Tikhomirov

- given $P \xrightarrow{\pi} B$ G-pbdl, V vsp. s.t. $G \xrightarrow{\varsigma} \mathrm{Aut}\,V$ rep., build assoc. vbdl $W = P \times_\varsigma V$.
- what is its connection?
- recall on pbdl, $A = \{\mathcal{H}_p\}$, $\omega_A \in \Omega^1(P, \mathfrak{g})$, $\mathcal{H}_p = \ker(\omega_{A,p} : TP_p \twoheadrightarrow \mathfrak{g})$
- on $W = P \times_\varsigma V$ we build a diff. operator

$$\nabla_A : \Omega^0(B, W) \longrightarrow \Omega^1(B, W)$$
$$s \longmapsto \nabla_A s, \quad (\nabla_A s)(\overset{TB}{\overset{\uparrow}{\tau}}) \in W|_B$$

by working locally (since diff. op. is local)

- let $s = \{[p(b), v(b)] \mid b \in \mathcal{U} \subseteq B\}$ where $p : \mathcal{U} \to \pi^{-1}(\mathcal{U})$, $v : \mathcal{U} \to V$ (don't take $\mathcal{U} = B$ unless P trivial!!) and let

$$(\nabla_A s)(\tau_b) := [p(b), \varsigma_*(\omega_A(p_* \tau_b))(v(b)) + \underbrace{\tau_b(v(b))}_{\frac{\partial v(b)}{\partial \tau_b}}]$$

where $\varsigma_* : \mathfrak{g} \to \mathrm{Gl}(V) = \mathrm{End}\,V$

- covariant differential

- for $f \in \Omega^0(B)$, $s \in \Omega^0(B, W)$ we check $\nabla_A(fs) = f \nabla_A s + df \otimes s$, Leibnitz rule

- extend by multilinearity to $\nabla_s : \Omega^k(B, W) \to \Omega^{k+1}(B, W)$ by taking $\eta \in \Omega^r(B)$, $\vartheta \in \Omega^{k-r}(B, W)$ and letting

$$\nabla_A(\eta \wedge \vartheta) = d\eta \wedge \vartheta + (-)^r \eta \wedge (\nabla_s \vartheta)$$

- we get $\Omega^0(B, W) \xrightarrow{\nabla_A} \Omega^1(B, W) \xrightarrow{\nabla_A} \Omega^2(B, W) \xrightarrow{\nabla_A} \dots$
but this is no cpx ?

$$\nabla_A^2(fs) = \nabla_A(f \nabla_A s + df \otimes s)$$
$$= f \cdot \nabla_A^2 s + df \otimes \nabla_A s + d^2 f \otimes s - df \otimes \nabla_A s$$
$$= f \cdot \nabla_A^2 s$$

so we define tensor $\nabla_A^2 s = s \otimes F_A \xrightarrow{ev} \Omega^2(B, W)$
where F_A is called curvature of ∇_A,
$F_A \in \Omega^2(B, \operatorname{End} W)$

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- let N fixed mfd, $\varphi \in \Omega^i(N, \mathfrak{g})$, $\psi \in \Omega^j(N, \mathfrak{g})$
- using Lie brckt $[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ we build $[\varphi, \psi] \in \Omega^{i+j}(N, \mathfrak{g})$ s.t.

$$[\varphi, \psi](X_1, \dots, X_{i+j}) := \frac{1}{i!\,j!} \sum_{\sigma \in S_{i+j}} [\varphi(X_{\sigma(1)}, \dots, X_{\sigma(i)}), \psi(X_{\sigma(i+1)}, \dots, X_{\sigma(i+j)})]$$

- properties:

1) $[\psi, \varphi] = -(-)^{i \cdot j} [\varphi, \psi]$

2) for $\vartheta \in \Omega^k(N, \mathfrak{g})$,

$$(-)^{ik}[[\varphi, \psi], \vartheta] + (-)^{kj}[[\vartheta, \varphi], \psi] + (-)^{j \cdot i}[[\psi, \vartheta], \varphi] = 0$$

- in particular, $[[\omega, \omega], \omega] = 0$

- for particular uses we take, e.g. $\mathrm{Aut}\,V = SO(2)$
so we're interested in matrix groups
- always think like this
- so, $\psi \wedge \varphi$ = combination of matrix product and wedge product
- in particular, $[\psi, \varphi] = \psi \wedge \varphi - (-)^{i \cdot j} \varphi \wedge \psi$

Curvature of conn. on a pbdl.

Def. For $\varphi \in \Omega^k(P, \mathfrak{g})$ and connection A, define $(D_A \varphi)(\tau_1, \ldots, \tau_{k+1}) := d\varphi(\tau_1^H, \ldots, \tau_{k+1}^H)$, where $\tau = \tau^H + \tau^V \in TP$ is the splitting.

