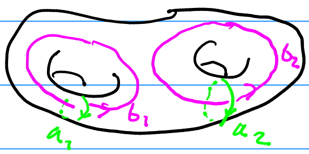
Bertola

- Tau & Theta functions, themselves given by matrices $\tau \in Mat_{g\times g}(\mathbb{C})$, $\tau = \tau^T$, $\operatorname{Im} \tau > 0$ ($\uparrow$ pos. def.)

$$\to \Theta(\vec{z}, \tau) = \sum_{\vec{n} \in \mathbb{Z}_g} e^{i\pi \vec{n}^T \tau \vec{n} + 2\pi i \vec{n}\cdot\vec{z}}$$

$\vec{z} = (z_1, \dots, z_g) \in \mathbb{C}^g$

- given a Riem. sfc $\mathcal{C}$ of genus g, Torelli marked (meaning we picked a sympl. basis $\{a_i, b_j\}_{i,j=1,\dots,g}$ so $H_1(\mathcal{C}, \mathbb{Z}) = \langle a_i, b_j \mid a_i \# b_j = \delta_{ij} \rangle$)



$$\begin{array}{c|cc} & a & b \\ \hline a & 0 & \mathbb{1}_g \\ b & -\mathbb{1}_g & 0 \end{array}$$

there is an $\perp$ basis of holom. differentials $\omega_j(z) = f_j(z)\,dz$ s.t.

$$\oint_{a_i} \omega_j(z) = \delta_{ij}, \text{ and let}$$

$$\tau_{jk} := \oint_{b_j} \omega_k$$

Integrable systems (∞-dim)

- e.g. KdV hierarchy

- set of evolutionary PDEs for a function $u = u(x, \underline{t})$ of the form ($t_1 = x$, $\underline{t} = t_3, t_5, \dots$)

$$\frac{\partial u}{\partial t_j} = P_j(u, u_x, u_{xx}, \dots, u_N)$$

- KdV: $u_t = u u_x + u_{xxx}$
- $\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial t_k}$ $\forall j k$

- $\tau(t_1, t_3, \ldots)$ a function s.t. $u = \frac{\partial^2}{\partial t_2} \log \tau$

- KdV eqns can be expressed as an isospectral deformation of an operator in $L^2(\mathbb{R}, dx)$ (Lax operator)
  - given:
    $L = \partial_x^2 + u(x)$, $L\psi = \lambda\psi$
    find dependence of $u(x)$ on params. t s.t. eigenvalues constant
    (and $u(x) \to 0$ at $x \to \infty$ fast enough, etc)
  - usually,
    $$\tau = \det\left(\mathrm{Id}_{L^2(\mathbb{R}, dx)} + \mathcal{E}(t_1, t_3, \ldots) \right)$$
    Fredholm (i.e. infinite) determinant
    $\mathcal{E} \in \mathcal{B}(L(\mathbb{R}, dx))$

- most often τ is a determinant ("determinant")
  - if it vanishes, probably unsolvable

Random matrices

- $M \in \mathcal{H}_N = \{ M = M^+ \}$
- pick prob. measure on $\mathcal{H}_N$ $d\mu(M)$

- ex 1 Wigner class: $M \in \mathcal{H}_N^{real}$ s.t.
  M_{ij} are i.i.d. with some $d\nu(x)$
  - classically, semicircle law

- not treated using int. systems, though
- usually, $N \to \infty$ is the question
- $d\mu(M) = \frac{1}{Z_N} \cdot e^{-\mathrm{tr}(V(M))} dM$

where dM = Lebesgue measure on $\mathcal{H}_N$
$= \prod_{i<j} \mathrm{Re}(dM_{ij}) \mathrm{Im}(dM_{ij}) \prod_{i \geq 1}^{N} dM_{ii}$

$V(x)$ = arbitrary
$= t_1 x^1 + t_2 x^2 + \dots + \underbrace{t_{2n}}_{>0} x^{2n}$

