Dabrowski

- recalling

Thm (Serre–Swan) smooth vbdls on M

$\leftrightarrow$ proj fin rk $C^\infty(M)$-modules

- $E \longrightarrow \Gamma(E)$ directly
- $\coprod_x p(x)\mathbb{C}^N =: E \longleftarrow \mathcal{E} = pA^N$ in other dir.

$\mathcal{E}/\mathcal{E}\cdot\ker(ev_x)$

- this is actually an equivalence of categories, with morphisms being bdle homs (induced by diffeos) $\longleftrightarrow$ module homs

- now, $\mathrm{Spin}_{\mathbb{C}}$-structure $\longrightarrow \Gamma^\infty(\Sigma = \tilde{F} \times_\gamma \mathbb{C}^{2^{[\frac{n}{2}]}})$

$(\tilde{F} \times_S \mathbb{R}^n = TM)$ Morita eq.

$\mathcal{E}$ $\Gamma^\infty(\mathbb{C}(M)) - C^\infty(M,\mathbb{C})$ - bimod

$\overset{?}{\longleftarrow}$

$$\begin{array}{ccc} \mathrm{Spin}_{\mathbb{C}}(n) & \xrightarrow{\gamma} & U(2^{[n/2]}) \\ s \downarrow & & \downarrow \pi = Ad \\ SO(n) & \xhookrightarrow{i} & PU(2^{[n/2]}) \end{array}$$

- by S.S., $\mathcal{E} = \Gamma(\Sigma)$ with Σ a $\mathbb{C}$-v.bdl$/M$

$\Gamma^\infty(\mathbb{C}(M)) = \mathrm{End}_{C^\infty(M,\mathbb{C})} \Gamma^\infty(\Sigma) = \Gamma^\infty(\mathrm{End}(\Sigma))$

- same transition functions, global sctns

$\pi(\gamma_{ab}) = Ad\, g_{ab}$

- note $J_\pm : \mathbb{C}^{2^{[n/2]}} \circlearrowleft$ induces

$$(\mathcal{J}_\pm \psi)(\hat{e}) := \underset{C_\pm \circ c.c.}{\overset{\shortparallel}{J_\pm}}(\psi(\hat{e}))$$

- this, if it exists, selects the real $\mathrm{Spin} \subset \mathrm{Spin}_\mathbb{C}$
- $\mathcal{J}_\pm$ shares properties of $J_\pm$, $\varepsilon, \varepsilon', \varepsilon''$

- for functions $Ad_\pm f = \mathcal{J}_\pm f \mathcal{J}_\pm^{-1}$

$$= (C_\pm \circ \underbrace{c.c.) f (c.c.}_{\bar{f}} \circ C_\pm)$$

$$= \bar{f}$$

- this is the $*$-antiinvolution seen from C^*-theoretic p.o.v.
  - but $Ad_\pm$ is an involution...
    - so this works for <u>commutative</u> setting

- we will instead define reality cond. as $[J A J^{-1}, A] = 0$, i.e. $J a J^{-1} b = b J a J^{-1}$ $\forall a, b \in A$

Spin-connection & cov. derivatives of <u>spinor fields</u>

- by connection on $\mathbb{K}$-lin vbdl $E \to M$ we mean $C^\infty(M, \mathbb{R})$-lin map $\nabla : \underset{\Gamma(TM)}{\overset{\cap}{X}} \mapsto \nabla_X$,

$\nabla_X : \Gamma^\infty(E) \to \Gamma^\infty(E)$
satisfying Leibniz rule

$$\nabla_X(f s) = (\overset{df(X)}{\overset{\shortparallel}{\mathcal{L}_X f}}) s + f \nabla_X s$$

- we use instead $\hat{\nabla}: \Gamma^\infty(\Sigma) \to \Gamma^\infty(\Sigma) \otimes_{C^\infty(M,\mathbb{R})} \Gamma^\infty(T^*M)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$ G.N.

