Костя

Batyrev mirror sym. construction & GLSM. Pt 1

- special Kähler geometry:

$$e^{-K(x,\bar{x})} = i \sum_i x^i \frac{\partial F(\bar{x})}{\partial \bar{x}^i} - \bar{x}^i \frac{\partial F(x)}{\partial x^i}, \quad F(x) \text{ local holom. fn.}$$

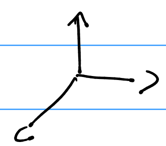
K : kähler pot.

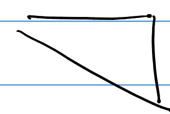
GLSM

- equipped w round metric: $Z_{S^2} \overset{\text{conj.}}{=} e^{-K(x,\bar{x})}$, for some CY
- we'll be needing us some spicy...

...Toric geometry

- X toric var, $(\mathbb{C}^*)^n \circlearrowright X$
- 3 constructions: i) fans ii) polytopes iii) factors



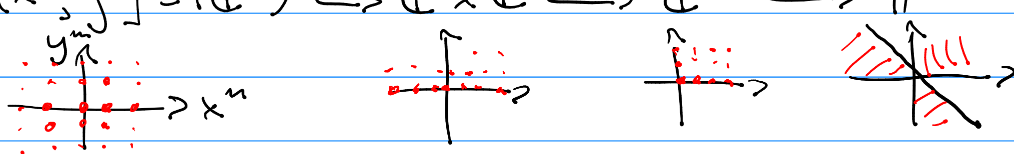


Batyrev

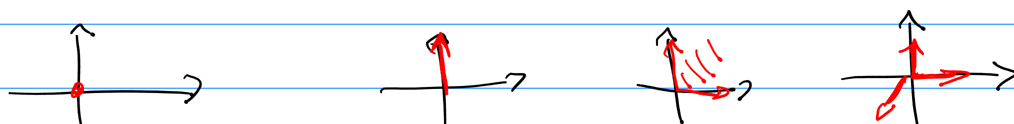
GIT quot, Ham. reduction <-> GLSM

- 1-dim toric varieties.
  - torus $\mathbb{C}^* = \operatorname{Spec} \mathbb{C}[x, x^{-1}]$, clearly $\mathbb{C}^* \circlearrowright$
    - M-lattice of exponents of functions on torus
  - add pt => $\mathbb{C} = \operatorname{Spec} \mathbb{C}[x]$,
  - projectify, $\mathbb{P}^1$, get N-lattice $= M^*$, n_i s.t. $\langle n_i, m_i \rangle \geq 0$
- 2-dim

$$\operatorname{Spec} \mathbb{C}[x^{\pm}, y^{\pm}] = (\mathbb{C}^*)^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C} \hookrightarrow \mathbb{C}^2 \hookrightarrow \mathbb{P}^2$$



duals:



- because $\langle n, m \rangle \overset{?}{\geq} 0$?

Def. N integral lattice, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

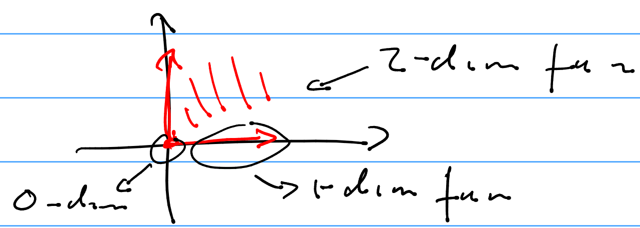
Rational (~) cone:

