Arbarello.

$C \subset \mathbb{P}^r$, $\deg C = d$, $p(t) = dt - g + 1$

$T_{[C]}(\mathrm{Hilb}_{\mathbb{P}^r, p}) = H^0(N_{C/\mathbb{P}^r})$

$$h^0(N) - h^1(N) \leq \dim_{\mathbb{C}}(\mathrm{Hilb}) \leq h^0(N_{C/\mathbb{P}^r})$$

$$\overset{\shortparallel}{\chi(N)} = 3g-3 + g - h^0(L)h^1(L) + (r+1)^2 - 1$$

Exercise:

$$\begin{tikzcd} & 0 \arrow[d] & & & \\ & \mathcal{O}_C \arrow[d] \arrow[r, equal] & \mathcal{O}_C \arrow[d] & & \\ 0 \arrow[r] & \Sigma_L \arrow[r] \arrow[d] & H^0(L)^\vee \otimes L \arrow[r] \arrow[d, dashed] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T_C \arrow[r] \arrow[d] & T_{\mathbb{P}}|_C \arrow[r] & N \arrow[r] & 0 \\ & 0 & & & \end{tikzcd}$$

diff operators of order ≤ 1 ...

$$H^0(L) \otimes H^0(KL^{-1}) \xrightarrow{\mu_0} H^0(K)$$

- check: μ_0 injective $\iff H^1(K) = 0$

$\rightarrow$ this implies $\exists$ a patch in moduli space where μ_0 inj, i.e. $H^1(K) = 0$, i.e. Hilb smooth at C

$X \subset \mathbb{P}^4$ cubic threefold, $F = \{ \ell \in Gr(2,5) \mid \ell \subset X \}$

$$N_{\ell/X} \nearrow \mathcal{O}_\ell(-1) \otimes \mathcal{O}_\ell(1) \quad \searrow \mathcal{O}_\ell \oplus \mathcal{O}_\ell$$

$, h^1(N) = 0 \rightarrow$ smooth

- in general, $X \subset \mathbb{P}^r$ fixed, $\mathrm{Hilb}_{X,P} = \{ Y \subset X \mid P_Y = P \}$, $T(\mathrm{Hilb}_{X,P}) = H^0(N_{Y/X})$

Mumford's example:

- X cubic in $\mathbb{P}^3$ $\Rightarrow$ 27 lines $\to$ pick one, L

$\omega_X = \mathcal{O}_X(-H)$

$H^2 = 3,\ H\cdot L = 1,\ L^2 = -1$

$|C|,\ C = 4H + 2L$

$\dim|C| = 37,\ g(C) = 24,\ \deg C = 14.\ |C|$ very ample

$p_C(t) = 14t - 23$

$\mathrm{Hilb}_{\mathbb{P}^3, p} \supset V = \left\{ \begin{array}{l} \text{smooth curves of genus 24,} \\ \text{degree 14, contained in} \\ \text{some smooth cubic} \end{array} \right\}$

$\to$ V irreducible by monodromy argument
  - take space of all cubics + a line
    - 27-sheeted cover
  - loop around singular cubic
    $\to$ obtain transitive S_n action on lines

- take $\mathcal{H}$ irred. component of Hilb containing V

- we want $\mathcal{H} = V$, $\dim \mathcal{H} = \dim V = 56$, $\forall C \in V$, $\dim T_{[C]}(\mathcal{H}) = 57$

$\to \dim V = \dim \text{cubics} + \dim|C| = 19 + 37 = 56$

$\underbrace{h^0(N) - h^1(N)}_{\chi(N) = 4d = 56} \leq \dim_{\mathbb{C}} \mathcal{H} \leq h^0(N_{C/\mathbb{P}^3})$

$$0 \to \underbrace{N_{C/X}}_{\mathcal{O}_C(C)} \to N_{C/\mathbb{P}^3} \to \underbrace{N_{X/\mathbb{P}^3}|_C}_{\mathcal{O}_C(3H)} \to 0$$

$\mathcal{H} = V$: $\Gamma \subset \mathcal{H}$ general pt

$\hookrightarrow$ smooth deg 14 deg 24 curve

- assume $\Gamma \not\subset$ smooth cubic (otherwise $\overline{V} = \mathcal{H}$).

  $\rightarrow$ lies on smooth quartic

- look at $Z = \{$ smooth quartic containing a conic $\}$

  $\dim Z \leq 34 - 9 + 3 + 5 = 33$

  $\dim Y \leq \dim Z + \dim |\Gamma| = 57$

  $\dim \mathcal{H} \leq 56 \Rightarrow \mathcal{H} = V$

$$
\begin{tikzcd}
0 \arrow[r] & \underset{37}{H^0(N_{C/X})} \arrow[r] & \underset{57}{H^0(N_{C/\mathbb{P}^3})} \arrow[r, "\alpha"] & \underset{20}{H^0(N_{X/\mathbb{P}^3}|_C)} \arrow[r] & 0 \\
& & & \underset{\substack{\cup \\ \langle X \rangle \subset}}{H^0(\mathbb{P}^3, \mathcal{O}(3H))} \; 20 \arrow[u, "\beta"] &
\end{tikzcd}
$$

$\rightarrow$ look at Luna slice étale