BRUZZO

Morphisms (locally) of finite type

- quasi-finite $\to \{f^{-1}(y)\}$ is finite for any $y \in Y$, $\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$
- ex: $\mathbb{A}^n_k = \operatorname{Spec} k[x_1 \dots x_n]$ is of finite type

  $\mathbb{A}^1_k \to \mathbb{A}^n_k$, $z \mapsto z^n$ $\to$ $k[x]$-module structure given by raising to n-th power $\to$ basis given by $1, x, \dots, x^{n-1}$

- ex: $k^* \to \mathbb{A}^1_k$

  $\mathbb{A}^1_k - \{0\} = D(x) = \operatorname{Spec} k[x, \frac{1}{x}]$ $\to$ quasi finite but not finite

- ex: $X = \operatorname{Spec} \dfrac{k[x,y]}{(x-y^2)}$ $\rightsquigarrow$ $\dfrac{k[x,y]}{(x-y^2)} = k[x] \oplus y k[x]$ $\hookrightarrow$ finitely gen. as a module $\Rightarrow$ finite

$\to$ remove a pt and no longer finite

- ex: $X = \operatorname{Spec} \dfrac{k[x,y,t]}{(ty-x^2)}$ $\to$ X, Y integral schemes of fin. type

  $f \downarrow$ $\to$ fibres: $t = a \neq 0$, integral variety

  $Y = \operatorname{Spec} k[t] \simeq \mathbb{A}^1_k$ $\quad t = 0 \Rightarrow f^{-1}(0) = \operatorname{Spec} \dfrac{k[x,y]}{(x^2)}$

  $\to$ NONREDUCED !

ERRATA from last time: $\operatorname{Proj} S = \{\text{prime ideals } p,\ S_+ \not\subset p\}$

Fiber product of top. spaces.

- standard def.

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "q"] \\ & X \arrow[r, "p"] & Z \end{tikzcd}$$

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ X \arrow[r, "g"] & S \end{tikzcd}$$

$X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$, $f \rightsquigarrow R \xrightarrow{f^\#} A$, $g \rightsquigarrow R \xrightarrow{g^\#} B$, $C = A \otimes_R B$

$\hookrightarrow (a \otimes b)(a' \otimes b') = (aa' \otimes bb')$

$\Rightarrow X \times_S Y = \operatorname{Spec} C$

$$\begin{tikzcd} C = A \otimes B & A \arrow[l, "\pi_1^\#"'] \\ B \arrow[u, "\pi_2^\#"] & R \arrow[l, "g^\#"] \arrow[u, "f^\#"'] \end{tikzcd}$$

$\pi_1^\#(a) = a \otimes 1$

$(-)^\#$ is a rather.. .. .. notation, $(-)^*$ is better...

- EXAMPLE

X, Y schemes over k: $X \times_k Y \equiv X \times_{\mathrm{Spec}\, k} Y$

- $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathrm{Spec}\, \underbrace{k[x] \otimes_k k[y]}_{k[x,y]} = \mathbb{A}^2_k$

→ note that closed sets in $\mathbb{A}^1_k$ are finite collections of points, but in $\mathbb{A}^2_k$ they could be curves, e.g. $x=y$
→ difference in topology

Def.

$$\begin{tikzcd}
X \arrow[dr, "\Delta_f"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

DIAGONAL MORPHISM Δ_f associated to $X \xrightarrow{f} Y$.

Def. f is a separated morphism if Δ_f closed immersion.
X scheme over k is separated (over k) if $X \to \mathrm{Spec}\, k$ separated.

Prop. All morphisms over affine schemes are separated.

Cor: All affine schemes over k are separated over k.

→ consider $X =$ —:— line w/ double origin
⇒ $X \times_k X =$ aff plane w/ 2 axes & 4 origins
$\Delta =$ line w/ double origin
⇒ all origins are in the closure of Δ ⇒ Δ ¬ closed ⇒ X not affine

Pf:
- note that on CRing we have the codiagonal $\Delta^\#: A \otimes A \to A$
$a \otimes a' \mapsto aa'$
- surjective ⇒ $\mathcal{O}_{X \times_Y X} \xrightarrow{\Delta^\#} \Delta_* \mathcal{O}_X$ surjective on stalks
- for $I \subset A$, $A \twoheadrightarrow A/I$, $\mathrm{Spec}\, A/I \to \mathrm{Spec}\, A$ morphism of Spec-s
→ take $I = \ker \Delta^\#$ ⇒ Δ closed immersion □

Prop. $X \xrightarrow{f} Y$ separated iff Δ_f closed in $X \times_Y X$

Pf ⇒ obvious, ⇐

$$\begin{tikzcd}
X \arrow[r, "\Delta_f"] \arrow[dr, bend right, "\cong"'] & X \times_Y X \arrow[d, "\pi_1"] \\
& X
\end{tikzcd}$$

Δ_f^{-1} may be identified with the identity ⇒ closed im. □

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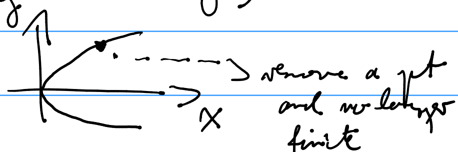
→ finitely gen. as a module ⟺ finite



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Δ_f^{-1} may be identified with the identity ⇒ closed im. □

Bruzzo

- recalling: f separated if Δ_f closed immersion.

$$\begin{tikzcd}
X \arrow[dr, "\Delta_f"] \arrow[drr, bend left, "\sim"] \arrow[ddr, bend right, "2"'] & & \\
& X \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Properties:

(i) open & closed immersions are separated

(ii) composition respects separatedness

(iii) so does base change

$$\begin{tikzcd}
X \times_Y Y' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

(iv) f is separated iff Y has an open cover $\{U_i\}$ s.t. all restrictions $f^{-1}(U_i) \xrightarrow{f} U_i$ are separated

Proper morphisms

Def. $f: X \to Y$ is proper if it is

- (i) of finite type
- (ii) separated
- (iii) universally closed (closed after any base change)

$X \times_Y Y' \xrightarrow[\text{closed}]{f'} Y'$

We call a scheme X over field k complete if $X \to \operatorname{Spec} k$ proper.

examples: - $\mathbb{P}^n_k$ is complete

- $\mathbb{A}^n_k$ is of finite type, separated,
but $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not universally closed:

$$\left.\begin{tikzcd}
\mathbb{A}^2_k = \mathbb{A}^1_k \times_k \mathbb{A}^1_k \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\
\mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k
\end{tikzcd}\right\} \text{base change}$$

not closed!? $\{xy - 1 = 0\}$ closed but image is not.

Towards QCoh.

$(X, \mathcal{O}_X)$ ringed space.

A sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is a sheaf of ab. grps s.t.

(i) $\forall U \subset X$ open, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module

(ii) restrictions $\rho_{UV}: \mathcal{M}(U) \to \mathcal{M}(V)$ are morphisms of $\mathcal{O}_X(U)$-modules.

Consider $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules & $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$.

We have a hom-set $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ which we turn into a sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \doteq \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ not $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$

We give these sheaves a monoidal structure:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \doteq \left[U \rightsquigarrow \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \right]^{\#?}$$

Rmk. $x \in X$, $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \neq \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right)_x$

$\mathcal{F}$ is free if $\mathcal{F} \simeq \bigoplus_{i \in I} \mathcal{O}_X$, $|I| < \infty$

$\mathcal{F}$ is locally free if $\forall x \in X$ $\exists$ open nbhd $U \ni x$ s.t. $\mathcal{F}|_U = \bigoplus_{i \in I} \mathcal{O}_U$

For $X \xrightarrow{f} Y$, the pushforward construction gives a functor $f_*: \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}$.

