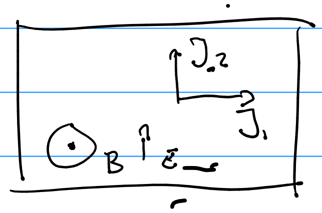
M. Porta.

Math. methods for cond. mat. systems.

Transport in cond mat systems.

- quantum Hall effect (QHE)
- thin surface, low T, high B



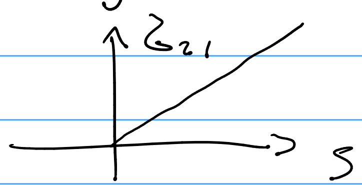
- weak E $\Rightarrow$ $J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \sigma E + O(E^2)$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \text{ conductivity matrix}$$

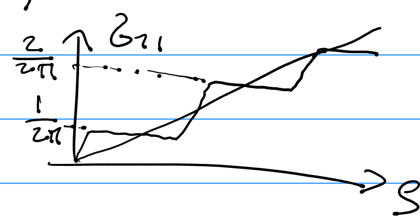
$\to \sigma_{21} = \sigma_{12}$ transverse (Hall) conductivities

$\to \sigma_{11} = \sigma_{22}$ longitudinal (symmetries: $\sigma_{11} = \sigma_{22}$)

- classically:



- but (v. Klitzing '80): σ_{21} takes values in $\mathbb{Z}$
- in $e^2 = \hbar = 1$ units,



$\sigma_{21} \in \frac{1}{2\pi}\mathbb{Z}$

Mathematical model

- $\Lambda \hookrightarrow \mathbb{Z}^2$ lattice
- Hilb. space $\mathfrak{h} = \ell^2(\Lambda \times \mathbb{C}^M)$, so

$\mathfrak{h} \ni \psi \equiv \psi(x, s)$, $x \in \Lambda$, $s = 1 \dots M$

and $\|\psi\|_2 = \sum_{x,s} |\psi(x,s)|^2 = 1$

- observables $\langle O \rangle_\psi := \langle \psi, O\psi \rangle$

- q. dynamics: $i\partial_t \psi(t) = H\psi(t)$, $\psi(t)|_{t=0} \in D(H)$

- Example: $H = -\Delta_{\mathbb{Z}^d}$ (lattice Laplacian)

$$(H\psi)(x) = -\sum_{y:\|y-x\|=1} (\psi(y) - \psi(x))$$

- more generally $H = -\Delta_{\mathbb{Z}^d} + V$

- more more gen. $(H\psi)(\overset{(x,f)}{z}) = \sum_{z'} \underset{\text{short ranged}}{H(z,z')}\psi(z')$

Rmk. Think of Bravais latt. as decorated square lattices.

Lemma. Given A on ℓ_2 with kernel $A(x,y)$ & $|A(x,y)| \le c$

then $\|A\|^2 \le a_1 \cdot a_2$, where $a_1 = \sup_x \sum_y |A(x,y)|$, $a_2 = \sup_y \sum_x |A(x,y)|$

Def. (resolvent set) H (s.a.) op, $S(H) := \{ z \in \mathbb{C} \mid (H-z): D(H) \to \mathcal{H} \text{ is bijective} \}$

Fact. $S(H)$ open

- let $R_z(H) = (z-H)^{-1}$ resolvent of H.
  $\to$ analytic for $z \in S(H)$

Spectrum: $\sigma(H) = \mathbb{C} \setminus S(H)$ (closed)

- if H s.a., $\sigma(H) \subset \mathbb{R}$.

- by spectral thm, $\forall$ s.a. op. H on $\mathcal{H}$ $\exists!$ projection valued measure s.t. (hold the thought)

- PVB: for $P: \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$:

i) $P(\Omega) = P(\Omega)^2 = P(\Omega)^*$

ii) $P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$

iii) if $\Omega = \bigsqcup_n \Omega_n$ then $P(\Omega)\psi = \lim_{N\to\infty} \sum_{n=0}^{N} P(\Omega_n)\psi$

- next define $\mu_\psi(\Omega) = \langle \psi, P(\Omega)\psi \rangle$

$\mu_{\psi,\varphi}(\Omega) = \langle \psi, P(\Omega)\varphi \rangle$

(resume) $H = \int \lambda \, dP(\lambda)$, $\langle \psi, H\varphi \rangle = \int d\mu_{\psi,\varphi}(\lambda)\lambda$

- now, given any Borel measure μ,

$$\underset{\substack{\uparrow \\ \text{abs. cont.}}}{\mu = \mu_{a.c.}} + \underset{\substack{\uparrow \\ \text{sing. cont.}}}{\mu_{s.c.}} + \underset{\substack{\uparrow \\ \text{pure pt.}}}{\mu_{p.p}}$$

where $d\mu_{a.c.} = f(\lambda)\,d\lambda$, $f \in L^1(\mathbb{R})$

$d\mu_{s.c.}$ supported on set of 0 Lebesgue-measure

$d\mu_{p.p.}$ supp. on countable set of pts

Spectral subspaces: $\mathcal{H} = \mathcal{H}_{a.c.} \oplus \mathcal{H}_{s.c.} \oplus \mathcal{H}_{p.p}$

where $\mathcal{H}_\# = \{ \psi \in \mathcal{H} \mid \mu_\psi \text{ is of type } \# \}$

- similarly $\sigma_\#(H) = \sigma(H|_{\mathcal{H}_\#})$

Examples:

i) $H = -\Delta_{\mathbb{Z}^d} \Rightarrow \sigma(H) = \sigma_{ac}(-\Delta_{\mathbb{Z}^d}) = [0, 4d]$

ii) $H = V$, $(V\psi)(x) = \omega(x)\psi(x)$

$\Rightarrow \sigma(V) = \sigma_{p.p.}(V) = \bigcup_{x \in \Lambda} \omega(x)$

- eigenstates: i) plane waves
  ii) $\delta_{x_0, x}$

Dynamical characterisation of the spectrum (RAGE thm):

- let H on $\mathfrak{h}$ s.a. op.

$\mathfrak{h}_c = \mathfrak{h}_{a.c.} \oplus \mathfrak{h}_{s.c.} = \{\psi \in \mathfrak{h} \mid \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \, \|\chi(|x|<L) e^{-iHt}\psi\|^2 = 0\}$

$\mathfrak{h}_{p.p.} = \{\psi \in \mathfrak{h} \mid \lim_{L\to\infty} \sup_{t\geq 0} \|(1-\chi(|x|<L))e^{-iHt}\psi\| = 0\}$

- in english: → states in $\mathfrak{h}_{a.c.}$ leave ball $|x|<L$ for good
→ $\mathfrak{h}_{s.c.}$ may come back on measure zero set (which is why we integrate $\frac{1}{T}\int_0^T dt$) but leave on avg
→ $\mathfrak{h}_{p.p.}$ stay

- ok... what about multiparticle systems?

- for 1 ptcl, $\langle O \rangle_\psi = \mathrm{tr}\, O P_\psi$, $P_\psi = |\psi\rangle\langle\psi|$

- Many ptcl: $\psi \in \ell^2(\underbrace{\Lambda \times \cdots \times \Lambda}_{N \text{ times}})$

$\psi \equiv \psi(x_1 \cdots x_N)$, x_i pos. of i-th ptcl.

- identical ptcls $\iff |\psi(x_1 \ldots x_N)| = |\psi(x_{\pi(1)} \cdots x_{\pi(N)})|$

$\hookrightarrow \psi(x_1 \cdots x_N) = \left\{ \begin{matrix} 1 \\ \text{or} \\ \mathrm{sgn}(\pi) \end{matrix} \right\} \psi(x_{\pi(1)} \cdots {\pi(N)})$

$\rightarrow \langle O_N \rangle = \mathrm{tr}\, O_N P_N$

- Marginals

$\Rightarrow$ 1 ptcl density mtrx $\gamma_{\psi_N} := N\, \mathrm{Tr}_{2 \cdots N}\, P_N$

$$\gamma_{\psi_N}(x, y) = \sum_{x_2 \cdots x_N} \psi_N(x, x_2 \cdots x_N) \overline{\psi_N(y, x_2 \cdots x_N)}$$

- label it $\gamma^{(1)}_{\psi_N}$

- now $\langle O_N \rangle_{\psi_N} = \mathrm{tr}\, O_N P_{\psi_N} = \mathrm{tr}\, O \gamma^{(1)}_{\psi_N}$ if $O_N = \sum O^{(i)}$ and $O^{(i)} = \mathbb{1}^{\otimes(i-1)} \otimes O \otimes \mathbb{1}^{\otimes(N-i)}$

- check: $\gamma^{(1)}_{\psi_N} \geq 0$ $\left(\begin{matrix}\text{bosons} \\ \text{and} \\ \text{fermions}\end{matrix}\right)$, $\gamma^{(1)}_{\psi_N} \leq \mathbb{1}$, (fermions)

- consider noninteracting Hamiltonian

$$H_N = \sum_{i=1}^{N} H^{(i)}, \quad H^{(i)} = \mathbb{1}^{\otimes(i-1)} \otimes H \otimes \mathbb{1}^{(N-i)}$$

- suppose $|\Lambda| < +\infty$

- eigenstates look like $\psi_N = f_{i_1} \wedge \cdots \wedge f_{i_N}$,

$f_i \rightsquigarrow$ i-th eigenstate of H.

