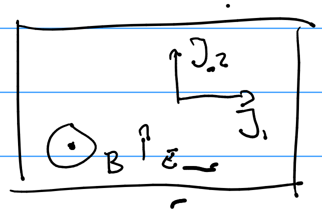
M. Porta.

Math. methods for cond. mat. systems.

Transport in cond mat systems.

- quantum Hall effect (QHE)
- thin surface, low T, high B



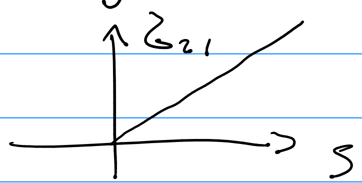
- weak $E \Rightarrow J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \sigma E + O(E^2)$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \text{ conductivity matrix}$$

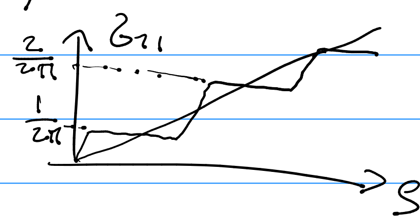
→ $\sigma_{21} = \sigma_{12}$ transverse (Hall) conductivities

→ $\sigma_{11} = \sigma_{22}$ longitudinal (symmetries: $\sigma_{11} = \sigma_{22}$)

- classically:



- but (v. Klitzing '80): σ_{21} takes values in $\mathbb{Z}$
- in $e^2 = \hbar = 1$ units,



$\sigma_{21} \in \frac{1}{2\pi}\mathbb{Z}$

Mathematical model

- $\Lambda \hookrightarrow \mathbb{Z}^2$ lattice
- Hilb. space $\mathcal{H} = \ell^2(\Lambda \times \mathbb{C}^M)$, so

  $\mathcal{H} \ni \psi \equiv \psi(x, s)$, $x \in \Lambda$, $s = 1 \dots M$

  and $\|\psi\|_2 = \sum_{x,s} |\psi(x,s)|^2 = 1$

- observables $\langle O \rangle_\psi := \langle \psi, O\psi \rangle$

- q. dynamics: $i\partial_t \psi(t) = H\psi(t)$, $\psi(t)\big|_{t=0} \in D(H)$

- Example: $H = -\Delta_{\mathbb{Z}^d}$ (lattice Laplacian)

$$(H\psi)(x) = -\sum_{y:\|y-x\|=1} (\psi(y) - \psi(x))$$

- more generally $H = -\Delta_{\mathbb{Z}^d} + V$

- more more gen. $(H\psi)(\underset{(x,t)}{z}) = \sum_{z'} \underset{\text{short ranged}}{H(z,z')}\psi(z')$

Rmk. Think of Bravais latt. as decorated square lattices.

Lemma. Given A on ℓ_2 with kernel $A(x,y)$ & $|A(x,y)| \le c$ then $\|A\|^2 \le a_1 \cdot a_2$, where $a_1 = \sup_x \sum_y |A(x,y)|$, $a_2 = \sup_y \sum_x |A(x,y)|$

Def. (resolvent set) H (s.a.) op, $S(H) := \{ z \in \mathbb{C} \mid (H - z) : D(H) \to \mathcal{H} \text{ is bijective} \}$

Fact. $S(H)$ open

- let $R_z(H) = (z - H)^{-1}$ resolvent of H.
  → analytic for $z \in S(H)$

Spectrum: $\sigma(H) = \mathbb{C} \setminus S(H)$ (closed)

- if H s.a., $\sigma(H) \subset \mathbb{R}$.

