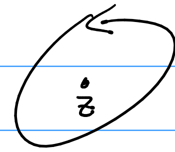
Bertola

- RHP $\begin{cases} \Gamma_+(z) = \Gamma_-(z) M(z) \\ \Gamma(\infty) = 1 \end{cases}$



- has ! sol'n iff $T_{M^{-1}} : \mathcal{H}_+ \to \mathcal{H}_+$, $\mathcal{H}_+ = \{\vec f(z) \mid \text{analytic inside}\}$ inverts

- $T_{M^{-1}}[\vec f] := C_+[\vec f M^{-1}]$

- claim: $T^{-1}_{M^{-1}}[\vec f] = C_+[\vec f\, \Gamma_-^{-1}]\,\Gamma_+ = \oint \frac{\vec f(w)\,\Gamma_-^{-1}(w)}{(w-z)} \frac{dw}{2\pi i}\, \Gamma_+(z)$

Pf. RHS has sol. $\Rightarrow T^{-1}_{M^{-1}} \circ T_{M^{-1}} = T_{M^{-1}} \circ T^{-1}_{M^{-1}} = id_{\mathcal{H}_+}$

T_S inv ($S = M^{-1}$) $\Rightarrow$ def. $G(z) = T_S^{-1}[\mathbb{1}] \in \mathcal{H}_+$
set $\Gamma(z) = \mathbb{1} - \oint \frac{G(w)(M^{-1}(w) - \mathbb{1})}{z - w} \frac{dw}{2\pi i}$
If $z \in D_+$, $\Gamma_+(z) = \mathbb{1} - T_S[G] + G = G$
so $\Gamma_+ - \Gamma_- = \Gamma_+ M^{-1} - \Gamma_+ \Rightarrow \Gamma_+ = \Gamma_- M$.

Check $\det \Gamma \neq 0$. Supp. $\exists c \in \mathbb{C}$ s.t.
$\det \Gamma(c) = 0$. $\Rightarrow \vec h^T \Gamma(c) = 0$.
Def. $\vec\phi(z) = \frac{\vec h^T}{z-c} \Gamma(z)$, $\phi_+(z) \in \mathcal{H}_+$, $\phi_-(z) \in \mathcal{H}_-$
But $T_S[\phi_+] = C_+\left[\frac{\vec h^T}{z-c} \Gamma_+ M^{-1}\right] = C_+[\phi_-] = 0$ ↯

Tau function

- Widom: $T_{M^{-1}} \circ T_M = Id_{\mathcal{H}_+} + tr.cl. \Rightarrow \exists \det T_{M^{-1}} \circ T_M$

(missed some notes)

- $(\Theta) :=$ "Malgrange divisor"
$= \{t \in \mathcal{S} \mid T_{M^{-1}} \text{ not invertible}\}$

Lemma $\mathcal{S} \setminus (\Theta)$ is open

Pf: Pick $t_0 \in \zeta$ s.t. $T_{M^{-1}(0;t_0)}$ inv.

Let t be near t_0

Study stability of $\begin{cases} \Gamma_+(z;t) = \Gamma_-(z;t) M(z;t) \\ M(\infty;t) = \mathbb{1} \end{cases}$

We know $\Gamma(z;t_0)$ exists, let

$R(z;t) := \Gamma(z;t)\Gamma(z;t_0)^{-1}$

$$R_+ = \Gamma_+ \Gamma_{0+}^{-1} = \Gamma_-(\Gamma_{0-}^{-1}\Gamma_{0-}) M M_0^{-1} \Gamma_{0-}^{-1}$$
$$= R_- \Gamma_{0-}(M M_0^{-1})\Gamma_{0-}^{-1}$$

Since M depends analytically on t,

$$M M_0^{-1} = \mathbb{1} + O(\delta t)$$

Let $S = M^{-1}(z, \underline{t})$, $dS = \sum_j \frac{\partial S}{\partial t_j}(z; \underline{t})\, dt_j$

and consider

$$\Omega := T_S^{-1} \cdot dT_S - T_{S^{-1}dS} : \mathcal{H}_+ \to \mathcal{H}_+$$

Lemma Ω is tr. class and

$$\omega_+(z) := \mathrm{tr}_{\mathcal{H}_+}\Omega = \oint \mathrm{Tr}\left(\Gamma_+^{-1}\Gamma_-' M^{-1} dM\right)\frac{dz}{2\pi i}$$
$$= \sum_j \oint \mathrm{Tr}\left(\Gamma_+^{-1}\Gamma_-' M^{-1}\frac{\partial M}{\partial t_j}\right)\frac{dz}{2\pi i}\, dt_j$$

$\frac{d}{dz}$

Rmk. $\omega_- = \sum_j \left(\oint \mathrm{tr}\left(\Gamma_-^{-1}\Gamma_-' \frac{\partial M}{\partial t_j} M^{-1}\right)\frac{dz}{2\pi i}\right) dt_j$

$$\omega_+ - \omega_- = \sum_j \oint \mathrm{tr}\left(\frac{d}{dz}M\, M^{-1}\, \frac{\partial M}{\partial t_j} M^{-1}\right)\frac{dz}{2\pi i}\, dt_j$$

Pf (of Lemma) Only one t, $(\dot{\ }) = \frac{d}{dt}$, $(\ )' = \frac{d}{dz}$.

