Bruzzo.

- $(X, \mathcal{O}_X)$ ringed space; $\mathcal{M}, \mathcal{N}$ $\mathcal{O}_X$-modules.
- $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ can be sheafified: $[U \rightsquigarrow \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)]^{4}$
- example: $\mathbb{Z}_X$ constant sheaf.
  - for U w/ 2?? connected components, $\mathbb{Z}_X(U) \simeq \mathbb{Z} \oplus \mathbb{Z}$

    $\mathbb{Z}_X(U) \otimes_{\mathbb{Z}_X(U)} \mathbb{Z}_X(U) \simeq \mathbb{Z} \oplus \mathbb{Z}$

- stalks: $(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})_x = \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x$

- A ring, M A-module.
  - tensoring gives a right-exact functor:

$$\begin{aligned} A\text{-mod} &\xrightarrow{-\otimes_A M} A\text{-mod} \\ N &\longmapsto N \otimes_A M \end{aligned}$$

Def. M is a *flat* A-module if $-\otimes_A M$ is exact.

- ex. free modules are flat
- counterex. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$.
  - tensor with $-\otimes_{\mathbb{Z}} \mathbb{Z}_2$. Note that $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \simeq \mathbb{Z}_2$

    $\Rightarrow \quad \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{\sim} \mathbb{Z}_2 \to 0$, not flat.

- $(X, \mathcal{O}_X)$ loc. ringed space, $\mu_x \subset \mathcal{O}_{X,x}$ max. ideal

  $\Rightarrow \quad 0 \to \mu_x \to \mathcal{O}_{X,x} \xrightarrow{ev} k(x) \to 0$
  - $k(x)$ is an $\mathcal{O}_{X,x}$-module.
  - tensor w $-\otimes_{\mathcal{O}_{X,x}} k(x)$:

$$\mu_x \otimes_{\mathcal{O}_{X,x}} k(x) \to k(x) \xrightarrow{\sim} k(x) \to 0.$$

- for $X \xrightarrow{f} Y$, $\mathrm{Sh}X \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Sh}Y$ form an adjoint pair

  f_* ← left-ex.

  f^{-1} ← exact

- we extend to $\mathcal{O}_X\text{-mod} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathcal{O}_Y\text{-mod}$

- $\mathcal{F} \in \mathcal{O}_X\text{-mod}$, $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$
  $\hookleftarrow$ module over $\mathcal{O}_Y(U)$

- $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

i) f is flat if $\forall x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(y)}$-module
ii) $\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{G})$

$\mathcal{F}(X) = \Gamma(X, \mathcal{F})$, $\Gamma : Sh_X \to Ab$ left exact
$\to H^i(X, \mathcal{F})$ right-derived functors of Γ

$X = \mathrm{Spec}\, A$, M A-module $\mid \rightsquigarrow$ sheaf $\tilde{M}$ on X
s.t. $\Gamma(X, \tilde{M}) \simeq M$
$\tilde{M}_p \simeq M_p$, $S_p^{-1}M = \frac{S_p \times M}{\sim} = M_p$

- trivial example is $\mathcal{O}_X$ itself.

- if $X = \mathrm{Spec}\, A$, an $\mathcal{O}_X$-module $\mathcal{F}$ is
  - quasi-coherent if $\mathcal{F} \simeq \tilde{M}$ for some A-module M
  - coherent if q.c. + finitely generated
- if X a scheme, demand above defs on affine open coverings
  $\to X = \bigcup_\alpha U_\alpha$, $\mathcal{F}|_{U_\alpha} \simeq \tilde{M}_\alpha$, ...

- define $j_!\mathcal{F} = \left[V \rightsquigarrow \begin{matrix} \mathcal{F}(V) & \text{if } V \subset U \\ 0 & \text{if not} \end{matrix} \right]$, $\mathcal{F}$ sheaf on top. sp. U.
- if $\mathcal{G}$ sheaf on X,
  $0 \to j_!(\mathcal{G}|_U) \to \mathcal{G} \to i_*(\mathcal{G}|_{X-U}) \to 0$, $i : X - U \to X$.

- X aff scheme, $U \subset X$ proper open : $\Gamma(X, j_!\mathcal{O}_U) = 0$, $(j_!\mathcal{O}_U)_x \simeq \mathcal{O}_{X,x}$
  - not quasi-coherent

- X integral noetherian scheme
  - $\exists!$ generic pt ξ. $\mathcal{O}_\xi$ is the fn. field $K(X)$ of X
    - $\to$ look at this as a constant sheaf
    - $\to$ q.c. but not coherent

$i: Y \hookrightarrow X$, $0 \to \mathcal{J}_Y \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0$

$\mathcal{J}_Y$: sheaf of ideals

- if $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I$, $I \subset A$ ideal,
  then $\mathcal{J}_Y \simeq \tilde{I}$, $i_* \mathcal{O}_Y \simeq \widetilde{A/I}$ $\to$ q.c.

- A ring, $X = \operatorname{Spec} A$: (i) $\sim : A\text{-mod} \to \mathcal{O}_X\text{-mod}$

$$M \mapsto \tilde{M}$$

$\to$ exact & fully faithful

i.e. $\operatorname{Hom}_A(M,N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$

(ii) $\widetilde{M \otimes_A N} \simeq \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$

$\widetilde{M \oplus N} \simeq \tilde{M} \oplus \tilde{N}$

(iii) $A \to B$. $N \in B\text{-mod}$, $\tilde{N} \in \mathcal{O}_{\operatorname{Spec} B}\text{-mod}$, $M \in (A \to B), \ldots$

then: $f_* \tilde{N} \simeq \tilde{N}_A$, $f^* \tilde{M} \simeq \widetilde{M \otimes_A B} \in \mathcal{O}_{\operatorname{Spec} B}\text{-mod}$

- $\mathcal{O}_X$-mod is an abelian cat.