Tanzini

- we saw: (susy) gauged linear σ-model (GLSM) $\longrightarrow$ nonlin σ-model (NLSM)

$\underline{\mathbb{C}P^{N-1}}$, $U(1)$ gauge theory w N chiral fields $\Phi_1 \dots \Phi_N$

$$L = \int d^4\vartheta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^V \Phi_i - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right) + \frac{1}{2}\left(-t \int d^2\tilde{\vartheta}\, \Sigma + c.c. \right)$$

$$= \sum_{j=1}^{N} \Big(-D^\mu \bar{\varphi}_j D_\mu^{(v_\mu)} \varphi_j + i \bar{\psi}_{j-} D_{\bar{z}}^{(v_\mu)} \psi_{j-} + i \bar{\psi}_{j+} D_{\bar{z}} \psi_{j-}$$

$$- |\sigma|^2 |\varphi_j|^2 - \bar{\psi}_{j-} \sigma \psi_{j+} - \bar{\psi}_{j+} \bar{\sigma} \psi_{j-} - i \bar{\varphi}_j \lambda_- \psi_{j+}$$

$$+ i \bar{\varphi}_j \lambda_- \psi_{j+} + i \bar{\psi}_{j+} \lambda_+ \varphi_j - i \bar{\psi}_{j-} \lambda_+ \varphi_j$$

$$+ \frac{1}{2e^2} \left(-\partial^\mu \bar{\sigma} \partial_\mu \sigma + i \bar{\lambda}_- \partial_{\bar{z}} \lambda_- + i \bar{\lambda}_+ \partial_z \lambda_+ + \bar{F}_{12} F^{12} \right)$$

$$+ \vartheta F_{12} - \frac{e^2}{2} \left(\sum_{i=1}^{N} |\varphi_i|^2 - r \right)^2$$

- vacua. $U = \sum_i |\sigma|^2 |\varphi_i|^2 + \frac{e^2}{2} \left(\sum |\varphi_i|^2 - r \right)^2$

- cases
$$\begin{cases} r > 0: & \sum |\varphi_i|^2 = r \qquad S^{2N-1},\ \sigma = 0 \\ r = 0: & \varphi_i \equiv 0\ \forall i,\ \sigma \text{ unconstrained} \\ r < 0: & \text{no solutions.} \end{cases}$$

- we focus on $r > 0$, $\mathbb{C}P^{N-1} \simeq S^{2N-1}/U(1)$

- for σ we have $\underbrace{\sum |\varphi_i|^2 |\sigma|^2}_{= r |\sigma|^2} - \frac{1}{2e^2} |\partial \sigma|^2 + \dots$

which means σ has mass $e\sqrt{2r}$

$\rightarrow$ and by susy same for $\lambda_\pm$, v

- $e \to +\infty$ limit:
  - φ_i span $\mathbb{CP}^{N-1}$, $\sum_{i=1}^{N} \bar{\varphi}_{i\pm}\varphi_i = \sum_i \bar{\varphi}_i \varphi_{i\pm} = 0$

$$v_\mu = \frac{i}{2} \frac{\sum_i \bar{\varphi}_i \partial_\mu \varphi_i - (\partial_\mu \bar{\varphi}_i)\varphi_i}{\sum |\varphi_i|^2}$$

$$\sigma = - \frac{\sum_i \bar{\varphi}_{i+} \varphi_{i-}}{\sum |\varphi_i|^2}$$

- $ds^2 = \frac{r}{2\pi} g^{FS}$

- on the tgt mfd $\mathbb{CP}^{N-1}$ $\exists$ gauge conn. A of a line bdl L whose $c_1(L)$ generates $H^2(\mathbb{CP}^{N-1}, \mathbb{Z})$
  $c_1(L) = -\frac{1}{2\pi} dA = -\frac{1}{2\pi}\omega_{FS}$

  $\frac{\theta}{2\pi} \int \underbrace{\varphi^*(dA)}_{\varphi^*(\omega_{FS})}$ B field coupling

