Тихомиров

Sobolev completions

- B cpt mfd, $\dim B = n$, (B, g) Riemannian
- $W \longrightarrow B$ vbdl, A any conn'n on W

$$s \in \Omega^k(B, W) = \Gamma(B, \Lambda^k T B^* \otimes W)$$

conn'n A

conn'n A_{LC}, Levi-Civita connection

- $\nabla(f \otimes g) = \nabla_A f \otimes g + f \otimes \nabla_{A'} g$ is extension
- we also want a scalar product (any pos. def.) on W, so we may talk abt $|s(x)| = \sqrt{(s(x), s(x))}$

- on $\Omega^\ell(B, W)$ define norms

$$\|s\|_{L^p_k} := \left[\int_B \left(|s(x)|^p + |\nabla_A(s)(x)|^p + \dots + |(\nabla_A)^k(s)(x)|^p \right) d\mathrm{Vol} \right]^{\frac{1}{p}}$$

with $p \geq 1$, $k \geq 0$

- this is a norm on $\Omega^\ell(B, W)$

- we denote by $L^p_k(\Omega^\ell(B, W))$ the completion of $\Omega^\ell(B, W)$ wrt $\|\cdot\|_{L^p_k}$
- notation: $L^p_k(W) := L^p_k(\Omega^0(W))$

- let $w(p, k) := k - \frac{n}{p}$

- let $W, W' \longrightarrow B$ 2 vbdls
- clearly $\Omega^0(W) \otimes \Omega^0(W') \xrightarrow{\text{mult.}} \Omega^0(W \otimes W')$

Thm (Sobolev multiplication thm)
$L^p_k(W) \otimes L^{p'}_{k'}(W') \xrightarrow{\text{mult.}} L^q(W \otimes W')$, if
$w(p,k) + w(p',k') > w(q,r)$, $\min(k,k') \geq r$,
is cont. (bounded) operation

Thm (Sobolev embedding thm)
$L^p_k(W) \xhookrightarrow{i} L^{p'}_{k'}(W)$ is bounded
if $w(k,p) > w(k',p')$, $k \geq k'$.
Moreover, if $k > k'$, i is compact.

- letting, $C^r(W)$ = space of all sectn's of W r times differentiable,
we get $L^p_k(W) \xhookrightarrow[\text{cpt.}]{} C^r(W)$ if $w(k,p) \geq r$.

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- armed with this ...
- recall $\mathcal{A}(P)$ = aff. space over $\Omega^1(\mathrm{ad}\, P)$,
where $P \xrightarrow{\pi} B$ G-pbdl, G = cpt mtrx gp,
B cpt mfd

- def. $\mathcal{A}_{L^p_k}(P) = \{ A + a \mid a \in L^p_k(\Omega^1(\mathrm{ad}\, P)) \}$

- Banach mfd. (with 1 chart ?)

- recall $\mathrm{Ad}\, P = P \times_{\mathrm{Ad}} G$,
$\mathcal{G}(P) = \Gamma(\mathrm{Ad}\, P)$ = gp of gauge transformation
↪ terminology note: G is called the "gauge gp", $\mathcal{G}(P)$ gp of gauge tr's

- recall, $\Gamma(\mathrm{Ad}P) \ni s : P \longrightarrow G$, $s(pg) = g^{-1}s(p)g$
  and for $G = SU(2)$, $Z(SU(2)) = \{\pm 1\}$,

$$\begin{tikzcd} \mathrm{Ad}P \arrow[r] & B \arrow[l, bend right, "\{-1\}"'] \arrow[l, bend left, "\{1\}"] \end{tikzcd}$$

- want to extend to Banach (mfd) Lie gp
  $\mathcal{G}_{L^p_k}(P) = L^p_k(\mathrm{Ad}P)$

- now, we Rmk. this is Hilbert sp.
  if $p = 2$ 1st good case
- however, we are interested in
  $k = 2$, $p > 2$, $n = 4$ (in $p = 2$, $\mathcal{G}_{L^2_2}$ not compact?)
- notice $L^3_2(W) \xhookrightarrow{\text{S.E.Th.}} C^0(W)$
- so $\mathcal{G}_{L^3_2}(P)$ will be gp under mult.
  of (cont.) sectn's

- let $T\mathcal{G}_{L^3_2}(P) := L^3_2(\mathrm{ad}P)$
  Lie alg. w fibrewise bracket

- $\exp : L^3_2(\mathrm{ad}P) \longrightarrow \mathcal{G}_{L^3_2}(P)$ defined
  fibrewise is bijection in nbhd of $\{1\}$-section

- we stop writing k, e.g. $\mathcal{A}_{L^2_2}(P) := \mathcal{A}_2(P)$

- action $\mathcal{A}_2(P) \times \mathcal{G}_3(P) \to \mathcal{A}_2(P)$
  - check: $\omega(2,2) > 0$ $\omega(3,2) > 0$ $\omega(2,2) > 0$
    - this works, and even is smooth (won't prove)

- note in passing that $E \xrightarrow{f} F$, E, F Banach, f is called
  diff. at $x \in E$ if $\exists$ bounded lin op $d_x f : E \to F$ s.t. $\lim_{\|h\| \to 0} \frac{\| f(x+h) - f(x) - (d_x f)(h) \|}{\|h\|} = 0$

- $\mathcal{A}_2(P) \twoheadrightarrow \mathcal{A}_2(P)/\mathcal{G}_3(P) =: \mathcal{B}(P)$,

moduli sp. of conn's

- q: is $\mathcal{B}(P)$ Hausdorff?
  - if $\{A_n\}, \{B_n\} \in \mathcal{A}_2(P)$ are s.t.

  i) $\exists\, g_n \in \mathcal{G}_3(P)$ with $g_n^* A_n = B_n$,

  ii) $A_n \to A$, $B_n \to B$, $A, B \in \mathcal{A}_2(P)$ as $n \to \infty$

  iii) $\exists\, g = \lim_{n\to\infty} g_n \in \mathcal{G}_3(P)$

  then $\mathcal{B}(P)$ Hsdff if $g^* A = B$

- take $A_n = A + a_n$, $B_n = A + b_n$, $a_n, b_n \in L^2_2$

$$\Rightarrow B_n = A + b_n = g_n^{-1} d g_n + g_n^{-1}(A + a_n) g_n$$

$$\Rightarrow \underbrace{d g_n - g_n A + A g_n}_{\nabla_A(g_n) = d g_n + [A, g_n]} = g_n b_n - a_n g_n$$