$\to$ g.s.? take N lowest eigenstates f_i

$$\to \gamma^{(1)}_{\psi_N} = \sum_{i=1}^{\mu} |f_i\rangle\langle f_i|$$

$$= \chi(H \le \mu_N) \text{ where } \operatorname{tr} \chi(H \le \mu_N) = N$$

Infinitely many part. $P := \chi(H \le \mu)$

FERMI PROJECTOR

$\to$ the transport properties depend heavily on properties of P

$\rightsquigarrow$ e.g. $|P(x,y)| \le C_1 e^{-C_2 \|x-y\|} \Leftrightarrow$ insulator

*) $\mu > \|H\| \Rightarrow P(x,y) = \delta_{xy}$

*) $|P(x,y)| \le C_1 e^{-C_2\|x-y\|}$

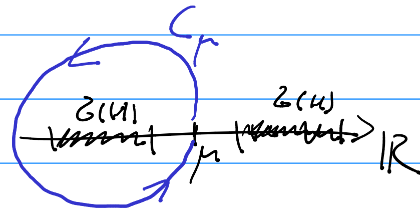
*) H_ω Random Schr. op (R.S.O.) $H_\omega = -\Delta + \lambda \underset{\text{random}}{V}$

$\to$ if $|\lambda| \gg 1$ then point spectrum

- Supp. $\mu \notin \sigma(H)$

$P = \chi(H \leq \mu)$

$= \int_{C_\mu} \frac{dz}{2\pi i} \cdot \frac{1}{z-H}$ where



Prop. Supp $z \notin \sigma(H)$, H s.a. finite ranged.
Then $|R_z(H)(x-y)| \leq C e^{-c|x-y|}$.

Pf. let $\alpha \in \mathbb{C}^d$, def $H_\alpha := e^{\alpha \cdot \hat{x}} H e^{-\alpha \cdot \hat{x}}$,
$H_\alpha(x,y) = e^{\alpha \cdot (x-y)} H(x,y)$

Then $\|H - H_\alpha\| \leq C|\alpha|$ so if $H-z$ inv,
$H_\alpha - z$ inv for small $|\alpha|$.

$$\frac{1}{z-H_\alpha} = e^{\alpha \cdot \hat{x}} \frac{1}{z-H} e^{-\alpha \cdot \hat{x}}$$

$$\Rightarrow |e^{\alpha(x-y)} R_z(H)(x,y)| \leq C.$$

M. Porta

Today: $H_\omega = H + \lambda V_\omega$, $(V_\omega \psi)(x) = \omega(x)\psi(x)$

$\underbrace{\text{not s.a.}\ \ \ }_{\text{s.a. together}}$ $\{\omega(x)\}_{x\in\Lambda}$ i.i.d. random vars., $\omega(x)\in\mathbb{R}$

- also assume $d\mu(\omega(x)) = \rho(\omega(x))\, d\omega(x)$
  $\rho(\cdot)$ bounded, cptly supp.

- $H(x,y) = \overline{H(y,x)}$ is finite ranged & bdded

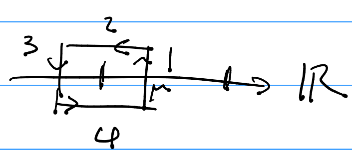
- $\|H_\omega\| \le C$ with $\mathbb{P} = 1$

Thm Let $0<\vartheta<1$. $\exists\, \lambda_0$ s.t. $|\lambda| > \lambda_0$, $\forall z \notin \mathbb{R}$

$$\mathbb{E}_\omega \left| \frac{1}{H_\omega - z}(x,y) \right|^{\vartheta} \le C e^{-c\|x-y\|}$$

Rmk for $\vartheta = 1$, $\mathbb{E}_\omega|\cdot|$ not bdded uniformly in $\operatorname{Im} z$.

Application: decay of $\mathbb{E}_\omega |P_\omega(x,y)|$, $P_\mu = \chi(H_\omega \le \mu)$



1) $$\oint dz \frac{1}{H_\omega - \mu}(x,y) = \int_{-\delta}^{+\delta} d\eta \frac{1}{H_\omega - \mu - i\eta}(x,y)$$

$$\le \int_{-\delta}^{+\delta} d\eta\, \mathbb{E}_\omega \left| \frac{1}{H_\omega - \mu - i\eta}(x,y) \right|$$

$$\le \int_{-\delta}^{+\delta} d\eta \frac{1}{\eta^{1-\vartheta}} \mathbb{E}_\omega \left| \frac{1}{H_\omega - \eta - i\eta}(x,y) \right|^{\vartheta} \le C_\vartheta e^{-c\|x-y\|}$$

- now write $H_\omega = tH + V_\omega$ with $|t| \ll 1$ instead of large λ (for easier notation)

- use resolvent identity

$$\frac{1}{H_\omega - z} = \frac{1}{V_\omega - z} - \frac{1}{H_\omega - z}\, tH\, \frac{1}{V_\omega - z}$$

in components:

$$\frac{1}{H_\omega - z}(x,y) = \frac{1}{\omega(x) - z}\delta_{xy} - \sum_a \frac{1}{H_\omega - z}(x,a)\, t\, H(a,y) \frac{1}{\omega(y) - z}$$

- iterate:

$$\frac{1}{H_\omega - z}(x,y) = \sum_{\gamma : y \to x} (-t)^{|\gamma|} H(x, \gamma(|\gamma|-1)) \cdots H(\gamma(2), \gamma(1))\, H(\gamma(1), y) \times \prod_{n=0}^{|\gamma|} \frac{1}{\omega(\gamma(n)) - z}$$

paths $\gamma(0) = y$, $\gamma(|\gamma|) = x$

↳ can be arb. large → so self ints. can be a problem

→ convergent if $\| tH / \operatorname{Im} z \| < 1$

- we'll use a different expansion

thm (Feenberg expansion)

$$\frac{1}{H_\omega - z}(x,y) = \sum_{\substack{\hat\gamma : y \to x \\ \text{self avoiding}}} (-t)^{|\hat\gamma|} H(x, \hat\gamma(|\hat\gamma|-1)) \cdots H(\hat\gamma(2), \hat\gamma(1)) \cdot H(\hat\gamma(1), y) \cdot \prod_{k=0}^{|\hat\gamma|} \frac{1}{H_{\hat\gamma,k} - z}(\hat\gamma(k), \hat\gamma(k))$$

$$H_{\hat\gamma,k} := H_\omega \big|_{\ell^2(\Lambda \setminus \bigcup_{\tau < k} \{\hat\gamma(\tau)\})}$$

Pf. We define $\hat{\mu}$ as

i) $\hat{\mu}(0) = \mu(0)$

ii) $\forall \tau \in [0, |\mu|)$

$$\hat{\mu}(\tau+1) = \mu(1+\max\{n \in [0, |\mu|) \mid \mu(n) = \hat{\mu}(\tau)\})$$

$\frac{1}{H_{\hat{\mu}, k} - z}(\hat{\mu}(n), \hat{\mu}(n))$ admits an expansion

in terms of loops, avoiding $\hat{\mu}(\tau < k)$.

Prop. (rank 1 formula) Let $z \in \mathbb{C}/\mathbb{R}$.

Let $H = H_0 + \omega(x)|x\rangle\langle x|$ where

H_0 s.a. op independent of $\omega(x)$.

Then $\frac{1}{H-z}(x,x) = \frac{1}{\omega(x) - \Sigma_0(z;x)}$

where $\Sigma_0(z;x) = \left(R_z(H_0)(x,x)\right)^{-1}$

Pf. Just use resolvent identity.

Consequence of frac. moment bound: $\Sigma(H\omega)$

$\overset{\shortparallel}{\Sigma_{p.p}}(H\omega)$, $\mathbb{P}$-1

Porta,

Quantum transport

- we need to introduce time dependence in order to discuss transport
  $H(t) = H + e^{\eta t} Q$, for $t \leq 0$, $\eta > 0$
- e.g. el. field, $Q = -\vec{E} \cdot \vec{x}$

Goal: evolution of $P = \chi(H \leq \mu)$

- remember: $\psi \in \ell^2(\Lambda)$ evolves as $i\partial_t \psi(t) = H(t)\psi(t)$ and $\psi(-\infty) = \psi$

$\rightarrow$ Liouville eq. $i\partial_t P(t) = [H(t), P(t)]$, $P(-\infty) = \chi(H \leq \mu)$

- adiabatic limit: small $\vec{E}$, as $\eta \searrow 0^+$

$$P(0) - P = \int_{-\infty}^{0} dt \, \frac{d}{dt} e^{iHt} P(t) e^{-iHt}$$

$$= \int_{-\infty}^{0} dt \, i \, e^{iHt} [\underbrace{H - H(t)}_{+e^{\eta t} E \cdot x}, P(t)] e^{-iHt}$$

$$= i \int_{-\infty}^{0} dt \, e^{\eta t} e^{iHt} [E \cdot x, P(t)] e^{-iHt}$$

$$= i \int_{-\infty}^{0} dt \, e^{\eta t} e^{iHt} [E \cdot x, P] e^{-iHt} + \text{``}O(E^2)\text{''}$$

- velocity $v_i = i[H(t), x_i] = i[H, x_i]$
- current: $J_i = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \operatorname{Tr} \mathbb{1}_{\Lambda_L} v_i (P(0) - P)$

$$\doteq \mathcal{T}(v_i (P_0 - P)) = \sum \sigma_{ij} E_j$$

where $|\Lambda_L| < \infty$ but we want $L \to \infty$ limit.

$\Rightarrow \sigma_{ij} = \lim_{\eta \searrow 0^+} \tau\left(\int_{-\infty}^{0} dt \; v_i e^{itH} [X_j, P] e^{-itH} e^{\eta t} \right)$

Thm Let $d=2$. Then $|P(x,y)| \leq C e^{-c\|x-y\|}$

Thm $\sigma_{12} \in \frac{1}{2\pi} \mathbb{Z}$, $\sigma_{11} = 0$

$K_2 \hookrightarrow \mathcal{L}(h)$, $K_2 \doteq \{ M \in \mathcal{L}(h) \mid \tau(M^* M) < +\infty \}$

- e.g., $|M(x,y)| \leq C e^{-c\|x-y\|} \Rightarrow M \in K_2$

- define $\langle A, B \rangle = \tau(A^* B)$ on K_2,
  so $(K_2, \langle \cdot, \cdot \rangle)$ Hilb. sp.
  ($\exists$ possible to show $\tau(AB) = \tau(BA)$).

