Tikhomirov

- M mfd, $\Omega^k(M) = \Gamma(M, \Lambda^k T^* M)$ space of diff k-forms on M
- E vbdl on M, $\Omega^k(M, E) := \Gamma(M, \Lambda^k T^* M \otimes E)$ sp. of E-valued k forms
- $\Gamma(TM) \otimes \Omega^1(E) \xrightarrow{ev} \Gamma(E)$

$$\underset{X}{\uplus} \otimes \underset{s}{\uplus} \longmapsto s(X)$$

- note that any $f : \Gamma(TM) \to \Gamma(E)$ satisfying $f(hX) = h f(X)$ for any $h \in C^\infty(M)$ can be realised using an evaluation

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- G Lie gp, $\mathfrak{g}$ Lie alg. of G, $\mathfrak{g} = TG_{id}$
- def $L_g, R_g : G \circlearrowleft$ by $h \mapsto gh$, $h \mapsto hg$ $\forall g \in G$ giving us $(L_g)_*, (R_g)_* : TG \circlearrowleft$, $\underset{TG_h}{\tau_h} \mapsto \underset{TG_{gh}}{g_* \tau_h}$, ...
- define $Ad(g) : \underset{TG_{id}}{\tau_{id}} \mapsto \underset{TG_{id}}{g_* \tau_{id} g_*^{-1}}$

$\Rightarrow Ad(g) : \mathfrak{g} \circlearrowleft$, $Ad : G \to Aut\, G$

- $\exists$ distinguished $\omega \in \Omega^1(G, \underline{\mathfrak{g}})$, Maurer-Cartan where by underline we mean $\underline{V} = M \times V$ if V vsp
- $\omega(\underset{TG_h}{\tau_h}) := \underset{TG_{\underline{id}}}{h_*^{-1} \tau_h}$

$$(L_g^* \omega)(\tau_h) = \omega(L_{g*} \tau_h) = \omega(\underset{TG_{gh}}{g_* \tau_h})$$
$$= (gh)_*^{-1} g_* \tau_h = h_*^{-1} \tau_h = \omega(\tau_h).$$

so ω left invariant

$$(R_g^* \omega)(\tau_h) = \omega(\tau_h g_*) = (hg)_*^{-1} \tau_h g_* = g_*^{-1} h_*^{-1} \tau_h g_*$$
$$= (Ad(g^{-1})\omega)(\tau_h)$$

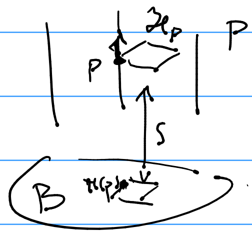
so $R_g^* \omega = Ad(g^{-1}) \omega$.

Connections on smooth pbdls

$P \xrightarrow{\pi} B$ $\quad G$-pbdl, $\dim B = n$

Def. A **connection** on P is a smooth family
$$\{ \mathcal{H}_p \subset TP_p \mid p \in P,\ (R_g)_* \mathcal{H}_p = \mathcal{H}_{pg} \}$$
s.t. $\pi_* : \mathcal{H}_p \xrightarrow{\sim} TB_{\pi(p)}$.

– since π submersion, $\pi_* : TP_p \to TB_{\pi(p)}$ surjects:
$$0 \to TP^v \to TP \to \pi^* TB \to 0 \qquad (1)$$
where $TP \supset TP^v := \{ v \in TP_p \mid p \in T,\ v \text{ is tang. to } \pi^{-1}(\pi(p)) \}$



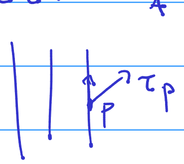
– $\underline{\mathcal{H}} = \{ \mathcal{H}_p \}$ yields the splitting of (1)

Prop The existence of a connection A (on a smooth G-pbdl $P \xrightarrow{\pi} B$) is equivalent to the existence of a 1-form $\omega_A \in \Omega^1(P, \mathfrak{g})$ satisfying

1) $\omega_A(\tau_h g^*) = Ad(g^{-1}) \circ \omega_A(\tau_h)$

11) $\forall\, b \in B \quad (\varphi_{\alpha, b}^{-1})^* \omega_A = \omega$, where $\varphi_{\alpha, b} := \varphi_\alpha|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \{b\} \times G$ (G is)

Pf. Let ω_A satisfy 1) $\wedge$ 11), $\omega_A : \tau \mapsto a \in \mathfrak{g}$.



