \mathcal{H}^k$, then $\mathcal{H}^\infty := \bigcap_{k \in \mathbb{N}} \mathcal{H}^k = \Gamma(\mathcal{E})$.
  - $\Gamma(\mathcal{E})$ is proj & fin. rk $C^\infty(M)$-mod ($\mathcal{C}^\infty(M)$-mod)

- recall:
  - $\exists \langle -, - \rangle : \Gamma(\mathcal{E})^2 \to \mathcal{C}^\infty(M)$ sesquil. s.t.

$$\begin{aligned}
\langle \psi, \varphi \rangle_M &:= \int_M \langle \psi, \varphi \rangle \, d\mathrm{Vol}_g \\
&= \mathrm{W}_{res}\left((\psi, \varphi) |D|^{-n} \right) \\
&= \mathrm{Tr}^{+}\left((\psi, \varphi) |D|^{-n} \right) \\
&= ⨍ (\psi, \varphi) |D|^{-n}
\end{aligned}$$

- which leads us to...

Axiom: Finiteness & A.C. prop

$\longrightarrow$ A is a pre-C^*-alg, $H^\infty = \bigcap_{k \in \mathbb{N}} \mathrm{Dom}(D^k)$
is a fin. gk. proj. module $/A$,
and $\exists$ hermitian A-valued form $\langle\cdot,\cdot\rangle$
on H^∞ s.t.

$$\langle \psi, \varphi \rangle_H = ⨍ (\psi, \varphi)\, |D|^{-n}$$

- last analytic axiom...

Regularity (smoothness)

operator	order	δ prime. op.
D	1	matrix
$\|D\|$	1	scalar
$[\|D\|, a], a \in A$	0	scalar
$[\|D\|, [D, a]]$	0	matrix

$\}$ bdd

$\Rightarrow$ iterating $[\underbrace{|D|, \ldots, [|D|}_{k \text{ times}}, \overset{}{b}]] \in \mathbb{B}(\mathcal{H})$ $\forall k \geq 1$, $b = a$ or $[D, a]$

- so $A \cup [D, A] \subset \mathrm{Dom}^\infty \delta := \bigcap_{k \in \mathbb{N}} \mathrm{Dom}\, \delta^k$

where $\mathrm{Dom}\, \delta^k := \{\beta \in \mathbb{B}(\mathcal{H}) \mid \delta^k(\beta) \in \mathbb{B}(\mathcal{H})\}$

and $\delta(\beta) := [|D|, \beta]$.

- equivalently:
$\mathbb{R} \ni t \longmapsto \| e^{it|D|} \beta e^{-it|D|} \|$ should be
C^∞ $\forall \beta \in A \cup [D, A]$ $(*)$

$\Downarrow$

Axiom: $(A, \mathcal{H}, D)$ is regular iff $(*)$

- exercise: why $[D,[D,a]] \notin \mathbb{B}(\mathcal{H})$

Rmk • possible to see A is largest subalg of cont. funcs. s.t. $(*)$

• can be char. as Frechet alg. complete in $\|a\|_k := \|\delta^k a\|$, $\|a\|'_k := \|\delta^k([D,a])\|$ $\forall k \in \mathbb{N}$

- last one...

Reality

- a S.T. is also called (unbdd) KR-cycle; in our case we have A, D, J, χ (n = even) on $\mathcal{H} = L^2(\Sigma)$ s.t. Ad_J implements involution τ on A

- if A N.C. alg, $JaJ^{-1} \neq a^*$ since $*$ is anti involution

- but $JAJ^{-1} \subset A' \,(\subset \mathbb{B}(\mathcal{H}))$ permits to define $*$-rep of A^{op} which commutes w rep of A

$\Rightarrow$ $*$-rep of $A \otimes A^{op}$ on $\mathcal{H}$,

$$\pi_{A\otimes A^{op}}(a \otimes b^{o})\psi := \underset{\substack{\parallel \\ (\pi_A(a),\text{ we don't write it})}}{a}\, J b^* J^{-1}\psi$$

- equivalently, $\mathcal{H}$ becomes an A-bimod by

$$a \psi b := \pi_{A \otimes A^{op}}(a \otimes b^{o}) \psi$$

- moreover, by $\mathrm{Ad}\, J$ we get an invol. on $A \otimes A^{op}$ by $a \otimes b^{op} \mapsto b^{*} \otimes a^{o}$