- let H s.a. on $\ell^2(\Lambda)$ finite ranged
  & let $\mathcal{L}_H \circlearrowright K_2$, $\mathcal{L}_H(M) = [H, M]$.
  - $\rightarrow$ fact: $\mathcal{L}_H$ s.a. on K_2
  - $\rightarrow$ so $t \mapsto e^{it\mathcal{L}_H}$ is unitary

$$e^{it\mathcal{L}_H} = \sum_{n \geq 0} \frac{(it)^n}{n!} \mathrm{ad}_H^n(M)$$

$$e^{it\mathcal{L}_H}(M) = e^{iHt} M e^{-iHt},$$

$$(\mathcal{L}_H - i\eta)^{-1} = i \int_{-\infty}^{0} dt \, e^{it\mathcal{L}_H} e^{\eta t}$$

- now $\sigma_{ij} = \lim_{\eta \downarrow 0^+} i\,\tau\left(\int_{-\infty}^{0} dt\; v_i e^{\eta t} e^{itH}[x_j, P] e^{-itH}\right)$

$$= \lim_{\eta \downarrow 0^+} \tau\Big(v_i\,(\mathcal{L}_H - i\eta)^{-1}\big(\underbrace{[x_j, P]}_{\in \mathcal{K}_2}\big)\Big)$$

Thm. Suppose $|P(x,y)| \le C e^{-c\|x-y\|}$.
Then $\sigma_{ij} = i\,\tau\, P\big[[x_i, P], [x_j, P]\big]$

Pf. Note $[x_i, P] = P[x_i, P]P^\perp + P^\perp[x_i, P]P$,
where of course $P^\perp = 1 - P$,
and note $\mathcal{L}_H(PM) = P\,\mathcal{L}_H(M)$.

$$(\mathcal{L}_H - i\eta)^{-1}\big(P[x_i, P]P^\perp + P^\perp[x_i, P]P\big)$$

$$= P(\mathcal{L}_H - i\eta)^{-1}[x_i, P]P^\perp + P^\perp(\cdots)^{-1}[x_i, P]P$$

- using cyclicity of τ,

$$\tau\big(v_i(\mathcal{L}_H - i\eta)^{-1}[x_i, P]\big)$$

$$= \tau\Big(\underbrace{(P^\perp v_i P + P v_i P^\perp)}_{i\mathcal{L}_H([[x_i, P], P])}(\mathcal{L}_H - i\eta)^{-1}([x_j; P])\Big)$$

, using $[H, P] = 0$

$$\sigma_{ij} = \lim_{\eta \downarrow 0^+} i\,\tau\Big(\mathcal{L}_H([[x_i, P], P])\,(\mathcal{L}_H - i\eta)^{-1}([x_j, P])\Big)$$

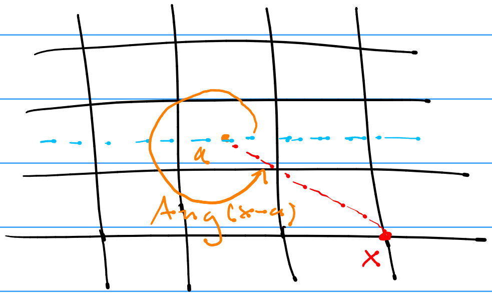
$$= \lim_{\eta \downarrow 0^+} i\,\big\langle \mathcal{L}_H([[x_i, P], P])^*, (\mathcal{L}_H - i\eta)^{-1}[x_j, P]\big\rangle$$

$$= \lim_{\eta \downarrow 0^+} i\,\big\langle \underbrace{(\mathcal{L}_H + i\eta)^{-1}\mathcal{L}_H}_{= 1 \text{ in the limit.}}([[x_i, P]P]), [x_j, P]\big\rangle$$

$$= i\,\tau\big([[x_i, P], P]\,[x_j, P]\big). \quad \square$$

$d=2$.

$a \in \mathbb{Z}^2 + \{1/2, 1/2\}$



$\psi \mapsto U_a\psi$,

$(U_a\psi)(x) = e^{i\vartheta_a(x)}\psi(x)$,

$\vartheta_a(x) = \mathrm{Arg}(x-a)$

- magnetic flux. $\Delta Q = \mathrm{Tr}(\overbrace{U_a P U_a^*}^{Q} - P)$

$\neq$ well-defined.

Def. let P, Q be 2 orth. proj. such that $P-Q$ is compact $\left(P-Q = \sum a_n |f_n\rangle\langle f_n|\right)$, $a_n \to 0$ as $n\to\infty$

Define: $\mathrm{Ind}(P,Q) = \dim\ker(P-Q-1) - \dim\ker(P-Q+1)$.

Rmk.
- $\ker(P-Q-1) = \{\psi\in\mathcal{H} \mid P\psi=\psi,\ Q\psi=0\}$
- $\ker(P-Q+1) = \{\psi\in\mathcal{H} \mid P\psi=0,\ Q\psi=\psi\}$
- by cptness of $P-Q$, $\dim\ker(P-Q-1)$ is finite, since in $\left(\sum (a_n-1)|f_n\rangle\langle f_n|\right.$, only finitely many a_n can $=1$ $\left.\right)$

Thm (quantization of σ_{12}) let $d=2$, $|P(x,y)| \le Ce^{-c\|x-y\|}$.

Then $\sigma_{12} = \frac{1}{2\pi}\mathrm{Ind}(P, U_a P U_a^*)$

Prop. let P, Q orth. proj. on $\mathcal{H}$. Supp $(P-Q)^{2n+1}$ is trace class for some n.

Then $\mathrm{Ind}(P,Q) = \mathrm{Tr}_{\mathcal{H}}(P-Q)^{2n+1}$

Pf. Show that $\sigma(P-Q)$ is given by pairs $(-\lambda,\lambda)$ with same multiplicities if $|\lambda|<1$.

So $\mathrm{Tr}(P-Q)^{2n+1} = \sum (b_m)^{2n+1} = (+1)^{2n+1}(\text{mult of } +1) + (-1)^{2n+1}(\text{mult. of } -1)$

dim ker(P−Q+1)

dim ker(P−Q−1)

- so, check that if λ eigenv. of $P-Q$, so is $-\lambda$;
- def $C = P-Q$, $S = P+Q$, check $S^2+C^2 = 1$, $SC+CS=0$.
- let $\lambda \in \sigma(C)\setminus\{0\}$, $\varphi \in h$ its eigenvector
  → claim, $S\varphi$ eigenvector w ev. $S\varphi$
- $CS\varphi = -SC\varphi = -\lambda S\varphi$.
- for $S : \ker(C-1) \to \ker(C+1)$ we claim

Injects Supp. $\ker(C-1)\setminus\{0\} \ni \varphi$.

$S\varphi = 0 \Rightarrow S^2\varphi = 0 \Rightarrow (1-C^2)\varphi = 0$

$(1-\underbrace{\lambda^2}_{\ne 1})\varphi = 0$ $\Rightarrow \varphi = 0$ ↯

Surjects given $\varphi \in \ker(C+1)$, $\exists\, \tilde\varphi \in \ker(C-1)$:

$$S\tilde\varphi = \frac{S^2\varphi}{(1-\lambda^2)} = \frac{(1-C^2)}{(1-\lambda)^2}\varphi = \varphi$$

so multiplicities same. □

- next we'll take $P = \chi(x \le 0)$, $Q = U_a P U_a^*$
  → $(P - U_a P U_a^*)^3$ will be tr. class...

Porta.

Conductivity quantization, cont'd.

- we saw if $(P-Q)^{2n+1}$ tr. class, $\mathrm{Ind}(P,Q) = \mathrm{Tr}(P-Q)^{2n+1}$

→ take $P = \chi(H \le \mu)$, $Q = U_a P U_a^*$

- claim: $(P - U_a P U_a^*)^3$ is tr. cl.

$$(P - U_a P U_a^*)(x+\alpha, x) = \underbrace{P(x+\alpha, x)}_{\le Ce^{-c\|\alpha\|}} \underbrace{\left(1 - e^{i\theta_a(x)} e^{-i\theta_a(x+\alpha)}\right)}_{\le \frac{c}{\|x\|}}$$

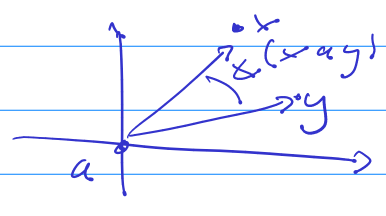
- fact: $\|T\|_1 = \mathrm{Tr}|T| \le \sum_a \sum_x |T(x+a, x)|$

→ so $(P - \ldots)^3$ is tr class because the bound becomes summable

Prop $\sigma_{12} = \frac{1}{2\pi} \mathrm{Tr}\,(P - U_a P U_a^*)^3$

Pf. $$\mathrm{Tr}(P - U_a P U_a^*)^3 = \sum_{xyz} P(x,y) P(y,z) P(z,x) \left.\begin{array}{l} \cdot\left(1 - e^{i(\theta_a(x) - \theta_a(y))}\right) \\ \cdot\left(1 - e^{i(\theta_a(y) - \theta_a(z))}\right) \\ \cdot\left(1 - e^{i(\theta_a(z) - \theta_a(x))}\right) \end{array}\right\}(*)$$

$$(*) = 2i\left(\sin \measuredangle(z,a,x) + \sin \measuredangle(x,a,y) + \sin\measuredangle(y,a,z)\right)$$



$$\mathrm{Tr}(P - U_a P U_a^*)^3 = \frac{2i}{L^2} \sum_{a \in \Lambda_L^*} \sum_{xyz} P(x,y) P(y,z) P(z,x)\, S_a(x,y,z)$$

$$= \frac{2i}{L^2} \sum_{a \in \Lambda_L^*} \sum_{x \in \Lambda_L} \sum_{y,z} P(x,y)P(y,z)P(z,x)\, S_a(x,y,z)$$

$$+ \frac{2i}{L^2} \underbrace{\sum_{a \in \Lambda_L^*} \sum_{x \in \mathbb{Z}^2 \setminus \Lambda_L} \sum_{y,z} (\;\;)\, \underbrace{S_a(x,y,z)}_{\leq \max\left(|a-x|^{-3}, |a-y|^{-3}, |a-z|^{-3}\right)}}_{\to 0 \text{ as } L \to \infty}$$

$$= \frac{2i}{L^2} \sum_{x \in \Lambda_L} \sum_{y,z} \sum_{a \in \mathbb{Z}^{2*}} (\ldots) + (\text{previous error})$$

$$- \frac{2i}{L^2} \sum_{x \in \Lambda_L} \sum_{y,z} \sum_{a \in \mathbb{Z}^{2*} \setminus \Lambda_L^*} (\ldots)$$

