Antonini.

Quick recap.

- let $H$ be Hilbert space, separated (i.e. $\cong \ell^2(\mathbb{N})$),
  our inner product is $\langle \cdot \mid \cdot \rangle$
  $\mathbb{C}$-linear, antilinear

**Thm** (Riesz-Frechet) $\varphi$ bounded lin. functional,
$\exists! \, \xi \in H$ s.t. $\varphi(\eta) = \langle \eta \mid \xi \rangle$, $\|\xi\| = \|\varphi\|$

- so we get $H \to H^*$
  1) norm-preserving
  ii) antilinear

- our goal will be to construct the adjoint

**Def.** sesquilinear form is $(\cdot \mid \cdot) : H \times H \to \mathbb{C}$
lin. antilin.

- bounded if $\exists k$ s.t. $|(x, y)| \leq k \|x\| \cdot \|y\| \quad \forall x, y \in H$
- its norm is the smallest such $k$

- there is a 1-1 correspondence:

bounded sesquil. forms $\longleftrightarrow$ bounded lin. operators
$(\cdot \mid \cdot) : X \times Y \to \mathbb{C}$ $\qquad T : X \to Y$, $T \in \mathcal{L}(X, Y)$

given by $(\xi, \eta) = \langle T\xi, \eta \rangle$ inner product on $Y$
such that $\|T\| = \|(\cdot \mid \cdot)\|$

- fix $T \in \mathcal{L}(X, Y)$. We construct $T \mapsto T^* \in \mathcal{L}(Y, X)$ such that $\langle T^* y, \xi \rangle_X = \langle y \mid T\xi \rangle_Y$

- given just $H$, $\mathcal{L}(H, H) = \mathbb{B}(H)$, we have a map
$$\mathbb{B}(H) \longrightarrow \mathbb{B}(H)$$
$$T \longmapsto T^*$$
such that
  i) $\|T\| = \|T^*\|$ isometric
  ii) $(\alpha T)^* = \bar{\alpha} T^*$
  iii) $T^{**} = T$
  iv) $\|T^* T\| = \|T\|^2$ $C^*$-identity

- $\ker T^* = (\text{Range } T)^\perp$

- $C^*$-id follows from $\|T\| = \sup_{\|\xi\| = \|y\| = 1} \langle T\xi, y \rangle$

- now let $P, T \in \mathbb{B}(H)$ s.t.:
  i) $T$ selfadjoint, $T = T^*$
  ii) $P$ projection, $P = P^* = P$
  iii) isometry $T^* T = I$
  iv) unitary $T^* T = T T^* = I$
  v) normal

Principles of func. analysis

① uniform boundedness.
- $X, Y$ Banach spaces
- $\mathcal{T} \subset \mathcal{L}(X, Y)$. If $\{\|T\xi\| \mid T \in \mathcal{T}\}$ is bounded $\forall \xi$, then $\mathcal{T}$ is bounded

② open mapping theorem
- $X, Y$ Banach, $T \in \mathcal{L}(X, Y)$
- if $T$ maps $X$ <u>onto</u> $Y$, then it is open

③ closed graph.
- $X, Y$ Banach, $T : X \to Y$ linear map
- then, if $\text{graph}(T) := \{ (\xi, T\xi) \in X \times Y \mid \forall \xi \in X \}$ is closed $\Rightarrow$ $T$ is bounded

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- $H$, $T \in \mathbb{B}(H)$
- $P \in \mathbb{C}[x] \rightsquigarrow P(T) \in B(H)$
- recall spectrum $\sigma(T) : \{ \lambda \in \mathbb{C} \mid \lambda - T \text{ not invertible} \}$
- then $\sigma(P(T)) = \{ P(\lambda) \mid \lambda \in \sigma(T) \}$
  - just use fund. thm. of algebra to write
    $$P(T) - \lambda I = \alpha (T - \mu_1 I) \cdots (T - \mu_r I)$$

- for $T \in \mathbb{B}(H)$, define numeric range
  $$W(T) := \{ (T\xi, \xi) \mid \|\xi\| = 1 \}$$
  - then $\sigma(T) \subset \overline{W(T)}$

- if $T$ selfadj., $\sigma(T) \subset \mathbb{R}$, so define
  $$\mathbb{R} \ni \begin{array}{l} m = \inf W(T) \\ M = \max W(T) \end{array}$$
  - then, $\sigma(T) \subset [m, M]$, $m, M \in \sigma(T)$
  - also note $\|T\| = \sup \{ |\lambda| \mid \lambda \in W(T) \}$
- recall (real) Stone-Weierstraß:
  polynomials w/ real coefficients are dense in the (Banach) space $\mathcal{C}([m, M], \mathbb{R})$ in the sup norm

polyn w coeff $\in \mathbb{R}$

uniformly by S-W

- take $g_n \longrightarrow g \in \mathcal{C}([m, M], \mathbb{R})$,
then $g_n(T) \longrightarrow g(T)$ uniformly in $\mathbb{B}(H)$

- we constructed an algebra morphism
$$\mathcal{C}(\sigma(T)) \xrightarrow{\mathcal{F}} \mathbb{B}(H)$$
i) can be extended to $\mathcal{C}(\sigma(T), \mathbb{C})$ (and from now on we do this)
ii) $\mathcal{F}(\mathcal{C}(\sigma(T))) \subset \mathbb{B}(H)$
$\cap$
$\{T\}''$ ⟵ double commutant

iii) $\forall f \in \mathcal{C}(\sigma(T))$, $\sigma(f(T)) = f(\sigma(T))$
spectral mapping theorem

Positive operators
- write $T \geq 0$ if $\langle Tx, x\rangle \geq 0 \ \forall x \in H$ $(T \in \mathbb{B}(H))$
- $T \geq 0 \Rightarrow T = T^*$ (use polarization identity
$$\langle x, y\rangle = 1/4 \sum_{\substack{z \in \mathbb{C} \\ z^4 = 1}} z \|x + zy\|^2 )$$

- for such $T$ $\exists$ continuous map
$$\mathbb{R}_+ \longrightarrow \mathbb{B}(H)$$
$$\alpha \longmapsto T^\alpha$$
such that $T^\alpha \cdot T^\beta = T^{\alpha+\beta}$, $T^1 = T$, $T^\alpha$ positive
- in particular $T^{1/2}$ is the unique positive square root of $T$
- a bit abstract, but e.g. if $\|T\| < 1$, $\sigma(T) \subset [0,1]$
and let $\{P_n\}$ polynomials s.t. $P_0 = 0$, $P_{n+1}(t) = P_n(t) + 1/2\,(t - P_n(t))^2$
$\rightarrow$ then $P_n \rightarrow t^{1/2}$

- an operator $U$ is a partial isometry if $\exists\, N \subset H$ closed s.t. $\begin{cases} U|_N \text{ is norm-preserving} \\ U|_{N^\perp} = 0 \end{cases}$

- it follows that $N = \overline{\text{range } U^*}$, $\text{Ker } U = N^\perp$

- now $P := U^* U \Rightarrow P: H \to H$ is orth. proj to $N$

- conversely if $U^* U := P$ is a projection then $U$ is partial isom. with $N = U(H)$

- how to show?

- for $x \in N$, $\langle Px | x \rangle = \langle Ux, Ux \rangle = \|Ux\|^2 = \|x\|^2$ and clearly $Px = 0$ for $x \notin N$

**Thm** (Polar decomposition, von Neumann)

i) $T \in \mathbb{B}(H)$. $\exists!$ positive $|T| \in \mathbb{B}(H)$ s.t. $\|Tx\| = \| |T| x \|$, furthermore $|T| = (T^* T)^{1/2}$

ii) $\exists!$ partial isometry $U$ s.t.

- $\text{Ker } U = \text{Ker } T$
- $T = U |T|$
- $U^* U |T| = |T|$

  $U^* T = |T|$

  $U U^* T = T$

- take field $K = \overline{K}$ (in particular $K = \mathbb{C}$), let $A$ algebra over $K$, unital ($1 \in A$)
- for $\lambda \in K$, write $\lambda \cdot 1 =: \lambda \in A$
- label $A^{-1} = \{$ inv. elements $\in A \}$

- for $x \in A$, define spectrum $Sp_A x := \{\lambda \in K \mid (x - \lambda) \notin A^{-1}\}$

- if $A$ not unital, $\exists$ an algebra $\tilde{A}$ called the unitalization, such that, only as vect. sp. $\tilde{A} = A \times K$, and $(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$
  - $\Rightarrow$ its unit is $(0, 1) = 1_{\tilde{A}}$
  - furthermore $\exists$ embedding of algebras
    $A \xhookrightarrow{i} \tilde{A}$
    $a \longmapsto (a, 0)$
  - identify $A \sim i(A)$
    $\tilde{A} \ni (a, \lambda) = i(a) + \lambda(0, 1) =: a + \lambda$
- we define $Sp'_A a := Sp_{\tilde{A}} a$

- note that $0 \in Sp'_A a$ for any $a \in A$ because $(a, 0)(a', \lambda) = (aa' + \lambda a, 0) \neq (0, 1)$

- if $A$ already unital w. $e \in A$ unit, then $\exists$ canonical iso $\tilde{A} \xrightarrow{\pi} \underbrace{A \times K}_{\text{as algebras}}$

- in this case $Sp'_A a = Sp_A a \cup \{0\}$

- $\pi(a, \lambda) = (a + \lambda e, \lambda)$

Autonomi.

