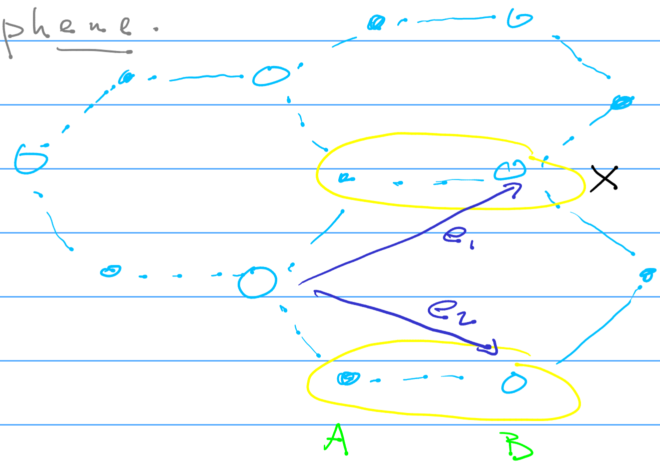
Porta.

graphene.



$\Lambda = \Lambda_A + \Lambda_B$

$\psi(x) = \begin{pmatrix}\psi_A(x)\\ \psi_B(x)\end{pmatrix}$

$$(H\psi)(x) = -t\begin{pmatrix}\psi_B(x)+\psi_B(x-e_1)+\psi_B(x-e_2)\\ \psi_A(x)+\psi_A(x+e_1)+\psi_A(x+e_1)\end{pmatrix}$$

$\hat H = \bigoplus_{\underline{k}\in\mathbb{T}^2}\hat H(\underline k)$, $\hat H(\underline k)$ called Bloch Hamiltonian

$\hat H(\underline k) = \begin{pmatrix}0 & -t\,\Omega(\underline k)\\ -t\,\overline{\Omega(\underline k)} & 0\end{pmatrix}$, $\Omega(\underline k) := 1+e^{-i\underline k\cdot e_1}+e^{-i\underline k\cdot e_2}$

- chemical pot. $\mu=0$
- Fermi sfc. $F_\mu := \{\underline k\in\mathbb T^2 \mid \hat H(\underline k)-\mu \text{ has } 0 \text{ eigenvalue}\}$
  $= \{k_F^+, k_F^-\}$

Transport: $Z_{12}=Z_{21}=0$, $Z_{11}=Z_{22}=1/4$ (universal, i.e. t-indep.)
- was found for nonint. systems
  - interacting ones?

Goal, construct $\langle\cdot\rangle_{\beta,L}$ with $H=H_0+\lambda V$, analyticity?

$$g(z,z') = \frac1\beta\sum_{k_0\in M_\beta} e^{-ik_0(z_0-z_0')}\int\frac{d^2\underline k}{(2\pi)^2}e^{-i\underline k(\underline z-\underline z')}\hat g(\underline k)$$

where $\hat g(k) = \frac{1}{-ik_0+\hat H(\underline k)}$ (writing $k:=(k_0,\underline k)$, $k_F^\omega := (0,\underline k_F^\omega)$)

$k = k'+k_F^\omega$, k' small $\Rightarrow \|\hat g(k)\|\simeq\frac{1}{\|k'\|}$ for $\|k'\|$ small

$\Rightarrow g\notin L^1$

IR multiscale analysis.

- $\hat{g}(K) = \hat{g}_{UV}(K) + \hat{g}_{IR}(K)$

where $\hat{g}_{UV}(k) = \hat{g}(k)\chi(\|k-k_F^\omega\|>\delta)$,

$\hat{g}_{IR}(k) = \hat{g}(k)\sum_{\omega=\pm}\chi(\|k-k^\omega_F\|<\delta)$

- decompose $\psi = \psi^{(UV)} + \psi^{(IR)}$

where $\psi^{(IR)\pm}(x) = \sum_\omega e^{\pm i k_F^\omega x}\psi^{(IR)\pm}_{\omega,x}$

$\|\hat{g}_\omega(\underset{k-k_F^\omega}{\underbrace{k'}})\| \simeq \frac{1}{\|k'\|}$ means $\|g_\omega(z-z')\| \simeq \frac{1}{\|z-z'\|^2}$