$\hookrightarrow$ again $O(L^{-2})$

Claim: $\frac{1}{2\pi} \sum_{a \in \mathbb{Z}^{2*}} S_a(x,y,z) = \underset{u}{Area}(x,y,z)$

$\frac{1}{2}(x-y) \wedge (y-z)$

$\rightarrow$ assuming that, we get

$$Tr(P - U_a P U_a^*)^3 = \frac{2\pi \cdot 2i}{L^2} \sum_{x \in \Lambda_L} \sum_{y,z} P(x,y)P(y,z)P(z,x) \cdot \left[(x_1-y_1)(y_2-z_2) - (1 \leftrightarrow 2)\right]$$

- note: $P(x,y)(x_1-y_1) = [x_1, P](x,y)$ ...

$$= \frac{2\pi i}{L^2} \sum_{x \in \Lambda_L} \Big(Tr[x_1, P][x_2, P]P - Tr[x_2, P][x_1, P]P + O(1) \Big).$$

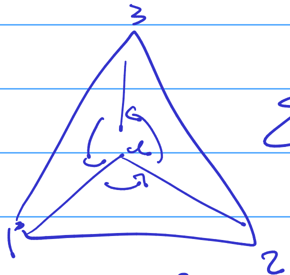
Prop let $g(\alpha)$ be bounded func., $g(\alpha) = -g(-\alpha)$, $g(\alpha) = \alpha + O(\alpha^3)$ as $\alpha \to 0$. Let $u_1, u_2, u_3 \in \mathbb{Z}^2$, $a \in \mathbb{Z}^{2*}$, let $\alpha_i(a) = \measuredangle(u_{i+1}, a, u_{i+2}) \in (-\pi, \pi]$ with convention $\measuredangle(\text{collinear}) = 0$.

Then $\sum_{a \in \mathbb{Z}^{2*}} \sum_{i=1,2,3} g(\alpha_i(a)) = 2\pi \, \text{Area}(u_1, u_2, u_3)$

Pf. Look first at $g(\alpha) = \alpha$, assume $\underset{u_1 \quad u_2}{\overset{u_3}{\triangle}}$ positively oriented,

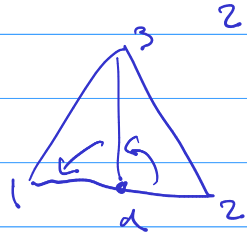
then $\sum_{i=1}^{3} \alpha_i(a) = 2\pi \cdot \begin{cases} 1 \\ 1/2 \\ 0 \end{cases}$ for $a \begin{cases} \text{inside } \triangle \\ \text{on } \partial\triangle \\ \text{outside } \triangle \end{cases}$

1°



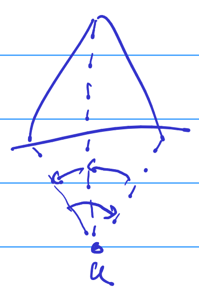
$\sum \alpha_i = 2\pi$

11°



$\alpha_3 = 0$
$\Rightarrow \sum \alpha_i = \pi$

111°



$\sum \alpha_i = 0$

So $\frac{1}{2\pi} \sum_{a \in \mathbb{Z}^{2*}} \sum_{i=1,2,3} \alpha_i(a) = \#$ of lattice pts of $\mathbb{Z}^{2*}$ inside $\triangle$ and $\frac{1}{2} \# \ldots$ on $\partial\triangle$

This $\#$ does not change if we translate $\triangle$ in $\mathbb{Z}^2$ or reflect it wrt sym. axis of $\mathbb{Z}^2$ and $\mathbb{Z}^{2*}$

Triangles we obtain in that manner tile the plane.

Conclude: $\frac{1}{2\pi} \sum_a \sum_{i=1,2,3} \alpha_i(a) = \text{Area}(\Delta)$

If $\neq \text{Area}(\Delta)$, let $\Lambda \subseteq \mathbb{Z}^2$ be a region obtained as tiling by copies of Δ.

$\Rightarrow |\{\#\text{ lattice pts of } \mathbb{Z}^{2*} \text{ in } \Lambda\}|$

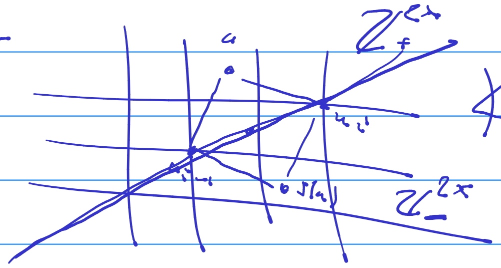
$= \sum_{\Delta \subset \Lambda} |\{\# \cdots \text{ in } \Delta\}|$,

as $|\Lambda| \to \infty$, $\frac{|\{\# \cdots\}|}{|\Lambda|} \to 1$, so we can get claim.

Now if $g(\alpha) \neq \alpha$, claim: $\underline{\sum_{a \in \mathbb{Z}^{2*}} \sum_{i=1,2,3} \left(g(\alpha_i(a)) - \alpha_i(a)\right) = 0}$.

Idea: 1-to-1 corr. between $a \in \mathbb{Z}^{2*}$ and $\sigma(a) \in \mathbb{Z}^{2*}$ s.t. $\alpha_i(\sigma(a)) = -\alpha_i(a)$.

- now



$\measuredangle(u_{i+1}, a, u_i)$

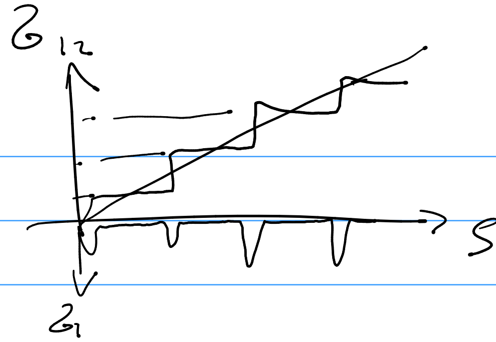
$= -\measuredangle(u_{i+1}, \sigma(a), u_i)$

- so, letting $f(\alpha_i(a)) = g(\alpha_i(a)) - \alpha_i(a)$,

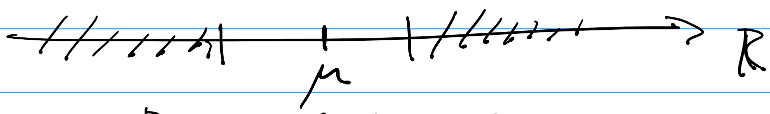
$$\sum_{a \in \mathbb{Z}^{2*}} f(\alpha_i(a)) = \sum_{a \in \mathbb{Z}^{2*}_+} \Big(f(\alpha_i(a)) + f(\underbrace{\alpha_i(\sigma(a))}_{-\alpha_i(a)}) \Big)$$

$= 0.$ $\square$

<u>Recall</u> IQHE

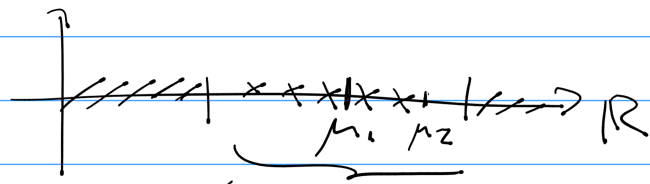


- H gapped:



$\Rightarrow P = \chi(H \leq \mu)$

- for $H_\omega = H + \lambda V_\omega$, $|\lambda| >> 1$,



"mobility gap"

$\Rightarrow \mathbb{E}\, \sigma_{12}\big|_{\mu_1} = \mathbb{E}\, \sigma_{12}\big|_{\mu_2}$

- we get the jumps when μ gets to the continuous spectra

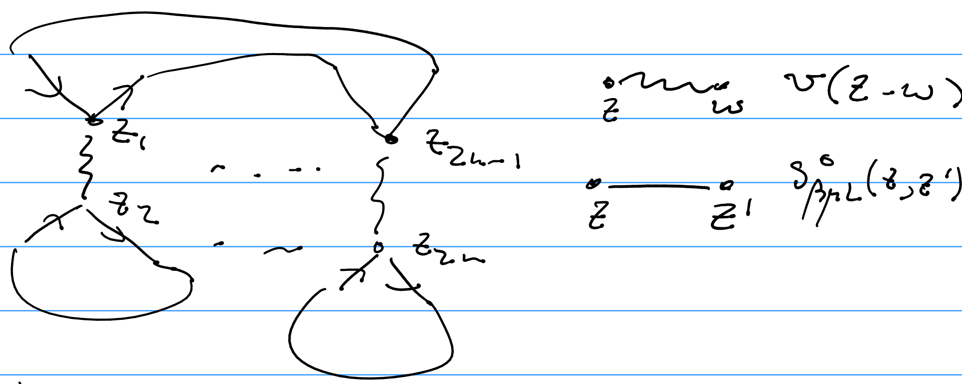
Part a.

