Stoppa.

Cor (to Bonnet-Myers) (M,g) complete, $Ric(x) \geq \delta > 0$, $\forall x$, $\|x\| = 1$ and some δ. Then $\pi_1(M)$ is finite.

Pf. Take $\tilde{M} \xrightarrow{\pi} M$ universal cover. Consider pullback metric $\pi^* g$ on $\tilde{M}$. We know $(\tilde{M}, \pi^* g)$ complete. Then π becomes an isometry.
So we get $Ric_{(\tilde{M}, \pi^* g)}(x) \geq \delta > 0$.
By Bonnet-Myers, $\tilde{M}$ cpt. In particular, π is a finite-to-one map, so $\#(\pi^{-1}(m)) = \#(\pi_1(M, pt))$.

Rmk. $K(\sigma) \geq \delta > 0 \Rightarrow Ric(x) \geq \delta' > 0$ (converse fails).

Rmk. Compact tori $\mathbb{T}^n$ do not admit a metric of strictly positive Ricci curvature, since $\pi_1(\mathbb{T}^n, pt) \simeq \mathbb{Z}^n$.
In 2d we see this directly from Gauss-Bonnet, since $\int_{\mathbb{T}^2} K(p)\, dVol = 0$ means $K(p) \not> 0$ on $\mathbb{T}^2$.

- we have structure thm for spaces of constant curvature $\Rightarrow$ universal cover is isometric to $\mathbb{H}^n$ (<0), $\mathbb{R}^n$ ($=0$), $\mathbb{S}^n$ (>0)

→ what about spaces of constant Ricci curvature?
→ $\forall p \in M$, $\forall x \in T_pM$, $\|x\| = 1$, $Ric_p(x) = \lambda$, $\lambda \in \mathbb{R}$ fixed
- equivalently, $Ric = \lambda g$ Einstein manifolds
- Q: find an example of a cpt (M,g) w $Ric = 0$.
  → solved in 80's by Yau.
  → in general, a hopeless task.

Idea: use cpx structure to understand Ricci better.

- on cpx mfd, pick special basis (eigenvectors of J^{dual}, $dz^k, d\bar{z}^k$) to get $\mathcal{A}^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(M)$

  → using natural projections & inclusions, it's enough to define exterior derivative on each summand → $d: \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{k+1}(M) \otimes \mathbb{C} = \bigoplus_{p'+q'=k+1} \mathcal{A}^{p',q'}(M)$

  → in particular $\partial := \pi^{p,q+1} \circ d$, $\bar{\partial} := \pi^{p+1,q} \circ d$

 → claim: $d = \partial + \bar{\partial}$, which follows from $df = \frac{\partial f}{\partial z^k} dz^k + \frac{\partial f}{\partial \bar{z}^k} d\bar{z}^k = (\partial + \bar{\partial}) f$ on functions, since $\alpha = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{I,J} \, dz_I \wedge d\bar{z}_J \mapsto d\alpha = \sum d\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J$

  → we used local coordinates, but this is in fact the type decomposition of J^{dual}, into $\pm\sqrt{-}$ eigenspaces

  → it can be shown that

$$d = \partial + \bar{\partial} \iff J \text{ integrable} \iff N(J) = 0$$

- for the metric, we extend it to $TM \otimes \mathbb{C}$ by $\mathbb{C}$-linearity
- the Hermitian condition: $g(JX, JY) = g(X, Y)$

Lemma $g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = g\left(\frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{z}_l}\right) = 0$

Pf. $g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = g\left(J\left(\frac{\partial}{\partial z_i}\right), J\left(\frac{\partial}{\partial z_j}\right)\right) = -g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$.

Cor. g can be written as $g = \sum g_{j\bar{k}} \, dz_j \odot d\bar{z}_k$

Cor. $\omega(X, Y) := g(JX, Y)$ as $\omega = \sqrt{-1} \sum g_{j\bar{k}} \, dz_j \wedge d\bar{z}_k$