Fantechi

- we will be following Kashiwara - Shapira

- let $\mathcal{A}, \mathcal{B}$ ab. cats, $F : \mathcal{A} \to \mathcal{B}$ right exact.
  - e.g.
    - A ring, $\mathcal{A} = \mathcal{B} = \mathrm{Mod}_A$, M A-mod $F(N) = N \otimes_A M$
    - X top. sp., $\mathcal{A} = \mathcal{B} = \mathcal{O}_X$-mod, $F(\mathcal{G}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$
    - X scheme, $\mathcal{A} = \mathcal{B} = (\mathrm{QCoh}_X) \vee (X \text{ noeth} \Rightarrow \mathrm{Coh}\,X)$

      $f : X \to Y \Rightarrow f^* : (\mathcal{O}_Y\text{-mod}, \mathrm{Qcoh}_Y, \mathrm{Coh}\,Y) \to (Y \leftrightarrow X)$

      - for QCoh, $f^*(\mathcal{F}) = f^{-1}(\mathcal{F}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$
    - $\mathcal{A}$ ab. cat, $\mathcal{B} = Ab$, $M \in \mathrm{ob}\,\mathcal{A}$

      $\mathrm{Hom}_{\mathcal{A}}(-, M) : \mathcal{A}^{op} \to Ab$
- recall $X \xrightarrow{f} Y$ proper (proj) mor of sch. f.g./ [illegible] k

  $n = \max\{\dim f^{-1}(y) \mid y \in Y\}$, $R^n f_* : \mathrm{Coh}\,X \to \mathrm{Coh}\,Y$

  is right exact.
  - for $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, $R^i f_*$ long ex. sq. ends with $\to R^n f_* \mathcal{F} \to R^n f_* \mathcal{F}'' \to 0 = R^{n+1} f_* \mathcal{F}'$.

Def. A full add. subcat P of $\mathcal{A}$ is called a cat of F-projectives or F-cyclics if

i) ever ob of $\mathcal{A}$ is the quotient of an ob in P

ii) given $0 \to M' \to M \to M'' \to 0$ (*) exact in $\mathcal{A}$, if $M, M'' \in \mathrm{ob}\,P$, so is M'

iii) if in (*) M, M', M'' in P, then $0 \to F(M') \to F(M) \to F(M'') \to 0$ exact.

Thm Assume an F-proj P exists. Then

1) $D^-(\mathcal{A})^P \to D^-(\mathcal{A})$ is eq of cats, where $D^-(\mathcal{A})^P$ full subcat of cpx's $A^\bullet$ s.t. $(A^i = 0 \ \forall i >> 0) \wedge (\forall i \in \mathbb{Z}\ A^i \in P)$

ii) the naive functor $F : \mathcal{C}^-(\mathcal{A})^P \to \mathcal{C}^-(\mathcal{B})$ induces a functor $LF : D^-(\mathcal{A})^P \to D^-(\mathcal{B})$ and hence $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$

iii) $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ doesn't depend on choice of P

Def When thm applies, we call $F : \mathcal{A} \to \mathcal{B}$ def'd by $\mathcal{A} \to D^-(\mathcal{A}) \xrightarrow{LF} D^-(\mathcal{B}) \xrightarrow{h^i} \mathcal{B}$ the i-th left der fctor of F.

$A \mapsto \cdots \to 0 \to A \to 0 \to \cdots$

- exercise: A noeth comm ring, M fixed f.g A-mod, $\mathcal{A} = \mathrm{FGMod}_A$, $F(N) = N \otimes_A M$
  i) show proj A-mod in $\mathcal{A}$ are a cat of F-proj (N proj $\iff \tilde{N}$ loc. free on $\mathrm{Spec}\, A$)
  ii) compute $L^i F$ where $A = \mathbb{C}[x,y]/\langle xy \rangle$, $M = \mathbb{C}$; $L^i F(N)$, $N = \Omega_{A/\mathbb{C}}$, M