$$\sigma = \{ \sum \lambda_j v_j \mid \lambda_j \in \mathbb{R}_+, v_j \in N \}$$

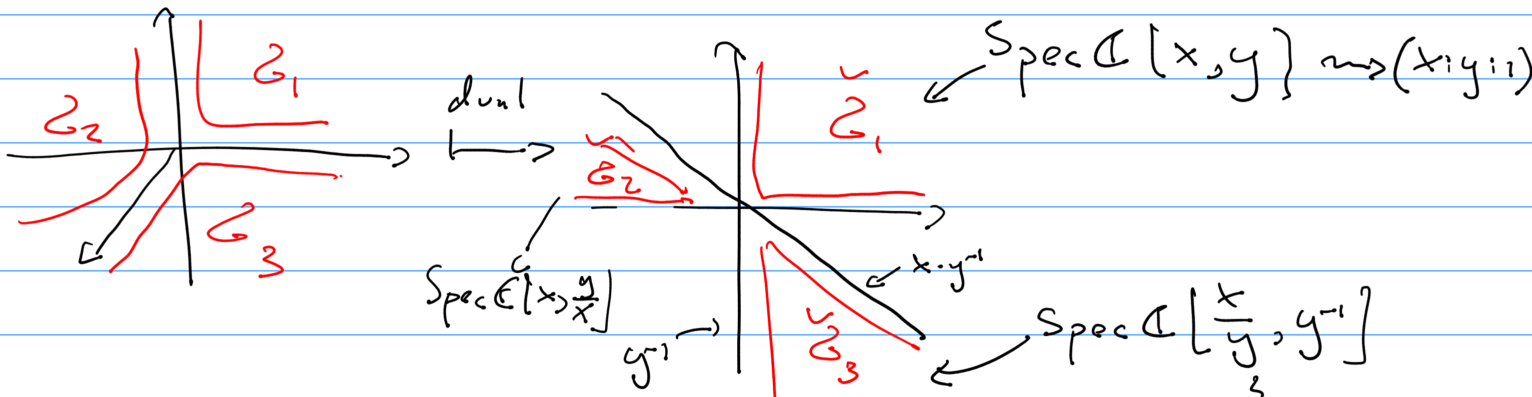
$\sigma \cap (-\sigma) = \{0\}$

A <u>fan</u> $\Sigma = \{\sigma_i\}_{i \in I}$ s.t. if $\sigma \in \Sigma$, then also all its faces $\in \Sigma$.

- e.g. in



- for $\mathbb{P}^2$ ~

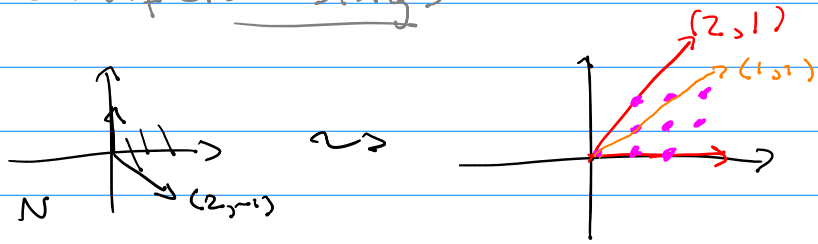


$$\left(\frac{x}{y} : 1 : y^{-1}\right) \sim (x : y : 1)$$

- we obtain toric var X_Σ by gluing the charts

Singularities on tor. var

Orbifold sings



we need 1 more vector to get whole int. lattice

$\rightsquigarrow \mathrm{Spec}\,\mathbb{C}[xy^2, x, \overset{(1,1)}{xy}]$

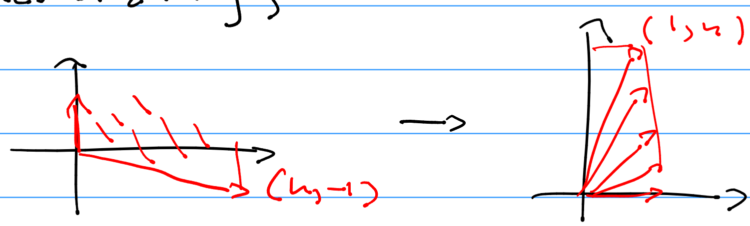
$= \mathrm{Spec}\,\mathbb{C}[u,v,w]/\langle u \cdot v = w^2\rangle$

- conic

$\Rightarrow x = 1/a,\ y = b/a \Rightarrow \mathbb{C}^2/\mathbb{Z}_2$

ALE (almost lin. Euclidean) space

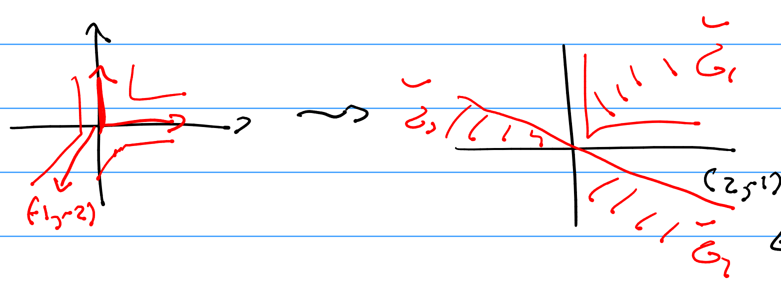
- generalizing,



$X_{\{\sigma\}} = \mathrm{Spec}\,\mathbb{C}[xy^n, xy^{n-1}, \dots, x]$

- let $x = a^{-n}$, $y = b/a$, gives $\mathbb{C}^2/\mathbb{Z}_n \ni e^{2\pi i/n}$

