Bertola

KdV & Malgrange

- solitonless soln's

RHP on $\mathbb{R}$

$$\Gamma_+ = \Gamma_- \begin{bmatrix} 1-|r|^2 & -\bar{r}(z)e^{-2i\overbrace{(-)}^{\vartheta\,!!}} \\ r(z)e^{2i(4tz^3+xz)} & 1 \end{bmatrix} =: \Gamma_- J(z)$$

$$\Gamma(z) = \begin{pmatrix} 1 & 1 \\ 2iz & -2iz \end{pmatrix}\left(\mathbb{1} + \frac{i}{z}a(x,t)\sigma_3 + \mathcal{O}(z^{-2})\right)$$
as $z \to \infty$

- also a symmetry $\Gamma(-z) = \sigma_1 \Gamma(z)\sigma_1$
- $u(x,t) = -2\partial_x a$ solves KdV, i.e.
  $u_t - 6uu_x + u_{xxx} = 0$
- (Dyson) if r is trans'n coeff. of fast decaying pot. then
  $u(x,t) = -2\partial_x^2 \ln\det\left[\mathrm{Id} - \mathbb{K}\big|_{[x,\infty)}\right]$
  where $\mathbb{K}$ has int. kernel
  $K(s,u) = F(s+u)$, $F(s) := \frac{1}{2\pi}\int_{\mathbb{R}} r(z)e^{8itz^3+izs}\,dz$

- recall Malg. form $\Theta = \int_{\mathbb{R}} \mathrm{tr}\, \Gamma_-^{-1}\Gamma_-'\,\delta J\, J^{-1}\,\hat{dz}$

$$\delta\Theta = \frac{1}{2}\int \mathrm{tr}\,\delta J\,J^{-1} \wedge \frac{d}{dz}\delta J\, J^{-1}\,\hat{dz}$$

$$= \frac{1}{2}\int_{\mathbb{R}}\left[\delta(re^{2\vartheta}) \wedge \frac{d}{dz}\left(\bar{r}(z)e^{-2\vartheta}\right) - \frac{d}{dz}\delta(re^{2\vartheta}) \wedge \delta(\bar{r}e^{-2\vartheta})\right]\hat{dz}$$

$$= -\int \delta\left(\frac{d}{dz}(re^{2\vartheta}) \wedge \delta(\bar{r}e^{-2\vartheta})\right)\hat{dz}$$

$$= -\delta\left(\int (re^{2\vartheta})' \cdot \delta(\bar{r}e^{-2\vartheta})\right)$$

$$\Rightarrow 2d\ln\tau := \Theta + \int re^{2\vartheta}\cdot\delta(\bar{r}e^{-2\vartheta})$$

- let's compute $\partial_x \ln \tau$

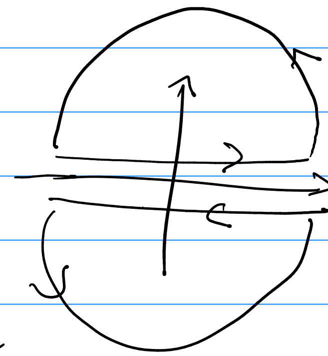
- $\omega(\partial_x) = \int \underbrace{\mathrm{tr}\, \Gamma_-^{-1}\Gamma_-' \, \partial_x J J^{-1}} = \dots$

$$= \int \Delta\, \mathrm{tr}(\Gamma^{-1}\Gamma' \vartheta_x \sigma_3)\, d\hat{z} + \partial_x(\bar{J} e^{-2\vartheta}) \frac{d}{dz}(\sigma e^{2\vartheta})$$

↳ jump $\Delta f = f_+ - f$

- $\omega(\partial_x) + \int(\cdot \;\dots) = \int \Delta(\dots)$

$$= \lim_{R\to\infty}\left[\int_{C_1} \mathrm{tr}\dots + \int_{C_2} \mathrm{tr}\dots\right]$$



"formal" residue

$$= -\text{"}\mathrm{res}_{\infty}\text{"}\; \mathrm{tr}(\Gamma^{-1}\Gamma'\sigma_3)\cdot iz$$

↳ not a real residue

$$= -a(x,t)$$

Fuchsian system of ODEs in $\mathbb{CP}^1$

- ODE for $n\times n$ mtrx $\Psi(z)$,

$$\begin{cases} \Psi'(z) = A(z)\Psi(z), & A(z) = \displaystyle\sum_{j=1}^{k} \frac{A_j}{z-t_j}, \\ \Psi(\infty) = \mathbb{1} & \end{cases}$$