→ f_* is left-exact.

- we also define $(f^{-1}\mathcal{G})(U) = \lim\limits_{V \supset f(U)} \mathcal{G}(V)$, + sheafification.

→ sheaf on X !

→ $f^{-1}: Sh_Y \to Sh_X$

① f^{-1} exact. (follows from $(f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$)

②
$$\begin{array}{ccc} X \times_Y \underline{\mathcal{Y}} & \to & \underline{\mathcal{Y}} \\ \downarrow & & \downarrow \\ X & \to & Y \end{array}$$

$Sh_X \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} Sh_Y$ are an adjoint pair of functors.

Pf. Construct $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ on opens, $\varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$

For

$$\begin{tikzcd} E \times_Y X \arrow[r] \arrow[d] & E \arrow[d, "\pi"'] \\ X \arrow[r, "f"] \arrow[u, bend left, "s'"] & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

where $\pi \circ s = id$,

$s'(x) = ((s \circ f)(x), x)$

$$\mathcal{G}(V) \ni s \longmapsto (f^{-1}\mathcal{G})(f^{-1}(V))$$
$$= (f_* f^{-1}\mathcal{G})(V). \quad \square$$

Bruzzo.

- $(X, \mathcal{O}_X)$ ringed space; $\mathcal{M}, \mathcal{N}$ $\mathcal{O}_X$-modules.
- $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ can be sheafified: $[U \rightsquigarrow \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)]^{G}$
- example: $\mathbb{Z}_X$ constant sheaf.
  - for U w/ ?? connected components, $\mathbb{Z}_X(U) \simeq \mathbb{Z} \oplus \mathbb{Z}$
    $\mathbb{Z}_X(U) \otimes_{\mathbb{Z}_X(U)} \mathbb{Z}_X(U) \simeq \mathbb{Z} \oplus \mathbb{Z}$

- stalks: $(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})_x = \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x$

- A ring, M A-module.
  - tensoring gives a right-exact functor:

$$\begin{aligned} A\text{-mod} &\xrightarrow{- \otimes_A M} A\text{-mod} \\ N &\longmapsto N \otimes_A M \end{aligned}$$

Def. M is a *flat* A-module if $- \otimes_A M$ is exact.
- ex. free modules are flat
- counterex. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$.
  - tensor with $- \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Note that $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \simeq \mathbb{Z}_2$
    $\Rightarrow$ $\mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{\sim} \mathbb{Z}_2 \to 0$, not flat.

- $(X, \mathcal{O}_X)$ loc. ringed space, $\mu_x \subset \mathcal{O}_{X,x}$ max. ideal
  $\Rightarrow$ $0 \to \mu_x \to \mathcal{O}_{X,x} \xrightarrow{ev} k(x) \to 0$
  - $k(x)$ is an $\mathcal{O}_{X,x}$-module.
  - tensor w $- \otimes_{\mathcal{O}_{X,x}} k(x)$:
    $$\mu_x \otimes_{\mathcal{O}_{X,x}} k(x) \to k(x) \xrightarrow{\sim} k(x) \to 0.$$
- for $X \xrightarrow{f} Y$, $Sh X \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} Sh Y$ form an adjoint pair
  left-ex. (f_*), exact (f^{-1})

- we extend to $\mathcal{O}_X\text{-mod} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathcal{O}_Y\text{-mod}$

- $\mathcal{F} \in \mathcal{O}_X\text{-mod}$, $(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$
  module over $\mathcal{O}_Y(U)$

- $f^* \mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

i) f is flat if $\forall x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(y)}$-module
ii) $\mathrm{Hom}_{\mathcal{O}_Y}(f_* \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f^* \mathcal{G})$

$\mathcal{F}(X) = \Gamma(X, \mathcal{F})$, $\Gamma : Sh_X \to Ab$ left exact
$\to H^i(X, \mathcal{F})$ right-derived functors of Γ

$X = \mathrm{Spec}\, A$, M A-module $\mid \rightsquigarrow$ sheaf $\tilde{M}$ on X
s.t. $\Gamma(X, \tilde{M}) \simeq M$
$\tilde{M}_p \simeq M_p$, $S_p^{-1} M = \frac{S_p \times M}{\sim} = M_p$

- trivial example is $\mathcal{O}_X$ itself.

- if $X = \mathrm{Spec}\, A$, an $\mathcal{O}_X$-module $\mathcal{F}$ is
  - quasi-coherent if $\mathcal{F} \simeq \tilde{M}$ for some A-module M
  - coherent if q.c. + finitely generated

- if X a scheme, demand above defs on affine open coverings
  $\to X = \bigcup_\alpha U_\alpha$, $\mathcal{F}|_{U_\alpha} \simeq \tilde{M}_\alpha$, ...

- define $j_! \mathcal{F} = \left[V \rightsquigarrow \begin{matrix} \mathcal{F}(V) & \text{if } V \subset U \\ 0 & \text{if not} \end{matrix} \right]$, $\mathcal{F}$ sheaf on top. sp. U.
- if $\mathcal{G}$ sheaf on X,
  $0 \to j_!(\mathcal{G}|_U) \to \mathcal{G} \to i_*(\mathcal{G}|_{X-U}) \to 0$, $i : X - U \to X$.

- X aff scheme, $U \subset X$ proper open : $\Gamma(X, j_! \mathcal{O}_U) = 0$, $(j_! \mathcal{O}_U)_x \simeq \mathcal{O}_{X,x}$
  - not quasi-coherent

- X integral noetherian scheme
  - $\exists!$ generic pt ξ. $\mathcal{O}_\xi$ is the fn. field $K(X)$ of X
    - $\to$ look at this as a constant sheaf
    - $\to$ q.c. but not coherent

$i: Y \hookrightarrow X$, $0 \to \mathcal{J}_Y \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0$

sheaf of ideals

- if $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I$, $I \subset A$ ideal,

then $\mathcal{J}_Y \simeq \tilde{I}$, $i_* \mathcal{O}_Y \simeq \widetilde{A/I}$ $\to$ q.c.

- A ring, $X = \operatorname{Spec} A$: (i) $\sim : A\text{-mod} \to \mathcal{O}_X\text{-mod}$

$$M \mapsto \tilde{M}$$

$\to$ exact & fully faithful

i.e. $\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$

(ii) $\widetilde{M \otimes_A N} \simeq \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$

$\widetilde{M \oplus N} \simeq \tilde{M} \oplus \tilde{N}$

(iii) $A \to B$. $N \in B\text{-mod}$, $\tilde{N} \in \mathcal{O}_{\operatorname{Spec} B}\text{-mod}$, $M \in (A \to B)$,..

then: $f_* \tilde{N} \simeq \tilde{N}_A$, $f^* \tilde{M} \simeq \widetilde{M \otimes_A B} \in \mathcal{O}_{\operatorname{Spec} B}\text{-mod}$

- $\mathcal{O}_X$-mod is an abelian cat.

Bruzzo.

Derived functors.