$\dot{T}_S = T_{\dot{S}}$, $T^{-1}_{S^{-1}} T_{\dot{S}} - T_{S^{-1}\dot{S}}$, $\Gamma_+ = \Gamma_- M$

$\Rightarrow \Gamma_-^{-1} = M\Gamma_+^{-1} = S^{-1}\Gamma_+^{-1}$

- $T_S^{-1} \circ T_{\dot S}[\vec f] = C_+[C_+(\vec f \dot S)\Gamma_-^{-1}]\Gamma_+$
$= C_+[\vec f \dot S S^{-1}\Gamma_+^{-1}]\Gamma_+$
$= \oint \frac{dw}{2\pi i} \frac{\vec f(w)\dot S S^{-1}\Gamma_+^{-1}(w)\Gamma_+(z)}{w-z}$

- $T_{S^{-1}\dot S}[f] = \oint \frac{dw}{2\pi i}\frac{\vec f(w)\dot S S^{-1}(w)}{w-z}$

- there may be a typo here...

- in general, for tr. class $\supset \mathcal{K}$,
s.t. $\mathcal{K}[\vec f](z) = \int dw\, \vec f(w)\, \mathbb{K}(w,z)$

$\xrightarrow{tr} \int dw\, \mathrm{tr}\, \mathbb{K}(z,z)$

- in our case, claim follows.

Thm Near $t_0 \in (\circledast)$ the malgrange form is a "logarithmic form" w/ Poincaré residue equal to $\dim \ker T_S$
("log. form" means $\omega_I = d(\ln f(\underline{t})) +$ analytic, $f(\underline{t})$ loc. anal at t_0)

Pf. (thm) Supp. $t_0 \in (\circledast)$. We know $T_S = 0$
$\Rightarrow \dim \underbrace{\ker T_{S_0}}_{\mathcal{K}} = \dim \underbrace{\mathrm{coker}\, T_{S_0}}_{\mathcal{C}} < +\infty$ $\quad \subset \mathcal{H}_+$

Let $H: \mathcal{K} \to \mathcal{C}$ be fixed, invertible, define
$$Q(t) := T_{S(t)} + H : \mathcal{H}_+ \to \mathcal{H}_+$$
$Q(t)$ inv, inv in nbd of t_0.

- $T_{S(t)} \circ Q^{-1}(t) = \mathrm{Id} - F(t)$, $F(t)$ of fin. rk (a parametrix)
- indeed, $T_{S(t)} = Q(t) - H$
  $\Rightarrow T_{S(t)} \circ Q^{-1}(t) = \mathbb{1} - H \circ Q^{-1}(t)$.
- so F is tr. class.

- let $f(t) := \det T_{S(t)} \circ Q^{-1}(t) = \det [\mathrm{Id}_{\mathcal{H}_+} - F(t)]$
  $= \exp\left[-\sum_j \frac{1}{j} \operatorname{tr} F^j(t) \right]$

- $f(t)$ anal. near t_0, also
  $f(0) = 0$ since $\forall \psi \in \mathcal{E}$,
  $0 = T_{S(0)} \psi = T_{S(0)} Q^{-1}(t_0) \psi = T_{S(0)} H^{-1} \psi$
  i.e. $\mathcal{K} = \ker (\mathrm{Id} - F(t_0))$

- so, let ψ, $F(t_0)\psi = \psi$, & pick basis for $\mathcal{K}$ s.t. $F_{jk}(t) = \delta_{jk} + O(\delta t)$, of $\dim K = r$
  $\Rightarrow f(t) = \det_{r \times r} \big(\overbrace{\mathbb{1} - F_{jk}(t)}^{O(\delta t)} \big)^{r}_{j,k=1}$
  $= C_i \, (t - t_0)^r$

- finally, $\frac{d}{dt} \ln f(t) = \operatorname{tr}\left[(T_S \circ Q^{-1})^{-1} (T_S \mathring{Q}^{-1}) \right]$

(from Jacobi's variational formula
$\frac{d}{dt} \ln \det A(t) = \operatorname{tr} A^{-1} \mathring{A}$)

