ICTP Seminars.

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mock modular forms

$\downarrow$

quantum modular forms (Witten-Reshetikin-Turaev invts) CS partition functs on M_3

$\downarrow$

false theta functions (homological block)

half index 3d $N=2$ theory.

Def. A (weakly hol.) mock modular form wrt Γ of weight $k \in \frac{1}{2}\mathbb{Z}$ $h: \mathbb{H} \to \mathbb{C}$ which is hol. w/ at most exp. growth at all cusps

(h, g) "shadow" g hol. mod. form of weight $(2-k)$

s.t. $\hat{h}(\tau) = h(\tau) + g^*(\tau)$ transforms like a mod. form of weight k wrt Γ, where $g^*(\tau) = c\int_{-\bar{\tau}}^{\infty} dz \frac{g(z)}{(\tau+z)^k}$

- $\vartheta_{m,r}(\tau, z) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell \equiv r \bmod 2m}} q^{\ell^2/4m} y^\ell$ $\quad q = e^{2\pi i \tau}$, $y = e^{2\pi i}$

$$g_{m,r}(\tau) = \frac{1}{2\pi} \partial_z \vartheta_{m,r}(\tau, z)\Big|_{z=0} = \sum_{\dots} \ell\, q^{\ell^2/2m}$$

Quantum modular forms

- defined on $\mathbb{Q} \cup \{\infty\}$

  $\to$ continuity, Γ-orbits etc. make little sense

- Zagier: $Q : \mathbb{Q} \to \mathbb{C}$

  $P_\gamma(x) = Q(x) - Q|_\gamma(x)$

  $\hookrightarrow (cx+d)^{-k} Q\left(\frac{ax+b}{cx+d}\right)$

- in fact, we can extend the domain of Q to $(\mathbb{C} \setminus \mathbb{R}) \cup \mathbb{Q}$, not uniquely, for so-called strong q.m.f.s

$$\lim_{t \searrow 0^+} \left(h(\tau) - G_j(\tau) \right) =: Q(x), \quad \tau = x - i\frac{t}{2\pi}$$

$$\shortparallel$$

$$\lim_{t \searrow 0^+} g^*\left(x + i\frac{t}{2\pi}\right)$$

False theta functions

- basically, put in some signs!

$$\sum_{\substack{n \in \mathbb{Z} \\ n \equiv r (2m)}} \operatorname{sgn}(n)\, q^{\ell^2/4m} = \tilde{g}(\tau)$$

- Eichler integral. $f(\tau)$ cusp form of $w \in \mathbb{N}$,

$$\tilde{f}(\tau) = \int_\tau^\infty dz \frac{f(z)}{(z-\tau)^{2-w}} = \sum_{n \geq 1} c(n)\, n^{1-w} q^n$$

where $f(z) = \sum_{n>0} c(n) q^n$

- now $Q(x) := \lim_{t \searrow 0^+} \tilde{g}(x+it)$

- in essence:

$$\begin{matrix} \tilde{g}(\tau) \\ \xrightarrow{\hspace{3em}} \mathbb{R} \\ g^*(\tau) \end{matrix}$