- suppose $F: \mathcal{A} \to \mathcal{B}$ left exact.
  - given $0 \to A' \to A \to A'' \to 0$ exact seq.

    we have $0 \to FA' \to FA \to FA'' \to$
    $\to R^1FA' \to R^1FA \to R^1FA''$
    $\to R^2FA' \to R^2FA \to R^2FA''$
    $\to \dots$ long exact. $(R^0F \simeq F)$
- suppose $(K^\bullet, d)$ cochain cplx
  - natural notion of cohomology, $H^n(K^\bullet, d) = \dfrac{\ker d: K^n \to K^{n+1}}{\operatorname{Im} d: K^{n-1} \to K^n}$.
- a morphism $f: K^\bullet \to L^\bullet$ induces

  a morphism $H(f): H(K) \to H(L)$,
  - explicitly, $\xi \in H^n(K)$, $\xi = [x] \Rightarrow H(f)(\xi) = [f(x)]$

- given a cochain exact seqn. $0 \to K' \xrightarrow{f} K \xrightarrow{g} K'' \to 0$,

  we get a long exact seqn

$$0 \to H^0(K') \xrightarrow{H^0(f)} H^0(K) \to H^0(K'')$$

$$\xrightarrow{\partial_0} H^1(K') \xrightarrow{H^1(f)} H^1(K) \to H^1(K'') \xrightarrow{\partial_1} \dots$$

where ∂_n is the connecting morphism

- let's define it.

  $\xi \in H^n(K'')$, $\partial \xi \in H^{n+1}(K')$?
  - $\xi = [x'']$, $g(x) = x''$ s.t. $g(dx) = d''g(x) = d''x'' = 0$
  - now $dx = f(x')$, $f(d'x') = d^2x = 0 \Rightarrow d'x' = 0$

  $\Rightarrow \partial_n(\xi) = [x']$.

- when do maps $(K^\bullet, d) \overset{f}{\underset{g}{\rightrightarrows}} (L^\bullet, d')$ induce the same cohomological morphism?
- define homotopy $s_{n+1}: K^{n+1} \to L^n$ $\forall n$,

i.e.

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r, "d_{n-1}"] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & K^n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & K^{n+1} \arrow[r] \arrow[dl, "s_{n+1}"'] \arrow[d, shift right] \arrow[d, shift left] & \cdots \arrow[dl] \\
\cdots \arrow[r] & L^{n-1} \arrow[r, "d'_{n-1}"] & L^n \arrow[r, "d'_n"] & L^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

s.t. $s_{n+1} \circ d_n - d'_{n-1} \circ s_n = f_n - g_n$, $\forall n$

- now $H(f) = H(g)$, since

$H(f)([x]) - H(g)([x])$

$= [f(x)] - [g(x)]$

$= [s \circ d(x) - d' \circ s(x)] = 0$.

- as a special case, $f = id$, $g = 0 \Rightarrow H^n(K) = 0$, $\forall n$.

- we say $(K^\bullet, d)$ and $(L^\bullet, d')$ are homotopy equivalent if $\exists$ morphisms $(K^\bullet, d) \underset{g}{\overset{f}{\rightleftarrows}} (L^\bullet, d')$ s.t. $g \circ f \sim id_K$, $f \circ g \sim id_L$.

Rmk. this means $H(g) \circ H(f) = H(f) \circ H(g) = id$, so $H(f) \cong H(g)$.

Abelian categories.

- $\mathcal{A} \in (Cat)_0$ is abelian if, for any $A, B \in \mathcal{A}_0$,
  1) $Hom(A, B)$ is an abelian group.
  ii) $\mathcal{A}$ has sums & products
  iii) $\mathcal{A}$ has (co)kernels and every mono is a kernel, epi is a cokernel

$$\begin{tikzcd}
& C \arrow[dl] \arrow[d, "g"] & \\
Ker f \arrow[r] & A \arrow[r, "f"] & B
\end{tikzcd}$$

Def. A **resolution** of $A \in \mathcal{A}$ is a pair $((L,d),\varepsilon)$
$\varepsilon : A \to L^0$ and the sequence
$$0 \to A \xrightarrow{\varepsilon} L^0 \xrightarrow{d_0} L^1 \xrightarrow{d_1} L^2 \xrightarrow{d_2} \cdots$$
is exact.

- now take $F : \mathcal{A} \to \mathcal{B}$ functor of ab. cats, additive & left-exact.
- take an *injective* resolution of $A \in \mathcal{A}_0$
$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$
and apply F to get the exact sequ.
$$FA \to FI^0 \to FI^1 \to FI^2 \to \cdots$$

Def. $H^n(FI^\bullet) =: R^n F A$ is the **n-th right derived functor of F.**

Rmk. Injective resolns aren't unique.

- what is an injective object?

Def. $I \in \mathcal{A}_0$ is **injective** if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is an exact functor.

Rmk. h_A is only left-exact for any $A \in \mathcal{A}_0$

- e.g. in (Ab), injectives are divisible groups, i.e. $G \in (Ab)_0$ s.t. $\forall g \in G \; \exists h$ s.t. $g = nh$
→ e.g. $\mathbb{Q}$.

Prop. TFAE:

i) $I \in \mathcal{A}$ is injective

ii)
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[dl, dashed, "g"] \\ & I & \end{tikzcd}$$
s.t. $g \circ f = id$

iii) every s.e.s. $0 \to I \to A \to B \to 0$ splits.

Def. $\mathcal{A}$ has enough injectives if every object can be embeded into an injective object, $0 \to A \dashrightarrow I$

- examples: Ab, A-mod, Sh_X, $\mathcal{O}_X$-mod
- note that in this case any object has an injective resolu

$$\begin{tikzcd} & & & 0 \arrow[d] & & \\ 0 \arrow[r] & A \arrow[r] & I_0 \arrow[r] \arrow[dr, dotted] & Q_0 \arrow[r] \arrow[d] & 0 & \\ & & & I_1 \arrow[d] \arrow[dr, dotted] & & \\ & & 0 \arrow[r] & Q_1 \arrow[r] \arrow[d] & I_2 \arrow[r] \arrow[dr, dotted] & \cdots \\ & & & 0 & & {} \end{tikzcd}$$

- note that for
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & L^0 \arrow[r] & L^1 \arrow[r] & \cdots \\ 0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd}$$

where $I^\bullet$ is injective ($L^\bullet$ not necessarily), f "lifts", ie $\exists\, g_n : L^n \to I^n$, $n = 0, 1, \ldots$.

These morphisms aren't unique, but any 2 such lifts f & g are homotopic.

Bruzzo.

- take $\mathcal{A}$ abelian cat, $I \in \mathcal{A}$
- equivalent conditions:
  - $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact
  - $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed] \\ & I & \end{tikzcd}$$
  - all exact sequences $0 \to I \to B \to C \to 0$ split

→ we call I an injective object

→ we say $\mathcal{A}$ has enough injectives if every object embeds into an injective object,

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to \cdots$$

- consider $F: \mathcal{A} \to \mathcal{B}$ additive & left exact,
  take $H^i(F(I^\bullet)) = R^iF(A)$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & L^0 \arrow[r] \arrow[d, dashed, "g_0"] & L^1 \arrow[r] \arrow[d, dashed, "g_1"] & \cdots \\ 0 \arrow[r] & B \arrow[r] & M^0 \arrow[r] & M^1 \arrow[r] & \cdots \end{tikzcd}$$

$\Rightarrow$ i) $f: A \to B$ lifts to $g: L^\bullet \to M^\bullet$

ii) g not unique, but all lifts homotopic

Pf (induction)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\varepsilon"] & B \arrow[r] & M^0 \\ & L^0 \arrow[urr, dashed, "g_0"'] & & \end{tikzcd}$$

, g_0 not necessarily unique.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varepsilon"] \arrow[d, "f"] & L^0 \arrow[r] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "h_0"] & L^1 \arrow[dl, dashed, "k_1"] \\ & B \arrow[r] & M^0 & \end{tikzcd}$$

, ...