- quantum level: $r(\mu') = r(\mu) - N \log \mu/\mu'$

$$\left.\begin{aligned} R^{FS}_{i\bar{j}} &= N g^{FS}_{i\bar{j}} \\ g_{i\bar{j}} &= \frac{r}{2\pi} g^{FS}_{i\bar{j}} \\ R_{i\bar{j}} &= \frac{r}{2\pi} R^{FS}_{i\bar{j}} \end{aligned}\right\} \quad g_{i\bar{j}}(\mu') = \frac{1}{2\pi}\left(r - N\log \frac{\mu}{\mu'}\right) g^{FS}_{i\bar{j}}$$

↳ form of metric survives (one loop) corrections

- If we started with $r=0$, we wouldn't have gotten $\mathbb{CP}^{N-1}$, but quantum corrections fix this.

- $\omega - iB = \frac{r - i\vartheta}{2\pi}\,\omega^{FS}$, the cpx kähler modulus is a twisted chiral field

Toric mfds

- $U(1)^k = \prod_a U(1)_a$, N chiral sfields, $Q_{1a} - Q_{Na}$, $1/e^2_{a,b} = \delta_{a,b}\, 1/e_a^2$
- $U = \sum_{i=1}^{N} |Q_{ia}\sigma_a|^2 |\varphi_i|^2 + \sum_{a=1}^{k} \frac{e_a^2}{2}\left(\sum_{i=1}^{N} |\varphi_i|^2 Q_{ia} - r_a\right)^2$

$r_a > 0$ ... $\sum_i Q_{ia} |\varphi_i|^2 = r$, $a = 1 \dots k$ $\quad (*)$

- $X_r = \{(\varphi_1 \dots \varphi_n) \mid (*) \text{ holds}\} / U(1)^k$
  symplectic quotient

- as cpx mfd $X_r \cong X_p = (\mathbb{C}^N \setminus P)/(\mathbb{C}^*)^k$
  GIT quotient

  where $P \subset \mathbb{C}^N$ is locus of $\mathbb{C}^N$ whose $(\mathbb{C}^*)^k$ orbit does not contain solns to $(*)$

- $X_r \cong X_p$ is a toric manifold
  - $U(1)^N \circlearrowright \mathbb{C}^N \rightsquigarrow U(1)^{N-k}$ free & trans. action on open dense sets of X_r
  - equiv., $(\mathbb{C}^*)^{N-k}$ .... of X_p

- example: conifold, tot. sp. of $(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$ bdl over $\mathbb{P}^1$

- GLSM: $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ $U(1)$ with
  charges $(1, 1, -1, -1)$

- condition is $|\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 - |\varphi_4|^2 = r$

  $r > 0$ ... φ_1, φ_2 hom. coords on $\mathbb{P}^1$
  $r < 0$ ... φ_3, φ_4 — " —
  $r = 0$ ... $x = \varphi_1\varphi_3$, $y = \varphi_1\varphi_4$
  $z = \varphi_2\varphi_3$, $w = \varphi_2\varphi_4$
  obey $xy = zw$, conifold

- if $c_1(X_r)$ $\begin{cases} > 0 & \text{Fano} \\ \geq 0 & \text{nef} \\ = 0 & \text{conformal} \end{cases}$

- B field v.e.v. is nonzero so it resolves the singularity

- turning on W i.e. F-terms induces hypersfcs on tgt mfd
- in $\mathbb{C}\mathbb{P}^{N-1}$, $G(\varphi_1 \cdots \varphi_N) = \sum_{i_1 \cdots i_d \in I} \alpha_{i_1 \cdots i_d} \varphi_{i_1} \cdots \varphi_{i_d}$

  genericity $\xrightarrow{\text{fails if}}$ $G = 0$, further $\varphi_1 = \cdots = \varphi_N = 0$

  smooth hyper sfc M, $G(\varphi_1 \cdots \varphi_N) = 0$, $c_1(M) = (N-d)[H]|_M$
- for $N = d$ $r > 0$ CY hypersfc
  $r < 0$ LG model w $\mathbb{Z}_N$ sym.