- $\pi : A \to B$ morphism of unital algs, $\pi(1_A) = 1_B$,
- $Sp_B \pi(x) \subset Sp_A x$, since $\pi(\lambda - x) = \lambda - \pi(x)$

- fix $\mathbb{k} = \overline{\mathbb{k}}$ ($\mathbb{C}$), let $\mathbb{k}(x)$ = rational funcs
- Poles. for $T \in \mathbb{k}(x)$ written as
 $T = \frac{a_0(a_1 - \lambda_1)\cdots(a_n - \lambda_n)}{b_0(b_1 - \mu_1)\cdots(b_m - \mu_m)}$, $P(T)$ set of poles

- for $R \subset \mathbb{k}$ any subset, define
 $\mathbb{k}_R(x) := \{T \in \mathbb{k}(x) \mid Pol(T) \cap R = \phi\}$

- for $x \in \mathbb{k}(x)$, $Sp_{\mathbb{k}_R(x)} x = R$, since $(\lambda - x)$ is not invertible for any $\lambda \in R$ in $\mathbb{k}_R(x)$

- let $A$ any unital alg, $y \in A$ s.t. $Sp_A y \subset \mathbb{k}$
- define $\mathbb{k}(x)_{Sp_A y} := \{$ rat. funcs. without poles on $Sp_A y\}$

 - for $\mathbb{k}_{Sp_A y}(x) \ni T = \frac{P}{Q}$, $Q(y) \in A^{-1}$
 $\Rightarrow T(y) = P(y)\, Q(y)^{-1}$

Prop. given $y \in A$, the map $T \mapsto T(y)$ is the unique morphism of unital algebras $\varphi : \mathbb{k}_{Sp_A y}(x) \to A$, $\varphi(x) = y$.

Rmk. $P \in \mathbb{k}(x) \Rightarrow Sp_A(P(y)) = P(Sp_A y)$

## Banach algebra.

- Banach space $B$ (complete wrt norm $\|\cdot\|_B$)
- $B$ will be a cpx algebra w bounded multiplication

$$\|xy\|_B \le \|x\|_B \cdot \|y\|_B$$

- if unital, $\|1_B\|_B = 1$

- example: $(E, \|\cdot\|)$ Banach space, then $\mathcal{L}(E,E) = \mathbb{B}(E)$ Banach alg

- if $1 \notin A$, embed $A \hookrightarrow \tilde{A} = A \times \mathbb{C}$ and set $\|(a,\lambda)\|_{\tilde{A}} = \|a\|_A + |\lambda|$

**Prop.** Let $(E, \|\cdot\|_E)$ Banach space w bounded multip. i.e. $\|x \cdot y\|_E \le c\|x\|_E \|y\|_E$ and with unit. Then there is an equivalent norm making $E$ into a unital Banach alg

($\|\cdot\|$ equiv to $|||\cdot|||$ iff $\exists\, c_1, c_2$ s.t. $(\|x\| \le c_1 |||x|||) \wedge (|||x||| \le c_2 \|x\|)$ )

**Pf.** $E \xrightarrow{L} \mathbb{B}(E)$ injective

$b \longmapsto L_b$ where $L_b(a) = ba$

Then $|||x||| := \|L_x\|_{\mathbb{B}(E)}$.

**Lemma.** Let $A$ unital Banach alg.

i) $a \in A$, $\|a\| < 1 \Rightarrow 1 - a \in A^{-1}$

and $(1-a)^{-1} = \sum_{n \in \mathbb{N}} a^n$ is abs. convergent

ii) Set $x \in A$. Then $\forall \lambda \in \mathbb{C}$ s.t. $\|x\| < |\lambda|$, $\lambda \notin Sp_A x$.

Furthermore, $\lim_{\substack{\lambda \to \infty \\ \lambda \notin Sp_A x}} \|\lambda \cdot (\lambda - x)^{-1} - 1\| = 0$

Pf. i) $\|a\|<1$ means $\|\sum a^n\| \le \sum \|a\|^n = \frac{1}{1-\|a\|}$,
and also $a \cdot \sum_{n\in\mathbb{N}} a^n = \sum_{n=1}^{\infty} a^n$.

ii) let $a := x\lambda^{-1}$, so $\|a\| = \|x\|/|\lambda| < 1$.
By i), $1-a \in A^{-1}$, and $1-a = \frac{1}{\lambda}(\lambda - x)$
$\Rightarrow \lambda - x \in A^{-1}$.
Also $\lambda(\lambda-x)^{-1} = (1-a)^{-1}$. i) again gives
$\|\lambda(\lambda-x)^{-1} - 1\| = \|(1-a)^{-1} - 1\| = \|\sum_{n=1}^{\infty} a^n\|$
$\le \frac{\|a\|}{1-\|a\|} = \frac{\|x\|}{|\lambda| - \|x\|}$

**Def.** $V, W$ normed spaces, $U \subset V$ open.
$f: U \to W$ is Fréchet-differentiable at $x \in U$ if $\exists$ bounded lin. op. $T: V \to W$ s.t.

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Th\|_W}{\|h\|_W} = 0$$

We call $T$ its Fréchet derivative (write $d_x f$).

**Prop.** $A$ unital Banach alg. Then $A^{-1} \subset A$ is open, and $\varphi: A^{-1} \circlearrowleft$ given by $x \mapsto x^{-1}$ is $C^1$ and satisfies $d_x\varphi(h) = -x^{-1}hx^{-1}$

**Thm.** $A$ cpx unital Banach alg. Fix $x \in A$.
Then

$Sp_A x$
- i) $\neq \emptyset$
- ii) is compact
- iii) the resolvent, i.e. map $\mathbb{C}\setminus Sp_A x \xrightarrow{\varphi} A$, $z \mapsto (x-z)^{-1}$ is holomorphic*

* "weakly holom.", meaning $\forall \ell \in A^*$, $\ell\circ\varphi: \mathbb{C}\setminus Sp_A x \to \mathbb{C}$ is holom.

Pf. ii) $\lambda \overset{\psi}{\longmapsto} (\lambda - x)$ is continuous so

$\mathbb{C} \setminus Sp_A x = \psi^{-1}(A^{-1})$ is open.

We know $Sp_A x \subseteq B(0, \|x\|)$.

So $Sp_A x$ cpt.

iii) fix any $\ell \in A^*$ and look at

$$\begin{aligned} \mathbb{C} \setminus Sp_A x &\longrightarrow \mathbb{C} \\ z &\longmapsto \ell((x-z)^{-1}) \end{aligned}$$

1) assume $Sp_A x = \phi$. Then $\ell((x-z)^{-1})$ is entire for any $\ell \in A^*$.

Since $\lim_{z \to \infty} \ell((x-z)^{-1}) = 0$, $\ell((x-z)^{-1}) = 0$.

But $A$ is not empty.

**Corollary.** (Gel'fand – Mazur)

$A$ Banach alg and a division ring (skew-field) $\Rightarrow A = \mathbb{C}$.

Pf. let $i : \mathbb{C} \to A$

$\lambda \mapsto \lambda \cdot 1_A$

Clearly $i$ surjects.

Further, for $x \in A$ $\exists \lambda \in \mathbb{C}$ st. $i(\lambda) - x \notin A^{-1}$

since $Sp_A x \neq 0$. Since $A^{-1} = \{0\}$,

$i(\lambda) = x$.

- fix $A$ commutative unital

**Def.** A *character* is a nonzero continuous morphism of unital algebras $\chi : A \to \mathbb{C}$

- recall that a principal ideal $I \subset A$ is maximal if $(I \subset J \neq A) \Rightarrow J = I$.