A_j fixed, $\sum A_j = 0$, ie $z=\infty$ is a regular pt

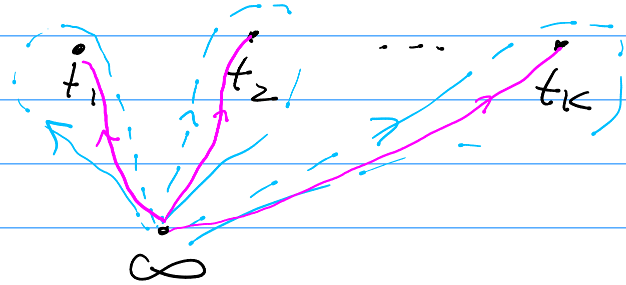
- technical requirements:
  $\forall i$, evals of A_i do not differ by nonzero integers (nonresonance of the Fuchsian sings.)

- $(\det \Psi)' = \operatorname{tr} A(z) \det \Psi \Rightarrow \det \Psi = e^{\int \operatorname{tr} A(z)}$

- If $[\gamma] \in \pi_1(\mathbb{CP}^1 \setminus \{t_1, \dots, t_k\}, \infty)$ and $\Psi(z\gamma)$ denotes analytic. cont. of $\Psi(z)$ around $[\gamma]$ then $\Psi(z\gamma) = \tilde{\Psi}(z)$ solves same ODE $\Rightarrow \tilde{\Psi}(z) = \Psi(z) \cdot M_{[\gamma]}^{-1}$ where $M : \pi_1 \to GL_n$ is representation;

$$\Psi(z\gamma \cdot \hat{\gamma}) = \Psi(z) M_{\hat{\gamma}}^{-1} M_{\gamma}^{-1} = \Psi(z) M_{\gamma \cdot \hat{\gamma}}^{-1}$$

- to make it sing-val, introduce branch cuts

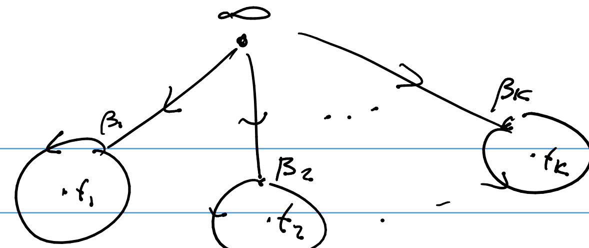


- Ψ is sing-val on $\mathbb{CP} \setminus$ cuts
- on the cut ℓ_j oriented towards t_j,
  $\Psi_+(z) = \Psi_-(z) M_j$
- since $[\gamma_1 \circ \gamma_2 \circ \dots \circ \gamma_n] = [pt] \Rightarrow M_1 \cdots M_k = \mathbb{1}$

Local behaviour

- $\Psi(z) = G_j \underbrace{(\mathbb{1} + O(z - t_j))}_{\text{analytic}} (z - t_j)^{-L_j} C_j$

where $A_j = G_j L_j G_j^{-1}$, $M_j = C_j e^{2\pi i L_j} C_j^{-1}$



- suggests RHP

$$\Gamma(z) = \begin{cases} \Psi(z) & \text{outside } \bigcup_j \mathbb{D}_j \\ \Psi(z)\, C_j (z - t_j)^{-L_j}, & z \in \mathbb{D}_j \end{cases}$$

$c_j e^{L_j \ln(z-t_j)}$

- near β_j,

$c_j e^{L_j(\ln(z-t_j)+2\pi i)}$

Isomonodromy deformations

- find how A_j's should depend on $t_1, \dots, t_K$ so that monodromy is $\underline{t}$-indep
- [Schlesinger]:

$$\begin{cases} \dfrac{\partial A_j}{\partial t_k} = \dfrac{[A_j, A_k]}{t_j - t_k} \\[2ex] \dfrac{\partial A_j}{\partial t_j} = -\displaystyle\sum_{k \neq j} \dfrac{[A_j, A_k]}{t_j - t_k} \end{cases}$$

(nonlin. nonauton. PDEs)

- given $\{A_j(\underline{t}^0) = A_j^0\}_{j=1,\dots,k}$, sol'n exists

- [80's Jimbo Miwa Ueno] define "tau fn"

$$d \ln \tau_{JMU} := \frac{1}{2} \sum_{j=1}^{k} \left(\operatorname*{res}_{z=t_j} \operatorname{tr} A^2(z) \right) dt_j$$

$$= \sum_{j=1}^{k} \left(\sum_{\ell \neq j} \frac{\operatorname{tr} A_j(t) A_\ell(t)}{t_j - t_\ell} \right) dt_j$$