- pick $\gamma>1$, $h\in\mathbb{Z}_-$ and set cutoff

$f_h(k') = \chi(\gamma^{-h}\|k'\|) - \chi(\gamma^{-(h+1)}\|k'\|)$

so

$$\hat{g}_\omega(k') = \sum_{h=h_\beta}^{0}\underbrace{\hat{g}^{(h)}_\omega(k')}_{:=f_h(k')\,\hat{g}_\omega(k')}$$

- $k_0 = \frac{2\pi}{\beta}(n+1)$, $k' = (k_0, \underline{k}')$, $\|k'\| \geq \frac{\pi}{\beta} =: \gamma^{h_\beta}$

$$|g^{(h)}_\omega(z,z')| \leq \int d^3k\,|\hat{g}^{(h)}_\omega(k')| \leq C\gamma^{2h}$$

$$(1+\|z-z'\|^M\gamma^{hM})\,|g^{(h)}_\omega(z,z')| \leq C_M\gamma^{2h}$$

$$\Rightarrow \|g^{(h)}_\omega(z,z')\| \leq \frac{C_M\gamma^{2h}}{1+(\gamma^h\|z-z'\|)^M}$$

- write $\psi^{(\leq 0)}_\omega = \sum_{h=h_\beta}^{0}\psi^{(h)}_\omega$

$\rightsquigarrow V^{(h)}(\psi^{(\leq h)}) = \sum_P \int dz\, W_P^{(h)}(z)\,\psi^{(\leq h)}_z(P)$

- powers-counting. $\sum_{s,s'}\lambda\int dx\,\psi^{(\leq 0)+}_{x,s}\psi^{(\leq 0)-}_{x,s}\psi^{(\leq 0)+}_{x,s'}\psi^{(\leq 0)-}_{x,s'}$

becomes, by writing $\psi^{(\leq 0)} = \psi^{(\leq -1)} + \psi^{(0)}$:

$$\longrightarrow \sum_{s,s'}\lambda\int dx\,\psi^{(\leq -1)+}_{x,s}\psi^{(\leq -1)-}_{x,s}\psi^{(\leq -1)+}_{x,s'}\psi^{(\leq -1)-}_{x,s'} + \text{h.o.t.}$$

- $g^{(\leq -1)} = \gamma^{-2}\, \tilde{g}^{(\leq 0)}(\gamma^{-1}\cdot) \quad (\gamma > 1)$
  and $\psi^{(\leq -1)} = \gamma^{-1}\, \tilde{\psi}^{(\leq 0)}_{\gamma^{-1}\cdot}$
- so our sum becomes

$$\sum_{s,s'} \underbrace{\lambda\, \gamma^{-4}\, \gamma^{3}}_{\lambda \gamma^{-1}} \int dx \left(\tilde{\psi}^{(\leq 0)}\right)^4$$

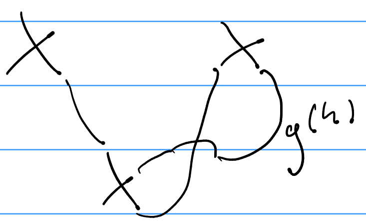
$\lambda \gamma^{-1} \to$ RG - irrelevant

$$\Rightarrow \| g^{(h)}_{\omega}(z,z') \| \leq \frac{C_n \gamma^{2h}}{1 + (\gamma^{h}|z-z'|)^n}$$

- ψ^2: RG - relevant

$$\mathbb{E}_{\leq 0}\left(e^{-V^{(0)}(\psi^{(\leq 0)})}\right) = e^{\sum_{T \geq h}^{-1} E_T}\, \mathbb{E}_{\leq h}\left(e^{-V^{(h)}(\psi^{(\leq h)})}\right)$$

where $V^{(h)}(\psi^{(\leq h)}) = \sum_P \int dz\, W^{(h)}_P(z)\, \psi^{(\leq h)}_z(P)$



$$\| g^{(h)} \|_1^{s-1}\, \| g^{(h)} \|_\infty^{s+1-\frac{|P|}{2}} \leq \gamma^{-h(s-1)}\, \gamma^{2h(s+1-\frac{|P|}{2})}$$

$$\gamma^{h(-|P|+3+s)} \leq \gamma^{h(-|P|+4)} \quad (*)$$

- however $>\!\!-\bullet\!-\!\bullet\!-\!\bullet\!-\cdots\!-$, bad! isolate $|P|=2$
  (γ^{-h} for each)