- another example:



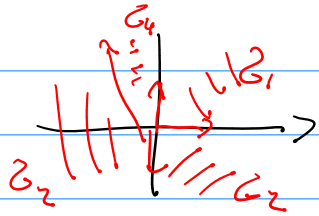
$\leftarrow \mathrm{Spec}\,\mathbb{C}[x,y] \rightsquigarrow (x:y:1)$

$\leftarrow \mathrm{Spec}\,\mathbb{C}[\frac{x^2}{y}, y^{-1}]$? weight 2

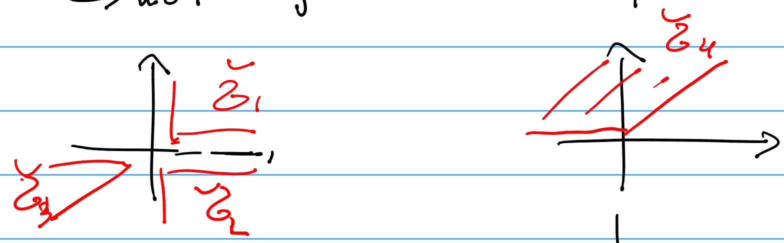
- we have $(x:y:1) \sim (\lambda x : \lambda^2 y : \lambda)$

$\rightsquigarrow \mathbb{P}^2_{(1,2,1)}$

$(\frac{x}{\sqrt{y}} : 1 : \frac{1}{\sqrt{y}})$

- for 

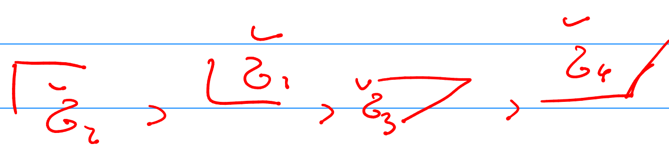
$\hookrightarrow$ not regular so we split



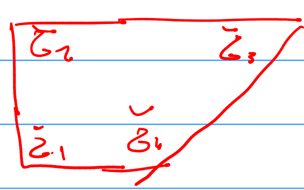
$\hookrightarrow \mathbb{P}$ $\qquad\qquad\qquad$ $\downarrow \mathbb{A}$

$$\Rightarrow X_\Sigma = \text{Proj}\, \mathbb{C}[x] \times \text{Spec}\, \mathbb{C}[y]$$

Polytopes

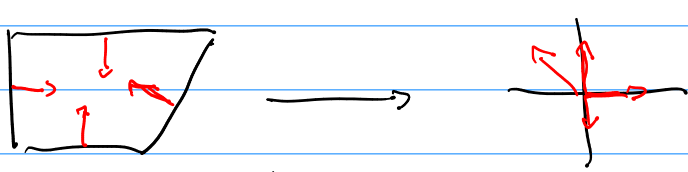


$\Big\}$ "glue"



$\to$ polytope, cnvx hull of some integral pts in M

- we can construct fans from polytopes,



just put (anti)normal vects in $\vec{0}$.

Divisors

- as far as I understood, they live on rays of Σ

D effective div $= \sum a_\rho [V_\rho]$, $[V_\rho]$ defined by rays $\vec{v}_\rho$.

$\Delta_D = \{ m \in M \mid \forall \sigma \in \Sigma \ \forall \langle v, m_\sigma \rangle \geq -a_\rho \}$

$\mathbb{P}^4 \supset \{ w = 0 \}$, $w = x_1^5 + \dots + x_5^5 + \sum \phi_{s_1 \dots s_5} x_1^{s_1} \dots x_5^{s_5} \in \mathcal{O}(5)$