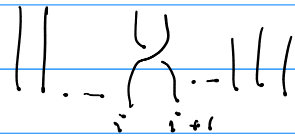
Srni

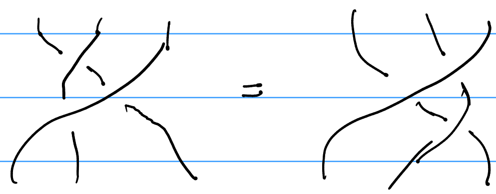
Ševera - From braids to quantization.

Braid groups. B_n = braids w/ n strands
$= \pi_1((\mathbb{C}^n/\Delta)/S_n)$

- generators: $s_i =$ 

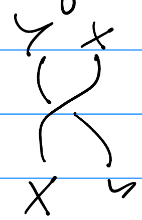
- relations $s_i s_j = s_j s_i$, $|i-j| \geq 2$

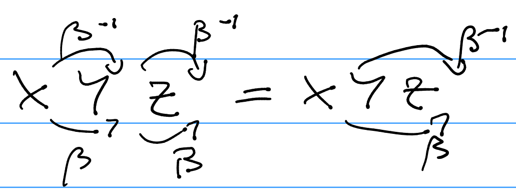
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$



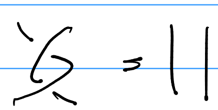
Monoidal cats

- $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, $1_{\mathcal{C}} \in \mathcal{C}$
- associativity nat. isos satisfying pentagram
- Braided MC-s: $\exists$ natural iso

$\beta_{X,Y}: X \otimes Y \xrightarrow{\sim} Y \otimes X$, $\beta_{X,Y} =$ 

s.t.  ← hexagon relns.

- Sym. monoidal cat:

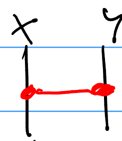
BMC s.t. 

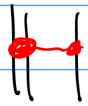
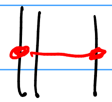
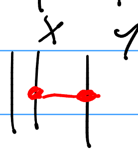
→ so we stop drawing over/under crossings

Monoidal functors.

- $F: \mathcal{C} \to \mathcal{D}$ monoidal if $F(X) \otimes F(Y) \xrightarrow[(*)]{\sim} F(X \otimes Y)$ s.t. (associativity diagram) holds and $F(1_{\mathcal{C}}) \xrightarrow[(*)]{\sim} 1_{\mathcal{D}}$
- F is lax monoidal if $(*)$ not necessarily isos
- similarly braided m.f.-s "commute" w. β

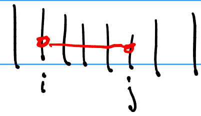
Infinitesimal braids or chord diagrams

- an inf. braided cat = linear SMC $\mathcal{C}$ w. nat. transf. $t_{X,Y}: X \otimes Y \to X \otimes Y$ s.t.
  $\beta_{X,Y} := \sigma_{X,Y} \circ (1 + \varepsilon t_{X,Y})$, $\varepsilon^2 = 0$
  is a braiding in $\mathcal{C}$, and $t_{X,Y} = t_{Y,X}$
- drawing: $t_{X,Y} =$ 

  s.t.  $=$  $+$ , Leibniz rule

- e.g. $\mathfrak{g}$ Lie alg, $t \in (S^2\mathfrak{g})^{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}$, $\mathcal{C} = U\mathfrak{g}$-mod
  $t_{X,Y} = S_X \otimes S_Y(t) \in \mathrm{End}(X \otimes Y)$.

- algebra of inf. pure braids $\mathcal{A}_n = \langle t_{ij} \rangle$,
  $1 \le i,j \le n$, $i \ne j$, $t_{ij} = t_{ji}$ with relations
  $[t_{ij}, t_{kl}] = 0$ if $ijkl$ distinct, $[t_{ij}, t_{ik} + t_{jk}] = 0$

- $t_{ij} =$ 

- $\mathcal{A}_n$ is a cocommutative Hopf algebra (t_{ij} primitive)

Drinfeld associators.

- problem: extend 1^{st} order deformation (iBMC) to true deformation

Thm (Drinfeld) $\exists\ \Phi \in \mathbb{Q}\langle\langle x, y\rangle\rangle$ such that
$$\beta^{new}_{X,Y} := \tau_{X,Y} \circ \exp\left(\frac{\hbar}{2} t_{X,Y}\right) \text{ and}$$
$$\gamma^{new}_{XYZ} := \gamma^{old}_{XYZ} \circ \Phi(\hbar t_{X,Y}, \hbar t_{Y,Z})$$

make any SMC into a BMC.

- recall, $\gamma_{XYZ} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$
- Φ is called a Drinfeld associator if satisfies $\Delta \Phi = \Phi \times \Phi$ for x, y primitive

Where do they come from?

