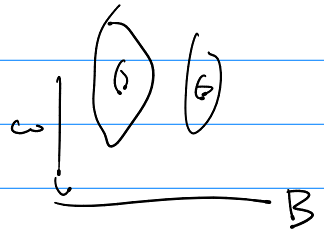
AG Seminar - Baranzini.



- information collected in $(\mathcal{M}, \omega, B)$, where we put transition functions $(t, f_{ij}(z,t))$ to be holomorphic & to agree with trans. fns. on $\mathcal{M}_t$ for fixed t's.
- we may consider the tangent space spanned by elements $\vartheta_{ij}(t) = \sum_{\alpha=1} \frac{\partial f^{\alpha}_{ij}(z,t)}{\partial t} \frac{\partial}{\partial z^{\alpha}_i}$,

or in fact $[\vartheta_t] \in H^1(\mathcal{M}_t, \Theta_t)$ as infinitesimal deformations
- setting $z_i = f_{ij}(z_j, t)$,

$$\vartheta^{\alpha}_{ij}(z_i,t) - \vartheta^{\alpha}_{ik}(z_i,t) + \frac{\partial z^{\beta}_i}{\partial z^{\alpha}_j} \vartheta^{\beta}_{jk}(z_j,t) = 0$$

where $\vartheta^{\alpha}_{ij} = \frac{\partial f^{\alpha}_{ij}(z_j,t)}{\partial t}$

→ differentiating by t gives

$$\dot{\vartheta}_{ij}(t) - \dot{\vartheta}_{ik}(t) + \dot{\vartheta}_{jk}(t) = [\vartheta_{ij}(t), \vartheta_{jk}(t)]$$

$\underset{\text{coboundary}}{\shortparallel}$

- we get:

<u>Thm</u> Suppose $\vartheta \in H^1(M_{t_0}, \vartheta_{t_0})$ and you have a family with $\vartheta(t_0) = \vartheta$. Then $[\vartheta, \vartheta] = 0$ in $H^2(M_{t_0}, \vartheta_t)$

<u>Thm</u> $H^2(M_{t_0}, \vartheta_{t_0}) = \{0\} \Rightarrow$ possible to build up cpx analytic family for M_{t_0}.

→ so we know something abt obstructions

- some definitions

1) $(\mathcal{M},\omega,B)$ is complete at a point $t_0 \in B$
if $\forall (\mathcal{N},\pi,D)$ with $\omega^{-1}(t_0) = \pi^{-1}(0)$
$\exists\ \varphi : \mathcal{N}_\Delta \to \mathcal{M}$ holomorphic
$\exists\ h : \Delta \to B$
and

$$\begin{tikzcd} N_\Delta \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathcal{M} \arrow[d, "\omega"] \\ \Delta \arrow[r, "h"] & B \end{tikzcd}$$

commutes.

ii) $(\mathcal{M},\omega,B)$ is effectively parametrized
if $\delta_t : T_t B \to H^1(M_t, \Theta_t)$ injects $\forall t \in B$.

iii) # moduli $= m = \dim B$ provided that
$(\mathcal{M},\omega,B)$ complete & $m \le \dim H^1(M_t, \Theta_t)$.

- Some assumptions

- $\{M_t\}_{t\in B}$ c.c.m. & $M_t \hookrightarrow W$, $\mathrm{codim}_W M_t = 1$ submfld.

- $(\mathcal{M},\omega,B)$ ⊕ $\Phi : \mathcal{M} \to W$

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\Phi"] & & W \\ & M_t \arrow[ul] \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

$\{U_j\}$ cover $\mathcal{M}$, $\varphi(U_j) \subseteq W_j$

$M_t \cap W_j = \{ S_j(z_j, t) = 0 \}$
$\to$ gives a line budl $F_t = [M_t]$
w transitions $S_i = F_{ij} S_j$ on U_{ij}
$\to$ noticing that $S_k = 0$ on M_t gives
a section $\delta(t) \underset{\text{on } W_j}{=} \frac{\partial S_j}{\partial t}$, w $\frac{\partial S_i}{\partial t} = F_{ij} \frac{\partial S_j}{\partial t}$