__Idea:__ $V^{(h)}(\psi^{(\leq h)}) = \underbrace{\mathcal{L}\, V^{(h)}(\psi^{(\leq h)})}_{|P|=2} + \underbrace{\mathcal{R}\, V^{(h)}(\psi^{(\leq h)})}_{|P|\geq 4}$

$$\mathcal{L}\, V^{(h)}(\psi^{(\leq h)}) = \left(\psi^{(\leq h)+},\, W^{(h)}_2\, \psi^{(\leq h)-}\right)$$

- $\mathcal{L}V^{(h)}$ in $P_{\leq h}(d\psi^{(\leq h)})$

$$\int P_{\le h}(d\psi^{(\le h)})\, e^{-(\psi^{(\le h)}, W_2^{(h)}\psi^{(\le h)-})}(\cdots)$$

$$= e^{t_h}\int \tilde{P}_{\le h}(d\psi^{(\le h)})(\cdots)$$

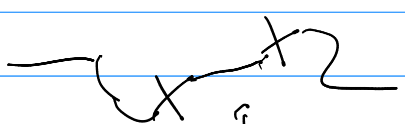
where $\tilde{P}_{\le h}(d\psi^{(\le h)}) = D\psi^{(\le h)} \exp\{-(\psi^{(\le h)+}, (g^{(\le h)-1} + W_2^{(h)})\psi^{(\le h)-})\}$

$$\frac{1}{H+V-z} = \frac{1}{H-z} - \frac{1}{H-z}V\frac{1}{H-z} + \cdots$$

$$-\hat{g}_\omega^{(\le h)}(k) = \chi_{\le h}\begin{pmatrix} ik_0 & v_h(ik_1'+\omega k_2') \\ v_h(-ik_1'+\omega k_2') & ik_0 \end{pmatrix}^{-1}\frac{1}{Z_h}$$

where v_h, Z_h given iteratively (from interaction)
and $Z_0 = 1$, $v_0 = \frac{3}{2}t$

$$-\hat{W}_2^{(h)}(k')\Big|_{k'=0} = 0,\quad k'\cdot\nabla_{k'}\hat{W}_2^{(h)}(k')\Big|_{k'=0}$$
$$= \begin{pmatrix} z_{h,0}\, ik_0 & z_{h,1}(+ik_1'+\omega k_2') \\ z_{h,1}(-ik_1'+\omega k_2') & z_{h,0}\, ik_0 \end{pmatrix}$$



$g = \frac{1}{k}$

odd
\# of g's

$$Z_{h-1} = Z_h + z_{0,h}$$
$$Z_{h-1}v_{h-1} = Z_h v_h + z_{1,h}$$
$$\begin{pmatrix} Z_{h-1} \\ v_{h-1} \end{pmatrix} = \begin{pmatrix} Z_h \\ v_h \end{pmatrix} + \hat{\beta}_h \longleftarrow$$ beta function from RG-flow

$$|\hat{\beta}_h| \le C|\lambda|\gamma^h \Rightarrow |Z_h - 1| \le \sum_{T\ge h}^{0} |\lambda|\gamma^T = O(|\lambda|)$$
$$\Rightarrow \|W_P^{(h)}\|_1 \le |\lambda|^{(1-\theta)(\frac{|P|}{2}-1)}\gamma^{h(-|P|+4)}$$
$$\|W_P^{(h)} - \cancel{X}\| \le |\lambda|^{2(1-\theta')}$$

- $|\lambda| \ll 6$, $\langle \cdot \rangle_{\beta,L}$ is analytic in β, L (result from 2010.)

Applications to transport.

