Du Pei

Brane quantization & reps. of DAHA

$$\begin{array}{ccc} & \text{QFTs} & \\ \text{geom.} \swarrow & & \searrow \text{alg.} \\ \text{symp/cpx/alg.geom moduli spaces} & & \text{alg. of defects} \end{array}$$

alg. of defects $\xrightarrow[\text{to}]{\text{promote}}$ (higher) cat. of defects

symp/cpx/alg.geom moduli spaces $\overset{\text{usually related}}{\longleftrightarrow}$ alg. of defects

- today: 4d $\mathcal{N}=2$ theories

$$\begin{array}{ccc} & \text{4d } \mathcal{N}=2 \text{ theories} & \\ \swarrow & & \searrow \\ \text{Coulomb branches} & \overset{\text{"brane quant."}}{\longleftrightarrow} & \text{operator algebras} \end{array}$$

- 4d $\mathcal{N}=2$ theories $\begin{cases} \text{on } \mathbb{R}^4 \\ \text{on } S^1 \times \mathbb{R}^3 \end{cases}$

Coulomb branch:
- $\mathcal{B}$ $\leftarrow$ special Kähler
- $\mathcal{M}_c$ $\leftarrow$ hyperkähler

- $\mathcal{M}_c$: cpx. str's I, J, K
  - sym. str's $\omega_I, \omega_J, \omega_K$
  - holom. sympl. forms $\Omega_I, \Omega_J, \Omega_K$, $\Omega_I = \omega_J + i\omega_K$ et cycl

$\rightarrow$ studying $(\mathcal{M}_c, I)$ gives us a map

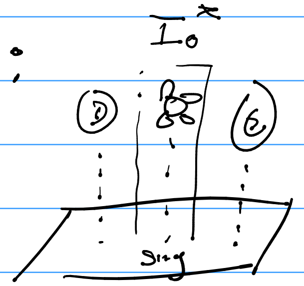
$$\begin{array}{ccc} F & \hookrightarrow & \mathcal{M}_c \\ & & \downarrow \\ & & \mathcal{B} \end{array}$$

with fibres F generically cpt Lagr. tori w.r.t. Ω_I

- physically: $\begin{cases} \mathcal{B} - \text{param. by v.e.v.s of loc. ops} \\ F - \quad \text{—''—} \quad \text{of line ops along } S^1 \end{cases}$

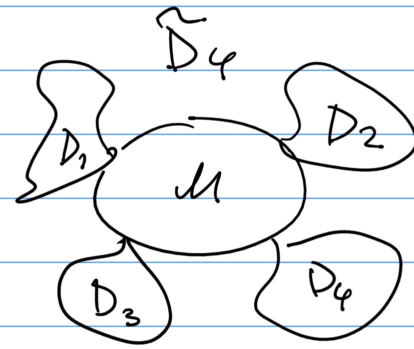
- Main example: 4d $\mathcal{N}=2^*$ theory ($G=SU(2)$)
  - $\to$ 4d $\mathcal{N}=2$ vect. multiplet
  - + 4d $\mathcal{N}=2$ hypermultiplet in adjoint
- $\mathcal{M}_C$ will be elliptic sfc.

• mass = 0:



$B =$

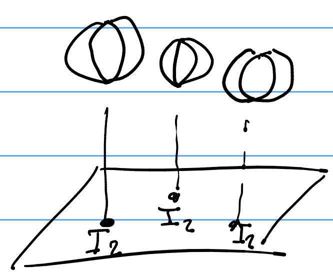
- sing fiber:



& kähler modulus controlls sizes of D_i

• mass $= \beta + i\gamma$

$B =$



- $(\mathcal{M}_C, J)$ will be a cubic sfc

$$x^2 + y^2 + z^2 - xyz = t^2 + t^{-2} + 2, \quad t = e^{2\pi(\gamma + i\alpha)}$$

- character variety

$$\{\pi_1(\quad) \to SL(2,\mathbb{C})\}/\text{conj}$$

$$\text{hol} = \begin{pmatrix} t^2 & 0 \\ 0 & t^{-2} \end{pmatrix}$$

→ same cubic (exercise)

– 6d (2,0) A_1 on $\bigcirc$

$\updownarrow$

4d $\mathcal{N}=2^*$ theory

$\circlearrowleft$

$SL(2,\mathbb{Z})$

– meaning of x, y, z?
  – params of hol. along curve
    $(1,0)$, $(0,1)$, $(1,1)$
  – physics pictr:
    Wilson, t'Hooft & dyonic loops along S^1

– algebra: $\mathbb{C}[x,y,z]$ / the cubic eqn.

  – its rep. theory:
    pts on $(\mathcal{M}_c, J)$

– now look at 4d $\mathcal{N}=2$ on $S^1 \times \mathbb{R}^2_\hbar \times \mathbb{R}$ (Ω-background)

$\circlearrowleft$ $\mathbb{C}^*$ $q = e^\hbar$

⇝ NC deformation of $(\mathcal{M}_c, J)$
– deformation quantization
– acts on Hilb. sp. of QM on the $\mathbb{R}$

Rmk quantization is hard
– 2 approaches → ① def. quant.
  → formal in $\hbar$
  → will not give Hilb. sp.

