Bertola

Malgrange's form

- $\begin{cases} \Gamma_+ = \Gamma_- M & , z \in \Sigma \text{ multicontour} \\ \Gamma(\infty) = \mathbb{1} \end{cases}$

- shorthand $\hat{dz} = \frac{dz}{2\pi i}$, $\Theta := \int_\Sigma \hat{dz} \, \mathrm{tr}\, \Gamma_-^{-1} \Gamma_-' \, dM M^{-1}$

Lemma The sol'n of $\begin{cases} \dot\Gamma_+ = \dot\Gamma_- M + \Gamma_- \dot M \\ \dot\Gamma(\infty) = 0 \end{cases}$ is

$$\dot\Gamma(z) = \int_\Sigma \frac{\Gamma_-(w)\dot M(w) M^{-1}(w)\Gamma_-^{-1}(w)}{w-z} \hat{d}w \, \Gamma(z)$$

- note that this works bcos $\frac{\hat{dw}}{w-z}$ unique kernel on Riem. sph.
- let $\partial = \partial/\partial t_a$, $\tilde\partial = \partial/\partial t_b$:
- $d\Theta(\partial, \tilde\partial) = \partial\Theta(\tilde\partial) - \tilde\partial\Theta(\partial)$

$$= \dots$$

$$= \int_\Sigma \mathrm{tr}\Big(\partial(\Gamma_-^{-1}\Gamma_-')\tilde\partial M M^{-1} - \tilde\partial(\quad)\partial M M^{-1} + \Gamma_-^{-1}\Gamma_-' [\partial M M^{-1}, \tilde\partial M M^{-1}]$$

- letting $\varphi(z,w) := \mathrm{tr}\big(\Gamma_-^{-1}(z)\Gamma_-(w) \cdot \partial M M^{-1}(w) \Gamma_-^{-1}(w)\Gamma_-(z)\tilde\partial M M^{-1}(z) - (\partial \leftrightarrow \tilde\partial) \big) = -\varphi(w,z)$

we get

$$d\Theta(\partial,\tilde\partial) = \int \hat{dz} \int \hat{dw} \frac{\varphi(z,w)}{(z-w)^2} + \int \hat{dz} \, \mathrm{tr}\, \Gamma_-^{-1}\Gamma_-' [\partial M M^{-1}, \tilde\partial M M^{-1}]$$

Lemma. γ contour w/o self.int,

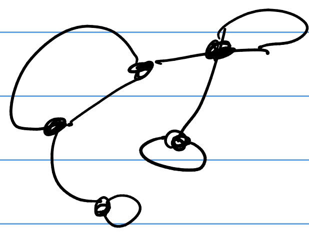
$$\int_\gamma \hat{dz} \int_\gamma \hat{dw} \frac{\varphi(z,w)}{(z_- - w)^2} = -\frac{1}{2} \int \hat{dz} \, \partial_w \varphi(w,z) \Big|_{w=z}$$

- consider Σ without self-int. contours

$$d\tilde{\Theta} = -\frac{1}{2}\int \hat{d}_z \underbrace{\partial_w \varphi(z,w)\Big|_{w=z}}_{\substack{2\,\mathrm{tr}\, \Gamma_-^{-1}\Gamma_-^{\sim}[\partial M M^{-1},\, \tilde{\partial} M M^{-1}] \\ +\mathrm{tr}\left[(\partial M M^{-1})^{\wedge}\, \tilde{\partial} M M^{-1} - (\partial - \tilde{\partial})\right]}} + \int \hat{d}_z\, \mathrm{tr} \ldots$$

$$= -\frac{1}{2}\int \mathrm{tr}\,(d M M^{-1})^{\wedge} \wedge (d M M^{-1})$$

- if Σ has vertices, first enlarge them to ε-disks, let $D_\varepsilon := \bigcup_{v\in V} D_\varepsilon(v)$



$$\int_\Sigma \int_\Sigma \frac{\varphi}{(z-w)^2} = \Big\{ \int_{\Sigma\setminus D_\varepsilon} \int_{\Sigma\setminus D_\varepsilon} \Big\} \xrightarrow{\varepsilon\to 0} -\frac{1}{2}\int_{\Sigma\setminus D_\varepsilon} \hat{d}z\, \partial_w \varphi\Big|_{w=z}$$

