Arbarello.

$X \hookrightarrow \mathbb{P}^r$, $N \in \mathbb{Z}_+$, $p(t) \in \mathbb{Q}[t]$

$\longrightarrow$ $\exists\, n_0 = n_0(X, N, p(t))$ Hilb. polyn.

s.t. $\forall \mathcal{F}$ sheaf on X, $p_{\mathcal{F}}(t) = p(t)$

& $0 \to \mathcal{G} \to \mathcal{O}_X^N \to \mathcal{F} \to 0$

- then:

1) $H^p(\mathcal{F}(n)) = H^p(\mathcal{G}(n)) = 0$, $p \geq n_0$
2) $\mathcal{G}(n)$ generated by global sections
3) $H^0(\mathcal{G}(n_0)) \otimes H^0(\mathcal{O}_X(s)) \longrightarrow H^0(\mathcal{G}(n_0+s)) \to 0$

$$\mathrm{Quot}_{X,N,P} = \{ \mathcal{F} \text{ on } X,\ \mathcal{O}_X^N \to \mathcal{F} \to 0,\ p_{\mathcal{F}}(t) = p(t) \}/\sim$$

$$\xrightarrow{\ \varphi\ } Gr(v - R, v)$$

$$[\mathcal{F}] \underset{\text{injection}}{\longmapsto} H^0(X, \mathcal{G}(n_0)) \subset V$$

- where: $0 \to H^0(\mathcal{G}(n_0)) \to H^0(\underbrace{\mathcal{O}_X^N(n_0)}_{V,\ \dim V = v}) \to H^0(\underbrace{\mathcal{F}(n_0)}_{p(n_0) = R}) \to 0$

- stopped taking notes for a bit...

Flatness.

- for comm. rings, M an A-mod is flat $\iff$ $- \otimes_A M$ exact
- schemes: $\mathcal{F}$

$$\begin{array}{c} \mathcal{F} \\ \downarrow \\ Z = X \times S \xrightarrow{\ \xi\ } S \end{array} \Rightarrow \mathcal{F} \text{ flat over } S$$

$\iff \forall\, z \in Z$ s.t. $\mathcal{F}_z$ is flat $\mathcal{O}_{\xi(z)}$-module

- $\forall\, U \subset Z$ open, $V \subset S$ open, $\xi(U) \subseteq V$,
$\mathcal{F}(U)$ is $\mathcal{O}_S(V)$-flat

Def. Flat family of sheaves on X parametrized by S is
$$\begin{tikzcd} \mathcal{F} \arrow[d] \\ X\times S \arrow[r] & S \end{tikzcd}$$
flat over S, $\mathcal{F}|_s = \mathcal{F}|_{X\times\{s\}}$

Thm A) 1) $\mathcal{F}$ as above flat $\Rightarrow$ $p_{\mathcal{F}_s}(t)$ locally constant
2) S reduced $\Leftarrow$ – " –

Thm B) 1) $\mathcal{F}$ as above flat $\Leftrightarrow$ $\xi_* \mathcal{F}(n)$ locally free, $n\gg 0$

$$\begin{tikzcd} \mathcal{F} \arrow[d] \\ \mathbb{P}^r\times S \arrow[r, "\xi"] & S \end{tikzcd}$$
$R^p\xi_*\mathcal{F}(U) = H^p(\mathbb{P}^r\times U, \mathcal{F}|_U)$ on $S\supset U$ open

$$\begin{tikzcd} (R^p\xi_*\mathcal{F})_s \doteq \lim\limits_{s\ni U} R^p\xi_*\mathcal{F}(U) = \lim\limits_{s\ni U} H^p(\mathbb{P}^r\times U, \mathcal{F}|_U) \arrow[d] \\ (R^p\xi_*\mathcal{F})_{(s)} \arrow[r, "\varphi_s"] & H^p(\mathbb{P}^r\times\{s\}, \mathcal{F}|_{\{s\}}) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{P}^r\times\{s\} \arrow[r, "g=1\times f"] \arrow[d, "\eta"] & \mathbb{P}^r\times S \arrow[d, "\xi"] \\ \{s\} \arrow[r, hook, "f"] & S \end{tikzcd}$$
$\Rightarrow f^* R^p\xi_*\mathcal{F} = R^p\eta_* g^*\mathcal{F}$

$\to$ take $\{s\} \leftrightarrow T$ more generally

Thm (Grothendieck) $\begin{tikzcd} \mathcal{F} \arrow[d] \\ \mathbb{P}^r\times S \arrow[r, "\xi"] & S \end{tikzcd}$ $\mathcal{F}$ flat over S.
Then $\exists$ a complex of locally free S-modules
$$K^0 \to K^1 \to K^2 \to \cdots$$
which computes $R^p\xi_*\mathcal{F}$ functorially on base ch:
$$f^*K^0 \to f^*K^1 \to f^*K^2 \to \cdots$$
meaning: $\mathcal{H}^p(f^*K^\bullet) = R^p\eta_* g^*\mathcal{F}$

- e.g. $f = id \Rightarrow \mathcal{H}^p(K^\bullet) = R^p\xi_*\mathcal{F}$
$T = \{s\} \Rightarrow \mathcal{H}^p(K^\bullet_{(s)}) = H^p(\mathbb{P}^r, \mathcal{F}_s)$

Universal family

$$\begin{tikzcd}
\mathcal{F}' \arrow[d] & \mathcal{F} := \mathcal{F}'|_{X \times Quot} \arrow[d] \\
X \times Gr \arrow[r, hook] & X \times Quot
\end{tikzcd}$$

$\xi_* \mathcal{F}(n)_s = H^0(X, \mathcal{F}_s(n))$ $\quad n \gg 0$

$P_{\mathcal{F}_s(n)}(t) = p(t+n)$

$\Rightarrow \xi_* \mathcal{F}(n)$ locally free $\Rightarrow \mathcal{F}$ flat over Quot.