Stoppa

Def ("4-index tensor")

$$(X,Y,Z,W) := g(R(X,Y)Z,W)$$

Rmk. it is induced by a global section of $(TM)^{\otimes 4}$

Prop. $\forall X Y Z W$ we have

a) $\sum_{cycl} (X\cdot Y Z W) = 0$

b) $(Y X Z W) = -(X Y Z W)$

c) $(X Y Z W) = -(X Y W Z)$

d) $(Z W X Y) = (X Y Z W)$

Pf. a) Bianchi; b) $R(X,Y) = -R(Y,X)$

c) since char $\neq 2$, we check $(X Y Z Z) = 0$.

$$(X Y Z Z) = \langle \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]} Z, Z\rangle$$

$$= Y\langle \nabla_X Z, Z\rangle - X\langle \nabla_Y, Z\rangle + \tfrac{1}{2}[X,Y]\langle Z,Z\rangle = 0.$$

d) $$\sum^{\circlearrowleft}_{XYZ} (X Y Z T) + \sum^{\circlearrowleft}_{YZT} (Y Z T X)$$

$$+ \sum^{\circlearrowleft}_{ZTX} (Z T X Y) + \sum^{\circlearrowleft}_{TXY} (T X Y Z) = 0+0+0+0$$

Sectional curvature

Def. Pick $p \in (M,g)$. Pick $\sigma \subset T_pM$ a 2-dim subspace.
Write $\sigma = \mathrm{Span}(X,Y)$.

$$\mathcal{K}(\sigma) := \frac{(X Y X Y)}{(\mathrm{Area}(X,Y))^2} \qquad \text{where } \mathrm{Area}(X,Y) = \sqrt{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$$

Lemma. This is well-posed.

Pf. Check basis (in)dependence.

Lemma. The statement "$\mathcal{K}(z)$ determines R, $\forall z$, at every pt of M^4" makes sense:
If R, R' two 3-linear forms on $(V, \langle -,- \rangle)$, and if $\mathcal{K}(z) = \mathcal{K}'(z)$

$$\frac{(XYXY)}{Ar(XY)^2} \qquad \frac{(XYXY)'}{Ar(XY)^2}$$

where $(XYZW) := \langle R(XY)Z, W \rangle$

$(XYZW)' := \langle R'(XY)Z, W \rangle$

$\forall z$, then $R = R'$.

Pf. $(XYXY)' - (XYXY) = 0$

$\Rightarrow \langle (R'(XY) - R(XY))X, Y \rangle = 0$

Since $\mathcal{K}(z)$ is basis-indep, pick $X \perp Y$. $\Rightarrow (R'(X,Y) - R(X,Y))X \perp Y$

$\Rightarrow (R'(X,Y) - R(X,Y))X = \lambda_Y X$, $\lambda_Y \in \mathbb{R}$.

$\Rightarrow \lambda_Y = \frac{1}{\langle X, X \rangle} \langle (R'(X,Y) - R(X,Y))X, X \rangle$

$= \frac{1}{\|X\|^2}\Big((XYXX)' - (XYXX) \Big) = 0$, by antisym. & $\forall Y$.

Prop. (2nd Bianchi id) $\displaystyle\sum_{ZWT}^{\circlearrowleft} \nabla_T (XYZW) = 0.$

Lemma. Let $\gamma(t)$ geodesic, $X(t)$ v.f. along $\gamma(t)$, $X(0) = 0$. Then

$$\frac{D}{dt} R(\dot\gamma, X)\dot\gamma = \nabla_{\dot\gamma} R(\dot\gamma, X)\dot\gamma \Big|_{t=0} = R(\dot\gamma, \tfrac{DX}{dt})\dot\gamma \Big|_{t=0}$$

Pf. By 2nd Bianchi id, $\forall Z$

$$0 = \nabla_{\dot\gamma}(\dot\gamma X \dot\gamma Z) = \frac{d}{dt}\langle R(\dot\gamma X)\dot\gamma, Z \rangle - \langle R(\dot\gamma, \tfrac{DX}{dt})\dot\gamma, Z \rangle - \langle R(\dot\gamma, X)\dot\gamma, \tfrac{DZ}{dt} \rangle$$

Evaluate at $t = 0$.

- recall the Jacobi equation $\frac{D^2}{dt^2}J + R(\dot{\gamma}, J)\dot{\gamma} = 0$.

Lemma. Jacobi fields along a fixed $\gamma(t)$ form a fin. dim. v.s, with $\dim = 2n = 2\dim M$.

Pf. Pick set of v.f.s $\{e_i(t)\}_{i \in \{1, \dots, n\}}$ along $\gamma(t)$. s.t. they form an onb for $T_{\gamma(t)}M$ at every t. and $\nabla_{\dot{\gamma}(t)} e_i(t) = 0$.
Well-def. since linear ODE given by initial conds.
Now write $J(t) = J^k(t) e_k(t) \Rightarrow n$ 2^{nd} order linear ODEs for $J^k(t)$ by Jacobi eqn.

Rmk There exist 2 obvious solns:

$$J(t) = \dot{\gamma}(t), \quad J(t) = t\,\dot{\gamma}(t).$$

- denote by $\mathcal{J}$ the space of Jacobi fields along $\gamma(t)$ with $J(0) = 0$.

Lemma. $\forall\, J(t) \in \mathcal{J}$, $J(t) = (d\exp_{\gamma(0)})_{t\dot{\gamma}(0)}\left(t \frac{DJ}{dt}(0)\right)$

Pf. By the previous lemma, we need to check that RHS is a Jacobi field w initial conds $(0, \frac{DJ}{dt}(0))$.
First, note that, analogueously to $\frac{\partial}{\partial s}(\exp_p(tv(s))|_{(t,0)}$, it is a Jacobi field.
Secondly, check vanishing at $t=0$.