- now consider

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] \arrow[d, shift right, "g_0"'] & I^1 \arrow[r] \arrow[d, shift right, "g_1"'] & \cdots \\ 0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] \arrow[u, shift right, "h_0"'] & J^1 \arrow[r] \arrow[u, shift right, "h_1"'] & \cdots \end{tikzcd}$$

$\Rightarrow$ $h \circ g \sim id_I$, $g \circ h \sim id_J$ $\Rightarrow F(I^\bullet) \sim F(J^\bullet) \Rightarrow H^i(F(I^\bullet))$ is $H^i(F(J^\bullet))$

Elementary properties.

1) $R^0 F \simeq F$

$\rightsquigarrow$ F left-exact: $0 \to F(A) \to F(I^0) \xrightarrow{F(d_0)} F(I^1)$ is exact

$\Rightarrow F(A) = \ker(F(d_0))$

$\Rightarrow H^0 F(A) = \ker(H(F(d_0))) \simeq \ker F(d_0) \simeq F(A)$

ii) if A injective, $R^i F(A) = 0$, $i > 0$

examples.

i) Sh_X sheaves of ab. groups on top space X

$\Rightarrow R^i \Gamma(X, \mathcal{F}) \equiv H^i(X, \mathcal{F})$

ii) $(X, \mathcal{O}_X)$. $\mathcal{O}_X$-mod has enough injectives $\Rightarrow H^i(X, \mathcal{F})$

iii) $Hom_R(M, -) : R\text{-mod} \to R\text{-mod}$

$\Rightarrow R^i Hom(M, N) = Ext^i(M, N)$

iv) $\mathcal{M} \in \mathcal{O}_X$-mod:

$Hom_{\mathcal{O}_X}(\mathcal{M}, -) : \mathcal{O}_X\text{-mod} \to Ab \rightsquigarrow Ext^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{V})$

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, -) : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod} \rightsquigarrow \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{V})$

- a Lie algebra L over a ring R is an R-module with a bilinear operation $[-,-] : L \times L \to L$
  $\to$ skew + Jacobi

Chevalley-Eilenberg cohomology

- direct definition.

$\xi \in C^p(L, M) = \Lambda^p L \otimes M$, $\rho : L \to \mathrm{End}_R(\overset{\overset{R\text{-module}}{\downarrow}}{M})$ representation

$$(d\xi)(x_1, \ldots, x_{p+1}) = \sum_{\substack{i,j=1 \\ i<j}}^{p+1} (-)^{i+j} \xi([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1})$$

- indirect definition.

$M^L = \{\text{invariants}\} = \{m \in M \mid \rho(x)(m) = 0 \ \forall x \in L\}$

Bruzzo.

Horse shoe lemma.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

, given $0\to A\to B\to C\to 0$ and the objects' resolutions

Pf. induction using the snake lemma.

Snake lemma.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker f \arrow[r, dashed] \arrow[d] & \ker g \arrow[r, dashed] \arrow[d] & \ker h \arrow[d] \arrow[dddll, dashed] & \\
 & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{Coker} f \arrow[r, dashed] \arrow[d] & \operatorname{Coker} g \arrow[r, dashed] \arrow[d] & \operatorname{Coker} h \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

, and $\dashrightarrow$ is exact.

- stopped writing for a bit.

- F-acyclic objects: $A\in\mathcal{A}$ is F-acyclic if $R^{>0}(F(A))=0$.
- a resolution $0\to A\to L^0\to L^1\to\cdots$ is an F-acyclic resolution if each L^i is F-acyclic

Thm. $0 \to A \to L^\bullet$ F-acyclic resol^n

$\Longrightarrow H^i(F(L^\bullet)) \xrightarrow{\sim} R^iF(A)$, $i \geq 0$, naturally.

Bruzzo.

Flabby sheaves (flasque)

- $\mathcal{F} \in Sh_X$, $V \overset{\text{open}}{\hookrightarrow} U$ any pair of opens
- $\mathcal{F}$ is flabby if $\mathcal{F}(U) \to \mathcal{F}(V)$ surjects

Examples

I) skyscrapers.

$\to G \in Ab$, $x \in X \Rightarrow G(x)(U) = \begin{cases} G & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$

II) if X irreducible, all constant sheaves are flabby

III) $\mathcal{F} \in Sh_X$, étalé space $\pi : \underline{\mathcal{F}} \to X$

$\to \mathcal{C}^0(\mathcal{F}) = \{ \underline{\text{all}} \text{ sections of } \pi \}$ is flabby

including noncontinuous

- every sheaf embeds into a flabby one:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{C}^0(\mathcal{F}) \arrow[r] \arrow[dr] & Q_0 \arrow[r] \arrow[d] & 0 \\
 & & & \mathcal{C}^1(\mathcal{F}) \arrow[d] \arrow[dr] & \\
 & & 0 \arrow[r] & Q_1 \arrow[r] & \mathcal{C}^2(\mathcal{F}) \arrow[dr] \\
 & & & & & \dots
\end{tikzcd}$$

Godement canonical flabby resolution

Thm $\mathcal{F}$ flabby $\Rightarrow H^i(X,\mathcal{F})=0, \forall i>0$.

Lemma. $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ s.e.s. of sheaves & $\mathcal{F}'$ flabby.
Then $0\to\mathcal{F}'(U)\to\mathcal{F}(U)\to\mathcal{F}''(U)\to 0$ exact.

Pf. (Godement)

- now Thm follows from $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to H^1(\mathcal{F}')$.

Lemma $\mathcal{F}',\mathcal{F}$ flabby $\Rightarrow$ quotient $\mathcal{F}''$ flabby

Pf.
$$\begin{array}{ccccccccc}
0 & \to & \mathcal{F}'(U) & \to & \mathcal{F}(U) & \to & \mathcal{F}''(U) & \to & 0\\
 & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}'(V) & \to & \mathcal{F}(V) & \to & \mathcal{F}''(V) & \to & 0
\end{array}$$

Lemma. Injective sheaves are flabby

Pf. $0\to\mathcal{I}\to\mathcal{C}^0(\mathcal{I})\to\mathcal{Q}\to 0 \Rightarrow \mathcal{C}^0(\mathcal{I})\simeq \mathcal{I}\oplus\mathcal{Q}$

flabby

Pf of Thm $0\to\mathcal{F}\to\mathcal{I}\to\mathcal{Q}\to 0$

$\mathcal{I}$ injective

$\Rightarrow 0\to H^0(X,\mathcal{F})\to H^0(X,\mathcal{I})\to H^0(X,\mathcal{Q})\to H^1(X,\mathcal{F})\to 0.$

$0\to H^1(X,\mathcal{Q})\to H^2(X,\mathcal{F})\to 0$

$\dots\ 0\to H^n(X,\mathcal{Q})\to H^{n+1}(X,\mathcal{F})\to 0$

$\to H^1(X,\mathcal{F})=0$ by 1^{st} lemma, $\mathcal{Q}$ flabby by 2^{nd}.

$\to$ induction

- A ring, $X = \operatorname{Spec} A$, M A-module $\Rightarrow \tilde{M} \in \mathcal{O}_X$-mod and $\tilde{M}_p = M_p$ i.e. quasi-coh.

Prop Quasi-coherent sheaves on the spectrum of a noetherian ring are acyclic.

Lemma. A noeth. ring, I injective A-module $\Rightarrow \tilde{I}$ flabby.

Pf of thm $\mathcal{F} \in Sh_{X = \operatorname{Spec} A}$, $\mathcal{F} \simeq \tilde{M}$.

