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Geometric rudiments

- spinors carry reps of $\mathcal{C}(V)$ and $Spin$, and either can be globalized
  - Spin is more traditional, but $\mathcal{C}(V)$ is better for NCG due to links w/ vbdls
- we start with Spin, and (anti)Euclidean case

- we take bdl of frames F with setups $\epsilon_i \in V$, $i=1,\dots,n$
- it carries a right $SO(n)$-action
  $F \times SO(n) \to F$, $(\epsilon, g) \mapsto \epsilon g$
  which is clearly free & transitive
- we can think of a real vector as $V = \sum v_i \epsilon_i$, $v_i \in \mathbb{R}$, and this gives us an equivalent characterisation $t: F \to W \subseteq \mathbb{R}^n$ which gives us coordinates
- we see that
  1) a (p,s)-tensor $\rightsquigarrow (\mathbb{R}^n)^{\otimes p} \otimes (\mathbb{R}^{n*})^{\otimes s}$
  2) $t(\epsilon g) = R(g^{-1}) t(\epsilon)$ where R rep. of $SO(n)$

- equivalently, work with assoc. vbdl to F with rep. R,

$$[\epsilon, w] \in F \times_R W := \frac{F \times W}{\sim},$$
$$(\epsilon, w) \sim (\epsilon g, R(g^{-1}) w)$$

- so we get $t \leftrightarrow [\epsilon, t(\epsilon)]$.

- analoguously for spinors
- $\tilde{F}$ equipped w/ free & trans. $Spin(n)$-action
- given rep $\tilde{R}: Spin \to L(S)$, S being e.g. $\mathbb{C}^{2^m}$, $m = \lfloor \frac{n}{2} \rfloor$
  call a spinor of type $\tilde{R}$
  an $\tilde{R}$-equivariant $\psi: \tilde{F} \to S$, $\psi(\tilde{e}\tilde{g}) = \tilde{R}(\tilde{g}^{-1})\psi(\tilde{e})$

- we link this to $SO(n)$ and F:

$$\begin{array}{ccc} \tilde{F} & \circlearrowleft & Spin(n) \\ \exists\, \eta \downarrow & & \downarrow s \\ F & \circlearrowleft & SO(n) \end{array} \qquad \text{s.t. } \eta(\tilde{e}\tilde{g}) = \eta(\tilde{e})s(\tilde{g})$$

Rmk $t \circ \eta: \tilde{F} \to W$ is $R \circ s$-equivariant by construction $\Rightarrow$ tensors of type R
are therefore spinors of type $\tilde{R} = R \circ s$.
- in particular, $R = id_V \Rightarrow \tilde{R} = s$.
  $\Rightarrow$ vectors are s-type spinors.

Spin structures

- we generalise to M oriented smooth Riem. mfld

Def. Spin structure is loc. trivial bdle of frames satisfying

$$\begin{tikzcd} \tilde{F} \arrow[d, dashed, "\eta"'] \arrow[dd, bend right=60, "\tilde{\pi}"'] & Spin(n) \arrow[l, hook'] \arrow[d, "s"] \arrow[dl, phantom, "\circlearrowleft"] \\ F \arrow[d, "\pi"] & SO(n) \arrow[l, hook'] \\ M & \end{tikzcd}$$

where the fibers have free, transitive $SO(n)$ / $Spin(n)$ - right action.

- we call such an M a spin mfd.
- not all are spin: $\mathbb{C}\mathbb{P}^2$ isn't.
- where is the obstruction?
- $\{\mathcal{U}_\alpha\}$ cover of M, F has transition functs $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \xrightarrow[\varphi_{\alpha\beta}]{} SO(n)$
  s.t. $e_\beta = e_\alpha \varphi_{\alpha\beta}$
- the spin mfd condition means this lifts to $\tilde{\varphi}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \longrightarrow Spin(n)$

- we have the Čech cocycle

$$K_{\alpha\beta\gamma} := \tilde{\varphi}_{\beta\gamma} \tilde{\varphi}_{\alpha\gamma}^{-1} \varphi_{\alpha\beta} : \mathcal{U}_{\alpha\beta\gamma} \longrightarrow Spin(n)$$

and always

$$(\partial K)_{\alpha\beta\gamma\delta} = K_{\beta\gamma\delta} K^{-1}_{\alpha\gamma\delta} K_{\alpha\beta\delta} K^{-1}_{\alpha\beta\gamma} = 1 \in \mathbb{Z}_2 \subset Spin(n)$$

so it's true that $[K] \in H^2(M, \underline{\underline{\mathbb{Z}_2}})$

- note, since $S(K_{\alpha\beta\gamma}) = +1$, $K_{\alpha\beta\gamma} \in \mathbb{Z}_2 \subset Spin(n)$