- KZ-connection $A_n \in \Omega^1(\mathbb{C}^n - \Delta s) \otimes \mathfrak{t}_n$,
$$A_n = \sum_{k<l} t_{kl} \frac{d(z_k - z_l)}{z_k - z_l} \text{ is flat}$$

- $\mathrm{hol}_{\circlearrowleft} A_2 = \exp(2\pi i t_{12})$

- $\Phi_{KZ}(t_{12}, t_{23}) := \lim_{\varepsilon \to 0^+} \varepsilon^{-t_{23}}\, \mathrm{hol}_{\circlearrowleft} A_3 \cdot \varepsilon^{t_{12}}$

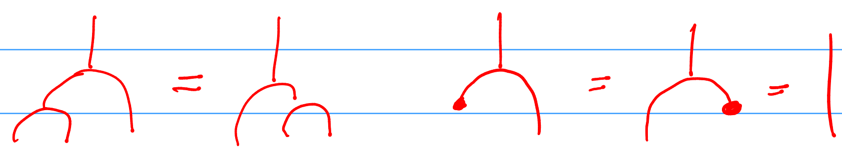
$\to$ then $\Phi(x, y) := \Phi_{KZ}\left(\frac{x}{2\pi i}, \frac{y}{2\pi i}\right) \in \mathbb{C}\langle\langle x, y\rangle\rangle$ is an associator (but over $\mathbb{C}$).

Second lecture:

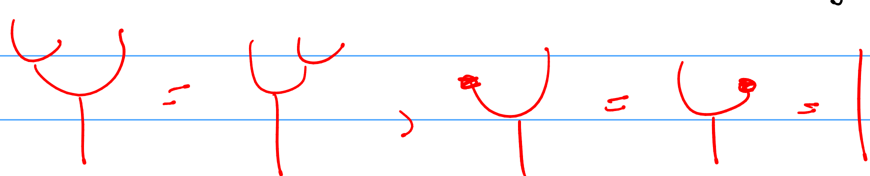
Quantization of Poisson–Hopf algebras

Hopf algebras

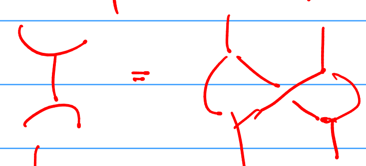
- a Hopf algebra H in a BMC $\mathcal{C}$ is
  1) a monoid: $m: H \otimes H \to H$, $\eta: 1_{\mathcal{C}} \to H$



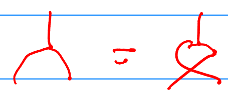
  11) a comonoid: $\Delta: H \to H \otimes H$, $\varepsilon: H \to 1_{\mathcal{C}}$



compatibility

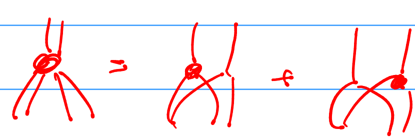


antipode



- a Poisson-Hopf algebra H in a linear SMC $\mathcal{C}$ is a comm. Hopf algebra with a Poisson bracket $\{-,-\}$ s.t. $\Delta: H \to H \otimes H$ is a Poisson algebra morphism

- Poisson brckt:



Quantization

- problem statement: given $(H, m_0, \Delta_0, \{,\}, \eta, \varepsilon)$ P.H. algebra, construct $m_\hbar = m_0 + \hbar m_1 + \dots$, $\Delta_\hbar = \dots$, $S_\hbar = S_0 + \dots$ s.t. $m_1 - m_1^{op} = \{-,-\}$ s.t. $(H, m_\hbar, \Delta_\hbar, S_\hbar, \eta, \varepsilon)$

is a Hopf algebra

- Etingof–Kazhdan '95 – solution for $H = (\mathcal{U}g)^\vee$
- method here: joint work w Ján Pulmann

Nerve of a group G

... is the functor $F : \mathrm{FinSet}^{op} \to \mathrm{Set}$

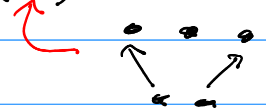
$$F(X) = \{ g : X \times X \to G \mid g(a,b)\,g(b,c) = g(a,c),\ g(a,a) = 1 \}$$

$$F(X) \cong G^{|X|-1} : \quad \underset{1}{\bullet} \xrightarrow{g(1,2)} \underset{2}{\bullet} \xrightarrow{g(2,3)} \underset{3}{\bullet} \xrightarrow{g(3,4)} \underset{4}{\bullet}$$

- for a general F: $F(X) \to F(\bullet\ \bullet)^{|X|-1}$ $(*)$

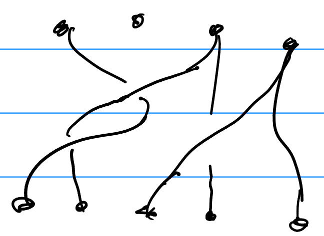
Prop. F nerve of group iff $(*)$ bijection $\forall X$.
In that case $G = F(\bullet\ \bullet)$, with product

$$F(\bullet\ \bullet) \times F(\bullet\ \bullet) \cong F(\bullet\ \bullet\ \bullet) \xrightarrow{F(\ \ )} F(\bullet\ \bullet)$$

