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§0. Where we are going

- X sm. proj. var $(X = G(k,n))$
  $\circlearrowleft$
  $T = (\mathbb{C}^\times)^d$

$$X \xrightarrow{q} pt$$

- want to compute $\int_X \gamma = q_*([X] \cap \gamma) \in \mathbb{Z}$

$H^*(X)$

H^*-module str. on H_*

$$\begin{tikzcd} X_G \arrow[r, hook, "i"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ BG & pt \arrow[l, hook', "b"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} H^*_G(X) \arrow[r, "i^*"] \arrow[d, dashed, "\int_X"'] & H^*(X) \arrow[d, "\int_X"] \\ H^*_G \arrow[r, "b^*"'] & \mathbb{Z} \end{tikzcd}$$

- lift γ to $\gamma^G \in H^*_G(X)$
- compute $\int_X \gamma^G$ by localisation
- apply b^* to get $\int_X \gamma$.

§1. Equivariant pushforwards

- $X \xrightarrow{f} Y$ G-equiv. map of cpt mfds,
  $n = \dim X$, $m = \dim Y$, $q = n - m$

$$\begin{tikzcd} H^p(X) \arrow[r, dashed, "f_*"] \arrow[d, "pd", "\wr"'] & H^{p-q}(Y) \\ H_{n-p}(X) \arrow[r, dashed, "f_*"] & H_{n-p}(Y) \arrow[u, "pd^{-1}"', "\wr"] \end{tikzcd}$$

- but $X_G \xrightarrow{f_G} Y_G$ ∞-dim'l so no Poincaré duality
  $X_G = EG \times^G X$, $Y_G = EG \times^G Y$

- use directed systems. G: cpt Lie gp, then $\exists$ $\{E_i \to B_i\}$ whose limit is $EG \to BG$, i-connected, ie $\pi_k(E_i)=0$, $1\le k\le i$ and let $X^i_G = E_i \times^G X$, $Y^i_G = E_i \times^G Y$ so $H^p(X^i_G) \xrightarrow{\sim} H^p_G(X)$ $\forall p \le i$?

- then

$$\begin{tikzcd} X^i_G \arrow[dr] & & Y^i_G \arrow[dl] \\ & B_i & \end{tikzcd}$$

fibres X fibres Y

$\dim B_i = \ell$

$\dim X^i_G = \ell + n$

$\dim Y^i_G = \ell + m$

$$\begin{tikzcd} H^p_G(X) \arrow[r, "\sim"] \arrow[rrr, bend left, "f^{G,p}_*"] & H^p(X^i_G) \arrow[r, dashed] \arrow[d, "s", "pd"'] & H^{p-q}(Y^i_G) \arrow[r, "\sim"] & H^{p-q}_G(Y) \\ & H_{\ell+n-p}(X^i_G) \arrow[r, "f^{G,p}_*"'] & H_{\ell+n-p}(Y^i_G) \arrow[u, "s", "pd^{-1}"'] & \end{tikzcd}$$

§2. Self-intersection formula

- X cpt mfd ; $E \to X$ oriented VBdl, rk $E = r$

$$\begin{array}{ccccccc} H^0(X) & \xrightarrow{\sim} & H^r(E, E \setminus X) & \to & H^r(E) & \to & H^r(X) \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} & & & & \\ 1 & & \eta & & & & \end{array}$$

Thom isom.

$H^r(X) \ni e(E) =$ image of η

- let $X \overset{i}{\hookrightarrow} Y$ cl. inclusion of cpts, $N = N_{X/Y} = TY|_X / TX$
- $\exists$ tubular nbhd $X \subset T \overset{op}{\subset} Y$, $H^i(N, N \setminus X) \xrightarrow{\sim} H^i(T, T\setminus X) \xrightarrow{\sim} H^i(Y, Y\setminus X)$

• say N is oriented compatibly wrt z,
and $d = \mathrm{codim}_Y X$

• then

$$\begin{tikzcd}
H^0(X) \arrow[r, "\mathrm{Thom}", "\sim"'] \arrow[rrr, bend left, "z_*"] & H^d(N, N\setminus X) \arrow[r, "\sim"] \arrow[d] & H^d(Y, Y\setminus X) \arrow[r] & H^d(Y) \arrow[d, "z^*"] \\
 & H^d(N) \arrow[rr, "\sim"] & & H^d(X) \ni e(N)
\end{tikzcd}$$