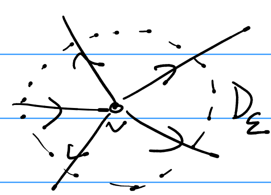
$$+ \Big\{ \int_{\Sigma\setminus D_\varepsilon} \int_{\Sigma\cap D_\varepsilon} + \int_{\Sigma\cap D_\varepsilon} \int_{\Sigma\setminus D_\varepsilon} \Big\} \to 0$$

$$+ \int_{\Sigma\cap D_\varepsilon} \int_{\Sigma\cap D_\varepsilon} := C(\varepsilon)$$

is interesting.

- $C(\varepsilon)$ computation

- around $v \in V$,



$$\varphi_{k,j}(w,z) := \varphi(w,z)\Big|_{\substack{w\in\gamma_k \\ z\in\gamma_j}}$$

- we do have $\varphi_{k,j}(w,z) = -\varphi_{j,k}(z,w)$, which says little abt $\varphi_{j,k} := \varphi_{j,k}(v,v)$

- $\prod_i M_i = \mathbb{1} \overset{\partial}{\Rightarrow} \ldots \Rightarrow \sum_{l=1}^{n} \varphi_{kl}(v,w) = 0$

$$\Rightarrow C(q) = \sum_{v} \int_{*} d\hat{z} \int_{*} dw \, \frac{\varphi(z,w)}{(z_- - w)^2}$$

- for each $v \in V$,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\gamma_j^z} d\hat{z} \int_{\gamma_k^z} d\hat{w} \, \frac{\varphi_{jk}(z,w)}{(z_- - w)} = \begin{cases} j = k & \to 0 \\ j \neq k & \end{cases}$$

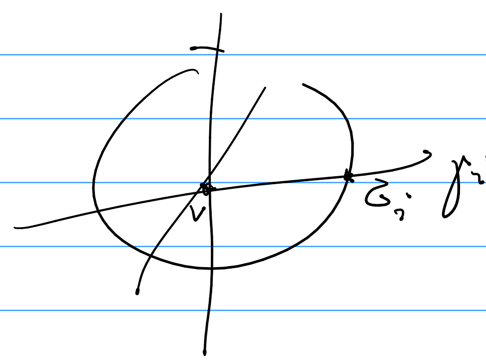
regularize

$$\sum_{j=1}^{n} \sum_{k \neq j} \left[\int d\hat{z} \int d\hat{w} \, \frac{\varphi_{jk}(z,w) - \varphi_{jk}}{(z-w)^2} + \varphi_{jk} \int d\hat{z} \int d\hat{w} \, \frac{1}{(z-w)^2} \right]$$

- using (anti) symmetry, the contribution becomes only the 2^{nd} term i.e.

$$\sum_{\ell} \sum_{m \neq \ell} \frac{\varphi_{m\ell}}{(2\pi i)^2} \ln_{(\ell)} \left(\frac{z_\ell - z_m}{v - z_m} \right)$$

branch along γ_ℓ



$$-d\,\Theta = \frac{1}{2} \int_{\Sigma} \operatorname{tr} dM M^{-1} \wedge \frac{d}{dz}(dM M^{-1}) + \sum_{v \in V(\Sigma)} \eta_v$$

where $\eta_v := -\frac{1}{4\pi i} \sum_{\ell \geq 2}^{n_v} \operatorname{tr}\left(M_{\ell-1}^{-1} dM_\ell \wedge d(M_{[\ell:n_v]}) M_{[\ell:n_v]} \right)$

Example KdV

- $\mathcal{L} = -\partial_x^2 + u(x)$, $\int_{\mathbb{R}} (1+|x|)|u| < \infty$

sufficient cond. for scattering theory

i.e. $-\psi'' + u\psi = z^2\psi$, $\psi \in L^2(\mathbb{R}, dx)$ will have both cont. and disc. spectrum

- e.g. $u = \dfrac{-3k^2}{\cosh^2 kx} \rightsquigarrow$ 1 e.v. $-k^2$ discr.
  - others cont
  - $\to$ but <u>no</u> reflection coeff!

