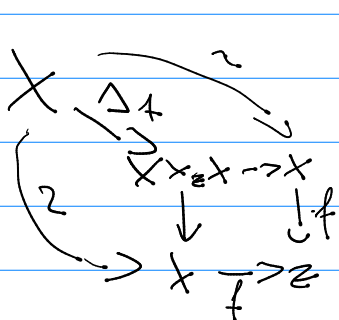


Bruzzo

- recalling: f separated if Δ_f closed immersion.



Properties:

- (i) open & closed immersions are separated
- (ii) composition respects separatedness
- (iii) so does base change

$$\begin{array}{ccc} X \times_Y Y' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

- (iv) f is separated iff Y has an open cover $\{U_i\}$ s.t. all restrictions $f^{-1}(U_i) \xrightarrow{f} U_i$ are separated

Proper morphisms

Def. $f: X \rightarrow Y$ is proper if it is

- (i) of finite type
- (ii) separated
- (iii) universally closed (closed after any base change)

$$X \times_Y Y' \xrightarrow{\text{closed}} Y'$$

We call a scheme X over field k complete if $X \rightarrow \text{Spec } k$ proper.

examples: - \mathbb{P}_k^n is complete

- A_k^n is of finite type, separated,

but $A_k^1 \rightarrow \text{Spec } k$ is not universally closed:

$$\begin{array}{ccc} A_k^2 = A_k^1 \times_k A_k^1 & \longrightarrow & A_k^1 \\ \downarrow & & \downarrow \\ A_k^1 & \longrightarrow & \text{Spec } k \end{array} \quad \left. \begin{array}{l} \text{not closed?} \\ \{x=y-1=0\} \text{ closed} \\ \text{but image is not.} \end{array} \right\} \text{base change}$$

Towards Gcoh.

(X, \mathcal{O}_X) ringed space. \mathcal{M}

A sheaf of \mathcal{O}_X -modules is a sheaf of ab. grps s.t.

(i) $\forall U \in \mathcal{X}$ open, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$ -module

(ii) restrictions $\mathcal{M}|_V: \mathcal{M}(U) \rightarrow \mathcal{M}(V)$ are morphisms of $\mathcal{O}_X(U)$ -modules.

Consider \mathcal{F}, \mathcal{G} \mathcal{O}_X -modules & $\mathcal{F} \otimes \mathcal{G}$.

We have a hom-set $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ which we

turn into a sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \doteq \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ not $\text{Hom}_{\mathcal{O}_X}(U)$
($\mathcal{F}(U), \mathcal{G}(U)$)

We give these sheaves a monoidal structure:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \doteq [U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)]$$

Remark. $x \in X$, $\text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \neq (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x$

\mathcal{F} is **free** if $\mathcal{F} \cong \bigoplus_{i \in I} \mathcal{O}_X$, $|I| < \infty$

\mathcal{F} is **locally free** if $\forall x \in X \exists$ open nbhd $U \ni x$
s.t. $\mathcal{F}|_U = \bigoplus_{i \in I} \mathcal{O}_U$

For $X \neq Y$, the pushforward construction gives
a functor $f_*: \mathcal{O}_X\text{-mod} \rightarrow \mathcal{O}_Y\text{-mod}$.

$\rightarrow f_*$ is **left-exact**.

- we also define $(f^{-1}\mathcal{G})(U) = \lim_{V \supset f(V)} \mathcal{G}(V)$, + **sheafification**.

\rightarrow sheaf on X ?

$\rightarrow f^{-1}: \text{Sh}_Y \rightarrow \text{Sh}_X$

① f^{-1} **exact**. (follows from $(f^{-1}\mathcal{G})_* \cong \mathcal{G}_{f,*}$)

$$\begin{array}{ccc} \textcircled{2} & X \times_Y \underline{Y} & \rightarrow \underline{Y} \\ & \downarrow & \downarrow \\ & X & \rightarrow Y \end{array}$$

$\mathcal{S}h_X \xrightleftharpoons[f^{-1}]{f_*} \mathcal{S}h_Y$ are an adjoint pair of functors.

Pf. Construct $f^{-1}f_* \mathcal{F} \rightarrow \mathcal{F}$ on opens, $\lim_{U \supset f(V)} \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(U)$

For $\begin{array}{ccc} \mathcal{E} \times_X X & \xrightarrow{\quad} & \mathcal{E} \\ s' \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array} \quad \text{where } \pi \circ s' = id, \quad s'(x) = (s \circ f)(x), x$

$$\mathcal{F}(U) \ni s \mapsto (f^{-1}f)(f^{-1}(U)) = (f_* f^{-1} \mathcal{F})(U) \quad \square$$