Gauge @ IGAP

$6d \Rightarrow$ invariants – Putrov, cont'd

- last time: • $VOA_{u(1)}[M^4]$, gluing, SW
  • ESW, multimonopole mus
- other $VOA_g[M^4]$?

① $g = u(N)$ [Nakajima]
$M^4 = \widetilde{\mathbb{C}^2/\Gamma_h}$, $\Gamma_h \subset SU(2)$ finite
h ← simply laced
$VOA_{u(N)}[M^4] = \hat{h}_N$, assoc. to affine Lie algebra

Recall. Affine Lie algebra
• $(h \otimes \mathbb{C}[[t, t^{-1}]]) \oplus \mathbb{C}\cdot K$
• for $a \in h$, $a_i := a \otimes t^i$, and
$[a_i, b_j] = i\,\delta_{ij}\,(a,b)\,K + [a,b]_{i+j}$
• corresponding VOA:
→ highest weight module,
$K|0\rangle = N|0\rangle$
$a_i|0\rangle = 0 \quad \forall i \geq 0$

①' $M^4 = \text{Bordism}(S^3/\Gamma_h \longrightarrow S^3/\Gamma_f)$

s.t. $M^4 \bigsqcup_{S^3/\Gamma_h} \widetilde{\mathbb{C}^2/\Gamma_h} = \widetilde{\mathbb{C}^2/\Gamma_f}$

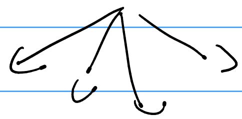
$VOA_{u(N)}[M^4] = \dfrac{\hat{f}_N}{\hat{h}_N}$ "coset VOA"

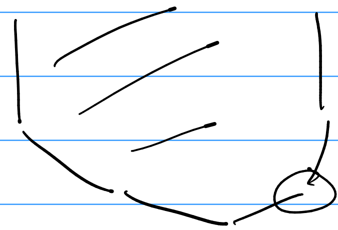
↳ not quotient, bad notation
• commutant of $\hat{h}_N$ in $\hat{f}_N$

(11) [Gukov-Feigin]

M^4 noncpt. toric sfc

Toric fan





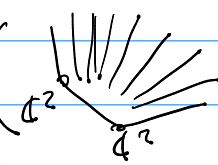
$\cong \mathbb{C}^2$

$VOA^{\Pi}_{SU(N)}(M^4)$ depending on $\varepsilon_1, \varepsilon_2$

$VOA^{\Pi}_{SU(N)}(\mathbb{C}^2) \cong W_N$-algebra

$$VOA^{\Pi}_{SU(2)}(\mathbb{C}^2) \cong Vir^{(\varepsilon_1,\varepsilon_2)} \cong \langle L_{-n_1} \cdots L_{-n_k}|0\rangle\rangle$$

$$c = 1 + 6\left(\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

$VOA^{\Pi}_{SU(2)}$(

)

$$= \left(Vir^{(\varepsilon_1,\varepsilon_2)} \otimes Vir^{(\varepsilon_1',\varepsilon_2')}\right) \oplus \text{certain modules}$$

$$\begin{pmatrix} \varepsilon_1' \\ \varepsilon_2' \end{pmatrix} = \underset{SL(2,\mathbb{Z})}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

III $M^4 = \mathbb{CP}^1 \times \Sigma_{g,n}$

$$\mathrm{VOA}_{\mathfrak{g}}[\mathbb{CP}^1 \times \Sigma_{g,n}]$$
$$\cup$$
$$\underbrace{\hat{\mathfrak{g}} \times \cdots \times \hat{\mathfrak{g}}}_{n} \times \widehat{\mathfrak{u}(1)}$$

- pick G, $\mathrm{Lie}\, G = \mathfrak{g}$

$$\mathrm{Tr}_{\mathrm{VOA}_{\mathfrak{g}}(\mathbb{CP}^1 \times \Sigma_{g,n})} (-)^F q^{L_0} \underset{\substack{\uparrow \\ G^{\times n} \times U(1)}}{\alpha} \in \mathrm{Rep}(G^n \times U(1))[[q]]$$

Conj. I) $\mathrm{VOA}_{\mathfrak{g}}[\mathbb{P}^1 \times \Sigma_{0,n}] \overset{\text{"naive"}}{=} \Gamma(D^{ch}_{\underbrace{X_{g,0,n}}_{\substack{\text{Higgs br.} \\ \text{of correspondy} \\ \text{class S}}}})$

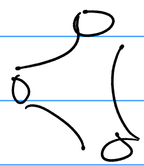
II) construct more by gluing

- what is D^{ch}_X? Let X sm. cpt. anal. mfd.

$\to D^{ch}_X$ is VOA-valued sheaf

- $D^{ch}_{\mathbb{C}^N}(\mathbb{C}^N) =$ N copies of β-γ
$$= \langle \beta^{i_1}_{-n_1} \cdots \beta^{i_p}_{-n_p} \gamma^{j_1}_{-m_1} \cdots \gamma^{j_q}_{-m_q} |0\rangle \rangle$$
$$=: V_N$$

$$\Rightarrow D^{ch}_{\mathbb{C}^N}(\underset{\substack{\shortparallel \\ \mathrm{Spec}\,\mathbb{C}[x_1,\dots,x_N] \\ f(0) \neq 0}}{\mathbb{C}^N \setminus \{f=0\}}) = V_N \otimes \mathbb{C}[\gamma^1_0,\dots,\gamma^N_0][\tfrac{1}{f(0)}]$$

$\mathrm{VOA}_{SU(2)}\Big($  $\times\ \mathbb{P}^1\Big)$

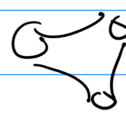
$= \beta$-γ valued in $\mathbb{C}^8$

$\circlearrowleft$

$SU(2)\times SU(2)\times SU(2)\times U(1)$

$$\chi_{SU(2)}(--) = \prod_{\pm} \frac{1}{\vartheta(u\, x^{\pm} y^{\pm} z^{\pm})}.$$

$$\vartheta(x) := \prod_{n\geq 0} (1-xq^n)(1-q^{n+1}/x)$$

$\mathrm{VOA}_{SU(3)}\Big($  $\times\, \mathbb{P}^1\Big) = \Gamma\big(D^{ch}_X\big)$

X - centered E_6 instanton m.sp.

$$\chi_{SU(3)}(\cdot -) \in R(E_6 \times U(1))[[q]]$$