Rmk. We can analogously define F-injs for F left exact

- back to foundational material
- functor $[n] : C(\mathcal{A}) \circlearrowleft$ for $n \in \mathbb{Z}$ s.t. for $A^\bullet \in \mathcal{C}(\mathcal{A})$, $(A[n])^i = A^{n+i}$, d maybe gets a sign

Prop i) $[n]$ is autoequivalence w inverse $[-n]$
  ii) $[n]$ sends $C^+(\mathcal{A}), C^b(\mathcal{A}), C^-(\mathcal{A})$ to their respective selves
  iii) $[n]$ induces fctor $K(\mathcal{A}) \circlearrowleft$
  iv) $h^i(A[n]) = h^{i+n}(A)$

v) if $A \xrightarrow{\varphi} B$ in $C(A)$, φ qiso iff $\varphi[n]$ qiso

Cor. $\forall n \in \mathbb{Z}$, $[n]$ induces $D(A) \circlearrowleft$

Pf.
$$\begin{tikzcd} K(A) \arrow[r, "{[n]}"] \arrow[d] & K(A) \arrow[d] \\ D(A) \arrow[r, dashed] & D(A) \end{tikzcd}$$
if φ qiso, $\varphi[n]$ qiso in $K(A)$
by univ property $\exists!$ $D(A) \circlearrowleft$

- let $\mathcal{A}, \mathcal{B}$ ab.cat $F: \mathcal{A} \to \mathcal{B}$ exact
- exercise: F exact $\iff$ F commutes w ker, coker and with (co)limits

Lemma Let $F: \mathcal{A} \to \mathcal{B}$ exact, $A^\bullet \in \mathcal{C}(\mathcal{A})$
$\forall n \in \mathbb{Z}$, $Z^n(A) = \ker(A^n \to A^{n+1})$
$B^n(A) = \operatorname{Im}(A^{n-1} \to A^n)$
$= \operatorname{coker}(\ker d_{n-1} \to A^n)$
$h^n(A) := \operatorname{coker}(B^n(A) \to Z^n(A))$
Then $\exists$ natural iso $F(Z^n(A)) \simeq Z^n(F(A))$,
$F(B^n(A)) \simeq B^n(F(A))$, $F(h^n(A)) \simeq h^n(F(A))$
where $F(A) \in D(\mathcal{B})$ $(F(A))^i = F(A^i)$, $d_i(F(A)) = F(d_i)$

Cor If F exact, $\varphi: A_1^\bullet \to A_2^\bullet$ mor in $\mathcal{C}(\mathcal{A})$,
then φ qis $\Rightarrow$ $F(\varphi)$ qis

Cor If $F: \mathcal{A} \to \mathcal{B}$ exact, F induces $K(A) \to K(B)$, $D(A) \to D(B)$.

- ex. 1 $X \xrightarrow{f} Y$ proj mor, assume factors as
$X \xrightarrow{i} X \times \mathbb{P}^n \xrightarrow{\pi} Y$, i cl. emb, π proj, $f_* = \pi_* \circ i_*$
$Rf_*: D(\mathrm{QCoh}\,X) \to D(\mathrm{QCoh}\,Y)$

- ex 2. X sch, $\mathcal{A} = \mathcal{O}_X$-mod or $QCoh_X$ or Coh_X, $\mathcal{L} \in Pic(X)$. Then $\otimes \mathcal{L} : \mathcal{A} \to \mathcal{A}$ is exact and induces autoequiv. of $D(\mathcal{A})$ w/ inv $\otimes \mathcal{L}^\vee$
- research q.: given X sch, $\mathcal{A}$ as above. Classify autoequivalences of $D(\mathcal{A})$.

as triang. cats.

- for R^iF, $0 \to M' \to M \to M'' \to 0$ gives long ex. sq. for F left exact if enough inj.
- what about L^iF? similar, but induced by structure of triang. cats