<u>Prop</u> i) $[K]$ is indep. of choice of lifts $\varphi_{\alpha\beta} \mapsto \tilde{\varphi}_{\alpha\beta}$, and indep. of choice of frames $e_\alpha \mapsto \tilde{e}_\alpha$.

ii) M is spin iff $[K] = 1$

<u>Rmk</u> $[K]$ is called the Stiefel-Whitney class of M

- examples:
  - $M \times SO(n) \longleftarrow \overbrace{M \times Spin(n)}^{=\tilde{F}}$
  - $S^n \cong SO(n+1)/SO(n) \Rightarrow F = SO(n+1)$,
    $\tilde{F} = Spin(n+1)$

Rmk $Spin_{\mathbb{C}}$-structure exists iff
$[K]$ is mod 2 reduction of some
class in $H^2(M, \mathbb{Z})$

- $\rho(Spin^c) \subset U(\mathbb{C}^{2^m} = S)$, but S has $\langle \cdot, \cdot \rangle_S$, so

$$\langle \underbrace{\psi \circ \tilde{e}_\alpha \tilde{g}}_{\tilde{R}(\tilde{g}^{-1})\psi(\tilde{e}_\alpha)}, \underbrace{\psi' \circ \tilde{e}_\alpha \tilde{g}}_{\tilde{R}(\tilde{g}^{-1})\psi'(\tilde{e}_\alpha)} \rangle_S$$

$$= \langle \psi(\tilde{e}_\alpha), \underbrace{\tilde{R}(\tilde{g}^{-1})^{+}\tilde{R}(\tilde{g}^{-1})}_{=1}\psi'(\tilde{e}_\alpha) \rangle_S$$

$$\Rightarrow \langle \psi, \psi' \rangle := \int_M \langle \psi(\tilde{e}_\alpha), \psi'(\tilde{e}_\alpha) \rangle_S \, Vol_g \in \mathbb{C}$$

is well def.

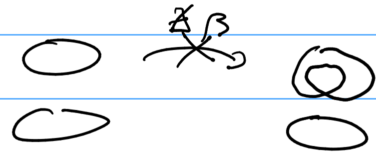
Rmk - spin sts. may not be unique.
- equivalent $\tilde{F}_1, \tilde{F}_2$ if

$$\begin{tikzcd} \tilde{F}_1 \arrow[rr, dashed, "\exists \beta"] \arrow[dr, "q_1"'] & \circlearrowleft & \tilde{F}_2 \arrow[dl, "q_2"] \\ & F \arrow[d] & \\ & M & \end{tikzcd}$$

Prop $\exists$ free & transitive action $H^1(M, \mathbb{Z}_2)$ on $\Pi(M) = \{\text{spin sts on } M\}/\sim$

- and since $H^1(M, \mathbb{Z}_2) = \mathrm{Hom}(\pi_1(M), \mathbb{Z}_2)$, we can think geometrically

- e.g. $M = S^1$, $SO(1) = 1$, $\mathrm{Spin}(1) = \pm 1$



$M = T^2 = S^1 \times S^1$, $F = T^2 \times SO(2)$

$$\begin{array}{ccc} \tilde{F} = T^2 \times \mathrm{Spin}(2) & \ni & (x, y, \underset{U(1)}{u}) \\ \downarrow \eta_{jk} & & \downarrow \\ F & & (x, y, R(jx + ky) S(u)) \end{array}$$

- claim $\nexists \beta$ s.t.

$$\begin{tikzcd} \tilde{F} \arrow[rr, "\beta"] \arrow[dr, "\eta_{jk}"'] & & \tilde{F} \arrow[dl, "\eta'_{j'k'}"] \\ & F \arrow[d] & \\ & M & \end{tikzcd}$$

- show: $\eta'_{j'k'} \circ \beta = \eta_{jk}$ only if $j' = j, k = k'$

$M = T^n \Rightarrow \exists\ 2^n$ inequiv. spin. str.

$\rightarrow$ corresponds to (anti) periodic bdry conditions