→ ② geom. quant.

→ gives Hilb. sp. -- no algebra

→ but here we get <u>both</u>.

- [A. Oblomkov] Def. quant. of $(\mathcal{M}_C, J)$ gives $S\ddot{H}$ (spherical double affine Hecke alg)

↪ abbr. DAHA

- subalg. of DAHA $\ddot{H}$

$$[x,y]_q := q^{-1/2}xy - q^{1/2}yx = (q^{-1}-q)z$$
$$[y,z]_q := (q^{-1}-q)z$$
$$[z,x]_q := (q^{-1}-q)y$$

- also, cubic eqn becomes

$$q^{-1}x^2 + qy^2 + q^{-1}z^2 - q^{1/2}xyz = (q^{-1/2}t - q^{1/2}t^{-1})^2 + (q^{1/2}+q^{-1/2})^2$$

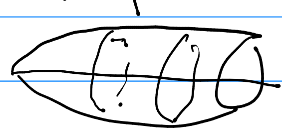
→ $q \to 1, \hbar \to 0$: cubic eqn

→ $t \to 1$ (4d $\mathcal{N}=4$ limit): Skein algebra

- $S\ddot{H} \overset{\text{Morita}}{\sim} \ddot{H}$

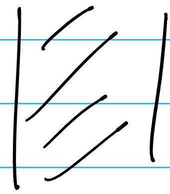
Brane quantization:

- 4d $\mathcal{N}=2^*$ on $\underbrace{S^1 \times \mathbb{R}^2_\hbar}_{\substack{\text{view as} \\ S^1\text{-fibered}}} \times \mathbb{R}$



cptify $\longrightarrow$ 2d (4,4) sigma model to $\mathcal{M}_C$

on



bdary conds.   in A-model ($\mathcal{M}_C$, $\mathrm{Im}\,\frac{\Omega_J}{i\hbar}$

- "A-branes"

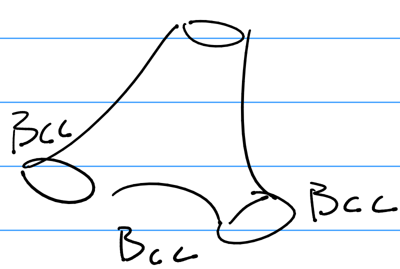
- familiar story: A-branes in CY_3 are Lagrangian

- but now, [Kapustin-Orlov] they are coisotropic branes

- $\left(\mathcal{M}_C, \frac{\Omega_J}{i\hbar}\right) \rightsquigarrow B_{cc} \Rightarrow$ A-brane wrt $\mathrm{Im}\,\frac{\Omega_J}{i\hbar}$ equipped w/a line bdl $\mathcal{L}$ + B-field s.t.

$$c_1(\mathcal{L}) + B = \mathrm{Re}\,\frac{\Omega_J}{i\hbar}$$

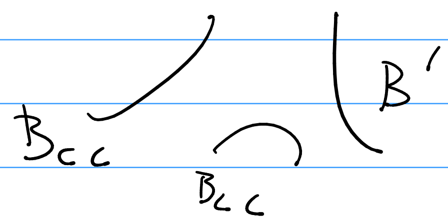
- $\mathrm{Hom}(B_{cc}, B_{cc})$ is an algebra by TQFT yoga



- NC algebras to be expected

→ the def. quant. of $\left(\mathcal{M}_c, \frac{\Omega_J}{i\hbar}\right)$

- Hom(B_{cc}, B'), B' another A-brane, has a natural module structure



- question: A-brane → $S\ddot{H}$-mod ...
  is it a map? Or a functor?
  → if so, does it induce equiv. of cats?

- maybe:

Step 1| cpt. A-branes $\overset{?}{\longleftrightarrow}$ finite reps of $S\ddot{H}$

$\Downarrow$

Step 2| look at morphisms:

$HF^*(B_1, B_2) \longleftrightarrow$ (higher) extensions of reps

$\Downarrow$

Step 3| ∞-dim reps

- 1 has been checked, 2 in progress

- for generic values of q, t, $S\ddot{H}$ has no fin dim reps

$SL(2,\mathbb{Z})$ action on $\mathcal{M}_c$

$\mathcal{B}'$	$\mathcal{B}=0$/ Shortening cond	$\dim R$	interpretation?
$\mathcal{F}$	$q=e^{-\pi i/k}$, $k\in\mathbb{Z}_{>}$	$2k$	Chern-Simons th
$\mathcal{M}$	$t^2=-q^n$, $n\in\mathbb{Z}_{>}$	$n+1$	Refine CS-th
$\mathcal{D}_i$	$t^2=q^{-(2l+1)}$ $l\in\mathbb{Z}_{>}$	1	$(2,2l+1)$ minimal model