Тихомиров

- $\begin{matrix} P \\ \downarrow \\ B \end{matrix}$, g, assume $\Omega^1_{L^2_2}(\mathrm{ad}P) \xrightarrow{P_{+,g}\circ\nabla_A} \Omega^2_{+,g}(\mathrm{ad}P)_{L^2_1}$ surjects (*)
- $A \in \mathcal{A}_2^*$
- $D_A = (\nabla_A^*, P_{+,g}\circ\nabla_A) : \Omega^1 \to \Omega^0 \oplus \Omega^2_{+,g}$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & \ker D_{A,g} \arrow[d] \arrow[r, phantom, "="] & \ker P_{+,g}\circ\nabla_A|_{\ker\nabla_A^{*,g}} \arrow[d] & \\
\begin{matrix}\ker\nabla_A \\ \| \\ 0\end{matrix} \arrow[r] & \Omega^0 \arrow[r, "\nabla_A"] \arrow[d, "S"', "\Delta_A=\nabla_A^*\nabla_A"] & \Omega^1 \arrow[r] \arrow[d, "D_{A,g}"] & \ker\nabla_A^{*,g} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^0 \arrow[r] \arrow[d] & \Omega^0\oplus\Omega^2_{+,g} \arrow[r, "pr_2"] \arrow[d] & \Omega^2_+ \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\Leftarrow$ if A irred, since $\ker\Delta_A = \{0\} \iff \mathrm{Stab}_A = \mathbb{Z}_2$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \mathbb{R} \arrow[d] \arrow[dl, equal] & \ker D_{A,g} \arrow[d] \arrow[r, phantom, "="] & \ker P_{+,g}\circ\nabla_A|_{\ker\nabla_A^{*,g}} \arrow[d] & \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Omega^0 \arrow[r, "\nabla_A"] \arrow[d, "\Delta_A=\nabla_A^*\nabla_A"] & \Omega^1 \arrow[r] \arrow[d, "D_{A,g}"] & \ker\nabla_A^{*,g} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \Omega^0 \arrow[r] \arrow[d] & \Omega^0\oplus\Omega^2_{+,g} \arrow[r, "pr_2"] \arrow[d] & \Omega^2_+ \arrow[r] \arrow[d] & 0 \\
 & & \mathbb{R} \arrow[r, equal] \arrow[d] & \mathbb{R} \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

$\Leftarrow$ for A red., $\ker\Delta_A = \mathbb{R}$

- recall: ind $D_{A,g} = 8c_2(P) - 3(1 - b_1 + b_2^+)$

- $\mathcal{C}$ param sp. of metrics (Banach mfd)
- $\underline{\Omega}^2 := \Omega^2(\mathrm{ad}\,P)_{L^2_1} \times \mathcal{C}$

$$\begin{array}{c} \Omega^2(\mathrm{ad}\,P)_{L^2_1} \times \mathcal{C} \\ \uparrow i \\ \bigcup_{g\in\mathcal{C}} \{ \Omega^2_{+,g}(\mathrm{ad}\,P)_{L^2_1}, g \} =: \Omega^2_+ \end{array}$$

$$\begin{tikzcd} & \Omega^2 & \underline{\Omega}^2 \arrow[l, "pr_1"'] \\ \mathcal{A}^*_2 \times \mathcal{C} \arrow[ur] \arrow[r, "\phi"] & \Omega^2_+ \arrow[ur, hook, "i"'] & \end{tikzcd}$$

$$(A, g) \longmapsto P_{+,g} F_A$$

$\widetilde{\mathcal{M}}^*(P, \mathcal{C}) := \phi^{-1}(0\text{-sect'n}) = \bar{\phi}^{-1}(0)$

Uhlenbeck:

1) $\forall (A, g) \in \widetilde{\mathcal{M}}^*(P, \mathcal{C})$, $(\phi_*)_{(A,g)} : \Omega^1_{L^2_2} \oplus T_g\mathcal{C} \twoheadrightarrow \Omega^2_{+,g}$ surjects
   - hence, $\widetilde{\mathcal{M}}^*(P, \mathcal{C})$ is smooth Banach mfd

ii) for generic $g \in \mathcal{C}$, $P_{+,g} \circ \nabla_A$ surjects, i.e. $(*)$ holds

iii) $\bar{\pi} : \mathcal{M}^*(P, \mathcal{C}) \to \mathcal{C}$ is a Fredholm map and $(\bar{\pi}_*)$ has at any pt $y = ([A], g)$ index $= \mathrm{ind}\, D_{A,g}$, and $\mathcal{M}^*(P, \mathcal{C})$ is sm. Ban. mfd.

