Arbarello.

$X \hookrightarrow \mathbb{P}^r$, $N \in \mathbb{Z}_+$, $p(t) \in \mathbb{Q}[t]$

$\longrightarrow$ $\exists\, n_0 = n_0(X, N, p(t))$

s.t. $\forall F$ sheaf on X, $p_F(t) = p(t)$ (Hilb. polyn.)

& $0 \to G \to \mathcal{O}_X^N \to F \to 0$

- then:

1) $H^p(F(n)) = H^p(G(n)) = 0$, $p \geq n_0$
2) $G(n)$ generated by global sections
3) $H^0(G(n_0)) \otimes H^0(\mathcal{O}_X(s)) \longrightarrow H^0(G(n_0+s)) \to 0$

$$\mathrm{Quot}_{X,N,p} = \{F \text{ on } X,\ \mathcal{O}_X^N \to F \to 0,\ p_F(t) = p(t)\}/\sim$$

$$\mathrm{Quot}_{X,N,p} \xrightarrow{\varphi} Gr(v - R, v)$$

$$[F] \xmapsto[\text{injection}]{} H^0(X, G(n_0)) \subset V$$

- where: $0 \to H^0(G(n_0)) \to H^0(\underbrace{\mathcal{O}_X^N(n_0)}_{V,\ \dim V = v}) \to H^0(F(n_0)) \to 0$, $H^0(F(n_0)) = p(n_0) = R$

- stopped taking notes for a bit...

Flatness.

- for comm. rings, M an A-mod is flat $\Leftrightarrow$ $- \otimes_A M$ exact
- schemes: $\mathcal{F}$ on $Z = X \times S \xrightarrow{\xi} S$ $\Rightarrow$ $\mathcal{F}$ flat over S

$\Longleftrightarrow$ $\forall\, z \in Z$ s.t. $\mathcal{F}_z$ is flat $\mathcal{O}_{\xi(z)}$-module

- $\forall\, U \subset Z$ open, $V \subset S$ open, $\xi(U) \subseteq V$,
  $\mathcal{F}(U)$ is $\mathcal{O}_S(V)$-flat

Def. Flat family of sheaves on X parametrized by S is
$$\begin{tikzcd} \mathcal{F} \arrow[d] \\ X\times S \arrow[r] & S \end{tikzcd}$$
flat over S, $\mathcal{F}|_s = \mathcal{F}|_{X\times\{s\}}$

Thm A) 1) $\mathcal{F}$ as above flat $\Rightarrow$ $P_{\mathcal{F}}(t)$ locally constant

2) S reduced $\Leftarrow$ – " –

Thm B) 1) $\mathcal{F}$ as above flat $\Leftrightarrow$ $\xi_* \mathcal{F}(n)$ locally free, $n \gg 0$

$$\begin{tikzcd} \mathcal{F} \arrow[d] \\ \mathbb{P}^r \times S \arrow[r, "\xi"] & S \end{tikzcd}$$
$R^p \xi_* \mathcal{F}(U) = H^p(\mathbb{P}^r \times U, \mathcal{F}|_U)$ on $S \supset U$ open

$$\begin{tikzcd} (R^p \xi_* \mathcal{F})_s \doteq \lim\limits_{s\ni U} R^p\xi_*\mathcal{F}(U) = \lim\limits_{s\ni U} H^p(\mathbb{P}^r\times U, \mathcal{F}|_U) \arrow[d] & \arrow[d] \\ (R^p\xi_*\mathcal{F})_{(s)} \arrow[r, "\varphi_s"] & H^p(\mathbb{P}^r\times\{s\}, \mathcal{F}|_{\{s\}}) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{P}^r\times\{s\} \arrow[r, "g = 1\times f"] \arrow[d, "\eta"] & \mathbb{P}^r\times S \arrow[d, "\xi"] \\ \{s\} \arrow[r, hook, "f"] & S \end{tikzcd}$$
$\Rightarrow f^* R^p \xi_* \mathcal{F} = R^p \eta_* g^* \mathcal{F}$

$\to$ take $\{s\} \leftrightarrow T$ more generally

Thm (Grothendieck) $\begin{tikzcd} \mathcal{F} \arrow[d] \\ \mathbb{P}^r\times S \arrow[r, "\xi"] & S \end{tikzcd}$ $\mathcal{F}$ flat over S.

Then $\exists$ a complex of locally free S-modules
$$K^0 \to K^1 \to K^2 \to \cdots$$
which computes $R^p\xi_*\mathcal{F}$ functorially on base ch:
$$f^*K^0 \to f^*K^1 \to f^*K^2 \to \cdots$$
meaning: $\mathcal{H}^p(f^*K^\bullet) = R^p\eta_* g^*\mathcal{F}$

- e.g. $f = id \Rightarrow \mathcal{H}^p(K^\bullet) = R^p\xi_*\mathcal{F}$

$T = \{s\} \Rightarrow \mathcal{H}^p(K^\bullet_{(s)}) = H^p(\mathbb{P}^r, \mathcal{F}_s)$

Universal family

$$\begin{tikzcd} \mathcal{F}' \arrow[d] & \mathcal{F} := \mathcal{F}'|_{X \times \mathrm{Quot}} \arrow[d] \\ X \times Gr & X \times \mathrm{Quot} \arrow[l, hook'] \end{tikzcd}$$

$\xi_* \mathcal{F}(n)_s = H^0(X, \mathcal{F}_s(n))$ $\quad n \gg 0$

$P_{\mathcal{F}_s(n)}(t) = p(t+n)$

$\Rightarrow \xi_* \mathcal{F}(n)$ locally free $\Rightarrow \mathcal{F}$ flat over Quot.

