Stoppa.

Ricci curvature.

- (M,g), $\{e_1,\dots,e_n\}$ ON frame at $p\in M$.
→ fix $x := e_n$

Def. (Ricci) $Ric_p(x) := \frac{1}{n-1}\sum_{i=1}^{n-1}\langle R(x,e_i)x, e_i\rangle$

Lemma. $Ric_p(x)$ is well-defined, independently of the choice of frame.

Pf. Consider $L\in End(T_pM)$, $L(Z) := R(X,Y)Z$ for $X,Z\in T_pM$ fixed. Trace in (FDVect) is just the usual one; define $Q(X,Y) = Tr\, L$
Not hard to see it is a symmetric quadratic form. In particular, if $\|X\|=1$, then
$$Q(X,X) = Tr\, L_{X,X} = \sum_{i=1}^{n-1}\langle R(X,e_i)X, e_i\rangle$$
$$= (n-1)\, Ric_p(X).$$

Rmk. Usually $Q(X,Y)$ is also called the Ricci curvature, denoted by $Ric_p(X,Y)$.

Scalar curvature.

Def. (Sc. curv.) $\forall p\in M$ given by $S(p) := \frac{1}{n}\sum_{j=1}^{n} Ric_p(e_j)$

Lemma. Well-defined.

Pf. $\forall p\in M$, T_pM possesses 2 symmetric quadratic forms, $\langle -,-\rangle_p$ & $Q_p(-,-)$.
By linear algebra $\exists!\, N\in End(T_pM)$ s.t.
$\langle N(X),Y\rangle_p = \langle X, N(Y)\rangle_p$ & $Q_p(X,Y) = \langle N(X),Y\rangle_p$ $\forall X,Y\in T_pM$.

Consider $\mathrm{Tr}\, N = \sum_{i=1}^{n} \langle N(e_i), e_i \rangle_p$
$= \sum_{i=1}^{n} Q_p(e_i, e_i) = n(n-1)\, S(p).$

Bonnet-Myers thm.

Thm. (M, g) complete. Suppose Ric uniformly bounded from below by a strictly positive constant;
$\forall p \in M, \forall v \in T_pM, \|v\|=1,\ \mathrm{Ric}_p(v) \geq \frac{1}{r^2} > 0.$
Then M is cpt & $\mathrm{diam}(M, g) \leq \pi r$.

Rmk. Paraboloid of revolution $\subset \mathbb{R}^3$: $\mathrm{Ric}_p(v) \geq 0$ & noncpt.

Energy functional.

$$E(c(t)) := \int_a^b \|\dot{c}(s)\|\, ds$$

- fix $p, q \in M$ for now.

Def. $\Omega_{p,q} :=$ { piecewise C^1 paths from p to q }

Def $c(t) \in \Omega_{p,q}$. $T_c \Omega_{p,q} :=$ { piecewise C^1 v.f.s along $c(t)$ vanishing at endpoints }

- given $V(t) \in T_c \Omega_{p,q}$, we get a variation (assuming $c: [0,a] \to M$.)
$$f(s,t) = \exp_{c(t)}(s\, V(t)), \quad (s,t) \in (-\varepsilon, \varepsilon) \times (0, a)$$
& conversely, given $f(s,t)$, we get
$$V(t) = \frac{\partial f}{\partial s}\Big|_{s=0}(t) = df_{(s=0,t)}\left(\frac{\partial}{\partial s}\right)$$

$\rightarrow$ we study E along these variations

$$\Rightarrow \quad E(s) := \int_0^a \left\| \frac{\partial f}{\partial t}(s,t) \right\|^2 dt$$

Lemma. $c(t)$ is a minimum of E iff $c(t)$ is a minimising geodesic parametrised by $\|\dot{c}(t)\| = \text{const}.$

Pf. By Schwarz ineq., $(\mathrm{Len}(c))^2 \leq a\,E(c)$, with equality iff $\|\dot{c}(t)\| = \text{const}$.
Take $\gamma(t)$ to be parametrised as such and also a minimising geodesic:

$$\underbrace{L(\gamma)^2}_{a E(\gamma)} \leq \underbrace{L(c)^2}_{\leq\, a E(c)}$$

Def. $c(t)$ is a critical pt of E if $\forall$ variations $f(s,t)$, $\frac{d}{ds}E(s)\big|_{s=0} = 0$.

Lemma. Suppose given $0 = t_0 < t_1 < \dots < t_{k+1} = a$, $c(t)$ is C^1 on each $[t_i, t_{i+1}]$.
Then
$$\frac{1}{2}\frac{d}{ds}E(s)\Big|_{s=0} = -\int_0^a \left\langle V(t), \frac{D}{dt}\dot{c}(t) \right\rangle dt - \sum_{i=1}^{k} \left\langle V(t_i), \dot{c}(t_i+0) - \dot{c}(t_i-0) \right\rangle$$

Cor. If $\frac{dE}{ds}\big|_{s=0} = 0$ $\forall$ variations $\Rightarrow c(t)$ geodesic.

Lemma. Everything as above. Fix $\gamma(t)$ crit. pt of E (geodesic)
Then
$$\frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_0 = -\int_0^a \left\langle V(t), \frac{D^2}{dt^2}V(t) + R(\dot{\gamma}(t), V(t))\dot{\gamma}(t) \right\rangle dt - \sum_{i=1}^{k} \left\langle V(t_i), \frac{DV}{dt}(t_i+0) - \frac{DV}{dt}(t_i-0) \right\rangle$$

Cor. Elements $J \in \mathcal{J}_{p,q}$ give $\frac{d^2E}{ds^2}(s)\big|_{s=0} = 0$.

Pf of Bonnet-Myers.

Fix $p, q \in M$. Claim: $d(p,q) \leq \pi \eta$.

M complete $\Rightarrow \exists\ \gamma(t)$ minimising geodesic, $\gamma \in \Omega_{p,q}$,

s.t. $\mathrm{len}(\gamma) = d(p,q)$ (Hopf-Rinow)

Normalise $\gamma: [0,1] \to M$, $\|\dot\gamma\| = \mathrm{len}(\gamma)$.

Assume $\mathrm{len}(\gamma) > \underline{\pi \eta}$.

Fix ON frame $\{e_1, \dots, e_{n-1}, e_n = \dot\gamma(t)/\mathrm{len}\,\gamma\}$

parallel along $\gamma(t)$.

Consider v.f.s $V_k(t) := \sin(\pi t)\, e_k(t)$, $k = 1, \dots, n-1$

$\to V_k(t) \in T_\gamma \Omega_{p,q}$.

$$\Rightarrow \frac{1}{2} \frac{d^2 \sigma_k}{ds^2}\Big|_{s=0} = -\int_0^1 \langle V_k, V_k'' + R(\dot\gamma, V_k)\dot\gamma \rangle\, dt$$

$$= \int_0^1 \sin^2(\pi t)\left(\pi^2 - \mathrm{len}^2\gamma\, K(e_n(t), e_k(t)\right) dt$$

$$\Rightarrow \frac{1}{2} \sum_{k=1}^{n-1} \frac{d^2 \sigma_k}{ds^2}\Big|_{s=0} = \dots = \int_0^1 \sin^2(\pi t)\big((n-1)\pi^2 - (n-1)\mathrm{len}^2\gamma\, \mathrm{Ric}_{\gamma(t)}(e_n(t))\big)\, dt < 0$$

$\to$ but $\mathrm{Ric}_{\gamma(t)}(v) \geq \frac{1}{\eta^2}$ $\lightning$.