Ulnxomnpolo

- $A \in \mathcal{A}_2$, $0 \to \Omega^0(\mathrm{ad}\,P) \xrightarrow{\nabla_A} \Omega^1(\mathrm{ad}\,P) \xrightarrow{P_+\circ\nabla_A} \Omega^2(\mathrm{ad}\,P) \to 0$
- on $\begin{matrix}P\\ \downarrow\\ B\end{matrix}$, $\dim B = 4$, (B,g) orientable, $*: \Omega^i(B) \xrightarrow{\simeq} \Omega^{4-i}(B)$
- for $F_A^+ = 0$ i.e. A ASD, this is a cpx

- from

$$\begin{tikzcd} 0 \arrow[r] & \Omega^0(B) \arrow[r, "d"] & \Omega^1(B) \arrow[r, "d"] \arrow[dr] & \Omega^2(B) \arrow[d, "P_+"] & \\ & & & \Omega^2_+(B) \arrow[r] & 0 \end{tikzcd}$$

we get for $\begin{matrix} T^*B \ni (x,v) \\ p\downarrow \\ B \end{matrix}$

$$\begin{tikzcd} 0 \arrow[r] & \overset{\mathbb{R}}{\overset{\cong}{\Lambda^0 T^*B}} \arrow[r, "\mathrm{Symbol}"] & \Lambda^1 T^*B \arrow[r, "\mathrm{Symbol}^+"] & \Lambda^2_+ T^*B \arrow[r] & 0 \end{tikzcd}$$

$$\begin{matrix} \underset{1}{\in} & \longmapsto & \underset{v}{\in} \end{matrix}$$

$$w \longmapsto \underset{P_+(v\wedge w)}{\overset{\shortparallel}{(v\wedge w)_+}}$$

- $\Lambda^2_\pm = \mathrm{span}(a_1^\pm, a_2^\pm, a_3^\pm)$

  where $a_1^\pm = e_1\wedge e_2 \pm e_3\wedge e_4$

  $a_2^\pm = e_1\wedge e_3 \pm e_4\wedge e_2$

  $a_3^\pm = e_1\wedge e_4 \pm e_2\wedge e_3$

  for some g-orth. basis $e_1, - e_4$
- note that in sequence $\dim \Lambda^1 T^*B = \dim \Lambda^0 T^*B + \dim \Lambda^2_+ T^*B$

  so only check exactness in middle

- $v = e_1$, $w = \sum_i \lambda_i e_i \Rightarrow (v\wedge w)_+ = \lambda_2(e_1\wedge e_2)_+ + \lambda_3(e_1\wedge e_3)_+ + \lambda_4(e_1\wedge e_4)_+$

  $= \frac{1}{2}(\lambda_2 a_1^+ + \lambda_3 a_2^+ + \lambda_4 a_3^+) = 0$

  $\Rightarrow \lambda_2 = \lambda_3 = \lambda_4 = 0$ since a_1^+, a_2^+, a_3^+ lin. indep

  $\Rightarrow v = \lambda \cdot w$, $\lambda \in \mathbb{R}$, so it's exact.

- note that locally $\nabla_A = d + 0$-order diff op,

  so $\mathcal{E}^\bullet(A)$: $0 \to \Omega^0(\mathrm{ad}\,P) \xrightarrow{\nabla_A} \Omega^1(\mathrm{ad}\,P) \xrightarrow{P_+\circ\nabla_A} \Omega^2_+(\mathrm{ad}\,P) \to 0$

is an elliptic cpx. (since symbols are exact)

-we will want to calculate $\chi(\mathcal{E}^\bullet(A)) = h^0 - h^1 + h^2$

-recall that $\ast : \Omega^2(B) \circlearrowleft s$ induces $\delta : \Omega^i \longrightarrow \Omega^{i-1}$
as $(d\alpha, \beta) = (\alpha, \delta\beta)$, $\delta = (-)^i \ast^{-1} \circ d \circ \ast$
-form Laplace-Beltrami $\Delta = d\circ\delta + \delta\circ d$
-define $\mathbb{H}^p := \{\alpha \in \Omega^p \mid \Delta\alpha = 0\}$

-$\forall \alpha \in \Omega^p$ $\exists!$ decomp. $\alpha = \underset{\substack{\uparrow \\ \mathbb{H}^p}}{h(\alpha)} \overset{\perp}{\oplus} \underset{\substack{\uparrow \\ d\Omega^{p-1}}}{d\beta} \overset{\perp}{\oplus} \underset{\substack{\rotatebox{90}{$\in$} \\ \delta\Omega^{p+1}}}{\delta\gamma}$

- Hodge theory:
$$\begin{array}{ccc} H^p_{dR}(B,\mathbb{R}) & \xrightarrow{\sim} & \mathbb{H}^p \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\alpha] & \longmapsto & h(\alpha) \end{array}$$

-further, $\mathbb{H}^p_\pm := \{\alpha \in \mathbb{H}^p \mid \ast\alpha = \pm\alpha\}$
so $\mathbb{H}^p = \mathbb{H}^p_+ \overset{\perp}{\oplus} \mathbb{H}^p_-$

-on H^2 $\exists$ pairing $H^2 \otimes H^2 \longrightarrow H^4(B,\mathbb{R}) \cong \mathbb{R}$
$\langle [\alpha], [\beta] \rangle \longmapsto \alpha \cup \beta := \int_B \alpha \wedge \beta$
symmetric, nondegenerate

"trivial", product connection

-now, for $\mathcal{E}^\bullet(A)$: $0 \to \Omega^0(B) \otimes \underline{g} \to \Omega^1(B) \otimes \underline{g} \to \Omega^2_+(B) \otimes \underline{g} \to 0$

