Scarpa.

- (M, J, g) cpt Kähler
- Kähler-Einstein problem, $\overset{Ric}{S}(\tilde{\omega}) = \lambda \tilde{\omega}$
- look at Riemann surfaces

$\Rightarrow \mathbb{R} \simeq H^2_{dR}(\Sigma, \mathbb{R}) \ni [S(\omega)], [\omega] \Rightarrow \lambda$ exists

- $\int_\Sigma S(\omega) = \int_\Sigma S_{1\bar{1}}\, dz^1 \wedge d\bar{z}^1$

$$= \int_\Sigma \underbrace{i g^{1\bar{1}} R_{1\bar{1}}}_{S(g)} \underbrace{g_{1\bar{1}}\, dz \wedge d\bar{z}}_{\omega}$$

$$= \int_\Sigma S(g)\, \omega \overset{\text{Gauss}}{\underset{\text{Bonnet}}{=}} 2\pi \chi(\Sigma)$$

$$\lambda \cdot \int_\Sigma \omega = \lambda\, vol_\omega(\Sigma)$$

$$\Rightarrow g = \begin{cases} 0 \longrightarrow \lambda > 0 \text{ Fano} \\ 1 \longrightarrow \lambda = 0 \text{ Cal-Yau} \\ > 1 \longrightarrow \lambda < 0 \text{ general case} \end{cases}$$

- if ω, ω' Kähler, they are cohomologous iff $vol_\omega(\Sigma) = vol_{\omega'}(\Sigma)$, on a Riem. s.

- set $K(\omega) = \{ e^f \omega =: \omega_f \mid \int_\Sigma e^f \omega = \int_\Sigma \omega \}$
- fix $S(\omega_f) = 1$
- $S(\omega_f) = -i \partial\bar{\partial} \log(e^f g)$

$= -i \partial\bar{\partial} f - S(\omega)$ (use $\Delta = -\operatorname{div}\circ\operatorname{grad}$ $= dd^* + d^* d$)

$\Rightarrow e^{-f} g^{11} Ric_{1\bar{1}}(\omega_f) = -e^{-f} \partial\bar{\partial} f - e^{-f} S(\omega)$

$S(\omega_f) = 1$

$\Rightarrow \Delta f + e^f = -S(\omega)$

- in the general case $\int_\Sigma -S(\omega)\, \omega > 0$

Claim: $\forall \psi$ s.t. $\int \psi > 0$, $\exists!\, f$ s.t. $\underbrace{\Delta f + e^f = \psi}_{\star}$.

Lemma 1. (uniqueness) $f_\pm$ satisfy $\Delta f_\pm + e^{f_\pm} \gtrless \psi$.
If f solves ☆ then $f_- < f < f_+$ (also $\leq, \leq$).

Pf. $\chi := f_+ - f$, $\Delta\chi + e^f(e^\chi - 1)$

$$= \Delta f_+ + e^{f_+} - \psi \geq 0$$

For $p \in \Sigma$ min. of χ, $\Delta\chi(p) \leq 0$, so
$\underbrace{e^{f(p)}}_{>0}(e^{\chi(p)} - 1) > 0$

$\Rightarrow \chi(p) > 1$. So $f_+ - f = \chi > 0$.

Others similarly.

Lemma. (regularity) $k \geq 2$, $\psi \in W^{2,k}$, $f \in W^{2,2}$ satisfies $\Delta f + e^f = \psi$. Then $f \in W^{2,4}$

- existence is a bit trickier wrt regularity
- in any case, notice $\psi = 1 \Rightarrow f = 0$.
→ look at $S = \{t \in [0,1] \mid ☆_t$ has a soln$\}$,

where $☆_t \rightsquigarrow \Delta f + e^f = t\psi + (1-t)$.

→ since $f = 0 \rightsquigarrow 0 \in S$, $S \neq \emptyset$.

→ we will show $S = [0,1]$ by showing it is both open and closed