- by spectral thm, $\forall$ s.a. op. H on $\mathcal{H}$ $\exists!$ projection valued measure s.t. (hold the thought)
- PVB: for $P: \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$:

i) $P(\Omega) = P(\Omega)^2 = P(\Omega)^*$

ii) $P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$

iii) if $\Omega = \bigsqcup_n \Omega_n$ then $P(\Omega)\psi = \lim_{N\to\infty} \sum_{n=0}^{N} P(\Omega_n)\psi$

- next define $\mu_\psi(\Omega) = \langle \psi, P(\Omega)\psi \rangle$

$\mu_{\psi,\varphi}(\Omega) = \langle \psi, P(\Omega)\varphi \rangle$

(resume) $H = \int \lambda \, dp(\lambda)$, $\langle \psi, H\varphi \rangle = \int d\mu_{\psi,\varphi}(\lambda)\lambda$

- now, given any Borel measure μ,

$$\mu = \underset{\substack{\uparrow \\ \text{abs. cont.}}}{\mu_{a.c.}} + \underset{\substack{\uparrow \\ \text{sing. cont.}}}{\mu_{s.c.}} + \underset{\substack{\uparrow \\ \text{pure pt.}}}{\mu_{p.p}}$$

where $d\mu_{a.c.} = f(\lambda)\,d\lambda$, $f \in L^1(\mathbb{R})$

$d\mu_{s.c.}$ supported on set of 0 Lebesgue-measure

$d\mu_{p.p.}$ supp. on countable set of pts

Spectral subspaces: $\mathcal{H} = \mathcal{H}_{a.c.} \oplus \mathcal{H}_{s.c.} \oplus \mathcal{H}_{p.p}$

where $\mathcal{H}_{\#} = \{ \psi \in \mathcal{H} \mid \mu_\psi \text{ is of type } \# \}$

- similarly $\sigma_{\#}(H) = \sigma(H|_{\mathcal{H}_{\#}})$

Examples:

i) $H = -\Delta_{\mathbb{Z}^d} \Rightarrow \sigma(H) = \sigma_{ac}(-\Delta_{\mathbb{Z}^d}) = [0, 4d]$

ii) $H = V$, $(V\psi)(x) = \omega(x)\psi(x)$

$$\Rightarrow \sigma(V) = \sigma_{pp}(V) = \bigcup_{x \in \Lambda} \omega(x)$$

- eigenstates: i) plane waves
  ii) $\delta_{x_0, x}$

Dynamical characterisation of the spectrum (RAGE thm):

- let H on $\mathcal{H}$ s.a. op.

$$\mathcal{H}_c = \mathcal{H}_{a.c.} \oplus \mathcal{H}_{s.c.} = \left\{ \psi \in \mathcal{H} \,\middle|\, \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \, \| \chi(|x|<L) e^{-iHt} \psi \|^2 = 0 \right\}$$

$$\mathcal{H}_{p.p.} = \left\{ \psi \in \mathcal{H} \,\middle|\, \lim_{L\to\infty} \sup_{t\geq 0} \| (1 - \chi(|x|<L)) e^{-iHt} \psi \| = 0 \right\}$$

- in english: → states in $\mathcal{H}_{a.c.}$ leave ball $|x|<L$ for good
→ $\mathcal{H}_{s.c.}$ may come back on measure zero set (which is why we integrate $\frac{1}{T}\int_0^T dt$) but leave on avg
→ $\mathcal{H}_{p.p.}$ stay

- ok... what about multiparticle systems?

- for 1 ptcl, $\langle O \rangle_\psi = \operatorname{tr} O P_\psi$, $P_\psi = |\psi\rangle\langle\psi|$

- Many ptcl: $\psi \in \ell^2(\underbrace{\Lambda \times \cdots \times \Lambda}_{N \text{ times}})$

$\psi \equiv \psi(x_1 \cdots x_N)$, x_i pos. of i-th ptcl.

- identical ptcls $\Longleftrightarrow$ $|\psi(x_1 \cdots x_N)| = |\psi(x_{\pi(1)} \cdots x_{\pi(N)})|$

$\hookrightarrow$ $\psi(x_1 \cdots x_N) = \left\{ \begin{matrix} 1 \\ \text{or} \\ \operatorname{sgn}(\pi) \end{matrix} \right\} \psi(x_{\pi(1)} \cdots \pi(N))$