$\chi(\mathcal{E}^\bullet(A)) = \underset{\substack{\nearrow \\ \dim g,\ g = SU(2)}}{3}(b_0(B) - b_1(B) + b_2(B))$

$-D_A = (\nabla_A^\ast, P_+ \circ \nabla_A) : \Omega^1(\mathrm{ad}\,P)_{L^2_2} \longrightarrow \Omega^0(\mathrm{ad}\,P)_{L^2_1} \oplus \Omega^2_+(\mathrm{ad}\,P)_{L^2_1}$

$\to$ $\mathcal{E}^\bullet(A)$ elliptic $\Longleftrightarrow$ D_A is elliptic

- here it is Fredholm
- further, it turns out,
  - $\chi(\mathcal{E}^\bullet(A)) = \operatorname{ind} D_A = \dim \ker D_A - \dim \operatorname{coker} D_A$

    $= \text{topological index} = f(c_2(P), b_i(B), \dim g)$

    $\overset{\text{in our}}{\underset{\text{case}}{=}} 8c_2(P) - \underbrace{3(b_0 - b_1 + b_2)}_{\chi(\text{trivial } \mathcal{E}^\bullet(A))}$

- $\left\{ \begin{matrix} P \\ \downarrow SU(2) \\ B \end{matrix} \right\}$ are classified by $c_2(P) = \frac{1}{8\pi^2}\int \operatorname{tr}(F_A \wedge F_A) \in H^4(B, \mathbb{Z}) \cong \mathbb{Z}$ for A any conn.

- excision principle:

i) let $\underset{\cup}{X^+}$, $\underset{\cup}{X^-}$ smooth cpt var

$\mathcal{U} = \mathcal{U}^+ \simeq \mathcal{U}^-$

ii) given 2 pairs of ell. ops $P^\pm$, $Q^\pm$ on $X^\pm$,

$P^\pm : \Gamma(E_1^\pm) \to \Gamma(F_1^\pm)$, $Q^\pm : \Gamma(E_2^\pm) \to \Gamma(F_2^\mp)$

s.t. $P^\pm = Q^\pm$ outside $\mathcal{U}^\mp$,

meaning (in P^+ coinc. w Q^+ on V^+ where

$X^\pm = \mathcal{U}^\pm_{\cong \mathcal{U}} \cup V^\pm$, $V^\pm \overset{\text{open}}{\hookrightarrow} X^\pm$)

$\exists$ isos α, β $E_1^+|_{V^+} \xrightarrow{\alpha} E_2^+|_{V^+}$, $F_1^+|_{V^+} \xrightarrow{\beta} F_2^+|_{V^+}$

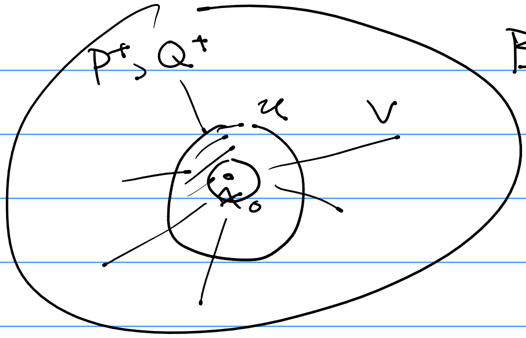
s.t.

$$\begin{tikzcd}
\Gamma(E_1^+|_{V^+}) \arrow[r, "P^+"] \arrow[d, "\alpha"'] & \Gamma(F_1^+|_{V^+}) \arrow[d, "\beta"] \\
\Gamma(E_2^+|_{V^+}) \arrow[r, "Q^+"] & \Gamma(F_2^+|_{V^+})
\end{tikzcd}$$

iii) $P^+|_{\mathcal{U}}$ coincides with $P^-|_{\mathcal{U}}$

$Q^+|_{\mathcal{U}}$ — " — $Q^-|_{\mathcal{U}}$

$\overset{\text{i, ii, iii}}{\Longrightarrow}$ $\operatorname{ind} P^+ - \operatorname{ind} Q^+ = \operatorname{ind} P^- - \operatorname{ind} Q^-$



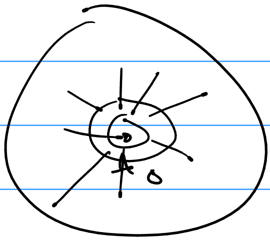
$B = X^+$

$V \simeq B \setminus \{\overset{A_0}{\overset{\shortparallel}{pt}}\}$

$P^+ = D_{A_1}$ on X

$Q^+ = D_{A_2}$ on X

$c_2(A_1) = c$, $c_2(A_2) = c+1$



$S^4 = X^-$

$\mathcal{U} = D^4$ open disk

$P^- = D_{A_1'}$ on S^4

$Q^- = D_{A_2'}$

$c_2(A_1') = c$, $c_2(A_2') = c+1$

- for Y open mfd (= not geodesically closed), $Y = X \setminus \{pt\}$ all $\begin{matrix} P \\ \downarrow SU(2) \\ Y \end{matrix}$ pbdls $\underline{\text{are trivial}}$

- follows from 3-connectedness of $BSU(2)$ i.e. $\pi_i(BSU(2)) = 0$ for $i \leq 3$ which follows from 2-conn. of $SU(2)$ and fibration

$$\begin{tikzcd} SU(2) \arrow[r, hook] & \overset{\text{contractible}}{ESU(2)} \arrow[d] \\ & BSU(2) \end{tikzcd}$$

long exact squc.

→ so excision postulates are satisfied and we only look at S^4 ?