**Prop.** In $A$ unital Banach alg, all maximal ideals are closed.

**Pf.** $I$ maximal $\Rightarrow$ $I = \overline{I} \neq A$,
since $I \neq A$, $I \cap A^{-1} = 0$
$\overline{I} \overset{\shortparallel}{\cap} A^{-1}$

- recall: $A$ comm. ring. $I \triangleleft A$ maximal $\Leftrightarrow$ $A/I$ field.

- for $A$ commutative unital Banach alg,
characters $\longleftrightarrow$ maximal ideals
$\chi \longleftrightarrow \ker \chi$

Spectrum.

- $A$ comm. unital Banach alg.

$Sp\, A :=$ space of characters with topology of pointwise convergence

weakest top making the family of maps $\{ \underset{Sp A}{\overset{\in}{\chi}} \mapsto \chi(x) \}_{x \in A}$ continuous

# Automini

- $\underset{\text{comm}}{\overset{\text{unital}}{\mathrm{Sp}\, A}} = \{\text{nonzero characters}\}$
- note that top. of pt-wise convergence is the weakest top. making $\{ev_a : \mathrm{Sp}\, A \to \mathbb{C},\ a \mapsto f(a)\}$ continuous

**Thm** (Gel'fand transform)

a) $\forall x \in A$, $\mathrm{Sp}_A\, x = \{\chi(x) \mid \chi \in \mathrm{Sp}\, A\}$

b) $\mathrm{Sp}\, A$ is cpt.

c) $\exists$ natural continuous algebra morphism

$$A \to C(\mathrm{Sp}\, A)$$
$$x \mapsto G(x) = (\chi \mapsto \chi(x))$$
$$G(x)(\chi) = \chi(x)$$

Pf. a) $\subseteq$. $\lambda \in \mathrm{Sp}_A\, x \implies ((x-\lambda)A$ is ideal$) \wedge ((x-\lambda)A \neq A)$. Zorn implies $\exists\, J$ s.t. $(J \supseteq (x-\lambda)A) \wedge (J$ maximal in $A)$.

$\exists!\ \chi : A \to \mathbb{C}$ s.t. $\ker \chi = J$, so $\chi(x-\lambda) = 0 \implies \chi(x) = \lambda$.

$\supseteq$. $\chi \in \mathrm{Sp}\, A$. $x - \chi(x) \cdot 1_A \in \ker \chi$, maximal ideal $\implies x - \chi(x) \cdot 1_A \notin A^{-1}$.

b) $\to$ b₁) $\mathrm{Sp}\, A \subset \underbrace{B_1(0, A')}_{\text{cpt. by Alaoglu}}$

b) $\to$ b₂) closed

b₁) $\chi \in \mathrm{Sp}\, A$, $x \in A \implies |\chi(x)| \le \|x\| \implies \|\chi\| \le 1$

b₂) $\mathrm{Sp}\, A = \underbrace{\{\ell \in A' \mid \ell(xy) = \ell(x)\ell(y)\}}_{\text{closed (use nets?)}} \cap \underbrace{\{\ell \in A' \mid \ell(1) = 1\}}_{\text{closed}}$

c) $\forall x \in A$, $G(x)$ is cont.

$G(xy)(\chi) = \chi(xy) = \chi(x)\chi(y) = (G(x) \cdot G(y))(\chi)$

Now apply a) to show boundedness.

**Cor.** If $a, b \in A$ commute, then
$Sp_A(a+b) \subset Sp(a) + Sp(b)$, $Sp_A(ab) \subset Sp(a)\,Sp(b)$

- if $A$ is not unital,
$Sp\, A = \{ \varphi \in Sp\, \tilde{A} \mid \varphi|_A \neq 0 \}$
- $Sp\, \tilde{A}$ is Alexandroff cptification of $Sp\, A$

Naturality
- let $\pi : A \to B$ morph. of unital, comm. Banach algs
- let $\pi^* : Sp B \to Sp A$, $\chi \mapsto \chi \circ \pi$
$\pi_* : C(Sp A) \to C(Sp B)$, $\pi_*(f) = f \circ \pi^*$

- the following commutes

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "G_A"'] & B \arrow[d, "G_B"] \\ C(Sp A) \arrow[r, "\pi_*"] & C(Sp B) \end{tikzcd}$$

**Def.** A Banach alg $A$ is rationally generated by $z$ if $A$ is the smallest closed subalgebra containing
$z$ and $\{ (z-\lambda)^{-1} \mid \lambda \notin Sp_A z \}$
- facts: $A$ comm., chars determined by values at this subalg.
- $Sp\, A \to Sp_A z \subset \mathbb{C}$, $\chi \mapsto \chi(z)$

- now fix $X \subset \mathbb{C}$ cpt. Look at

$$R_X = \left\{ \begin{matrix} \text{rat. functions} \\ \text{with no poles in } X \end{matrix} \right\} \hookrightarrow C(X)$$

- $Ch(X) := \overline{R_X}^{C(X)}$

$\shortparallel$ rat. gen by $z : X \to \mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, phantom, "\shortparallel"] & Sp(A) \\ Sp_A z \arrow[ur] & \end{tikzcd}$$

where $X \ni y \to ev_y$, $\xi(z) \leftarrow \xi$

- $\varphi$ homeo
- we identify Gel'fand transform with $Ch(X) \overset{G}{\hookrightarrow} C(X)$
- why?
  - $X \xrightarrow{\varphi} Sp\,A$ homeomorphism
    $y \mapsto ev_y$
  - here $A = Ch(X) \subset C(X)$
  - now $A = Ch(X) \xrightarrow{G} C(Sp(A))$
    $\downarrow \varphi^*$
    $C(X)$
- let's compute it.
- let $y \in X$. $\varphi^*(G(f))(y) \overset{\varphi^*(\cdot) = \cdot \circ \varphi}{=\!=} G(f)(\varphi(y))$
  $= G(f)(ev_y) = f(y)$

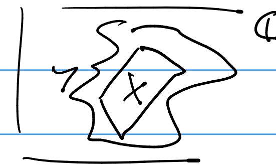
so indeed it injects.

- In general $G$ does not surject
- if $\operatorname{Int} X \neq \emptyset$, take $y \in \operatorname{Int} X$,

consider $\underset{Ch(X)}{\underset{\uparrow}{f}} \mapsto f'(y)$ ← continuous by Cauchy formula

- sometimes, in abstract settings, not injective, e.g. if $x \cdot y = 0$ $\forall x, y \in E$, $E$ Banach

- let $\mathrm{Hol}(X) := \left\{ \begin{array}{c} \text{germs of hol. funcs} \\ \text{defined around } X \end{array} \right\}$




- algebra
- let $Y \supseteq X$ cpt neighbourhood

$$\mathring{i}_{X,Y} : \mathrm{Ch}(Y) \longrightarrow \mathrm{Hol}(X)$$

rat. func $\longmapsto$ its germ around $X$

**Thm** (Hol. functional calc.) $A$ unital Ban. alg., fix $x \in A$. $\exists!$ morph. of unital algs

$$\varphi_x : \mathrm{Hol}(\mathrm{Sp}_A x) \longrightarrow A$$

i) $\varphi_x(z) = x$, $z : \mathrm{Sp}_A x \longrightarrow \mathbb{C}$, $z(y) = y$

ii) $\forall\, Y \supset \mathrm{Sp}_A x$ cpt nbhd,

$\varphi_x \circ \mathring{i}_{\mathrm{Sp}_A x, Y}$ is continuous

**Rmk.** Suppose we constructed $\varphi_x$. If $g$ rat func w/o poles in $\mathrm{Sp}_A x$, then $\varphi_x(g) = g(x) \in A$.

