Dąbrowski

Introduction to NCG

- 40 hrs, Mon & Fri 11am for now

- NCG ⇝ hybrid field
- we focus on recent "layer", Riemannian (mainly spin) NCG
  - encoded in spec. 3ple $(\mathcal{A}, \mathcal{H}, D)$
- describe some props + Connes' reconstr.

- while describing the S.T. we encounter also some old friends, e.g NC topology:

  (loc). cpt top. sp. $\Leftrightarrow$ comm. (non) unital C^*-alg

  loc. trivial vbdls $\Leftrightarrow$ fin. proj modules over ↑
- K-theory, Hochschild cyclic cohomology

- we describe: coupling to gauge fields, products, quotients, ...

- examples: $\mathcal{H} \simeq L^2(\Omega(M))$, $D = d + d^*$

  $\mathbb{T}^2_\theta$, S^2_q, N.C.S.M.

- won't discuss:
  - index th.
  - symmetries (isometries, diffeos, q-groups & Hopf algs)

- presentation style: $M \cap \phi$ (or $M \cup \phi$?)
  - proofs usually sketched

Spinors

- "implicit" in Euclid's pf. of Pythagoras' thm

⋮

1770 Euler: "cover" of $SO(3)$

1878 Clifford: "geom" algebra

Lipschitz: representation th

1913 E. Cartan: reps of rotational Lie algs. which do ~~not~~ exponentiate

1935 Brauer & Weyl: rep theory for "proj" reps of $O(n)$

- in physics, Dirac introduces (1928) RQM

$$(p_0 - \alpha_1 p_1 - \alpha_2 p_2 - \alpha_3 p_3 - \beta)\psi = 0$$

where $p_\mu = i\hbar\partial_\mu$, α_i, β 4×4 $\mathbb{C}$-matrices s.t. $\alpha_j^2 = 1$, $\{\alpha_i, \alpha_j\} = 0$ for $i \neq j$,

$\beta^2 = m^2c^2\,1$, $\{\alpha_i, \beta\} = 0$

$$\Rightarrow \quad p_0^2 - p_1^2 - p_2^2 - p_4^2 - m^2c^2 = 0.$$

- set $m = c = \hbar = 1$, write $\gamma^0 := \beta$, $\gamma^i := -\gamma^0\alpha_i$

$$\Rightarrow \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \begin{cases} +2 & \mu = \nu = 0 \\ -2 \quad \text{if} & \mu = \nu \in \{1,2,3\} \\ 0 & \mu \neq \nu \end{cases}$$

and $D\psi = \psi$ in $\mathbb{R}^{3|1}$, where $D = i\gamma^m\partial_m$ Dirac op.

- coupling to em-field minimally, $\partial_m \mapsto \partial_m + e A_m$ was an enormous success
  - described: "spin", antiptcls
- extended to any mfd M → spin mfds
  - math: index theory (M Riemannian, elliptic ops)
  - phys: dim ≥ 4, Kaluza-Klein, susy, str. theory

Alg. preliminaries

- fix V vsp (linear space) / $\mathbb{K} = \mathbb{R}, \mathbb{C}$
- let $\eta : V \times V \to \mathbb{K}$ sym., bilin. form

- <u>claim</u>: we can reconstruct η from $Q : V \to \mathbb{K}$, $Q(v) := \eta(v,v)$
  - since $2\eta(v,w) = Q(v+w) - Q(v) - Q(w)$

Thm (Sylvester) $\exists$ lin. basis $\{\varepsilon_j\}_{j=1,\dots,n}$ of V s.t. $\eta_{jk} := \eta(\varepsilon_j, \varepsilon_k)$

$$= \begin{cases} \mathrm{diag}(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q}, 0, \dots, 0) & \text{if } \mathbb{K} = \mathbb{R} \\ \mathrm{diag}(1, \dots, 1, 0, \dots, 0) & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

(p,q) is called the signature of η (or Q).

- let $T(V) = \mathbb{K} \oplus V \oplus V \otimes V \oplus \dots = \bigoplus_{k \in \mathbb{N}} V^{\otimes k}$ be full tensor alg. of V
  - unital, $1 \in \mathbb{K}$.
  - multipl. is $\otimes$, i.e. concatenation

Def $\mathcal{C}(V,Q)$ "$=\mathcal{C}(V)$" $:= T(V)/I(V)$

where $I(V) := \langle v\otimes v - Q(v) \mid v\in V\rangle$

$= \langle v\otimes w + w\otimes v - 2\eta(v,w) \mid v,w\in V\rangle$

$= \langle \varepsilon_j\otimes\varepsilon_k + \varepsilon_k\otimes\varepsilon_j - \eta_{jk} \mid j,k=1,\dots,n\rangle$

- as notation, we omit $\otimes$ in $\mathcal{C}(V)$

- note: $\mathbb{K} \overset{i}{\hookrightarrow} \mathcal{C}(V) \hookleftarrow V$, $i(v)^2 = Q(v)$

Rmk A ass. unit. alg, $j: V\to A$ s.t. $j(v)^2 = Q(v)\cdot 1_A$,
then

$$\begin{tikzcd} & A \\ V \arrow[ur, "j"] \arrow[r, hook, "i"'] & \mathcal{C}(V) \arrow[u, dashed, "\exists!\,\tilde{j}"'] \end{tikzcd}$$

commutes.

Application $T: V\circlearrowleft$ isometry induces $\tilde{T}: \mathcal{C}(V)\circlearrowleft$ automorphism,

since

$$\begin{tikzcd} V \arrow[r, hook] & \mathcal{C}(V) \\ V \arrow[u, "T"] \arrow[ur] \arrow[r, hook] & \mathcal{C}(V) \arrow[u, dashed, "\tilde{T}"'] \end{tikzcd}$$

- $\mathcal{C}(V)$ univ. unital ass. alg. gen by ε_i: $\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 2\eta_{ij}$

$\to$ basis $\{1, \varepsilon_j, \underset{j<k}{\varepsilon_j\varepsilon_k}, \dots, \underset{\substack{j_1\le\dots\le j_k \\ 0\le k\le n}}{\varepsilon_{j_1}\cdots\varepsilon_{j_k}}, \dots\}$

$\to \dim_{\mathbb{K}}\mathcal{C}(V) = 2^n$

- if $\eta\equiv 0$, $\mathcal{C}(V) = \Lambda V$

- from now on η nondegenerate

$\Rightarrow$ $p=n$, $\mathcal{C}(V)\cong\mathcal{C}(\mathbb{C}^n)$ if $\mathbb{K}=\mathbb{C}$

$p+q=n$, $\mathcal{C}(V)\cong\mathcal{C}(\mathbb{R}^{p|q})$ if $\mathbb{K}=\mathbb{R}$

Def $\mathcal{C}(V,\eta)$, η nondeg, we call *Clifford algebra*.