For τ_p vertical, $\omega_A|_{TP^v_p} : TP^v_p \xrightarrow{\sim} \mathfrak{g}$,
so $TP_p = TP^v_p \oplus \mathcal{H}_p$, where
$\mathcal{H}_p := \ker(\omega_A|_{TP_p})$.

Equivariance follows from $\mathcal{H}_{pg} = \ker(\omega_A|_{TP_{pg}}) \overset{(1)}{=} \mathcal{H}_p g_*$.

Conversely, given $A = \{\mathcal{H}_p\}_{p \in P}$, define $\omega_A \in \Omega^1(P, \mathfrak{g})$ by

$$\omega_A : TP_p \xrightarrow[\mathrm{pr}_{\mathcal{H}_p}]{} TP_p^v \xrightarrow[(\varphi_\alpha, \pi(p))_*]{\ \ ^{-1}\ \ } TG \xrightarrow[\omega]{\sim} \mathfrak{g}$$

$\hookrightarrow$ proj. w/ kernel $\mathcal{H}_p$

- existence?
- note that it exists for trivial
  $P = B \times G \xrightarrow[\pi = \mathrm{pr}_1]{} B$, $A = \{\mathcal{H}_p = T_{\mathrm{pr}_2^{-1}(\mathrm{pr}_2(p))}\}$
  - so take $\varphi_\alpha : P|_{\mathcal{U}_\alpha} \xrightarrow{\sim} \mathcal{U}_\alpha \times G$ local trivs and $\{\lambda_\alpha\}$ part. of unity subordinate to a cover $\{\mathcal{U}_\alpha\}$.
  - then $\omega = \sum_\alpha \lambda_\alpha \omega_\alpha$, where ω_α prod. conns

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- notice that $\vartheta = \omega_A - \omega_{A'}$ restricts to 0 along fibers & satisfies $\vartheta(\tau g_*) = Ad(g^{-1})\vartheta(\tau)$
so look at $\Omega^1(P, \mathfrak{g})$
$$\cup$$
$$\Omega^1_{Ad}(P, \mathfrak{g}) = \{\vartheta \in \Omega^1(P, \mathfrak{g}) \mid \vartheta \text{ satisfies above}\}$$

Rmk The set of connections on P is an affine space over $\Omega^1_{Ad}(P, \mathfrak{g})$

- now let $s = \mathrm{Ad} : G \to \mathrm{Aut}\,\mathfrak{g}$, look at

$$P \times_s G = P \times_{\mathrm{Ad}} G =: \mathrm{ad}\, P$$

$$[(p,v)] = (p,v) \bmod \sim \;\big|\; (p,v) \sim (pg, \mathrm{Ad}(g^{-1})v)$$

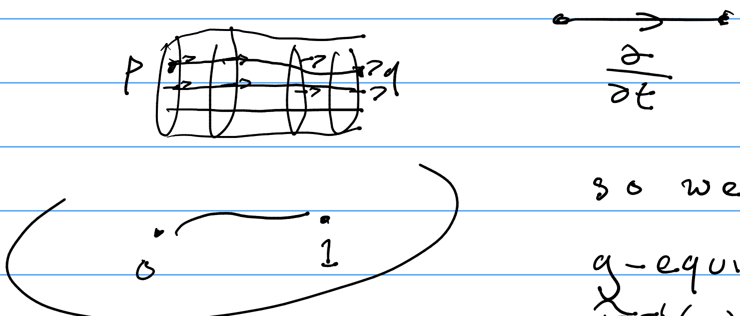
adjoint vbdl

- claim: $\Omega^1_{\mathrm{Ad}}(P, \mathfrak{g}) \xrightarrow{\sim} \Omega^1(B, \mathrm{ad}\, P)$

- note that connections survive pullbacks, just pull the 1-form back

$$\begin{tikzcd} P \arrow[d] & f^*P \arrow[l] \arrow[d] \\ B & A \arrow[l, "f"] \end{tikzcd}$$

- take $A = [0,1] \xhookrightarrow{f} B$



$$\frac{\partial}{\partial t} \longrightarrow \widetilde{\frac{\partial}{\partial t}}$$

so we get, integrating along $\widetilde{\frac{\partial}{\partial t}}$ g-equiv maps $p \mapsto q$, $\tilde{\pi}^{-1}(0) \longrightarrow \tilde{\pi}^{-1}(1)$