Tanzini.

SQM.

- on a manifold M, odd elements of H.lb. superspace
$$\wr S$$
differential forms

→ so in particular we use that iso to write

$$\langle \varphi^{(1)} | \varphi^{(2)} \rangle = \int \overline{\varphi^{(1)}} \wedge * \varphi^{(2)}$$

$$= \sum_{F=0}^{m} \int d^m x \sqrt{g}\, g^{I_1 J_1} \cdots g^{I_F J_F}\, \overline{\varphi}^{(1)}_{I_1 \cdots I_F}\, \varphi^{(2)}_{J_1 \cdots}$$

- free ptcl ($h=0$) $\Rightarrow$ $Q = \overline{\psi}^I \nabla_I$
$$\wr S$$
$$d_{dR}$$

thus $H = \frac{1}{2}\{Q, Q^+\} \xrightarrow{\sim} \frac{1}{2}\Delta = \frac{1}{2}\{d, d^+\}$

- susy g.s. → $Q|\psi\rangle = Q^+|\psi\rangle = 0$
$\wr S$
harmonic forms on M.
- given $b_F(M) = \dim H^F(M)$ ← harmonic forms
- therefore
$$\Omega = \sum_{F=0}^{m} (-)^F b_F = \chi(M)$$

Digression. Let $P \to M$ $SO(2n)$-principal bdl.
- define Euler class $e(P) = \frac{1}{(2\pi)^n} Pf(F)$ ← curvature

- fact: $\chi(P) = \int_M e(P)$. In particular for $P = TM$,
$(2\pi)^n \chi(M) = \int_M Pf(R)$ gen. Gauß-Bonnet

- susy path int:

$$Z = \int [d\varphi\, d\psi\, d\bar{\psi}]\, e^{-S(\varphi,\psi,\bar{\psi})} = \mathrm{Tr}(-)^F e^{-\beta H}$$

$\{\varphi : S^1_\beta \to M,$
$\varphi(0) = \varphi(\beta)$
$\psi(0) = \psi(\beta)$
$\bar{\psi}(0) = \bar{\psi}(\beta)\}$ ← we could pick antiperiodic conditions for fermions since we always deal w/ bilinears, but then we do not reproduce $\mathrm{Tr}(-)^F e^{-\beta H}$.

$$-\mathcal{L} = \frac{1}{2} g_{IJ}\dot{\varphi}^I\dot{\varphi}^J + g_{IJ}\bar{\psi}^I \underbrace{\overset{\varphi^*(\nabla_M)}{\overset{\|}{D_\tau}}\psi^J}_{\partial_\tau\psi^I + \Gamma^I{}_{JK}\partial_\tau\varphi^J\psi^K} + \frac{1}{2} R_{IJKL}\psi^I\bar{\psi}^K\psi^J\bar{\psi}^L$$

$$\delta\mathcal{L} = d_\tau(\cdots)$$

$$\delta\varphi^I = \epsilon\bar{\psi}^I - \bar{\epsilon}\psi^I$$
$$\delta\psi^I = \epsilon(-\dot{\varphi}^I - \Gamma^I{}_{JK}\psi^J\bar{\psi}^K)$$
$$\delta\bar{\psi}^I = \epsilon(\dot{\varphi}^I - \Gamma^I{}_{JK}\bar{\psi}^J\psi^K)$$

$$\delta(-) = 0 \Rightarrow \dot{\varphi}^I = 0 \to \text{constant maps } (\varphi_0, \psi_0, \bar{\psi}_0)$$

$\underbrace{\Gamma^I{}_{JK} = 0}_{\text{first order}}$

$$Z = \int D\varphi_0 D\psi_0 D\bar{\psi}_0 \prod_{n\neq 0} D\varphi_n D\psi_n D\bar{\psi}_n \exp\Big[-\oint d\tau\Big(\frac{1}{2} g_{IJ}\dot{\varphi}^I\dot{\varphi}^J - g_{IJ}\bar{\psi}^I\partial_\tau\psi^J + \frac{1}{2}R_{IJKL}\psi_0^I\bar{\psi}_0^K\psi_0^J\bar{\psi}_0^L\Big)\Big]$$

where $\varphi(\tau) = \sum_{n\in\mathbb{Z}} \varphi_n e^{in\tau}$ etc.