$\Rightarrow M = \Gamma(X, \mathcal{F})$ A-module

$\rightarrow$ take resoln.:

$0 \to M \to I^0 \to I^1 \to \cdots$ inj. resoln

$\Rightarrow 0 \to \tilde{M} = \mathcal{F} \to \tilde{I}^0 \to \tilde{I}^1 \to \cdots$ flabby resoln

$\Rightarrow$ now $H^i(X, \mathcal{F}) \simeq H^i(\Gamma(X, \tilde{I}^\bullet)) \simeq H^i(I^\bullet) = 0, \; i > 0$

$H^0(X, \mathcal{F}) \simeq H^0(I^\bullet) \simeq M$.

Thm (Serre) X noetherian scheme. TFAE:

1) X is affine

ii) $H^i(X, \mathcal{F}) = 0 \;\; \forall i > 0 \;\; \forall \mathcal{F}$ quasi-coherent

iii) $H^i(X, \mathcal{F}) = 0 \;\; \forall$ coherent sheaves of ideals of $\mathcal{O}_X$

Čech cohomology review:

- $(X, \mathcal{F}, \mathfrak{U})$, $C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p})$, ...

- if $\mathcal{F}$ sheaf, $H^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$

Thm X noetherian separated scheme,
$\mathcal{F}$ quasi-coherent sheaf on X,
$\mathfrak{U}$ open cover of affine sets.
Then $H^p(\mathfrak{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$ $\forall p \geq 0$.

$j_{i_0 \dots i_p} : U_{i_0 \dots i_p} \hookrightarrow X$. $\check{\mathcal{C}}(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_p} (j_{i_0 \dots i_p, *} \mathcal{F}|_{U_{i_0 \dots i_p}})$

→ these are sheaves

→ exact in positive degree

→ $\exists$ $\epsilon : \mathcal{F} \to \check{\mathcal{C}}^0(\mathfrak{U}, \mathcal{F})$

→ $\check{\mathcal{C}}^0(\mathfrak{U}, \mathcal{F}) = \prod_k j_{k,*} \mathcal{F}|_{U_k}$

$$j_{k,*} \mathcal{F}|_{U_k}(U) = \mathcal{F}|_{U_k}(i_k^{-1}(U)) = \mathcal{F}(U \cap U_k)$$

$$\epsilon(s) = \prod s(U \cap U_k)$$

→ $0 \to \mathcal{F} \to \check{\mathcal{C}}^\bullet(\mathfrak{U}, \mathcal{F})$ is a resoln (Čech resoln)

→ $\Gamma(X, \check{\mathcal{C}}(\mathfrak{U}, \mathcal{F})) \simeq C^p(\mathfrak{U}, \mathcal{F})$

Bruzzo

Lie algebra cohomology

- L Lie algebra over R, M rep. of L

$M \otimes_R \Lambda^\bullet L^\vee \qquad \mathrm{Hom}_R(\Lambda^\bullet L, M)$

$$\xi \in C^p(L,M),\quad d\xi(x_1,\dots,x_{p+1}) = \sum_{i<j} (-)^{i+j}\, \xi([x_i,x_j], \overset{\text{omit } x_i, x_j}{\dots}, x_p) + \sum (-)^{i-1} S(x_i)\,\xi(x_1,\dots,\hat{x}_i,\dots,x_p)$$

- X scheme, noetherian separated, $\mathcal{F}$ q.c. sheaf on X, $\mathcal{U}$ open affine cover

$$H^p(\mathcal{U},\mathcal{F}) \xrightarrow{\varphi} H^p(X,\mathcal{F})$$

$$\begin{array}{ccccccc} 0 & \to & \mathcal{F} & \to & \check{\mathcal{C}}^0(\mathcal{U},\mathcal{F}) & \to & \cdots \\ & & \| & & \downarrow & & \\ 0 & \to & \mathcal{F} & \to & \mathcal{Y}^0 & \to & \cdots \end{array}$$

Lemma. X noeth scheme, $\mathcal{F} \in \mathrm{QCoh}_X$.
$\mathcal{F}$ embeds into a flabby q.c. sheaf.

Pf. $\mathcal{U} = \{U_i\}$ aff open cover, $\mathcal{F}|_{U_i} = \tilde{M}_i$, M_i an A_i-module (where $X|_{U_i} = \mathrm{Spec}\, A_i$).
Put $0 \to M_i \to I_i \Rightarrow 0 \to \tilde{M}_i \to \tilde{I}_i$
Denote $f_i : U_i \hookrightarrow X$ and put $\mathcal{G} := \oplus f_{i*}(\tilde{I}_i)$.
For $s \in \mathcal{F}(U)$, $s|_{U \cap U_i} \in \mathcal{F}(U_i \cap U) \longmapsto f_{i*}(\tilde{I}_i)$.
So by construction, $\mathcal{F} \hookrightarrow \mathcal{G}$, and $\mathcal{G}$ is flabby, q.c.

- so now $0 \to \mathcal{F} \to \overset{\text{q.c. flabby}}{\mathcal{G}} \to \overset{\text{q.c}}{\mathcal{R}} \to 0$

$\Rightarrow 0 \to \mathcal{F}(U_{i_0 \dots i_p}) \to \mathcal{G}(U_{i_0 \dots i_p}) \to \mathcal{R}(U_{i_0 \dots i_p}) \to H^1(U_{i_0 \dots i_p}, \mathcal{F}) = 0$

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{R}) \to 0$$

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, "\wr"] & H^0(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d, "\wr"] & H^0(\mathcal{U}, \mathcal{R}) \arrow[r] \arrow[d, "\wr"] & H^1(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & H^1(X, \mathcal{F}) \arrow[r] & H^0(X, \mathcal{G}) \arrow[r] & H^0(X, \mathcal{R}) \arrow[r] & H^1(X, \mathcal{F}) \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow \psi$ iso $\Rightarrow H^1(\mathcal{U}, \mathcal{F}) \simeq H^1(X, \mathcal{F})$

and
$$\begin{tikzcd} 0 \arrow[r] & H^p(\mathcal{U}, \mathcal{R}) \arrow[r, "\sim"] \arrow[d, "\wr"] & H^{p+1}(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & H^p(X, \mathcal{R}) \arrow[r, "\sim"] & H^{p+1}(X, \mathcal{F}) \arrow[r] & 0 \end{tikzcd} \qquad p \geq 1$$

Thm. (Leray) $X \in \mathrm{Top}$, $\mathcal{F} \in \mathrm{Sh}_X$, $\mathcal{U} = \{U_i\}$ open cover and $H^q(U_{i_0 \dots i_p}, \mathcal{F}) = 0$ $\forall q > 0$, for all intersections.
Then $H^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$, $p \geq 0$.

- put $\check{H}^p(X, \mathcal{F}) := \underset{\overleftarrow{\{\mathcal{U}\}}}{\lim} H^p(\mathcal{U}, \mathcal{F})$ where the covers are ordered by refinement

$$\Rightarrow \begin{tikzcd} H^p(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & H^p(X, \mathcal{F}) \\ \check{H}^p(X, \mathcal{F}) \arrow[ur, dashed, "\psi_p"'] & \end{tikzcd}$$

Thm If X paracompact, ψ_p iso $\forall p$.

Def. An open cover is *locally finite* if every pt of X has a nbhd intersecting only a finite # of sets $\in \mathcal{U}$.

Def. X *paracompact* if Hausdorff & every open cover admits a loc. fin. refin.

- given $s \in \mathcal{F}(X)$, put $\mathrm{supp}(s) = \{x \in X \mid s_x \neq 0\}$ (closed)
- take sheaf of rings R on X
- a partition of unity subordinated to a loc. fin. open cover $\mathcal{U} = \{U_i\}_{i \in I}$ is a collection $\{s_i \in R(X)\}_{i \in I}$ s.t. i) $\mathrm{supp}(s_i) \subset U_i$, ii) $\sum_{i \in I} s_i = 1$.