$-s_{d,t}\left(\frac{\partial}{\partial s}\right) = -\frac{\partial S}{\partial s}$ infinitesimal displacement

$s_{d,t} : T_t B \to H^0(M_t, \mathcal{O}(F_t))$

$$0 \to \Theta_t \to \Xi_t \to F_t|_{M_t} \to 0$$

$$\Rightarrow \cdots \to H^0(M_t, F_t) \xrightarrow{\delta^*} H^1(M_t, \Theta_t) \to 0$$

$$\begin{tikzcd} H^0(M_t,F_t) \arrow[rr, "\delta^*"] & & H^1(M_t,\Theta_t) \\ & T_t B \arrow[ul, "s_{d,t}"] \arrow[ur, "s_t"'] & \end{tikzcd}$$

Lemma this commutes.

- for $W = \mathbb{CP}^{n+1}$,

$$0 \to \ker(M_t) \to \Xi \to \Xi_t \to 0$$

$$\ker(M_t) \cong \Xi \otimes F_t^\vee$$

- write $d = \deg S$, for homogenous grading

$$H^a(\mathbb{P}^{n+1}, \Xi \otimes F_t^\vee) \overset{\text{Serre duality}}{\cong} H^{n+1-a}(\mathbb{P}^{n+1}, \Omega^{n+1}(\Xi^\vee \otimes F_t))$$

$$\Omega^{n+1}(\Xi^\vee \otimes F_t) \cong K \otimes \Xi^\vee \otimes E^d \cong \Xi^\vee \otimes E^{d-n-2} = \Omega^1(E^{d-n-2})$$

$\underline{\text{Bott}} \Rightarrow$ $H^{\tilde{q}}(\mathbb{CP}^{n+1}, \Omega^{\tilde{p}}(E^k)) \neq 0$ iff

1) $\tilde{p} = \tilde{q}$, $k = 0$

ii) $\tilde{q} = 0$, $k > \tilde{p}$

iii) $\tilde{q} = n+1$

$\to$ so we get (o) for $q = 0,1,2$ (except $(n,d) = (2,4)$)

$$\Rightarrow H^0(\mathbb{P}^{n+1}, \Xi) \cong H^0(M_t, \Xi_t)$$

$$H^1(\mathbb{P}^{n+1}, \Xi) \cong H^1(M_t, \Xi_t)$$

$$H^1(\mathbb{P}^{n+1}, \Xi) \cong 0$$

- write $S(t,\xi) = \xi_0^d + \dots + t_\mu \xi_\mu^d$, $\mu + 1 = \binom{n+d}{d}$
  and $\mathrm{Aut}(\mathbb{CP}^{n+1}) \circlearrowright B$, $(\gamma, t) \mapsto \gamma \cdot t$
- we must have $S(\gamma \cdot t, \xi) = S(t, \gamma^{-1}\xi)$

- on B 1-parameter $\rightsquigarrow$ $\xi_\lambda(t) \dfrac{\partial}{\partial t_\lambda} = \breve{\xi}$

$$\xi_\mu(w) \frac{\partial}{\partial w_\mu} = \bar{\xi}$$

$$\breve{\xi}(S) = -\bar{\xi}(S)$$

- all together we get the nice diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(M_t, \Xi_t) \arrow[r] \arrow[d] & H^0(M_t, \mathcal{O}(F_t)) \arrow[r, "\delta^*"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & H^1(M_t, \Theta_t) \arrow[r] & 0 \\
0 \arrow[r] & \underbrace{H^0(\mathbb{P}^{n+1}, \Xi)}_{(n+2)^2-1} \arrow[r] & \underbrace{T_t B}_{\mu} \arrow[u, "\rho_{0,t}"] \arrow[ur, "\rho_t"'] & &
\end{tikzcd}$$

Lemma $H^0(M_t, \mathcal{O}(F_t)) = \langle V_1, \dots, V_\mu \rangle$

$$\Longrightarrow \dim H^1(M_t, \Theta_t) = \mu - (n+2)^2 + 1$$
$$= \binom{n+d}{n} - (n+2)^2 = \dim T_t \hat{B}$$

Thm If ρ_t surj. at t_0 then the family is complete at t_0.