$\rightarrow d \ln \tau_{JMU}$ is closed if Schlesinger equns hold for $\{A_j\}$

Rmk consider $t_j \mapsto t_j' = \frac{a t_j + b}{c t_j + d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$

$$\Rightarrow \tau(\underline{t}') = \tau(\underline{t}) \cdot \prod_{j=1}^{k} (c t_j + d)^{2\Delta_j}$$

where the conf. weights are $\Delta_j = \frac{1}{2} \operatorname{tr} A_j^2$
(note the Schl. equn's preserve spec A_j)

$\rightarrow$ CFT link: $\tau(\underline{t}) = \langle \mathcal{O}^{\Delta_1}(t_1) \cdots \mathcal{O}^{\Delta_k}(t_k) \rangle$

- note that RHP depends on

$\underline{t}, \vec{L}, \vec{C}$ subject to $\prod_{j=1}^{k} C_j e^{2\pi i L_j} C_j^{-1} = \mathbb{1}$

$\rightarrow$ "deformation space"

$$\mathcal{M} = \{ \vec{t}, \underbrace{\vec{L}, \vec{C}}_{\substack{\text{keep fixed} \\ \text{to obtain isomon.}}} \mid \prod C e^{2\pi i L} C^{-1} = \mathbb{1} \}$$

Prop $\Theta\left(\frac{\partial}{\partial t_\nu}\right) = \frac{\partial}{\partial t_\nu} \ln \tau_{JMU}$

Pf. $J_j(z) = C_j (z - t_j)^{-L_j} \Rightarrow \partial_{t_j} J \cdot J^{-1} = C_j \frac{(-L_j)}{z - t_j} C_j^{-1}$,
$\partial_{t_{k \neq j}} J_j^{-1} J_j = 0$

$$\Rightarrow \odot(\partial_{t_j}) = -\oint_{t_j} \operatorname{tr} \Psi_-^{-1} \underbrace{\Psi_-' (\Psi_-^{-1}\Psi_-)}_{A(z)} \frac{C_j L_j C_j^{-1}}{z-t_j}$$

$$= -\oint \operatorname{tr} A(z) \cdot \underbrace{\frac{\Psi(z) C_j L_j C_j^{-1} \Psi^{-1}(z)}{z-t_j}}$$

$$\frac{G_j (\mathbb{1}+O_j) L_j (\mathbb{1}-O_j) G_j^{-1}}{z-t_j}$$

$$\|$$

$$\frac{A_j}{z-t_j} + G_j [O_j, L_j] G_j^{-1} + O(z, t_j)$$

$$= -\operatorname{res}_{z=t_j} \operatorname{tr} \frac{A(z) A_j}{z-t_j} = -\sum_{\ell \neq j} \frac{\operatorname{tr} A_j A_\ell}{t_j - t_\ell} = \text{claim}$$

- differential:

$$-4\pi i \, \delta \odot = \sum_{\ell=2}^{k} \operatorname{tr} \delta M_{\ell-1} M_{\ell-1}^{-1} \wedge \delta M_{[\ell,k]} M_{[\ell,k]}^{-1})$$

$$+ \sum_{j=1}^{k} \operatorname{tr} \left(e^{2\pi i L_j} \delta C_j C_j^{-1} \wedge e^{-2\pi i L_j} \delta C_j C_j^{-1} + 4\pi i \, \delta L_j \wedge \delta C_j C_j^{-1} \right)$$

- rmks: no $\underline{t}$-dependence

→ closed 2-form on

$$\mathcal{M}_0 = \left\{ \vec{C}, \vec{Q} = e^{2\pi i \vec{L}} \;\middle|\; \prod C_j Q_j C_j^{-1} = \underline{\mathbb{1}} \right\}$$

- $(-4\pi i) \, \delta \odot =: \omega_{\mathcal{M}}$ is inv under $C_j \mapsto S C_j$, i.e. $M_j \mapsto S M_j S^{-1}$, $S \in GL_n$ arbitrary
- $\omega_{\mathcal{M}}$ descends on $\mathcal{M}_0 / GL_n$

$\xRightarrow{\text{Thm}}$ $\omega_{\mathcal{M}}$ is symplectic on quotient
- to define τ, look at $\odot - \int \omega_{\mathcal{M}} \Rightarrow d \ln \tau$