- then $Z_N(t_1, t_2, \dots, t_{2N}) = \int_{\mathcal{H}_N} e^{-\mathrm{tr}\sum t_j M^j} dM$

is a τ-function for the KP hierarchy

- what? these are not evolutionary PDEs, but e.g. $u_{yy} = (u_t - u u_x - u_{xxx})_x$ (*)
(shallow waves in comoving frame, hence no spatial symmetry)
- try plugging in $N=1$, $\tau = \int_{\mathbb{R}} dx\, e^{-t_1 x - t_2 x^2 - t_3 x^3 - t_4 x^4}$
(Pearsey integral), $u = -\left(\frac{\partial}{\partial t_1}\right)^2 \ln \tau(\underset{x}{t_1}, \underset{y}{t_2}, \underset{t}{t_3}, \dots)$
into (*) $\to$ very unpleasant

Gap probabilities of Determinantal random point processes (DRPP)

- point process? example:

- consider eigenvalues of RM $M = U X U^+$
where $U \in U(N)$, $X = \mathrm{diag}(x_1, \dots, x_N)$

- the j.p.d.f. of eigenvals is

$$\frac{d\nu(x_1,\dots,x_N)}{dx_1\dots dx_N} = \frac{1}{\tilde{Z}_N}\prod_{i<j}(x_i-x_j)^2 e^{-\sum_{i=1}^N V(x_i)}$$

- notice the Vandermonde $\Delta(x) = \det\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & & x_2^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^{N-1}\end{bmatrix}$

- Dyson: the marginals

$$S_k(x_1,\dots,x_k) := \int_{x_{k+1},\dots,x_N\in\mathbb{R}} d\nu$$

are determinants

$$= \frac{N!}{(N-k)!}\det\underbrace{\left(K_N(x_i,x_j)\right)_{i,j=1}^{k}}_{\begin{bmatrix} K_N(x_1,x_1) & K_N(x_1,x_2) & \dots \\ K_N(x_2,x_1) & \ddots & \\ \vdots & & \ddots \end{bmatrix}}$$

where $K_N(x,y) := e^{\frac{-V(x)-V(y)}{2}}\sum_{j=0}^{N-1}\frac{p_j(x)\,p_j(y)}{\|p_j\|^2}$

where $p_j(x)$ are $\perp$ polynomials for $L^2(\mathbb{R}, e^{-V(x)}dx)$ i.e. $\int p_j(x)p_k(x)e^{-V(x)}dx = \|p_j\|^2\delta_{jk}$
s.t. $p_j(x) = x^j +$ subleading.

- Gap probability: $\mathbb{P}$(there are no particles in $[a,b]$)

- ex. Tracy-Widom

$$K(x,y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y}$$

where $Ai(x)$ Airy func, (part.) soln
to $f'' - xf = 0$, $Ai(x) = \text{const.} \int_{\mathbb{R}} e^{i\frac{t^3}{3} - itx} \frac{dx}{2\pi i}$

- it is a tot. positive kernel,
  meaning $\det\left[K(x_i, x_j)\right]_{i=1}^{n} \geq 0$ for any n

  - so it is a prob measure... sort of?
- defines DRPP (∞ # of ptcls)
  - gap prob
$$\mathbb{P}(\text{none in } [z, \infty)) = F_2(z)$$
(Tracy-Widom)
- it is τ-func. of Painlevé $\underline{\mathrm{II}}$
$$\left(\frac{d}{dz}\right)^2 \ln F_2(z) = u^2(z)$$

  then $u(z)$ solves 2^{nd} Painlevé
$$u'' - zu = u^3(z), \quad u(z) \sim Ai(z) \text{ as } z \to \infty$$
  unique

- $F_2(z) = \det\left(\mathrm{Id}_{L^2} - \underbrace{K|_{[z,\infty)}}_{\text{whatever this means}}\right)$

- book: Intro. to int. sys., Babelon, Bernard, Talon

Example: Euler top

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \quad \vec{J} = I\vec{\omega}, \quad \dot{\vec{J}} = -\omega \wedge \vec{J}$$