The curvature 2-form is $\Omega_A := D_A \omega_A \in \Omega^2(P, \mathfrak{g})$.

Prop $\Omega_A = d\omega_A + \frac{1}{2}[\omega_A, \omega_A]$ (*)

- first recall some things. For $A \in \mathfrak{g}$, $A^* \in \Omega^0(TP)$ given by $A^* = \frac{\partial}{\partial t} p\exp(tA)|_{t=0}$ satisfies $[A^*, B^*] = [A, B]^*$ (fundamental field)
- further, for $X \in \Omega^0(T\mathcal{U})$ denote by $\tilde{X} \in \Omega^0(T\pi^{-1}\mathcal{U})$ its horizontal lift, $R_{g*}\tilde{X} = \tilde{X}$
- then $[A^*, \tilde{X}] = 0$

Pf. 1° τ_1, τ_2 horizontal. By def, $\Omega(\tau_1, \tau_2) = d\omega(\tau_1, \tau_2)$ and $\frac{1}{2}[\omega, \omega](\tau_1, \tau_2) \equiv 0$.
11° τ_1 vert, τ_2 horiz.
$\exists A^*$ s.t. $A^*_p = \tau_1$, $\exists \tilde{X}$ for some X s.t. $\tilde{X}_p = \tau_2$.

Then $\Omega(\tau_1, \tau_2) = 0$. Also,

$$d\omega(\tau_1, \tau_2) = \underbrace{A^*(\underbrace{\omega(\tilde{X})}_{=0 \text{ by horizontality}})_p}_{=0} - \tilde{X}\overbrace{(\overbrace{\omega(A^*)}^{=A})_p}^{=0} - \omega(\overbrace{[A^*, \tilde{X}]}^{=0})_p$$

and $[\omega, \omega](\tau_1, \tau_2) = 0$

III° τ_1, τ_2 vertical.

Now by def $\Omega(\tau_1, \tau_2) = d\omega(0,0) = 0$,

and $d\omega(\tau_1, \tau_2) = A^* B - B^* A - [A, B]$,

and $\frac{1}{2}[\omega, \omega](A^*, B^*) = [A, B]$. $\square$

- given $\{\mathcal{U}_\alpha\}$, we get sections

$\mathcal{U}_\alpha \xrightarrow{\sigma_\alpha} P|_{\mathcal{U}_\alpha}$ from local trivialisations

$\varphi_\alpha : P|_{\mathcal{U}_\alpha} \xrightarrow{\sim} \mathcal{U}_\alpha \times G \hookleftarrow \mathcal{U}_\alpha \times \{1\} \simeq \mathcal{U}_\alpha$

- we have also $\omega_\alpha = \sigma_\alpha^* \omega$, $\Omega_\alpha = \sigma_\alpha^* \Omega$,
forms on $\mathcal{U}_\alpha$.

- what happens on overlaps $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$?
  - with local trivs, $\varphi_\alpha = g_{\alpha\beta} \varphi_\beta$,
    while for $\{\omega_\alpha\}, \{\Omega_\alpha\}$

$$\omega_\beta = g_{\alpha\beta}^{-1} d g_{\alpha\beta} + g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta}$$

$$\Omega_\beta = g_{\alpha\beta}^{-1} \Omega_\alpha g_{\alpha\beta} \implies F_A = \{\Omega_\alpha\} \in \Omega^2(\mathrm{ad}\, P)$$

- we have $\pi^* F_A = \Omega$.

- It can be shown that for $\varphi \in \Omega^k(P, \mathfrak{g})$ s.t.
$\varphi(\tau_1 g_*, \ldots, \tau_k g_*) = g^{-1} \varphi(\tau_1, \ldots, \tau_k)\, g$
and $\varphi(\tau_1, \ldots, \tau_k) = 0$ if τ_1 vertical,
then $D_A \varphi = d\varphi + [\omega_A, \varphi]$

Prop $D_A \Omega = d\Omega + [\omega, \Omega] = 0$. (Bianchi id)

$D_A F_A = 0$.

- let G matrix gp

- notice $\operatorname{tr}(\underbrace{F_A \wedge \ldots \wedge F_A}_{k \text{ times}}) = \operatorname{tr}(g^{-1} F_A g \wedge \ldots \wedge g^{-1} F_A g)$ $\in \Omega^{2k}(B, \mathbb{R})$

- also $d \operatorname{tr}(F_A \wedge \ldots) = \operatorname{tr}(D_A F_A \wedge \ldots) = 0$

- further, if A' another conn., the difference of these traces will be exact

- so $[\operatorname{tr} F_A \wedge \ldots \wedge F_A] \in H^{2k}(B, \mathbb{R})$