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Morse theory

Def. $f : M \to \mathbb{R}$ ($\dim_{\mathbb{R}} M = m$) is Morse if $\mathrm{Hess}(f)$ is nondeg. at all crit. pts.

Lemma (Morse) In the nbhd of a crit. pt. x_c $\exists$ coordinates $\{x\}$ s.t.

$$f(x) - f(x_c) = -x_1^2 \cdots - x_{\lambda(p)}^2 + x_{\lambda(p)+1}^2 \cdots + x_n^2$$

where $\lambda(p) = \#$ neg. eigenvalues of $\mathrm{Hess}(f)|_{x_c}$

Cor. i) crit. pts are isolated
ii) M cpt $\Rightarrow$ $\#$ crit. pts finite

Exercise: prove corollary.

- topology: Poincaré polynomial $P_t(M) = \sum_p b_p t^p$,
$P_{-1}(M) = \chi(M)$

- let $M_p := \#$ cr. pts w/ p neg. eigenvalues (of Hessian)

$\to$ then: i) $\chi(M) = \sum_p (-)^p M_p$

ii) $M_p \geq b_p$ (weak Morse ineq.)

iii) $\sum_p M_p t^p - P_t(M) = (1+t) \sum_p Q_p t^p$, $Q_p \geq 0$ (strong Morse ineq.)

- back to SQM.

- $\varphi : S^1 \longrightarrow M$, susy fixed pts $\dot{\varphi} = 0$
  $\varphi_c / \nabla h(\varphi_c) = 0$

$$\delta \psi^I = \varepsilon\left(\dot{\varphi}^I + \lambda g^{IJ} \partial_J h\right) = 0$$

$$0 \leq \int_{S^1} dt \, \frac{1}{2} \left| \frac{d\varphi^I}{dt} \pm \lambda g^{IJ} \partial_J h \right|^2$$

$$= \int_{S^1} dt \, \frac{1}{2} g_{IJ} \dot{\varphi}^I \dot{\varphi}^J + \frac{\lambda^2}{2} g^{IJ} \partial_I h \partial_J h$$

$$\underbrace{\pm \lambda \int_{S^1} \underbrace{\partial_I h(\varphi) \, \dot{\varphi}^I \, dt}_{\frac{dh}{dt}}}_{\text{vanishes for } S^1 \text{ but for } \Sigma \text{ w } \partial\Sigma \neq \phi \text{ doesn't}}$$

- for $\varphi : \mathbb{R} \to M$,

$$\lambda \int_{\mathbb{R}} dh = S(+\infty) - S(-\infty)$$
$$= \lambda \left| h(\varphi_c(+\infty)) - h(\varphi_c(-\infty)) \right|$$

- so $S_{SQM} \geq \lambda \left| h(\varphi_c(+\infty)) - h(\varphi_c(-\infty)) \right|$

$$\dot{\varphi}^I \pm \lambda g^{IJ} \partial_J h(\varphi) = 0$$

(anti) instantons or gradient flow lines

$\to e^{-\lambda}$, nonperturbative wrt $\mathcal{O}\left(\frac{1}{\lambda}\right)$

- $Q = e^{-\lambda h} d\, e^{\lambda h}$, $Q^{+} = e^{\lambda h} d^{+} e^{-\lambda h}$

$$H = \frac{1}{2}\{Q, Q^{+}\} = \frac{1}{2}\Delta + \frac{1}{2}\lambda \nabla_I \partial_J h\, [\bar{\psi}^I, \psi^J]$$
$$+ \frac{\lambda^2}{2} g^{IJ} \partial_I h\, \partial_J h$$

- $\lambda \to +\infty$:

$$H(x_c) = \underbrace{\sum_{I=1}^{m} \frac{1}{2} p_I^2 + \frac{1}{2}\lambda^2 c_I^2 (x^I)^2 + \frac{1}{2}\lambda c_I [\bar{\psi}^I, \psi_I]}_{H_0} + O\left(\frac{1}{\lambda}\right)$$

- at i-th critical pt:

$$|a_i\rangle = e^{-\lambda \sum_I |c_I| x_I^2} \prod_{J: c_J < 0} \bar{\psi}^J |0\rangle$$ ← N.B. this means that at x_c we get $\mu(x_c)$-forms!

- $\# J =: \mu(p)$ Morse index

$$|a_i\rangle \in \Omega^{\mu(p_i)} \otimes \mathbb{C} =: X_{\mu_i}$$

$\underline{Q|a_i\rangle \neq 0}$, $0 \neq \langle a_j\, Q\, a_i\rangle$

no longer a ground state!

$$\langle a_j\, Q\, a_i\rangle = \int_M \bar{a}_j \wedge * \underbrace{(d + dh \wedge) a_i}_{\mu_i + 1}$$

$\bar{a}_j$: μ_j

$\Rightarrow \mu_j - \mu_i = 1$

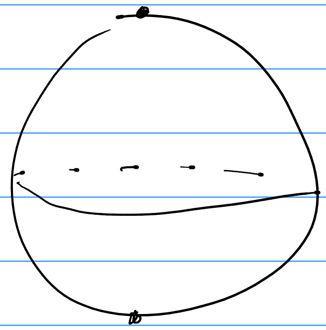
$\rightarrow \langle a_i Q a_j\rangle \neq 0$ gives $\mu_i - \mu_j = 1$

