Bruzzo.

# Flabby sheaves (flasque)

- $\mathcal{F} \in Sh_X$, $V \overset{\text{open}}{\hookrightarrow} U$ any pair of opens
- $\mathcal{F}$ is **flabby** if $\mathcal{F}(U) \to \mathcal{F}(V)$ surjects

## Examples

1) skyscrapers.
   $\to$ $G \in Ab$, $x \in X$ $\Rightarrow$ $G(x)(U) = \begin{cases} G & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$

II) if $X$ irreducible, all constant sheaves are flabby

III) $\mathcal{F} \in Sh_X$, étalé space $\pi : \underline{\mathcal{F}} \to X$

$\to$ $\mathcal{C}^0(\mathcal{F}) = \{ \underline{\text{all}}$ sections of $\pi \}$ is flabby
including noncontinuous

- every sheaf embeds into a flabby one:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{C}^0(\mathcal{F}) \arrow[r] \arrow[dr] & Q_0 \arrow[r] \arrow[d] & 0 \\
 & & & \mathcal{C}^1(\mathcal{F}) \arrow[d] \arrow[dr] & \\
 & & 0 \arrow[r] & Q_1 \arrow[r] & \mathcal{C}^2(\mathcal{F}) \arrow[dr] & \\
 & & & & & \cdots
\end{tikzcd}$$

Godement canonical flabby resolution

**Thm** $\mathcal{F}$ flabby $\Rightarrow H^i(X,\mathcal{F})=0, \forall i>0$.

**Lemma.** $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ s.e.s. of sheaves & $\mathcal{F}'$ flabby.
Then $0\to\mathcal{F}'(U)\to\mathcal{F}(U)\to\mathcal{F}''(U)\to 0$ exact.

Pf. (Godement)

- now Thm follows from $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to H^1(\mathcal{F}')$.

**Lemma** $\mathcal{F}',\mathcal{F}$ flabby $\Rightarrow$ quotient $\mathcal{F}''$ flabby

Pf.
$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F}'(U) & \to & \mathcal{F}(U) & \to & \mathcal{F}''(U) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{F}'(V) & \to & \mathcal{F}(V) & \to & \mathcal{F}''(V) & \to & 0 \end{array}$$

**Lemma.** Injective sheaves are flabby

Pf. $0\to\mathcal{J}\to\mathcal{C}^0(\mathcal{J})\to\mathcal{Q}\to 0 \Rightarrow \mathcal{C}^0(\mathcal{J})\simeq\mathcal{J}\oplus\mathcal{Q}$

flabby

**Pf of Thm** $0\to\mathcal{F}\to\mathcal{J}\to\mathcal{Q}\to 0$

$\mathcal{J}$ injective

$\Rightarrow 0\to H^0(X,\mathcal{F})\to H^0(X,\mathcal{J})\to H^0(X,\mathcal{Q})\to H^1(X,\mathcal{F})\to 0$.

$0\to H^1(X,\mathcal{Q})\to H^2(X,\mathcal{F})\to 0$

$\dots\ 0\to H^n(X,\mathcal{Q})\to H^{n+1}(X,\mathcal{F})\to 0$

$\Rightarrow H^1(X,\mathcal{F})=0$ by $1^{st}$ lemma, $\mathcal{Q}$ flabby by $2^{nd}$

$\Rightarrow$ induction

- $A$ ring, $X = \operatorname{Spec} A$, $M$ $A$-module $\Rightarrow \tilde{M} \in \mathcal{O}_X$-mod
and $\tilde{M}_p = M_p$ i.e. quasi-coh.

**Prop** Quasi-coherent sheaves on the spectrum of a noetherian ring are acyclic.

**Lemma.** $A$ noeth. ring, $I$ injective $A$-module $\Rightarrow \tilde{I}$ flabby.

**Pf of thm** $\mathcal{F} \in Sh_{X = \operatorname{Spec} A}$, $\mathcal{F} \simeq \tilde{M}$.

$\Rightarrow M = \Gamma(X, \mathcal{F})$ $A$-module

$\to$ take resoln.:

$0 \to M \to I^0 \to I^1 \to \dots$ inj. resoln

$\Rightarrow 0 \to \tilde{M} = \mathcal{F} \to \tilde{I}^0 \to \tilde{I}^1 \to \dots$ flabby resoln

$\Rightarrow$ now $H^i(X, \mathcal{F}) \simeq H^i(\Gamma(X, \tilde{I}^\bullet)) \simeq H^i(I^\bullet) = 0$, $i > 0$

$H^0(X, \mathcal{F}) \simeq H^0(I^\bullet) \simeq M$.

**Thm (Serre)** $X$ noetherian scheme. TFAE:

i) $X$ is affine

ii) $H^i(X, \mathcal{F}) = 0$ $\forall i > 0$ $\forall \mathcal{F}$ quasi-coherent

iii) $H^i(X, \mathcal{F}) = 0$ $\forall$ coherent sheaves of ideals of $\mathcal{O}_X$

Čech cohomology review:

- $(X, \mathcal{F}, \mathfrak{U})$, $C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p})$, ...

- if $\mathcal{F}$ sheaf, $H^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$

Thm $X$ noetherian separated scheme, $\mathcal{F}$ quasi-coherent sheaf on $X$, $\mathfrak{U}$ open cover of affine sets. Then $H^p(\mathfrak{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$ $\forall p \geq 0$.

$j_{i_0 \dots i_p} : U_{i_0 \dots i_p} \hookrightarrow X$. $\check{\mathcal{C}}(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_p} (j_{i_0 \dots i_p, *} \mathcal{F}|_{U_{i_0 \dots i_p}})$

→ these are sheaves

→ exact in positive degree

→ $\exists$ $\epsilon : \mathcal{F} \to \check{\mathcal{C}}^0(\mathfrak{U}, \mathcal{F})$

→ $\check{\mathcal{C}}^0(\mathfrak{U}, \mathcal{F}) = \prod_k j_{k,*} \mathcal{F}|_{U_k}$

$$j_{k,*} \mathcal{F}|_{U_k}(U) = \mathcal{F}|_{U_k}(i_k^{-1}(U)) = \mathcal{F}(U \cap {}_k)$$

$$\epsilon(s) = \prod s(U \cap U_k)$$

→ $0 \to \mathcal{F} \to \check{\mathcal{C}}^\bullet(\mathfrak{U}, \mathcal{F})^k$ is a resoln (Čech resoln)

→ $\Gamma(X, \check{\mathcal{C}}(\mathfrak{U}, \mathcal{F})) \simeq C^p(\mathfrak{U}, \mathcal{F})$