Algebraic geometry, exercises

- let $\mathbb{k}$ field, $A := \mathbb{k}[t]/t^2$, $T = \operatorname{Spec} A$, X a $\mathbb{k}$-scheme
- look at maps $\operatorname{Map}_{\mathbb{k}}(T, X)$

$$
\begin{array}{c}
A \\
\uparrow \\
m = (\varepsilon),\ \varepsilon := t + (t^2) \\
\uparrow \\
0
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
\emptyset \\
\downarrow \\
\operatorname{Spec} A/m = \operatorname{Spec} \mathbb{k} =: S \\
\downarrow \\
T
\end{array}
$$

closed subscheme

- for $f : T \to X \underset{\text{compose with } S \hookrightarrow T}{\rightsquigarrow} S \to X$

X $\mathbb{k}$-scheme $\iff \mathcal{O}_X$ sheaf of $\mathbb{k}$-algebras
$\iff \forall x \in X$, $\mathcal{O}_{X,x}$ $\mathbb{k}$-algebra
$\iff \forall x \in X$, $\mathbb{k}(x)$ is a field extension of $\mathbb{k}$

Def. $x \in X$ is **$\mathbb{k}$-rational** if $\mathbb{k} \hookrightarrow \mathbb{k}(x)$ is iso.

Exercise $\operatorname{Map}_{\mathbb{k}}(S, X) \simeq \{\forall x \in X \mid x \ \mathbb{k}\text{-rational}\}$

$f : T \to X$. Since $T = \{m\}$, $f(m) = x$ for some $x \in X$.
$\mathcal{O}_{X,x} \to \mathcal{O}_{T,m}$ induces
$\Rightarrow m_{X,x}/m_{X,x}^2 \to m_{T,m}/m_{T,m}^2 = (\varepsilon)/(\varepsilon)^2 \simeq (\varepsilon)$

In the other way, define $f : T \to X$ s.t. $f(m) := x \in X$.
We need $\mathcal{O}_X \to f_* \mathcal{O}_T$ or $f^{-1}\mathcal{O}_X \to \mathcal{O}_T$.
Since $\Gamma(T, \mathcal{O}_T) \simeq A$, $\Gamma(T, f^{-1}\mathcal{O}_X) \simeq \mathcal{O}_{X,x}$
Note that $0 \to m_{X,x} \to \mathcal{O}_{X,x} \to \mathbb{k} \to 0$ splits
by assumption.
Therefore we need to give $\operatorname{Hom}_{\mathbb{k}}(\mathcal{O}_{X,x}, A) \simeq \operatorname{Hom}_{\mathbb{k}}(m_{X,x}, A) \oplus \operatorname{Hom}_{\mathbb{k}}(\mathbb{k}, A)$
$\Rightarrow \mathbb{k} \hookrightarrow A$ ✓, $m_{X,x} \to m_{X,x}/m_{X,x}^2 \to \mathbb{k} \xrightarrow{\sim} \mathbb{k}[\varepsilon] \hookrightarrow A$
Explicitly, $\varphi : \mathcal{O}_{X,x} \to A$, $f \mapsto f(x) + \underset{(m_{X,x}/m_{X,x}^2)^\vee}{\underset{\uparrow}{v}}(f - f(x) + m_{X,x}^2)\,\varepsilon$

Fibres of morphisms

- $X \xrightarrow{f} Y$, $y \in Y$. $\exists!$ map $\mathrm{Spec}(\Bbbk(y)) \xhookrightarrow{i_y} Y$

$X_y := X \times_{f, Y, i_y} \mathrm{Spec}(\Bbbk(y))$ ← scheme theoretic fiber of f over y.

- let $f^{-1}(\{y\}) := \{x \in X \mid f(x) = y\}$

Exercise Top. space of X_y is homeo. to $f^{-1}(\{y\})$

- By definition

$$\begin{tikzcd} X_y \arrow[r, "g"] \arrow[d, "z"'] & X \arrow[d, "f"] \\ \mathrm{Spec}\,\Bbbk(y) \arrow[r, "i_y"'] & Y \end{tikzcd}$$

- now $g(X_y) \subseteq f^{-1}(y)$ since for $z \in X_y$

$f(g(z)) = i_y(g(z)) = i_y(0) = y$

- also, $f^{-1}(y) \subseteq g(X_y)$. Take $x \in f^{-1}(y)$ i.e. $f(x) = y$.

