Dabrowski

$\|\cdot\|_T^2 = \|\cdot\|^2 + \|T\cdot\|^2$

- $D \subseteq \overline{D} \subseteq D^*$, $\mathrm{Dom}\, D = \Gamma_c(\Sigma)$

Lemma $\Gamma_c(\Sigma)$ dense in $\|\cdot\|_{D^*}$-norm in $\mathrm{Dom}\, D^*$
$\Rightarrow \overline{D} = D^*$. ($D$ is ess. s.a.)

Lemma The antecedent is true.

Pf. Using $\{f_\alpha\}_\alpha$ p.o.1 sub. to $\{U_\alpha\}_\alpha$, $\Sigma|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^{2^{n/2}}$.
Pick $h \in C^\infty(\mathbb{R}_+)$, $h : \mathbb{R}_+ \to \mathbb{R}_+$, $\int_{\mathbb{R}^n} h(x)\,dx = 1$,
$\varepsilon > 0$, let $h_\varepsilon(x \in \mathbb{R}^n) := \frac{1}{\varepsilon^n} h\left(\frac{\|x\|}{\varepsilon}\right)$.
Then $h_\varepsilon \to$ "δ_x" as $\varepsilon \to 0$.
$\mathrm{Dom}\, D^* \ni \psi = \sum \psi_\alpha$, $\psi_\alpha := \psi f_\alpha \in \mathrm{Dom}\, D^*$.
which can be seen by looking at
$\langle \varphi, D(f\varphi)\rangle = \langle \varphi, f\cdot D\varphi + \underbrace{df}_{\text{bounded}}\cdot\varphi\rangle$.

$C^\infty(\mathbb{R}^n)$

Now $(\psi_\alpha * h_\varepsilon)(x) = \int_{\mathbb{R}} dy\, \psi_\alpha(y) h_\varepsilon(x-y) \to \psi_\alpha(x)$

$D(\psi_\alpha * h_\varepsilon)(x) = \int dy\, \psi(y)\, D h_\varepsilon(x-y)$
$= (D^*\psi_\alpha) * h_\varepsilon(x) \longrightarrow D^*\psi_\alpha$
$\Rightarrow \psi_\alpha * h_\varepsilon \xrightarrow{\|\cdot\|_{D^*}} \psi_\alpha$, so
$\psi_\varepsilon = \sum(\psi_\alpha * h_\varepsilon) \to \sum \psi_\alpha = \psi \in \mathrm{Dom}\, D^*$. □

- from now on M cpt, $\partial M = \emptyset$, $D = \overline{D} = D^* \Rightarrow \mathrm{Spec}\, D \subseteq \mathbb{R}$

- let $\|\psi\|_{H^1}^2 = \|\psi\|^2 + \underbrace{\|\nabla\psi\|^2}_{\text{norm on } \Gamma(\Lambda^1 M)\otimes\Gamma(\Sigma)}$, 1^{st} Sobolev norm

Thm (Gonday) The norms $\|\cdot\|_D$ and $\|\cdot\|_{H^1}$ are equivalent.

- follows: $D : \mathrm{Dom}\, D \to L^2$ is bdd ($\Rightarrow$ cont.)
on $H_1 \xrightarrow{\text{completion}} L^2$, $\overline{\Gamma_c(\Sigma)}^{\|\cdot\|_{H^1}}$

- check: $\|D\psi\|^2 = \sum_{j,k} \langle e_j \cdot \nabla_{e_j}\psi, e_k \cdot \nabla_{e_k}\psi \rangle$

$$\overset{\text{Schwartz}}{\leq} \sum \|e_j \cdot \nabla_{e_j}\psi\| \cdot \|e_k \cdot \nabla_{e_k}\psi\|$$

$$\leq \sum \|\nabla_{e_j}\psi\| \cdot \|\nabla_{e_k}\psi\|$$

$$\overset{0 \leq a^2 - b^2 - 2ab}{\leq} \frac{1}{2} \sum_{j,k} \left(\|\nabla_{e_j}\psi\|^2 + \|\nabla_{e_k}\psi\|^2 \right)$$

$$= \frac{1}{2} \cdot 2 \sum_{j,k} \|\nabla_{e_j}\psi\|^2$$

$$= n \cdot \sum_j \underbrace{\|\nabla_{e_j}\psi\|^2}_{= \|e^j \otimes \nabla_{e_j}\psi\|^2}$$

$$= n \|\nabla\psi\|^2 \leq n \|\psi\|^2_{H^1}$$

by definition

- J, χ are bdd, extend to $H = L^2(\Sigma)$,
  $J f^* J^{-1} = f$, $J^2 = \varepsilon$, $\chi^2 = id$, $\chi^* = \chi$, $\chi D = -D\chi$

- Spinor laplacian $\Delta := \nabla^* \circ \nabla$
  - we can show $\sigma_\Delta^{\text{princ.}} = \sigma_{D^2}^{\text{princ}} = (\xi \cdot)^2 = -\|\xi\|^2$
  so $D^2 = \Delta +$ lower order diff. op.

