

# A.G. exercises.

Exercise,  $X \xrightarrow{f} Y$  morphisms of  $S$ -schemes

Prove  $f=g$  if

- i)  $\exists U \subseteq X$  open, dense,  $f|_U = g|_U$
- ii)  $X$  reduced
- iii)  $Y$  separated /  $S$

Preliminaries:

①  $X \xrightarrow{f} Y$ ,  $Z \xrightarrow{i} Y$  closed subscheme

$\Rightarrow$  scheme-theoretic preimage of  $Z$  under

$f$  is  $f^{-1}(Z) := Z \underset{f \circ i}{\times} X$

- being separated over  $S$  means  $Y^2 := Y \times_S Y$

closed imbedding under  $\Delta_{Y/S} := (id_Y, id_Y)$

- look at:

$$\begin{array}{ccc} E & \xrightarrow{i} & X \\ \pi \downarrow \ulcorner & & \downarrow (f, g) \\ Y & \xrightarrow{\Delta} & Y^2 \end{array} \quad \begin{array}{l} 1) E \xrightarrow{i} X \text{ closed imb} \\ 2) E \xrightarrow{i} X \xrightarrow{f, g} Y \text{ commutes} \end{array}$$

$$\begin{array}{ccc} & & Y \\ Y^2 & \xrightarrow{\pi_1} & \\ & \searrow \pi_2 & \\ & & Y \end{array}$$

$$(f, g) \circ i = \Delta \circ \pi$$

$\Downarrow$

$$\pi_1 \circ (f, g) \circ i = \pi_1 \circ \Delta \circ \pi$$

$\Downarrow$

$$f \circ i = \pi$$

3) given  $Z$ ,  $Z \xrightarrow{h} X$  s.t.  
 $Z \rightarrow X \xrightarrow{f, g} Y$  commutes  
 $\Rightarrow \exists! Z \xrightarrow{f, g} Y$ , etc.

- back to exercise:

→ let  $E \xrightarrow{i} X \xrightarrow{f} Y$  be the equaliser,  
 $j: U \rightarrow X$  open imbedding

$$\Rightarrow \begin{array}{ccc} U & & \\ \downarrow i & \searrow j & \\ E & \xrightarrow{i} & X \xrightarrow[f]{f} Y \end{array} \quad \begin{array}{l} \text{commutes since} \\ f|_U = g|_U \Leftrightarrow f \cdot j = g \cdot j \end{array}$$

$\Rightarrow$  topologically,  $U \subseteq E \Rightarrow \overline{U} \subseteq \overline{E}$   
 $\Rightarrow X \subseteq E$

Remk.  $X$  reduced,  $Z \hookrightarrow X$  closed subscheme  
s.t.  $Z = X$  as sets. Then  $Z = X$  as schemes.

Pf. Affine case. Let  $X = \text{Spec } A$ ,  $A$  reduced  
 $\Rightarrow Z \cong \text{Spec } A/I$  for some  $I \subseteq A$   
since closed subscheme.

Topologically,  $Z = V(I) = V(0) = X$   
 $\Rightarrow \sqrt{I} = \sqrt{0} \stackrel{A \text{ reduced}}{=} 0 \Rightarrow I = (0)$   
 $\Rightarrow Z \cong \text{Spec } A/(0) = \text{Spec } A = X$ .

Counterexamples.

- ①  $X$  nonreduced
- ②  $Y$  nonseparated

① Take  $k$  alg. cl.,  $S = \text{Spec } k$ ,  $X = \text{Spec } A = \text{Spec } \frac{k[t,y]}{(y^2, xy)}$   
 $Y = A^1_k \supset X, Y$   $k$ -schemes.

If  $y \neq 0$  in  $t$ ,  $y \in (y^2, xy) \Leftrightarrow y = ay^2 + bx \cdot y$   
 $\Rightarrow 1 = ay + bx \rightarrow 1 = 0$  for  $(y, x) = (0, 0)$ .

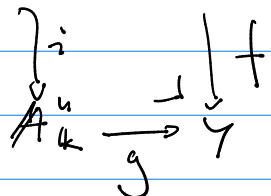
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- now  $f, g \in \text{Map}_{\text{Spec } A} (X, Y = \text{Spec } k[t]) \cong \text{Map}_{k\text{-alg}} (k[t], A) \cong A$   
Take  $y_0 \in t$

-  $y \neq 0$ , but  $\exists$  open dense  $U \subseteq X$  s.t.  $\overbrace{y|_U = 0|_U}^{f|_U = g|_U}$   
 $\Rightarrow U = X_\times \Rightarrow f|_U = g|_U = y|_{y \neq 0} = 0 = 0|_U$ .

② ( $\gamma$  not separated)

- look at  $U \xrightarrow{i} A^n_k \supset \gamma = A^n \coprod_{i \circ j \circ i} A^n$



$\rightarrow \gamma$  affine  $n$ -space /  $k$  with doubled origin

$\rightarrow$  not separated,  $X := A^n_k$  reduced  $k$ -scheme,

$\rightarrow$  but,  $f|_U = f \circ i = g \circ i = g|_U$  by definition

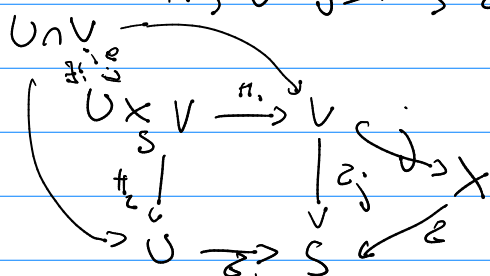
yet  $f \neq g$  since  $f(0) = 1^{\text{st}} \text{ origin}$   
 $g(0) = 2^{\text{nd}} \text{ origin} \Rightarrow$

Exercise.  $S$  affine,  $X$   $S$ -scheme,  $U, V \subseteq X$  open.  
 $(U, V \text{ affine} \ \& \ X \text{ separated}) \Rightarrow (U \cap V \text{ affine})$

- first show  $U \cap V$  closed subscheme of an affine scheme

$\rightarrow U, V, S$  affine  $\Rightarrow U \times_S V$  affine,

$U \xrightarrow{i} X, V \xrightarrow{j} X, z: X \rightarrow S$



$$U \times_S V \xrightarrow{i \circ \pi_1} X \xleftarrow{j \circ \pi_2}$$

Exercise: Prove  $U \cap V \xrightarrow{e} U \times_S V \xrightarrow{i \circ \pi_1} X \xleftarrow{j \circ \pi_2}$  equaliser in  $(Sch)$ ,  
 which implies  $e$  is closed embedding.

$$\left( \begin{array}{c} \text{from} \\ \gamma \xrightarrow{\Delta} \gamma^2 \end{array} \downarrow (f, g) \right)$$

