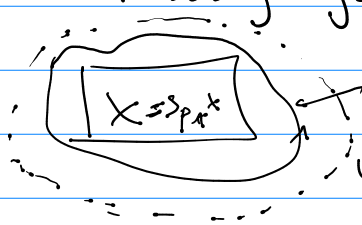
Antonini

- recall Prop B (Cauchy formulas) $\Rightarrow$ Prop A saying i) continuity $R_\gamma \to A$, $p \mapsto p(x)$ ii) $\mathrm{Hol}(x) = \bigcup_n i_{x,Y_n}(Ch(Y_n))$
- for i), note that $p(x) = \frac{1}{2\pi i}\int p(z)(z-x)^{-1}\, d\varphi \wedge dz$ is continuous, automatically
- for ii), $Y_n := \{z \in \mathbb{C} \mid d(z, X) \le \frac{1}{n}\}$. Take f holomorphic on $V \supset X$ open, take $\varphi \in C^1_c(V)$ on some Y_n. Fix $z \in V \setminus Y_n$, def $h_z \in Ch(Y_n)$, $(\lambda \mapsto (z-\lambda)^{-1} =: h_z(\lambda))$
- now $f|_{Y_n} = \frac{1}{2\pi i}\int_{K := \mathrm{supp}\,\varphi} f(z)\, h_z\, dz \wedge d\varphi$

so we integrate $V \setminus Y_n \mapsto Ch(Y_n)$.
Then set $f(x) =: \varphi_x(f)$, $f \in \mathrm{Hol}\,X$.

- the usual way goes like this (contour int.)



γ cycle $\subset U \setminus sp_A x$

$$I_\gamma(z) = \begin{cases} 1, & z \in sp_A x \\ 0, & z \notin U \end{cases}, \quad I_\gamma(z) = (2\pi i)^{-1}\int_\gamma (z'-z)^{-1}\, dz'$$

$$f(x) = (2\pi i)^{-1}\int_\gamma f(z)(z-x)^{-1}\, dz \in A$$

Prop a) $g \in \mathrm{Hol}(sp_A x)$. $sp_A\, g(x) = g(sp_A x)$. Further, $g' \in \mathrm{Hol}(sp_A g(x))$, $(g' \circ g)(x) = g'(g(x))$

b) $\pi : A \to B$. $g(\pi(x)) = \pi(g(x))$, $g \in \mathrm{Hol}(sp_A x)$.

Pf. a) $g = [f]$ germ of $f \in \mathrm{Hol}(U)$, $sp_A x \subset U$. If $\lambda \notin f(sp_A x)$, $z \mapsto (\lambda - f(z))^{-1}$ is holom. around $sp_A\, g(x) \Rightarrow \lambda \notin sp_A\, g(x)$.

...

Lemma $\{u_n\}$ sqn in $\mathbb{R} \cup \{-\infty\}$ s.t. $\forall n, p$, $(n+p)u_{n+p} \leq n u_n + p \cdot u_p$. Then it converges to its infimum.

Pf. Fix $m \geq 1$. By induction on k, $u_{km} \leq u_m$.
For $n = km + r$, $0 \leq r < m$, $u_n \leq n^{-1}(k \cdot m\, u_m + r u_1)$
$$= u_m + \frac{r}{n}(u_1 - u_m)$$
$\Rightarrow \limsup u_n \leq \inf u_n$.

Prop A unital Banach algebra, $x \in A$. Then
$$\|x^n\|^{1/n} \xrightarrow{n\to\infty} \sup\{|\lambda| : \lambda \in Sp_A x\} =: r(x),$$
Spectral radius, $= \inf\{\|x^n\|^{1/n}\} =: s(x)$.

