Informal deformation theory (Michele ♡)

- deforming smth like $F(\underline{x}) = \underline{a_0}\underline{x}^{d_0} + \underline{a_1}\underline{x}^{d_1} + \dots$
  → deforming coefficients is not a good idea, it might no longer be a curve
- infinitesimal family:
  $\Bbbk = \Bbbk(\mathbb{C})$, $A \cong \Bbbk[t_1, \dots, t_r]/\mathfrak{a}$, $\sqrt{\mathfrak{a}} = (t_1, \dots, t_n)$
  $X = \operatorname{Spec} B \hookrightarrow \mathbb{A}^n_{\Bbbk}$, $T = \operatorname{Spec} A$
  - family $(\underset{\text{scheme}}{\Xi}, \varphi)$ s.t.

$$\begin{tikzcd} X \cong \operatorname{Spec}\Bbbk \times_T \Xi \arrow[r] \arrow[d] & \Xi \arrow[d, "\text{flat}"] \\ \operatorname{Spec}\Bbbk \arrow[r] & T \end{tikzcd}$$

Lemma. $Z_0 \hookrightarrow Z$, $N \subseteq \mathcal{O}_Z$ nilpotent.
Z_0 affine implies Z affine.
→ so we can take $\Xi = \operatorname{Spec} \dots$

Rmk.
$$\begin{tikzcd} \Bbbk \otimes_A A_{\mathfrak{a}} & \text{“}A_{\mathfrak{a}}\text{”} \arrow[l] & ? \\ \Bbbk \arrow[u] & A \arrow[l] \arrow[u] & \end{tikzcd}$$

- why flatness? it "preserves relations":

$$P^n \to P^m \to P \to P/I \to 0, \quad P^m \to I \to P, \quad 0 \to I \to 0$$

$$P_A^n \to P_A^m \to P_A \to P_A/I_A \to 0, \quad P_A^m \to I_A \to P_A, \quad 0 \to I_A \to 0$$

$\otimes_A \Bbbk$

Def. $X \hookrightarrow \mathbb{A}^n$ is a complete intersection if $I_X = (f_1, \dots, f_m)$ where $\operatorname{codim}_{\mathbb{A}^n}(X) = m$.

- we would like to deform these!
- start with 1^{st} order deformations: $A = \mathbb{k}[t]/(t^2) = \mathbb{k}[\varepsilon]$

$\Rightarrow I_X = (f_1, \dots, f_m)$

$\Rightarrow I_A = ((f_1 + \varepsilon g_1), \dots, (f_m + \varepsilon g_m), \cancel{\varepsilon g_{m+1}})$

forbidden by flatness!

- if $J_A = ((f_1 + \varepsilon \tilde{g}_1), \dots, (f_m + \varepsilon \tilde{g}_m))$ also a deformation - is it the same? when?

CLAIM. $I_A = J_A \Leftarrow g_i - \tilde{g}_i \in I_X$.

- we will not define a "change of coordinates" explicitly beyond this, but use the term:

$x_i \rightsquigarrow x_i' = x_i + \varphi_i(x_1, \dots, x_n)\varepsilon$

$\Rightarrow f_i(\underline{x}) + g_i(\underline{x})\varepsilon \rightsquigarrow f_i(\underline{x}) + \varepsilon\Big[\underbrace{\sum_j \frac{\partial f_i}{\partial x_j} \varphi_j(\underline{x})}_{\text{Jacobian ideal, } J} + g_i\Big]$

- now look, $\mathbb{k}[x_1, \dots, x_n] \downarrow$

$$\underbrace{P_{\mathbb{k}} \times \dots \times P_{\mathbb{k}}}_{n-d \text{ times}} \Big/ I_X^{\,n-d} = \mathcal{O}_X \oplus \dots \oplus \mathcal{O}_X / J$$

Rmk. for locally complete intersections, if X is smooth $\Rightarrow X$ is RIGID.
$\rightarrow$ because not all derivatives vanish, $J \ni 1$.

- now look at d-vector spaces (d = length) in $H^0(\xi, \mathcal{O}_\xi)$

		dim	length
$\to \mathbb{k}[\varepsilon]$	$= \langle 1, \varepsilon \rangle$	0	2
$\mathbb{k}[t]/(t^3)$	$= \langle 1, t, t^2 \rangle$	1	3
$\mathbb{k}[x,y]/(x^2, y^2, xy)$	$= \langle 1, x, y \rangle$	2	3

$\xi \hookrightarrow \mathbb{P}^{n \geq 3}$, $d \geq 4$.

- e.g. let's deform $y^2 - x^2(x-1)$

$\Rightarrow A[x,y] / (y^2 - x^2(x-1) + \varepsilon P(x,y))$

$\Rightarrow (I, J) = (y^2 - x^2(x-1), y, -3x^2 - 2x)$

chinese remainder $= (x, y)(x^2(x-1), y, -3x-2)$

$\Rightarrow \underset{\substack{\uparrow \\ \text{sing. locus}}}{P_{\mathbb{k}}/(x,y)} \times \underbrace{P_{\mathbb{k}}/(\quad)}_{=0} \simeq \mathbb{C}$

$= 0$, since $-3x-2$ & $x^2(x-1)$ aren't $= 0$ at the same time?

$\to$ so $P[x,y] = \lambda \in \mathbb{C}$, just a scale.

ask about obstructions $k[t]/(t^n) \to k[t]/(t^{n+1})$

+ Thm 16

$$P^{\ell} \xrightarrow{(r_{ij})} P^m \xrightarrow{(f_i)} P \to P/I \to 0 \quad \text{exact}$$

$$\xrightarrow{\text{deform}} P_A^{\ell} \xrightarrow{(r_{ij}')} P_A^m \xrightarrow{(f_i')} P_A \to P_A/I_A \to 0$$

s.t. $r_{ij}' = r_{ij} + \varepsilon s_{ij}$

$f_i' = f_i + \varepsilon g_i$

and exact

$$\Rightarrow r'f' = \underbrace{rf}_{=0 \text{ by exactness}} + \varepsilon(gr + fs) \overset{\text{?}}{=} 0$$

$$\Rightarrow gr = -fs$$

$$g : I \to P/I \in \operatorname{Hom}_P(I, P/I)$$

- note that $\operatorname{Hom}_{P/I}(I/I^2, P/I) \simeq N_{X/\mathbb{A}^n}$

- for smooth X, $0 \to T_X \to T_{\mathbb{A}^n}|_X \to N_{X/\mathbb{A}^n} \to 0$
- for nonsmooth, $\cdots \cdots \to T_X^1 \to 0$

Schlessinger sheaf.