- facts: finite # of e.vals $0 > -k_N^2 > \cdots > -k_1^2$ on imag. axis

- Jost funcs $\begin{cases} \psi_\pm(x;z) = e^{\pm ixz}(1+o(1)), & x \to +\infty \\ \varphi_\pm(x;z) = e^{\pm ixz}(1+o(1)), & x \to -\infty \end{cases}$

analytic in z

$$\begin{pmatrix} \varphi_- \\ \varphi_+ \end{pmatrix} = \underbrace{\begin{bmatrix} a(z) & b(z) \\ \bar{b}(z) & \bar{a}(z) \end{bmatrix}}_{\text{scat. mtrx}} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$$

$|a|^2 - |b|^2 \leq 1$

- $a(z)$ has analytic ext to $\mathbb{H}^+ \ni z$
- $a(z) = 0 \iff z$ is eigenvalue, say z_j

$$\varphi_-(x, iz_j) = b_j \psi_+(x, iz_j)$$

$$\underline{\text{RHP}} \quad \Gamma(z;x) = \begin{cases} \begin{bmatrix} \frac{\varphi_-(x,z)}{a(z)} & \psi_+ \\ \frac{\partial_x \varphi_-}{a(z)} & \partial_x \psi_+ \end{bmatrix} e^{izx\sigma_3}, & \operatorname{Im} z > 0 \\ \begin{bmatrix} \psi_- & \frac{\varphi_+}{\bar{a}(z)} \\ \partial_x \psi_- & \frac{\partial_x \varphi_+}{\bar{a}(z)} \end{bmatrix} e^{izx\sigma_3}, & \operatorname{Im} z < 0 \end{cases}$$

- for $z \in \mathbb{R}$,

$$\Gamma_+(z,x) = \Gamma_-(z,x) \begin{bmatrix} 1 - |r(z)|^2 & -\bar{r} e^{-2ixz} \\ r e^{2ixz} & 1 \end{bmatrix}$$

where $r(z) := \frac{b(z)}{a(z)}$ "refl. coeff."

with

$$\Gamma(z,x) = \begin{bmatrix} 1 & 1 \\ -iz & +iz \end{bmatrix} \left(\mathbb{1} + \frac{i \int_x^\infty ds\, u(s)}{z} \sigma_3 + O(z^{-2}) \right)$$

as $z \to \infty$

- note that if we solve this RHP, and look at $\Gamma(z \to \infty, x)$, we obtain from the σ_3 term the potential
  $\to$ inverse scattering problem

- for KdV, additional t-dependence,
  $u_t = 6uu_x - u_{xxx}$
  $\to$ but in scatt. RHP, just let $e^{-2ixz} \mapsto e^{-2ixz + 8itxz^3}$

Thm (Dyson '76)

$$u(x,t) = -2\,\partial_x^2 \ln \det\left(\mathrm{Id}_{L^2([x,\infty),ds)} - \mathcal{K}\big|_{L^2([x,\infty),ds)} \right)$$

where $\mathcal{K}$ integral op with kernel

$$(\mathcal{K}f)(s) = \int_{\mathbb{R}} K(s,v) f(v)\, ds,$$

$$K(s,v) = F(s+v),$$

$$F(s) = \sum_{n=1}^{N} \gamma_n(t) e^{-\kappa_n s} + \frac{1}{2\pi} \int_{\mathbb{R}} r(z,t) e^{izs}\, dz$$

where $r(z,t) = r_0(z)\, e^{-8itz^3 + \dots}$

$\gamma_n(t) = \gamma_n(0) \cdot (\quad)$