Dąbrowski

- note we get intertwiners from

$$\begin{array}{ccc} id: SO(n) & \to & SO(n) \\ \uparrow \varrho & & \| \\ \varrho: Spin(n) & \to & SO(n) \end{array}$$

which gives us:

$\exists$ s.s. $\Rightarrow \tilde{F} \times_\varrho \mathbb{R}^n \simeq F \times_{id} \mathbb{R}^n \simeq TM$

Clifford bdls & spinor fields

- let $\mathbb{C}(M) = \coprod_{x \in M} \mathcal{C}(T_x M) \otimes \mathbb{C}$ vbdl of algebras

$$\| \quad \tilde{F} \times_{\bigoplus_{k=0} \varrho^{\wedge k}} (\Lambda \mathbb{R}^n)$$

$$\mathcal{C}(T_x M) \quad M \quad x$$

- $(\mathbb{C}(M))_x = End_{\mathbb{R}}(\Sigma_x)$

$$\| \quad \tilde{F} \times_\gamma \mathbb{C}^{2^m}$$

- fibrewise multiplication $[\tilde{e}, a] \cdot [\tilde{e}, s] = [\tilde{e}, \gamma(a)s]$ ♡

with $\tilde{e} \in \tilde{F}$, $a \in \mathcal{C}(\mathbb{R}^n)$

- reformulate in terms of modules ..
  - $\Gamma(\mathbb{C}(M))$ is an algebra and ♡ defines left action $\Gamma(\mathbb{C}(M)) \times \Gamma(\Sigma) \to \Gamma(\Sigma)$

- we also insist on $\Gamma(\Sigma)$ being a $C^\infty(M)$-module on the right

- after completion we have $\Gamma_0(\Sigma)$ a $\Gamma_0(\mathbb{C}(M)) - \mathcal{C}_0(M)$-bimodule and $\Gamma_0(\mathbb{C}(M))$ and $\mathcal{C}_0(M)$ are strongly Morita equivalent

Obs. $\exists$ s.s. $\Leftrightarrow \Gamma(\mathbb{C}(M)) \overset{\text{mos.}}{\simeq} \mathcal{C}(M)$

Digression on Serre-Swan

- M cpt, $E \to M$ finite rk vbdl.
  $\Rightarrow \Gamma^\infty(E)$ is $\underbrace{\text{fin. gen.}}_{(=\text{finite})}$ proj. module over $C^\infty(M) =: A$.

- we construct a projector

- let $\{\mathcal{U}_\alpha\}_{\alpha=1,\dots,r}$ open covering of M, $\{f_\alpha\} \subset C^\infty(M)$ s.t.
  $f_\alpha \geq 0$, $\operatorname{supp} f_\alpha \subset \mathcal{U}_\alpha$, $\sum_\alpha f_\alpha^2 = 1$ "part. of unity"

  - write sections as $s|_{\mathcal{U}_\alpha} = \sum_{j=1}^{N} \underbrace{\sigma^j_\alpha}_{s_\alpha} s_{\alpha j}$ local coordinates

    and let $\check{s}_\alpha := \begin{cases} f_\alpha s_\alpha & \text{in } \mathcal{U}_\alpha \\ 0 & \text{in } M \setminus \mathcal{U}_\alpha \end{cases}$

    so $\check{s}_\alpha \in \mathcal{C}^\infty(M, \mathbb{C}^N)$

    $\to$ we got a map $\Gamma(E) \xrightarrow{K} A^{\oplus r \cdot N}$, of A-modules

  - define $\mathcal{H}: A^{\oplus r \cdot N} \to \Gamma(E)$
    $$t \equiv \{t_\alpha\} \mapsto \hat{t}_\alpha := \sum_\beta \varphi_{\alpha\beta} f_\beta t_\beta$$

  - check: $\hat{t}_\alpha = \sum_\beta \varphi_{\alpha\beta} f_\beta t_\beta$
    $$= \sum_\beta \varphi_{\alpha\gamma} \varphi_{\gamma\beta} f_\beta t_\beta = \varphi_{\alpha\gamma} \hat{t}_\alpha$$

    so truly $\hat{t}_\alpha \in \Gamma(E|_{\mathcal{U}_\alpha})$

- compute $\mathcal{H} \circ K(s)|_{U_\alpha} = b_\alpha \sum_\beta \underbrace{\varphi_{\alpha\beta} f_\beta f_\beta s_\beta}_{f_\beta^2 \overbrace{\varphi_{\alpha\beta} s_\beta}^{= s_\alpha}}$
$= s_\alpha b_\alpha \cdot 1 = s$

$\Rightarrow \mathcal{H} \circ K = id_{\Gamma(\mathcal{E})}$

- for $p = K \circ \mathcal{H}$ we get

$p^2 = K \circ \overbrace{\mathcal{H} \circ K}^{= id_{\Gamma(\mathcal{E})}} \circ \mathcal{H} = p$

- can be shown to be projection

$\Rightarrow \Gamma(\mathcal{E}) \simeq p A^{\oplus, N}$

- explicitly, $p_{\alpha i, \beta j} = f_\alpha (\varphi_{\alpha\beta})_{ij} f_\beta \in A$, $p \in Mat_{\times N}(A)$

- vice-versa let $\mathcal{E} = pA^N$
where $p = p^2 \in Mat_N(\underset{C^\infty(M)}{\underset{\shortparallel}{A}}) \simeq C^\infty(M, Mat_N(\mathbb{C}))$

and $E_x := \mathcal{E} / \ker ev_x$
$= p(x)\, \mathbb{C}^N$
$\rightarrow p(x) = tr \circ p \in C^\infty(M, \mathbb{N})$
locally constant
- so $\dim E_x$ loc. constant
- we need to see $E := \coprod_{x \in M} E_x$ is loc. triv.
- this is basically automatic
- pick $e_1, \ldots, e_n \in E_x \subseteq \mathbb{C}^N$
let $s_1, \ldots, s_n$ sect'n's of E s.t. $s_i(x) = e_i$,
$i \in 1, \ldots, \dim E_x$, collections of $k \times N$ matrices
with a $k \times k$ minor with $\det \neq 0$ (??)
so we can define trivialisations

Thm (Serre – Swan) $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ vbdl of fin. rk

$\Longleftrightarrow$ finite proj. modules over $C^\infty(M)$.

$$\begin{tikzcd} Spin^{\mathbb{C}} \arrow[r, hook] \arrow[d, "s"] & U \arrow[d, "\pi"] \\ SO \arrow[r, hook, "i"] & PU = U/U(1)\cdot\mathbb{1} \end{tikzcd}$$

– gives us $Spin^{\mathbb{C}}$ – structures $\begin{array}{c} Spin^{\mathbb{C}} \dashrightarrow \tilde{F} \\ \downarrow \varphi \\ F \end{array}$