Thm (Schrödinger-Lichnerowicz-Weitzenberg)

$$D^2 = \Delta + \frac{1}{4} R + \underbrace{\frac{1}{2} dA \cdot}_{}$$

we ignore this, it's bdd

$\to$ asymptotically eigenvalues same

- on $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $(\Delta - \lambda)^{-1}$, $(D^2 - \lambda)^{-1}$ exist and are bdd

- further, they are compact
  → eigenvalues converge to zero as a sequence ? (no "endless" repetitions)

$n=$ "2"

- check $(D-\lambda)^{-1}$ is cpt, we need

$\|D\psi\|^2 = \langle D^2\psi, \psi\rangle = \|\nabla\psi\|^2 + \int_M d\mathrm{vol}\, \langle\psi,\psi\rangle \frac{1}{4}R$

$\Rightarrow \|\psi\|^2_{H^1} + \left(\frac{R_{min}}{4} - 1\right)\|\psi\|^2 \le \|D\psi\|^2 \le \|\psi\|^2_{H^1} + \left(\frac{R_{max}}{4} - 1\right)\|\psi\|^2$

Prop $(D-\lambda)^{-1}$ is cpt for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Pf. $\|\psi\|^2_{H^1} \le \|(D-\lambda)\psi\|^2 + \left(|\lambda|^2 + 1 - \frac{R_{min}}{4}\right)\|\psi\|^2$

Set $\varphi = (D-\lambda)\psi \in \mathrm{Ran}(D-\lambda) = H = L^2(\Sigma)$.

So, substituting,

$\|(D-\lambda)^{-1}\varphi\|^2_{H^1} \le \|\varphi\|^2 + \mathrm{const}_1 \underbrace{\overset{\text{bdd.}}{\|(D-\lambda)^{-1}\varphi\|^2}}_{\le \mathrm{const}_2 \cdot \|\varphi\|^2}$

$\le \mathrm{const}_3 \|\varphi\|^2$

so cpt by Rellich thm.

Rmks
- $\exists$ complete O.N.B $\{\psi_n\}$ of D-eigenv. in $L^2(\Sigma) = H$
- $D\psi_n = \lambda_n\psi_n$ and $|\lambda_n| \nearrow \infty$ as $n \to \infty$
- $\varphi = \sum \underset{\ell^2}{\underset{\in}{\alpha_n}} \psi_n \in H^k$ iff $\sum |\alpha_n| (1 + |\lambda_n|^k) < \infty$
  ↳ $\|\cdot\|^2_{H^k} = \|\cdot\|^2 + \|\nabla\cdot\|^2 + \dots + \|\nabla^k\cdot\|^2$
- $D^k\varphi = \sum \alpha_n (\lambda_n)^k \psi_n$

Boundedness of commutators

- $C^\infty(M,\mathbb{C})$ action on $\Gamma(\mathcal{E})$ extends to $L^2(\mathcal{E}) = H$ by bounded operators, as a $*$-representation
  - meaning multiplicative, $*$-preserving, inv. to scalar mult., e.g. $(z f^* + f^1 \cdot f^2)\triangleright - = z\cdot(f\triangleright -)^* + (f^1\triangleright -)(f^2\triangleright -)$
- exercise: $\|f\triangleright -\| = \|f\|_\infty$ on $L^2(\mathcal{E}, \mathrm{vol}_g)$

$\|[D, f\triangleright]\| = \|df\cdot\| = \|\gamma(df)\| \leq \mathrm{const}\cdot \sup_M |df|$

- so commutators bdd even though D is <u>not</u>.
- unfortunately, f cannot only be continuous, but...

Rmk $C^\infty(M,\mathbb{C})$ is not the biggest subalg. of $C(M,\mathbb{C})$ s.t. $\|[D,f]\| < \infty$. Indeed, this holds for Lipschitz functions, i.e. $|f(x) - f(y)| < C\cdot \mathrm{dist}(x,y)$

Def A **spectral triple** $(A, \mathcal{H}, D)$ consists of
- a $*$-algebra A
- a Hilb. sp. $\mathcal{H}$ (separable), carrying a faithful, bounded $*$-rep π of A
- $D = D^*$ with bdd comm. with $\pi(a)$, $\forall a \in A$ with a cpt. resolvent

A s.t. is **even** if $\exists \chi = \chi^*, \chi^2 = \mathrm{id}, [\chi, \pi(a)] = 0$ $\forall a \in A$, $\{\chi, D\}_+ = 0$. If not, s.t. is **odd**.

A s.t. is real if $\exists$ antiunitary J s.t. that
$J\pi(a^*)J^{-1}$ commutes w/ $\pi(b)$ $\forall b \in A$
and $J^2 = \varepsilon$, $JD = \varepsilon' D J$, $J\chi = \varepsilon'' \chi J$
where $\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$.

A s.t. is finite if $\dim \mathcal{H} < \infty$
— " — commutative if A is.

Prop Let M spin, cpt w/o boundary, $\Sigma \to M$ Dirac spinor vbdl.
Then $(C^\infty(M), L^2(\Sigma), \not{D})$ is
a comm. s.t., even if $\dim M$ = even,
and real if M is spin.