**Prop A** i) $\forall$ cpt nbds $Y$ as in theorem, the map $R_Y \longrightarrow A$, $p \mapsto p(x)$ is cont. in uniform. conv.

ii) $\exists$ decreasing sqn $\{Y_n\}$ cpt nbds of $X$ s.t.

$$\mathrm{Hol}(X) = \bigcup_{n \in \mathbb{N}} \mathring{i}_{X, Y_n} \mathrm{Ch}(Y_n)$$

**Prop B** Let $\varphi \in C^1_c(\mathbb{C})$, $\varphi \equiv 1$ on open $U \supset X$. $K := \mathrm{supp}\,\varphi$.

a) $\forall$ rat func $P$ w poles$(P) \cap K \neq 0$,

$$P \in R_K \Rightarrow p(x) = \frac{1}{2\pi i} \int_K P(z)(z - x)^{-1} \, dz \wedge d\varphi$$

Bochner integral

b) $\forall$ holom. $f$ around $K$ and any pt $\lambda \in U$,

$$f(\lambda) = \frac{1}{2\pi i} \int_K \frac{f(z)}{z-\lambda} \, dz \wedge d\varphi$$

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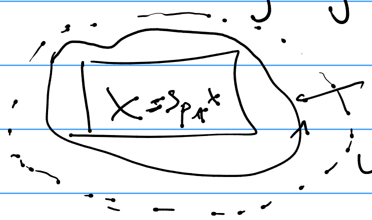
- recall Prop B (Cauchy formulas) $\Rightarrow$ Prop A saying i) continuity $\underset{p \mapsto p(x)}{R_Y \to A}$ ii) $\mathrm{Hol}(X) = \bigcup_n i_{X,Y_n}(Ch(Y_n))$
- for i), note that $P(x) = \frac{1}{2\pi i} \int P(z)(z-x)^{-1} d\varphi \wedge dz$ is continuous, automatically
- for ii), $Y_n := \{ z \in \mathbb{C} \mid d(z, X) \le \frac{1}{n} \}$. Take $f$ holomorphic on $V \supset X$ open, take $\varphi \in C^1_c(V)$ on some $Y_n$. Fix $z \in V \setminus Y_n$, def $h_z \in Ch(Y_n)$, $(\lambda \mapsto (z-\lambda)^{-1} =: h_z(\lambda))$
- now $f|_{Y_n} = \frac{1}{2\pi i} \int_{K \supset \mathrm{supp}\, \varphi} f(z) h_z \, dz \wedge d\varphi$

so we integrate $V \setminus Y_n \longmapsto Ch(Y_n)$.
Then set $f(x) =: \varphi_x(f)$, $f \in \mathrm{Hol}\, X$.

- the usual way goes like this (contour int.)




$\gamma$ cycle $\subset U \setminus Sp_A x$

$$I_\gamma(z) = \begin{cases} 1, & z \in Sp_A x \\ 0, & z \notin U \end{cases}, \quad I_\gamma(z) = (2\pi i)^{-1} \int_\gamma (z' - z)^{-1} dz'$$

$$f(x) = (2\pi i)^{-1} \int_\gamma f(z)(z-x)^{-1} dz \in A$$

**Prop** a) $g \in \mathrm{Hol}(Sp_A x)$. $Sp_A\, g(x) = g(Sp_A x)$. Further, $g' \in \mathrm{Hol}(Sp_A g(x))$, $(g' \circ g)(x) = g'(g(x))$

b) $\pi : A \to B$. $g(\pi(x)) = \pi(g(x))$, $g \in \mathrm{Hol}(Sp_A x)$.

Pf. a) $g = [f]$ germ of $f \in \mathrm{Hol}(U)$, $Sp_A x \subset U$. If $\lambda \notin f(Sp_A x)$, $z \mapsto (\lambda - f(z))^{-1}$ is holom. around $Sp_A\, g(x) \Rightarrow \lambda \notin Sp_A\, g(x)$.

...

**Lemma** $\{u_n\}$ squ in $\mathbb{R}\cup\{-\infty\}$ s.t. $\forall n, p$, $(n+p)u_{n+p} \le n u_n + p\cdot u_p$. Then it converges to its infimum.

Pf. Fix $m \ge 1$. By induction on $k$, $u_{km} \le u_m$.
For $n = km + r$, $0 \le r < m$, $u_n \le n^{-1}(k\cdot m\, u_m + r\, u_1)$
$= u_m + \frac{r}{n}(u_1 - u_m)$

$\Rightarrow \limsup u_n \le \inf u_n$.

**Prop** $A$ unital Banach algebra, $x \in A$. Then
$\|x^n\|^{1/n} \xrightarrow{n\to\infty} \sup\{|\lambda| : \lambda \in Sp_A x\} =: r(x)$,
Spectral radius, $= \inf\{\|x^n\|^{1/n}\} =: s(x)$.

Pf. Apply lemma to $u_n := \log \|x^n\|^{1/n} \Rightarrow$ it converges.
Note that $\boxed{r \le s}$, because if $\lambda \in Sp_A x$, $|\lambda| \le \|x\|$,
so $|\lambda^n| \in Sp_A x^n$ and $|\lambda^n| \le \|x^n\| \Rightarrow |\lambda| \le \|x^n\|^{1/n}$
$\forall n$, so $|\lambda| \le \inf \|x^n\|^{1/n}$.
But we claim $\boxed{r \ge s}$. We prove $\forall R > r \Rightarrow R \ge s$.
$x^n = (2\pi i)^{-1}\int_{|z|=R} z^n(x-z)^{-1}\, dz \wedge d\varphi \Rightarrow \|x^n\| \le k R^n$, $k>0$
$\Rightarrow \|x^n\|^{1/n} \le (k)^{1/n} R \Rightarrow s \le R$.

## $C^*$ algebras.

**Def.** Banach algebra with involution $*$ s.t.

$x \mapsto x^*$ is involutive, antilinear, antisym.

isometric, and

$$\|x^* x\| = \|x\|^2 \qquad (\Delta)$$

$C^*$-identity.

**Rmk.** $(\Delta)$ holds iff $\|x^* x\| \geq \|x\|^2$

- we call $x$ self-adjoint iff $x = x^*$
- every element decomposes uniquely

$x = h + ik$, where $h = \frac{1}{2}(x + x^*)$, $k = \frac{i}{2}(x^* - x)$

- example
  - $X$ cpt Hausdorff, $f \in C(X, \mathbb{C})$,

    $f^* = (m \mapsto \overline{f(m)})$, $\|f\| := \sup_{t \in X} |f(t)|$

  - $\mathbb{B}(\underset{\text{H.l6.sp.}}{H})$

- we call $u \in A$ unitary iff $u^* u = u^* u = 1$

**Prop** $A$ unital $C^*$-alg.

- $u$ unitary $\Rightarrow Sp_A u \subset U(1)$
- $a$ self-adjoint $\Rightarrow Sp_A a \subset \mathbb{R}$
- for any $x$, $Sp_A x^* = \{ \overline{\lambda} \mid \lambda \in Sp_A x \}$

Pf. 1) by $C^*$-id, $|\lambda| \leq 1 = \|u\|$, $\frac{1}{|\lambda|} \leq 1 = \|u^{-1}\|$.

**Prop.** $A$ unital $C^*$-alg, $B \subset A$ unital closed involutive subalgebra with $1_A \in B$. Then $\forall x \in B$,

$$Sp_A x = Sp_B x.$$

Pf sketch: $\lambda \in \mathbb{C} - Sp_A x$, $y = (x - \lambda)^*(x - \lambda)$. Then

$Sp_B y - Sp_A y = \{ \mu \in \mathbb{C} \setminus Sp_A y \mid (y - \mu)^{-1} \in A \setminus B \}$

is an open set in $\mathbb{C}^2$ subset of $\mathbb{R}$, since $y$ self-adj.

**Prop.** $A$ $C^*$-alg, $x \in A$ normal, i.e. $x^*x = xx^*$.
Then $r(x) = \|x\|$.

**Pf.** If $h = h^*$, $\|h^{2^n}\| = \|h\|^{2^n} \Rightarrow \rho(h) = \|h\|$.
If $x$ normal, $\|(x^*x)^n\| = \|x^n\|^2 \Rightarrow \rho(x^*x) = \rho(x)^2$
But $x^*x$ is selfadj, so $\rho(x^*x) = \|x^*x\| = \|x\|^2$.

Gelfand - Naimark.

**Thm** (2nd Gel'fand's thm) For a commutative $C^*$-alg $A$, the Gel'fand transform
$$\mathcal{G} : A \longrightarrow C(\mathrm{Sp}\, A)$$
is iso.

**Thm** (cpx. Stone-Weierstraß) Let $X$ cpt. Hsdff, $S \subset C(X, \mathbb{C})$ subset which separates points. Then the unital cpx $*$-algebra generated by $S$ is dense in the $C^*$-alg $C(X, \mathbb{C})$
("separates pts" means $(\forall x, y \in X)(\exists f \in S)\ f(x) \neq f(y)$)

Antonini

Pf (of Gelfand thm)

- statement: $A \xrightarrow{G} C(Sp\,A)$,
  i. $G$ is $*$-preserving
  ii. $G$ isometric $\Rightarrow$ injective — $G(A)$ closed
  - so by Stone-Weierstraß $G(A) = C(Sp\,A)$

i) $h = h^* \Rightarrow G(h)^* = G(h)$, want.
$G(h)(\chi) = \chi(h) \in Sp_A\, h \underbrace{\subset \mathbb{R}}_{h \text{ s.a.}}$, so $G(h)(\chi)^* = G(h)(\chi)$.

ii) $\forall h \in A$, $\|G(h)\| = \sup_{\chi \in Sp\,A} \{ |\chi(h)| \}$

$= \sup \{ |\lambda| \mid \lambda \in Sp_A\, x \}$

$= \|h\|$ for $C^*$-alg. □

Cor $A$ comm. $\Rightarrow$ any $\chi$ is s.a., $\chi(h^*) = \overline{\chi(h)}$

Pf. $\chi(x^*) = G(x^*)(\chi) = G(x)^*(\chi) = \overline{\chi(h)}$

- $U$ unital $C^*$-alg
- state on $U$: lin, cont. functional $f: U \to \mathbb{C}$
  - $f(x^* x) \geq 0$, positive
  - $f(1) = 1$, unital
- the space of states on $U$ is convex
  - $\exists$ extremal pts

Thm (Krein-Milman) Every cpt convex set $S$ in a locally convex v.sp. is the convex hull of its extremal pts.

