Guzzetti

- we studied systems given by $\frac{dY}{dz} = A(z)Y$ and found possible isolated singularities only at points coinciding with sings. of $A(\cdot)$.
- non-linear problem?

$y = f(z, y) \to$ sings. can depend on z and y (initial conds.)

- e.g. $y' = \frac{1}{z} + \frac{1}{y}$

- e.g. (Riccati) $y' = -y^2 \Rightarrow y(z) = \frac{1}{z-a}$, a determined by initial conds.

- e.g. $y' = \frac{y^2}{z^2} \Rightarrow y = \frac{az}{a-z} \to$ now y $\exists$ at $z \to 0$, but not f.

- e.g. $y' = \frac{1}{y^{\alpha-1}} \Rightarrow y(z, a) = (\alpha(z-a))^{\frac{1}{\alpha}}$

$\to \alpha$ irrational $\Rightarrow$ log-branch pt since $y \to \ln(z-a)$

- e.g. $\left[\left(\frac{y'}{y}\right)'\right]^2 + 4\left(\frac{y'}{y}\right)^3 = 0 \Rightarrow y = c\, e^{1/z-a}$, "movable" ess. sing

- e.g. $y' = -y^2/z \to y(z) = \frac{1}{\ln\left(\frac{z}{a}\right)} \to$ $z = a$ movable pole, $z = 0$ branch. pt

- lesson: impossible to find singularities from looking at the equation.

- XIX century: define new functions from nonlin. eqns.
  - requirement: branching pts are fixed by equation, at least of the general soln (i.e., solution depending on # constants = order of eqn).
- "Painlevé property" $\Leftrightarrow$ general soln does not have movable branch pts and ess. sing.

Thm [L. Fuchs - Picard] The only polynomial 1^{st} order eqns satisfying the Painlevé property are i) $g=0$: $y' = a(z)y^2 + b(z)y + c(z)$ (Riccati) a, b, c rational

ii) $g=1$: $(y')^2 = 4y^3 - g_2 y - g_3$, $g_{2,3} \in \mathbb{C}$

$\Rightarrow y = \wp(z-\alpha, g_2, g_3)$.

- in general, $P(z, y, y', y'')$ intractable.
  - $\to$ look @ $y'' = F(z, y, y')$, y, y' meromorphic in z.

- It can be shown that there are 53 rationally distinct equations $\to$ 47 can be solved as classical functions of rational functions (i.e. $\sqrt{R(z)}$, $\int R(z)$, etc.), but 6 cannot (Painlevé transcendents)
  - $\to$ to be precise, this is the classification of $y'' = L(z,y)(y')^2 + M(z,y)y + N(z,y)$ <u>only</u>
    - $\to (y')^3$ is an open problem
  - $\to$ these 6 are called Painlevé, n, $n \in \{1, \dots, 6\}$
- how to "solve"?
  - look at branch pts, asymptotics, ...

- e.g. Painlevé, I: $y'' = 6y^2 - z$
  - Boutroux's approach: $y(z) = \sqrt{z}\, u(z)$, $z = \frac{4}{5} s^{5/4}$

$$\Rightarrow u' \frac{d^2 u}{d s^2} = 6u^2 - 1 \underbrace{- \cancel{\frac{1}{s}\frac{du}{ds}} + \cancel{\frac{4}{24}\frac{u}{s^2}}}_{s \to \infty}$$

$$\Rightarrow (u')^2 = 4u^3 - 2u - g_3$$

$$\Rightarrow y(z) = \sqrt{z}\, \wp\left(\sqrt{\tfrac{4}{5}}\, z^{5/4} - a, 2, g_3\right) + 6(1).$$