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- conventions. M smooth mfd, $TM = \bigcup TM_x$ tgt bdl $\ni v : C^\infty(M) \to \mathbb{R}$, $f \mapsto v(f)$ satisfying Leibniz

- we are given X top. sp (paracpt, e.g. cpt), B top. sp., G Lie gp (structure gp of pbdl)

Def. A top. sp. P w surj cont. map $\pi : P \to B$, and a cont. map $P \times G \to P$ called right G-action, $(p,g) \mapsto pg$ s.t. $(pg_1)g_2 = p(g_1 g_2)$ satisfying the following properties:

i) π is G-invariant, i.e. $\pi(pg) = \pi(p)$ $\forall g \in G$
ii) G-action is free: $pg = p \Rightarrow g = id_G$

Further, $\exists$ open cover $\{U_\alpha\}_{\alpha \in A}$ of B and

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"', "\text{homeo}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & U_\alpha \times G \arrow[d, "pr_1"] \\ U_\alpha \arrow[r, equal] & U_\alpha \end{tikzcd}$$

$\to$ transitivity follows from $(x,g)h := (x, gh)$ on $U_\alpha \times G$.

G-equivariant homeomorphisms

- B, P smooth mfds, π should be smooth submersion so $\pi^{-1}(b)$ will be smooth mfds $\forall b \in B$
- recall, $f : X \to Y$ is submersion if smooth and $f_*|_{TX_x} : TX_x \to TY_{f(x)}$ is surj $\forall x \in X$, where $f_* : TX \xrightarrow{df} TY$, $\tau \to f_*(\tau)$, $f_*(\tau)(\varphi) = \tau(\varphi \circ f)$.

Examples

- B sm. mfd, $\dim B = n$,

$$\begin{array}{l} p \in P = \{ (b, \underbrace{v_1, \dots, v_n}_{=p}) \mid b \in B,\ v_1, \dots, v_n \text{ basis of } TB_b \} \\ \downarrow \\ b \in B \end{array}$$

- $G = GL(n,\mathbb{R})$-action as $pA = (b, w_1, \dots, w_n)$
  where $w_i = v_j A_{ji}$
- P is called the frame bdl

- $\mathbb{C}P^n = \mathbb{P}(\underset{(z_1, \dots, z_{n+1})}{V^{n+1}}) = B$,

$$\begin{tikzcd} V^{n+1} \arrow[dr, "\text{taut.}"'] & \underset{\text{restr.}}{\supset} S^{2n+1} = \{ (z_1, \dots, z_{n+1}) \mid \sum |z_i|^2 = 1 \} \arrow[d, "\pi"] \\ & \mathbb{C}P^n \end{tikzcd}$$

$G = U(1) = \{ z \mapsto z e^{i\varphi} \mid \varphi \in [0, 2\pi) \}$

So set $P = S^{2n+1}$. Hopf bdl

- transition funcs for pbdl $P \to B$, $\{ \mathcal{U}_\alpha \}$
  are $(\varphi_\alpha |P_{\alpha\beta}) \circ (\varphi_\beta | P_{\alpha\beta})^{-1} =: g_{\alpha\beta}$

where
$$\begin{tikzcd} \mathcal{U}_{\alpha\beta} \times G & \pi^{-1}(\mathcal{U}_{\alpha\beta}) \arrow[l, "\varphi_\beta|P_{\alpha\beta}"', "\sim"] \arrow[r, "\varphi_\alpha|P_{\alpha\beta}", "\sim"'] & \mathcal{U}_{\alpha\beta} \times G \\ & P_{\alpha\beta} \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$
$$\mathcal{U}_{\alpha\beta} \times G \xrightarrow[\sim]{g_{\alpha\beta}} \mathcal{U}_{\alpha\beta} \times G$$

$g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \longrightarrow G$, equivariance saves us from dependence of $- \times G$.

- cocycle condition on $\mathcal{U}_{\alpha\beta\gamma}$, $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{id}_{\mathcal{U}_{\alpha\beta\gamma}}$
- classified by $H^1(B, G)$

$$\begin{tikzcd} P \arrow[r, "\pi"] & B \\ f^*P \arrow[u, "\tilde{f}"] \arrow[r, "\tilde{\pi}"'] & A \arrow[u, "f"'] \end{tikzcd}$$

G-pbdl

$$f^*P = P \times_B A = \{(p,a) \in P \times A \mid \pi(p) = f(a)\}$$

pullback bdl

- $P \xrightarrow{\pi} B$ G-pbdl, F top sp with G-left action
  $G \times F \to F$, $(g, v) \mapsto gv$

- define $P \times_G F = P \times F/\sim$,
  $(p, v) \sim (pg, g^{-1}v)$

so

$$\begin{tikzcd} P \times_G F \arrow[d, two heads, "\lambda"'] & {[p,v]} \arrow[d, mapsto] \\ B & \pi(b) \end{tikzcd}$$

pbdl w fiber F

associated bdl

- e.g. $F = V$ vsp., $\rho: G \to \mathrm{Aut}\, V$ exact

  $\mathcal{V} := P \times_G V \xrightarrow{\lambda} B$ is
  G-vbdl assoc. to P

- taking its frame bdl gives back P!
  - denote it by $P(\mathcal{V})$

Universal bdls.