- introduce differential $\delta : X^{\mu} \to X^{\mu+1}$

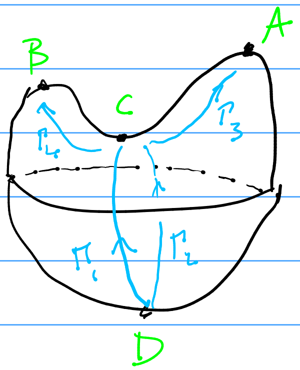
$$n(a_i, a_j) = \sum_{\Gamma} n_\Gamma, \quad n_\Gamma = \pm 1$$ ← Γ = path connecting cr.pts differing by 1 dimension, n_Γ is direction of path

$$Q|a_i\rangle = \sum_{a_j} n(a_i, a_j) |a_j\rangle$$

$$\langle a_i \, Q \, a_j \rangle = \sum_{\Gamma : i \to j} n_\Gamma \, e^{-\lambda |h(i) - h(j)|}$$



$$h^0(S^2) = 1$$
$$h^2(S^2) = 1$$
$$h^1(S^2) = 0$$



$$Q|D\rangle = n^{CD}_{\Gamma_1} + n^{CD}_{\Gamma_2} = 0$$

$$Q|C\rangle = e^{-\lambda} n^{AC}_{\Gamma_3} |A\rangle + e^{-\lambda} n^{BC}_{\Gamma_4} |B\rangle$$
$$= e^{-\lambda} \left(|A\rangle - |B\rangle \right)$$

$$Q|A\rangle = Q|B\rangle = 0$$

$$Q^\dagger |A\rangle \sim |C\rangle, \quad Q^\dagger |B\rangle \sim -|C\rangle$$
$$Q^\dagger \left(|A\rangle + |B\rangle \right) = 0$$

→ so in fact we get, again, 2 g.s.

→ $|D\rangle$ same as before, but the two new maxima are actually only admissable in superposition.

$$\varphi^I(t) = \gamma^I(t) + \xi^I$$
$$\hookrightarrow \dot{\gamma}^I - \lambda g^{IJ}\partial_J h(\gamma) = 0$$

$$S = \lambda |h(x_i) - h(x_j)| + \int_{\mathbb{R}} \left(\tfrac{1}{2} |D_- \xi|^2 - D_- \bar{\psi}^I \psi_I \right) dt$$

$$D_- \xi^I := D_t \xi^I - \lambda^{IJ} D_J \partial_K h \, \xi^K$$

- so in the path integral we will have a situation $\int (\ker D_-)(\ker D_-)^+$
- we also note $\operatorname{Ind} D_- = \dim \ker D_- - \dim \operatorname{coker} D_-$
$= \mu_i - \mu_j = +1$
- also assume $\dim \ker D_+ = 0$ (genericity assumption)

$$\int dt_0 \prod_I d\bar{\psi}_0^I \prod_{n\neq 0} d\xi_n^I \, d\psi_n^I \, d\bar{\psi}_n^I \, \exp^{-S}$$

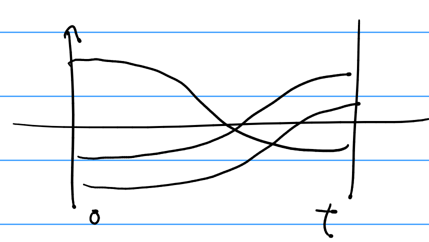
$$Q(\partial_I h) = D_J \partial_I h \, \bar{\psi}^I \psi^J = 0$$

$$\frac{\langle \partial_I h \bar{\psi}^I \rangle}{|h(\varphi_i) - h(\varphi_j)|} = \frac{1}{|\cdots|} \int dt_0 \prod_I d\psi_0^I \, \partial_I h \bar{\psi}_0^I \frac{\det{}' D_-}{\sqrt{\det D_-^+ D_-}} e^{-\lambda |h(\varphi_i) - h(\varphi_j)|}$$
$$= \frac{h(\varphi_i) - h(\varphi_j)}{|-\,\text{''}\,-|} = \pm 1$$

$$H(h) : T_x M \circlearrowleft, \quad v^I \to g^{IJ} D_J \partial_K h \, v^K$$

$$D_- = D_t - H(h)$$
- for $f_{I_\mp}(t) := e_I(t) \exp\left[\pm \int_0^t \lambda_I(t')\, dt'\right]$

we get $D_- f_I = 0$



Def. Morse func. is perfect if ct. pts have Morse index differing by at least 2
$\Longleftrightarrow$ cbdry op $\delta \equiv 0$.

- example of p.M.f: moment map of circle action
$i_v \omega = df$

Rmk. Instead of looking at $Q \rightsquigarrow$ deRham, getting Witten index $= \chi(M)$, we can look at Atiyah-Singer index thm by putting $\bar{\psi}^I = \psi^I$.

$$S(\varphi, \psi) = \frac{1}{2} g_{IJ} \dot{\varphi}^I \dot{\varphi}^J + \psi^I D_t \psi^J g_{IJ}$$

Exercise: Prove that path int. of this action gives $\mathrm{Ind}(\not{D}) = \int_M \hat{A}(M)$
where $\hat{A}(M) = \prod_i \frac{x_i}{\mathrm{sh}\, x_i}$

- N.B. $\langle a_j Q a_i \rangle \underset{\lambda \to \infty}{\sim} \frac{1}{h(\varphi_i) - h(\varphi_j) + O(1/\lambda)} \langle a_j \underbrace{[Q, h]}_{\partial_I h \bar{\psi}^I} a_i \rangle$

and this localises

$\rightarrow$ this is why we computed $\langle \partial_I h \bar{\psi}^I \rangle$, it is precisely the overlap! (besides being susy inv)