Fantechi

- cohom & base change.
X, Y loc. f.t. $/ k = \bar{k}$, $f: X \to Y$ proj mor,
$\mathcal{F} \in Coh_X$ flat over Y
0) $\forall y \in Y(\bar{k})$ $\forall i \in \mathbb{Z}$ $\exists$ nat map $R^i f_* \mathcal{F} \otimes k(y) \xrightarrow{\varphi_{i,y}} H^i(X_y, \mathcal{F}|_{X_y})$
i) if $\varphi_{i,y}$ surj then iso & same holds $\forall$ pts in nbhd of y in Y
ii) $\varphi_{i,y}$ iso then
$\varphi_{i-1,y}$ surj $\iff$ $R^i f_* \mathcal{F}$ loc. free near y

Rmk thm is local in Y, so all proj versions can be reduced to $X \overset{i}{\hookrightarrow} \mathbb{P}^N \times Y \xrightarrow{\pi} Y$, $f = \pi \circ i$,
i cl. emb, π projection

Rmk Grothendieck spec. seq. $R^q \pi_* R^p i_* \Rightarrow R^n f_*$
But i_* exact $\Rightarrow$ $R^n \pi_* \circ i_* \Rightarrow R^n f_*$ is eq.
i.e. $R^n f_* \mathcal{F} \underset{can}{\overset{\sim}{\leftarrow}} R^n \pi_* i_* \mathcal{F}$

Rmk $\mathcal{F}$ flat over $Y \overset{def.}{\iff} \forall x \in X$, $\mathcal{F}_x$ flat as $\mathcal{O}_{Y, f(x)}$-mod

Exercise $\exists$ nat iso $\mathcal{F}_x \xrightarrow{\sim} (i_* \mathcal{F})_x$, $\mathcal{O}_{Y, f(x)}$-linear
- hint: local, $X = \operatorname{Spec} R$, $\mathbb{P}^N \times Y = \operatorname{Spec} P$, $Y = \operatorname{Spec} A$
- R is an A-alg, $P = R/I$, $m = m_x \subseteq P$,

$$\begin{array}{ccccccc} k & \to & A & \to & R & \to & P \\ & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid \\ & & m_A & \to & m_R & \to & m \end{array} \Rightarrow k \hookrightarrow A/m_A \hookrightarrow R/m_R \hookrightarrow P/m \cong k$$

maximal so all isos

$i_* \mathcal{F} \hookrightarrow M^{=R}$ as R-mod, $\mathcal{F}_x = M_m \simeq M^R{}_{m_R}$ as R_{m_R} mod
- reduce thm to case $X = \mathbb{P}^N \times Y \to Y$, Y affine

Rmk. Let $\mathcal{F} \in Coh_{\mathbb{P}^N \times Y}$, Y affine.
Then Groth spec. sequ. $\Gamma(X, -) = \Gamma(Y, -) \circ \pi_*$
gives $H^p(Y, R^q \pi_* \mathcal{F}) \Rightarrow H^n(X, \mathcal{F})$.
But Y affine and $\underbrace{R^q \pi_* \mathcal{F}}_{\text{coherent}}$ so we have
a vanishing thm, $\forall \mathcal{G} \in Qcoh_Y$, $H^p(Y, \mathcal{G}) = 0$
$\forall p > 0$. So $\Gamma(Y, R^n \pi_* \mathcal{F}) \underset{\text{canot}}{\cong} H^i(X, \mathcal{F})$

- fix $y \in Y$. Serre vanishing to $\mathcal{F}|_{X_y}$ says
$\exists n_0$ s.t. $\forall n > n_0$ $H^i(X_y, \mathcal{F}(n)|_{X_y}) = 0$ $\forall i > 0$

Rmk. [Restriction, tensoring] = 0.

means $\mathcal{F}(n)|_{X_y}$ gen. by global sections
- apply thm with $i = 1$.
$\varphi_{1,y}$ surj. because $H^1(X_y, \mathcal{F}(n)|_{X_y}) = 0$

$\overset{\text{thm}}{\Longrightarrow}$ φ_{1y} iso $\Rightarrow$ $\underbrace{R^1 \pi_* \mathcal{F}(n)}_{\text{coh}} \otimes k(y) = 0$