- $A$ comm. unital, $\underset{A}{\text{chars of}}$ = pure states

Conj (NC Stone-Weierstraß) $\mathfrak{A}$ unital, $B \subset \mathfrak{A}$
$C^*$-subalg. which separates pure states
Then $\mathfrak{A} = B$.

## Continuous functional calc.

- $A$ unital $C^*$-alg, $x \in A$ normal

Claim $\exists!$ $*$-morph of unital algebras
(our funct. calc.)

1) $\Phi_x = C(Sp_A x) \longrightarrow A$
$z \mapsto \Phi_x(z) = x$

11) $Sp_A x \subset \mathbb{R} \Rightarrow x = x^*$

Pf. $A \supset B := \overline{\{P(x,x^*) \mid P \in \mathbb{C}[x,x^*]\}}^A$
comm. due to normality
$Sp\, B \ni \chi \longmapsto \chi(x) \in Sp_B x = Sp_A x$
$\psi$ homeo

$\chi(x^*) = \overline{\chi(x)}$, $\chi(P(x,x^*)) = P(\chi(x), \overline{\chi(x)})$

- now
$$\begin{tikzcd} C(Sp_A x) \arrow[r, "\psi^*"] \arrow[ddr, "\Phi_x"'] & C(Sp\, B) \arrow[d, "G_B^{-1}"] \\ & B \arrow[d, phantom, "\cap"] \\ & A \end{tikzcd}$$

- uniqueness can be shown from denseness

$C^*$-alg

**Prop** i) $B$ involutive Banach alg, $\pi : B \to A$ involutive morphism is contractive, $\|\pi(x)\|_A \leq \|x\|_B$

ii) any involutive injective morphism $\pi : A_1 \to A_2$ between $C^*$-algs is isometric

**Prop** (Spectral mapping) $x \in A$ normal

i) $\forall f \in C(Sp_A x)$, $Sp_A f(x) = f(Sp_A x)$

Since $f(x)$ normal, for $g \in C(Sp_A f(x))$, $(g \circ f)(x) = g(f(x))$

ii) $\forall \pi : A \to B$, $f(\pi(x)) = \pi(f(x))$

- $A$ Banach $*$-alg, nonunital

$\longrightarrow \tilde{A} := \{ a + \lambda, \lambda \in \mathbb{C} \}$

- $(a+\lambda)^* = a^* + \bar{\lambda}$

- $A \hookrightarrow \tilde{A}$ involutive closed

- recall $Sp_A x = Sp_{\tilde{A}} x$

**Prop** i) $a \in A$ unitary (i.e. it's unitalisation is unitary) $\Rightarrow Sp\, x \subset U(1)$

ii) $Sp_A x^* = \{ \bar{\lambda} \mid \lambda \in Sp_A x \}$

iii) $x \in A$ normal $\Rightarrow \|x\| = r(x)$

iv) for $x \in A$ normal, $f \in C(Sp_A x)$, $f(x) \in \tilde{A}$. If $f(0) = 0$. Then $f(x) \in A$

- iv) follows from $\tilde{A} \ni x + \lambda \xmapsto{\pi} \lambda$ and functorality $\pi(f(x)) = f(\underbrace{\pi(x)}_{0})$

- for nonunital commutative case,

$$A \xrightarrow[G]{\sim} C_0(\mathrm{Sp}\, A),$$

where for any $X$ loc. cpt Hsdff, $C_0(X) :=$ cont. funcs. on $X$ vanishing at infty

## Enveloping $C^*$-algs

- $\mathcal{A}$ involutive alg., $p$ $C^*$-seminorm, i.e. seminorm with $p(x^* x) = p(x)^2$
- $C^*$-seminorms $\longleftrightarrow$ seminorms on $\mathcal{A}$ s.t. for a $*$-morphism $i : \mathcal{A} \to A$, where $A$ $C^*$-alg, $\|a\| = \|i(a)\|$

- let $\Lambda$ = set of seminorms on $\mathcal{A}$,
- let $\mathcal{A}_\Lambda := \{ x \in \mathcal{A} \mid \sup p(x) < \infty, p \in \Lambda \}$
  - involutive subalg. with $C^*$-seminorm

$$x \mapsto \sup\{ p(x) \mid p \in \Lambda \}$$

**Def** If $\mathcal{A}$ has maximal seminorm $p_{max}$, then we say $\mathcal{A}$ has enveloping $C^*$-algeb

$$C^*(\mathcal{A}) := \text{Hausdorff completion of } \mathcal{A} \text{ wrt } p_{max}$$

- note univ. property

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] \arrow[d, "f"'] & C^*(\mathcal{A}) \arrow[dl, dashed] \\ B & \end{tikzcd} \quad \exists! g,\ f = g \circ i$$

- take locally cpt group
  - take left Haar measure $\mu(\overset{G}{\overset{\in}{x}}\overset{\text{Borel}}{\overset{\in}{E}}) = \mu(E)$

- define left action $L_y f(x) = f \circ y^{-1}(x)$ for $f \in C_c^+(G)$, note $L_x L_y = L_{xy}$

- if $\mu_1, \mu_2$ left Haar measures, $\exists! \alpha > 0$ s.t. $\mu_1 = \lambda \mu_2$

## Modular function

- fix $\lambda$ left Haar m., $x \in G$.
- then $\lambda_x(E) := \lambda(Ex)$ is another left m.
  $\Rightarrow \exists \Delta(x) > 0$, $\lambda_x = \Delta(x)\lambda$
  - measures discrepancy between L & R

- look at $L^1(G) = \{$int. funcs wrt Haar$\}$

$$(f * g)(x) := \int_G f(y)\, g(y^{-1}x)\, dx$$

$$\left( \text{''} = \int_{y \circ z = x} f(y)\, g(z) \text{''} \right)$$

$$f^*(x) := \Delta(x^{-1})\, \overline{f(x^{-1})}$$

$\rightarrow (L^1(G), *)$ is $*$-Banach alg

## Antonini

- $\hat{G}$ Pontryagin dual of $G$ LCA (loc. cpt. a.G.) grp
- $G$ discrete $\Rightarrow$ $\hat{G}$ cpt; $G$ cpt. $\Rightarrow$ $\hat{G}$ discrete
- e.g. $G = \mathbb{Z}$, $\pi : \mathbb{Z} \to U(1)$ characters
  - take $\pi(1) =: \zeta \in U(1)$, so $\pi(n) = \zeta^n$ and $\hat{\mathbb{Z}} = U(1)$
- keeping $\pi = \pi_\zeta$, take $f \in L^1(\mathbb{Z})$, $(f_n)_{n \in \mathbb{Z}}$ and put
$$\hat{\pi}_\zeta(f) = \int_{\mathbb{Z}} f_n \pi_\zeta(n)\, \delta_{\mathbb{Z}} = \sum_{n \in \mathbb{Z}} f_n \zeta^n$$
so $\hat{\pi}_\zeta(f) = \hat{f}(\zeta)$ where $\hat{f} : U(1) \to \mathbb{C}$ given as $\hat{f}(z) = \sum_{n \in \mathbb{Z}} f_n z^n$

- $\|f\|_* = \sup_{\zeta \in U(1)} \|\pi_\zeta(f)\| = \|\hat{f}\|_\infty$

## Order & positivity

- $C \subset V$, $V$ vsp.
- $C$ cone if $x \in C$, $a > 0 \Rightarrow ax \in C$.
- $C$ convex cone if $x, y \in C$, $a, b > 0 \Rightarrow ax + by \in C$
- $C$ flat if $\exists x \in C$ nonzero s.t. $-x \in C$
- $C$ salient if not flat

$\to$ a convex cone is salient iff $C \cap (-C) \subseteq \{0\}$.

