Bruzzo

Divisors

- natural notions:
  - Weil - codim 1 subschemes, $D = \sum_n \alpha_n D_n$, $\alpha_n \in \mathbb{Z}$
  - Cartier $\rightsquigarrow$ line bdles

Dimension of a scheme X:

- supremum n of integers such that $\exists$ a chain of nested closed irred. subsets $Z_0 \subset \cdots \subset Z_n = X$
- while $\mathrm{codim}_X Z$ is supremum of $Z = Z_0 \subset \cdots \subset Z_n = X$
- if X irred, of fin. type over a field, Z irred, $\mathrm{codim}_X Z + \dim Z = \dim X$

- A ring, $p \in \mathrm{Spec}\, A$. Define the height of p as the supremum of n s.t. $p_0 \subset \cdots \subset p_n = p$.
  $\rightarrow \dim A$ = supremum of all heights (Krull dimension)

- now suppose A local ring, m max ideal
  $\rightarrow$ consider $\dim_{\mathbb{k}} m/m^2$

Def. A local ring A is regular if $\dim A = \dim_{\mathbb{k}} m/m^2$.

Def. A scheme is regular if all local rings are regular. It is regular in codimension 1 if all local rings of dimension 1 are regular.

- the last property, $\dim \mathcal{O}_{X,p} = 1$, has an important consequence on the structure of singularities.
  $\rightarrow$ $p \rightsquigarrow$ hypersurface X_p, so this means its generic pt is regular $\Leftrightarrow$ regular pts lie on dense opens
  $\Rightarrow \mathrm{codim}_X \mathrm{Sing}(X) \geq 2$.
  $\rightarrow$ so divisors, codim 1 objects, play nice

- take field K. A map $v: K^* \to \mathbb{Z}$ such that
  1) $v(xx') = v(x) + v(x')$
  11) $v(x+x') \geq \min(v(x), v(x'))$

is called a discrete valuation.
$R = \{x \in K^* \mid v(x) \geq 0\} \cup \{0\}$ is called the valuation ring of (K, v).
$\to$ it is in fact local with $m = \{x \in K^* \mid v(x) > 0\} \cup \{0\}$

Lemma. Set A noetherian local int. domain of dimension 1. TFAE
 1) A regular
 11) A valuation ring of some field

- example: $D = \mathrm{Spec}\, \frac{k[x,y]}{(x)}$. D integral, ξ generic pt, $j: D \hookrightarrow \mathbb{A}^2_k$, $j(\xi) = D$. $A = \mathcal{O}_{\mathbb{A}^2_k, j(\xi)} = k[x,y]_{(x)} = \left\{ \frac{P(x,y)}{Q(x,y)} \,\middle|\, x \nmid Q \right\}$
  $\to v: K(\mathbb{A}^2_k) \to \mathbb{Z}$, $v|_A\left(\frac{P}{Q}\right) = m$ if $P(x,y) = x^m f(x,y)$.

- now assume X integral, separated, noetherian, regular in codim 1 scheme.

Def. A prime Weil divisor is a closed integral subscheme of codim 1.
$\mathrm{Div}(X)$ = free group generated by prime divisors over $\mathbb{Z}$

- since D integral, it has a unique generic pt ξ, $\mathcal{O}_{X, j(\xi)}$ is a discrete valuation ring for the field of rational functions

Lemma. If $f \in K^*$, $v_D(f) \neq 0$ for a finite # of pts.

→ we construct principal divisors
$$(f) := \sum_{\substack{\text{prime divisors} \\ Y \subseteq X}} v_Y(f)\, Y \in \mathrm{Div}\, X$$

→ linear equivalence $D_1 \sim D_2$ if $D_1 - D_2 = (f)$ for some $f \in K$
→ $\mathrm{Div}(X)/\sim \; = \mathrm{Cl}(X)$

→ recall that an element of a ring is irreducible if $y = xz$ means x or z unit.
→ a UFD is a domain s.t. $x = x_1 \cdots x_n$, with each x_i irreducible, uniquely up to units.

Lemma. If A UFD & $X = \operatorname{Spec} A \Rightarrow \mathrm{Cl}(X) = 0$

Lemma. i) X as usual, Z proper closed irred. subscheme, $\operatorname{codim}_X Z = 1$, $U = X - Z$.
We have a map
$$\mathrm{Div}(X) \to \mathrm{Div}(U)$$
$$\sum a_i Y_i \mapsto \sum a_i (Y_i \cap U)$$
and $\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$ is exact.

ii) $\operatorname{codim}_X Z \geq 2 \Rightarrow \mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(U)$