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Kähler geometry: Intro 2 the Calabi conj.

Prelims.

Let M cpx cpt mfd, $\dim_{\mathbb{C}} M = n$.
Let $J: TM \to TM$, $J^2 = -Id$ be the cpx structure.
Note that $\mathbb{C}^n \cong \mathbb{R}^{2n} \cong T\mathbb{C}^n$ so $J_{\mathbb{C}^n}: T_p\mathbb{C}^n \to T_p\mathbb{C}^n$ is well-def., and that a map is biholo. if its differential commutes w $J_{\mathbb{C}^n}$.

- pick $\vec{z} = (z_1, \dots, z_n)$ cpx coords on M
  $T_{\mathbb{C}} M := TM \otimes_{\mathbb{R}} \mathbb{C}$
  $T^{1,0}M := \langle \partial_{z_1}, \dots, \partial_{z_n} \rangle_{\mathbb{C}}$
  $T^{0,1}M := \langle \partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_n} \rangle_{\mathbb{C}}$
  - $T^{1,0}$ = i-eigenspace of J, $T^{0,1} = -i$ - ...
- define $\Lambda^{p,q} T^*M := \Lambda^p T^{1,0*}M \wedge \Lambda^q T^{0,1*}M$
  as well as a sheaf of (p,q)-forms $\mathcal{A}^{p,q}$
- we get naturally differentials $d, \partial, \bar{\partial}$ ...

Kähler structure & facts

- ask for compatibility w metric, $g \circ (J \otimes J) = g$
- define 2-form $\omega := g(J\_, \_)$

Def. (M, J, g) is Kähler if $d\omega = 0$.

- note that we get orientability since $vol_g := \frac{1}{n!} \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}}$
- Kähler $\Rightarrow$ Symplectic.

Lemma. Kähler $\Leftrightarrow \nabla^g J = 0$. Also, Kähler $\Rightarrow \nabla^g$ perserves T^{10}, T^{01}.

- if Kähler, $\Gamma^\gamma_{\alpha\beta} \mapsto \Gamma^c_{ab}$ & $\Gamma^{\bar{c}}_{\bar{a}\bar{b}}$ (no mixing of indices)
  $\to$ comes from: $d\omega = 0 \Leftrightarrow \partial_a g_{b\bar{c}} = \partial_b g_{a\bar{c}}$.

- cute fact: $f \in C^\infty(M)$, $\Delta_g f := -\mathrm{div}(\mathrm{grad}\, f)$

$$= -g^{ij}\nabla_i\nabla_j f$$

$$\xRightarrow{\text{Kähler}} = -2g^{a\bar{b}}\partial_a\partial_{\bar{b}} f$$

Curvature of Kähler metric.

- $R(\partial_\alpha\partial_\beta)(J\partial_\gamma) = JR(\partial_\alpha,\partial_\beta)\partial_\gamma$
  - $\Rightarrow R$ perserves $T^{10} \oplus T^{01}$
    - $\Rightarrow R_{\alpha\beta}{}^\gamma{}_\delta$ but γ, δ have same type.
- only possible coefficients $(\neq 0)$ are $\underset{\text{"}}{R_{a\bar{b}}{}^c{}_d}$, $R_{\bar{a}b}{}^c{}_d$ and conj.

  $-\partial_{\bar{b}}\Gamma^c{}_{ad}$

- not much to say abt. Ricci tensor usually, but there is in the Kähler setting.
  - $Ric = Ric_{a\bar{b}}\, dz^a \otimes d\bar{z}^b$
  - $Ric(J-, J-) = Ric(-,-)$
    - $\Rightarrow \rho(g) := Ric(J-,-)$ is called the Ricci form

Lemma. $\rho(g) = -i\partial\bar{\partial}\log\det g$

Pf. $\rho_{a\bar{b}} = i\,Ric_{a\bar{b}} = -i\partial_{\bar{b}}(g^{c\bar{d}}\partial_a g_{c\bar{d}})$ by the Leibniz formula $\partial\det A = \mathrm{Tr}(\mathrm{adj}(A)\cdot\partial A)$.

- $\rho(g)$ is closed, and $\rho(g) - \rho(g') = -i\partial\bar{\partial}\log\frac{\det g}{\det g'}$ is exact.

- $\frac{1}{2\pi}[\rho(g)]$ is a well-def class in $H^2(M)$, 1^{st} Chern class

$$c_1(K^*_M) = c_1(\Lambda^{top}T^{10}M)$$

Calabi conjecture.

- define Kähler class of ω, $K(\omega) := \{\omega' \in [\omega] \mid \omega' \text{ Kähler}\}$
- we have a map $K(\omega) \to c_1(M)$

$$\omega' \mapsto \frac{1}{2\pi} S(\omega')$$

  - conjecture: "this map is a bijection"
    - $\forall$ real 1,1-form $\eta \in 2\pi c_1(M)$ $\exists!$ $\omega' \in K(\omega)$ s.t. $S(\omega') = \frac{1}{2\pi}\eta$

$\partial\bar\partial$-Lemma

Let α real exact p,q-form. Then $\exists$ real $(p-1),(q-1)$-form η s.t.

$$\alpha = i\partial\bar\partial\eta$$

$$K(\omega) = \{\underbrace{\omega + i\partial\bar\partial\varphi}_{=\omega_\varphi} \mid \varphi \text{ real}, \omega_\varphi > 0\}$$

- if η real 1,1-form & $[\eta] = [S(\omega)]$,
  $\eta = S(\omega) - i\partial\bar\partial F = -i\partial\bar\partial \log\det g - i\partial\bar\partial F$
- we impose $S(\omega_\varphi) = -i\partial\bar\partial \log\det\omega_\varphi \overset{!}{=} \eta$
  - equivalent to solving

$$\frac{\det(g_{a\bar b} + \partial_a\partial_{\bar b}\varphi)}{\det g_{a\bar b}} = e^F$$

Calabi conjecture

Assume $K(\omega) = \lambda c_1(M)$. Is there $\omega' \in K(\omega)$ s.t. $\omega' = \lambda S(\omega')$?

Kähler-Einstein problem: $g = \lambda \mathrm{Ric}$.

Either:

i) $K^*_M > 0$ FANO $\lambda > 0$

ii) K^*_M deg 0 CALABI-YAU $\lambda = 0$

iii) $K^*_M < 0$ GENERAL TYPE $\lambda < 0$

- example.
  - $\mathbb{P}^n$ is kähler
    - → Fubini-Study $\omega_{FS} = i\partial\bar{\partial}\log(|z_0|^2 + \dots + |z_n|^2)$.
    - → clearly $V \subseteq \mathbb{P}^n$ also kähler
    - → by the adj. formula for smooth hypersurface of deg d: $\deg K^*_V = n+1-d$

- fix $\lambda = \pm 1, 0$. By assumption, $\exists \omega_0$ st $\lambda\omega_0 \in c_1(M)$

  $S_0 - \lambda\omega_0 = i\partial\bar{\partial}F$

  - we want ω_φ s.t. $S(\omega_\varphi) = \lambda\omega_0$

  ---- ⇒ $$\frac{\det(g_{a\bar{b}} + \partial_a\partial_{\bar{b}}\varphi)}{\det g_{a\bar{b}}} = e^{F_0 - \lambda\varphi}$$

Pf (for $\lambda = -1$) Put RHS $= e^{tF + \varphi}$ and def

$S = \{t \in [0,1] \mid \exists \text{ soln}\}$

- S is nonempty: $t=0$, $\varphi \equiv 0$ is a soln.
- S is open (impl. fn. th. argument).
- S is closed (boundedness arguments)