$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces $\Bbbk(y) \to \Bbbk(x)$ induces $\mathrm{Spec}\,\Bbbk(x) \to \mathrm{Spec}\,\Bbbk(y)$.

$$\Rightarrow \quad \begin{tikzcd} \mathrm{Spec}\,\Bbbk(x) \arrow[r, dashed, "\exists! h"] \arrow[rr, bend left] \arrow[dr, bend right, "f"'] & X_y \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}\,\Bbbk(y) \arrow[r] & Y \end{tikzcd}$$

- write $z := h(0) \in X_y \Rightarrow g(z) = g(h(0)) = x$.
- therefore $g(X_y) = f^{-1}(y)$
- g is a topological immersion. (Claim)

→ check on affine case (local property)

- put $X = \mathrm{Spec}\,B$, $Y = \mathrm{Spec}\,A$, $y =: P \subseteq A$ prime ideal

→ $\Bbbk(y) = \Bbbk(p) \simeq A_p / pA_p$

- by def $X_y = \mathrm{Spec}\,B \times_{\mathrm{Spec}\,A} \mathrm{Spec}\,A_p/pA \overset{(*)}{\simeq} \mathrm{Spec}(B \otimes_A A_p/pA_p) \xrightarrow{g} \mathrm{Spec}\,B$.

(*) $B\otimes_A \overbrace{A_p/pA}^{\simeq A_p\otimes_A A/p} \simeq \overbrace{(B\otimes_A A_p)}^{\simeq B_p}\otimes_A A/p \simeq B_p/pB_p$

- now note that in general:

1) A ring, $A \supseteq I$ ideal: $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is top. embedding
2) $S \subseteq A$ multp. system $\Rightarrow$ $\operatorname{Spec} A_S \to \operatorname{Spec} A$ — " —

<u>Lemma</u> $X \xrightarrow{f} Y$ cont., (V_α) family of opens in Y s.t. $f(X) \subseteq \bigcup_\alpha V_\alpha$.
Then: f top. embedding
$\Leftrightarrow \forall \alpha$, $f: f^{-1}(V_\alpha) \to V_\alpha$ top. embedding.

<u>Exercise</u>. Compute fibres of $X \xrightarrow{f} Y$, where $X = \operatorname{Spec}(\mathbb{k}[x,y]/(x-y^2))$, $Y = \mathbb{A}^1_{\mathbb{k}}$.

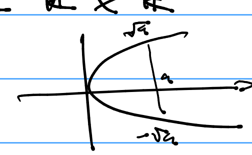
$\forall a \in \mathbb{k} \Rightarrow (x-a) \in \mathbb{A}^1_{\mathbb{k}}$ closed pt (assuming $\mathbb{k}$ alg closed).
$\Rightarrow \mathbb{k}(a) = \mathbb{k}[x]/(x-a) \simeq \mathbb{k}$

$\eta = (0) \in \mathbb{A}^1$, $\mathbb{k}(\eta) = \mathcal{O}_{\mathbb{A}^1,\eta} = \operatorname{Frac}(\mathbb{k}[x]) = \mathbb{k}(x)$.
→ only pts in $\mathbb{A}^1_{\mathbb{k}}$!

- pick $a \in \mathbb{k}$. $X_a := X \times_Y \operatorname{Spec} \mathbb{k}(a)$

$$= \operatorname{Spec}\left(\underbrace{\frac{\mathbb{k}[x,y]}{(x-y^2)} \otimes_{\mathbb{k}[x]} \frac{\mathbb{k}[x]}{x-a}} \right)$$
$$\simeq \mathbb{k}[x,y]/(x-y^2, x-a)$$
$$\simeq \mathbb{k}[y]/(y^2-a)$$

→ 2 cases: 1°, $a \neq 0 \Rightarrow (y^2-a) = (y-\sqrt{a})\cdot(y+\sqrt{a})$
$\overset{CRT}{\Rightarrow} \mathbb{k}[y]/(y^2-a) \simeq \mathbb{k}\times\mathbb{k}$
2°, $a = 0 \Rightarrow$ dual #s!



- now look @ fiber over η:
$X_\eta = \operatorname{Spec}\left(\frac{\mathbb{k}[x,y]}{(x-y^2)} \otimes_{\mathbb{k}[x]} \operatorname{Frac}(\mathbb{k}(x)) \right)$
- put $D = \mathbb{k}(y)$, $K = \operatorname{Frac} D$
$= \operatorname{Spec}\left(\frac{D[y]}{y^2-x} \otimes_D K \right) = \operatorname{Spec}\left(\overbrace{\frac{K[y]}{y^2-x}}^{\simeq K[\sqrt{x}]} \right) \simeq \operatorname{Spec} \mathbb{k}(x, \sqrt{x})$