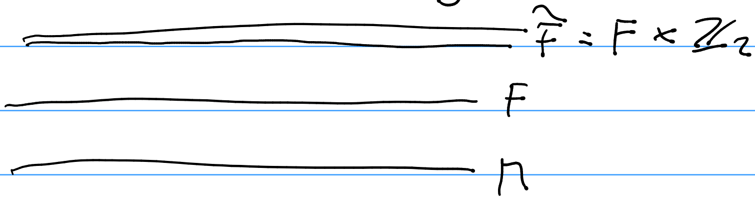
Dąbrowski

- recall:
  - we started with $\underset{\text{Dirac}}{\gamma} : \mathcal{C}(X) \longrightarrow L(\mathbb{C}^{2^{[n/2]}})$
  - passed to (M, g) spin mfd w/ Σ $\mathbb{C}$-vbdl of spinors s.t. for $x \in M$, $\check{\gamma} : \mathcal{C}(T_x M) \times \Sigma_x \longrightarrow \Sigma_x$,
    $$\check{\gamma}(\alpha, \psi) := \gamma(\alpha)\psi.$$
  and built $\not{D} = \gamma \circ \nabla \in \mathrm{Aut}\,\Gamma(\Sigma)$

$\Rightarrow$ $(\mathcal{C}^\infty(M), L^2(\Sigma, \mathrm{vol}_g), \not{D})$ is S.T.
- $\dim M = \text{even}$ $\Rightarrow$ $\exists\, \chi = \gamma(\mathrm{vol}) = \mathrm{vol}\cdot$
- M spin $\Rightarrow$ $\exists\, J$ real structure

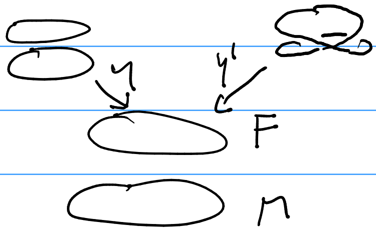
Examples

- $M = \mathbb{R}$, "x" = "id", $g = dx^2 = dx \otimes dx$, $g(e, e) = 1$



$\partial_1 = i$, $D = i\partial_x$, $J = \text{c.c.}$
$\Gamma(\Sigma) =$ smooth cpx functions

- $M = \mathbb{R}/2\pi\mathbb{Z} = S^1$, same x mod 2π, g



$-\Gamma(\Sigma) = \begin{cases} \text{periodic} \\ \text{antiperiodic} \end{cases}$

$\Downarrow$

$\psi_m = \text{const}\cdot e^{imx}$, $m \in \begin{cases} \mathbb{Z} \\ \mathbb{Z} + 1/2 \end{cases}$

$\not{D} = i\partial_x$, $\not{D}\psi_m = \pm\sqrt{m^2}\,\psi_m$
$\longrightarrow$ spectrum depends on sp. str. (e.g., $\ker \not{D}|_{\text{antip.}} = \{0\}$)

- $M = \mathbb{R}^2$, $g = dx \otimes dx + dy \otimes dy$
  $\exists$ global frame $(x, y) = e$
  $\Rightarrow F = M \times SO(2)$, $\tilde{F} = M \times Spin(2)$ trivial,
  so $\Sigma = M \times \mathbb{C}^2$, $\Gamma(\Sigma) = C^\infty(M, \mathbb{C}^2)$

$$D = i \underset{\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)}{\sigma_1} \partial_x + i \underset{\left(\begin{smallmatrix} & -i \\ i & \end{smallmatrix}\right)}{\sigma_2} \partial_y = i \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}, \quad z = x + iy$$

$$\chi = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad J_- = \begin{pmatrix} & -i \\ i & \end{pmatrix} \circ c.c., \quad J_+ = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \circ c.c.$$

- $M = \mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{T}^2 = S^1 \times S^1$
  same x, y, g

$$\begin{array}{l} \tilde{F} = M \times Spin(2) \\ \downarrow \eta_{jk} \\ F = M \times SO(2) \\ \downarrow \uparrow e_{jk} \\ M \end{array} \qquad \tilde{e}_{jk}$$

- rotation $\begin{pmatrix} \cos(jx + ky) & \sin(\ ) \\ -\sin(\ ) & \cos(\ ) \end{pmatrix}$
- $j, k = 0, 1$

$\Rightarrow$ different sp. str.

$\Rightarrow \psi$ (anti) periodic in x, y

$$\not{D} \psi_{m,n} = \pm \sqrt{m^2 + n^2}\, \psi_{m,n}, \quad \begin{array}{l} m \in \mathbb{Z} + j/2 \\ n \in \mathbb{Z} + k/2 \end{array}$$

$$\psi_{m,n} = e^{imx + iny} \begin{pmatrix} a \\ b \end{pmatrix}$$

[this is for $e_{jk}, \tilde{e}_{\tilde{j}\tilde{k}}$ with $\tilde{j} = j$, $\tilde{k} = k$
- if we take $\tilde{j} \neq j, \tilde{k} \neq k$, the lifts $\tilde{e}_{\tilde{j}\tilde{k}}$ and $\eta_{\tilde{j}\tilde{k}}^{-1} \circ e_{jk}$ differ by rotation, which is "local gauge transf."