- $G = GL(n, \mathbb{R})$, $Gr(n, n+k)$ Grassmannian of n-dim subspaces in $\mathbb{R}^{n+k}$, smooth, dim $= nk$

- we have
$$\cdots \subset \underset{(x_1,\dots,x_{n+k}) \mapsto (x_1,\dots,x_{n+k},0)}{\mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}} \subset \cdots \subset \bigcup_{k \geq 0} \mathbb{R}^{n+k} =: \mathbb{R}^{\infty}$$
inducing
$$\cdots \subset Gr(n, n+k) \subset Gr(n, n+k+1) \subset \cdots$$
$$\cdots \subset \bigcup_{k \geq 0} Gr(n, n+k) =: Gr(n, \infty)$$
as inductive or direct limit

- now, for every k $\exists$ tautological vbdl

$$\begin{tikzcd} \tau(n, n+k) \arrow[d] & V \arrow[l, hook'] \arrow[d] \\ Gr(n, n+k) & \{V\} \arrow[l, phantom, "\ni"] \end{tikzcd}$$
, $\dim V = n$, $V \subset \mathbb{R}^{n+k}$

$\operatorname{rk} \tau(n, n+k) = n$

- so we also get
$$\begin{tikzcd} \tau(n) \arrow[d] \\ Gr(n, \infty) \end{tikzcd}$$

such that $\tau(n)|_{Gr(n, n+k)} = \tau(n, n+k)$

- B paracpt w countable cover $\{\mathcal{U}_\alpha\}_{\alpha\in\mathbb{Z}_+}$
in particular cpt., $\exists$ part of 1 $\{\lambda_\alpha\}$ subord to $\{\mathcal{U}_\alpha\}$

$$V \xrightarrow{f} B \text{ vbdl of rk } n, \quad \begin{tikzcd} V|_{\mathcal{U}_\alpha} \arrow[r, "\varphi_\alpha"] & \mathcal{U}_\alpha \times \mathbb{R}^n \arrow[d, "pr_2"] \\ & \mathbb{R}^n \end{tikzcd}$$

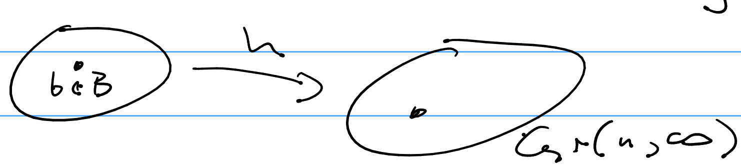
$$\mu_\alpha := pr_2 \circ (\lambda_\alpha \varphi_\alpha) : V|_{\mathcal{U}_\alpha} \longrightarrow \mathbb{R}^n =: \mathbb{R}^n_\alpha$$

$$\mu : V \longrightarrow \bigoplus_{\alpha\in\mathbb{Z}_+} \mathbb{R}^n_\alpha, \quad \mu := \sum_\alpha \mu_\alpha$$

$\exists\, h : B \to Gr(n,\infty)$ s.t.

$$\begin{tikzcd} \underline{\underline{V}} \cong h^*\tau(n) \arrow[r] \arrow[d, "f"'] & \tau(n) \arrow[d] \\ B \arrow[r, "h"] & Gr(n,\infty) \end{tikzcd}$$

$V_b \longrightarrow \tau(n)$, via μ only finitely many coordinates $\neq 0$.



$$\begin{array}{l} P = P(\mathcal{V}) \text{ frame pbdl of } \mathcal{V} \\ \downarrow \pi \\ B \end{array} \quad \to \text{then } \exists h \text{ s.t. } P = h^* P(\tau(n)) \; ?$$

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- let $I = [0,1] \subseteq \mathbb{R}$, we naively get $\begin{array}{c} P \times I \\ \downarrow \pi \times id \\ B \times I \end{array}$

Prop For B paracpt., $\exists$ an isom. of G-bdls
$$\begin{array}{c} P \\ \downarrow \pi \\ B \times I \end{array} \quad \text{and} \quad \begin{array}{c} (P|_{B \times \{1\}}) \times I \\ \downarrow \\ B \times I \end{array}$$

Cor $P|_{B \times \{1\}} \simeq P|_{B \times \{0\}}$

- $B \xrightarrow{f_1, f_0} Gr(n, \infty)$ homotopic maps via $B \times I \to Gr(n,\infty)$
  - then $f_1^* P(\tau(n)) \simeq f_0^* P(\tau(n))$

- so, homotopy classes of $[B, Gr(n,\infty)]$
$$\Downarrow$$
isom. classes of G- pbdls.
where $G = GL(n, \mathbb{R})$

- in fact, for B paracpt w count. basis, it injects.