- the signs $\epsilon, \epsilon', \epsilon'' \equiv n = \dim(A, \mathcal{H}, D) \bmod 8$

Dabrowski

1st order

- $a \in A = C^\infty(M)$, $[D, a] = i\, da$.
- $B = \Gamma(\mathcal{Cl}(M))$, $A \subset Z(B)$, B gen. by A, $[D, A]$
$\Rightarrow$ $\mathcal{H}$ is A-B-bimodule
$\Rightarrow$ thus, $[[D, a], b] = 0 \quad \forall a, b \in A$
$\hookrightarrow$ all of this in comm. case.

- can be formulated NC
- we know $Ad_J : A \to A' \hookrightarrow \mathbb{B}(\mathcal{H})$,
  $[a, JbJ^{-1}] = 0$

- first order cond'n $[[D, a], JbJ^{-1}] = 0$

Orientability

- in NCG diff calculus is done "algebraically",
[Hochschild-Kostant-Connes thm]
- e.g. $vol_\alpha = e^1_\alpha \wedge \dots \wedge e^n_\alpha$, $\dim M = n$, indep. of frame
  - in terms of coords c^j_α, $j = 1, \dots, n$,
    $$vol_\alpha = \det(\vartheta_\alpha)\, dc^1_\alpha \wedge \dots \wedge dc^n_\alpha$$
  where $e_\alpha{}^j = \sum_k (\vartheta_\alpha)^j{}_k\, dc^k_\alpha$
  - pick p.o.1. $\sum f_\alpha = 1$, define
    $$c^0_\alpha := i^{n-m} f_\alpha \cdot \det \vartheta_\alpha$$
  - let $c := \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma \sum_\alpha c^0_\alpha \otimes c^{\sigma_1}_\alpha \otimes \dots \otimes c^{\sigma_n}_\alpha \in A^{\otimes(n+1)}$
   Hochschild n-chains w/ coeffs in A

- $bc := \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma \sum_\alpha b(\cdot \dots)$
  $= \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma \sum_\alpha \Big[c^0_\alpha c^{\sigma_1}_\alpha \otimes \dots \otimes c^{\sigma_n}_\alpha + \sum_{j=1}^n (-)^j c^0_\alpha \otimes \dots \otimes c^{\sigma_j}_\alpha \cdot c^{\sigma_{j+1}}_\alpha \otimes \dots \otimes c^{\sigma_n}_\alpha$

$$+ (-)^n c_\alpha^{\sigma_n} c_\alpha^0 \otimes \cdots \otimes c_\alpha^{\sigma_{n-1}} \big] \in A^{\otimes n}$$

$\Rightarrow$ $bc = 0$, and $b^2 = 0$

- define Hochschild homology

- given $(A, \pi, \mathcal{H}, D)$ S.T., emphasising rep π, define for any $c^0 \otimes \cdots \otimes c^n \in A^{\otimes(n+1)}$

$$\pi_D(c^0 \otimes \cdots \otimes c^n) := \pi(c^0)[D, \pi(c^1)] \cdots [D, \pi(c^n)]$$

omitting writing π

$$\Rightarrow \pi_D(c_{vol}) = \frac{1}{n!} \sum_{\sigma \in S_n} (-)^\sigma \sum_\alpha c_\alpha^0 \underbrace{[D, c_\alpha^{\sigma_1}]}_{\gamma(d c_\alpha^{\sigma_1}) \,=\, \gamma\left(\sum_{i_1} (x^{-1})^{\sigma_1}_{i_1} e_\alpha^{i_1}\right)} \cdots \underbrace{[D, c_\alpha^{\sigma_n}]}_{\gamma(d c_\alpha^{\sigma_n}) \,=\, \cdots}$$

$$= \cdots$$

$$= i^{-m} \sum_\alpha f_\alpha \, \gamma(e_\alpha^1 \cdots e_\alpha^n)$$

$$= \gamma(\omega) = \begin{cases} \chi & \text{for } n = \text{even} \\ 1 & \text{for } n = \text{odd} \end{cases}$$

Axiom (Orientability) $\exists$ Hochschild n-cycle $c \in Z_n(A, A)$ s.t. $\pi_D(c) = \begin{cases} \chi \\ 1 \end{cases}$ for $n \begin{cases} \text{even} \\ \text{odd} \end{cases}$.

