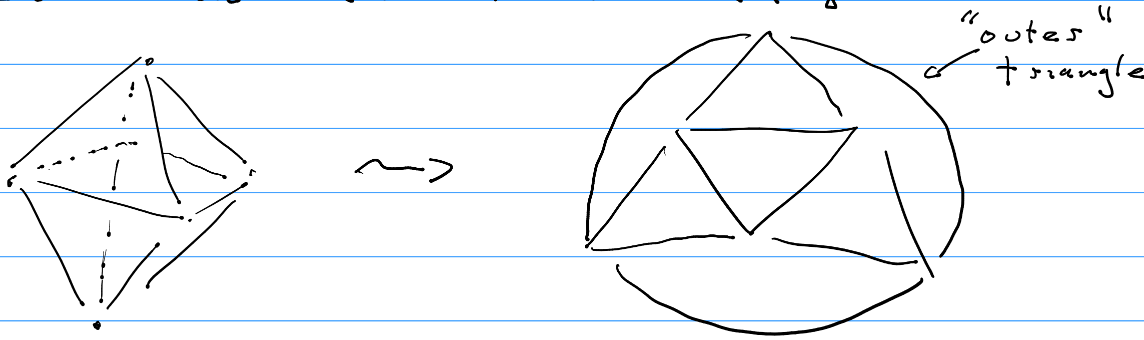
Fantechi

- (TR5) octahedron
- recall what an octahedron is:



- start with $X \xrightarrow{f} Y \xrightarrow{g} Z$, $g \circ f =: h$

and draw

$$\begin{tikzcd}
M_f \arrow[dr, "+1"] \arrow[drr, bend right=60, blue] & & Y \arrow[ll] \arrow[dr, "g"] & & M_g \arrow[ll, "+1"'] \arrow[llll, bend right, blue, "+1"'] \\
& X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \arrow[ur] \arrow[dl] & \\
& & M_h \arrow[ul, "+1"] \arrow[uurr, bend right, blue] & &
\end{tikzcd}$$

we demand that all triangles commute, and that the outer blue triangle is distinguished

$$M_f \to M_h \to M_g \to M_f[1]$$

Rmk We proved if $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \to X[1]$ d.t then $\psi \circ \varphi = 0$. But in $C(A)$, $X \xrightarrow{\varphi} Y \to M(\varphi)$ is nonzero

Lemma In a triang. cat., given d.t. morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\varphi"] & Y \arrow[r] \arrow[d, "\psi"] & Z \arrow[r] \arrow[d, "\vartheta"] & X[1] \arrow[d, "\varphi"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd},$$

if φ and ψ iso, then ϑ iso.

Rmk By axioms,
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "\exists\vartheta"] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

Rmk However, this ϑ is not unique or canonical.

Pf. By coYoneda embedding,
$\mathrm{Hom}(W, Z) \to \mathrm{Hom}(W, Z')$, $\varphi \mapsto \vartheta \circ \varphi$
is bijective. Also remembering $[1]$
is auto eq., φ, ψ iso $\Rightarrow$ $\varphi[1], \psi[1]$ iso.
$\mathrm{Hom}(W, -)$ is cohomological, so
gives exact seq.

$$\begin{tikzcd} h^W(X) \arrow[r] \arrow[d, "\wr\ \varphi\circ -"] & h^W(Y) \arrow[r] \arrow[d, "\wr\ \psi\circ -"] & h^W(Z) \arrow[r] \arrow[d, "\vartheta"] & h^W(X[1]) \arrow[r] \arrow[d, "\wr\ \varphi[1]\circ -"] & h^W(Y[1]) \arrow[d, "\wr\ \psi\circ -"] \\ h^W(X') \arrow[r] & h^W(Y') \arrow[r] & h^W(Z') \arrow[r] & h^W(X'[1]) \arrow[r] & h^W(Y'[1]) \end{tikzcd}$$

so ϑ iso by diagram chasing (5-lemma).

Recall For $\mathcal{A}$ ab. cat. we "defined" $D(\mathcal{A})$
as localisation of $K(\mathcal{A})$ at qisos.
Categorical nonsense says it exists,
but it's not simple to describe.

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{Y} \\ D(\mathcal{A}) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

- there <u>are</u> some assumptions we are given, tho

Def Let $\mathcal{C}$ cat. A multiplicative system S is a collection of mor in $\mathcal{C}$ s.t.

i) all identities are in S

ii) $f \in S \wedge g \in S \wedge \exists g \circ f \Rightarrow g \circ f \in S$

iii) given
$$\begin{tikzcd} X \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$
in S, we can add Y' and this commutes
$$\begin{tikzcd} & Y' \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$

iv) assuming $f, g : X \to Y$ in S, TFAE

 i) $\exists h : T \to X$ in S s.t. $f \circ h = g \circ h$

 ii) $\exists l : Y \to W$ in S s.t. $l \circ f = l \circ g$

- informally, a mor in the localisation of $\mathcal{C}$ at S looks like
$$f = f_1 \circ s_1^{-1} \circ f_2 \circ s_2^{-1} \circ \cdots \qquad \begin{array}{l} f_i \in \operatorname{mor} \mathcal{C} \\ s_i \in \operatorname{mor} S \end{array}$$

