Tanzini

SUSY or Top. q. mech.

- Hilb. sp. $\rightsquigarrow$ complete unitary vect. sp.
  - $\to |\alpha\rangle \in \mathcal{H}$, $\langle\beta| \in \mathcal{H}^*$
  - $\to$ bra-ket $\langle\_|\_\rangle : \mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C}$ is
    i) sesquilinear
    ii) $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$
    iii) $\langle\alpha|\alpha\rangle \geq 0$ with $=$ iff $|\alpha\rangle = 0$

- supersym. Hilb. sp. is $\mathbb{Z}_2$-graded $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$
  i) Q degree 1 operator (supercharge), $Q^2 = 0$
  $\to$ also $Q^\dagger$
  ii) $H := \frac{1}{2}\{Q, Q^\dagger\}$
  iii) $(-)^F$ is $\begin{matrix} +1 \\ -1 \end{matrix}$ on $\begin{matrix} \mathcal{H}_B \\ \mathcal{H}_F \end{matrix}$, $[(-)^F, Q] = -Q$.

- consequences:
  i) $[(-)^F, H] = [Q, H] = [Q^\dagger, H] = 0$

  $\to$ follows from graded Jacobi and $\{Q,Q\}=0$

  ii) $H \geq 0$. Also, $H|\alpha\rangle = 0$ iff $Q|\alpha\rangle = Q^\dagger|\alpha\rangle = 0$.

  $\to$ let $\beta = Q|\alpha\rangle$, $\gamma = Q^\dagger|\alpha\rangle$. Then
  $\langle\alpha|H|\alpha\rangle = \frac{1}{2}(\|\beta\|^2 + \|\gamma\|^2) \geq 0$.

- note that $Q_1 \equiv Q + Q^\dagger$ is iso $\mathcal{H}_B^{E>0} \xrightarrow[\sim]{Q_1} \mathcal{H}_F^{E>0}$
  so states come in boson-fermion pairs
  $\to$ not true for ground states.

- Witten index: $\Omega := \dim \mathcal{H}_B^{E=0} - \dim \mathcal{H}_F^{E=0}$
  - invariant as long as deformations don't change boundary conds!
- claim: $\Omega = \mathrm{tr}_{\mathcal{H}} (-)^F e^{-\beta H}$, easy to see.

Example.

- 1D: $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})\bar{\psi} = \mathcal{H}_B \oplus \mathcal{H}_F$

- $[p, x] = -i$, $\{\psi, \bar{\psi}\} = 1$
- representation $p = -i\partial_x$, $\psi = \partial_{\bar{\psi}}$

$Q := \bar{\psi}(ip + h'(x))$

$\rightarrow Q^{\dagger} := \psi(-ip + h'(x))$, $h \in C^\infty(\mathbb{R})$ superpotential

$$2H = p^2 + (h')^2 + i\bar{\psi}\psi[p, h'] - i\psi\bar{\psi}[p, h']$$
$$= p^2 + (h')^2 + h''(\bar{\psi}\psi - \psi\bar{\psi})$$

- susy g.s.? $Q|\alpha\rangle = Q^{\dagger}|\alpha\rangle = 0$

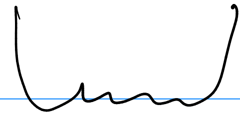
$\rightarrow$ write it as $\varphi = \varphi_B(x) + \varphi_F(x)\psi$

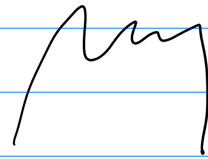
$\rightarrow$ write that as $\varphi = \begin{pmatrix} \varphi_B \\ \varphi_F \end{pmatrix}$

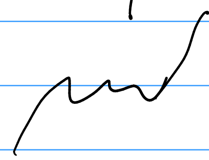
$\rightarrow$ now $Q = \begin{pmatrix} 0 & 0 \\ \partial_x + h' & 0 \end{pmatrix}$, $Q^{\dagger} = \begin{pmatrix} 0 & -\partial_x + h' \\ 0 & 0 \end{pmatrix}$