Rmk $[a, J b J^{-1}] = 0$ $\rightarrow$ 0^{th} order cond.

$[[D, a], J[D, b]J^{-1}] = 0$ $\rightarrow$ 2^{nd} ord. cond., satisfied by Hodge-de Rham S.T.

Poincaré duality ($/\mathbb{R}$ or $/\mathbb{C}$)

- classically, $H^k_{dR}(M) \times H^{n-k}_{dR}(M) \longrightarrow \mathbb{C}$
  perfect pairing $\langle[\alpha],[\beta]\rangle := \int_M \bar{\alpha}\wedge\beta$
  hermitian
- well-def. on classes
- nondeg. due to existence of volume form i.e. Hodge duality

$$\forall \alpha \neq 0, \int_M \bar{\alpha}\wedge *\alpha = \int_M \langle\alpha,\alpha\rangle_g \,\mathrm{vol}_g > 0$$

where $\langle \alpha_1\wedge\dots\wedge\alpha_k, \beta_1\wedge\dots\wedge\beta_l\rangle_g = \delta_{kl}\det\langle\alpha_i,\beta_j\rangle_k$
where $\langle\alpha,\beta\rangle_g = g^{-1}(\bar{\alpha},\beta)$ for $\alpha,\beta\in\Omega^1(M)$

- duality comes from isomorphism
  $K^\bullet(C(M))$ and $K_\bullet(C(M)) = K^\bullet(M)$ in K-th
  and Chern isomorphism $K^\bullet(M)\otimes\mathbb{C}\simeq H^\bullet_{dR}(M)\otimes\mathbb{C}$

Axiom (Poincaré) The pairing

$$K_0(A)\otimes K_0(A)\longrightarrow\mathbb{Z},$$
$$([p],[q])\longmapsto \mathrm{ind}(pJqJ^{-1}\, \overset{D|_{\mathcal{H}^+}}{\overset{\shortparallel}{D_+}}\, pJqJ^{-1})$$

and

$$K_1(A)\otimes K_1(A)\longrightarrow\mathbb{Z}$$
$$([u],[v])\longmapsto \frac{1}{4}\mathrm{ind}\left(\left(1+\frac{D}{|D|}\right)uJvJ^{-1}\left(1+\frac{D}{|D|}\right)\right)$$

are nondegenerate.

- note, we use the fact that if D Fredholm,
  $P(D\otimes \mathrm{id}_N)P'$ is also Fredholm, P, P' projs.

Dąbrowski

Reconstruction

- D encodes $\begin{cases}\to \text{top.}\\ \to \text{geom.}\end{cases}$, generates K-homology
  $ds = (D + i\varepsilon)^{-1}$
- claim: S.T. + 7 axioms encodes all geom. info of M
  - assuming irreducibility: $\nexists$ p proj. $\in Z(A)$ commuting w/ D, J, X
    - this is slightly stronger than needed
  - was first shown for $A = C^\infty(M)$ [Connes, '95] where M closed.

Thm (Reconstruction I, for $A = C^\infty(M)$)

a) $\exists!$ Riemannian metric g on M s.t. the geodesic distance is $d_g(x,y) = d_D(x,y) := \sup\left\{ |a(x) - a(y)| \;\middle|\; a \in A, \|[D,a]\| \leq 1 \right\}$

b) g only depends on $[D] = \{ UDU^+ \mid U \in U(\mathcal{H}),\ U \text{ comm. w/ } A, J, X \}$, which form a fin. coll'n of affine sp.s Ω_σ, labelled by spin str'cts σ on M.

c) the action functional

$$S : D \mapsto W\text{res}(|D|^{2-n}) := \frac{1}{n(2\pi)^n} \int_{S^*M} \operatorname{tr}_S \sigma^{-n}_{|D|^{2-n}}(x,\xi)\, dx\, d\xi$$

is a pos. quad. form on each Ω_σ, with a unique min. attained at the canonical S.T. of (M,g), and its value is equal to $\int_M R \operatorname{vol}_g$