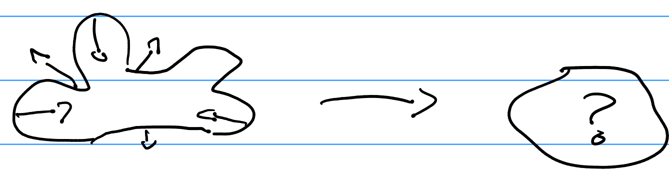
Bellettini

- strictly introductory course, a more advanced will follow

Problem. Given closed sfc $\Sigma \subset \mathbb{R}^n$, $\Sigma = \partial E$, $x \in \Sigma$, $t \in [0,T)$ time during which we move Σ, s.t.
(*) normal vel. $\vec{v}(t,x) =$ mean curv $\vec{H}_{\Sigma_t}(t,x)$



- (*) is vague — parametric approach:
  - although $x, \vec{v}, \vec{H}$ indep of param., we can calculate therein
- set of pts approach
  - a mfd can be realised as a level-set, graph, ... → pick best-suited one
- it would be best to use both, somehow

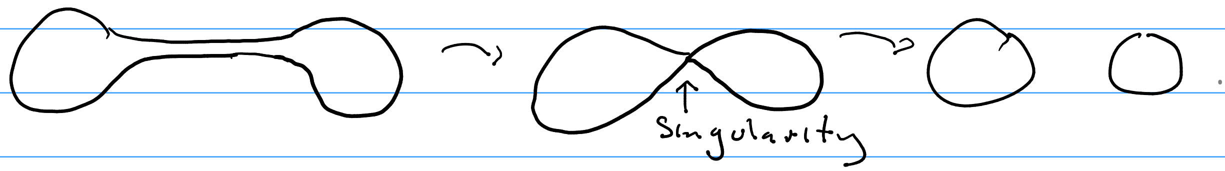
Def (sm. flow, parametric) Let M smooth $(n-1)$-mfd either cpt w/o bdry or complete w/o bdry.
Let $T>0$. A *smooth flow* starting from $\Sigma_0 \subset \mathbb{R}^n$ is a map $F: [0,T] \times M \to \mathbb{R}^n$
- smooth
- each $F(t,\cdot)$ is a smooth immersion
- $\Sigma_0 = F(0,M)$

Rmk While self-intersecting objects are interesting, we often assume $F(t,\cdot)$ are (local) embeddings

comment: is Σ_0 connected? complete? a bit unclear...

Def (normal velocity) Let F smooth par. flow of embedding, $p \in M$, $t \in [0, T]$.
Define $v(t,p) := \mathrm{pr}_{N_{F(t,p)}}(\partial_t F(t,p))$

- by def. of F there is no "topology change"
  - observe the possible evolution



- hard to prove, yet reasonable
  - F fails to describe it & "fixes topology"

Def F sm. flow as above is **normal** if $\frac{d}{dt} F(t,p)$ normal.

- so the projection becomes obsolete

Thm "Any sm. flow admits normal reparametrization"

- slogan? param-free version?

Def A sm. flow F is a **mean curvature flow** if

$$\frac{d}{dt} F(t,p) = \vec{H}(t,p)$$

$\forall t \in [0,T]$, $\forall p \in M$, where $\vec{H}(t,\cdot)$ is mean curvature of $F(t,M)$

- in particular it is normal
- parabolic, nonlinear system of PDEs

Rmk $\exists!$ short time solution (not trivial.)

- what kind of sing. develops as $t \nearrow T_{max}$?
  - not yet fully classified
- weak solns after $t > T_{max}$?
  - there are around 8 different versions of this.

- clearly F mean curv flow $\Rightarrow \frac{d}{dt} F(t,p)^{\perp} = \vec{H}(t,p)$ normal proj
  - suppose $\tilde{F}$ sm. flow s.t. $\frac{d}{dt}\tilde{F}(t,p)^{\perp} = \vec{H}(t,p)$
    - can I find mean curv flow F from $\tilde{F}$?

- let $F(t,p) := \tilde{F}(t, \varphi(t,p))$ with
  $\varphi(t,\cdot): M \to M$ diffeo satisfying

  $$D_\varphi F(t,\varphi(t,p)) \frac{\partial \varphi}{\partial t}(t,p) = -\left(\frac{\partial \tilde{F}}{\partial t}(t,\varphi(t,p))\right)^{Tang}$$
  has soln for $t \in [0,\delta]$

- then $\frac{d}{dt} F = \frac{d}{dt}\tilde{F} + D_\varphi \tilde{F}\, \varphi_t = \frac{d}{dt}\tilde{F}^{\perp} = \vec{H}$

  - horrible degeneracy ☹ disgusting really

Def • $\mathrm{II}_{f.f.}$ ass. to F (at fixed t): $T_pM \times T_pM \to N_{F(t,p)}\Sigma_t$

$$h_{ij}(p) := \left\langle \nu(p), \frac{\partial^2 F}{\partial u^i \partial u^j}(p)\right\rangle = -\left\langle \frac{\partial \nu}{\partial u^i}(p), \frac{\partial F}{\partial u^j}(p)\right\rangle = h_{ji}(p)$$

where ν interior, $\{\frac{\partial}{\partial u^i}\}$ local basis.

• $w(p)$ Weingarten op $\in \mathrm{Aut}\, T_pM$,
  $\langle w(p)(X), Y\rangle_g := \mathrm{II}_{f.f.}(p)(X,Y)$

$-w(p)$ has real eigenvalues $\lambda_1 \leq \dots \leq \lambda_{n-1}$,
fix this notation

Def $H = \lambda_1 + \dots + \lambda_{n-1}$, $\vec{H} = -H\nu$, ν interior

Thm $\Delta F = g^{ij}(\partial_i \partial_j F - \Gamma^k{}_{ij}\partial_k F) = -g^{ij}h_{ij}\nu = \vec{H}$