- Hamiltonian wrt $\{J_i, J_l\} = \varepsilon_{ilk} J_k$
  with $H = \frac{1}{2} \sum \frac{J_e^2}{I_e}$
- the P.B. has Casimir $C = |\vec{J}|^2 = J_1^2 + J_2^2 + J_3^2$
- define $J := I\,\Omega + \Omega\, I = \begin{bmatrix} 0 & (I_1+I_2)\omega_3 & -(I_1+I_3)\omega_2 \\ - & 0 & (I_2+I_3)\omega_1 \\ - & - & 0 \end{bmatrix}$

  $L(\lambda) := I^2 + \frac{J}{\lambda}, \quad M(\lambda) := \lambda I + \Omega$

- then: eq. of motion $\Leftrightarrow$ $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$
  $= [I, J] + [\Omega, I^2] + \frac{[\Omega, J]}{\lambda^2}$

Lemma Let L be a matrix, M matrix dep. on t and entries of L.
Then if $\dot{L} = [M, L]$, e.vals of L are const.

Pf. $\det(\mu \mathbb{1} - L(t)) =: P(\mu, t)$.
- Jacobi: $\frac{d}{dt} \det G(t) = \operatorname{tr}(\tilde{G}(t) \cdot \dot{G}(t))$
  where $\tilde{G}$ adjugate ($G^{-1} = \tilde{G}/\det G$)

$$\frac{d}{dt}\det(\mu\mathbb{1} - L(t)) = -\operatorname{tr}\, \widetilde{(\mu\mathbb{1} - L)} \cdot \underbrace{\dot{L}(t)}_{[M, L]}$$

$$= \operatorname{tr} \widetilde{(\mu\mathbb{1} - L)}\, [L - \mu\mathbb{1}, M]$$

$$\overset{\text{cycl}}{=} \operatorname{tr} M\, [\widetilde{L - \mu\mathbb{1}}, L - \mu\mathbb{1}]$$

$$= 0. \quad \text{since } [\tilde{G}, G] = 0.$$

- $\operatorname{tr} L = I_1^2 + I_2^2 + I_3^2$

$\operatorname{tr} L^2 = \operatorname{tr} I^4 - \frac{2}{\lambda^2} |\vec{J}|^2$

$\operatorname{tr} L^3 = \operatorname{tr} I^6 - 3/\lambda^2 \left(\frac{1}{I_1} \operatorname{tr} I^2 \cdot |\vec{J}|^2 - I_1 I_2 I_3 H \right)$

The Z.S. construction

- for evol. hier. of ODEs for $L(\lambda)$ = r×r b. mtx w/ prescribed pole structure $= L_0 + \sum_{j=1}^{k} \sum_{\ell=1}^{n_j} \frac{L_{j,\ell}}{(\lambda - a_j)^\ell}$

- stick to r×r, dim = n r²

$$L(\lambda) = \frac{L_{-n}}{\lambda^n} + \frac{L_{-n+1}}{\lambda^{n-1}} + \cdots + \frac{L_1}{\lambda}$$

- look for $M_1, M_2, \_\_$ s.t.

$$\frac{\partial}{\partial t_j} L(\lambda) = [M_j(\lambda), L(\lambda)]$$

are commuting vfs, $\partial_j \partial_k L = \partial_k \partial_j L$

$\Rightarrow \partial_j [M_k, L] = [M_k, [M_j, L]] + [\partial_{t_j} M_k, L]$

$\quad \partial_k [M_j, L] = [M_j, [M_k, L]] + [\partial_{t_k} M_j, L]$

- look at one of these $\dot{L} = [M, L]$

i) M must have same pole structure (only $\lambda = 0$) as L, since $\dot{L}$ has

- take $M = (M)_-$, Laurent tail

ii) assume L_{-n} is regular semisimple (simple and distinct e.vals)

Lemma $\exists\, g(\lambda),\ g = g_0 + \lambda g_1 + \cdots,\ \det g_0 \neq 0$
s.t.

$$L(\lambda) = g(\lambda) A(\lambda) g^{-1}(\lambda),\quad A(\lambda) \text{ diag.}$$

Prop $M(\lambda)$ has the form

$$M(\lambda) = \left(g(\lambda) B(\lambda) g^{-1}(\lambda) \right)_-$$

where $B(\lambda)$ diag mtx w pole of arb. order at $\lambda = 0$