Pf. Apply lemma to $u_n := \log \|x^n\|^{1/n} \Rightarrow$ it converges.
Note that $\boxed{r \leq s}$, because if $\lambda \in Sp_A x$, $|\lambda| \leq \|x\|$, so $|\lambda^n| \in Sp_A x^n$ and $|\lambda^n| \leq \|x^n\| \Rightarrow |\lambda| \leq \|x^n\|^{1/n}$ $\forall n$, so $|\lambda| \leq \inf \|x^n\|^{1/n}$.
But we claim $\boxed{r \geq s}$. We prove $\forall R > r \Rightarrow R \geq s$.
$x^n = (2\pi i)^{-1} \int_{|z|=R} z^n (x - z)^{-1}\, dz \wedge dy \Rightarrow \|x^n\| \leq K R^n, K > 0$
$\Rightarrow \|x^n\|^{1/n} \leq (K)^{1/n} R \Rightarrow s \leq R$.

C^* algebras.

Def. Banach algebra with involution $*$ s.t.
$x \mapsto x^*$ is involutive, antilinear, antisym.
isometric, and
$$\|x^*x\| = \|x\|^2 \qquad (\Delta)$$
C^*-identity.

Rmk. (Δ) holds iff $\|x^*x\| \geq \|x\|^2$

- we call x self-adjoint iff $x = x^*$
- every element decomposes uniquely
  $x = h + ik$, where $h = \frac{1}{2}(x + x^*)$, $k = \frac{i}{2}(x^* - x)$

- example • X cpt Hausdorff, $f \in C(X, \mathbb{C})$,
  $f^* = (m \mapsto \overline{f(m)})$, $\|f\| := \sup_{t \in X} |f(t)|$

  • $\mathbb{B}(\underset{\text{H.16.sp.}}{H})$
- we call $u \in A$ unitary iff $u^*u = u^*u = 1$

Prop A unital C^*-alg.
- u unitary $\Rightarrow$ $Sp_A u \subset U(1)$
- a self-adjoint $\Rightarrow$ $Sp_A a \subset \mathbb{R}$
- for any x, $Sp_A x^* = \{\bar{\lambda} \mid \lambda \in Sp_A x\}$

Pf. 1) by C^*-id, $|\lambda| \leq 1 = \|u\|$, $\frac{1}{|\lambda|} \leq 1 = \|u^{-1}\|$.

Prop. A unital C^*-alg, $B \subset A$ unital closed involutive subalgebra with $1_A \in B$. Then $\forall x \in B$,
$$Sp_A x = Sp_B x.$$

Pf sketch: $\lambda \in \mathbb{C} - Sp_A x$, $y = (x - \lambda)^*(x - \lambda)$. Then
$Sp_B y - Sp_A y = \{\mu \in \mathbb{C} \setminus Sp_A y \mid (y - \mu)^{-1} \in A \setminus B\}$
is an open set in $\mathbb{C}^2$ subset of $\mathbb{R}$, since y self-adj.

Prop. A C^*-alg, $x \in A$ normal, i.e. $x^* x = x x^*$.
Then $r(x) = \|x\|$.

Pf. If $h = h^*$, $\|h^{2^n}\| = \|h\|^{2^n} \Rightarrow \rho(h) = \|h\|$.
If x normal, $\|(x^* x)^n\| = \|x^n\|^2 \Rightarrow \rho(x^* x) = \rho(x)^2$
But $x^* x$ is selfadj, so $\rho(x^* x) = \|x^* x\| = \|x\|^2$.

Gelfand - Naimark.

Thm (2nd Gel'fand's thm) For a commutative C^*-alg A, the Gel'fand transform
$$\mathcal{G} : A \longrightarrow C(\mathrm{Sp}\, A)$$
is iso.

Thm (cpx. Stone-Weierstraß) Let X cpt. Hsdff, $S \subset C(X, \mathbb{C})$ subset which separates points.
Then the unital cpx $*$-algebra generated by S is dense in the C^*-alg $C(X, \mathbb{C})$
("separates pts" means $(\forall x, y \in X)(\exists f \in S)\ f(x) \neq f(y)$)