$= \operatorname{tr} \left[Q T_S^{-1} \left(T_{\mathring{S}} Q^{-1} - T_S Q^{-1} \mathring{Q} Q^{-1} \right) \right]$
$= \operatorname{tr} \left[Q \left(T_S^{-1} T_{\mathring{S}} - Q^{-1} \mathring{Q} \right) Q^{-1} \right]$
$= \operatorname{tr} \left[T_S^{-1} T_{\mathring{S}} - Q^{-1} \mathring{Q} \right]$
$= \operatorname{tr} \left[\underbrace{\left(T_S^{-1} T_{\mathring{S}} - T_{S^{-1}\mathring{S}} \right)}_{\operatorname{tr-cl}} + \left(T_{S^{-1}\mathring{S}} - Q^{-1} \mathring{Q} \right) \right] = \omega_t + \overbrace{\operatorname{tr} \left(T_{S^{-1}\mathring{S}} - Q^{-1} \mathring{Q} \right)}^{\text{analytic at } t = t_0}$ □

Tau functions

- $\omega = \oint \operatorname{tr}\left(\Gamma_-^{-1}\Gamma_-' \, dM\, M^{-1}\right)$, d = tot. der wrt deformations

- but $d\omega \neq 0$

Prop $d\omega = \frac{1}{2}\oint \frac{dz}{2\pi i}\operatorname{tr}\left(dM M^{-1} \wedge \frac{d}{dz}(dM M^{-1})\right)$

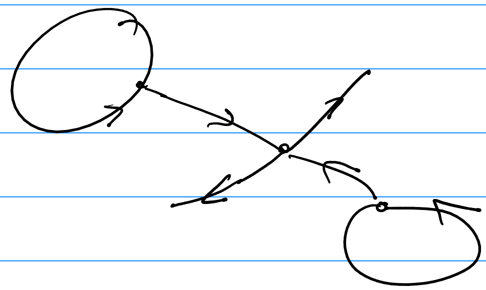
$= \frac{1}{2}\sum_{j,k} dt_j \wedge dt_k \oint \frac{dz}{2\pi i}\operatorname{tr}(\partial_j M M^{-1})(\partial_k M M^{-1})'$

Pf next time.

Observe.
- $d\omega$ 2-form on $\mathcal{G}$, holomorphic everywhere on $\mathcal{G}$.
- On every open $\mathcal{U}_\alpha \subseteq \mathcal{G}$, $\exists\, \vartheta_\alpha$ s.t. $d\vartheta_\alpha = d\omega$, ϑ_α hol.
- define $d\ln\tau_\alpha = \omega - \vartheta_\alpha$, on each $\mathcal{U}_\alpha$
- τ_α vanishes on $(\Theta) \cap \mathcal{U}_\alpha$
- $\tau_\alpha / \tau_\beta = \exp\int(\vartheta_\alpha - \vartheta_\beta)$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $\vartheta_\alpha - \vartheta_\beta =: g_{\alpha\beta}$
- so could be our τ, but has some nonuniqueness.

General RHPs

- let $\Sigma = \bigcup \gamma_j$ be cpt. oriented "multicontour"



- let $J(z): \Sigma \to GL_n(\mathbb{C})$, holom. dep on t
s.t.

Properties 1) $J|_{\gamma_i}$ = restriction of matrix function analytic in z to γ_i

ii) for each vertex $v \in V(\Sigma)$,
defining $J_e(z) := J(z)^{\pm}|_{\gamma_e}$, $e = 1, \dots, n_v$, $+1$ if orient. out, -1 if orient. is in

we impose $J_1(z) \cdots J_{n_v}(z) = \mathbb{1}$ ("no monodromy cond.")

- define Malgrange form

$$\Theta := \int_\Sigma \mathrm{tr}\left(\Gamma_-^{-1} \Gamma_-' \, dJ \, J^{-1}\right) \frac{dz}{2\pi i}$$

$$= \sum_j \oint_{\gamma_j} \mathrm{tr}\left(\Gamma_-^{-1} \Gamma_-' \, dJ_j \, J_j^{-1}\right) \frac{dz}{2\pi i}$$

Thm $$d\Theta = \frac{1}{2} \int_\Sigma \frac{dz}{2\pi i} \mathrm{tr}\left[dJ \, J^{-1} \wedge (dJ \cdot J^{-1})'\right] + \sum_{v \in V(\Sigma)} \eta_v$$

where $$\eta_v = -\frac{1}{4 i \pi} \sum_{e=2}^{n_v} \mathrm{tr}\left[J_{e-1}^{-1} dJ_e \wedge dS_e \cdot S_e\right]$$

where $J_e = J_e(v)$, $S_e = J_e J_{e+1} \cdots J_{n_v}$.