$\rightarrow$ $\langle O_N \rangle = \operatorname{tr} O_N P_N$

- Marginals

$\Rightarrow$ 1 ptcl density mtrx $\gamma_{\psi_N} := N \operatorname{Tr}_{2 \cdots N} P_N$

$$\gamma_{\psi_N}(x, y) = \sum_{x_2 \cdots x_N} \psi_N(x, x_2 \cdots x_N) \overline{\psi_N(y, x_2 \cdots x_N)}$$

- label it $\gamma^{(1)}_{\psi_N}$

- now $\langle O_N \rangle_{\psi_N} = \operatorname{tr} O_N P_{\psi_N} = \operatorname{tr} O \gamma^{(1)}_{\psi_N}$ if
$O_N = \sum O^{(i)}$ and $O^{(i)} = \mathbb{1}^{\otimes(i-1)} \otimes O \otimes \mathbb{1}^{\otimes(N-i)}$

- check: $\gamma^{(1)}_{\psi_N} \geq 0$ $\left(\begin{matrix}\text{bosons} \\ \text{and} \\ \text{fermions}\end{matrix}\right)$, $\gamma^{(1)}_{\psi_N} \leq \mathbb{1}$, (fermions)

- consider noninteracting Hamiltonian

$$H_N = \sum_{i=1}^{N} H^{(i)}, \quad H^{(i)} = \mathbb{1}^{\otimes(i-1)} \otimes H \otimes \mathbb{1}^{(N-i)}$$

- suppose $|\Lambda| < +\infty$

- eigenstates look like $\psi_N = f_{i_1} \wedge \cdots \wedge f_{i_N}$,

$f_i \rightsquigarrow$ i-th eigenstate of H.

$\rightarrow$ g.s.? take N lowest eigenstates f_i

$$\rightarrow \gamma^{(1)}_{\psi_N} = \sum_{i=1}^{N} |f_i\rangle\langle f_i|$$

$$= \chi(H \leq \mu_N) \quad \text{where } \operatorname{tr}\chi(H \leq \mu_N) = N$$

Infinitely many part. $P := \chi(H \leq \mu)$

FERMI PROJECTOR

$\rightarrow$ the transport properties depend heavily on properties of P

$\rightsquigarrow$ e.g. $|P(x,y)| \leq C_1 e^{-C_2\|x-y\|} \Leftrightarrow$ insulator

*) $\mu > \|H\| \Rightarrow P(x,y) = \delta_{xy}$

*) $|P(x,y)| \leq C_1 e^{-C_2\|x-y\|}$

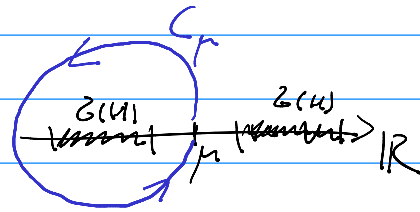
*) H_ω Random Schr. op (R.S.O.) $H_\omega = -\Delta + \lambda \underset{\text{random}}{V}$

$\rightarrow$ if $|\lambda| \gg 1$ then point spectrum

- Supp. $\mu \notin \Sigma(H)$

$$P = \chi(H \le \mu)$$

$$= \int_{C_\mu} \frac{dz}{2\pi i} \frac{1}{z-H} \quad \text{where}$$



Prop. Supp $z \notin \Sigma(H)$, H s.a. finite ranged.
Then $|R_z(H)(x-y)| \le C e^{-c|x-y|}$.

Pf. let $\alpha \in \mathbb{C}^d$, def $H_\alpha := e^{\alpha \cdot \hat{x}} H e^{-\alpha \cdot \hat{x}}$,

$$H_\alpha(x,y) = e^{\alpha \cdot (x-y)} H(x,y)$$

Then $\|H - H_\alpha\| \le C|\alpha|$ so if $H - z$ inv,
$H_\alpha - z$ inv for small $|\alpha|$.

$$\frac{1}{z-H_\alpha} = e^{\alpha \cdot \hat{x}} \frac{1}{z-H} e^{-\alpha \cdot \hat{x}}$$

$$\Rightarrow |e^{\alpha(x-y)} R_z(H)(x,y)| \le C.$$