but iii) tells us we can write $\tilde{s}_1^{-1} \circ f_1 \circ \cdots$ etc

Thm If S mult sys, then the local. of $\mathcal{C}$ at S, $S^{-1}\mathcal{C}$, will be defined as
$$\operatorname{Mor}(X, Y) = \{ \overbrace{X \xleftarrow{s} Z \xrightarrow{f} Y}^{f \circ s^{-1}} \mid s \in S, f \in \operatorname{Hom}(Z, Y) \} / \sim$$
where $X \xleftarrow{s} Z \to Y \sim X \xleftarrow{s} Z_1 \to Y$ iff $\exists$ comm. diag.
$$\begin{tikzcd} & Z \arrow[dl, "s_0"'] \arrow[dr] & \\ X & Z_2 \arrow[l, "s"'] \arrow[u, "\in S"] \arrow[d, "\in S"'] \arrow[r] & Y \\ & Z \arrow[ul, "s"] \arrow[ur] & \end{tikzcd}$$

Rmk We can also define it using $X \to Z \leftarrow_s Y/\sim$

Rmk Let $A \xrightarrow{\varphi} B$ in $C(\mathcal{A})$.
Then φ qiso $\iff h^i(\mathcal{M}(\varphi)) = 0$
$\forall i \in \mathbb{Z} \Rightarrow A \xrightarrow[\varphi]{} B \to C \xrightarrow{+1}$ d.t. in $K(\mathcal{A})$
so, φ iso $\iff \forall i \in \mathbb{Z}\ h^i(C) = 0$

Def. Let $\mathcal{T}$ triang cat. A **null system** $\mathcal{N}$ is a subset of ob $\mathcal{T}$ s.t.
(N1) $0 \in \mathcal{N}$
(N2) for $X \in$ ob $\mathcal{T}$, $X \in \mathcal{N} \iff X[1] \in \mathcal{N}$
(N3) given $X \to Y \to Z \xrightarrow{+1}$ d.t., $(X, Y \in \mathcal{N}) \Rightarrow Z \in \mathcal{N}$

Rmk. If $\mathcal{T} = K(\mathcal{A})$, $\mathcal{N} = \{A \mid h^i(A) = 0\ \forall i\}$

Def. For $\mathcal{N}$ null system, define $S(\mathcal{N}) \subset$ mor $\mathcal{T}$
by $\varphi \in S(\mathcal{N}) \iff \exists$ d.t. $X \xrightarrow[\varphi]{} Y \to Z \xrightarrow{+1}$
where $Z \in \mathcal{N}$

Prop $\mathcal{N}$ null sys $\Rightarrow S(\mathcal{N})$ mult. sys

Exercise If $X \underset{\sim}{\xrightarrow{\varphi}} Y$, $\varphi \in S(\mathcal{N})$ [use TR5 ?]

Notation $\mathcal{T}$ Δcat, $\mathcal{N}$ nullsys, then $\mathcal{T}/\mathcal{N} := S(\mathcal{N})^{-1}\mathcal{T}$

Prop. i) $\mathcal{T}/\mathcal{N}$ is Δcat where d.t. are images of d.t in $\mathcal{T}$
ii) the image of an object in $\mathcal{N}$ is (isom to) zero in $\mathcal{T}/\mathcal{N}$.
iii) for any other Δcat $\mathcal{T}'$, any Δcat functor $F: \mathcal{T} \to \mathcal{T}'$ factors uniquely via $\mathcal{T}/\mathcal{N}$ iff $F(X) = 0\ \forall X \in \mathcal{N}$

Cor For any $\mathcal{A}$ ab. cat, $D(\mathcal{A}) = K(\mathcal{A}) / {}^{\text{cpxs } \simeq}_{\text{zero coh}}$
has a natural Δ cat structure,
same for $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$

Cor let $n \in \mathbb{Z}$, recall $\tau_{\geq n}, \tau_{\leq n} : C(\mathcal{A}) \circlearrowleft$.
i) $\tau_{\geq n}, \tau_{\leq n}$ send homotopic mor. into homotopic mor, i.e. induce fctors on $K(\mathcal{A})$
ii) by iii) in Prop., since if $A \in C(A)$ has $h^i(A) = 0$ $\forall i$, same holds for $\tau_{\geq n} A$, $\tau_{\leq n} A$, so $\tau_{\geq n}, \tau_{\leq n}$ induce functors on $D(A)$

– so
$$\begin{array}{ccc} C(A) & \xrightarrow{\tau_{\geq n}} & C(A) \\ \downarrow & & \downarrow \\ K(A) & \xrightarrow{\tau_{\geq n}} & K(A) \\ \downarrow & & \downarrow \\ D(A) & \longrightarrow & D(A) \end{array}$$

Prop. Let $\mathcal{A}$ ab. cat w enough injectives,
let $K^+(\mathcal{J}) \subseteq K^+(A)$ full subcat of cpx of inj. objects.
Then
$$K^+(\mathcal{J}) \to K^+(\mathcal{A}) \to D^+(\mathcal{A})$$
is eq. of cats.
Similarly with enough projs P,
$$K^-(P) \to K^-(A) \to D^-(A)$$

– so we don't need $D(A)$ in these cases