Dąbrowski

- $\mathcal{L}^1 \subset \mathcal{L}^{1+} \subset \mathcal{L}^p, \forall p > 1 \subset \mathbb{K}(\mathcal{H}) \subset \mathbb{B}(\mathcal{H})$
- $\mathrm{Tr}_\omega|_{\text{measurable}} =: \mathrm{Tr}^+$ Dixmier trace

  if $\exists \lim \frac{\sum_{i=1}^N \lambda_i}{\log N}$.

- $\mathrm{Tr}^+(T) := \lim \tau_e(T)$, smeared versions of
- if $\dim M = n$, $|D|^{-n}$ measurable, $\in \mathcal{L}^{1+}$,

  $\mathrm{Tr}^+ |D|^{-n} = 2^m v_{n-1}/n$
- defining $S_\pm = \frac{1}{2}(|S| \pm S)$ which

  takes pos/neg "part", we get a "polarization identity"

$$\mathrm{Tr}^+ T = \frac{1}{2}\Big[\mathrm{Tr}^+(T+T^*)_+ - \mathrm{Tr}^+(T+T^*)_- + i\,\mathrm{Tr}^+\big[(iT - iT^*)_+\big] - i\,\mathrm{Tr}^+\big[(iT-iT^*)_-\big]\Big]$$

- can be related to Wodzicki residue, unique (if $n \geq 2$) tracial state on ΨDO (pseudodiff. op) on $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, M cpt. dim n

$$W_{res}(P) := \int_M \int_{\|\xi\|=1} \mathrm{tr}\, \sigma_P^{-n}(x,\xi)\, \nu_\xi\, d^n x$$

where $\nu_\xi := \sum \xi^j \partial_{\xi_j} \lrcorner\, d^n \xi$

$= \sum (-)^j \xi^j d\xi^1 \wedge \dots \wedge \widehat{d\xi^j} \wedge \dots \wedge d\xi^n$

- before showing connection, compute for $P = f|D|^{-n}$, $f \in C^\infty(M)$

$$\sigma^{-n}_{f|D|^{-n}}(\xi) = [f|D|^{-n}, h] \quad \hookrightarrow dh = \xi$$

$$= f \underbrace{|\xi|_g^{-n}}_{g(\xi,\xi)^{-n/2}} \mathbb{1}_{2^m}$$

$$\Rightarrow W_{res} = 2^m \int f(x) \int_{|\xi|=1,} \underbrace{\left(g(x)^{ij}\xi_i\xi_j\right)^{-n/2} \nu_k}_{\text{check: closed form}} d^n x \,.$$

usual length on $\mathbb{R}^n$

- so change coords

$y = \psi' \xi$ s.t. $\psi' = \sqrt{g(x)}$

$$\Rightarrow \int_{y \in \text{ellipsoid}} |y|^{-n/2} \nu_y$$

$\shortparallel$

sphere by Stokes

$$\Rightarrow W_{res} = 2^m v_{n-1} \int_M f(y) \underbrace{(\det g)^{1/2}}_{\text{Jacobian}} d^n y$$

$$= 2^m v_{n-1} \int f \, d\mathrm{vol}_g$$

Thm (trace; Connes) Any ΨDO P of order $-n$ on $\mathbb{C}$-vbdl $E \to M$, M cpt, dim n, belongs to $\mathcal{L}^{1+}$ and is measurable if

$$\mathrm{Tr}^+ P = \frac{1}{n(2\pi)^n} W_{res}(P)$$

Pf, conceptual

- change of metric on E $\Leftrightarrow$ unitary transformation
  - trace doesn't care
- suffices to do locally using p.o.1,
  since $f \cdot -$ is bdd, so $fP \in \mathcal{L}^{1+}$,
  pick $U \overset{\text{open}}{\subset} M$, $U \cong \mathbb{R}^n \subset \frac{S^n}{\Pi}n$ so $E|_U = U \times \mathbb{C}^{\dim E}$
- any P of order $-n$ write as $P = T\Delta^{-n/2}$, $T = P\Delta^{n/2}$
  T: $(\Psi DO)_0$ — or something like $\sqrt{1+\Delta}$ if $\operatorname{Ker}\Delta \neq \{0\}$
- $\mathcal{L}^{1+}$ is an ideal, measurable ones maybe... so $P = T\Delta^{-n/2} \in \mathcal{L}^{1+}$
- since $(T\Delta^{-s/2})_{s>n} \in \mathcal{L}^{1} \Rightarrow \operatorname{Tr}_\omega T\Delta^{-s/2} = 0$
  $\Rightarrow \operatorname{Tr}_\omega$ depends only on $\sigma^{-n}(P)$?

- now $E|_U \cong U \times \mathbb{C}^{\dim E}$
- $\{\sigma_\Delta^{-n}\} = \{\mathcal{C}^\infty(T^*M), (-n)\text{-homog.}\} = \mathcal{C}^\infty(S^*M)$
- $\Delta^{-n/2}$ scalar ΨDO on line bdl
- $\forall \omega$, Tr_ω is cont. functional on $\mathcal{C}^\infty(S^*M)$
- since $P \geq 0$, $\sigma^{-n}(x,\xi) \geq 0$

- now $U \subset S^n \xrightarrow{\text{emb.}} \mathbb{R}^{n+1}$, sym. gp $SO(n+1)$,
  lift to unitary U_g on $L^2(E, \langle\cdot,\cdot\rangle)$
- trans. action on S^n extends to
  — " — S^*S^n
$\rightsquigarrow \operatorname{Tr}_\omega P = \underset{P\text{-indep}}{\text{const.}} \int_{S^*M} \sigma^{-n}(P)\, \mathrm{vol}_{S^*M}$
  - take $P = |D|^{-n}$ to compute const
  $\rightarrow c = 1$, indep. of ω
  $\Rightarrow P$ measurable. □

- trace thm applies to $f|D|^{-n}$, since $\mathrm{Tr}^+ f|D|^{-n} \neq 0 \Rightarrow |D|^{-n} \notin \mathcal{L}_0^{1+} = \{T \in \mathcal{B}(H) \mid \frac{\sigma_N(T)}{\log N} \to 0\} \supset \mathcal{L}^1$

$$\mathrm{Tr}^+ f|D|^{-n} = \mathrm{Wres}\, f|D|^{-n} = \int_M f\, \mathrm{vol}_g$$

we generalize

Def When $(A, \mathcal{H}, D)$, A noncomm., we define $⨍ a|D|^{-n}$ as LHS.

$$\begin{aligned}
\mathcal{L}^{p+} &:= \{T \in \mathcal{B}(\mathcal{H}) \mid k\text{-th eigenval.}(|T|) = O(k^{-1/p})\} \\
&= \{ \quad " \quad \mid \sigma_N(|T|) = O(N^{p-1/p})\} \\
&= \{ \quad " \quad \mid N_e = O(e^p)\}
\end{aligned}$$

$\to O(\log N)$ for $p=1$

$$\mathcal{L}_0^{p+} := \overline{\text{finite rank}}^{\|\cdot\|_{p+}}$$

$$\|T\|_{p+} := \lim \frac{\sigma_N(|T|)}{N^{\frac{p-1}{p}}}$$

Axiom (Dimension) $\exists n \in \mathbb{N}$ s.t. $|D|^{-1} \in \mathcal{L}^{n+}$, $|D|^{-n}$ measurable, $D^{-n} \notin \mathcal{L}_0^{1+}$.

We call such n the dimension of the S.T., and it equals the dimension in the canonical case.