• Self-int. formula: $z^* z_* \mathbb{1} = e(N_{X\setminus Y})$

§ 3. Trivial torus action

• X : sm. T-var.
$\underset{T}{\circlearrowleft}$ trivial

• $E \xrightarrow{\pi} X$ equiv. vb, $E = \bigoplus_\chi E_\chi$ where $\chi \in \mathrm{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^{\dim T}$

$$\begin{tikzcd}
T \times E_\chi \arrow[r, "t\circ e = \chi(t)e"] & E_\chi \\
T \times E \arrow[u, phantom, "\cap" rotate=90] \arrow[r] \arrow[d, "id \times \pi"] & E \arrow[u, phantom, "\cap"] \arrow[d, "\pi"] \\
T \times X \arrow[r, "pr_2"] & X
\end{tikzcd}$$

$$\chi \Downarrow$$

$$\begin{tikzcd}
V_\chi \supset V_{E_\chi} \arrow[d] & & \\
ET \times^T X \cong BT \times X \arrow[d, "p_B"'] \arrow[dr, "p_X"] & \\
BT & X
\end{tikzcd}$$

• exercise: show $V_{E_\chi} \cong V_\chi \boxtimes E_\chi = p_B^* V_\chi \otimes p_X^* E_\chi$

• $c_i^T(E_\chi) = c_i(V_{E_\chi}) \overset{Ex}{=} \sum_{k=0}^{i} \binom{\mathrm{rk}\, E_\chi - k}{i-k} \underbrace{c_k(E_\chi)\cdot \chi^{i-k}}_{H^*(X)\otimes H_T^* \cong H_T^*(X)}$

• $X^T = \{x \in X \mid t\cdot x = x, \forall t \in T\} \supset F$ conn. comp.
$\rightsquigarrow E \to F \subset X^T \subset X$

$\Rightarrow E = \bigoplus_{\chi} \mathcal{O}_\chi \implies H^*_\pi(F) = H^*_\pi \otimes H^*(F)$

- but $H^*(F)$ is an ordinary coh. ring, so $\forall j \geq i$, $\alpha \in H^{2j}(F)$, α is nilpotent in $H^*_\pi(F)$, by virtue of $H^{2k}(F) = 0$

- thus $c_i^\pi(E_\chi) \in H^{2i}_\pi(F)$ invertible iff $\binom{\mathrm{rk}\, E_\chi}{i} \chi^i$ invertible in H^*_π

- $E = N_{F\setminus X} = N = \bigoplus_{i=1}^{s} N_{\chi_i}$

$$\begin{array}{l} E \\ \downarrow \\ F \subset X^\pi \subset X \\ \circlearrowleft \text{trivial} \\ \pi \end{array}$$

- **Goal:** $e^\pi(N)$ invertible after inverting $\chi_1, \dots, \chi_s$

- **Main tool:** (Fogarty) $x \in F \Rightarrow T_x F = (T_x X)^\pi$

$$\Rightarrow N_x^\pi = \frac{(T_x X)^\pi}{(T_x F)^\pi} \twoheadrightarrow \frac{(T_x X)^\pi}{T_x F} = (0)$$

$\Rightarrow \pi \curvearrowright N_x$ nontrivial, so weights $\neq 0$

$$\Rightarrow e^\pi(N) = \prod_{i=1}^{s} \underbrace{e^\pi(N_{\chi_i})}_{\neq 0} \neq 0 \in H^*_\pi(F)$$

$$\Rightarrow e^\pi(N) \in H^*_\pi(F) \longrightarrow H^*_\pi(F)[\chi_1^{-1}, \dots, \chi_s^{-1}]$$