Def. R is fine if, $\forall$ loc. fin. open cover $\mathcal{U}$, there is a partition of unity $\underbrace{\ }_{\text{of } R}$ subordinated to $\mathcal{U}$.

- e.g. diff. mfds, paracpt. top. sp.

- M R-module, R fine, $\mathcal{U}$ f.o.c.
  $\Rightarrow H^p(\mathcal{U}, R) = 0$, $p > 0$

Pf. take partition of unity $\{s_i\}$, $\alpha \in C^q(\mathcal{U}, M)$, $\alpha = \{\alpha_{i_0 \dots i_q}\}$.
Define $(K\alpha)_{i_0 \dots i_{q-1}} := \sum_{k=0}^{q} (-)^k \sum_{i_{k-1} < j < i_k} s_j \alpha_{i_0 \dots j \dots i_{q-1}}$

so $K : C^q(\mathcal{U}, M) \to C^{q-1}(\mathcal{U}, M)$.
Now show $K \circ \delta + \delta \circ K = \mathrm{id} - 0$, i.e. the cohomology is homotopic to that of a point.

Thm If M a module over a fine sheaf of rings over a paracpt. space $X \Rightarrow \check{H}^p(X, M) = 0$, $p > 0$.

- note that for $0 \to \mathbb{R} \to \Omega_X^0 \to \Omega_X^1 \to \dots$, every Ω_X^p is a C_X^∞-module, i.e. it is fine
  $\Rightarrow H^p(X, \Omega_X^q) = 0$, $\forall p > 0$, $\forall q \geq 0$ $\Rightarrow H^p(\mathbb{R}, X) \simeq H^p_{dR}(X)$
- if X, Y homotopic diff mfd $\Rightarrow H^p_{dR}(X) \simeq H^p_{dR}(Y)$

Mayer-Vietoris sequence

- $X = U \cup V$.

$$0 \to H^0(\mathcal{U}, \Omega^k) \to C^0(\mathcal{U}, \Omega^k) \to C^1(\mathcal{U}, \Omega^k) \to H^1(\mathcal{U}, \Omega^k) \to 0$$

$$0 \to \Omega^\bullet(X) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V) \to 0$$

$$\Rightarrow \quad 0 \to H^0_{dR}(X) \to H^0_{dR}(U) \oplus H^0_{dR}(V) \to H^0_{dR}(U \cap V)$$

$$\to H^1_{dR}(X) \to \dots \qquad \text{MV sequence}$$

Bruzzo

Divisors

- natural notions:
  - Weil - codim 1 subschemes, $D = \sum_n \alpha_n D_n$, $\alpha_n \in \mathbb{Z}$
  - Cartier ⇝ line bdles

Dimension of a scheme X:

- supremum n of integers such that $\exists$ a chain of nested closed irred. subsets $Z_0 \subset \cdots \subset Z_n = X$
- while $\operatorname{codim}_X Z$ is supremum of $Z = Z_0 \subset \cdots \subset Z_n = X$
- if X irred, of fin. type over a field, Z irred,
  $\operatorname{codim}_X Z + \dim Z = \dim X$

- A ring, $p \in \operatorname{Spec} A$. Define the height of p as the supremum of n s.t. $p_0 \subset \cdots \subset p_n = p$.
  $\to \dim A$ = supremum of all heights (Krull dimension)

- now suppose A local ring, m max ideal
  $\to$ consider $\dim_{\mathbb{K}} m/m^2$

Def. A local ring A is regular if $\dim A = \dim_{\mathbb{K}} m/m^2$.

Def. A scheme is regular if all local rings are regular. It is regular in codimension 1 if all local rings of dimension 1 are regular.

- the last property, $\dim \mathcal{O}_{X,p} = 1$, has an important consequence on the structure of singularities.
  $\to$ p ⇝ hypersurface X_p, so this means its generic pt is regular $\iff$ regular pts lie on dense opens
  $\Rightarrow \operatorname{codim}_X \operatorname{Sing}(X) \geq 2$.
  $\to$ so divisors, codim 1 objects, play nice

- take field K. A map $v: K^* \to \mathbb{Z}$ such that
  1) $v(xx') = v(x) + v(x')$
  2) $v(x+x') \geq \min(v(x), v(x'))$

is called a discrete valuation.

$R = \{x \in K^* \mid v(x) \geq 0\} \cup \{0\}$ is called the valuation ring of (K, v).

$\to$ it is in fact local with $m = \{x \in K^* \mid v(x) > 0\} \cup \{0\}$

Lemma. Set A noetherian local int. domain of dimension 1. TFAE

1) A regular
2) A valuation ring of some field

- example: $D = \operatorname{Spec} \frac{k[x,y]}{(x)}$. D integral, ξ generic pt.,
  $j: D \hookrightarrow \mathbb{A}^2_k$, $j(\xi) = D$. $A = \mathcal{O}_{\mathbb{A}^2_k, j(\xi)} = k[x,y]_{(x)} = \left\{ \frac{P(x,y)}{Q(x,y)} \,\middle|\, x \nmid Q \right\}$
  $\to v: K(\mathbb{A}^2_k) \to \mathbb{Z}$, $v|_A\left(\frac{P}{Q}\right) = m$ if $P(x,y) = x^m f(x,y)$.

- now assume X integral, separated, noetherian, regular in codim 1 scheme.

Def. A prime Weil divisor is a closed integral subscheme of codim 1.
$\operatorname{Div}(X)$ = free group generated by prime divisors over $\mathbb{Z}$

- since D integral, it has a unique generic pt ξ, $\mathcal{O}_{X, j(\xi)}$ is a discrete valuation ring for the field of rational functions

Lemma. If $f \in K^*$, $v_D(f) \neq 0$ for a finite # of pts.

→ we construct principal divisors

$$(f) := \sum_{\substack{\text{prime divisors} \\ Y \subseteq X}} v_Y(f)\, Y \in \operatorname{Div} X$$

→ linear equivalence $D_1 \sim D_2$ if $D_1 - D_2 = (f)$ for some $f \in K$

→ $\operatorname{Div}(X)/\sim \; = \operatorname{Cl}(X)$

→ recall that an element of a ring is irreducible if $y = xz$ means x or z unit.

→ a UFD is a domain s.t. $x = x_1 \cdots x_n$, with each x_i irreducible, uniquely up to units.

Lemma. If A UFD & $X = \operatorname{Spec} A \Rightarrow \operatorname{Cl}(X) = 0$

Lemma. i) X as usual, Z proper closed irred. subscheme, $\operatorname{codim}_X Z = 1$, $U = X - Z$.

We have a map $\operatorname{Div}(X) \longrightarrow \operatorname{Div}(U)$

$$\sum a_i Y_i \longmapsto \sum a_i (Y_i \cap U)$$

and $\mathbb{Z} \to \operatorname{Cl}(X) \to \operatorname{Cl}(U) \to 0$ is exact.

ii) $\operatorname{codim}_X Z \geq 2 \Rightarrow \operatorname{Cl}(X) \xrightarrow{\sim} \operatorname{Cl}(U)$

Bruzzo.

- the proof that $Cl(\text{double cone}) \simeq \mathbb{Z}_2$ was fleshed out → unfortunately I arrived late.

Cartier divisors.