**Thm** $A$ $C^*$-alg, $h \in A$ s.a. TFAE

a)
$$\left.\begin{array}{ll} \text{i)} & Sp_A h \subseteq \mathbb{R}_+ \\ \text{ii)} & \exists k \in A,\ k = k^* \text{ s.t. } h = k^2 \\ \text{iii)} & \exists x \in A \text{ s.t. } x^* x = h \end{array}\right\} \text{definition of positive element}$$

b) the positive elements form a salient convex cone (containing zero)

Pf. 1a) $\iff (\exists t \in \mathbb{R}_+)\ \|t-h\| \leq t$ (assuming $A$ unital)

- now we have an induced order on $A$:
  $a \geq b$ if $a-b \in A_+$ ← pos. elements of $A$
- this is compatible w. vsp-struct.
  and of course $a \geq 0$ iff $a \in A_+$

- $(\forall a \in A_+)(\exists!\, b \in A_+)\ b^2 = a$
- $-t \leq a \leq t$ for $t \geq 0$ iff $\|a\| \leq t$

- let $a, b \in A$:
  i) $0 \leq a \leq b \Rightarrow \|a\| \leq \|b\|$
  ii) $a \leq b \Rightarrow x^* a x \leq x^* b x$

- if $A$ unital:
  i) $x \in A$, $y \in A^{-1}$. $x^* x \leq y^* y \iff \|x y^{-1}\| \leq 1$,
  ii) $x, y \in A^{-1} \cap A_+$. $x \leq y \iff y^{-1} \leq x^{-1}$

## Approximate units

- $B$ Banach alg, a (bilat.) approx. id.
  is a family (net) $(u_i)_{i \in \Lambda} \subset B$, $\Lambda$ filtered (directed) set
  s.t. $\lim_{i \in \Lambda} u_i x = x = \lim_{i \in \Lambda} x u_i$ $(*)$

- directed set $\Lambda \neq \emptyset$ with relation $(\leq)$,
  reflexive & transitive
  s.t. $a, b \in \Lambda \Rightarrow \exists c \;\; a \leq c \wedge b \leq c$

- $(*)$: $\lim_{i \in \Lambda} u_i = u$ if $\forall \varepsilon > 0 \; \exists c \in \Lambda$ s.t.
  $j \geq c \Rightarrow \|u - u_j\| \leq \varepsilon$

**Def.** $C^*$alg $A$ is called *σ-unital*
when $\exists$ an approx unit which
is a sequence.

$\not\Leftrightarrow$

- unital $\Rightarrow$ σ-unital $\Leftarrow$ separable
  $\not\Leftarrow$ $\qquad\qquad\qquad\qquad$ $\not\Rightarrow$

- both $H$ $\infty$-dim $\Rightarrow$ $\mathbb{B}(H)$ nonsep. $\wedge$ unital
  $\mathbb{K}(H)$ sep. $\wedge$ nonunital
  $\mathbb{B}(H)/\mathbb{K}(H)$, Calkin alg.,
  unital $\wedge$ nonsep
- $C_0(X)$ is σ-unital iff $X$ σ-cpt.

- for any $C^*$ $\Lambda := \{a \in A_+ \mid \|a\| < 1\}$ is directed
  and is a bounded approx. unit

**Cor.** $I \subset A$ closed bilateral ideal.
Then $a \in I \Rightarrow a^* \in I$.

- $I$ closed ideal $\Rightarrow$ $A/I$ is $C^*$-alg
  with quot. norm $\|a + I\|_{A/I} := \inf_{z \in a+I} \{\|z\|\}$

# Representations

- let $A$ $*$-alg.

- a representation is a $*$-morphism
  $A \xrightarrow{\pi} \mathbb{B}(H_\pi)$
  - $\pi(a)\pi(b) = \pi(ab)$, $\pi(a^*) = \pi(a)^*$

- Invariant subspaces
  $F \subset H_\pi$ linear subspace is invariant
  if $\pi(a)x \in F$ $\forall x \in F$ $\forall a \in A$

- $F \subset H_\pi$ invariant
  $\Rightarrow$ $\overline{F}$ is invariant ($\pi(a)$ is cont., for fixed $a \in A$)
  $\Rightarrow$ $F^\perp := \{ y \in H_\pi \mid \langle y, x \rangle = 0 \ \forall x \in F \}$
  is invariant

- if $F$ is closed inv. subspace, by restriction
  we get $\pi_F : A \to \mathbb{B}(F)$

- direct sums: $\bigoplus_{i \in I} H_i$ = collection of $(x_i)_{i \in I}$
  s.t. $\sum_{i \in I} \langle x_i, x_i \rangle < \infty$.

  - label $\sum_{i \in I} \langle x_i, y_i \rangle =: \langle (x_i), (y_i) \rangle$

- now $\forall i$ define $A \xrightarrow{\pi_i} \mathbb{B}(H_{\pi_i})$ assuming
  $\forall a \in A$, $\sup\{ \|\pi_i(a)\| \} < \infty$

  - $\exists!$ rep. $\pi := \bigoplus_{i \in I} \pi_i : A \to \mathbb{B}(\bigoplus H_{\pi_i})$
    s.t. $\pi(a)((x_i)_{i \in I}) = (\pi_i(x_i))_{i \in I}$, and $\|\pi(a)\| := \sup_{i \in I} \|\pi_i(a)\|$

- commutant: $S \subset \mathbb{B}(H)$, then
$$S' := \{ T \in \mathbb{B}(H) \mid Tx = xT \ \forall x \in S \}$$

- intertwining operators.
$\pi_i : A \to \mathbb{B}(H_{\pi_i})$, $i \in \{1,2\}$
$$\mathrm{Hom}(\pi_1, \pi_2) := \{ T \in \mathcal{L}(H_{\pi_1}, H_{\pi_2}) \mid T\pi_1(a) = \pi_2(a) T \ \forall a \in A \}$$
$\mathrm{Hom}(\pi, \pi) =: \mathrm{End}(\pi) = (\pi(A))'$
$T \in \mathrm{Hom}(\pi_1, \pi_2)$, $T^* \in \mathrm{Hom}(\pi_2, \pi_1)$,
$t \in \mathrm{Hom}(\pi_1, \pi_2)$, $s \in \mathrm{Hom}(\pi_2, \pi_3)$
$\Rightarrow s \circ t \in \mathrm{Hom}(\pi_1, \pi_3)$
$\mathrm{Hom}(\pi_1, \pi_2) \subset \mathcal{L}(H_{\pi_1}, H_{\pi_2})$ weakly closed
- we can see this by fixing $x \in H_{\pi_1}$, $y \in H_{\pi_2}$
and writing $\langle \pi_2(a^*) x, Ty \rangle = \langle x, T\pi_1(a) y \rangle$

Prop $A \xrightarrow{\pi} \mathbb{B}(H)$, $E \subset H$ closed subsp. with its projection $P : H \circlearrowleft$.
Then $E$ invariant $\iff$ $P \in (\pi(A))' = \mathrm{End}(\pi)$

Pf. $P \in \mathrm{End}(\pi) \Rightarrow \forall T \in \pi(A)$, $TP = PT$
$\Rightarrow TP = PTP \iff E$ inv. for $T$
Conversely, if $E$ inv then also $E^\perp$ inv
so $\begin{cases} (1-P)TP = 0 \\ (1-(1-P))T(1-P) = 0 = -PT(1-P) \end{cases}$
$\Rightarrow TP = PTP = PT \ \forall T$ $\square$

**Def** $\pi_1 \sim \pi_2$ if $\exists$ a unitary $\in \mathrm{Hom}(\pi_1, \pi_2)$

- if $E$ closed and inv for $\pi$
  then $\pi \sim \pi_E \oplus \pi_{E^\perp}$

Antonini

- $E$ invariant closed, $\pi \Rightarrow \pi \sim \pi_E \oplus \pi_{E^\perp}$
- $T \in Hom(\pi_1, \pi_2)$, polar decomp. $T = u|T|$ with $u \in Hom(\pi_1, \pi_2)$, $|T| \in End(\pi_1)$

$$\begin{tikzcd} H_1 \arrow[r, bend left, "T"] & H_2 \arrow[l, bend left, "T^*"] \end{tikzcd}$$

- consequence: $\pi_1|_{\overline{T^* H_2}} \sim \pi_2|_{\overline{TH_1}}$

Def $\pi_1, \pi_2$ are disjoint if they don't have equivalent subrepresentations

Thm $\pi_1, \pi_2$ are disjoint iff $Hom(\pi_1, \pi_2) = \{0\}$.