$$\frac{Z_{\beta,\mu,L}}{Z^0_{\beta,\mu,L}} = 1 + \sum_{n\geq 1} \frac{(-)^n}{n!} \int_{[0,\beta]^n} d\underline{t} \, \langle \mathbf{T} V_{t_1} \cdots V_{t_n} \rangle^0_{\beta,\mu,L}$$

where $V_t = e^{t(\mathcal{H}_0 - \mu \mathcal{N})} \, V \, e^{-t(\mathcal{H}^0 - \mu \mathcal{N})}$

$$= \sum_{n\geq 0} \frac{(-)^n}{n!} \int d\underline{Z}_n \, d\underline{W}_n \sum_{\xi \in G_n(\underline{Z}_n, \underline{W}_n)} \mathrm{Val}(\xi) .$$

$\xi \rightsquigarrow$



$$\mathrm{Val}(\xi) = \mathrm{sign}(\xi) \prod_{i=1}^n v(z_{2i-1} - z_{2i}) \prod_{e\in\xi} g_e$$

where $g_e = S^0_{\beta\mu L}(z(e), z'(e))$

Goal. Prove analiticity of $f_{\beta\mu L} = -\frac{1}{\beta|\Lambda_L|} \log Z_{\beta\mu L}$

Problems. (*) estimate of $\mathrm{Val}(\xi)$
(*)

- if gapped:

$$\frac{1}{\beta|\Lambda_L|} \int d\underline{Z} \, |\mathrm{Val}(\xi)| \leq \|v\|_1^n \, \|g_e\|_1^{n-1} \, \underbrace{\|g_e\|_\infty^{n+1}}_{\text{loops}}$$

since $|S^0_{\beta\mu L}(z,z')| \leq C e^{-c\|z-z'\|}$ for gap

- but we're completely ignoring signs

- also, # graphs $\simeq n!$ at $O(n)$, and series has only $\frac{1}{n!}$.

- Brydges - Battle - Federbush:

- let $f(n) = \sum_{\xi \in G_c(n)} Val(\xi)$ $[\prod v(z_{2i} - z_{2i-1})]$

- then $f(n) = \sum_{T \in T_n} \left[\prod_{e \in T} g_e\right] \times \int d\mu_T(t) \det G_T(t)$

where:

1) T is a tree between n vertices (by int. out $\xi \xrightarrow{\int} \bullet$ interactions)

ii) $G_T(t)$ is a $(2n-(n-1)) \times (2n-(n-1))$ matrix with entries

$$[G_T(t)]_{(j,i)(j',i')} = t_{j,j'}\, g\big(\underbrace{x(j,i) - x(j',i')}_{\ell}\big)$$

where $1 \le j, j' \le n$, $1 \le i, i' \le 2$,
$x(j,1) = z_{2j-1}$, $x(j,2) = z_{2j}$

and the element is only nonzero provided the vertices are connected by propagators

iii) $t_{j,j'} \in [0,1]$, $d\mu_T(t)$ is a ? measure on $\{t_{jj'}\}$

Rmk # of $T \sim n!$

Problem. estimate det?

Gram-Hadamard ineq. let M $n \times n$ matrix such that $M_{ij} = \langle A_i, B_j \rangle$.
Then $|\det M| \le \prod_{i=1}^{n} \|A_i\| \cdot \|B_i\|$

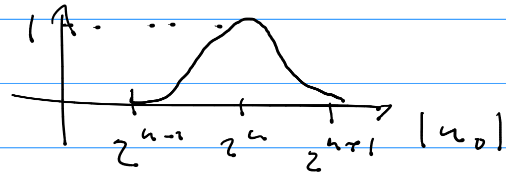
- it turns out we get UV divergence

Multiscale analysis (RG)

- write $g^{(\leq N)} = g^{(\leq 0)} + \sum_{k=0}^{N} g^{(k)}$

$$g^{(h)}(\vec z, \vec z') = \frac{1}{\beta} \sum_{\substack{k_0 \in \mathbb{M}_\beta \\ \frac{2\pi}{\beta}(n+\frac{1}{2})}} \frac{e^{-ik_0(z_0-z_0')}}{-ik_0 + H - \mu}(\vec z, \vec z')\, f_h(k_0)$$

where $f_h(k_0) = \chi(2^{-k}|k_0|) - \chi(2^{-(k-1)}|k_0|)$



Graßmann variables.

-> finite set $(\psi^+_\alpha, \psi^-_\alpha)_{\alpha \in \Lambda}$ s.t. $\{\psi^\varepsilon_\alpha, \psi^{\varepsilon'}_{\alpha'}\} = 0 \quad \forall \alpha\, \alpha'\, \varepsilon\, \varepsilon'$

-> let $(d\psi^+_\alpha, d\psi^-_\alpha)_{\alpha \in \Lambda}$ s.t. $\{d\psi^\varepsilon_\alpha, d\psi^{\varepsilon'}_{\alpha'}\} = 0$, $\{d\psi^\varepsilon_\alpha, \psi^{\varepsilon'}_{\alpha'}\} = 0$

- def. $\int d\psi^\varepsilon_\alpha = 0$, $\int d\psi^\varepsilon_\alpha\, \psi^\varepsilon_\alpha = 1$

- $F(\psi)$ polynomial $\Rightarrow F(\psi) = \sum_X f(X)\, \underbrace{\psi(X)}_{\prod_{(\alpha,\varepsilon)\in X} \psi^\varepsilon_\alpha}$

- $\int \psi(X) \prod_{\alpha \in \Lambda} d\psi^+_\alpha d\psi^-_\alpha = \mathrm{sign}(\pi(X))$

$\Rightarrow \psi(X) = \mathrm{sign}(\pi(X))\, \psi^-_{|\Lambda|} \psi^+_{|\Lambda|} \cdots \psi^-_1 \psi^+_1$

Prop (Gaussian int.) let M be $|A|\times|A|$ mtx.

Then
$$\frac{\int D\psi\, e^{-\sum_{\alpha\beta}\psi_\alpha^+ M_{\alpha\beta}\psi_\beta^-}\,\psi_{\beta'}^-\psi_{\alpha'}^+}{\int D\psi\, e^{-\sum_{\alpha,\beta}\psi_\alpha^+ M_{\alpha\beta}\psi_\beta^-}} = (M^{-1})_{\beta'\alpha'}$$

- given g $|A|\times|A|$ invertible (& diagonizable) w $\{g_\alpha\}_{\alpha\in|A|}$: eigenvalues, let $C=g^{-1}$ and define

$$P(d\psi)=\left[\prod_{\alpha\in A} d\psi_\alpha^+ d\psi_\alpha^- g_\alpha\right] e^{-\sum_{\alpha,\beta}\psi_\alpha^+ C_{\alpha\beta}\psi_\beta^-}$$

- immediately: $\int P(d\psi)=1$, $\int P(d\psi)\,\psi_\alpha^-\psi_\beta^+ = g_{\alpha\beta}$

Prop (Grass. Wick rule)

$$\int P(d\psi)\,\psi_{\alpha_1}^-\cdots\psi_{\alpha_n}^-\psi_{\beta_1}^+\cdots\psi_{\beta_m}^+$$
$$=\delta_{n,m}\sum_\pi \mathrm{sign}(\pi)\prod_{i=1}^n g_{\alpha_i,\pi(\beta_i)}$$

Rmk. $P_{g_1+g_2}(d\psi) = P_{g_1}(d\psi_1)\,P_{g_2}(d\psi_2)$
(addition principle)

$\rightarrow$ very important & useful!
- we can integrate out things one by one..

Porta.

- $A = \mathbb{M}_\beta^{(\leq N)} \times \{1, \dots, \dim(\mathcal{Y})\}$, $\mathbb{M}_\beta^{(\leq N)} = \{k_0 \in \mathbb{M}_\beta \mid \chi_N(k_0) > 0\}$
- $g_\alpha = \dfrac{\chi_N(u_0)}{-iu_0 + e_\alpha - \mu}$, $Hf_\alpha = e_\alpha f_\alpha$, $\chi_N(u_0) = \chi(\gamma^{-N}|u_0|)$
- $g^{(\leq N)}(z, z') = \dfrac{1}{\beta} \sum_{u_0 \in \mathbb{M}_\beta^{(\leq N)}} \sum_\alpha e^{-iu_0(z_0 - z_0')} \dfrac{f_\alpha(\vec{z})\overline{f_\alpha(\vec{z}')}}{-iu_0 + e_\alpha - \mu}$

$P_{\leq N}(d\psi) = \left[\prod_{\alpha \in A} g_\alpha\right] D\psi \exp\{-\sum_\alpha \psi_\alpha^+ g_\alpha^{-1} \psi_\alpha^-\}$.

$\psi_z^+ = \dfrac{1}{\sqrt{\beta}} \sum_{(u_0,\alpha) \in A} e^{iu_0 z_0} \overline{f_\alpha(\underline{z})}\, \psi^+_{(u_0,\alpha)}$

$\psi_z^- = \dfrac{1}{\sqrt{\beta}} \sum_{(u_0,\alpha) \in A} e^{-iu_0 z_0} f_\alpha(\underline{z})\, \psi^-_{(u_0,\alpha)}$

$V(\psi) := \lambda \int dz\, dz'\, \psi_z^+ \psi_{z'}^+ \psi_{z'}^- \psi_z^- \, v(z - z')$

$\langle \vartheta(\psi) \rangle_{\beta, L, \mu}^{(\leq N)} = \int P_{\leq N}(d\psi) e^{-V(\psi)} \vartheta(\psi)$

- free energy $f_{\beta,\mu,L,N} = -\dfrac{1}{\beta|\Lambda_L|} \log \int P_{\leq N}(d\psi) e^{-V(\psi)}$

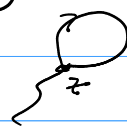
- our model: $\int P_{\leq N}(d\psi) \psi_{z_1}^- \cdots \psi_{z_n}^- \psi_{w_1}^+ \cdots \psi_{w_n}^+$
$= \sum_\pi \operatorname{sign}(\pi) \prod_{i=1}^n g^{(\leq N)}(z_i - w_{\pi(i)})$.