- X integral, $U \subseteq X$ nonempty open
- $\mathcal{K}_X$ (constant) sheaf of rational functions, $\mathcal{K}_X^* \subset \mathcal{K}_X$ the nonzero —"—
  → $\mathcal{O}_X$ and $\mathcal{O}_X^*$ are their respective subsheaves
→ Cartier divisors are elements of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$
  - for $\{U_i\}_i$ cover of X, these are given by $\{(U_i, f_i)\}$ with $f_i/f_j \in \mathcal{O}_X^*(U_{ij})$

- noting that $1 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 1$ s.e.s.:

Def. A Cartier divisor is principal if it lies in the image of $\mathcal{K}_X^* \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$
Put $CaCl(X) = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)/\sim$

Def. A scheme is locally factorial if all loc. rings UFD.

Thm. Let X integral separated noetherian loc. fact. scheme. Then $\exists$ a 1-1 correspondence between Weil & Cartier divisors.

Pf. Let $D \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ be represented by $\{(U_i, f_i)\}$ so write $D = \sum v_Y(f_i)\, Y$.
Now take $D \in Div(X)$, $x \in X$. We are interested in local behaviour, so take $X = Spec\, A$.
Write $\mathcal{O}_{X,x} =: A_x$. $A \to A_x^{UFD} \Rightarrow Spec\, A_x \to Spec\, A$.

Now look at
$$\begin{array}{ccc} D_x & \longrightarrow & \operatorname{Spec} A_x \\ \downarrow & & \downarrow \\ D_x & \longrightarrow & \operatorname{Spec} A \end{array}$$
. $D_x = (f_x)$.

- now look at $(X, \mathcal{O}_X)$ ringed space, X loc. of rk r.
- let $\mathcal{L} \in \mathcal{O}_X$-mod.
- we say $\mathcal{L}$ is free if $\mathcal{L} \simeq \overbrace{\mathcal{O}_X \otimes \cdots \otimes \mathcal{O}_X}^{r}$, and $\mathcal{L}$ is locally free if every pt $\in X$ has a nbhd U s.t. $\mathcal{L}(U)$ is free of rank r over $\mathcal{O}_X(U)$.
- if $rk = 1$, we call locally free $\mathcal{O}_X$-modules line bdls or invertible sheaves.

$\to$ given $\mathcal{L}_1, \mathcal{L}_2$ line bdls, so is $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$

$\to$ given $\mathcal{L}$, $\exists\, \mathcal{L}^{-1}$ s.t. $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$

$\to$ given open cover $\mathfrak{U} = \{U_i\}$, $\mathcal{L}(U_i) \simeq \mathcal{O}_X(U_i)$ and $g_{ij} \in \mathcal{O}_X^*(U_{ij})$ transition functions,, $g_{ij} g_{jk} g_{ki} = 1$, which is just the Čech 1-cocycle condition written in multiplikative notation.

$\Rightarrow \operatorname{Pic}(X) = H^1(\mathfrak{U}, \mathcal{O}_X^*) \simeq H^1(X, \mathcal{O}_X^*)$, noting that iso. line bdles are related by coboundaries

- back to divisors.

$\to$ take Cartier div. $\{(U_i, f_i)\}$ and let $f_i/f_j \in \mathcal{O}_X^*(U_{ij})$ be transition fns.

$\rightsquigarrow$ line bdle $\mathcal{O}_X(D)$.

$\to$ subsheaf of K_X generated by f_i^{-1} on U_i over $\mathcal{O}_{U_i}$

Props. i) $\mathcal{O}_X(D)$ line bundle & 1-1 corr. $D \Leftrightarrow \mathcal{O}_X(D)$

ii) $D_1 \sim D_2 \iff \mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$. This gives $\operatorname{CaCl}(X) \to \operatorname{Pic}(X)$.

iii) $\mathcal{O}_X(D_1 - D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$

Bruzzo.

- S graded ring, $S = \bigoplus_{d \in \mathbb{N}} S_d$, $S_i S_j \subseteq S_{i+j}$,
  S generated by S_0, S_1
- we also defined $\mathrm{Proj}\, S$, $S_{(f)} = (S_f)_{(0)}$ for $f \in S$
  and $D_+(f) = \mathrm{Spec}\, S_{(f)}$
- M graded S-module, $\tilde{M}$ an $\mathcal{O}_X$-module.
- $S(n) = \bigoplus_{d \geq n} S_d$, $\widetilde{S(n)} =: \mathcal{O}_X(n)$
  - fact: $\mathcal{O}_X(n)$ is locally free of rank 1.
    - to check: i) $\mathcal{O}_X(n)|_{D_+(f)} \simeq \widetilde{S(n)_{(f)}}$
      ii) $S(n)_{(f)} \simeq S_{(f)}$
  - e.g. $X = \mathrm{Proj}\, \mathbb{k}[x_0, \dots, x_n] = \mathbb{P}^n_{\mathbb{k}}$, $\overline{\mathbb{k}} = \mathbb{k}$
    $\to U_i := D_+(x_i)$
    $\Rightarrow \mathcal{O}_X(1)|_{U_i}$ generated by x_i
    so $g_{ij} \in \mathcal{O}_X^*(U_i \times_X U_j)$, $g_{ij} = \frac{x_j}{x_i}$

- look at $H_i = \{x_i = 0\}$ prime divisors, $i = 0, \dots, n$,
  but not principal since rational fns have 0 degree.
  $\to H_i - H_j = \left(\frac{x_i}{x_j}\right) \Rightarrow H_i \sim H_j \Rightarrow$ write $H = [H_i]$ in $Cl(\mathbb{P}^n_{\mathbb{k}})$
  - recall $D = \sum n_i Y_i$ divisor where $\{Y_i\}$ integral
    codimension 1 subschemes, $\deg D = \sum n_i \deg D_i$

Prop. a) $[D] = dH$, $d = \deg(D)$
b) $\deg(f) = 0 \quad \forall f \in K^*$
c) $\deg : Cl(\mathbb{P}^n_{\mathbb{k}}) \to \mathbb{Z}$ iso

Pf. a) $D = D_1 - D_2$, $D_{1,2}$ effective, $D_{1,2} = (g_{1,2})$
Since irred. hypersurfaces in $\mathbb{P}$ correspond to prime ideals
of height 1, principal. So $D - dH = \left(\frac{g_1}{x_0^d g_2}\right) = 0$, $d = \deg D_1 - \deg D_2$

- $\mathcal{O}_{\mathbb{P}^n_k}(H) = \mathcal{O}_{\mathbb{P}^n_k}(1)$, $\mathcal{O}_{\mathbb{P}^n_k}(D) = \mathcal{O}_{\mathbb{P}^n_k}(d)$, $d = \deg D$,
$\mathcal{O}_{\mathbb{P}^n_k}(-1) = \mathcal{O}_{\mathbb{P}^n_k}(1)^\vee$

Kähler differentials

- set A ring, B A-algebra, M B-module,
A, B commutative with unit

Def. An *A-derivation of B into M* is a map
$d: B \to M$ satisfying
i) $d(bb') = b\,db' + b'\,db$, $b, b' \in B$
ii) $da = 0$, $a \in A$

Def. A module of relative differentials $\Omega_{B/A}$
is a B-module with an A-derivation $d: B \to \Omega_{B/A}$
such that any $d': B \to M$ factors through $\Omega_{B/A}$
uniquely.

- construct it as $\Omega_{B/A} = B\langle db \rangle / \tilde{\Omega}_{B/A}$,
$\tilde{\Omega}_{B/A}$ generated by $d(b+b') - db - db'$, $d(bb') - b\,db' - b'\,db$,
da, put $f(db) = d'b$.

