Gauge invariants @ IG+P

3d N=4 Coulomb branch – Nakajima

- w/ Braverman, Finkleberg
- G a cpx reductive gp, e.g. $GL(n)$
  - G_c: max. cpt. subgp, $U(n)$
- M: a symp. rep of G, e.g. $N \oplus N^*$
  = quaternionic rep of G_c

- 4d $\mathcal{N}=2$ SYM $\overset{S^1}{\rightsquigarrow}$ 3d $\mathcal{N}=4$ SYM
  $\supset$ G_c-conn'n + M-valued spinors

$\nearrow$ Higgs br. $\rightarrow$ Coulomb br. $\Big\}$ hyperkähler w/ S^1-action, possibly w/ singularities

- Higgs – hyperkähler quot of M by G,
  $M///G = \mu^{-1}(0)//G = \mu^{-1}(0)/G_c$
- Coulomb – harder to define
  - as affine alg var w/ holo symp. str on reg. locus, w/ $\mathbb{C}^*$ action

- by product of construction $\to$ quantization
  - X: aff. alg var w/ holo symp. str (more gen. Poisson str.)
    $\Rightarrow \mathbb{C}[X]$ has $\{\cdot,\cdot\}$
  - $\mathcal{A}_\hbar$: alg. over $\mathbb{C}[X]$ is a quantization

$\overset{\text{def.}}{\Longleftrightarrow}$ i) $\mathcal{A}_\hbar / \hbar \mathcal{A}_\hbar \simeq \mathbb{C}[X]$

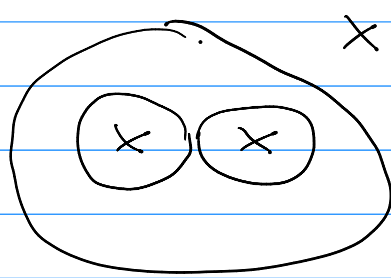
ii) $\dfrac{\tilde{f}\tilde{g}-\tilde{g}\tilde{f}}{\hbar} \equiv \{f,g\} \mod \hbar \mathcal{A}_\hbar$

where $f,g \in A \overset{\text{lift}}{\rightsquigarrow} \tilde{f},\tilde{g} \in \mathcal{A}_\hbar$

- e.g. for $M = N \oplus N^*$, $D_\hbar :=$ ring of $\hbar$-diff ops on N
  - $\longrightarrow$ quantum hamiltonian reduction
    - $\rightsquigarrow$ quant. of $M///G$

- denote Coulomb br. by $\mathcal{M}_C$
  - look instead at $\mathbb{C}[\mathcal{M}_C]$ ($\mathcal{M}_C = \operatorname{Spec} \mathbb{C}[\mathcal{M}_C]$)
  - in phys: chiral ring, ring of monopole operators
- take a pt x in $\mathbb{R}^3$ and its small nbhd B_x, $\partial B_x \simeq S^2$
  - Atiyah-Segal: $\mathcal{H}$ - quantum Hilb. sp. for S^2 $= Z(S^2)$
  - $\longrightarrow$ $\mathcal{H}$ is comm. ring
  - what is multiplication?

3 pts:



$\Rightarrow \partial X = S^2 \sqcup S^2 \sqcup S^2$

$Z(X) \in \mathrm{Hom}(\underbrace{Z(S^2)\otimes Z(S^2), Z(S^2)}_{\mathcal{H}\otimes\mathcal{H},\, \mathcal{H}})$

multiplication

- commutative since no ordering $\exists$
- task: define $Z(S^2) = \mathcal{H}$ rigourosly
- conventional approach (Donaldson / Casson th.)
  - generalised SW equations (☆)
  - A: G_c-conn
    s: M-val spinor
    $\mu: M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$

$$(\star)\begin{cases} \not{D}_A s = 0 \\ \mu(s) = *F_A \end{cases}$$

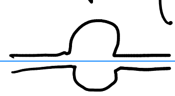
closed

- set $Z(X^3) = \#$ solns to $\star$

$Z(\Sigma^2)$ = homology of moduli space of solutions of 2d-reduction of $\star$

- usually conditions imposed to get rid of singularities of m.sp.
  -> but, $\Sigma = S^2 \Rightarrow$ all solns are reducible by dim. counting argument
- also, if this works:
  -> $Z(S^2)$ – fin. dim.
  $\Rightarrow$ Spec $Z(S^2)$ – 0. dim ?