$\overset{\text{NAK}}{\Longrightarrow}$ $(R^1 \pi_* \mathcal{F}(n))_y = 0 \Rightarrow \exists$ open nbhd U of y in Y

where $R^1 \pi_* \mathcal{F}(n)|_U = 0$

2) $\varphi_{0,y}$ surj $\Leftrightarrow$ $R^1 \pi_* \mathcal{F}(n)$ loc. free near y,
which is true. But $\varphi_{0,y}$ surj $\Rightarrow$ $\pi_* \mathcal{F}(n)$ loc. free
- replace Y by affine open nbhd of $y \in Y$ s.t. $\pi_* \mathcal{F}(n)$
free on U, $\mathcal{G} := \pi_* \mathcal{F}(n)$, by adj, $\pi^* \mathcal{G} \to \mathcal{F}(n)$
is surjection at every pt of X_y

Exercise Use Nakayama + π proper to show
$\exists V \subseteq Y$ open nbhd of y s.t. $\pi^* \mathcal{G} \to \mathcal{F}(n)$ surj on $\mathbb{P}^N \times V$

- hint. Let $A = \mathrm{coker}(\pi^* \mathcal{E} \to \mathcal{F}(n))$
coh $\Rightarrow$ supp A = supp A in $\mathbb{P}^N \times Y$ & disjoint from $X_y = \mathbb{P}^N \times \{y\}$ by Nak. properness
Replace Y by smaller open affine
$\Rightarrow \pi^*(\pi_* \mathcal{F}(n)) \twoheadrightarrow \mathcal{F}(n) \mid \otimes \mathcal{O}(-n)$
$\Rightarrow \underbrace{\pi^* \mathcal{E}(-n)}_{\downarrow} \twoheadrightarrow \mathcal{F}$
flat over Y since loc. free and $\mathbb{P}^N \times Y$ flat over Y.

- repeating this, we get

$$0 \to \mathcal{H} \to \pi^* \mathcal{E}_{N-1}(-n_{N-1}) \to \cdots \to \pi^* \mathcal{E}_0(-n_0) \to \mathcal{F} \to 0$$

where $\mathcal{E}_i$ loc free$/Y$, by induction $\mathcal{H}$ flat $/Y$ coh
So apply coh. & base change thm

$$R^i \pi_*(\pi^* \mathcal{E}_a(-n_a)) \simeq \underbrace{R^i \pi_* \mathcal{O}(-n_a)}_{\text{is}} \otimes \mathcal{E}_a$$

$R^N \pi_* \mathcal{E}_a(-n_a)$ loc. free, all other $i \neq N$ are zero, $H^i(\mathbb{P}^N \times Y, \mathcal{O}_{\mathbb{P}^N \times Y}(-n_a)) = 0$
- exercise: $R^i \pi_* \mathcal{H} = 0 \ \forall i \neq N$ (build resn. of $\mathcal{F}$ by sheaves s.t. $R^i \pi_* = 0 \ \forall i \neq N$)
- exercise: $\forall m \geq 0$

$$R^m \pi_* \mathcal{F} = h^{m-N}\big(R^N \pi_* \mathcal{H} \to R^N_{\pi_*} \pi^* \mathcal{E}_{N-1}(-n_{N-1}) \to \cdots \to R^N \pi_* \pi^* \mathcal{E}_0(-n_0)\big)$$

$$\begin{tikzcd} X_y \arrow[r, "j"] \arrow[d, "\alpha"] & \mathbb{P}^N \times Y = X \arrow[d, "\pi"] \\ y \arrow[r, "i"] & Y \end{tikzcd}$$

$\pi^* \mathcal{E}_a(-n_a)$ has vanishing $L^i j^*$.
j^* has van. $L^i(R^N \alpha_*)$,
$R^N \pi_*$ has van. $LP(i^*)$,
same for $\mathcal{H}$

$$\begin{tikzcd} \mathrm{Coh}_{\mathbb{P}^N \times Y} \arrow[r, "j^*"] \arrow[d, "R^N\pi_*"'] \arrow[dr, "\beta"] & \mathrm{Coh}\, \mathbb{P}^N \times \{y\} \arrow[d, "R^N\alpha_* = H^N(\_\_)"] \\ \mathrm{Coh}_Y \arrow[r, "i^*"] & \mathrm{Coh}_y = k\text{-f.d.vsp} \end{tikzcd}$$

- commutes by coh. & base ch.
- $L^i\beta(\mathcal{F})$ computable by given resn

- why is coh. & b.ch. like it is?