$\quad\quad \Gamma(\Sigma)$

$(\hat{\nabla} \overset{\omega}{z})(X) := \nabla_X z \quad\quad\quad\quad\quad \Gamma^\infty(\Sigma \otimes T^*M)$

- conn.'s form an affine sp. over $\Gamma^\infty(\operatorname{End}\Sigma \otimes T^*M)$
- we ask to preserve hermitian structure h on E, $\mathcal{L}_X h(z, z') = h(\nabla_X z, z') + h(z, \nabla_X z')$

- for $E = TM$, we get metric connection which can preserve g,

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

- extend it to arbitrary tensors by

$$\nabla_X (A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B$$

- note that the ideal $\{X \otimes Y + Y \otimes X - g(X,Y)\}$ is preserved by ∇.
  - so it descends to quotient by it, and becomes $\nabla_X(\alpha \cdot \beta) = \nabla_X \alpha \cdot \beta + \alpha \cdot \nabla_X \beta$ wrt Clifford multiplication

- we also ask for $\nabla_X Y - \nabla_Y X = [X, Y]$

$\to$ Prop $\exists!$ Levi-Civita conn.

- for $\{e_n\}$ o.n.b., $[e_i, e_j] = c_{ijk} e_k$, $\nabla_{e_i} e_j = \alpha_{ijk} e_k$

$\xrightarrow{\text{claim}}$ $\alpha_{ijk} = \frac{1}{2}(c_{ijk} + c_{kij} + c_{kji})$

- check antisymmetry in $(i \leftrightarrow j)$

- for $Y = Y_i e_i \Rightarrow (\nabla_{e_i} Y) = (\mathcal{L}_{e_i} Y_j + \alpha_{ijk} Y_k) e_j$

- $\mathring{\xi} : spin(n) \to so(n)$ lets us def.

$\alpha^{(\tilde e)} := \mathring{\xi}^{-1} \circ \alpha^{(e)}$, where $\eta(\tilde e) = e$

and

$$(\nabla_X \psi)\circ(\tilde e) = \left(\mathcal{L}_X + \gamma \circ \tilde{\alpha}^{(\tilde e)}(X)\right)(\psi \circ \hat e)$$

or equiv. $(\nabla \psi)\circ \hat e = \left(d + \gamma \circ \mathring{\xi}^{-1}\alpha^{(\hat e)}\right)\tilde\psi(\hat e)$

- $\nabla_X(\underbrace{Y \cdot \psi}_{=\gamma(Y)\psi}) = \nabla_X Y \cdot \psi + Y \cdot \nabla_X \psi$

$$\mathcal{L}_X \langle \psi, \varphi \rangle = \langle \nabla_X \psi, \varphi \rangle + \langle \psi, \nabla_X \varphi \rangle$$

where $\langle \psi, \varphi \rangle = \int_M \langle \psi(\hat e), \varphi(\hat e) \rangle_{\mathbb{C}^{2^{\lfloor n/2 \rfloor}}} \cdot vol$

- for different spin structs,

$\tilde F \ni \tilde e$

$\psi' \circlearrowleft \circlearrowright \psi$

$e' = e \circ g,\ g \in SO(n)$

$\alpha^{(eg)} = g^{-1} \circ \alpha^{(e)} \circ g - g^{-1} dg$

for same but with "rotated source"

$\tilde e' = \tilde e \tilde g$

$\psi' \circlearrowleft_{e} \circlearrowright \psi \Rightarrow \alpha^{(\tilde e \tilde g)} = \tilde g^{-1} \alpha^{(\tilde e)} \tilde g - \tilde g\, d\tilde g$

where $s(\tilde g) = g$

Def. (Curvature) $R_{ijkl} e_l = \nabla_{e_i} \nabla_{e_j} e_k - \nabla_{e_j} \nabla_{e_i} e_k - \nabla_{[e_i, e_j]} e_k$

- $R_{ijij} =: R$ scalar curvature

Def (Dirac operator) $\not{D} = \gamma \circ \nabla : \Gamma(B) \to \Gamma(T^*M) \otimes \Gamma(B) \xrightarrow{\gamma} \Gamma(B)$
where $\nabla = \sum_{j=1}^{n} e_j \otimes \nabla_{e_j}$ where
$\{e_j\}$ dual basis.
Define $\not{D}\psi = \sum e_j \cdot \nabla_j \psi = \sum \gamma_j \nabla_{e_j} \psi$.