$\neq$ correct Coulomb branch
($\dim M_C = 2 \cdot rk\, G$)

- their approach: replace S^2 by raviolo  $= D \sqcup_{D^\times = D \setminus \{0\}} D$, D = formal disk

- in alg. geom. language, affine Grassmannian
  $Gr_G = G(K)/G(\mathcal{O})$, $K = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$
  (Spec $K = D^\times$, Spec $\mathcal{O} = D$)
  $= Map_{polynomial}(S^1, G_c)/G_c$
  $= \{ f \in Map_{polyn}(S^1, G_c) \mid f(1) = 1_{G_c} \}$

- when $M = N \oplus N^*$, consider R w/ $G(\mathcal{O})$-action
  $[G(\mathcal{O}) \backslash R]$ = moduli stack of G-bdles + N-val. sections on R
  = moduli of (P_1, s_1) on D, (P_2, s_2) on D

and $\varphi: \rho_1|_{D^*} \to \rho_2|_{D^*}$ (respecting $s_1 \mapsto s_2$)

- $H_*([G(\mathcal{O})\backslash R]) = H_*^{G(\mathcal{O})}(R)$

has conv. product ↗ ↖ ρ_2 = triv bdle

$$D \underset{D^*}{\sqcup} D \underset{D^*}{\sqcup} D \to D \underset{D^*}{\sqcup} D$$

$$c_1 * c_2 = p_{13*}(p_{12}^*(c_1) \cap p_{23}^*(c_2))$$

Thm $(H_*([G(\mathcal{O})\backslash R]), *)$ is a comm. ring

$\Rightarrow$ Hence, $\mathcal{M}_C := \operatorname{Spec} H_*([G(\mathcal{O})\backslash R])$

- quantize: $\mathbb{C}^\times \curvearrowright D, D^*, R$ loop rotation

$$H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(R) = \mathcal{A}_\hbar$$

Examples

1) $G = \mathbb{C}^*$, $N = 0$

$$Gr_G = G(K)/G(\mathcal{O}) \simeq \operatorname{Map}_{\text{Polyn}}(S^1, G_c)/G_c$$
$$= \{ z^n \mid n \in \mathbb{Z} \}$$

$\cdots \times \quad \times \quad \times \quad \times \cdots$

$$H_*([G(\mathcal{O})\backslash Gr_G]) = \bigoplus_{n \in \mathbb{Z}} H_*^{\mathbb{C}^*(\mathcal{O})}(\{z^n\})$$

$\shortparallel$

$\mathbb{C}[w] \cdot r^n$ ↪ fund cl. of $\{z^n\}$

- product induced from G_c

$\{z^n\} \times \{z^m\} \mapsto \{z^{n+m}\} \Rightarrow r^n * r^m = r^{n+m}$

$\Rightarrow H_*(\cdots) = \mathbb{C}[w, r, r^{-1}] = \operatorname{Spec} \mathbb{C} \times \mathbb{C}^\times$
$= \operatorname{Spec} \mathbb{R}^3 \times S^1$ } toroidal $\mathcal{M}_C$

2) $G = \mathbb{C}^\times$ $\quad N = \mathbb{C}$

$R = \{ (z^n, \underset{\mathbb{C}[[z]]}{\underset{\cap}{s_1}}) \mid z^n s_1 \in \mathbb{C}[[z]] \}$

→ if $n \geq 0$ automatic
if $n < 0$, $s_1 \in z^{-n}\mathbb{C}[[z]]$

(p_1, s_1)

(p_2, s_2) $\quad \varphi = z^n$

p_2 ↓ trivial

$\cdots \quad z^2\mathcal{O} \quad z\mathcal{O} \quad \mathcal{O} \quad \mathcal{O} \quad \cdots$

$\underset{n-2}{\times} \quad \underset{n-1}{\times} \quad \times \quad \times$

'r^n' = fund. cl. of the fibre (z^n

'r^1' $*$ 'r^{-1}' $= w$ → gen. for $H_*^{\mathbb{C}^\times}(\{z^0\})$

$\Rightarrow H_*([G(\mathcal{O}) \backslash R]) = \mathbb{C}[w, \underset{\text{'}r^1\text{'}}{\overset{\shortparallel}{x}}, \underset{\text{'}r^{-1}\text{'}}{\overset{\shortparallel}{y}}] / (xy = w)$

$= \mathbb{C}[x, y]$

$\Rightarrow M_C = \mathbb{C}^2$