$\underline{N \to \infty}$? $\lim_{N\to\infty} g^{(\leq N)}(z, z') = g(z, z')$ $(z_0 \neq z_0')$.

- but at coinciding pts:

$\Delta := \underbrace{\lim_{N \to \infty} g^{(\leq N)}(z, z)}_{\frac{1}{2}\left(g(z,z)|_{z_0 - z_0' = 0^+} + g(z,z)|_{z_0 - z_0' = 0^-}\right)} - g(z, z) \neq 0$

$= \frac{1}{2}\left(g(z,z)|_{z_0 - z_0' = 0^+} - g(z,z)|_{z_0 - z_0' = 0^-} \right)$
$= \frac{1}{2}\left(\langle a_{\underline{z}}^+ a_{\underline{z}}^- \rangle - (-\langle a_{\underline{z}}^- a_{\underline{z}}^+ \rangle)\right)$
$= \frac{1}{2} \langle \{a_{\underline{z}}^+, a_{\underline{z}}^-\} \rangle = \frac{1}{2}$

- this discontinuity is only visible in tadpole diagrams  $\rightsquigarrow g(z,z)$

- "fix" it by adding $-1/2$ to number operat.:

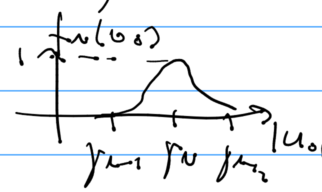
$$V = \lambda \sum_{z,z'} (a^+_z a^-_z - 1/2)\, v(z-z')\, (a^+_{z'} a^-_{z'} - 1/2)$$
$$= V_{old} + c(v)\lambda/2 \cdot N \quad \leftarrow \text{just a shift in } \mu$$

- multiscale analysis

$$\psi^{(\leq N)} = \psi^{(\leq N-1)} + \psi^{(N)}$$

$$g^{(N)}(z,z') \overset{\text{cov.}}{=} 1/\beta \sum_{k_0 \in \mathbb{M}_\beta^{(\leq N)}} e^{-ik_0(z_0 - z_0')} \frac{f_N(k_0)}{-ik_0 + H - \mu}(\underline{z},\underline{z}')$$

where $f_N(k_0) = \chi_N(k_0) - \chi_{N-1}(k_0)$



Claim: $\left| g^{(N)}(z,z') \right| \leq \dfrac{C_M}{1 + (2^N \underbrace{|z_0 - z_0'|_\beta}_{=\min(|z_0-z_0'|,\, |z_0-z_0'-\beta|)})^M} e^{-c\|\underline{z}-\underline{z}'\|} \quad \forall M \in \mathbb{N}$

UV-multiscale analysis

- goal: $\int P_{\leq N}(d\psi^{(\leq N)})\, e^{-V(\psi^{(\leq N)})} \overset{\text{notation}}{=} \mathbb{E}_{\leq N}\left(e^{-V(\psi^{(\leq N)})}\right)$

$$= \mathbb{E}_{\leq N-1}\, \mathbb{E}_N\left(e^{-V(\psi^{(\leq N-1)} + \psi^{(N)})}\right)$$
$$= e^{E_{N-1}}\, \mathbb{E}_{\leq N-1}\left(e^{-V^{(N-1)}(\psi^{(\leq N-1)})}\right)$$

where $E_{N-1} = \log \int P_N(d\psi^{(N)})\, e^{-V(\psi^{(N)})}$

$V^{(N-1)}(\psi^{(\leq N-1)}) = -E_{N-1} + \log \int P_N(d\psi^{(\leq N)})\, e^{-V(\psi^{(\leq N-1)} + \psi^{(N)})}$

Rewrite: $V(\psi) = \sum_P \int dz\, W_P^{(N)}(z)\, \psi_z(P)$.

$\psi_z(P) = \prod_{f\in P} \psi_{z(f)}^{\varepsilon(f)}$, $W_P^{(N)} = 0$ unless $\overbrace{|P|=4}^{\text{quartic. int.}}$

$W_P^{(N)} = \lambda\, \delta(z_1-z_2)\,\delta(z_2-z_3)\,\delta(z_3-z_4)$
for $v(z-z') = \lambda\delta(z_0-z_0')\,\delta_{\underline{z}\underline{z}'}$ (of course $|P|=4$)

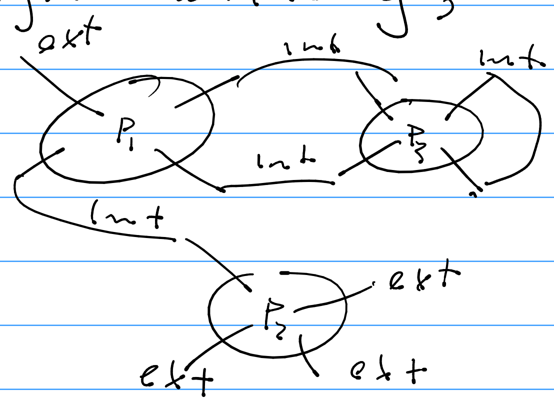
Cumulants: $\mathbb{E}_N^c(A_1(\psi^{(N)}),\dots,A_s(\psi^{(N)}))$
$:= \partial^s_{\varepsilon_1\cdots\varepsilon_s} \log \mathbb{E}_N(e^{\sum_i \varepsilon_i A_i(\psi)})\Big|_{\varepsilon=0}$
- conn. Feyn. diagrams

$\Rightarrow V^{(N-1)}(\psi^{(\le N-1)}) = \sum_{|P|\geq 2} P \int dz\, W_P^{(N-1)}(z)\, \psi_z^{(\le N-1)}(P)$

where $W_P^{(N-1)}(z) = \sum_{s\geq 1} \frac{1}{s!} \sum_{\substack{P_1\cdots P_s \\ \cup_i P_i^{ext}=P \\ |P_i|\geq 2}} \int dz_1^{int}\cdots dz_s^{int}$

$\mathbb{E}_N^c(\psi^{(N)}_{z_1^{int}}(P_1^{int}),\dots,\psi^{(N)}_{z_s^{int}}(P_s^{int}))\cdot \prod_{i=1}^s W_{P_i}^{(N)}(z_i)$

diagrammatically,



$P_i = P_i^{ext} \cup P_i^{int}$

Porta.

- what we did:

*) $\langle\ \rangle_{\beta L \mu}$ analytic for $|\lambda| < \lambda_{\beta, L}$

*) Improve? using info on H (like gap)

→ pert. theory. Problem? combinatorics causes $O(n!)$ growth

BBF formula - good if $g(z,z') = \langle A_z, B_{z'} \rangle$

- not really the case due to disc. in T

*) solution? regularized theory $\lim_{N\to\infty} g^{(\leq N)} = g$

- however, now $\|A_z^{(\leq N)}\| \cdot \| B_z^{(\leq N)}\| \leq C\, 2^{3N}$

- maybe not an optimal bound. try:

Task: multiscale analysis $g^{(\leq N)} = g^{(\leq 0)} + \sum_{h>0}^{N} g^{(h)}$ instead of all scales $(\leq N)$ at once

- free energy $f_{\beta,L,N} = -\frac{1}{\beta|\Lambda_L|} \log \int P_{\leq N}(d\psi^{(\leq N)}) e^{-V(\psi^{(\leq N)})}$

- now, by fermion magic,

$$\int P_{\leq N}(d\psi^{\leq N}) e^{-V(\psi^{(\leq N)})} = \overbrace{\int P_N(d\psi^N) e^{-V(\psi^{(N)})}}^{e^{\epsilon_{N-1}}} \cdot \int P_{\leq N-1}(d\psi^{(\leq N-1)}) e^{-V^{(N-1)}(\psi^{(\leq N-1)})}$$

$$-V^{(N-1)}(\psi^{(\leq N-1)}) = \sum_{\substack{|P|\geq 2 \\ \text{even}}} \int dz\, W_P^{(N-1)}(z)\, \psi_z^{(\leq N-1)}(P)$$

$$\psi_z^{(\leq N-1)}(P) = \prod_{g\in P} \psi_{z(g)}^{(\leq N-1)\, \varepsilon(g)}$$

- inductively, $\mathbb{E}_{\leq N}(e^{-V(\psi^{(\leq N)})}) = e^{\sum_{T=h}^{N-1} \epsilon_T}\, \mathbb{E}_{\leq h}(e^{-V^{(h)}(\psi^{(\leq h)})})$

where $V^{(h)}(\psi^{(\leq h)}) = \sum_P \int dz\, W_P^{(h)}(z)\, \psi_z^{(\leq h)}(P)$

- some estimates were being made, did not write
- in conclusion, $f_{\beta,L,N} = f_{\beta,L,0} + \sum_{T=0}^{N-1} \frac{\mathcal{E}_T}{\beta|\Lambda_L|}$

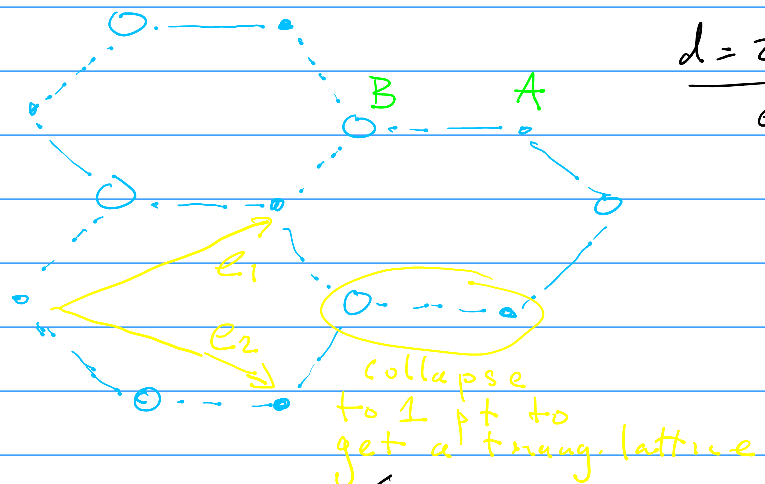
uniform in β, L, N, analytic for $|\lambda| < \lambda'$,

the limit $\beta, L, N \to \infty$ exists

- however, the existance of a gap was crucial
  - serious infrared problems

- let's look at a specific model

Interacting graphene



$\underline{d=2}$, H = Laplacian on honeycomb lattice

$$\Lambda \simeq \mathbb{Z}^2 \times \{A, B\}, \quad \psi(x) = \begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix}$$

$$(H\psi)(x) = \begin{pmatrix} \psi_B(x) + \psi_B(x+e_1) + \psi_B(x+e_2) \\ \psi_A(x) + \psi_A(x-e_1) + \psi_A(x-e_2) \end{pmatrix} \begin{matrix} \leftarrow A \\ \leftarrow B \end{matrix}$$

$$\widehat{(H\psi)}(\underset{\mathbb{T}^2}{k}) = \hat{H}(k)\hat{\psi}(k), \quad \hat{H}(k) = \begin{pmatrix} 0 & -t\Omega(k) \\ -t\overline{\Omega(k)} & 0 \end{pmatrix}$$