- $$\sigma_{ij} = \lim_{\eta \searrow 0^+} \lim_{\beta, L \to \infty} \frac{i}{L^2} \int_{-\infty}^{0} dt\, e^{\eta t} \langle [J_i(t), X_j] \rangle_{\beta,L}$$

in 2^{nd} quantization, i.e. $X = \sum_x x\, a_x^+ a_x$, $J = i[H, x]$

- one can prove universality:

$\sigma_{ij} = \sigma_{ij}\big|_{\lambda=0}$ → for graphene-like models (w/ Dirac cones)
→ for gapped models

→ graphene, $\sigma_{11} = 1/4$

→ gapped H, $\sigma_{12} \in \mathbb{Z}/2\pi$, $\sigma_{11} = 0$

Wick rotation

$$\int_{-\infty}^{0} dt\, e^{\eta t} \langle [J_i(t), X_j] \rangle$$

$J_i(t) = e^{iHt} J_i e^{-iHt}$

$$= \frac{1}{\eta} \left(\int_{-\infty}^{0} dt\, e^{\eta t} \langle [J_i(t), J_j] \rangle_{\beta,L} - \langle [J_i, X_j] \rangle_{\beta,L} \right)$$

Claim: $$\lim_{\beta, L \to \infty} \frac{1}{L^2} \int_{-\infty}^{0} dt\, e^{\eta t} \langle [J_i(t), J_j] \rangle_{\beta,L}$$

$$= \lim_{\beta, L \to \infty} \frac{-i}{L^2} \int_{-\beta/2}^{+\beta/2} dt\, e^{-i\eta t} \langle T\, J_i(-it) J_j \rangle_{\beta L}$$

- idea: suppose $\eta = \frac{2\pi}{\beta} \cdot n$, $n \in \mathbb{Z}$ ("bosonic Matsubara")

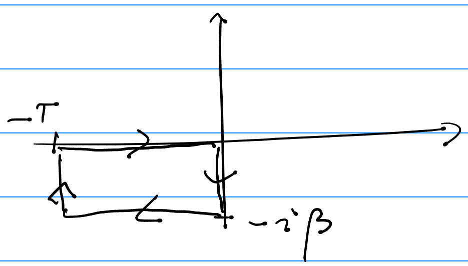
- write $\int_{-T}^{0} dt\, e^{\eta t} \langle [J_i(t), J_j] \rangle_{\beta L}$

$$= \int_{-T}^{0} dt\, e^{\eta t} \langle J_i(t) J_j \rangle_{\beta L} - \int_{-T}^{0} dt\, e^{\eta t} \langle J_j J_i(t) \rangle$$

- $\mathrm{tr}\, e^{-\beta H} J_j J_i(t) = \mathrm{Tr}\, e^{-\beta H} \underbrace{\left(e^{\beta H} J_i(t) e^{-\beta H} \right)}_{J_i(t - i\beta)} J_j$

- so, we get

$$\int_{-T}^{0} dt\, e^{\eta t} \langle J_i(t) J_j \rangle_{\beta L} - \int_{-T}^{0} dt\, \underbrace{e^{\eta t}}_{\simeq e^{\eta(t - i\beta)}} \langle J_i(t - i\beta) J_j \rangle_{\beta L}$$



$\oint = 0$

$$= -\int_{0}^{-i\beta} dt\, e^{\eta t} \langle J_i(t), J_j \rangle_{\beta L} + \cancelto{0}{\text{error}_T}$$

$$= i \int_{0}^{\beta} dt\, e^{-it\eta} \langle J_i(-it), J_j \rangle_{\beta L}$$

$$= i \int_{-\beta/2}^{\beta/2} dt\, e^{-it\eta} \langle T J_i(-it), J_j \rangle_{\beta L}$$

- so $\sigma_{ij} = \lim_{\eta \to 0^+} \lim_{\beta L \to \infty} \frac{1}{\eta} \frac{i}{L^2} \int dt \left(e^{-it\eta} - 1 \right) \langle T J_i(-it) J_j \rangle_{\beta L}$

"=" $\partial_\eta K_{ij}(\eta) \big|_{\eta = 0}$

where $K_{ii}(-\eta) = K_{ii}(\eta)$ if $K \in C^1$ $\sigma_{ii} = 0$

$$K_{ij}(\eta) = K^{\text{rel.}}_{ij}(\eta) + K^{\text{even}}_{ij}(\eta) \to \partial_\eta(\ldots)\big|_{\eta=0} = 0$$

↳ gives $1/4$