becomes invertible ☒

§4. Localisation formula

- (Atiyah–Bott): M cpt mfd, $\mathbb{T} \circlearrowright M$,
$M^{\mathbb{T}} \xhookrightarrow{i} M$, $M^{\mathbb{T}} = \bigsqcup_\alpha M_\alpha$ conn cpts, $M_\alpha \xhookrightarrow{i_\alpha} M$,
$N_\alpha = N_{M_\alpha \setminus M}$ normal bdl. $\mathcal{H}_{\mathbb{T}}$ = fraction field of $H^*_{\mathbb{T}}$
then Thm (A–B.) $H^*_{\mathbb{T}}(M^{\mathbb{T}}) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}} \xrightarrow{i_*} H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}$
is isom., w/ inverse

$$\varphi \longmapsto \sum_\alpha \frac{i_\alpha^* \varphi}{e^{\mathbb{T}}(N_\alpha)} \in \bigoplus_\alpha H^*_{\mathbb{T}}(M_\alpha) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}$$

- so every $\varphi \in H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}$ writes uniquely as

$$\varphi = \sum_\alpha i_{\alpha *} \frac{i_\alpha^* \varphi}{e^{\mathbb{T}}(N_\alpha)}$$

- so using
$$\begin{tikzcd} M_\alpha \arrow[r, hook, "i_\alpha"] \arrow[dr, "q_\alpha"'] & M \arrow[d, "q"] \\ & pt \end{tikzcd}$$
we get the

integration formula

$$\int_M \varphi = \sum_\alpha \int_{M_\alpha} \frac{i_\alpha^* \varphi}{e^{\mathbb{T}}(N_\alpha)} \in \mathcal{H}_{\mathbb{T}}$$

not $\mathbb{Z}$.

Example. M cpt mfd (sm. pr. var), $\mathbb{T}$-action, $M^{\mathbb{T}} = \{p_1, \ldots, p_s\}$. Then $\chi(M) = s$.

$$\Rightarrow \chi(M) = \int_M e(TM) = \int_M e^{\mathbb{T}}(TM)$$

$$= \sum_{i=1}^{s} \frac{e^{\mathbb{T}}(TM)|_{p_i}}{e^{\mathbb{T}}(T_{p_i}M)} = \sum_{i=1}^{s} 1 = s.$$

Example How many lines $\ell \in \mathbb{P}^2$ pass through 2 general pts?

- goal: $\int_{\mathbb{P}^2} c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2$

$$\underset{\cup}{V} = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)), \quad 0 \to S \to V \otimes_{\mathbb{C}} \mathcal{O} \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0$$

$V_{jk} = \mathbb{C} \cdot x_j \oplus \mathbb{C} x_k$, linear forms vanishing at p_i

$X = \mathbb{P}^2 \circlearrowleft T = (\mathbb{C}^*)^3$, $t \cdot x_i = t^{w_i} x_i$, $0 \le i \le 2$

$$X^T = \left\{ \begin{array}{l} p_1 = (1:0:0) \\ p_2 = (0:1:0) \\ p_3 = (0:0:1) \end{array} , \ w_i \text{ distinct} \right\}$$

$$\int_{\mathbb{P}^2} c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = \int_{\mathbb{P}^2} c_1^T(\mathcal{O}_{\mathbb{P}}(1))^2$$

$$= \sum_{i=1}^{3} \frac{c_1^T(\mathcal{O}_{\mathbb{P}^2}(1))^2|_{p_i}}{e^T(T_{p_i}\mathbb{P}^2)}$$

$$\mathcal{O}_{\mathbb{P}^2}(1)|_{p_i} = V/S|_{p_i} = V/V_{jk} = \underset{\substack{\uparrow \\ \text{weight } -w_i}}{\mathbb{C} \cdot x_i}$$

$$e^T(T_{p_i}\,\cdot) = \prod_{\substack{1 \le j \le 3 \\ j \ne i}} (w_i - w_j)$$

$$\to \text{so}: \ \sum \frac{c_1^T(\cdots)^2|_{p_i}}{e^T(T_p \cdots)} = \frac{(-w_1)^2}{(w_1 - w_2)(w_1 - w_3)} + \cdots$$

$$\left. \begin{array}{l} w_1 = 0 \\ w_2 = 1 \\ w_3 = -1 \end{array} \right\} \Rightarrow 0 + \frac{1}{2} + \frac{1}{(-1)(-2)} = 1$$