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Introductory notes on

Mechanics and Symmetry

Notes from a mathematical approach

Contents

Chapter 1

Hamiltonian Systems on Linear Symplectic Spaces

1.1 Introduction

Newton's second law for a particle moving in Euclidean three-dimensional space $\mathbb{R}^3$, under the influence of a *potential energy* $V(q)$, is

$$F = ma,$$

where $q \in \mathbb{R}^3$, $F(q) = -\nabla V(q)$ is the *force*, m is the mass of the particle, and

$$a = \frac{d^2 q}{dt^2} = \ddot{q}$$

is the acceleration (assuming we start in a postulated privileged coordinate frame called an *inertial frame*. The potential energy V is introduced through the notion of work and the assumption that the force field is conservative. The introduction of the *kinetic energy*

$$K = \frac{1}{2} m \left\| \dot{q} \right\|^2$$

is through the *power*, or *rate of work equation*:

$$\frac{dK}{dt} = m\langle \dot{q}, \ddot{q} \rangle = \langle \dot{q}, F \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the innter product on $\mathbb{R}^3$.

The *Lagrangian* associated to this system is defined by

$$L(q^i, \dot{q}^i) = \frac{m}{2} \|\dot{q}\|^2 - V(q)$$

and one can check that the Newton's second law is equivalent to the *Euler-Lagrange equations*:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0,$$

In fact, it follows from the definition of the Lagrangian that

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = m \ddot{q}_i - \frac{\partial V}{\partial q_i} = (ma + \nabla V)_i.$$

The Euler-Lagrange equation is a second-order differential equation in q. Those equations are worthy of independent study for a general L, they are the equations for stationary values of the *action integral*:

$$\delta \int_{t_1}^{t_2} L(q^i, \dot{q}^i) dt = 0,$$

as will be detailed later. These *varitational principles* play a fundamental role throughout mechanics (both in particle mechanics and field theory).

A simple computation shows that $dE/dt = 0$, where E is the *total energy*

$$E = \frac{!}{2} m \|\dot{q}\|^2 + V(q).$$

Lagrange and Hamilton observed that it is