- so $\begin{matrix} (\partial_x + h')\varphi_B = 0 \\ (-\partial_x + h')\varphi_F = 0 \end{matrix} \Rightarrow \varphi(x) = \begin{pmatrix} c_B e^{-h(x)} \\ c_F e^{+h(x)} \end{pmatrix}$

$\rightarrow$ but $\varphi \in L^2(\mathbb{R}) \oplus \bar{\psi} L^2(\mathbb{R})$

- if $h(x) \xrightarrow[x\to\pm\infty]{} +\infty$  1 bosonic g.s., $\Omega = +1$

- if $h(x) \xrightarrow[x\to\pm\infty]{} -\infty$  1 fermionic g.s., $\Omega = -1$

- if $h(x) \xrightarrow[x\to\pm\infty]{} \mathrm{sgn}(x)\infty$  <u>no</u> susy vacua, $\Omega = 0$

- now put $h \mapsto \lambda h$ and $\lambda >> 0$.

$$h(x) = \frac{1}{2}\omega x^2 \Rightarrow V(x) = \frac{1}{2}\omega^2 x^2 = h'(x)^2$$

$\to$ so $\varphi_{\omega>0} = e^{-\frac{1}{2}\omega x^2}$ (standard h.o.)

$$\varphi_{\omega<0} = e^{-\frac{1}{2}|\omega| x^2}\,\overline{\psi}$$

- $E_{bos} = (n + 1/2)|\omega|$

- $H_f = \frac{1}{2}\omega[\overline{\psi}, \psi] = \frac{\omega}{2}\begin{pmatrix} -1 & \\ & +1 \end{pmatrix}$

so, $E_{ferm} = \pm \frac{|\omega|}{2}$

$\omega > 0$: B: $0, |\omega|, 2|\omega|, \ldots$
F: $\times, |\omega|, 2|\omega|, \ldots$

$\omega < 0$: B: $\times, |\omega|, 2|\omega|, \ldots$
F: $0, |\omega|, 2|\omega|, \ldots$

$$\mathrm{Tr}_B\, e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)|\omega|}$$

$$\mathrm{Tr}_F\, e^{-\beta H} = e^{-\beta\omega/2} + e^{\beta\omega/2}$$

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} = \frac{e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}}}{e^{\frac{\beta|\omega|}{2}} - e^{-\frac{\beta|\omega|}{2}}} = \mathrm{cth}\left(\frac{\beta|\omega|}{2}\right)$$

- but $\Omega = \mathrm{Tr}_{\mathcal{H}} (-)^F e^{\beta H} = \dfrac{e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}}}{e^{\frac{\beta|\omega|}{2}} - e^{-\frac{\beta|\omega|}{2}}} = \dfrac{\omega}{|\omega|}$

- $\mathbb{R}^{N|N}$, $\varphi(\bar\psi, x) = \sum_{\substack{b_1 \ldots b_n \\ \in \{0,1\}^n}} \phi_{b_1 \ldots b_n}(x) (\bar\psi^1)^{b_1} \ldots (\bar\psi^n)^{b_n}$

$Q = \sum_I \bar\psi^I (i p_I + \partial_I h)$

$H = \sum_I \frac{1}{2} p_I^2 + \frac{1}{2} (\partial_I h)^2 + \frac{1}{2} \sum_{I,J} [\bar\psi_I, \psi_J] \partial_I \partial_J h$

- cannot solve in general due to couplings

$h \mapsto \lambda h$, $\lambda \to \infty$ gives

$\varphi_0 = \exp\left[-\frac{1}{2} \sum_I |\omega_I| (x^I)^2 \right] \prod_{I : \omega_I < 0} \bar\psi^I$

- $\# I : \omega_I < 0$ is called Morse index $\lambda(p)$
  → h Morse function

$\Rightarrow \Omega = (-)^{\lambda(p)}$

$\Omega = \sum_{p,\, h'(p) = 0} (-)^{\lambda(p)} = \chi(Q - cpx)$