Dąbrowski

- recall, $\varsigma : \mathcal{C}^*(V) \to L(\mathcal{C}(V))$, $\varsigma(a) = \bar{a} b a^{-1}$
- $\ker \varsigma = \mathbb{k}^*$
- in particular $\forall v \in V \ \exists v^{-1} = \frac{v}{\eta(v,v)}$

so $\forall w \in V$,

$$\varsigma(v) w = \bar{V} w v^{-1} = -\overbrace{V w}^{-wV + \{w,v\}} V^{-1}$$
$$= w - 2\frac{\eta(v,w)}{\eta(v,v)} v \leftarrow \text{reflection wrt } \perp v \text{ hyperplane}$$

- we can't always get all reflections, e.g. in $\mathbb{R}^{1,0}$
- say: v covers or v is lift of reflection

Def. $\Gamma(\{0\}) := \mathbb{k}^*$ and for $|V| \geq 1$,
$\Gamma(V) := \{ v_1 \cdots v_r \mid r \in \mathbb{N}, v_k \in V, \eta(v_k, v_k) \neq 0\} \subseteq \mathcal{C}^*(V)$

Rmk $\Gamma(V)$ is gp
Further, $\varsigma(\Gamma(V)) \hookrightarrow O(V) :=$ isometries of V
Cartan–Dieudonné: any $a \in O(V)$ is prod of such reflctns

- indeed, looking at inclusion as $\Gamma(V) \xrightarrow{\varsigma} O(V)$
also means $\ker \varsigma|_{\Gamma(V)} = \mathbb{k}^*$
- to confirm, $a \in \Gamma(V)$, $a = \prod_{i=1}^{r} v_i$, r odd
$a = a a a^{-1} = -\varsigma(a) a = \prod(-\varsigma(a) v_i) = -\prod \varsigma(a) v_i = -\prod v_i = -a$
so no such odd elements, while for even reduce to $\mathbb{k}^*$ using Eli Hord alg.

$$1 \to \mathbb{k}^* \to G(V) \to \varsigma(\Gamma(V)) \to 1$$

Rmks $G(V)$:

- not conn.
- when $(p=0)\vee(q=0)$, $G(\mathbb{K}^{n,0}) \cong G(\mathbb{R}^{0,n}) \cong G(n)$
- conn. comp. is $SO(n)$

$$\begin{matrix} \mathcal{C}\ell^{+}(V) \\ \cup \\ \Gamma(V) \\ \cup \end{matrix}$$

Defs $\mathrm{Pin}(V) := \{ a \in \Gamma(V) \mid q(v_k, v_k) = \pm 1 \}$

$\cup$

$\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap \mathcal{C}\ell^{+}(V)$

$\cup$

$\mathrm{Spin}_0(V) := \left\{ a \in \mathrm{Spin}(V) \;\middle|\; \begin{matrix} \# v_k \text{ with } q(v_k,v_k) = +1 \text{ even}, \\ \# v_k \text{ with } q(v_k,v_k) = -1 \text{ even} \end{matrix} \right\}$

- unfortunately, $\mathrm{Spin}_0(V)$ is not always conn.

Rmk
$$1 \to H \to \mathrm{Pin}(V) \xrightarrow{s} O(V) \to 1$$
$$1 \to H \to \mathrm{Spin}(V) \xrightarrow{s} SO(V) \to 1$$
$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_0(V) \xrightarrow{s} SO_0(V) \to 1$$
where $H := \mathbb{Z}_2$ for $\mathbb{K} = \mathbb{R}$,
$H := \mathbb{Z}_4 = \{\pm 1, \pm i\}$ for $\mathbb{K} = \mathbb{C}$

Prop $\Gamma(V) \subseteq \{ a \in \mathcal{C}\ell^{*}(V) \mid s(a) V \subset V \}$

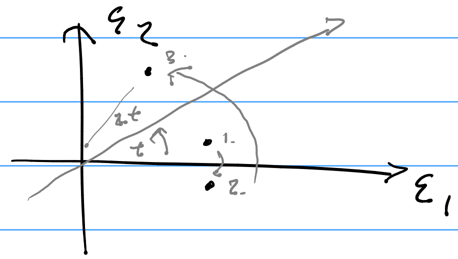
Pf. If $s(a)v \in V$ for some $v \in V$, using $v = -\bar{v}$ gives
$\|\bar{a} v a^{-1}\| = -\overline{\bar{a} v a^{-1}}\, \bar{a} v a^{-1} = a\,\overline{(-v)}\,\bar{a}^{-1} \bar{a} v a^{-1} = v \cdot v = Q(v)$, so $\Rightarrow$ □.