- take $\pi$, $H_\pi$ and look at, $S \subset H_\pi \rightsquigarrow$
  $\{\pi(a) S \mid a \in A\} \subset H_\pi$, stable by $\pi$

Def. $S$ is totalising for $\pi$ if $\overline{\{\pi(a) S\}} = H$

Def. If $S = \{x\}$ a singlet, $x \in H_\pi$, we say $x$ is cyclic for $\pi$, if $\overline{\pi(a) x} = H_\pi$.

Def If $H_\pi$ is totalising for $\pi$, $\pi$ is called nondegenerate.

- facts:
  - sum of disj. cyclic representations is cyclic $\longrightarrow$ so a cyclic rep need not be irred
  - any nondeg. rep of $*$-alg is a direct sum of cyclic reps.

# GNS

- any $f : A \to \mathbb{C}$ linear we call a linear form
- it is positive if $f(A_+) \subseteq \mathbb{R}_+$
  - it can be shown pos. forms are continuous
- the form $\mathbb{B}(H) \to \mathbb{C}$ given by $x \in H$ and $T \mapsto \langle Tx, x\rangle =: f_x(T)$ is positive

- Claim:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, bend right, "\text{positive form}"'] & \mathbb{B}(H) \arrow[d, "f_x"] \\ & \mathbb{C} \end{tikzcd}$$

, where $\pi(a^*a) = \pi(a)^*\pi(a)$.

- define sesquilinear nonneg. form
  $A \times A \to \mathbb{C}$
  $(a, b) \mapsto f(a^*b) =: \langle a | b \rangle_f$
- $\langle a | a \rangle_f \geq 0$
- $\ker \langle \cdot | \rangle_f = \{ a \in A \mid \langle a | b \rangle = 0 \ \forall b \in A \}$

- completion:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi_f"] & A/\ker\langle \cdot | \rangle_f \arrow[dl, bend left] \\ H_f & \end{tikzcd}$$

$$\langle \psi_f(a) | \psi_f(b) \rangle = f(a^*b)$$

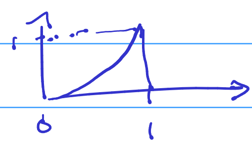
**Thm** Let $f : A \to \mathbb{C}$ positive form. $\exists!$ involutive rep. $\pi_f : A \to \mathbb{B}(H_f)$ such that $\forall\, a, b \in A$

$$\eta_f(ab) = \pi_f(a)\,\eta_f(b)$$

**Rmk.** Think of this as a left $A$-mod map ${}_A A \to {}_A \mathbb{B}(H_f)$, $a \cdot v = \pi_f(a) v$.

**Pf.** $a \in A$, $\|a\| \le 1$, $\operatorname{Spec}(a^* a) \subset [0,1]$ implies we can use continuous functional calculus to define $c := h(a^* a)$, where $h$ is the map

$$s \overset{h}{\longmapsto} 1 - (1-s)^{1/2}$$



and $c \in A$. The following holds:

$$\begin{cases} c = c^* \in A \\ (1-c)^2 + a^* a = 1 \text{ in } \tilde{A} \end{cases}$$

$\forall\, b \in A$, $(b - cb)^*(b - cb) + b^* a^* a b = b^* b$ follows, and so

$$\|\eta_f(b)\|^2 = \|\eta_f(ab)\|^2 + \|\eta_f(b - cb)\|^2$$

This tells us that for $a$ fixed, the map $\underset{\in A}{b} \mapsto \eta_f(ab) \in H_f$ satisfies $\|\eta_f(ab)\| \le \|\eta_f(b)\|$. So it extends to the statement of thm. $\square$

**Prop** $A$ unital. Any pos. form $f$ is bounded, with $\|f\| = f(1)$

Pf. $|f(a)| = |f(1^* a)| = |\langle \eta_f(1) | \eta_f(a) \rangle_{H_f}|$

$\leq \underbrace{\|\eta_f(1)\|}_{=\|\xi\|} \cdot \|\eta_f(a \cdot 1)\|$

$\overset{GNS}{=} \|\xi\| \cdot \|\pi_f(a)\| \, \|\eta_f(1)\|$

$= \underbrace{\|\xi\|^2}_{f(1)} \cdot \|a\|$ $\square$

- recall we defined approx unit as directed set $\Lambda := \{x \in A_+ | \ \|x\| < 1\}$ under increasing order, so that $\lim_{x \in \Lambda} xa = \lim_{x \in \Lambda} ax = a$

**Thm** $a_\lambda \longleftarrow \lambda \in I$, $z = \lim_{\lambda \in I} a_\lambda$ if $\forall \mathcal{U} \ni z$

**Prop** $A$ $C^*$-alg.

1) any pos. form $f$ is continuous, with $\lim_{a \in \Lambda} f(a) = \|f\|$.

11) $f$ is positive iff ($f$ cont.) $\wedge$ $\left(\sup\{\operatorname{Re} f(a) : a \in \Lambda\} = \|f\|\right)$

**Thm.** $f : A \to \mathbb{C}$ positive form.
$\exists$ a vector $\xi_f \in H_f$, normalised as $\|\xi_f\|^2 = \|f\|^2$, such that

- $f(x) = \langle \xi_f | \pi_f(x) \xi_f \rangle$
- $\forall x \in A$, $\eta_f(x) = \pi_f(x) \xi_f$.

**Pf.** $a, b \in \Lambda$, $b \geq a \Rightarrow (b-a)^2 \leq b-a$

$$\|\eta_f(b) - \eta_f(a)\|^2 = f((b-a)^2) \leq f(b) - f(a)$$

$$\leq \|f\| - f(a)$$

So $\underset{\Lambda}{a} \mapsto \eta_f(a) \in H_f$ converges,

let $\xi_f := \lim_{a \in \Lambda} \eta_f(a)$.

- $\forall x \in A$, $\eta_f(x) = \lim_{a \in \Lambda} \eta_f(xa)$

$$= \lim_{a \in \Lambda} \pi_f(x) \eta(a) = \pi_f(x) \xi_f$$

- $\forall x \in A$, $f(x) \overset{f \text{ cont.}}{=} \lim_{a \in \Lambda} f(ax) = \lim_{a \in \Lambda} \langle \eta_f(a) | \eta_f(x) \rangle$

$$= \lim_{a \in \Lambda} \langle \eta_f(a) | \pi_f(x) \xi_f \rangle = \langle \xi_f | \pi(x) \xi_f \rangle.$$

It follows that $\|f\| \leq \|\xi_f\|^2$, but
$\|\xi_f\|^2 = \lim_{a \in \Lambda} f(a^2) \leq \|f\|$ $\square$

- we completed the GNS construction

$$f \rightsquigarrow (H_f, \pi_f, \xi_f)$$

- start with $A$ invol. algebra with cyclic rep. $\pi : A \to \mathbb{B}(H)$, $x$ cyclic vect.

- we define positive form $f$ by $a \mapsto \langle x, \pi(a) x \rangle$

$$\begin{array}{ccc} A & \xrightarrow{\pi} & \mathbb{B}(H) \\ {\scriptstyle \psi_f}\downarrow & & \\ H_f & & \end{array}$$

$H_f :=$ Hsdff completion of $A$,

$$\langle \psi_f(a) | \psi_f(b) \rangle = f(a^* b) = \langle \pi(a) x, \pi(b) x \rangle$$

- both $\{\pi(A)x\} \subset H$ & $\{\psi_f(a)\} \subset H_f$ are dense

- $\exists !\ U \in \mathcal{U}(H, H_f)$ s.t. $\left\{ \begin{array}{l} U(\pi(a)x) = \psi_f(a) \\ \forall a \in A \end{array} \right.$

- call $a \mapsto U \pi(a) U^*$ $\pi_f : A \to \mathbb{B}(H_f)$

- then $(H_f, \pi_f, Ux)$ is $GNS(f)$

- we more or less have

**Prop** i) a rep. is cyclic iff equiv to a GNS

ii) any nondeg. rep is sum of GNS's.

- let $A$ $C^*$-alg

$$X := \left\{ \begin{array}{c} A \xrightarrow{f} \mathbb{C} \text{ positive lin forms} \\ \|f\| \leq 1 \end{array} \right\}$$

- $X$ is cpt in weak convergence

$C(X, \mathbb{R})$ = real ordered Banach space

$$\boxed{\begin{array}{l} A_h = \text{real ordered} \\ \quad \text{Ban. sp. of} \\ \quad \text{selfadj. elements} \\ \quad \text{of } A \end{array}} \xrightarrow[\text{Isometry}]{\text{Embedding}} C(X, \mathbb{R})$$

$$a \longmapsto \left(f \mapsto f(a)\right)$$

- tensor products:

  - Wegge-Olsen: K-theory & $C^*$-algebras
    - appendix (C) tensor products
  - define min, max, show properties.