Arbarello.

$C \subset \mathbb{P}^r$, $\deg C = d$, $p(t) = dt - g + 1$

$T_{[C]}(\mathrm{Hilb}_{\mathbb{P}^r, p}) = H^0(N_{C/\mathbb{P}^r})$

$\underbrace{h^0(N) - h^1(N)}_{\parallel} \le \dim_{\mathbb{C}}(\mathrm{Hilb}) \le h^0(N_{C/\mathbb{P}^r})$

$\chi(N) = 3g - 3 + g - h^0(L)h^1(L) + (r+1)^2 - 1$

Exercise:

diff operators of order ≤1 ...

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \mathcal{O}_C \arrow[r, equal] \arrow[d] & \mathcal{O}_C \arrow[d] & & \\
0 \arrow[r] & \Sigma_L \arrow[r] \arrow[d] & H^0(L)^\vee \otimes L \arrow[r] \arrow[d, dashed] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_C \arrow[r] \arrow[d] & T_{\mathbb{P}}|_C \arrow[r] & N \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

$H^0(L) \otimes H^0(KL^{-1}) \xrightarrow{\mu_0} H^0(K)$

- check: μ_0 injective $\Longleftrightarrow H^1(K) = 0$

$\to$ this implies $\exists$ a patch in moduli space where μ_0 inj, i.e. $H^1(K) = 0$, i.e. Hilb smooth at C

$X \subset \mathbb{P}^4$ cubic threefold, $F = \{ \ell \in Gr(2,5) \mid \ell \subset X \}$

$$\begin{tikzcd}[row sep=small]
 & \mathcal{O}_\ell(-1) \otimes \mathcal{O}_\ell(1) \\
N_{\ell/X} \arrow[ur] \arrow[dr] & \\
 & \mathcal{O}_\ell \oplus \mathcal{O}_\ell
\end{tikzcd}$$

, $h^1(N) = 0 \to$ smooth

- in general, $X \subset \mathbb{P}^r$ fixed, $\mathrm{Hilb}_{X,P} = \{ Y \subset X \mid P_Y = P \}$, $T(\mathrm{Hilb}_{X,P}) = H^0(N_{Y/X})$

Mumford's example:

- X cubic in $\mathbb{P}^3 \Rightarrow 27$ lines $\to$ pick one, L

  $\omega_X = \mathcal{O}_X(-H)$

  $H^2 = 3$, $H \cdot L = 1$, $L^2 = -1$

  $|C|$, $C = 4H + 2L$

  $\dim |C| = 37$, $g(C) = 24$, $\deg C = 14$. $|C|$ very ample

$p_C(t) = 14t - 23$

$\mathrm{Hilb}_{\mathbb{P}^3, p} \supset V = \left\{ \begin{array}{l} \text{smooth curves of genus } 24, \\ \text{degree } 14, \text{ contained in} \\ \text{some smooth cubic} \end{array} \right\}$

$\to$ V irreducible by monodromy argument
  - take space of all cubics + a line
    - 27-sheeted cover
  - loop around singular cubic
    $\to$ obtain transitive S_n action on lines

- take $\mathcal{H}$ irred. component of Hilb containing V

- we want $\mathcal{H} = V$, $\dim \mathcal{H} = \dim V = 56$, $\forall C \in V$, $\dim T_{[C]}(\mathcal{H}) = 57$

$\to \dim V = \dim \text{cubics} + \dim |C| = 19 + 37 = 56$

$\underbrace{h^0(N) - h^1(N)}_{\chi(N) = 4d = 56} \leq \dim_{\mathbb{C}} \mathcal{H} \leq h^0(N_{C/\mathbb{P}^3})$

$$0 \to \underset{\mathcal{O}_C(C)}{N_{C/X}} \to N_{C/\mathbb{P}^3} \to \underset{\mathcal{O}_C(3H)}{N_{X/\mathbb{P}^3}|_C} \to 0$$

$\mathcal{H} = V$: $\Gamma \subset \mathcal{H}$ general pt

$\hookrightarrow$ smooth deg 14 deg 24 curve

- assume $\Gamma \not\subset$ smooth cubic (otherwise $\overline{V} = \mathcal{H}$)

  $\to$ lies on smooth quartic

- look at $Z = \{$smooth quartic containing a conic$\}$

  $\dim Z \leq 34 - 9 + 3 + 5 = 33$

  $\dim Y \leq \dim Z + \dim |\Gamma| = 57$

  $\dim \mathcal{H} \leq 56 \Rightarrow \mathcal{H} = V$

$$
\begin{tikzcd}
0 \arrow[r] & \underset{37}{H^0(N_{C/X})} \arrow[r] & \underset{57}{H^0(N_{C/\mathbb{P}^3})} \arrow[r, "\alpha"] & \underset{20}{H^0(N_{X/\mathbb{P}^3}|_C)} \arrow[r] & 0 \\
& & & \underset{\substack{\cup \\ \langle X \rangle \supset C}}{H^0(\mathbb{P}^3, \mathcal{O}(3H))}\ 20 \arrow[u, "\beta"] &
\end{tikzcd}
$$

$\to$ look at <u>Luna slice étale</u>