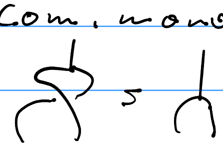


Category BrSet

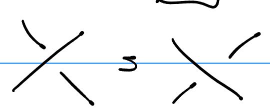
- motivation: nerves of Hopf algs.
  → BrSet replaces FinSet
- BrSet: BMC with morphisms



the BMC freely generated by a com. monoid

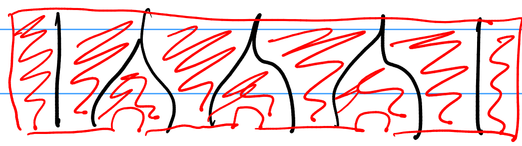


$\boxed{A} \otimes \boxed{B} = \boxed{A \mid B}$

- imposing  gives FinSet^{op}

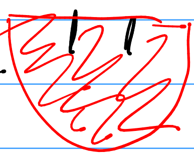
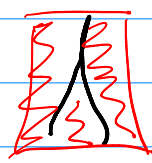
The nerve of a Hopf algebra.

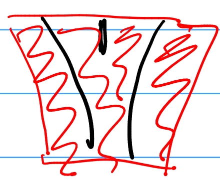
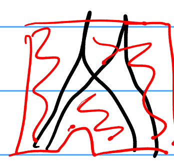
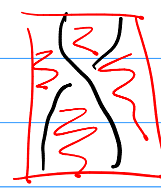
Thm. Hopf algs in a BMC $\mathcal{C}$ are equivalent to braided lax monoidal functors $F : \mathrm{BrSet} \to \mathcal{C}$ s.t.

 $: F(\bullet\,\bullet)^n \xrightarrow[\varphi_n]{} F(\bullet^{n+1})$ $(n\geq 1)$

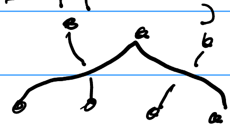
is an iso $(\forall n)$ and
$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r] \arrow[rr, bend right, "\psi"'] & F(\emptyset) \arrow[r] & F(\bullet) \end{tikzcd}$$
are isos.

- getting H from F: $H = F(\bullet\,\bullet)$

$\eta =$  , $\varepsilon = \psi^{-1} \circ$ 

$\Delta = \varphi_2^{-1} \circ$  $m =$  $S =$ 

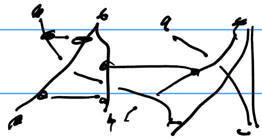
Constructing it.

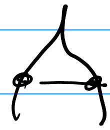
- objects $F(\bullet^n) = H^{n-1}$, $F(\emptyset) = 1_{\mathcal{C}}$
- morphisms $F($  $) : H^3 \to H^2$

- F should be monoidal, $F(\bullet^n)\,F(\bullet^m) \to F(\bullet^{n+m})$
so we insert 1 to get $H^{n-1} \otimes H^{m-1} \to H^{m+n-1}$

Nerve of Poisson-Hopf algebra.

- CSet: $\hat{\imath}$-braided version of Fin/BrSet



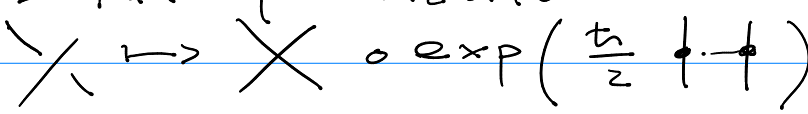
 $= 0$

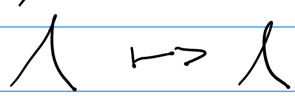
Quantization of Poisson-Hopf algebras

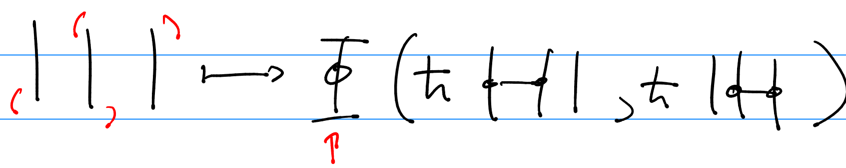
$$
\begin{tikzcd}
(\mathrm{Pa})\,\mathrm{BrSet} \arrow[r, "Q"] \arrow[rr, bend right, "\mathrm{Hopf}"'] & \mathrm{CSet} \arrow[r, "\text{Poisson-Hopf}"] & \mathcal{C}
\end{tikzcd}
$$

- Q is the quantization

Q :







Drinfeld associator

Rmks. Φ grouplike $\Rightarrow$ Q cptible w tensor products of Hopf algebras

nerves can be defined on groupoids
- quantization of "semi-comm" Hopf algs