Ramanujan Prize Ceremony @ ICTP
Awardee: dr. Ritabrata Munshi
Title: The subconvexity problem for L-functions
Talk by R.M

Summary: basic properties of $\zeta(s)$.

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s = \sigma + it \in \mathbb{C}$, $\sigma > 1$, $\zeta(s)$ abs. convergent.

- analytic cont.

$\rightarrow$ set $\vartheta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$

$$\Rightarrow \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}\int_0^{\infty} (\vartheta(x)-1)\frac{dx}{x^{1-s/2}}$$

- makes sense for $s \neq 1$.

$\rightarrow$ Poisson summ. $\Rightarrow$ $\vartheta(1/x) = \sqrt{x}\,\vartheta(x)$

$\Rightarrow$ $x \mapsto 1/x$ in the integral gives fn. eqn $\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$

L-fns: Defining props.

- Dirichlet series rep $L(s) = \sum_{n=1}^{\infty} \frac{A_n}{n^s}$
- abs. conv. in $\sigma > \sigma_0$
- analytic cont.
- fn. eqn $L(s) \rightsquigarrow \overline{L(1-\bar{s})}$

- ex 1: $\chi \in \mathrm{Hom}\left((\mathbb{Z}/M\mathbb{Z})^*, \mathbb{C}^*\right)$
  - $\chi(n) = 0$ if $(n,M) \neq 1$, $\chi(n) = \chi(n \bmod M)$ otherwise
- ex 2: $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$, $q = e(z) = e^{2\pi i z}$, $z \in \mathbb{H}$
  - modular form of weight 12.

Ramanujan's conj.:
- $\tau(mn) = \tau(m)\tau(n)$ for $(m,n)=1$
- $\tau(p^{j+1}) = \tau(p)\tau(p^j) - p^{11}\tau(p^{j-1})$
- $|\tau(p)| < 2p^{11/2}$

Hecke L-fn $L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^s}$, $\tau_0(n) = \tau_n / n^{11/2}$

Automorphic L-fn $L(s,\pi)$ ...

Constructing higher degree L-fns:

→ Twists: $L(s,\Delta\otimes\chi) = \sum \frac{\tau_0(n)\chi(n)}{n^s}$

→ Symm. square $L(s,\mathrm{Sym}^2\Delta) = \zeta(2s)\sum \frac{\tau_0(n^2)}{n^s}$

→ Rankin-Selberg $L(s,\mathrm{Sym}^2\Delta\otimes\Delta) = L(2s,\Delta)$ ...

Main conjecture.

- Grand Riemann Hypothesis
  All nontr. zeros of $L(s,\pi)$ are on $\frac{1}{2}+it$

- a consequence:
  - Def. conductor $:= q_\pi \prod_{i=1}^{d} (1+|\mu_i+it|)$
  - e.g.: $L(1/2+it,\chi)$
    → cond is $M(1+|t|)$
  - Lindelöf Hypothesis $L(1/2, \ldots) \ll [\text{conductor}]^{\varepsilon}$

- Consequences of L.H.
  - arithmetic chaos $\mu_f(z) = \frac{|f(z)y^{k/2}|^2}{\langle f,f \rangle} \frac{dz\,d\bar{z}}{y^{\cdots}}$

Subconvexity bound: $L(1/2, \pi) \ll \chi^{\frac{1}{4}-\delta}$

Distinguished lecture by Brian Conrey.

- Conrey has a beautiful tie.

#Team Zagier