$$\begin{tikzcd} \widetilde{\mathcal{M}}^*(P, \mathcal{C}) \arrow[r, hook] \arrow[d, "/\mathcal{G}_3"] & \mathcal{A}^*_2 \times \mathcal{C} \arrow[d, "/\mathcal{G}_3"] \\ y = ([A], g) \in \mathcal{M}^*(P, \mathcal{C}) \arrow[r, hook] \arrow[d, "\bar{\pi}"] & \mathcal{B}^*_2 \times \mathcal{C} \arrow[d, "pr_2"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

A reducible

$\Updownarrow$

$\exists$ S^1-pbdl $Q \to B$ with $c_1(Q) = x \in H^2(B, \mathbb{Z})$

s.t. $P = Q \times_{S^1} SU(2)$

$\Updownarrow$

cpx. dual

$\xi = P \times_{SU(2)} \mathbb{C}^2 = L \oplus L^\vee$ where $L = Q \times_{S^1} \mathbb{C}^1$ line bdl

with $c_1(L) = c_1(Q) = x$

$-c_2(\xi) = c_2(P) = -(x, x)$

Prop Fix $g \in \mathcal{C}$. $A \in \mathcal{A}_2$ is reducible ASD-conn. with $P = Q \times_{S^1} SU(2)$ where $x = c_1(Q)$, $(x^2) = -c_2(P)$

.

$\Updownarrow$

A is obtained from a conn. A_Q on S^1-pbdl $\begin{matrix} Q \\ \downarrow \\ B \end{matrix}$ with $c_1(Q) = x$,

$(x^2) = -c_2(P)$ s.t. $h(x) \in \mathcal{H}^-_g$

$[x] = h(x) \oplus d\alpha \oplus \delta\beta$

$\{\alpha \in \Omega^2_{dR}(B, \mathbb{R}) \mid \Delta_g \alpha = 0, *\alpha = -\alpha\}$

- how is it obtained?

$\nabla_{A_Q} : \Gamma(L) \longrightarrow \Gamma(L \otimes T^*B)$

$\rightsquigarrow \nabla_A : \Gamma(L \oplus L^\vee) \longrightarrow \Gamma((L \oplus L^\vee) \otimes T^* B)$,

$\nabla_A = \nabla_{A_Q} \oplus \nabla_{A_Q}^\vee$,

$d\langle s, t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A^\vee t\rangle$

Pf. $\Rightarrow$ Show $h(x) \in \mathbb{H}_{-,g}$. Since $\xi = L \oplus L^{\vee}$,

$F_A = \begin{pmatrix} F_{A_Q} & 0 \\ 0 & -F_{A_Q} \end{pmatrix}$ where F_{A_Q} is ASD,

so F_A is also. But $x = c_1(Q) = \frac{i}{2\pi} F_{A_Q}$.

Note Bianchi $0 = \nabla_{A_Q} F_{A_Q} = d F_{A_Q} + [\omega_{A_Q}, F_{A_Q}]$

$= d F_{A_Q}$ since $\mathcal{L}ie(S^1) = \mathbb{R}$

$\Rightarrow \delta F_{A_Q} = 0$.

Prop Take $x \in H^2(B, \mathbb{Z})$ w/ $(x^2) = -c_2(P)$.

Then $\mathbb{R}x \subset H^2(B,\mathbb{R})$. Consider

$$f : \mathcal{C} \longrightarrow Gr(b_2^-, H^2(B,\mathbb{R}))$$

$$g \longmapsto \mathbb{H}_{-,g} \subset \mathbb{H}_g$$

$$N_x := \{ V^{b_2^-} \in Gr \mid V^{b_2^-} \ni \mathbb{R}x \} \subset Gr(b_2^-, H^2(B,\mathbb{R}))$$

is $\cong$ $Gr(b_2^- - 1, b_2 - 1)$, $\parallel$ $Gr(b_2^-, b_2)$

So $\operatorname{codim}_{Gr} N_x = b_2^+ \leftarrow b_2(\underbrace{b_2 - b_2^-}_{b_2^+}) - (b_2 - 1)(\underbrace{b_2 - b_2^-}_{b_2})$

Outcome: if $b_2^+ > 0$, then for

$g \in \mathcal{C} \setminus f^{-1}\left(\bigcup_{(x^2) = -c_2} N_x \right)$ ($\leftarrow$ dense in $\mathcal{C}$)

we have $\mathcal{M}^*(A,g) = \mathcal{M}(A,g)$.