- now consider $B \otimes_A B \xrightarrow{m} B$
$\to$ obviously surjective ($1 \otimes b \mapsto b$):
$$0 \to I \to B \otimes_A B \xrightarrow{m} B \to 0$$
Claim: now $(I/I^2, d)$, $d: B \to I/I^2$ will be
a module of relative differentials
- $d: B \xrightarrow{\tilde{d}} I \twoheadrightarrow I/I^2$, $b \xmapsto{\tilde{d}} 1 \otimes b - b \otimes 1$
- ...

Example: M diff. mfd, $p \in M$, $m_p \subset C^\infty(M)$
$\Rightarrow m_p / m_p^2 \simeq T_p^* M$
$\rightarrow A \rightsquigarrow \mathbb{R}$, $B \rightsquigarrow C_p^\infty(M)$, $M \rightsquigarrow \mathbb{R} \Rightarrow \mathrm{Der}_{\mathbb{R}}(C_p^\infty, \mathbb{R}) \simeq T_p M$

Example: $f: X \to Y$ morphism of schemes
- not separated in general, but $\Delta(X)$ is locally a closed $\rightsquigarrow$ sheaf of ideals $\mathcal{J}$
$\Rightarrow \Delta^* \mathcal{J}/\mathcal{J}^2 \simeq \Omega_{X/Y}$

Example: $B = A[x_1, \dots, x_n] \rightsquigarrow \Omega_{B/A}$ free of rank n, generated by $dx_1, \dots, dx_n$
$\mathrm{Der}_A(B, B) \ni \frac{\partial}{\partial x_j} : x_i \longmapsto \delta_{ij}$
$\exists f_j : \Omega_{B/A} \to B$, $f_j(dx_i) = \delta_{ij}$
$0 = \sum P_i \, dx_i \Rightarrow f_j(\sum P_i \, dx_i) = P_j = 0$

Properties.

- A', B A-algebras, let $B' = A' \otimes_A B$
  Then $\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_B B'$.
  $\Omega_{B/A} \otimes_B B' \to \Omega_{B'/A'}$ given by $db \otimes 1 \mapsto d'(b \otimes 1)$
- B A-algebra, $I \subset B$ ideal, $C = B/I$. $\exists$ maps:
  i) $\Omega_{B/A} \otimes_B C \to \Omega_{C/A}$
  $db \otimes 1 \mapsto \bar{d}[b]$
  ii) $\delta : I/I^2 \to \Omega_{B/A} \otimes_B C$
  $[b] \longmapsto db \otimes 1$

  The sqn $I/I^2 \to \Omega_{B/A} \otimes C \to \Omega_{C/A} \to 0$ is exact.

Bruzzo.

- A c. ring w/ unity, B A-algebra, $I \subset B$ ideal
  $C = B/I$
- $I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \to 0$ exact,
  $\delta[b] = db \otimes 1$, $\alpha(db \otimes 1) = d[b]$

Corollary. If B fin. gen. over A, $\Omega_{B/A}$ fin. gen. over B.

Pf. Use that α is injective with $C \twoheadrightarrow B = \frac{A[x_1, \dots, x_n]}{I}$:
$B \twoheadrightarrow A[x_1, \dots, x_n]$, so that
$\Omega_{A[x_1, \dots, x_n]/A} \otimes B \to \Omega_{B/A} \to 0$

Example $A = \mathbb{k}$, $\bar{B}$ $\mathbb{k}$-alg, $X = \operatorname{Spec} \bar{B}$
$B = \mathcal{O}_{X,x}$, $x \in X$ is a $\mathbb{k}$-alg, $\mathcal{O}_{X,x} = \bar{B}_x$

- Suppose we are given rings A, B, C,
  $A \to B \xrightarrow{\chi} C$, not necessarily exact
- we get $\Omega_{B/A} \otimes_B C \xrightarrow{\psi} \Omega_{C/A} \xrightarrow{\varphi} \Omega_{C/B}$
  with $\varphi : dc \mapsto dc$, $\psi : db \otimes 1 \to d\chi(b)$
  $\to$ so it's exact and moreover φ surjects.
- assume B ring, maximal ideal m, $\mathbb{k} = B/m$

Prop Assume $\mathbb{k} \to B$ injects.
Then $\delta : m/m^2 \to \Omega_{B/\mathbb{k}} \otimes_B \mathbb{k}$ is an isomorphism

Pf. Use sequence before with $C \twoheadrightarrow \mathbb{k}$ to get surjectivity of δ. Look at
$\delta^* : \operatorname{Hom}_{\mathbb{k}}(\Omega_{B/\mathbb{k}} \otimes_B \mathbb{k}, \mathbb{k}) \to \operatorname{Hom}_{\mathbb{k}}(m/m^2, \mathbb{k})$
is
$\operatorname{Hom}_{\mathbb{k}}(\Omega_{B/\mathbb{k}}, \mathbb{k}) \simeq \operatorname{Der}_{\mathbb{k}}(B, \mathbb{k})$

We claim δ^* surjects. $0 \to m \to B \rightleftarrows \mathbb{k} \to 0$,
so pick $\underset{B}{b} = \underset{\mathbb{k}}{\lambda} + \underset{m}{c}$.

Now let $\varphi \in \mathrm{Hom}(m/m^2, \mathbb{k})$ and put $d\, b = \varphi([c])$, and check that $\delta^* \circ d = \varphi$ and d derivation.

Thm. B local ring. Assume

i) $\mathbb{k} = B/m \hookrightarrow B$

ii) $\mathbb{k} = \overline{\mathbb{k}}$, $\mathrm{char}\, \mathbb{k} = 0$ (more generally, $\mathbb{k}$ perfect)

iii) B is the localization of a fin. gen. $\mathbb{k}$-alg.

Then B is regular iff $\Omega_{B/\mathbb{k}}$ free of rank $= \dim B$.

- we had $\Omega_{X/Y} = \Delta^* \mathcal{J}/\mathcal{J}^2$, where $\mathcal{J}$ ideal sheaf of $\Delta(X)$.

$\rightsquigarrow$ Put $Y = \mathrm{Spec}\, A$, $X = \mathrm{Spec}\, B$, $X \times_Y X = \mathrm{Spec}\, B \otimes_A B$

Properties

- compatibility w base change

$\rightsquigarrow$ $g: Y \to Y'$, $X' = X \times_Y Y' \xrightarrow{g'} X$ $\Rightarrow \Omega_{X'/Y'} \simeq g'^* \Omega_{X/Y}$

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

- $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow f^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$
- $X \xrightarrow{f} Y$, $Z \subset X$ closed subscheme w ideal sheaf $\mathcal{J}$

$Z \overset{j}{\hookrightarrow} X \to Y$, $\mathcal{O}_X \to j_* \mathcal{O}_Z$,

$\mathcal{J}/\mathcal{J}^2 \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \Omega_{Z/Y} \to 0$

Def. A variety over $\mathbb{k}$ is an irreducible separated scheme of finite type over any closed field $\mathbb{k}$.

Thm. A variety X over $\mathbb{k}$ is regular (smooth) iff $\Omega_{X/\mathbb{k}}$ is locally free of $rk = \dim X$.

- suppose X smooth variety over $\mathbb{k}$, $Y \subset X$ irred. closed subscheme
  → then Y is nonsingular iff
  i) $\Omega_{Y/\mathbb{k}}$ locally free
  ii) $0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{X/\mathbb{k}} \otimes \mathcal{O}_Y \to \Omega_{Y/\mathbb{k}} \to 0$ is exact
- when this happens, $\mathcal{J}/\mathcal{J}^2$ is locally free of $rk = \operatorname{codim}_X Y$