- we use $\nabla = d + \alpha^{e \cdot g}$, $\alpha^e = 0$,
  $\alpha^{e \cdot g} = g^{-1} \alpha^e g + g^{-1} dg$
  and $g = R = \begin{pmatrix} \cos(jx + ky) & \sin(\cdot) \\ -\sin(\text{—}) & \cos(\cdot) \end{pmatrix}$

$\rightarrow \nabla^{spin} = d + \rho \circ \dot{s}^{-1}(R^{-1} \circ dR)$, $\dot{s}$: $spin(2) \rightarrow so(2)$
$= d + i/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (j\,dx + k\,dy)$

- but spec is the same modulo discrete shift

- $M = \mathbb{T}^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n = S^1 \times \text{—} \times S^1$
  - 2^n - sp. str, $t = \{t_j\}$, $j = 1, \text{—}, n$, $t_j \in \{0, 1\}$
  - $F, \tilde{F}, \Sigma$ trivial, $\not{D}^2 = \Delta \cdot \mathbb{1}_{2^{[n/2]}}$

  - eigenspinors labelled by $\lambda = (\lambda_j) \in \Lambda_t := \mathbb{Z}^n + \frac{t}{2}$, $\lambda_j \in \mathbb{Z}^n + \frac{t_j}{2}$
    - t spin index $\in \{1, \text{—}, 2^{[n/2]}\}$, $j \in 1, \text{—}, n$
    $\sim e^{i(\lambda_1 x_1 + \dots + \lambda_n x_n)} \begin{pmatrix} 1 \\ \vdots \end{pmatrix}$

  $\rightarrow$ eigenvalues $l = |\lambda|$, degenerate

- in general, for curved g, impossible to compute exactly
  - perhaps on homogenous spaces, etc.
  - what if g has torsion? etc.

- Hodge-deRham S.T.

$\Lambda^* T_x^* M \otimes \mathbb{C}$ ($\bigcup_{x \in M}$)

- (M,g) oriented, $(C^\infty(M), L^2(\Lambda M, vol_g), d+d^*)$

- basis $\{e^{j_1}\wedge \dots \wedge e^{j_k} \mid j_1 < \dots < j_k, k=0,\dots,n\}$

- check: $d+d^* = \lambda \circ \nabla_{\text{Levi-Civita}}$,
  where $\lambda(v)\cdot\alpha = (v\wedge - v\lrcorner)\alpha$
  $v \in \Gamma(T^*M)$

- $(\lambda(v))^2 = -|v|^2$
- λ rep. of $Cl(-)$ of $\dim = 2^n$ $\rightarrow$ reducable, but who cares
- check: $(\sigma_{d+d^*}(\xi))^2 = -|\xi|^2$

- further, we have gradings:
  1) $\gamma_\Lambda = (\pm)^k$ on $\alpha \in \Gamma(\Lambda^k M)$
  11) $\gamma'_\Lambda = \text{const.} \circ *$, $*$ = Hodge star
  s.t. $\gamma'_\Lambda(e^{j_1}\wedge\dots\wedge e^{j_k}) = i^{k(k-1)+n} e^{j_{k+1}}\wedge\dots\wedge e^{j_n}$ $\forall 0\le k\le n$
  where $e^{j_1},\dots,e^{j_k},e^{j_{k+1}},\dots,e^{j_n}$ is
  an even perm. of $e^1,\dots,e^n$
- check: $\gamma_\Lambda, \gamma'_\Lambda$ commute w $C^\infty(M,\mathbb{C})$,
  anticomm. w $d+d^*$

- index $d+d^* |_{\Lambda^{even}, \Lambda^{SD}} = \begin{cases} \text{Euler ind.} \\ \text{Signature} \end{cases}$ of M
  $\hookrightarrow$ +1 eigenspace of $\gamma_\Lambda, \gamma'_\Lambda$

- $\exists\ J$ = c.c.

- using λ, $\Gamma(\Lambda M)$ is $Cl(V)$-bimod
  - $\lambda_R = (v\wedge + v\lrcorner)\circ\gamma_\Lambda$

- as vsp, $\Lambda V = \mathcal{C}(V)$, so we get Morita equiv. over itself
- recalling $[\not{D}, f] = df \cdot$, $(\mathcal{C}^\infty(M), [d + d^*, f]) \simeq \Gamma(\mathcal{C}(M))$

  - $[J f J, \alpha] = 0 \quad \forall \alpha \in Cl, f \in \mathcal{C}^\infty(M)$

- $\exists\, J_\Lambda' = (-)^{k(k-1)/2}$ a.c.c. $\to$ intertwines λ_R, λ_L

- R. Plymen

- so we can build $\overset{NC}{\text{diff.}}$ forms from any $(A, H, D) \longmapsto \mathcal{C}_D(A)$ in this way

- to close, operator D on $\Gamma(\mathcal{E})$ is of Dirac-type if $[D, f]^2 = -g(df, df)$
$\Rightarrow \mathcal{C}(df) := [D, f]$
$\to$ then $(\mathcal{C}^\infty(M), L^2(\mathcal{E}), D)$ is a S.T.