- look at

$$0 \to \mathcal{H}_{-N} \to \mathcal{H}_{-N-1} \to \cdots \to \mathcal{H}_0 \to \mathcal{F} \to 0$$

s.t. $\mathcal{H}_i$ loc free, coh, $R^i\pi_*\mathcal{H}_j = 0$
$\forall\, i \neq N \Rightarrow R^N\pi_*\mathcal{H}_j = M_j$ ((loc) free mod

$Y = \operatorname{Spec} A$, $y \leftrightarrow m_y$ max ideal

$\Rightarrow M_{-N} \to \cdots \to M_0$, M_i free mod

$$(h^i(\_\_)) \otimes k(y)$$
$$\downarrow \exists \text{ nat map}$$
$$h^i(\_\_ \otimes_{\pi} k(y))$$

this is (0) of coh & b.ch. thm, which is clear since we just mod out some things

$$A^{\oplus a} \xrightarrow[(\quad)]{d_2} A^{\oplus b} \xrightarrow[(\quad)]{d_1} A^{\oplus c}$$, A domain,

matrices

rks of d_1, d_2 loc. const. near y

$$\text{cohom} = \frac{\ker d_1}{\operatorname{coker} d_2}$$

$\vdots$

$$\begin{array}{ccc}
\ker d_1 \otimes A/m_y & \twoheadrightarrow & \ker(d_1 \otimes A/m_y) \\
\uparrow & & \uparrow \\
(\operatorname{coker} d_2) \otimes A/m_y & \xrightarrow{\sim} & \operatorname{coker}(d_2 \otimes A/m_y)
\end{array}$$

– so it all boils down to matrices, algebra

Rmk Assuming $\pi: X \to Y$ is flat proj of rel dim N, we don't need to pass to $\mathbb{P}^N \times Y$, $R^N \pi_*$ is okay. Recall, we used flatness to get $\pi^* \xi(n) \twoheadrightarrow \mathcal{F}$, with $\pi^* \xi(n)$ becoming flat. We used these to build the resoln, with now everything flat. But if $\pi: X \to Y$ flat, then this is automatic and we don't need to be passing to $\mathbb{P}^N \times Y$ etc.

Lemma Let $f: X \to Y$ be proj, $\mathcal{F} \in \mathrm{Coh}_X$ flat$/Y$.
Consider cartesian diag

$$\begin{tikzcd} \tilde{X} \arrow[r, "\varphi"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ \tilde{Y} \arrow[r, "\psi"'] & Y \end{tikzcd}$$

[Ha § III] $\forall n \geq$ $\exists$ nat map
$$\psi^* R^n f_* \mathcal{F} \xrightarrow{\alpha} R^n \tilde{f}_* \varphi^* \mathcal{F}$$

Assume $y \in Y$ s.t. $\varphi_{n,y}$ surj.

Let $\tilde{y} \in \tilde{Y}$ st. $\psi(\tilde{y}) = y$.

Then α also iso near $\tilde{y}$.

Cor Let $\mathcal{F} \in \mathrm{Coh}_X$, $X \xrightarrow{f} Y$ proj, $\mathcal{F}$ flat$/Y$
locally on Y $Rf_*\mathcal{F} = [R^N \pi_* \mathcal{H}_{-N} \to \cdots \to R^N \pi_* \mathcal{H}_0][N]$

$Rf_*\mathcal{F} \in D^b(\mathrm{Coh}_Y)$ is loc. isom to a finite cpx of loc. free sheaves

Def. Let X sch, $A \in D^b(\mathrm{coh}_Y)$ or $\in D^b_{coh}(Y\text{-mod})$,
$a, b \in \mathbb{Z}$, $a \leq b$. We call A perfect of
perfect amplitude contained in $[a,b]$
if loc on X, A isom in the der. cat.
to $[\mathcal{E}^a \to \cdots \to \mathcal{E}^b]$, $\mathcal{E}^i$ loc free of fin. rk.

Thm $X \xrightarrow{f} Y$ proj (f.t./$k = \bar{k}$). Let $\mathcal{F} \in \mathrm{Coh}_X$.
Then $\mathcal{F}$ is flat $/Y$ $\iff$ $Rf_*\mathcal{F}$ perfect

[Ha III flatness] Y red ~~& irred~~ not necess. $\mathcal{F}$ flat $\iff$ $\forall n \gg 0$ $f_* \mathcal{F}(n)$ is loc. free on Y.