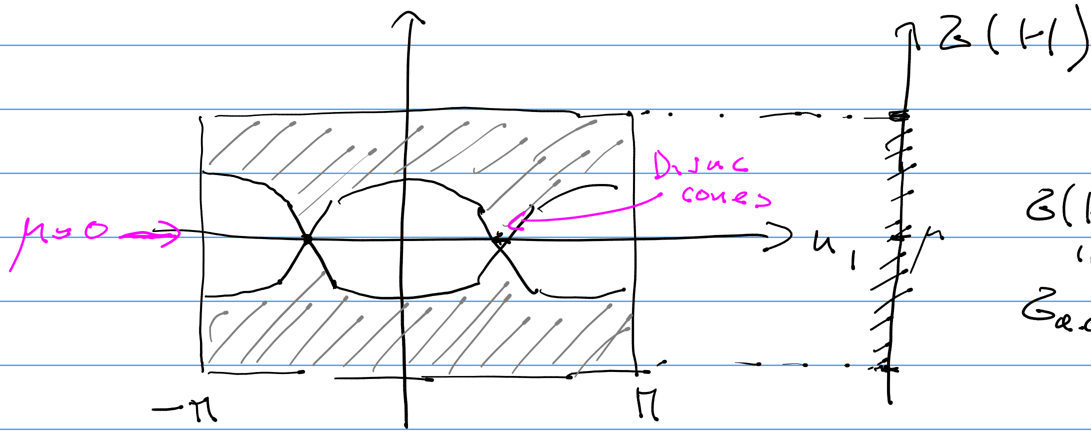
$$\Omega(k) = 1 + e^{-ik\cdot e_1} + e^{-ik\cdot e_2}$$

- eigenvalues $\varepsilon_\pm(k) = \pm t|\Omega(k)|$

$$\Omega(\underline{k}) = 0 \iff \underline{k} = \underline{k}_F^{\omega}, \ \omega = \pm$$

$$\Omega(\underline{k}' + \underline{k}_F^{\omega}) = \frac{3}{2}\left(i k_1' + \omega k_2'\right) + O(|k'|^2)$$

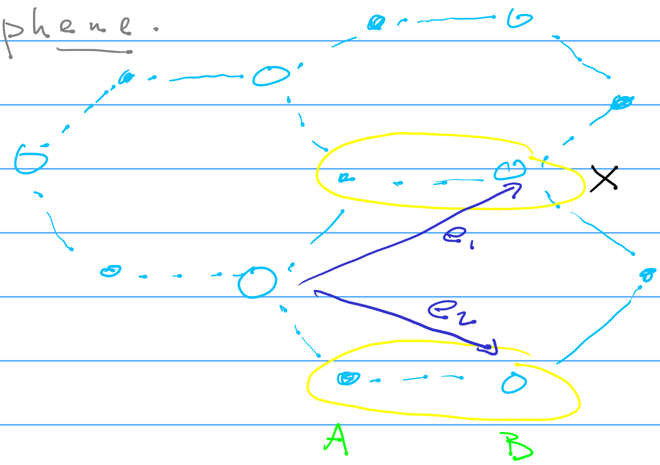
$$\bigcup_{k_2} \sigma(\hat{H}(k_1, k_2))$$



$\sigma(H)$
"
$\sigma_{a.c.}(H)$

Posta.

graphene.



$$\Lambda = \Lambda_A + \Lambda_B$$

$$\psi(x) = \begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix}$$

$$(H\psi)(x) = -t \begin{pmatrix} \psi_B(x) + \psi_B(x-e_1) + \psi_B(x-e_2) \\ \psi_A(x) + \psi_A(x+e_1) + \psi_A(x+e_1) \end{pmatrix}$$

$\hat{H} = \bigoplus_{\underline{k}\in\mathbb{T}^2} \hat{H}(\underline{k})$, $\hat{H}(\underline{k})$ called Bloch Hamiltonian

$$\hat{H}(\underline{k}) = \begin{pmatrix} 0 & -t\,\Omega(\underline{k}) \\ -t\,\overline{\Omega(\underline{k})} & 0 \end{pmatrix}, \quad \Omega(\underline{k}) := 1 + e^{-i\underline{k}\cdot e_1} + e^{-i\underline{k}\cdot e_2}$$

- chemical pot. $\mu = 0$
- Fermi sfc. $F_\mu := \{ \underline{k} \in \mathbb{T}^2 \mid \hat{H}(\underline{k}) - \mu$ has 0 eigenvalue$\}$
  $= \{ k_F^+, k_F^- \}$

Transport: $\sigma_{12} = \sigma_{21} = 0$, $\sigma_{11} = \sigma_{22} = 1/4$ (universal, i.e. t-indep.)
- was found for nonint. systems
  - interacting ones?

Goal. construct $\langle \cdot \rangle_{\beta,L}$ with $H = H_0 + \lambda V$, analyticity?

$$g(z,z') = \frac{1}{\beta} \sum_{k_0 \in M_\beta} e^{-ik_0(z_0 - z_0')} \int \frac{d^2\underline{k}}{(2\pi)^2} e^{-i\underline{k}(\underline{z}-\underline{z}')} \hat{g}(\underline{k})$$

where $\hat{g}(k) = \dfrac{1}{-ik_0 + \hat{H}(\underline{k})}$ (writing $k := (k_0, \underline{k})$, $k_F^\omega := (0, \underline{k}_F^\omega)$)

$k = k' + k_F^\omega$, k' small $\Rightarrow \|\hat{g}(k)\| \simeq \dfrac{1}{\|k'\|}$ for $\|k'\|$ small

$\Rightarrow g \notin L^1$

IR multiscale analysis.

- $\hat{g}(k) = \hat{g}_{UV}(k) + \hat{g}_{IR}(k)$

  where $\hat{g}_{UV}(k) = \hat{g}(k)\,\chi(\|k-k^\omega_F\| > \delta)$,

  $\hat{g}_{IR}(k) = \hat{g}(k) \sum_{\omega=\pm} \chi(\|k-k^\omega_F\| < \delta)$

- decompose $\psi = \psi^{(UV)} + \psi^{(IR)}$

  where $\psi^{(IR)\pm}(x) = \sum_\omega e^{\pm i k_F^\omega x}\, \psi^{(IR)\pm}_{\omega,x}$

$\|\hat{g}_\omega(\underbrace{k'}_{k-k_F^\omega})\| \simeq \frac{1}{\|k'\|}$ means $\|g_\omega(z-z')\| \simeq \frac{1}{\|z-z'\|^2}$

- pick $\gamma > 1$, $h \in \mathbb{Z}_-$ and set cutoff

  $f_h(k') = \chi(\gamma^{-h}\|k'\|) - \chi(\gamma^{-(h+1)}\|k'\|)$

  so

$$\hat{g}_\omega(k') = \sum_{h=h_\beta}^{0} \underbrace{\hat{g}^{(h)}_\omega(k')}_{:= f_h(k')\,\hat{g}_\omega(k')}$$

- $k_0 = \frac{2\pi}{\beta}(n+1)$, $k' = (k_0, \underline{k}')$, $\|k'\| \geq \frac{\pi}{\beta} =: \gamma^{h_\beta}$

$$|g^{(h)}_\omega(z,z')| \leq \int d^3k\, |\hat{g}^{(h)}_\omega(k')| \leq C\gamma^{2h}$$

$$(1 + \|z-z'\|^M \gamma^{hM})\,|g^{(h)}_\omega(z,z')| \leq C_M \gamma^{2h}$$

$$\Rightarrow \|g^{(h)}_\omega(z,z')\| \leq \frac{C_M \gamma^{2h}}{1 + (\gamma^h \|z-z'\|)^M}$$

- write $\psi^{(\leq 0)}_\omega = \sum_{h=h_\beta}^{0} \psi^{(h)}_\omega$

  $\rightsquigarrow V^{(h)}(\psi^{(\leq h)}) = \sum_P \int dz\, W_P^{(h)}(z)\, \psi_z^{(\leq h)}(P)$

- powers-counting. $\sum_{s,s'} \lambda \int dx\, \psi^{(\leq 0)+}_{x,s} \psi^{(\leq 0)-}_{x,s} \psi^{(\leq 0)+}_{x,s'} \psi^{(\leq 0)-}_{x,s'}$

  becomes, by writing $\psi^{(\leq 0)} = \psi^{(\leq -1)} + \psi^{(0)}$:

$$\longrightarrow \sum_{s,s'} \lambda \int dx\, \psi^{(\leq -1)+}_{x,s} \psi^{(\leq -1)-}_{x,s} \psi^{(\leq -1)+}_{x,s'} \psi^{(\leq -1)-}_{x,s'} + \text{h.o.t.}$$