Def Spinor norm $N(a) := \bar{a}^\tau a$ where $(v_1 \cdots v_k)^\tau = v_k \cdots v_1$ main anti-involution

Rmks • $N(v) = -\eta(v,v)$ for $v \in V$
• $N(\rho(a)v) = N(v)$ for $a \in \Gamma(V)$
• unfortunately, $N(a) \notin \mathbb{k}$ always!
• alternate definitions possible:

$$Pin(V) = \{a \in \Gamma(V) \mid N(a) = \pm 1\}$$
$$Spin(V) = Pin(V) \cap \mathcal{C}^+(V)$$
$$Spin_0(V) = \{a \in Spin(V) \mid N(a) = +1\}$$

- examples:
• $SO(1) = 1$, $Spin(1) = \mathbb{Z}_2$
• $SO(1,1) = \mathbb{R}$, rapidity, $Spin(1,1) = \mathbb{R} \sqcup \mathbb{R}$
• $p > 1$, nontrivial, e.g.
$a(t) = \varepsilon_1 (\cos t\, \varepsilon_1 + \sin t\, \varepsilon_2)$, $t \in [0, \pi)$



$\rho(a(t))$ = rot by $2t$
$\to$ so $\rho(a(\pi)) = id$,
but $a(\pi) = -1$

- topological reason: $SO(p,q)$ not conn.,
$SO_0(p,q) \supset SO(p) \times SO(q)$
$\Rightarrow \pi_1(SO_0(p,q)) = \pi_1(SO(p) \times SO(q))$
$\simeq \pi_1(SO(p)) \times \pi_1(SO(q))$
and $\pi_1(SO_0(n)) = \begin{cases} 1, & n = 1 \\ \mathbb{Z}, & n = 2 \\ \mathbb{Z}_2, & n \geq 3 \end{cases}$

- exercise: $SO(2) \simeq \mathbb{RP}^1 \simeq \mathbb{S}^1/\mathbb{Z}_2 \simeq \mathbb{S}^1$
$SO(3) \simeq \mathbb{RP}^3 \simeq \mathbb{S}^3/\mathbb{Z}_2$

- complexification:

$$\mathrm{Spin}_{\mathbb{C}}(n) := \frac{\mathrm{Spin}(n) \times U(1)}{\{(1,1), (-1,-1)\}} \xrightarrow{2:1} SO(n) \times U(1)$$

$$\underset{(a,\lambda) \sim (-a,-\lambda)}{[a, \lambda],} \longmapsto (S(a), \lambda^2)$$

$i_1 : \mathrm{Spin} \hookrightarrow \mathrm{Spin}_{\mathbb{C}}$, $\mathrm{Spin}_{\mathbb{C}} \xrightarrow{p_1} SO$, $[a,\lambda] \mapsto S(a)$
$i_2 : U(1) \hookrightarrow \mathrm{Spin}_{\mathbb{C}}$, $\mathrm{Spin}_{\mathbb{C}} \xrightarrow{p_2} U(1)$, $[a,\lambda] \mapsto \lambda^2$

Lie algebras

- $\mathrm{Lie}(\mathcal{C}^*(V)) = \mathcal{C}(V)$ with $[a,b] := ab - ba$.

- claim: Lie(conn. comp of Spin(V))
$:= \mathfrak{spin}(V) = \{[v,w] \mid v, w \in V\}$
- has good dim, $n(n-1)/2$

- define $\mathring{S}(\underset{\mathrm{Lie}\,\mathcal{C}^*(V)}{\alpha}) = (ev_{t=0} \circ \frac{\partial}{\partial t} \circ \exp)(t\alpha)$

$\Rightarrow \mathring{S}(\alpha) \underset{\mathrm{Lie}\,\mathcal{C}^*(V)}{u} = \frac{\partial}{\partial t} e^{t\alpha} u e^{t\alpha} \Big|_{t=0} = [\alpha, u]$

- $\mathring{s}([v,w])u = 4\big(v\eta(w,u) - w\eta(v,u)\big)$

- introduce basis $\epsilon^{jk} := \frac{1}{4}[\epsilon^j, \epsilon^k] = \frac{1}{2}\epsilon^j\epsilon^k$, $j<k$
  then $\mathring{s}(\epsilon^{jk}) = E^{(jk)}$
  where $E^{(jk)}_{jk} = \eta_{jj}$, $E^{(jk)}_{kk} = \eta_{kk}$, other entries zero

- note that $[\mathring{s}(M), u] = Mu$ for $M \in so(p,q)$

Representations

- we focus on cpx reps, $\mathbb{D} = \mathbb{C}$, $T(iu) = iT(u)$
  and $\mathcal{C}(\mathbb{C}^{n=p+q}) = \mathcal{C}(\mathbb{R}^{p,q}) \otimes_{\mathbb{R}} \mathbb{C}$
- think of $\mathbb{C}^{2n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

Prop $\exists!$ irred. faithful cpx rep γ
(for n even)

$1 \le j \le \frac{n}{2}$: $\gamma(\epsilon_j) =: \gamma_j = \underbrace{1\!\!1_2 \otimes \cdots \otimes 1\!\!1_2}_{j-1} \otimes \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3$

$\frac{n}{2}+1 \le j \le n$: $\gamma(\epsilon_j) =: \gamma_j = \underbrace{1\!\!1_2 \otimes \cdots \otimes 1\!\!1_2}_{j-1} \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3$