BRUZZO

Morphisms (locally) of finite type

- quasi-finite $\to \{f^{-1}(y)\}$ is finite for any $y \in Y$, $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$
- ex: $\mathbb{A}^n_k = \operatorname{Spec} k[x_1 \dots x_n]$ is of finite type

$$\begin{array}{c} \mathbb{A}^1_k \to \mathbb{A}^n_k \\ z \mapsto z^n \end{array}$$

$\to$ $k[t]$-module structure given by raising to n-th power

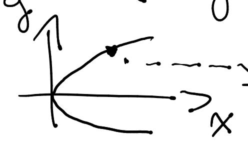
$\to$ basis given by $1, x, \dots, x^{n-1}$

- ex: $k^* \to \mathbb{A}^1_k$

$\mathbb{A}^1_k - \{0\} = D(x) = \operatorname{Spec} k[x, \frac{1}{x}]$ $\to$ quasi finite but not finite

- ex: $X = \operatorname{Spec} \dfrac{k[x,y]}{(x-y^2)}$ $\rightsquigarrow$ $\dfrac{k[x,y]}{(x-y^2)} = k[x] \oplus y\, k[x]$

$\hookrightarrow$ finitely gen. as a module $\Leftrightarrow$ finite



$\to$ remove a pt and no longer finite

- ex: $X = \operatorname{Spec} \dfrac{k[x,y,t]}{(ty - x^2)}$, $f \downarrow$, $Y = \operatorname{Spec} k[t] \simeq \mathbb{A}^1_k$

$\to X, Y$ integral schemes of fin. type

$\to$ fibres: $t = a \neq 0$, integral variety

$t = 0 \Rightarrow f^{-1}(0) = \operatorname{Spec} \dfrac{k[x,y]}{(x^2)}$

$\to$ NONREDUCED !

ERRATA from last time: $\operatorname{Proj} S = \{\text{prime ideals } p,\ S_+ \not\subset p\}$

Fiber product of top. spaces.

- standard def.

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "q"] \\ & X \arrow[r, "p"'] & Z \end{tikzcd}$$

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & S \end{tikzcd}$$

$X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$, $f \rightsquigarrow R \xrightarrow{f^\#} A$, $g \rightsquigarrow R \xrightarrow{g^\#} B$, $C = A \otimes_R B$

$\hookrightarrow (a \otimes b)(a' \otimes b') = (aa' \otimes bb')$

$\Rightarrow X \times_S Y = \operatorname{Spec} C$

$(-)^\#$ is a rather ... notation, $(-)^*$ is better...

$$\begin{tikzcd} C = A \otimes B & A \arrow[l, "\pi_1^\#"'] \\ B \arrow[u, "\pi_2^\#"] & R \arrow[l, "g^\#"] \arrow[u, "f^\#"'] \end{tikzcd}$$

$\pi_1^\#(a) = a \otimes 1$

- EXAMPLE

X, Y schemes over k: $X \times_k Y \equiv X \times_{\mathrm{Spec}\, k} Y$

- $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathrm{Spec}\, \underbrace{k[x] \otimes_k k[y]}_{k[x,y]} = \mathbb{A}^2_k$

→ note that closed sets in $\mathbb{A}^1_k$ are finite collections of points, but in $\mathbb{A}^2_k$ they could be curves, e.g. $x = y$

→ difference in topology

Def.

$$
\begin{tikzcd}
X \arrow[dr, "\Delta_f"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

DIAGONAL MORPHISM Δ_f associated to $X \xrightarrow{f} Y$.

Def. f is a separated morphism if Δ_f closed immersion.

X scheme over k is separated (over k) if $X \to \mathrm{Spec}\, k$ separated.

Prop. All morphisms over affine schemes are separated.

Cor: All affine schemes over k are separated over k.

→ consider X = —:— line w/ double origin

⇒ $X \times_k X$ = aff plane w/ 2 axes & 4 origins

Δ = line w/ double origin

⇒ all origins are in the closure of Δ ⇒ Δ ¬ closed ⇒ X not affine

Pf:

- note that on CRing we have the codiagonal $\Delta^\#: A \otimes A \to A$, $a \otimes a' \mapsto aa'$
- surjective ⇒ $\mathcal{O}_{X \times_Y X} \xrightarrow{\Delta^\#} \Delta_* \mathcal{O}_X$ surjective on stalks
- for $I \subset A$, $A \twoheadrightarrow A/I$, $\mathrm{Spec}\, A/I \to \mathrm{Spec}\, A$ morphism of Spec-s

→ take $I = \ker \Delta^\#$ ⇒ Δ closed immersion □

Prop. $X \xrightarrow{f} Y$ separated iff Δ_f closed in $X \times_Y X$

Pf ⇒ obvious, ⇐

$$
\begin{tikzcd}
X \arrow[r, "\Delta_f"] \arrow[dr, "\cong"'] & X \times_Y X \arrow[d, "\pi_1"] \\
& X
\end{tikzcd}
$$

Δ_f^{-1} may be identified with the identity ⇒ closed im. □