Gaussian approx: lowest order

$$\Big[\det{}'\Big(-\frac{d^2}{d\tau^2}\Big)\Big]^{-1/2}\Big(\det{}'\frac{d}{d\tau}\Big) = 1$$

$$\Big[\prod_{n\neq 0} n^2\Big]^{-1/2} = \Big[\prod_{n>0} n^2\Big]^{-1}$$

$$\prod_{n\neq 0} in = \prod_{n>0}(in)(-in) = \prod_{n>0} n^2$$

Digression 2. (ζ-func. regularization)

- G hermitian op. w. discrete spectrum $\{\lambda_n\}$

$$\zeta_G(s) := \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{dt}{t} t^s \operatorname{Tr}(P_\perp e^{-tG})$$

$$\overset{(\text{formally})}{=} \sum_n \frac{1}{\lambda_n{}^s}$$

- then $\det' G := e^{-\zeta'_G(0)}$

Exercise: do this for spectrum of $-\frac{d^2}{dz^2}$, $i\frac{d}{dz}$.

$$Z = \int_{\Pi(TM)} \prod_{I=1}^n D\varphi_0^I D\chi_0^I D\bar{\psi}_0^I \, e^{-\frac{1}{2} R_{IJKL} \bar{\psi}^I \bar{\psi}^J \psi^K \psi^L}$$

$$= \int_M \operatorname{Pf}(R) = \chi(M)$$

Adding a superpotential.

- $\varphi: S^1 \to M$, $h: M \to \mathbb{R}$.

- operator formalism: $Q_h \doteq e^{-h} Q e^h = \bar{\psi}(\partial + h')$

$$\Big\downarrow \text{Ts}$$

$$d_h \doteq e^{-h} d e^h$$

- on the Hilb space, $|\alpha\rangle \mapsto e^{-h} |\alpha\rangle$

- we will be adding $\mathcal{L}_h \doteq \frac{1}{2} g^{IJ} \partial_I h \partial_J h + \nabla_I \partial_J h \bar{\psi}^I \psi^J$
to our previous Lagrangian

- new fixed pts: constant maps to the crit pts of h

$$\Delta_B = -g_{IJ}\frac{d^2}{d\tau^2} + (D_I\partial_K h\, D_J\partial^K h)$$

$$\Delta_F = g_{IJ}\frac{d}{d\tau} + D_I\partial_J h$$

$n\neq 0$:

$\to$ B : $\prod_{n\neq 0}\left(g_{IJ}n^2 + (D_I\partial_K h\, D_J\partial^K h)|_{\varphi_0}\right)^{-1/2}$

$$= \prod_{n>0}\left(\text{—''—}\right)^{-1}$$

F : $\prod_{n\neq 0}\left(i n\, g_{IJ} + D_I\partial_J h|_{\varphi}\right) = \prod_{n>0}\left(i n\, g_{IJ} + \ldots\right)\left(-i n\, g_{IJ} + \ldots\right)$

$$= \prod_{n>0}\left(n^2 g_{IJ} + D_I\partial_K h\, D_J\partial^K h\right)$$

$n=0$:

$$\frac{\det D_I\partial_J h}{\left(\det(D_I\partial_K h\, D_J\partial^K h)\right)^{1/2}} = \operatorname{sign}\det \operatorname{Hess} h|_{\varphi_0}$$

$$\Rightarrow Z_{h\neq 0} : \sum_{\{\varphi_c :\, \partial h|_{\varphi_c}=0\}} \operatorname{sign}\det\operatorname{Hess} h = \chi(M) \text{ by Poincaré–Hopf}$$