$- \; g^{(\le -1)} = \gamma^{-2}\,\tilde{g}^{(\le 0)}(\gamma^{-1}\cdot) \qquad (\gamma > 1)$

and $\psi^{(\le -1)} = \gamma^{-1}\,\tilde{\psi}^{(\le 0)}_{\gamma^{-1}\cdot}$

- so our sum becomes

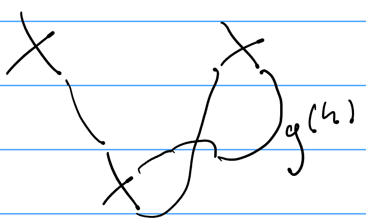
$$\sum_{s,s'} \underbrace{\lambda\, \gamma^{-4}\gamma^{3}}_{\lambda \gamma^{-1} \to \text{RG-irrelevant}} \int dx \left(\tilde{\psi}^{(\le 0)}\right)^4$$

$$\Rightarrow \; \| g^{(h)}_{\omega}(z,z') \| \le \frac{C_M\, \gamma^{2h}}{1 + (\gamma^{h}|z-z'|)^M}$$

- ψ^2: RG-relevant

$$\mathbb{E}_{\le 0}\left(e^{-V^{(0)}(\psi^{(\le 0)})}\right) = e^{\sum_{T \ge h}^{-1} E_T}\, \mathbb{E}_{\le h}\left(e^{-V^{(h)}(\psi^{(\le h)})}\right)$$

where $V^{(h)}(\psi^{(\le h)}) = \sum_P \int dz\, W^{(h)}_P(z)\, \psi^{(\le h)}_z(P)$



$$\| g^{(h)} \|_1^{s-1}\, \| g^{(h)} \|_\infty^{s+1-\frac{|P|}{2}} \le \gamma^{-h(s-1)}\, \gamma^{2h(s+1-\frac{|P|}{2})}$$

$$\gamma^{h(-|P|+3+s)} \le \gamma^{h(-|P|+4)} \quad (*)$$

- however  γ^{-h} for each, bad! isolate $|P|=2$

<u>Idea</u>: $V^{(h)}(\psi^{(\le h)}) = \underbrace{\mathcal{L}\, V^{(h)}(\psi^{(\le h)})}_{|P|=2} + \underbrace{\mathcal{R}\, V^{(h)}(\psi^{(\le h)})}_{|P|\ge 4}$

$$\mathcal{L}\, V^{(h)}(\psi^{(\le h)}) = \left(\psi^{(\le h)+},\, W^{(h)}_2\, \psi^{(\le h)-}\right)$$

- $\mathcal{L}V^{(h)}$ in $P_{\le h}(d\psi^{(\le h)})$

$$\int P_{\le h}(d\psi^{(\le h)})\, e^{-(\psi^{(\le h)}, W_2^{(h)} \psi^{(\le h)-})}(\cdots)$$

$$= e^{t_h}\int \tilde{P}_{\le h}(d\psi^{(\le h)})(\cdots)$$

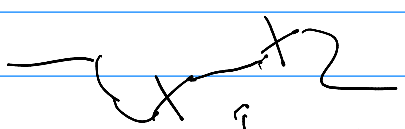
where $\tilde{P}_{\le h}(d\psi^{(\le h)}) = D\psi^{(\le h)} \exp\{-(\psi^{(\le h)+}, (g^{(\le h)-1} + W_2^{(h)})\psi^{(\le h)-})\}$

$$\frac{1}{H+V-z} = \frac{1}{H-z} - \frac{1}{H-z} V \frac{1}{H-z} + \cdots$$

$$-\hat{g}_\omega^{(\le h)}(k) = \chi_{\le h}\begin{pmatrix} ik_0 & v_h(ik_1'+\omega k_2') \\ v_h(-ik_1'+\omega k_2') & ik_0 \end{pmatrix}^{-1} \frac{1}{Z_h}$$

where v_h, Z_h given iteratively (from interaction)
and $Z_0 = 1$, $v_0 = \frac{3}{2}t$

$$-\hat{W}_2^{(h)}(k')\Big|_{k'=0} = 0, \quad k'\cdot\nabla_{k'}\hat{W}_2^{(h)}(k')\Big|_{k'=0}$$
$$= \begin{pmatrix} z_{h,0}\, ik_0 & z_{h,1}(+ik_1'+\omega k_2') \\ z_{h,1}(-ik_1'+\omega k_2') & z_{h,0}\, ik_0 \end{pmatrix}$$



$g = \frac{1}{k}$

odd # of g's

$$Z_{h-1} = Z_h + z_{0,h}$$
$$Z_{h-1}v_{h-1} = Z_h v_h + z_{1,h}$$
$$\begin{pmatrix} Z_{h-1} \\ v_{h-1} \end{pmatrix} = \begin{pmatrix} Z_h \\ v_h \end{pmatrix} + \hat{\beta}_h \leftarrow$$ beta function from RG-flow

$$|\hat{\beta}_h| \le C|\lambda|\gamma^h \Rightarrow |Z_h - 1| \le \sum_{T\ge h}^{0} |\lambda|\gamma^T = O(|\lambda|)$$
$$\Rightarrow \|W_P^{(h)}\|_1 \le |\lambda|^{(1-\vartheta)(\frac{|P|}{2}-1)}\gamma^{h(-|P|+4)}$$
$$\|W_P^{(h)} - \times\| \le |\lambda|^{2(1-\vartheta')}$$

- $|\rho| \geq 6$, $\langle \cdot \rangle_{\beta,L}$ is analytic in β, L (result from 2010.)

Applications to transport.

- $$\sigma_{ij} = \lim_{\eta \searrow 0^+} \lim_{\beta,L\to\infty} \frac{i}{L^2} \int_{-\infty}^{0} dt\, e^{\eta t} \langle [J_i(t), X_j] \rangle_{\beta,L}$$

in 2^{nd} quantization, i.e. $X = \sum_x x a_x^+ a_x$, $J = i[H,x]$

- one can prove universality:

$\sigma_{ij} = \sigma_{ij}\big|_{\lambda=0}$ → for graphene-like models (w/ Dirac cones)
→ for gapped models

→ graphene, $\sigma_{11} = 1/4$

→ gapped H, $\sigma_{12} \in \mathbb{Z}/2\pi$, $\sigma_{11} = 0$

Wick rotation

$$\int_{-\infty}^{0} dt\, e^{\eta t} \langle [J_i(t), X_j] \rangle$$

$J_i(t) = e^{iHt} J_i e^{-iHt}$

$$= \frac{1}{\eta}\left(\int_{-\infty}^{0} dt\, e^{\eta t} \langle [J_i(t), J_j] \rangle_{\beta,L} - \langle [J_i, X_j] \rangle_{\beta,L} \right)$$

Claim. $$\lim_{\beta,L\to\infty} \frac{1}{L^2} \int_{-\infty}^{0} dt\, e^{\eta t} \langle [J_i(t), J_j] \rangle_{\beta,L}$$

$$= \lim_{\beta,L\to\infty} \frac{-i}{L^2} \int_{-\beta/2}^{+\beta/2} dt\, e^{-i\eta t} \langle T\, J_i(-it) J_j \rangle_{\beta,L}$$

- idea: suppose $\eta = \frac{2\pi}{\beta} \cdot n$, $n \in \mathbb{Z}$ ("bosonic Matsubara")

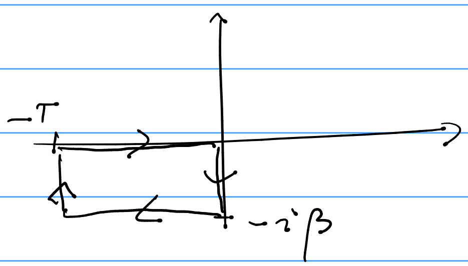
- write $\int_{-T}^{0} dt\, e^{\eta t} \langle [J_i(t), J_j] \rangle_{\beta L}$

$$= \int_{-T}^{0} dt\, e^{\eta t} \langle J_i(t)\, J_j \rangle_{\beta L} - \int_{-T}^{0} dt\, e^{\eta t} \langle J_j\, J_i(t) \rangle$$

- $\mathrm{tr}\, e^{-\beta H} J_j J_i(t) = \mathrm{Tr}\, e^{-\beta H} \underbrace{\left(e^{\beta H} J_i(t)\, e^{-\beta H}\right)}_{J_i(t-i\beta)} J_j$

- so, we get

$$\int_{-T}^{0} dt\, e^{\eta t} \langle J_i(t)\, J_j \rangle_{\beta L} - \int_{-T}^{0} dt\, \underbrace{e^{\eta t}}_{\simeq e^{\eta(t-i\beta)}} \langle J_i(t-i\beta)\, J_j \rangle_{\beta L}$$



$\oint = 0$

$$= -\int_{0}^{-i\beta} dt\, e^{\eta t} \langle J_i(t), J_j \rangle_{\beta L} + \cancel{\mathrm{error}_T} \to 0$$

$$= i \int_{0}^{\beta} dt\, e^{-it\eta} \langle J_i(-it), J_j \rangle_{\beta L}$$

$$= i \int_{-\beta/2}^{\beta/2} dt\, e^{-it\eta} \langle T\, J_i(-it), J_j \rangle_{\beta L}$$

- so $\sigma_{ij} = \lim_{\eta \to 0^+} \lim_{\beta L \to \infty} \frac{1}{\eta} \frac{i}{L^2} \int dt \left(e^{-it\eta} - 1\right) \langle T\, J_i(-it)\, J_j \rangle_{\beta L}$

"=" $\partial_\eta K_{ij}(\eta) \big|_{\eta = 0}$

where $K_{ii}(-\eta) = K_{ii}(\eta)$ if $K \in C^1$ $\sigma_{ii} = 0$

$$K_{ii}(\eta) = \underbrace{K^{\mathrm{rel.}}_{ii}(\eta)}_{\hookrightarrow \text{ gives } 1/4} + \overbrace{K^{\mathrm{even}}_{ii}(\eta)}^{\to\ \partial_\eta(\dots)|_{\eta=0} = 0}$$