# Antonini

- $X$ cpt case
→ $C(X)$ separable iff $X$ metrisable
- take $L: C(X) \to \mathbb{C}$ pos. form
  ($L(f) \geq 0 \ \forall f \in C(X)$ s.t. $f(t) \geq 0 \ \forall t$)

Thm (Riesz repr.) $\exists!$ Borel measure $\mu$
  s.t. $L(f) = \int f \, d\mu$

- now look at $\pi: C(X) \to L^2(X,\mu)$, $\pi(f)g = f \cdot g$.
- notice that $L^\infty(X,\mu)$ is a von Neumann algebra, meaning a weakly closed, nondegenerate as a representation, $*$-subalgebra of $\mathbb{B}(H)$

- $L$ extends to $L^\infty(X,\mu)$

$$\tilde{L}\Big(\sum_{\alpha \in I} e_\alpha\Big) = \sum_{\alpha \in I} \tilde{L}(e_\alpha)$$

for any family $(e_\alpha)_{\alpha \in I}$ of mutually $\perp$ projs

- so we get $\overline{\pi_f(A)}^{W.O.T.}$ ← weak operator topology in $\mathbb{B}(H_f)$

Def. A unital. States $S := \left\{ \begin{array}{l} \varphi \in A^*_+ \\ \varphi(1) = 1 \end{array} \right\}$

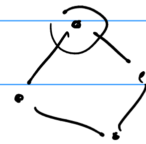
- properties:
  - $S \neq \emptyset$
  - convex
  - separates pts.
  - the corresponding GNS is isometric

- last prop: $\|\pi_f(a)\|^2 \geq \|\pi_f(a)\xi_f\|^2 = f(a^*a) \geq \|a\|^2$ contractive

but $\|\pi_f(a)\|^2 \leq \|a\|^2$ since $*$-morphisms are

- cptness in weak $*$-top, i.e. $\sigma(A^*, A)$
- Klein-Milman says

$S$ = closure of the cnvx envelope of its $\underbrace{\text{extremal pts}}_{=\text{irreducible reprs.}}$



- note: weak op. top. is the weakest topology making continuous

$\forall x \in H$, $\mathbb{B}(H) \longrightarrow (H, \text{weak top})$

$T \longmapsto Tx$

or, equiv., $\forall x, y \in H$, $T \mapsto \langle y, Tx \rangle$.

- If we had put $(H, \text{norm top})$, this would be strong op. top.

- $X = \{\text{pos lin forms on } A, \|f\| \leq 1\}$
- compact (weak top.) ptwise convgnc

$A_h \xleftrightarrow{F} C(X, \mathbb{R})$

self adj $\in A$, real ordered Ban. sp.

$F(a)(f) = f(a)$

- note we induce order, since
  $a \leq b$ in $A_h$ means $\forall x \in X$,
  $F(a)(x) = x(a) \leq x(b) = F(b)(x)$
- $F$ is also isometric (no pf. given)

- recall $f: A \to \mathbb{C}$ Hermitian if $f(a^*) = \overline{f(a)}$

Prop $f: A \to \mathbb{C}$ Hermitian cont. lin. form.
Then $\exists$ pos. forms $f^+, f^-: A \to \mathbb{C}$
s.t. $(f = f^+ - f^-) \wedge (\|f\| = \|f^+\| + \|f^-\|)$.

Cor $(\forall x \in A)$ $\exists$ a representation
$$\pi: A \to \mathbb{B}(H_\pi)$$
s.t. $\|\pi(x)\| = \|x\|$ and $\pi$ is a GNS .

Pf: $\exists$ pos. form $f: A \to \mathbb{C}$ s.t.
$$\begin{cases} \|f\| \leq 1 \\ f(x^* x) = \|x\|^2 \end{cases}$$
Then $(H_f, \pi_f, \xi_f)$ GNS, $\|\xi_f\|^2 = \|f\| \leq 1$
and $\|\pi_f(x)\|^2 \geq \|\pi_f(x)\xi_f\| = f(x^* x) = \|x\|^2$.
The obverse always holds. $\square$

- so we get many representations, but

Thm (Gel'fand-Naimark) Any $C^*$-alg
has an isometric repr. on a Hilb. sp.

Pf. $\forall a \in A$ $\exists \pi_a$. But $\bigoplus_{a \in A} \pi_a$ is isometric,
since $\forall b \in A$, $\|\bigoplus_{a \in A} \pi_a(b)\| = \sup_{a \in A} \|\pi_a(b)\| = \|\pi_b(b)\|$
$$= \|b\|.$$

- but if $A$ separable, we can do better

**Def** A pos. form $f$ is *faithful* if
$$\{a \in A \mid f(a^*a) = 0\} = \{0\}.$$

**Prop** $A$ separable. Then:

i) $\exists$ pos. form on $A$ which is faithful

ii) $\exists$ isometric reps. on a separable Hilbert space.

**Pf.** $\overline{(a_n)_n} = A$. $\forall n$ $\exists f_n : A \to \mathbb{C}$ positive

and
$$\begin{cases} f_n(a_n^* a_n) = \|a_n\|^2 \\ \|f_n\|^2 \le 1 \end{cases}$$

and all $f_n$ have $(H_{f_n}, \pi_{f_n}, \xi_{f_n})$.

Define $f := \sum 2^{-n} f_n$, so that $a$ nonzero, $\exists n$ s.t. $\|a - a_n\| < \frac{\|a\|}{2}$, $\|a - a_n\| < \|a_n\|$ gives
$$\|\pi_{f_n}(a)\xi_{f_n}\| \ge \|\pi_f(a_n)\xi_f n\| - \|a - a_n\|$$
$$= \|a_n\| - \|a - a_n\| > 0.$$
But $2^n f(a^*a) \ge f_n(a^*a) = \|\pi_{f_n}(a)\xi_{f_n}\|^2 > 0$.

$H$ will be separable since $(a_n)_n$ dense by $A \to A/\mathrm{Ann}^{\{0\}} \to$ completion.

$\pi$ inj. since $\|\pi_f(a)\xi_f\|^2 = f(a^*a) > 0$ for $a \neq 0$. $\square$

- let $U \subset \mathbb{B}(H)$ any subset

**Def.** $U' := \{ T \in \mathbb{B}(H) \mid TS = ST \ \forall S \in U \}$
is called commutant.

- always a subalgebra, regardless of $U$
- always weakly closed, since we can see $U'$ as $\forall x, y \in H$, all $T$ s.t.

$$\langle S^* x, Ty \rangle = \langle Tx, S^* y \rangle$$

$\forall S \in U$, so take net and it works due to continuity

- if $U$ is $*$-symmetric (i.e. $x^* \in U$ if $x \in U$) then $U$ is a unital $C^*$-algebra

- $A \subset B \Rightarrow B' \subset A'$
- $A \subset A'' =: (A')'$

$\rightarrow$ these give $A''' = A'$, so we only really have $A$, $A'$ and $A''$.

**Def.** A subalg $B \subset \mathbb{B}(H)$ is called nondeg. if its nat. repr $B \hookrightarrow \mathbb{B}(H)$ is nondeg.

**Lemma.** $A$ nondeg. subalg., $T \in A'$ fixed. Then

i) $T \in \overline{A}^{\text{strong}}$

ii) $T = T^* \Rightarrow T \in \overline{\{ S \in A \mid S^* = S \}}^{\text{strong}}$

iii) $\|T\| \leq 1 \Rightarrow T \in \overline{\{ S \in A \mid \|S\| \leq 1 \}}^{\text{strong}}$

**Thm** (Double commutant, von Neumann)

$H$ Hilbert, $A$ involutive nondeg subalg of $\mathbb{B}(H)$. TFAE

1) $A = A''$
2) $A$ is weakly closed in $\mathbb{B}(H)$
3) — " — strongly — " —
4) the unit ball of $A$ is weakly cl. in $\mathbb{B}(H)$
5) — " — strongly — " —

- von Neumann algs are (almost) never separable
  - in some properties they resemble measure theory
  - also Borel functional calculus

**Def.** A repr. of a $*$-alg is irreducible if the only invariant closed subspaces are $\{0\}$ and $H_\pi$.

**Thm** For a nontrivial representation of an involutive algebra $A$ TFAE

1) $\pi$ is irred.
2) any $H_\pi \ni v \neq 0$ is cyclic (or totalising)
3) $\pi(A)' = \mathbb{C}$
4) $\pi(A) \subset \mathbb{B}(H)$ is strongly dense

**Cor** For a $C^*$-alg, $\pi$ is irred. $\Rightarrow$ $\pi$ algebraically irreducible.